# Beauville-Bogomolov form and generalized Kummer manifolds

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## 1 Recall on the theory of integral cohomology of quotients

Let $G = \langle \iota \rangle$ be the group generated by an involution $\iota$ on a complex manifold $X$. We denote by $\mathcal{O}_K$ the ring $\mathbb{Z}$ with the following $G$-module structure: $\iota \cdot x = -x$ for $x \in \mathcal{O}_K$. For $a \in \mathbb{Z}$, we also denote by $(\mathcal{O}_K, a)$ the module $\mathbb{Z} \oplus \mathbb{Z}$ whose $G$-module structure is defined by $\iota \cdot (x, k) = (-x + ka, k)$. We also denote by $N_2$ the $\mathbb{F}_2[G]$-module $(\mathcal{O}_K, a) \otimes \mathbb{F}_2$. We recall Definition-Proposition 2.2.2 of [13].

**Definition-Proposition 1.1.** *Assume that $H^*(X, \mathbb{Z})$ is torsion-free. Then for all $0 \leq k \leq 2 \dim X$, we have an isomorphism of $\mathbb{Z}[G]$-module:*

$$H^k(X, \mathbb{Z}) \simeq \bigoplus_{i=1}^{r} (\mathcal{O}_K, a_i) \oplus \mathcal{O}_K^{\oplus s} \oplus \mathbb{Z}^{\oplus t},$$

*for some $a_i \notin 2\,\mathbb{Z}$ and $(r, s, t) \in \mathbb{N}^3$. It follows the following isomorphism of $\mathbb{F}_2[G]$-module:*

$$H^k(X, \mathbb{F}_2) \simeq N_2^{\oplus r} \oplus \mathbb{F}_2^{\oplus(s+t)}.$$

*We denote $l_2^k(X) := r$, $l_{1,-}^k(X) := s$, $l_{1,+}^k(X) := t$, $\mathcal{N}_2 := N_2^{\oplus r}$ and $\mathcal{N}_1 := \mathbb{F}_2^{\oplus s+t}$.*

**Remark 1.2.** *These invariants are uniquely determined by $G$, $X$ and $k$.*

Let $\pi : X \to X/G$ be the quotient map. We denote by $\pi^*$ and $\pi_*$ respectively the pull-back and the push-forward of $\pi$. We recall the commutativity behaviour of $\pi_*$ with cup product (Lemma 3.3.7 of [13]).

**Proposition 1.3.** *Let $X$ be a compact complex manifold of dimension $n$ and $\iota$ an involution. Assume that $H^*(X, \mathbb{Z})$ is torsion free. Let $0 \leq k \leq 2n$, $q$ an integer such that $kq \leq 2n$, and let $(x_i)_{1 \leq i \leq q}$ be elements of $H^k(X, \mathbb{Z})^\iota$. Then*

$$\pi_*(x_1) \cdot \ldots \cdot \pi_*(x_q) = 2^{q-1} \pi_*(x_1 \cdot \ldots \cdot x_q).$$

We also recall Definition 3.3.4 of [13].

**Definition 1.4.** *Let $X$ be a compact complex manifold and $\iota$ be an involution. Let $0 \leq k \leq 2n$, and assume that $H^k(X, \mathbb{Z})$ is torsion free. Then if the map $\pi_* : H^k(X, \mathbb{Z}) \to H^k(X/G, \mathbb{Z})/\operatorname{tors}$ is surjective we say that $(X, \iota)$ is $H^k$-*normal.

**Remark 1.5.** *$H^k$-normal property is equivalent to the following property.*

*For $x \in H^k(X, \mathbb{Z})^\iota$, $\pi_*(x)$ is divisible by 2 if and only if there exists $y \in H^k(X, \mathbb{Z})$ such that $x = y + \iota^*(y)$.*

We also need to recall Definition 3.5.1 of [13] about fixed locus.

**Definition 1.6.** *Let $X$ be a compact complex manifold of dimension $n$ and $G$ an automorphism group of prime order $p$.*

*1) We will say that* $\operatorname{Fix} G$ *is negligible if the following conditions are verified:*

- $H^*(\mathrm{Fix}\, G, \mathbb{Z})$ *is torsion-free.*
- $\mathrm{Codim}\,\mathrm{Fix}\, G \geq \frac{n}{2} + 1$.

*2) We will say that* $\mathrm{Fix}\, G$ *is almost negligible if the following conditions are verified:*

- $H^*(\mathrm{Fix}\, G, \mathbb{Z})$ *is torsion-free.*
- $n$ *is even and* $n \geq 4$.
- $\mathrm{Codim}\,\mathrm{Fix}\, G = \frac{n}{2}$, *and the purely* $\frac{n}{2}$*-dimensional part of* $\mathrm{Fix}\, G$ *is connected and simply connected. We denote the* $\frac{n}{2}$*-dimensional component by* $\Sigma$.
- *The cocycle* $[\Sigma]$ *associated to* $\Sigma$ *is primitive in* $H^n(X, \mathbb{Z})$.

Now, we are ready to provide Corollary 2.65 of [13] that we will our key tool in the next of this article.

**Corollary 1.7.** *Let* $G = \langle \varphi \rangle$ *be a group of prime order* $p = 2$ *acting by automorphisms on a Kähler manifold* $X$ *of dimension* $2n$. *We assume:*

*i)* $H^*(X, \mathbb{Z})$ *is torsion-free,*

*ii)* $\mathrm{Fix}\, G$ *is negligible or almost negligible,*

*iii)* $l_{1,-}^{2k}(X) = 0$ *for all* $1 \leq k \leq n$, *and*

*iv)* $l_{1,+}^{2k+1}(X) = 0$ *for all* $0 \leq k \leq n-1$, *when* $n > 1$.

*v)* $l_{1,+}^{2n}(X) + 2\left[\sum_{i=0}^{n-1} l_{1,-}^{2i+1}(X) + \sum_{i=0}^{n-1} l_{1,+}^{2i}(X)\right] = \sum_{k=0}^{\dim \mathrm{Fix}\, G} h^{2k}(\mathrm{Fix}\, G, \mathbb{Z})$.

*Then* $(X, G)$ *is* $H^{2n}$*-normal.*

We will also need a proposition from Section 7 of [3] about Smith theory. Let $T$ be a topological space and let $G = \langle \iota \rangle$ be an involution acting on $T$. Let $\sigma := 1 + \iota \in \mathbb{F}_2[G]$. We consider the chain complex $C_*(T)$ of $T$ with coefficients in $\mathbb{F}_2$ and its subcomplexes $\sigma C_*(T)$. We denote also $X^G$ the fixed locus of the action of $G$ on $T$.

**Proposition 1.8.** *(1) ([4], Theorem 3.1). There is an exact sequence of complexes:*

$$0 \longrightarrow \sigma C_*(T) \oplus C_*(T^G) \xrightarrow{f} C_*(T) \xrightarrow{\sigma} \sigma C_*(T) \longrightarrow 0\ ,$$

*where* $f$ *denotes the sum of the inclusions.*

*(2) ([4], (3.4) p.124). There is an isomorphism of complexes:*

$$\sigma C_*(T) \simeq C_*(T/G, T^G),$$

*where* $T^G$ *is identified with its image in* $T/G$.

