PLANES IN SYMPLECTIC VECTOR SPACES

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1. Symplectic linear algebra

Let V be a symplectic vector space of dimension $n \in 2\mathbb{N}$ over a field F with a nondegenerate symplectic form $\omega : \Lambda^2 V \to F$. A line is a one-dimensional subspace ov V, a plane is a two-dimensional subspace of V. A plane $P \subset V$ is called isotropic, if $\omega(x, y) = 0$ for any $x, y \in P$, otherwise non-isotropic. The symplectic group $\operatorname{Sp} V$ is the set of all linear maps $\phi : V \to V$ with the property $\omega(\phi(x), \phi(y)) = \omega(x, y)$ for all $x, y \in V$.

Proposition 1.1. *The symplectic group* $\operatorname{Sp} V$ *acts transitively on the set of non-isotropic planes as well as on the set of isotropic planes.*

Proof. Given two planes P_1 and P_2, we may choose vectors v_1, v_2, w_1, w_2 such that v_1, v_2 span P_1, w_1, w_2 span P_2 and $\omega(u_1, u_2) = \omega(w_1, w_2)$. We complete $\{v_1, v_2\}$ as well as $\{w_1, w_2\}$ to a symplectic basis of V. Then define $\phi(v_1) = w_1$ and $\phi(v_2) = w_2$. It is now easy to see that the definition of ϕ can be extended to the remaining basis elements to give a symplectic morphism. $\square$

Remark 1.2. The set of planes in V can be identified with the simple tensors in $\Lambda^2 V$ up to multiples. Indeed, given a simple tensor $v \wedge w \in \Lambda^2 V$, the span of v and w yields the corresponding plane. Conversely, any two spanning vectors v and w of a plane give the same element $v \wedge w$ (up to multiples).

Proposition 1.3. *If* $\phi \in \operatorname{Sp} V$ *acts through multiplication of a scalar,* $\phi(v) = \alpha v$, *then* $\alpha = \pm 1$ *(this is immediate from the definition). Moreover, if* $\phi(v) \wedge \phi(w) = \alpha v \wedge w$, *then* $\alpha = 1$.

Proof. We may assume that V is two-dimensional, generated by v and w. Our condition on ϕ reads then $\det \phi = \alpha$. But the condition on ϕ being symplectic is $\det \phi = 1$, because on a two-dimensional vector space there is only one symplectic form up to scalar multiple. $\square$

Remark 1.4. If F is the field with two elements, then the set of planes in V can be identified with the set $\{\{x, y, z\} \mid x, y, z \in V \backslash \{0\},\ x + y + z = 0\}$. Observe that for such a $\{x, y, z\}$, $\omega(x, y) = \omega(x, y) = \omega(y, x)$ and this value gives the criterion for isotropy.

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Proposition 1.5. *Assume that F is finite of cardinality q.*

$$\text{The number of lines in } V \text{ is } \frac{q^n-1}{q-1}, \tag{1}$$

$$\text{the number of planes in } V \text{ is } \frac{(q^n-1)(q^{n-1}-1)}{(q^2-1)(q-1)}, \tag{2}$$

$$\text{the number of isotropic planes in } V \text{ is } \frac{(q^n-1)(q^{n-2}-1)}{(q^2-1)(q-1)}, \tag{3}$$

$$\text{the number of non-isotropic planes in } V \text{ is } \frac{q^{n-2}(q^n-1)}{q^2-1}. \tag{4}$$

Proof. A line ℓ in V is determined by a nonzero vector. There are q^n-1 nonzero vectors in V and $q-1$ nonzero vectors in ℓ. A plane P is determined by a line $\ell_1 \subset V$ and a unique second line $\ell_2 \in V/\ell_1$. We have $\frac{q^2-1}{q-1}$ lines in P. The number of planes is therefore

$$\frac{\frac{q^n-1}{q-1}\cdot\frac{q^{n-1}-1}{q-1}}{\frac{q^2-1}{q-1}} = \frac{(q^n-1)(q^{n-1}-1)}{(q^2-1)(q-1)}.$$

