INTEGRAL COHOMOLOGY OF $K_2(A)$

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Abstract. What we know already

1. Preliminaries

Definition 1.1. Let n be a natural number. A partition of n is a decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_k),\ \lambda_1 \geq \ldots \geq \lambda_k > 0$ of natural numbers such that $\sum_i \lambda_i = n$. Sometimes it is convenient to write $\lambda = (\ldots, 2^{m_2}, 1^{m_1})$ with multiplicities in the exponent. We define the weight $\|\lambda\| := \sum m_i i = n$ and the length $|\lambda| := \sum_i m_i = k$. We also define $z_\lambda := \prod_i i^{m_i} m_i!$.

Definition 1.2. Let $\Lambda_n := \mathbb{Q}[x_1, \ldots, x_n]^{S_n}$ be the graded ring of symmetric polynomials. There are canonical projections : $\Lambda_{n+1} \to \Lambda_n$ which send x_{n+1} to zero. The graded projective limit $\Lambda := \varprojlim \Lambda_n$ is called the ring of symmetric functions. Let m_λ and p_λ denote the monomial and the power sum symmetric functions. They are defined as follows: For a monomial $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \ldots x_{i_k}^{\lambda_k}$ of total degree n, the (ordered) sequence of exponents $(\lambda_1, \ldots, \lambda_k)$ defines a partition λ of n, which is called the shape of the monomial. Then we define m_λ being the sum of all monomials of shape λ. For the power sums, first define $p_n := x_1^n + x_2^n + \ldots$. Then $p_\lambda := p_{\lambda_1} p_{\lambda_2} \ldots p_{\lambda_k}$. The families $(m_\lambda)_\lambda$ and $(p_\lambda)_\lambda$ form two $\mathbb{Q}$-bases of Λ, so they are linearly related by $p_\lambda = \sum_\mu \psi_{\lambda\mu} m_\mu$. It turns out that the base change matrix $(\psi_{\lambda\mu})$ has integral entries, but its inverse $(\psi^{-1}_{\lambda\mu})$ has not.

2. Cohomology of Hilbert schemes of points on a torus surface

Let A be a complex projective torus of dimension 2. Its first cohomology $H^1(A, \mathbb{Z})$ is freely generated by four elements a_1, a_2, a_3, a_4, corresponding to the four different circles on the torus. The cohomology ring is isomorphic to the exterior algebra:

$$H^*(A, \mathbb{Z}) = \Lambda^* H^1(A, \mathbb{Z}).$$

We abbreviate for the products $a_i \cdot a_j =: a_{ij}$ and $a_i \cdot a_j \cdot a_k =: a_{ijk}$. We write $a_1 \cdot a_2 \cdot a_3 \cdot a_4 =: x$ for the class corresponding to a point on A. We choose the a_i such that $\int_A x = 1$. We set $a_{\bar{i}}$ for the dual class of a_i, *i.e.* $a_i \cdot a_{\bar{i}} = x$. The bilinear form, given by $(a_{ij}, a_{kl}) \mapsto \int_A a_{ij} a_{kl}$ gives $H^2(A, \mathbb{Z})$ the structure of a unimodular lattice, isomorphic to $U^{\oplus 3}$, three copies of the hyperbolic lattice.

Let $A^{[n]}$ the Hilbert scheme of n points on the torus, *i.e.* the moduli space of finite subschemes of A of length n. Their rational cohomology can be described in

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terms of Nakajima's operators. First consider the direct sum

$$\mathbb{H} := \bigoplus_{n=0}^{\infty} H^*(A^{[n]}, \mathbb{Q})$$

This space is bigraded by cohomological *degree* and the *weight*, which is given by the number of points n. The unit element in $H^0(A^{[0]}, \mathbb{Q}) \cong \mathbb{Q}$ is denoted by $|0\rangle$, called the *vacuum*.