# 2 Symplectic involution on $K_2(A)$

Let $X$ be an irreducible symplectic manifold. Let

$$\nu : \mathrm{Aut}(X) \to H^2(X, \mathbb{Z})$$

the natural morphism. Hassett and Tschinkel (Theorem 2.1 in [8]) have shown that $\mathrm{Ker}\,\nu$ is a deformation invariant. Let $X$ be an irreducible symplectic fourfold of Kummer type. Then Oguiso in [19] has shown that $\mathrm{Ker}\,\nu = (\mathbb{Z}/3\,\mathbb{Z})^3 \rtimes \mathbb{Z}/2\,\mathbb{Z}$.

Let $A$ be an abelian variety and $g$ an automorphism of $A$. Let denote by $T_{A[3]}$ the group of translation of $A$ by elements of $A[3]$. If $g \in T_{A[3]} \rtimes \mathrm{Aut}_{\mathbb{Z}}(X)$ then $g$ induces a natural automorphism on $K_2(A)$. We denote the induced automorphism by $g^{[[3]]}$. If there is no ambiguity, we also denote the *induced automorphism* by the same letter $g$.

When $X = K_2(A)$, we have more precisely, by Corollary 3.3 of [2],

$$\mathrm{Ker}\,\nu = T_{A[3]} \rtimes (-\mathrm{id}_A)^{[[3]]}.$$

## 2.1 Uniqueness and fixed locus

Corollary 2.1 to put after Proposition 3.18 ?? saying that it will be use in Part 2

**Corollary 2.1.** *Let $A$ be an abelian surface and $g$ be an automorphisms on $A$. Let $g^{[[3]]}$ be the automorphisms induced by $g$ on $K_2(A)$. We have $H^3(K_2(A),\mathbb{Z}) \simeq H^1(A,\mathbb{Z}) \oplus H^3(A,\mathbb{Z})$ and the action of $g^{[[3]]}$ on $H^3(K_2(A),\mathbb{Z})$ is given by the action of $g$ on $H^1(A,\mathbb{Z}) \oplus H^3(A,\mathbb{Z})$.*

**Proof.** Let $g^{[3]}$ be the involution on $A^{[3]}$ induced by $g$. We have $g^{[3]*}(a_i^{(1)}) = (g^*a_i)^{(1)}$ and $g^{[3]*}(a_{\bar{i}}^{(0)}) = (g^*a_{\bar{i}})^{(0)}$. Moreover, we have by definition, $g^{[[3]]*} \circ \theta^* = \theta^* \circ g^{[3]*}$. The result follows from Proposition 3.18 ??. □

**Theorem 2.2.** *Let $X$ be an irreducible symplectic fourfold of Kummer type and $\iota$ a symplectic involution on $X$ then:*

*(1) We have $\iota \in \operatorname{Ker}\nu$.*

*(2) Let $A$ be an abelian surface then the couple $(X,\iota)$ is deformation equivalent to $(K_2(A), t_\tau \circ (-\operatorname{id}_A)^{[[3]]})$, where $t_\tau$ is the morphism induced on $K_2(A)$ by the translation by $\tau \in A[3]$.*

*(3) The fixed locus of $\iota$ is given by a K3 surface and 36 isolated points.*

**Proof.** (1) If $\iota \notin \operatorname{Ker}\nu$, by Section 5 of [14], the unique possible action of $\iota$ on $H^2(X,\mathbb{Z})$ is given by $H^2(X,\mathbb{Z})^\iota = U \oplus A_1^2 \oplus (-6)$. We will show that it is impossible. Let assume that $H^2(X,\mathbb{Z})^\iota = U \oplus A_1^2 \oplus (-6)$, we will find a contradiction.

As done in Section 3 of [16], consider a local universal deformation space of $X$:

$$\Phi : \mathcal{X} \to \Delta,$$

where $\Delta$ is a small polydisk and $\mathcal{X}_0 = X$. By eventually restricting $\Delta$, we can assume that $\iota$ extends to an automorphisms $M$ on $\mathcal{X}$ and $m$ on $\Delta$ such that, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "M"] \arrow[d] & \mathcal{X} \arrow[d] \\ \Delta \arrow[r, "m"] & \Delta \end{tikzcd}$$

Moreover, the differential of $m$ at 0 is given by the action of $\iota$ on $H^1(T_X)$ which is the same as the action on $H^{1,1}(X)$ since the symplectic holomorphic 2-form induces an isomorphism between those two and the symplectic holomorphic 2-form is preserved by the action of $\iota$. The morphism $m$ is linearizable, then $\Delta^m$ is smooth and $\dim \Delta^m = \operatorname{rk} H^2(X,\mathbb{Z})^\iota - 2 = 3$. Moreover, by [10] that we can find $x \in \Delta^m$ such that $\mathcal{X}_x$ is bimeromorphic to a Kummer fourfold $K_2(A)$. Since $H^2(X,\mathbb{Z})^\iota = U \oplus A_1^2 \oplus (-6)$, $\iota_x := M_{\mathcal{X}_x}$ induces a bimeromorphic involution $i$ on $K_2(A)$ with $H^2(K_2(A),\mathbb{Z})^i = U \oplus A_1^2 \oplus (-6)$.

Necessary, we have $\operatorname{NS}(K_2(A)) \supset A_1(-1)^2 \oplus (-6)$. It follows that $\operatorname{NS}(A) \supset A_1(-1)^2$. Let consider the involution $g$ defined by $-\operatorname{id}$ on $A_1(-1)^2$ and id on $(A_1(-1)^2)^\perp$. By Corollary 1.5.2 of [18], $g$ can be extend to an involution on $H^2(A,\mathbb{Z})$. Then by Theorem 1 of [21], $g$ provides a symplectic automorphism on $A$ with: $H^2(A,\mathbb{Z})^g = (A_1(-1)^2)^\perp = U \oplus A_1^2$. It follows from classification of Section 4 of [15], that $A = \mathbb{C}/\Lambda$ with $\Lambda = \langle (1,0),(0,1),(x,-y),(y,x)\rangle$, $(x,y) \in \mathbb{C}^2 \smallsetminus \mathbb{R}^2$ and $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Let also denote $g$ the automorphism on $K_2(A)$ induces by $g$. By construction, $g \circ i$ acts trivially on $H^2(K_2(A),\mathbb{Z})$. Hence by Corollary 3.3 and Lemma 3.4 of [5], $g \circ \iota$ extends to an automorphism of $K_2(A)$. In particular, $i$ extends to a symplectic involution on $K_2(A)$. Then $g \circ i \in \operatorname{Ker}\nu$.

By Corollary 2.1, $t_\tau$ acts trivially on $H^3(K_2(A),\mathbb{Z})$. Hence by Corollary 3.3 of [2], we have necessary:

$$g^*_{|H^3(K_2(A),\mathbb{Z})} = i^*_{|H^3(K_2(A),\mathbb{Z})} \circ (-\operatorname{id}_A)^*_{|H^3(K_2(A),\mathbb{Z})} \text{ or } g^*_{|H^3(K_2(A),\mathbb{Z})} = i^*_{|H^3(K_2(A),\mathbb{Z})}.$$

But $g^*_{|H^3(K_2(A),\mathbb{Z})}$ has order 4 and $i^*_{|H^3(K_2(A),\mathbb{Z})} \circ (-\operatorname{id}_A)^*_{|H^3(K_2(A),\mathbb{Z})}$ and $i^*_{|H^3(K_2(A),\mathbb{Z})}$ have order 2, which is a contradiction.

(2) It follows from (1), Theorem 2.1 of [8] and Corollary 3.3 of [2].

(3) It follows from (2) and Section 1.2.1 of [23].

