1 Prompt

Show that $\sum_{i=1}^{n} \frac{i}{2^i} < 2$. (Hint: Try to bound this sum term by term with a geometric progression.)

2 Discussion

Running the script written using **Horner's method** (see ex 3.50) shows that this sum rapidly converges to 2 from below. It hits the limits of double float precision somewhere between $n = 32$ and $n = 64$.

Unfortunately, although **Horner's method** allows us to efficiently demonstrate the relationship empirically in $O(n)$ time, it offers little insight into how to rigorously prove the relationship (at least I wasn't inspired by it). However, based on a $O(n^2)$ algorithm, we may be inspired to start with the following observations:

$$\sum_{i=1}^{n} \frac{i}{2^i} = \sum_{k=1}^{n} \sum_{i=k}^{n} \frac{1}{2^i}$$

$$\sum_{i=k}^{n} \frac{1}{2^i} = \sum_{i=0}^{n} \frac{1}{2^i} - \sum_{i=0}^{k-1} \frac{1}{2^i}$$

By **Proposition 3.5** on page 121:

$$\sum_{i=0}^{n} \frac{1}{2^i} - \sum_{i=0}^{k-1} \frac{1}{2^i} = \frac{\frac{1}{2}^{n+1} - \frac{1}{2}^{k}}{\frac{1}{2} - 1} = \frac{1}{2}^{k-1} - \frac{1}{2}^{n} < \frac{1}{2}^{k-1}$$

Therefore:

$$\sum_{k=1}^{n} \sum_{i=k}^{n} \frac{1}{2^i} < \sum_{k=1}^{n} \frac{1}{2}^{k-1} = \sum_{k=0}^{n-1} \frac{1}{2}^{k}$$

Apply **Proposition 3.5** on page 121 again:

$$\sum_{k=1}^{n} \sum_{i=k}^{n} \frac{1}{2^i} < \frac{\frac{1}{2}^{n} - 1}{\frac{1}{2} - 1} = 2 - \frac{1}{2}^{n-1} < 2$$

Hence, $\sum_{i=1}^{n} \frac{i}{2^i} < 2$. Awareness of a non-optimal algorithm will sometimes help you write a proof.