There are linear operators $\mathfrak{p}_m(\alpha)$, for each $m \in \mathbb{Z}$, $\alpha \in H^*(A, \mathbb{Q})$, acting on $\mathbb{H}$ which have the following properties: They depend linearly on α, and if $\alpha \in H^k(A, \mathbb{Q})$ is homogeneous, the operator $\mathfrak{p}_{-m}(\alpha)$ is bihomogeneous of degree $k + 2(|m| - 1)$ and weight m:

$$\mathfrak{p}_{-m}(\alpha) : H^l(A^{[n]}) \to H^{l+k+2(|m|-1)}(A^{[n+m]})$$

They satisfy the following commutation relations for $\alpha \in H^k(A, \mathbb{Q})$, $\beta \in H^{k'}(A, \mathbb{Q})$:

$$\mathfrak{p}_m(\alpha)\mathfrak{p}_{m'}(\beta) - (-1)^{k \cdot k'}\mathfrak{p}_{m'}(\beta)\mathfrak{p}_m(\alpha) = -m\, \delta_{m,-m'} \int_A \alpha \cdot \beta.$$

Every element in $\mathbb{H}$ can be decomposed uniquely as a linear combination of products of operators $\mathfrak{p}_m(\alpha)$, $m < 0$, acting on the vacuum. We abbreviate for a partition $\lambda = (\lambda_1, \dots, \lambda_k)$:

$$\mathfrak{q}_\lambda(\alpha) := \prod_{i=1}^{k} \mathfrak{p}_{-\lambda_i}(\alpha) \tag{1}$$

$$\mathfrak{q}_{*\lambda}(\alpha) := \left(\prod_{i=1}^{k} \mathfrak{p}_{-\lambda_i}\right) \left(\Delta_{(k)}(\alpha)\right) \tag{2}$$

For the study of integral cohomology, first note that if $\alpha \in H^*(A, \mathbb{Z})$ is an integral class, then $\mathfrak{p}_{-m}(\alpha)$ maps integral classes to integral classes. Moreover, there is the following theorem:

Theorem 2.1. [11] *The following operators map integral classes in $\mathbb{H}$ to integral classes:*

- $\frac{1}{z_\lambda}\mathfrak{q}_\lambda(1)$
- $\mathfrak{m}_\lambda(\alpha)$ *for* $\alpha \in H^2(A, \mathbb{Z})$

Here, $\mathfrak{m}_\lambda$ *is defined as* $\mathfrak{m}_\lambda(\alpha) := \sum_\mu \psi^{-1}_{\lambda\mu}\mathfrak{q}_{-\mu}(\alpha)$ *(see Definition 1.2)*

To obtain the multiplicative structure of $\mathbb{H}$, given by the cup-products, there is a description in [6] and [7] in terms of multiplication operators $\mathfrak{G}_k(a)$, $a \in H^*(A)$ [7, Def. 5.1], related to chern characters. There is the following commutation relation:

$$[\mathfrak{G}_k(a), \mathfrak{q}_1(b)] = \frac{1}{k!}\,\mathrm{ad}(\mathfrak{d})^k(\mathfrak{q}_1(ab)),$$

where the operator $\mathfrak{d}$ means multiplication with the first Chern class of the tautological sheaf. We set $a^{(k)} := \mathfrak{G}_k(a)(1)$.

Next we focus on the structure of $H^2(A^{[n]}, \mathbb{Z})$ for $n \geq 2$. It has rank 13, and there is a basis consisting of:

- $\frac{1}{(n-1)!}\mathfrak{p}_{-1}(1)^{n-1}\mathfrak{p}_{-1}(a_{ij})|0\rangle$, $1 \leq i < j \leq 4$,
- $\frac{1}{(n-2)!}\mathfrak{p}_{-1}(1)^{n-2}\mathfrak{p}_{-1}(a_i)\mathfrak{p}_{-1}(a_j)|0\rangle$, $1 \leq i < j \leq 4$,
- $\frac{1}{2(n-2)!}\mathfrak{p}_{-1}(1)^{n-2}\mathfrak{p}_{-2}(1)|0\rangle$. We denote this class by δ.