□

**Remark 2.3.** *(1) We also remark that the K3 surface fixed by $(t_\tau \circ (-\operatorname{id}_A))$ is given by the sub-manifold $Z_\tau$ defined in Section 4 of [8].*

*(2) Considering the involution $-\operatorname{id}_A$, the set*

$$\mathcal{P} := \{\xi \in K_2(A) |\ \operatorname{Supp}\xi = \{a_1, a_2, a_3\},\ a_i \in A[2] \smallsetminus \{0\}, 1 \leq i \leq 3\}$$

*provides 35 fixed points and the vertex of*

$$W_0 := \{\xi \in K_2(A) |\ \operatorname{Supp}\xi = \{0\}\}$$

*supplies the 36th point. We denote by $p_1, ..., p_{35}$ the points of $\mathcal{P}$ and by $p_{36}$ the vertex of $W_0$.*

## 2.2 Action on the cohomology

From Theorem 2.2, we can assume that $X = K_2(A)$ and $\iota = -\operatorname{id}_A$. To consider $t_\tau \circ (-\operatorname{id}_A)$ instead of $-\operatorname{id}_A$ will only has the effect to exchange the role of $[Z_0]$ and $[Z_\tau]$. Hence we do not lose any generality assuming that $\iota = -\operatorname{id}_A$.

**Proposition 2.4.** *(1) The involution $\iota$ acts trivially on $H^2(K_2(A), \mathbb{Q})$. It follows $l_2^2(K_2(A)) = l_{1,-}^2(K_2(A)) = 0$ and $l_{1,+}^2(K_2(A)) = 7$.*

*(2) The involution $\iota$ acts as $-\operatorname{id}$ on $H^3(K_2(A), \mathbb{Q})$. It follows $l_2^3(K_2(A)) = l_{1,+}^3(K_2(A)) = 0$ and $l_{1,-}^3(K_2(A)) = 8$.*

*(3) By Proposition 3.13 **??**, we have:*

$$H^4(K_2(A), \mathbb{Q}) = \operatorname{Sym}^2 H^2(K_2(A), \mathbb{Q}) \oplus^\perp \Pi' \otimes \mathbb{Q},$$

*where $\Pi' = \langle Z_\tau - Z_0,\ \tau \in A[3] \smallsetminus \{0\}\rangle$. The involution $\iota^*$ fixes $\operatorname{Sym}^2 H^2(K_2(A), \mathbb{Z})$ and $\iota^*(Z_\tau - Z_0) = Z_{-\tau} - Z_0$. It follows that $l_{1,-}^4(K_2(A)) = 0$, $l_{1,+}^4(K_2(A)) = 28$ and $l_2^4(K_2(A)) = 40$.*

**Proof.** (1) It follows from Corollary 3.3 of [2].

(2) It follows from Corollary 2.1.

(3) Let $\mathcal{S}$ be the over-lattice of $\operatorname{Sym}^2 H^2(K_2(A), \mathbb{Z})$ where we add all the classes divisible by 2 in $H^4(K_2(A), \mathbb{Z})$. We can calculate that the discriminant of $\Pi'$ is not divisible by 2. Since $H^4(K_2(A), \mathbb{Z})$ is unimodular, it follows that the discriminant of $\mathcal{S}$ is also not divisible by 2. Hence, we have:

$$H^4(K_2(A), \mathbb{F}_2) = \mathcal{S} \otimes \mathbb{F}_2 \oplus \Pi' \otimes \mathbb{F}_2 .$$

Moreover, we have:

$$\iota^*(Z_\tau - Z_0) = Z_{-\tau} - Z_0,$$

for all $\tau \in A[3] \smallsetminus \{0\}$. Hence $\operatorname{Vect}_{\mathbb{F}_2}(Z_\tau - Z_0, Z_{-\tau} - Z_0)$ is isomorphic to $N_2$ as a $\mathbb{F}_2[G]$-module. Moreover $H^2(K_2(A), \mathbb{Z})$ is invariant by the action of $\iota$, hence $\operatorname{Sym}^2 H^2(K_2(A), \mathbb{Z})$ and $\mathcal{S}$ is also invariant by the action of $\iota$. It follows that $\mathcal{S} \otimes \mathbb{F}_2 = \mathcal{N}_1$ and $\Pi' \otimes \mathbb{F}_2 = \mathcal{N}_2$. Since $\operatorname{rk}\mathcal{S} = 28$, we have $l_{1,+}^4 + l_{1,-}^4 = 28$. However, $\mathcal{S}$ is invariant by the action of $\iota$, it follows that $l_{1,-}^4 = 0$ and $l_{1,+}^4 = 28$. On the other hand $\operatorname{rk}\Pi' = 80$, it follows that $l_2^4 = 40$.

□

## 3 Application to Beauville–Bogomolov form

### 3.1 Statement of the main theorem

In [17], Namikawa propose a definition of the Beauville-Bogomolov form for some singular irreducible symplectic varieties. He Assumes that the singularities are only $\mathbb{Q}$-factorial singularities with a singular locus of codimension of $\geq 4$. Under these assumptions, he proves a local Torelli theorem. This result was completed by a generalization of the Fujiki formula by Matsushita in [11].

**Theorem 3.1.** *Let $Z$ be a projective irreducible symplectic variety of dimension $2n$ with only $\mathbb{Q}$-factorial singularities, and* $\operatorname{Codim}\operatorname{Sing} Z \geq 4$. *There exists a unique indivisible integral symmetric bilinear form $B_Z \in S^2(H^2(Z,\mathbb{Z}))^*$ and a unique positive constant $c_Z \in \mathbb{Q}$, such that for any $\alpha \in H^2(Z,\mathbb{C})$,*

$$\alpha^{2n} = c_Z B_Z(\alpha,\alpha)^n. \qquad (1)$$

*For $0 \neq \omega \in H^0(\Omega_U^2)$*

$$B_Z(\omega+\overline{\omega},\omega+\overline{\omega}) > 0. \qquad (2)$$

*Moreover the signature of $B_Z$ is $(3, h^2(Z,\mathbb{C})-3)$.*

*The form $B_Z$ is proportional to $q_Z$ and is called the Beauville–Bogomolov form of $Z$.*

**Proof**. The statement of the theorem in [11] does not say that the form is integral. However, let $Z_s$ be a fiber of the Kuranishi family of $Z$, with the same idea as Matsushita's proof, we can see that $q_Z$ and $q_{Z_s}$ are proportional. Then, it follows using the proof of Theorem 5 a), c) of [1]. □

We can also consider its polarizeed form.

**Proposition 3.2.** *Let $X$ be a projective irreducible symplectic variety of dimension $2n$ with* $\operatorname{Codim}\operatorname{Sing} X \geq 4$. *The equality (1) of Theorem 3.1 implies that*

$$\alpha_1 \cdot \ldots \cdot \alpha_{2n} = \frac{c_X}{(2n)!} \sum_{\sigma \in S_{2n}} B_X(\alpha_{\sigma(1)},\alpha_{\sigma(2)}) \ldots B_X(\alpha_{\sigma(2n-1)},\alpha_{\sigma(2n)}).$$

*for all $\alpha_i \in H^2(X,\mathbb{Z})$.*

These results were then generalized by Kirschner for complex spaces in [9]. In [12] (Theorem 2.5) was propose the first concrete example of Beauville-Bogomolov lattice for a singular irreducible symplectic variety. The variety studied in [12] is a partial resolution of the quotient of the Hilbert scheme of 2 points on a K3 surface quotiented by a symplectic involution. The objective of this section is to provide a new example of Beauville-Bogomolov lattice replacing Hilbert schemes of 2 points on a K3 surface by generalized Kummer fourfolds. Knowing the integral basis of the cohomology group of the generalized Kummer provided in Part 2 **??**, this calculation becomes possible. Moreover the calculation will be much more simple as in [12] because of the general techniques for calculating integral cohomology of quotients developed in [13] and the new technique using monodromy developed in Lemma 3.14. The other techniques developed in [12] are also in [13], so to simplify the reading, we will only cite [12] in the rest of the section.

