# PLANES IN SYMPLECTIC VECTOR SPACES

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## 1. Symplectic linear algebra

Let $V$ be a symplectic vector space of dimension $n \in 2\mathbb{N}$ over a field $F$ with a nondegenerate symplectic form $\omega : \Lambda^2 V \to F$. A line is a one-dimensional subspace ov $V$, a plane is a two-dimensional subspace of $V$. A plane $P \subset V$ is called isotropic, if $\omega(x, y) = 0$ for any $x, y \in P$, otherwise non-isotropic. The symplectic group $\operatorname{Sp} V$ is the set of all linear maps $\phi : V \to V$ with the property $\omega(\phi(x), \phi(y)) = \omega(x, y)$ for all $x, y \in V$.

**Proposition 1.1.** *The symplectic group* $\operatorname{Sp} V$ *acts transitively on the set of non-isotropic planes as well as on the set of isotropic planes.*

*Proof.* Given two planes $P_1$ and $P_2$, we may choose vectors $v_1, v_2, w_1, w_2$ such that $v_1, v_2$ span $P_1$, $w_1, w_2$ span $P_2$ and $\omega(u_1, u_2) = \omega(w_1, w_2)$. We complete $\{v_1, v_2\}$ as well as $\{w_1, w_2\}$ to a symplectic basis of $V$. Then define $\phi(v_1) = w_1$ and $\phi(v_2) = w_2$. It is now easy to see that the definition of $\phi$ can be extended to the remaining basis elements to give a symplectic morphism. □

*Remark* 1.2. The set of planes in $V$ can be identified with the simple tensors in $\Lambda^2 V$ up to multiples. Indeed, given a simple tensor $v \wedge w \in \Lambda^2 V$, the span of $v$ and $w$ yields the corresponding plane. Conversely, any two spanning vectors $v$ and $w$ of a plane give the same element $v \wedge w$ (up to multiples).

**Proposition 1.3.** *If* $\phi \in \operatorname{Sp} V$ *acts through multiplication of a scalar,* $\phi(v) = \alpha v$*, then* $\alpha = \pm 1$ *(this is immediate from the definition). Moreover, if* $\phi(v) \wedge \phi(w) = \alpha v \wedge w$*, then* $\alpha = 1$*.*

*Proof.* We may assume that $V$ is two-dimensional, generated by $v$ and $w$. Our condition on $\phi$ reads then $\det \phi = \alpha$. But the condition on $\phi$ being symplectic is $\det \phi = 1$, because on a two-dimensional vector space there is only one symplectic form up to scalar multiple. □

*Remark* 1.4. If $F$ is the field with two elements, then the set of planes in $V$ can be identified with the set $\{\{x, y, z\} \mid x, y, z \in V \backslash \{0\},\ x + y + z = 0\}$. Observe that for such a $\{x, y, z\}$, $\omega(x, y) = \omega(x, y) = \omega(y, x)$ and this value gives the criterion for isotropy.

From now on we assume that $F$ is finite of cardinality $q$.

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**Proposition 1.5.**

$$\text{(1)} \qquad \textit{The number of lines in } V \textit{ is } \frac{q^n-1}{q-1},$$

$$\text{(2)} \qquad \textit{the number of planes in } V \textit{ is } \frac{(q^n-1)(q^{n-1}-1)}{(q^2-1)(q-1)},$$

$$\text{(3)} \qquad \textit{the number of isotropic planes in } V \textit{ is } \frac{(q^n-1)(q^{n-2}-1)}{(q^2-1)(q-1)},$$

$$\text{(4)} \qquad \textit{the number of non-isotropic planes in } V \textit{ is } \frac{q^{n-2}(q^n-1)}{q^2-1}.$$

*Proof.* A line $\ell$ in $V$ is determined by a nonzero vector. There are $q^n-1$ nonzero vectors in $V$ and $q-1$ nonzero vectors in $\ell$. A plane $P$ is determined by a line $\ell_1 \subset V$ and a unique second line $\ell_2 \in V/\ell_1$. We have $\frac{q^2-1}{q-1}$ lines in $P$. The number of planes is therefore

$$\frac{\frac{q^n-1}{q-1} \cdot \frac{q^{n-1}-1}{q-1}}{\frac{q^2-1}{q-1}} = \frac{(q^n-1)(q^{n-1}-1)}{(q^2-1)(q-1)}.$$

For an isotropic plane we have to choose the second line from $\ell_1^\perp/\ell_1$. This is a space of dimension $n-2$, hence the formula. The number of non-isotropic planes is the difference of the two previous numbers. □

**Conjecture 1.6.** *There are* $6q$ *orbits of the induced action of* $\mathrm{Sp}(4,q)$ *on* $\Lambda^2\mathbb{F}_q^4$.

## 2. Orthogonal sums of lattices

Assume now that $V$ is a four-dimensional vector space over $\mathbb{F}_q$. Consider the free $\mathbb{Z}$-module $\mathbb{Z}[V]$ with basis $\{X_i \mid i \in V\}$. It carries a natural ring structure given by $X_i \cdot X_j := X_{i+j}$.