It is clear that these classes form a basis of $H^2(A^{[n]}, \mathbb{Q})$. By [11, Thm. 4.6,Lemma 5.2], they also form a basis for $H^2(A^{[n]}, \mathbb{Z})$. TODO: refine this argument

The first 6 classes give an injection $j : H^2(A, \mathbb{Z}) \to H^2(A^{[n]}, \mathbb{Z})$.

3. Generalized Kummer varieties

Definition 3.1. Let A be a complex projective torus of dimension 2 and $A^{[n]}$, $n \geq 1$, the corresponding Hilbert scheme of points. Denote $\Sigma : A^{[n]} \to A$ the summation morphism, a smooth submersion that factorizes via the Hilbert–Chow morphism : $A^{[n]} \to \mathrm{Sym}^n(A) \to A$. Then the generalized Kummer $K^{n-1}A$ is defined as the fiber over 0:

$$\begin{array}{ccc} K^{n-1}A & \xrightarrow{\theta} & A^{[n]} \\ \downarrow & & \downarrow \Sigma \\ \{0\} & \longrightarrow & A \end{array} \tag{3}$$

Our first objective is to collect some information about this pullback diagram. We recall Theorem 2 of [12].

Theorem 3.2. *The cohomology of $K_2(A)$ is torsion free.*

Or main reference is [1] where it is shown, that K^{n-1} is an irreducible holomorphically symplectic manifold. So $H^2(K_{n-1}(A), \mathbb{Z})$ admits an integer-valued nondegenerated quadratic form (called Beauville–Bogomolov form) q which gives $H^2(K_{n-1}(A), \mathbb{Z})$ the structure of a lattice isomorphic to $U^{\oplus 3} \oplus \langle -2n \rangle$, for $n \geq 3$. We have the following formula for $\alpha \in H^2(K_{n-1}(A), \mathbb{Z})$:

$$\int_{K_{n-1}(A)} \alpha^{2n-2} = n \frac{(2n-2)!}{2^{n-1}(n-1)!} q(a)^{n-1} \tag{4}$$

The morphism θ induces a homomorphism of graded rings

$$\theta^* : H^*(A^{[n]}, \mathbb{Z}) \longrightarrow H^*(K_{n-1}(A), \mathbb{Z}). \tag{5}$$

Proposition 3.3. *Let $n \geq 3$.*

(1) *θ^* maps $H^1(A^{[n]}, \mathbb{Z})$ to zero.*
(2) *θ^* is surjective on $H^2(A^{[n]}, \mathbb{Z})$ with kernel $\Lambda^2 H^1(A^{[n]}, \mathbb{Z})$.*

Proof. The first statement is clear since $H^1(K_{n-1}(A))$ is always zero [1, Thm. 3]. Furthermore, by [1, Sect. 7], $\theta^* : H^2(A^{[n]}, \mathbb{C}) \to H^2(K_{n-1}(A), \mathbb{C})$ is surjective. The second Betti numbers of $A^{[n]}$ and $K_{n-1}(A)$ are 13 and 7, respectively. It is clear that $\Lambda^2 H^1(A^{[n]}, \mathbb{Z})$ is contained in the kernel, and since the dimension of the kernel has to be 6, it must be all.