Concretely, let $X$ be an irreducible symplectic fourfold of Kummer type and $\iota$ a symplectic involution on $X$. By Theorem 2.2 the fixed locus of $\iota$ is the union of 36 points and a K3 surface $Z_0$. Then the singular locus of $K := X/\iota$ is the union of a K3 and 36 points. The singular locus is not of codimension four. We will lift to a partial resolution of singularities, $K'$ of $K$, obtained by blowing up the image of $Z_0$. By Section 2.3 and Lemma 1.2 of [6], the variety $K'$ is an irreducible symplectic V-manifold which has singular locus of codimension four.

All Section 3 is devoted to prove the following theorem.

**Theorem 3.3.** *Let $X$ be an irreducible symplectic fourfold of Kummer type and $\iota$ a symplectic involution on $X$. Let $Z_0$ be the K3 surface which is in the fixed locus of $\iota$. We denote $K = X/\iota$ and $K'$ the partial resolution of singularities of $K$ obtained by blowing up the image of $Z_0$. Then the Beauville–Bogomolov lattice $H^2(K',\mathbb{Z})$ is isomorphic to $U(3)^3 \oplus \begin{pmatrix} -5 & -4 \\ -4 & -5 \end{pmatrix}$, and the Fujiki constant $C_{K'}$ is equal to* 8.

The Beauville-Bogomolov form is a topological invariant, hence from Theorem 2.2 we can assume that $X$ is a generalized Kummer fourfold and $\iota = -\operatorname{id}_A$. As it will be useful to prove Lemma 3.14, we can assume even more. All generalized Kummer fourfolds are deformation equivalent, hence, we can assume that $A = E \times E$ with

$$E := \frac{\mathbb{C}}{\left\langle 1, e^{\frac{2i\pi}{6}} \right\rangle}.$$

## 3.2 Overview on the proof of Theorem 3.3

We first provide all the notation that we will need during the proof in Section 3.3. Then the proof is divided in the following steps:

(1) First Proposition 2.4 and Corollary 1.7 will provide the $H^4$-normality in Section 3.4.

(2) The knowledge of the element divisible by 2 in $\operatorname{Sym}^2 H^2(K_2(A), \mathbb{Z})$ from Part 2 **??** and the $H^4$-normality allow us to prove the $H^2$-normality in Section 3.5.

(3) An adaptation of the $H^2$-normality (Lemma 3.8) and several lemmas in Section 3.6 will provide an integral basis of $H^2(K', \mathbb{Z})$ (Theorem 3.9).

(4) Knowing an integral basis of $H^2(K', \mathbb{Z})$, we end the calculation of the Beauville–Bogomolov form in Section 3.7 using intersection theory and the generalized Fujiki formula (Theorem 3.1).

## 3.3 Notation

Let $K_2(A)$ be a generalized Kummer fourfold endowed with the symplectic involution $\iota$ induced by $-\operatorname{id}_A$. We denote by $\pi$ the quotient map $K_2(A) \to K_2(A)/\iota$. From Theorem 2.2, we know that the singular locus of the quotient $K_2(A)/\iota$ is the K3 surface, image by $\pi$ of $Z_0$, and 36 isolated points. We denote $\overline{Z_0} := \pi(Z_0)$. We consider $r' : K' \to K_2(A)/\iota$ the blow-up of $K_2(A)/\iota$ in $\overline{Z_0}$ and we denote by $\overline{Z_0}'$ the exceptional divisor. We also denote by $s_1 : N_1 \to K_2(A)$ the blowup of $K_2(A)$ in $Z_0$; and denote by $Z_0'$ the exceptional divisor in $N_1$. Denote by $\iota_1$ the involution on $N_1$ induced by $\iota$. We have $K' \simeq N_1/\iota_1$, and we denote $\pi_1 : N_1 \to K'$ the quotient map.

Consider the blowup $s_2 : N_2 \to N_1$ of $N_1$ in the 36 points $p_1, ..., p_{36}$ fixed by $\iota_1$ and the blowup $\widetilde{r} : \widetilde{K} \to K'$ of $K'$ in its 36 singulars points. We denote the exceptional divisors by $E_1, ..., E_{36}$ and $D_1, ..., D_{36}$ respectively. We also denote $\widetilde{\overline{Z_0}} = \widetilde{r}^*(\overline{Z_0}')$ and $\widetilde{Z_0} = s_2^*(Z_0')$. Denote $\iota_2$ the involution induced by $\iota$ on $N_2$ and $\pi_2 : N_2 \to N_2/\iota_2$ the quotient map. We have $N_2/\iota_2 \simeq \widetilde{K}$. We collect this notation in commutative diagram

$$\begin{tikzcd}
\widetilde{K} \arrow[r, "\widetilde{r}"] & K' \arrow[r, "r'"] & K_2(A)/\iota \\
N_2 \arrow[u, "\pi_2"] \arrow[r, "s_2"] \arrow[loop below, "\iota_2"] & N_1 \arrow[u, "\pi_1"] \arrow[r, "s_1"] \arrow[loop below, "\iota_1"] & K_2(A) \arrow[u, "\pi"] \arrow[loop below, "\iota"]
\end{tikzcd}$$

To finish, we denote $V = K_2(A) \smallsetminus \operatorname{Fix}\iota$ and $U = V/\iota$. Also, we set $s = s_2 \circ s_1$ and $r = \widetilde{r} \circ r'$. We denote also $\delta$ the half of the class of the diagonal in $H^2(K_2(A), \mathbb{Z})$.

We can commute the push-forward maps and the blow-up maps as proved in Lemma 3.3.21 of [13].

**Remark 3.4.** *Let $x \in H^2(N_1, \mathbb{Z})$, $y \in H^2(K_2(A), \mathbb{Z})$, we have:*

$$\pi_{2*}(s_2^*(x)) = \widetilde{r}^*(\pi_{1*}(x)),$$

$$\pi_{1*}(s_1^*(y)) = r'^*(\pi_*(y)),$$

## 3.4 The couple $(K_2(A), \iota)$ is $H^4$-normal

**Proposition 3.5.** *The couple $(K_2(A), \iota)$ is $H^4$-normal.*

**Proof**. We apply Corollary 1.7.

i) By Theorem 2 of [22], $H^*(K_2(A), \mathbb{Z})$ is torsion-free.

ii) From Remark 2.3 (1), we know that the connected component of dimension 2 of $\operatorname{Fix}\iota$ is given by $Z_0$. We know that $Z_0$ is a K3 surface, hence is simply connected. Moreover by Proposition 4.3 of [8] $Z_0 \cdot Z_\tau = 1$ for all $\tau \in A[3] \smallsetminus \{0\}$. Hence the class of $Z_0$ in $H^4(K_2(A), \mathbb{Z})$ is primitive. It follows that $\operatorname{Fix}\iota$ is almost negligible (Definition 1.6).