For an isotropic plane we have to choose the second line from $\ell_1^\perp/\ell_1$. This is a space of dimension $n-2$, hence the formula. The number of non-isotropic planes is the difference of the two previous numbers. $\square$

Conjecture 1.6. *There are $6q$ orbits of the induced action of $\mathrm{Sp}(4,q)$ on $\Lambda^2\mathbb{F}_q^4$.*

2. Symplectic vector spaces as index sets

Assume now that V is a four-dimensional vector space over $F=\mathbb{F}_q$. Consider the free F-module $F[V]$ with basis $\{X_i \mid i \in V\}$. It carries a natural F-algebra structure, given by $X_i \cdot X_j := X_{i+j}$ with unit $1 = X_0$. Let $\mathfrak{m}$ be the ideal generated by all elements of the form $(X_i - 1)$. Since $F[V]/\mathfrak{m} = F$, it is a maximal ideal.

Conjecture 2.1. *We define*

$$L := \left\{ \sum_{i\in\ell} X_i =: S_\ell \,\middle|\, \ell \subset V \textit{ line} \right\}$$

Then the ideal generated by L has dimension $q^n - \binom{q+n-2}{n}$.

We introduce an action of $\mathrm{Sp}(4,F)$ on $F[V]$ by setting $\phi(X_i) = X_{\phi(i)}$. Furthermore, the underlying additive group of V acts on $F[V]$ by $v(X_i) = X_{i+v} = X_iX_v$.

Definition 2.2. We define subsets of $F[V]$:

$$B_N := \left\{ \sum_{i\in P} X_i \,\middle|\, P \subset V \text{ non-isotropic plane} \right\},$$

$$B_I := \left\{ \sum_{i\in P} X_i \,\middle|\, P \subset V \text{ isotropic plane} \right\}.$$

Denote by $\langle B_\alpha\rangle$ and by (B_α) the linear span of B_α and the ideal generated by B_α, respectively. Note that (B_α) is the linear span of $\{v \cdot b \mid b \in B, v \in V\}$. Further, let D_α be the linear span of $\{v(b) - b \mid b \in B, v \in V\}$. Then D_α is in fact an ideal, namely the product of ideals $\mathfrak{m} \cdot (B_\alpha)$.

The following table illustrates the dimensions of these objects:

F	$\dim_F \langle B_N \rangle$	$\dim_F (B_N)$	$\dim_F D_N$	$\dim_F \langle B_I \rangle$	$\dim_F (B_I)$	$\dim_F D_I$
$\mathbb{F}_2$	10	11	5	10	10	10
$\mathbb{F}_3$	30	50	31	?	45	25
$\mathbb{F}_5$	121	355	270	?	?	91

Example 2.3. We give the 31 classes spanning D_N over $\mathbb{F}_3$: First, define for a plane $P \subset V$

$$S_P := \sum_{\tau \in P} X_\tau .$$

Then D_N is spanned by the classes

$$S_{P+\tau} - S_P$$

$$\text{for } P = \left\langle \left(\begin{smallmatrix}1\\0\\0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\\0\\0\end{smallmatrix}\right) \right\rangle \text{ and } 0 \neq \tau \in P^\perp = \left\langle \left(\begin{smallmatrix}0\\0\\1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\0\\0\\1\end{smallmatrix}\right) \right\rangle \tag{5}$$

$$\text{for } P = \left\langle \left(\begin{smallmatrix}0\\0\\1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\0\\0\\1\end{smallmatrix}\right) \right\rangle \text{ and } 0 \neq \tau \in P^\perp = \left\langle \left(\begin{smallmatrix}1\\0\\0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\\0\\0\end{smallmatrix}\right) \right\rangle \setminus \left(\begin{smallmatrix}1\\0\\0\\0\end{smallmatrix}\right) \tag{6}$$