It remains to show that θ^* is surjective for integral coefficients, too. We do it only for $n = 3$. We use a formula in [4, p. 8], namely:

$$\int_{A^{[3]}} j(a)^6 = \frac{5}{3} \int_A a^2 \int_{K_2(A)} \theta^* j(a)^4 \tag{6}$$

for all $a \in H^2(A)$. One computes $\int_{A^{[3]}} j(a)^6 = 15 \left(\int_A a^2\right)^3$. Comparing this with (4), we see that the sublattice given by the image of $\theta^* \circ j$ is unimodular. Secondly, we must show that $q(\theta^* \delta) = -6$. TODO: show this! Remark: $\theta^* \delta$ seems to be indivisible (because of (4)), but every product with $\theta^* \delta$ is divisible by 3. Indeed, the value of (4) for $\alpha = \theta^* \delta$ is 324. $\square$

Proposition 3.4. *We have* $a_i^{(0)} = \frac{1}{2}\mathfrak{q}_1(1)^2\mathfrak{q}_1(a_i)|0\rangle$. *The class of* $K_2(A)$ *in* $H^4(A^{[3]},\mathbb{Q})$ *is given by*

$$a_1^{(0)} \cdot a_2^{(0)} \cdot a_3^{(0)} \cdot a_4^{(0)}.$$

Conjecture: This is true for all n, *not only* $n = 3$.

Proof. We know that for all i and all $\beta \in H^7(A^{[3]})$, we have $\int_{K_2(A)} \theta^*(\alpha_i \cdot \beta) = \int_{A^{[3]}} \alpha_i \cdot \beta \cdot [K_2(A)] = 0$ and for a basis (γ_i) of $H^2(A^{[3]})$,

$$\int_{A^{[3]}} \gamma_i \cdot \gamma_j \cdot \gamma_k \cdot \gamma_l \cdot [K_2(A)] = 3\left(\langle\gamma_i,\gamma_j\rangle\langle\gamma_k,\gamma_l\rangle + \langle\gamma_i,\gamma_k\rangle\langle\gamma_j,\gamma_l\rangle + \langle\gamma_i,\gamma_l\rangle\langle\gamma_j,\gamma_k\rangle\right)$$

These equations admit a unique solution. □

Remark 3.5. This allows us to better understand the morphism θ^*. Since the Poincaré pairing is nondegenerated, $[K_{n-1}(A)] \cdot \alpha = 0$ implies $\theta^*\alpha = 0$.

Let us summarize our results on θ^* for the case $n = 3$:

Theorem 3.6. *The homomorphism* $\theta^* : H^*(A^{[3]},\mathbb{Q}) \to H^*(K_2(A),\mathbb{Q})$ *of graded rings is surjective in every degree except* 4. *Moreover, the image of* $H^4(A^{[3]},\mathbb{Q})$ *is equal to* $\mathrm{Sym}^2(H^2(K_2(A),\mathbb{Q}))$. *The kernel of* θ^* *is the ideal generated by* $H^1(A^{[3]},\mathbb{Q})$.

Now we focus on classes of cohomological degree 4.

Proposition 3.7. *The classes* $\theta^*\left(\mathfrak{p}_{-2}(a_{ij})\mathfrak{p}_{-1}(1)|0\rangle\right)$ *and* $\theta^*\left(\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_{ij})|0\rangle\right)$ *are linearly dependent.*

Proof. We can compute the product of these two classes with $[K_2(A)]$ in $H^*(A^{[3]})$. The two results are linearly dependent. Is there a direct geometric proof? □

Proposition 3.8. $\theta^*\left(\mathfrak{p}_{-3}(x)|0\rangle\right) = 0$

Corollary 3.9. $\theta^*\left(\mathfrak{p}_{-2}(a_{ij})\mathfrak{p}_{-1}(1)|0\rangle\right) = \frac{1}{4}\theta^*\left(\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_{ij})|0\rangle\right)$

Proof. Let a_{ij} and a_{kl} be complementary, *i.e.* $a_{ij}a_{kl} = 1$. We have $a_{kl}^{(1)} = -\frac{1}{2}\mathfrak{p}_{-2}(a_{kl})\mathfrak{p}_{-1}(1)|0\rangle$. Then:

$$\theta^*\left(a_{ij}^{(1)} \cdot a_{kl}^{(1)}\right) = \theta^*\left(\mathfrak{p}_{-3}(1)|0\rangle + \frac{1}{2}\mathfrak{p}_{-1}(x)^2\mathfrak{p}_{-1}(1)|0\rangle\right)$$