iii) By Proposition 2.4, we have $l^2_{1,-}(K_2(A)) = l^4_{1,-}(K_2(A)) = 0$.

iv) By Proposition 2.4, we have $l^3_{1,+}(K_2(A)) = 0$. Moreover $H^1(K_2(A)) = 0$, so $l^1_{1,+}(K_2(A)) = 0$.

v) We have to check the following equality:

$$\begin{aligned} &l^4_{1,+}(K_2(A)) + 2\left[l^1_{1,-}(X) + l^3_{1,-}(X) + l^0_{1,+}(X) + l^2_{1,+}(X)\right] \\ &= 36h^0(pt) + h^0(Z_0, \mathbb{Z}) + h^2(Z_0, \mathbb{Z}) + h^4(Z_0, \mathbb{Z}). \end{aligned}$$

By Proposition 2.4:

$$l^4_{1,+}(K_2(A)) + 2\left[l^1_{1,-}(X) + l^3_{1,-}(X) + l^0_{1,+}(X) + l^2_{1,+}(X)\right] = 28 + 2(8 + 1 + 7) = 60.$$

Moreover since $Z_0$ is a K3 surface, we have:

$$36h^0(pt) + h^0(Z_0, \mathbb{Z}) + h^2(Z_0, \mathbb{Z}) + h^4(Z_0, \mathbb{Z}) = 36 + 2 + 22 = 60.$$

It follows from Corollary 1.7 that $(K_2(A), \iota)$ is $H^4$-normal. □

As explained in Proposition 3.5.20 of [13], the proof of Corollary 1.7 provide first that $\pi_{2*}(s^*(H^4(K_2(A), \mathbb{Z})))$ is primitive in $H^4(\widetilde{K}, \mathbb{Z})$ and then the $H^4$ normality.

**Remark 3.6.** *The lattice $\pi_{2*}(s^*(H^4(K_2(A), \mathbb{Z})))$ is primitive in $H^4(\widetilde{K}, \mathbb{Z})$.*

## 3.5 The couple $(K_2(A), \iota)$ is $H^2$-normal

**Proposition 3.7.** *The couple $(K_2(A), \iota)$ is $H^2$-normal.*

**Proof**. We want to prove that the pushforward $\pi_* : H^2(K_2(A), \mathbb{Z}) \to H^2(K_2(A)/\iota, \mathbb{Z})/\operatorname{tors}$ is surjective. By Remark 1.5, it is equivalent to prove that for all $x \in H^2(K_2(A), \mathbb{Z})^\iota$ $\pi_*(x)$ is divisible by 2 if and only if there exist $y \in H^2(K_2(A), \mathbb{Z})$ such that $x = y + \iota^*(y)$.

Let $x \in H^2(K_2(A), \mathbb{Z})^\iota = H^2(K_2(A), \mathbb{Z})$ such that $\pi_*(x)$ is divisible by 2, we will show that there exists $y \in H^2(K_2(A), \mathbb{Z})$ such that $x = y + \iota^*(y)$. Then, by Proposition 1.3, $\pi_*(x^2)$ is divisible by 2. However, $x^2 \in H^4(K_2(A), \mathbb{Z})^\iota$; since $(K_2(A), \iota)$ is $H^4$-normal by Proposition 3.5, it means that there is $z \in H^4(K_2(A), \mathbb{Z})$ such that $x^2 = z + \iota^*(z)$.

Let $\mathcal{S}$ be, as before, the over-lattice of $\operatorname{Sym}^2 H^2(K_2(A), \mathbb{Z})$ where we add all the classes divisible by 2 in $H^4(K_2(A), \mathbb{Z})$. By Proposition 3.13 **??**, there exist $z_s \in \mathcal{S}$, $z_p \in \Pi'$ and $\alpha \in \mathbb{N}$ such that: $5 \cdot 3^\alpha \cdot z = z_s + z_p$. Hence, we have:

$$5 \cdot 3^\alpha \cdot x^2 = 2z_s + z_p + \iota^*(z_p).$$

Since $x^2 \in \operatorname{Sym}^2 H^2(K_2(A), \mathbb{Z})$, by Proposition 3.13**??**, $z_p + \iota^*(z_p) = 0$. It follows:

$$5 \cdot 3^\alpha \cdot x^2 = 2z_s. \tag{1}$$

We denote by $(u_1, u_2, v_1, v_2, w_1, w_2, \delta)$ the integral basis of $H^2(K_2(A), \mathbb{Z})$ given by Proposition 3.3 **??**, where $(u_1, u_2)$, $(v_1, v_2)$ and $(w_1, w_2)$ are basis of the hyperbolic plans $U$ and $\delta$ is half the diagonal of $K_2(A)$. We can write:

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \beta_1 v_1 + \beta_2 v_2 + \gamma_1 w_1 + \gamma_2 w_2 + d\delta.$$

Then

$$x^2 = \alpha_1^2 u_1^2 + \alpha_2^2 u_2^2 + \beta_1^2 v_1^2 + \beta_2^2 v_2^2 + \gamma_1^2 w_1^2 + \gamma_2^2 w_2^2 + d^2\delta^2 \mod 2H^4(K_2(A),\mathbb{Z}).$$

We also have:

$$5\cdot 3^{\alpha}\cdot x^2 = \alpha_1^2 u_1^2 + \alpha_2^2 u_2^2 + \beta_1^2 v_1^2 + \beta_2^2 v_2^2 + \gamma_1^2 w_1^2 + \gamma_2^2 w_2^2 + d^2\delta^2 \mod 2H^4(K_2(A),\mathbb{Z}).$$

It follows by (1) that $\alpha_1^2 u_1^2 + \alpha_2^2 u_2^2 + \beta_1^2 v_1^2 + \beta_2^2 v_2^2 + \gamma_1^2 w_1^2 + \gamma_2^2 w_2^2 + d^2\delta^2$ is divisible by 2. However by Proposition 3.11 **??** and Proposition 3.16 **??**, we have:

$$\mathcal{S} = \left\langle \operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z}); \frac{u_1\cdot u_2 + v_1\cdot v_2 + w_1\cdot w_2}{2}; \frac{u_i^2 - \frac{1}{3}u_i\cdot\delta}{2}; \frac{v_i^2 - \frac{1}{3}v_i\cdot\delta}{2}; \frac{w_i^2 - \frac{1}{3}w_i\cdot\delta}{2}, i\in\{1,2\}\right\rangle.$$

It follows that all the coefficients of $\alpha_1^2 u_1^2 + \alpha_2^2 u_2^2 + \beta_1^2 v_1^2 + \beta_2^2 v_2^2 + \gamma_1^2 w_1^2 + \gamma_2^2 w_2^2 + d^2\delta^2$ are divisible by 2. It means that $x$ is divisible by 2. It is what we wanted to prove. □

With exactly the same proof working in $H^4(\widetilde{K},\mathbb{Z})$ and using Remark 3.6, we provide the following lemma.

**Lemma 3.8.** *The lattice $\pi_{2*}(s^*(H^2(K_2(A),\mathbb{Z})))$ is primitive in $H^2(\widetilde{K},\mathbb{Z})$.*

## 3.6 Calculation of $H^2(K',\mathbb{Z})$

This section is devoted to prove the following theorem.

