

PLANES IN SYMPLECTIC VECTOR SPACES

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1. SYMPLECTIC LINEAR ALGEBRA

Let V be a symplectic vector space of dimension $n \in 2\mathbb{N}$ over a field F with a nondegenerate symplectic form $\omega : \Lambda^2 V \rightarrow F$. A line is a one-dimensional subspace of V , a plane is a two-dimensional subspace of V . A plane $P \subset V$ is called isotropic, if $\omega(x, y) = 0$ for any $x, y \in P$, otherwise non-isotropic. The symplectic group $\mathrm{Sp} V$ is the set of all linear maps $\phi : V \rightarrow V$ with the property $\omega(\phi(x), \phi(y)) = \omega(x, y)$ for all $x, y \in V$.

Proposition 1.1. *The symplectic group $\mathrm{Sp} V$ acts transitively on the set of non-isotropic planes as well as on the set of isotropic planes.*

Proof. Given two planes P_1 and P_2 , we may choose vectors v_1, v_2, w_1, w_2 such that v_1, v_2 span P_1 , w_1, w_2 span P_2 and $\omega(u_1, u_2) = \omega(w_1, w_2)$. We complete $\{v_1, v_2\}$ as well as $\{w_1, w_2\}$ to a symplectic basis of V . Then define $\phi(v_1) = w_1$ and $\phi(v_2) = w_2$. It is now easy to see that the definition of ϕ can be extended to the remaining basis elements to give a symplectic morphism. \square

Remark 1.2. The set of planes in V can be identified with the simple tensors in $\Lambda^2 V$ up to multiples. Indeed, given a simple tensor $v \wedge w \in \Lambda^2 V$, the span of v and w yields the corresponding plane. Conversely, any two spanning vectors v and w of a plane give the same element $v \wedge w$ (up to multiples).

Proposition 1.3. *If $\phi \in \mathrm{Sp} V$ acts through multiplication of a scalar, $\phi(v) = \alpha v$, then $\alpha = \pm 1$ (this is immediate from the definition). Moreover, if $\phi(v) \wedge \phi(w) = \alpha v \wedge w$, then $\alpha = 1$.*

Proof. We may assume that V is two-dimensional, generated by v and w . Our condition on ϕ reads then $\det \phi = \alpha$. But the condition on ϕ being symplectic is $\det \phi = 1$, because on a two-dimensional vector space there is only one symplectic form up to scalar multiple. \square

Remark 1.4. If F is the field with two elements, then the set of planes in V can be identified with the set $\{\{x, y, z\} \mid x, y, z \in V \setminus \{0\}, x + y + z = 0\}$. Observe that for such a $\{x, y, z\}$, $\omega(x, y) = \omega(x, y) = \omega(y, x)$ and this value gives the criterion for isotropy.

From now on we assume that F is finite of cardinality q .

Proposition 1.5.

- (1) The number of lines in V is $\frac{q^n - 1}{q - 1}$,
- (2) the number of planes in V is $\frac{(q^n - 1)(q^{n-1} - 1)}{(q^2 - 1)(q - 1)}$,
- (3) the number of isotropic planes in V is $\frac{(q^n - 1)(q^{n-2} - 1)}{(q^2 - 1)(q - 1)}$,
- (4) the number of non-isotropic planes in V is $\frac{q^{n-2}(q^n - 1)}{q^2 - 1}$.

Proof. A line ℓ in V is determined by a nonzero vector. There are $q^n - 1$ nonzero vectors in V and $q - 1$ nonzero vectors in ℓ . A plane P is determined by a line $\ell_1 \subset V$ and a unique second line $\ell_2 \in V/\ell_1$. We have $\frac{q^2 - 1}{q - 1}$ lines in P . The number of planes is therefore

$$\frac{\frac{q^n - 1}{q - 1} \cdot \frac{q^{n-1} - 1}{q - 1}}{\frac{q^2 - 1}{q - 1}} = \frac{(q^n - 1)(q^{n-1} - 1)}{(q^2 - 1)(q - 1)}.$$

For an isotropic plane we have to choose the second line from ℓ_1^\perp/ℓ_1 . This is a space of dimension $n - 2$, hence the formula. The number of non-isotropic planes is the difference of the two previous numbers. \square

Conjecture 1.6. *There are $6q$ orbits of the induced action of $\mathrm{Sp}(4, q)$ on $\Lambda^2 \mathbb{F}_q^4$.*

2. CONSTRUCTION OF AN ALGEBRA

Assume now that V is a four-dimensional vector space over $F = \mathbb{F}_q$. Consider the free F -module $F[V]$ with basis $\{X_i \mid i \in V\}$. It carries a natural F -algebra structure given by $X_i \cdot X_j := X_{i+j}$ with unit $1 = X_0$.

We introduce an action of $\mathrm{Sp}(4, F)$ on $F[V]$ by setting $\phi(X_i) = X_{\phi(i)}$. Furthermore, the underlying additive group of V acts on $F[V]$ by $v \cdot X_i = X_{i+v}$.

For two elements $v, w \in V$ with $\omega(v, w) = 1$, let $P \subset V$ be the non-isotropic plane spanned by v and w . Then set

$$y := \sum_{i \in P} X_i \in F[V].$$

Proposition 2.1. *Consider the images of y under the action of the symplectic group: $B := \{\phi(y) \mid \phi \in \mathrm{Sp} V\} = \{\sum_{i \in P} X_i \mid P \subset V \text{ non-isotropic plane}\}$. Then B spans a subspace of Π' of dimension*

- 10, if $F = \mathbb{F}_2$,
- 30, if $F = \mathbb{F}_3$,
- 121, if $F = \mathbb{F}_5$.

Proposition 2.2. *Now consider the span of $\{v \cdot b \mid b \in B, v \in V\}$. Note that this is the ideal in $F[V]$ generated by B . It is a space of dimension*

- 11, if $F = \mathbb{F}_2$,
- 50, if $F = \mathbb{F}_3$,
- 355, if $F = \mathbb{F}_5$.

Proposition 2.3. *Let $D := \{v \cdot b - b \mid b \in B, v \in V\}$. The span of D is a space of dimension*

- 5, if $F = \mathbb{F}_2$,
- 31, if $F = \mathbb{F}_3$,
- 270, if $F = \mathbb{F}_5$.