QUESTION ON ODD COHOMOLOGY OF $A^{[2]}$

SIMON KAPFER

Let A be a complex torus of dimension 2 and $A^{[2]}$ the Hilbert scheme of 2 points. It can be constructed as follows: Consider the direct product $A \times A$. Denote

$$b : \widetilde{A \times A} \to A \times A$$

the blow-up along the diagonal $\Delta \cong A$ with exceptional divisor E, so we have $i : E \to \widetilde{A \times A}$. Since the normal bundle of Δ in $A \times A$ is trivial, we have:

$$E \cong \Delta \times \mathbb{P}^1.$$

The action of $\mathfrak{S}_2$ on $A \times A$ lifts to an action on $\widetilde{A \times A}$. We have the pushforward $i_* : H^*(E, \mathbb{Z}) \to H^*(\widetilde{A \times A}, \mathbb{Z})$.

The quotient by the action of $\mathfrak{S}_2$ is

$$\pi : \widetilde{A \times A} \to A^{[2]}.$$

Now, $A^{[2]}$ is a compact complex manifold with torsion-free cohomology. By [1], we have an exact sequence

$$0 \to \pi_*(H^k(\widetilde{A \times A}, \mathbb{Z})) \to H^k(A^{[2]}, \mathbb{Z}) \to \left(\frac{\mathbb{Z}}{2Z}\right)^{\alpha_k} \to 0.$$

Question 0.1. *What is α_k for $k = 1, \ldots, 8$?*

I think, the pushforward of a class $a \otimes 1 \in H^k(E, \mathbb{Z})$ is given by

$$\pi_* i_*(a \otimes 1) = \mathfrak{q}_2(a)|0\rangle \in H^{k+2}(A^{[n]}, \mathbb{Z})$$

We know, that $\mathfrak{q}_2(1)|0\rangle$ is divisible by 2.

Problem 0.2. Let $a \in H^1(A, \mathbb{Z})$ and $b \in H^3(A, \mathbb{Z})$ basis elements. We can interpret $\mathfrak{q}_2(a)|0\rangle \in H^3(A^{[2]}, \mathbb{Z})$ and $\mathfrak{q}_2(b)|0\rangle \in H^5(A^{[2]}, \mathbb{Z})$ as classes concentrated on the exceptional divisor, that is, as elements of $\pi_* i_* H^*(E, \mathbb{Z})$. By Poincaré duality, the intersection matrix between $H^3(A^{[2]}, \mathbb{Z})$ and $H^5(A^{[2]}, \mathbb{Z})$ has determinant one. Now

$$\int_{A^{[2]}} (\mathfrak{q}_2(a)|0\rangle \cdot \mathfrak{q}_2(b)|0\rangle) = 2 \int_A (a \cdot b)\,.$$

Moreover, $\mathfrak{q}_2(a)|0\rangle$ and $\mathfrak{q}_2(b)|0\rangle$ are orthogonal to any other element of the form $\mathfrak{q}_1(x)\mathfrak{q}_1(y)|0\rangle$ under the intersection pairing. It follows, that one of $\mathfrak{q}_2(a)|0\rangle$ and $\mathfrak{q}_2(b)|0\rangle$ should be divisible by 2. But which one?

Maybe [2] is useful to answer this question, but I don't yet understand all of it.

References

1. G. Menet, *On the integral cohomology of quotients of complex manifolds.*
2. B. Totaro, *The integral cohomology of the Hilbert scheme of two points.*

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