$$\text{for } P = \left\langle \left(\begin{smallmatrix}1\\0\\0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\\2\\1\end{smallmatrix}\right) \right\rangle \text{ and } \tau \in \left\{ \left(\begin{smallmatrix}0\\1\\1\\2\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\\0\\2\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\1\\1\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}2\\2\\2\\2\end{smallmatrix}\right) \right\} \tag{7}$$

$$\text{for } P = \left\langle \left(\begin{smallmatrix}1\\0\\0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\\0\\1\end{smallmatrix}\right) \right\rangle \text{ and } \tau \in \left\{ \left(\begin{smallmatrix}0\\0\\0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}2\\0\\1\\2\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\\2\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\\2\\1\end{smallmatrix}\right) \right\} \tag{8}$$

$$\text{for } P = \left\langle \left(\begin{smallmatrix}1\\0\\0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\\1\\1\end{smallmatrix}\right) \right\rangle \text{ and } \tau \in \left\{ \left(\begin{smallmatrix}0\\0\\1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\\0\\1\end{smallmatrix}\right) \right\} \tag{9}$$

$$\text{for } P = \left\langle \left(\begin{smallmatrix}1\\0\\1\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\\0\\1\end{smallmatrix}\right) \right\rangle \text{ and } \tau \in \left\{ \left(\begin{smallmatrix}0\\1\\0\\2\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\\2\\2\end{smallmatrix}\right) \right\} \tag{10}$$

$$\text{for } P = \left\langle \left(\begin{smallmatrix}1\\0\\1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\\0\\1\end{smallmatrix}\right) \right\rangle \text{ and } \tau \in \left\{ \left(\begin{smallmatrix}0\\1\\0\\2\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\\2\\1\end{smallmatrix}\right) \right\} \tag{11}$$

$$\text{for } P = \left\langle \left(\begin{smallmatrix}1\\0\\0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\\0\\2\end{smallmatrix}\right) \right\rangle \text{ and } \tau = \left(\begin{smallmatrix}1\\0\\1\\0\end{smallmatrix}\right) \tag{12}$$

$$\text{for } P = \left\langle \left(\begin{smallmatrix}1\\0\\1\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\\2\\2\end{smallmatrix}\right) \right\rangle \text{ and } \tau = \left(\begin{smallmatrix}1\\1\\0\\2\end{smallmatrix}\right) \tag{13}$$

So we get $8+7+4+4+2+2+2+1+1 = 31$ linear independent classes spanning D_N.

Conjecture 2.4. *For $F = \mathbb{F}_q$, $\dim_F D_I = \frac{(q+2)(q^2+1)}{2}$.*

3. Orthogonal sums

Set $S := \mathrm{Sym}^2(\Lambda^2 V)$. Take two vectors $v, w \in V$ with $\omega(v,w) = 1$ and set $x := (v \wedge w)^2 \in S$. Denote P the plane spanned by v and w and set $y := \sum_{i \in P} X_i \in F[V]$. We set $Y' := y \cdot \mathfrak{m} = \{\sum_{i \in P} X_{i+j} - X_i \mid j \in V\}$.

We consider now the action of $\mathrm{Sp}\, V$ on $S \oplus F[V]$.

Proposition 3.1. *The elements $\phi(x) \oplus \phi(z)$, for $\phi \in \mathrm{Sp}\, V$, $z \in (y)$ span a vector space of dimension*

- *11, if $F = \mathbb{F}_2$,*
- *51, if $F = \mathbb{F}_3$,*

- *375, if* $F = \mathbb{F}_5$.

Proposition 3.2. *The elements* $\phi(x) \oplus \phi(y')$, *for* $\phi \in \mathrm{Sp}\, V$, $y' \in Y'$ *span a vector space of dimension*

- *10, if* $F = \mathbb{F}_2$,
- *50, if* $F = \mathbb{F}_3$,
- *289, if* $F = \mathbb{F}_5$.

Remark 3.3. If $\omega(v, w) = 0$, we would have the dimensions 10, 25, 105 instead.