But on the other hand, $\delta \cdot j(a) = \frac{1}{2}\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_{ij})|0\rangle + \mathfrak{p}_{-2}(a_{ij})\mathfrak{p}_{-1}(1)|0\rangle$, and

$$\theta^*\left(a_{kl}^{(1)} \cdot \delta \cdot j(a)\right) = \theta^*\left(-3\mathfrak{p}_{-3}(1)|0\rangle - 3\mathfrak{p}_{-1}(x)^2\mathfrak{p}_{-1}(1)|0\rangle\right).$$

□

Corollary 3.10. $\theta^*\left(\delta \cdot j(a_{ij})\right) = \theta^*\left(\frac{3}{4}\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_{ij})|0\rangle\right)$ *is divisible by 3.* □

Proposition 3.11. *The classes* $\theta^*\left(j(a_{ij})^2 - \frac{1}{3}j(a_{ij}) \cdot \delta\right)$ *are divisible by 2.*

Proof. By [11], the classes $\frac{1}{2}\mathfrak{p}_{-1}(a_{ij})^2\mathfrak{p}_{-1}(1)|0\rangle - \frac{1}{2}\mathfrak{p}_{-2}(a_{ij})\mathfrak{p}_{-1}(1)|0\rangle$ are integral in $H^4(A^{[n]})$. But $j(a_{ij})^2 = \mathfrak{p}_{-1}(a_{ij})^2\mathfrak{p}_{-1}(1)|0\rangle$ and $\theta^*\left(\frac{1}{3}j(a_{ij}) \cdot \delta\right) = \theta^*\left(\mathfrak{p}_{-2}(a_{ij})\mathfrak{p}_{-1}(1)|0\rangle\right)$. □

Proposition 3.12. *The class* $\theta^*\left(\delta^2 + j(a_{12}) \cdot j(a_{34}) - j(a_{13}) \cdot j(a_{24}) + j(a_{14}) \cdot j(a_{23})\right)$ *is divisible by 3.*

Proof. It is equal to $\theta^*\left(\mathfrak{p}_{-3}(1)|0\rangle - \frac{3}{2}\mathfrak{p}_{-1}(x)\mathfrak{p}_{-1}(1)^2|0\rangle\right)$. □

Proposition 3.13. *We have:*

$$H^4(K_2(A),\mathbb{Q}) = \operatorname{Sym}^2 H^2(K_2(A),\mathbb{Q}) \oplus^{\perp} \Pi' \otimes \mathbb{Q}.$$

Proof. In Section 4 of [5], we can find the following formula:

$$Z_\tau \cdot D_1 \cdot D_2 = 2 \cdot q(D_1, D_2), \tag{7}$$

for all D_1, D_2 in $H^2(K_2(A),\mathbb{Z})$, $\tau \in A[3]$ and q the Beauville-Bogomolov form. It follows that $\Pi' \subset \operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z})^{\perp}$. Since the cup-product is non-degenerated and by Proposition 4.3 of [5], we have:

$$\begin{aligned}\operatorname{rk}\left(\operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z}) \oplus \Pi'\right) &= \operatorname{rk}\operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z}) + \operatorname{rk}\Pi' \\ &= 28 + 80 \\ &= \operatorname{rk} H^4(K_2(A),\mathbb{Z}).\end{aligned}$$

It follows that

$$H^4(K_2(A),\mathbb{Q}) = \operatorname{Sym}^2 H^2(K_2(A),\mathbb{Q}) \oplus^{\perp} \Pi' \otimes \mathbb{Q}.$$