**Theorem 3.9.** *Let $K'$, $\pi_1$, $s_1$ and $\overline{Z_0}'$ be respectively the variety, the maps and the class defined in Section 3.3. We have*

$$H^2(K',\mathbb{Z}) = \pi_{1*}(s_1^*(H^2(K_2(A),\mathbb{Z})))\oplus\mathbb{Z}\left(\frac{\pi_{1*}(s_1^*(\delta)) + \overline{Z_0}'}{2}\right)\oplus\mathbb{Z}\left(\frac{\pi_{1*}(s_1^*(\delta)) - \overline{Z_0}'}{2}\right).$$

First we need to calculate some intersections.

**Lemma 3.10.** *(1) We have $E_l\cdot E_k = 0$ if $l\neq k$, $E_l^4 = -1$ and $E_l\cdot z = 0$ for all $(l,k)\in\{1,...,28\}^2$ and for all $z\in s^*(H^2(K_2(A),\mathbb{Z}))$.*

*(2) We have $\delta^4 = 324$.*

**Proof.** (1) Same proof as Proposition 4.6.16 1) of [13].

(2) It follows directly from the Fujiki formula.

□

We already have properties of primitivity.

**Lemma 3.11.** *(1) $\pi_{1*}(s_1^*(H^2(K_2(A),\mathbb{Z})))$ is primitive in $H^2(K',\mathbb{Z})$,*

*(2) The group $\widetilde{\mathcal{D}} = \left\langle \widetilde{\overline{Z_0}}, D_1, ..., D_{36}, \frac{\widetilde{\overline{Z_0}}+D_1+...+D_{36}}{2}\right\rangle$ is primitive in $H^2(\widetilde{K},\mathbb{Z})$.*

*(3) $\overline{Z_0}'$ is primitive in $H^2(K',\mathbb{Z})$,*

**Proof.** (1) By Lemma 3.8, $\pi_{2*}(s^*(H^2(K_2(A),\mathbb{Z})))$ is primitive in $H^2(\widetilde{K},\mathbb{Z})$. Then by Remark 3.4, $r'^*(\pi_*(H^2(K_2(A),\mathbb{Z})))$ is primitive in $H^2(K',\mathbb{Z})$. Using again Remark 3.4, we get the result.

(2), (3) See proof of Lemma 4.6.14 of [13].

□

With Lemma 3.11 (1) and (3), it only remains to prove that $\pi_{1*}(s_1^*(\delta)) + \overline{Z_0}'$ is divisible by 2 which will be gone in Lemma 3.15. To prove this lemma, we first prove that $\pi_{2*}(s^*(\delta)) + \widetilde{\overline{Z_0}}$ is divisible by 2. Knowing that $\widetilde{\overline{Z_0}} + D_1 + ... + D_{36}$ is divisible by 2, we have only to show that $\pi_{2*}(s^*(\delta)) + D_1 + ... + D_{36}$ is divisible by 2 which is done by Lemma 3.13 and 3.14.

First we need to know the group $H^3(\widetilde{K}, \mathbb{Z})$.

**Lemma 3.12.** *We have* $H^3(\widetilde{K}, \mathbb{Z}) = 0$.

**Proof**. We have the following exact sequence:

$$H^3(K_2(A), V, \mathbb{Z}) \to H^3(K_2(A), \mathbb{Z}) \xrightarrow{f} H^3(V, \mathbb{Z}) \to H^4(K_2(A), V, \mathbb{Z}) \xrightarrow{\rho} H^4(K_2(A), \mathbb{Z}).$$

By Thom isomorphism, $H^3(K_2(A), V, \mathbb{Z}) = 0$ and $H^4(K_2(A), V, \mathbb{Z}) = H^0(Z_0, \mathbb{Z})$. Moreover $\rho$ is injective, so $H^3(V, \mathbb{Z}) = H^3(K_2(A), \mathbb{Z})$.

Hence by Proposition 2.4 (2) and Proposition 3.2.8 of [13], we find that $H^3(U, \mathbb{Z}) = 0$. Since $H^3(K_2(A), \mathbb{Z})^\iota = 0$, $H^3(\widetilde{K}, \mathbb{Z})$ is a torsion group. Hence the result follows from the exact sequence

$$H^3(\widetilde{K}, U, \mathbb{Z}) \to H^3(\widetilde{K}, \mathbb{Z}) \to H^3(U, \mathbb{Z})$$

and from the fact that $H^3(\widetilde{K}, U, \mathbb{Z}) = 0$ by Thom isomorphism. □

**Lemma 3.13.** *There exists $D_\delta$ which is a linear combination of the $D_i$ with coefficient 0 or 1 such that $\pi_{2*}(s^*(\delta)) + D_\delta$ is divisible by 2.*

**Proof**. First, we have to use Smith theory as in Section 4.6.4 of [13].

Look at the following exact sequence:

$$\begin{aligned} 0 &\to H^2(\widetilde{K}, \widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2)) \to H^2(\widetilde{K}, \mathbb{F}_2) \to H^2(\widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k, \mathbb{F}_2)) \\ &\to H^3(\widetilde{K}, \widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) \longrightarrow 0. \end{aligned}$$

First, we will calculate the dimension of the vector spaces $H^2(\widetilde{K}, \widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2)$ and $H^3(\widetilde{K}, \widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2)$. By (2) of Proposition 1.8, we have

$$H^*(\widetilde{K}, \widetilde{Z_0} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) \simeq H^*_\sigma(N_2).$$

The previous exact sequence gives us the following equation:

$$h^2_\sigma(N_2) - h^2(\widetilde{K}, \mathbb{F}_2) + h^2(\widetilde{Z_0} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) - h^3_\sigma(N_2) = 0.$$

As $h^2(\widetilde{K}, \mathbb{F}_2) = 8 + 36 = 44$ and $h^2(\widetilde{Z_0} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) = 23 + 36 = 59$, we obtain:

$$h^2_\sigma(N_2) - h^3_\sigma(N_2) = -15.$$

Moreover by 2) of Proposition 1.8, we have the exact sequence

$$\begin{aligned} 0 &\to H^1_\sigma(N_2) \to H^2_\sigma(N_2) \to H^2(N_2, \mathbb{F}_2) \to H^2_\sigma(N_2) \oplus H^2(\widetilde{Z_0} \cup (\cup_{k=1}^{36} E_k), \mathbb{F}_2) \\ &\to H^3_\sigma(N_2) \longrightarrow \mathrm{coker} \longrightarrow 0. \end{aligned}$$

By Lemma 7.4 of [3], $h^1_\sigma(N_2) = h^0(\widetilde{Z_0} \cup (\cup_{k=1}^{36} E_k), \mathbb{F}_2) - 1$. Then we get the equation

$$\begin{aligned} &h^0(\widetilde{Z_0} \cup (\cup_{k=1}^{36} E_k), \mathbb{F}_2) - 1 - h^2_\sigma(N_2) + h^2(N_2, \mathbb{F}_2) \\ &- h^2_\sigma(N_2) - h^2(\widetilde{Z_0} \cup (\cup_{k=1}^{36} E_k), \mathbb{F}_2) + h^3_\sigma(N_2) - \alpha = 0, \end{aligned}$$

where $\alpha = \dim \operatorname{coker}$. So
$$21 - \alpha - 2h^2_\sigma(N_2) + h^3_\sigma(N_2) = 0.$$
From the two equations, we deduce that
$$h^2_\sigma(N_2) = 36 - \alpha, \qquad h^3_\sigma(N_2) = 51 - \alpha.$$