$\square$

Next we look at the Chern classes of the tangent sheaves. Since the morphism Σ from the defining pullback diagram (3) is a submersion, the normal bundle of $K_{n-1}(A)$ in $A^{[n]}$ is trivial. Hence $c(K_2(A)) = \theta^* c(A^{[3]})$. Looking in [2, Sect. 8], we find a general formula for Chern classes of Hilbert schemes of points on surfaces. So we deduce

$$\begin{aligned} c_2(A^{[3]}) &= \left(\tfrac{3}{2}\mathfrak{q}_{*(1,1)}(1)\mathfrak{q}_1(1) - \tfrac{1}{3}\mathfrak{q}_3\right)|0\rangle \\ &= 10(1_{(4)}^{[\bullet]}) - 2(1_{(2)}^{[\bullet]})^2 \\ c_4(A^{[3]}) &= \tfrac{4}{3}\mathfrak{q}_{*(1,1,1)}(1)|0\rangle = 4(1_{(4)}^{[\bullet]})^2. \end{aligned}$$

Proposition 3.14. *We have:*

$$c_2(K_2(A)) = \theta^*\left(4j(a_{12})\cdot j(a_{34}) - 4j(a_{13})\cdot j(a_{24}) + 4j(a_{14})\cdot j(a_{23}) - \frac{1}{3}\delta^2\right).$$

In particular $c_2(K_2(A)) \in \operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z})$.

Proof. We can write:

$$c_2(K_2(A)) = a + b,$$

with $a \in \operatorname{Sym}^2 H^2(K_2(A),\mathbb{Q})$ and $b \in \Pi'$. First, we prove that $b = 0$. We have $c_2(K_2(A)) \in \Pi'^{\perp}$ and also $a \in \Pi'^{\perp}$, it follows that $b \in \Pi'^{\perp}$. Since the cup-product is non-degenerated, it follows that b is of torsion. Then by Theorem 3.2, $b = 0$.

By (7) and Proposition 5.1 of [5], we can see that for all D_1 and D_2 in $H^2(K_2(A),\mathbb{Z})$, we have:

$$c_2(K_2(A))\cdot D_1 \cdot D_2 = 54 \cdot q(D_1, D_2),$$

where q is the Beauville-Bogomolov form. Then we can calculate that:

$$c_2(K_2(A)) = \theta^*\left(4j(a_{12})\cdot j(a_{34}) - 4j(a_{13})\cdot j(a_{24}) + 4j(a_{14})\cdot j(a_{23}) - \frac{1}{3}\delta^2\right).$$

$\square$

Corollary 3.15. *The class $\theta^*\delta^2$ is divisible by 3.*

Proposition 3.16. *The element*

$$\theta^* \left(j(a_{12}) \cdot j(a_{34}) - j(a_{13}) \cdot j(a_{24}) + j(a_{14}) \cdot j(a_{23})\right)$$

is divisible by 6. More precisely, it is equal to $6\,Y_p$ *(see* [5]*).*

Proof. Again by Section 4 of [5], we have:

$$W = \frac{3}{8}(c_2(K_2(A)) + 3\theta^*(\delta)^2).$$

It follows:

$$W = \frac{3}{8}\theta^* \left(4j(a_{12}) \cdot j(a_{34}) - 4j(a_{13}) \cdot j(a_{24}) + 4j(a_{14}) \cdot j(a_{23}) + \frac{8}{3}\delta^2\right). \tag{8}$$

It follows that

$$\theta^*(j(a_{12}) \cdot j(a_{34}) - j(a_{13}) \cdot j(a_{24}) + j(a_{14}) \cdot j(a_{23})).$$

is divisible by 2. For the divisibility by 3, combine Proposition 3.12 with Corollary 3.15. □

Remark 3.17. We also have the following formulas:

$$W = \theta^* \left(\mathfrak{p}_{-3}(1)|0\rangle\right) \tag{9}$$

$$Y_p = -\tfrac{1}{9}\,\theta^* \left(\mathfrak{p}_{-1}(1)L_{-2}(1)|0\rangle\right) \tag{10}$$

Let us now look at cohomology classes of odd degree. Since $H^1(K_2(A)) = H^7(K_2(A)) = 0$, we only need to consider the degrees 3 and 5.

Proposition 3.18. *The map* $\theta^* : H^*(A^{[3]}, \mathbb{Q}) \to H^*(K_2(A), \mathbb{Q})$ *is surjective in degrees 3 and 5. If we set*

$$B_3 := \{a_{\bar{i}}^{(0)},\ 1 \le i \le 4\} \cup \{a_i^{(1)},\ 1 \le i \le 4\} \tag{11}$$

$$B_5 := \{a_{\bar{i}}^{(1)},\ 1 \le i \le 4\} \cup \{a_i^{(2)},\ 1 \le i \le 4\}, \tag{12}$$

then the images of B_3 *and* B_5 *give bases of* $H^3(K_2(A), \mathbb{Q})$ *and* $H^5(K_2(A), \mathbb{Q})$ *that are orthogonal under the intersection pairing. We have*

$$\int \theta^* \left(a_{\bar{i}}^{(0)} \cdot a_i^{(2)}\right) = \pm\frac{3}{2} \tag{13}$$

$$\int \theta^* \left(a_i^{(1)} \cdot a_{\bar{i}}^{(1)}\right) = \pm\frac{1}{2}. \tag{14}$$

Moreover, $a_{\bar{i}}^{(0)} \cdot [K_2(A)]$ *and* $\frac{2}{3}a_i^{(2)} \cdot [K_2(A)]$ *are integral classes. This implies (by Poincaré duality) that* $\theta^* a_{\bar{i}}^{(0)}$ *and* $\frac{2}{3}\theta^* a_i^{(2)}$ *are integral.*

Question: Which of $\theta^* a_i^{(1)}$ *and* $\theta^* a_{\bar{i}}^{(1)}$ *is not integral?*

References

1. A. Beauville, *Varits khleriennes dont la premire classe de Chern est nulle*, J. Differential geometry, 18 (1983) 755-782
2. S. Boissière and M. Nieper-Wißkirchen, *Generating series in the cohomology of Hilbert schemes of points on surfaces*, LMS J. of Computation and Mathematics 10 (2007), 254–270 .
3. S. Boissière, M. Nieper-Wißkirchen, and A. Sarti, *Smith theory and Irreducible Holomorphic Symplectic Manifolds*, Journal of Topology 6 (2013), no. 2, 361390.
4. M. Britze, *On the cohomology of generalized Kummer varieties*, (2003)

5. B. Hassett and Y. Tschinkel, *Hodge theory and Lagrangian planes on generalized Kummer fourfolds*, Moscow Math. Journal, 13, no. 1, 33-56, (2013)
6. M. Lehn and C. Sorger, *The cup product of Hilbert schemes for $K3$ surfaces*, Invent. Math. **152** (2003), no. 2, 305–329.
7. W. Li, Z. Qin and W. Wang, *Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces* (2002)
8. E. Markman, *Integral generators for the cohomology ring of moduli spaces of sheaves over Poisson surfaces*, Adv. Math. **208** (2007), no. 2, 622–646.
9. E. Markman, *Integral constraints on the monodromy group of the hyperKähler resolution of a symmetric product of a $K3$ surface*, Internat. J. Math. **21** (2010), no. 2, 169–223.
10. H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Ann. of Math. (2) **145** (1997), no. 2, 379–388.
11. Z. Qin and W. Wang, *Integral operators and integral cohomology classes of Hilbert schemes*, Math. Ann. **331** (2005), no. 3, 669–692.
12. E. Spanier, *The homology of Kummer manifolds*, Proc. Amer. Math. Soc. 7, (1956), 155-160.

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