Come back to the exact sequence
$$0 \longrightarrow H^2(\widetilde{K}, \widetilde{\widetilde{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) \longrightarrow H^2(\widetilde{K}, \mathbb{F}_2) \xrightarrow{\varsigma^*} H^2(\widetilde{\widetilde{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2),$$
where $\varsigma : \widetilde{\widetilde{Z_0}} \cup (\cup_{k=1}^{36} D_k) \hookrightarrow \widetilde{K}$ is the inclusion. Since $h^2(\widetilde{K}, \widetilde{\widetilde{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) = h^2_\sigma(N_2) = 36 - \alpha$, we have $\dim_{\mathbb{F}_2} \varsigma^*(H^2(\widetilde{K}, \mathbb{F}_2)) = (8+36) - 36 + \alpha = 8 + \alpha$. We can interpret this as follows. Consider the homomorphism
$$\begin{aligned} \varsigma^*_{\mathbb{Z}} : H^2(\widetilde{K}, \mathbb{Z}) &\to H^2(\widetilde{\widetilde{Z_0}}, \mathbb{Z}) \oplus (\oplus_{k=1}^{36} H^2(D_k, \mathbb{Z})) \\ u &\to (u \cdot \widetilde{\widetilde{Z_0}}, u \cdot D_1, ..., u \cdot D_{36}). \end{aligned}$$
Since this is a map of torsion free $\mathbb{Z}$-modules (by Lemma 3.12 and universal coefficient formula), we can tensor by $\mathbb{F}_2$,
$$\varsigma^* = \varsigma^*_{\mathbb{Z}} \otimes \mathrm{id}_{\mathbb{F}_2} : H^2(\widetilde{K}, \mathbb{Z}) \otimes \mathbb{F}_2 \to H^2(\widetilde{\widetilde{Z_0}}, \mathbb{Z}) \oplus (\oplus_{k=1}^{36} H^2(D_k, \mathbb{Z})) \otimes \mathbb{F}_2,$$
and we have $8+\alpha$ independent elements such that the intersection with the $D_k$ $k \in \{1, ..., 36\}$ and $\widetilde{\widetilde{Z_0}}$ are not all zero. But, $\varsigma^*(\pi_{2*}(H^2(N_2, \mathbb{Z}))) = 0$ and $\varsigma^*(\widetilde{\widetilde{Z_0}}, \langle D_1, ..., D_{36} \rangle)$, (it follows from Proposition 1.3). By Lemma 3.11 (2), the element $\widetilde{\widetilde{Z_0}} + D_1 + ... + D_{36}$ is divisible by 2. Hence necessary, it remains $7 + \alpha$ independent elements in $H^2(\widetilde{K}, \mathbb{Z})$ of the form $\frac{u+d}{2}$ with $u \in \pi_{2*}(s^*(H^2(K_2(A), \mathbb{Z})))$ and $d \in \langle D_1, ..., D_{36} \rangle$.

Let denote by $u_1, ..., u_{7+\alpha}$ the $7 + \alpha$ elements in $\pi_{2*}(s^*(H^2(K_2(A), \mathbb{Z})))$ provided above. By Lemma 3.11 (2) $\langle D_1, ..., D_{36} \rangle$ is primitive in $H^2(\widetilde{K}, \mathbb{Z})$. Hence necessary, the element $u_1, ..., u_{7+\alpha}$ view as element in $\pi_{2*}(s^*(H^2(K_2(A), \mathbb{F}_2)))$ are linearly independent. Since $\dim_{\mathbb{F}_2} \pi_{2*}(s^*(H^2(K_2(A), \mathbb{F}_2))) = 7$, it follows that $\alpha = 0$ and $\mathrm{Vect}_{\mathbb{F}_2}(u_1, ..., u_7) = \pi_{2*}(s^*(H^2(K_2(A), \mathbb{F}_2)))$. Hence there exists $D_\delta$ which is a linear combination of the $D_i$ with coefficient 0 or 1 such that $\pi_{2*}(s^*(\delta)) + D_\delta$ is divisible by 2. □

**Lemma 3.14.** *We have:*
$$D_\delta = D_1 + ... + D_{36}.$$

**Proof**. The know that the monodromy acts on $A[2]$ as the symplectic group $\operatorname{Sp} A[2]$. Hence the monodromy action extends naturally to an action on the divisors $D_1, ..., D_{35}$. Also this monodromy action represented by $SpA[3]$ acts trivially on $D_{36}$ and on $\pi_{2*}(s^*(\delta))$. As explaining in Remark 1.4 **??** the 2 orbits of the action of $\operatorname{Sp} A[2]$ on the set $\mathfrak{D} := \{D_1, ..., D_{35}\}$ correspond to the two sets of isotropic and non-isotropic planes in $A[2]$. Hence by Proposition 1.5 (3), (4) **??** the action of $\operatorname{Sp} A[2]$ on the set $\mathfrak{D}$ has 2 orbits: one of 15 elements and another of 20 elements.

On the other hand, as we mentioned in the end of Section 3.1, we can assume that $A = E \times E$ where $E = \frac{\mathbb{C}}{\left\langle 1, e^{\frac{2i\pi}{6}} \right\rangle}$. Hence there is the following automorphism group acting on $A$:
$$G := \left\langle \begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle,$$
where $\rho = e^{\frac{2i\pi}{6}}$. The group $G$ extends naturally to an automorphism group of $N_2$ which we denote also $G$. Moreover the action of $G$ restricts to the set $\mathfrak{D}$. Then by Proposition 5.5 **??** the action of $G$ on $\mathfrak{D}$ has 2 orbits: one of 5 elements and one of 30 elements. Also the group $G$ acts trivially on $D_{36}$ and on $\pi_{2*}(s^*(\delta))$.

Hence the composition of the action of $G$ and the action of $\operatorname{Sp} A[2]$ acts transitively on $\mathfrak{D}$. Since $\pi_{2*}(s^*(\delta))$ is fixed by the action of $G$ and $\operatorname{Sp} A[2]$, $D_\delta$ has also to be fixed by the action of $G$ and $\operatorname{Sp} A[2]$ or it will contradict Lemma 3.11 (2). It follows that there are only 3 possibilities for $D_\delta$:

(1) $D_\delta = D_{36}$,

(2) $D_\delta = D_1 + ... + D_{35}$,

(3) or $D_\delta = D_1 + ... + D_{36}$.

Let $d$ be the number of $D_i$ with coefficient equal to 1 in the linear decomposition of $D_\delta$. The number $d$ can be 1, 35 or 36.

Then from Lemma 3.10 (1), (2) and Proposition 1.3

$$\left(\frac{\pi_{2*}(s^*(\delta)) + D_\delta}{2}\right)^4 = \frac{324 - d}{2}.$$

Hence $d$ has to be divisible by 2. It follows that $D_\delta = D_1 + ... + D_{36}$. □

**Lemma 3.15.** *The class $\pi_{1*}(s_1^*(\delta)) + \overline{Z_0}'$ is divisible by 2.*

**Proof**. We know that $\pi_{2,*}(s^*(\delta)) + \widetilde{\overline{Z_0}}$ is divisible by 2. Indeed by Lemma 3.11 (2), $\widetilde{\overline{Z_0}} + D_1 + ... + D_{36}$ is divisible by 2 and by Lemma 3.13 and 3.14, $\pi_{2,*}(s^*(\delta)) + D_1 + ... + D_{36}$ is divisible by 2.

We can find a Cartier divisor on $\widetilde{K}$ which corresponds to $\frac{\pi_{2*}(s^*(\delta)) + \widetilde{\overline{Z_0}}}{2}$ and which does not meet $\cup_{k=1}^{36} D_k$. Then this Cartier divisor induces a Cartier divisor on $K'$ which necessarily corresponds to half the cocycle $\pi_{1*}(s_1^*(\delta)) + \overline{Z_0}'$. □

## 3.7 Calculation of $B_{K'}$

We finish the proof of Theorem 3.3, calculating $B_{K'}$. We use the notation provided in Section 3.3.

**Lemma 3.16.** *We have*

$$\overline{Z_0}'^2 = -2r^*(\overline{Z_0}).$$

**Proof**. We use the same technique as Lemma 4.6.12 of [13]. Consider the following diagram:

$$\begin{tikzcd} Z_0' \arrow[r, hook, "l_1"] \arrow[d, "g"] & N_1 \arrow[d, "s_1"] \\ Z_0 \arrow[r, hook, "l_0"] & K_2(A), \end{tikzcd}$$

where $l_0$ and $l_1$ are the inclusions and $g := s_{1|Z_0'}$. By Proposition 6.7 of [7], we have:

$$s_1^* l_{0*}(Z_0) = l_{1*}(c_1(E)),$$

where $E := g^*(\mathscr{N}_{Z_0/K_2(A)})/\mathscr{N}_{Z_0'/N_1}$. Hence

$$s_1^* l_{0*}(Z_0) = c_1(g^*(\mathscr{N}_{Z_0/K_2(A)})) - Z_0'^2.$$

Since $K_2(A)$ is hyperkähler and $Z_0$ is a K3 surface, we have $c_1(\mathscr{N}_{Z_0/K_2(A)}) = 0$. So

$$Z_0'^2 = -s_1^* l_{0*}(Z_0).$$

Then the result follows from Proposition 1.3. □

**Proposition 3.17.** *We have the formula*

$$B_{K'}(\pi_{1*}(s_1^*(\alpha), \pi_{1*}(s_1^*(\beta))) = 6\sqrt{\frac{2}{C_{K'}}} B_{K_2(A)}(\alpha, \beta),$$

*where $C_{K'}$ is the Fujiki constant of $K'$ and $\alpha$, $\beta$ are in $H^2(K_2(A), \mathbb{Z})^\iota$ and $B_{K_2(A)}$ is the Beauville–Bogomolov form of $K_2(A)$.*

**Proof**. An easy use of the Fujiki formula provide the following proposition (same proof as Proposition 4.6.10 of [13]).

By (1) of Theorem 3.1, we have

$$(\pi_{1*}(s_1^*(\alpha)))^4 = C_{K'}B_{K'}(\pi_{1*}(s_1^*(\alpha), \pi_{1*}(s_1^*(\alpha)))^2.$$

And

$$\alpha^4 = 9B_{K_2(A)}(\alpha,\alpha)^2.$$

Moreover, by Proposition 1.3,

$$(\pi_{1*}(s^*(\alpha)))^4 = 8s^*(\alpha)^4 = 8\alpha^4.$$

By Point (2) of Theorem 3.1, we get the result. $\square$

In particular, it follows:

$$B_{K'}(\pi_{1*}(s_1^*(\delta), \pi_{1*}(s_1^*(\delta))) = -36\sqrt{\frac{2}{C_{K'}}} \tag{2}$$

**Lemma 3.18.**

$$B_{K'}(\pi_{1*}(s_1^*(\alpha)), \overline{Z_0}') = 0,$$

*for all* $\alpha \in H^2(S^{[2]},\mathbb{Z})^\iota$.

**Proof**. We have $\pi_{1*}(s_1^*(\alpha))^3 \cdot \overline{Z_0}' = 8s_1^*(\alpha)^3 \cdot \Sigma_1$ by Proposition 1.3, and $s_{1*}(s_1^*(\alpha^3)\cdot Z_0') = \alpha^3 \cdot s_{1*}(Z_0') = 0$ by the projection formula. We conclude by Proposition 3.2. $\square$

**Lemma 3.19.** *We have:*

$$B_{K'}(\overline{Z_0}', \overline{Z_0}') = -4\sqrt{\frac{2}{C_{K'}}}.$$

**Proof**. We have:

$$\begin{aligned}
\overline{Z_0}'^2 \cdot \pi_{1*}(s_1^*(\delta))^2 &= \frac{C_{K'}}{3}B_{M'}(\overline{Z_0}', \overline{Z_0}') \times B_{K'}(\pi_{1*}(s_1^*(\delta)), \pi_{1*}(s_1^*(\delta)))\\
&= \frac{C_{M'}}{3}B_{K'}(\overline{Z_0}', \overline{Z_0}') \times \left(-36\sqrt{\frac{2}{C_{K'}}}\right)\\
&= -12\sqrt{2C_{K'}}B_{K'}(\overline{Z_0}', \overline{Z_0}') \qquad (3)
\end{aligned}$$

By Proposition 1.3, we have

$$\overline{Z_0}'^2 \cdot \pi_{1*}(s_1^*(\delta))^2 = 8Z_0'^2 \cdot (s_1^*(\delta))^2. \tag{4}$$

By the projection formula, $Z_0'^2 \cdot (s_1^*(\delta))^2 = s_{1*}(Z_0'^2)\cdot\delta^2$. Moreover by lemma 3.16, $s_{1*}(Z_0'^2) = -Z_0$. Hence

$$Z_0'^2 \cdot (s_1^*(\delta))^2 = -Z_0 \cdot \delta^2. \tag{5}$$

It follows from (3), (4) and (5) that

$$-8Z_0 \cdot \delta^2 = -12\sqrt{2C_{K'}}B_{K'}(\overline{Z_0}', \overline{Z_0}'). \tag{6}$$

Moreover from Section 4 of [8], we have:

$$Z_0 \cdot \delta^2 = -12. \tag{7}$$

So by (6) and (7):

$$B_{K'}(\overline{Z_0}', \overline{Z_0}') = -8\sqrt{\frac{1}{2C_{K'}}}.$$

$\square$

Now we are able to finish the calculation of the Beauville–Bogomolov form on $H^2(K', \mathbb{Z})$. By (2), Propositions 3.17, Lemma 3.18, 3.19 and Theorem 3.9, the Beauville–Bogomolov form on $H^2(K', \mathbb{Z})$ gives the lattice:

$$U^3\left(6\sqrt{\frac{2}{C_{K'}}}\right) \oplus -\frac{1}{4}\sqrt{\frac{2}{C_{K'}}}\begin{pmatrix} 40 & 32 \\ 32 & 40 \end{pmatrix}$$
$$= U^3\left(6\sqrt{\frac{2}{C_{K'}}}\right) \oplus -\sqrt{\frac{2}{C_{K'}}}\begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix}$$

It follows that $C_{K'} = 8$, and we get Theorem 3.3.

### 3.8 Betti numbers and Euler characteristic of $K'$

**Proposition 3.20.** *We have:*

- $b_2(K') = 8$,
- $b_3(K') = 0$,
- $b_4(K') = 90$,
- $\chi(K') = 108$.

**Proof.** It is the same proof as Proposition 4.7.2 of [13]. From Theorem 7.31 of [24] and Proposition 2.4, we get the betti numbers. Then $\chi(K') = 1 - 0 + 8 - 0 + 90 - 0 + 8 - 0 + 1 = 108$. $\square$

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