PLANES IN SYMPLECTIC VECTOR SPACES

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1. Symplectic linear algebra

Let V be a symplectic vector space of dimension $n \in 2\mathbb{N}$ over a field F with a nondegenerate symplectic form $\omega : \Lambda^2 V \to F$. A line is a one-dimensional subspace ov V, a plane is a two-dimensional subspace of V. A plane $P \subset V$ is called isotropic, if $\omega(x, y) = 0$ for any $x, y \in P$, otherwise non-isotropic. The symplectic group $\operatorname{Sp} V$ is the set of all linear maps $\phi : V \to V$ with the property $\omega(\phi(x), \phi(y)) = \omega(x, y)$ for all $x, y \in V$.

Proposition 1.1. *The symplectic group* $\operatorname{Sp} V$ *acts transitively on the set of non-isotropic planes as well as on the set of isotropic planes.*

Proof. Given two planes P_1 and P_2, we may choose vectors v_1, v_2, w_1, w_2 such that v_1, v_2 span P_1, w_1, w_2 span P_2 and $\omega(u_1, u_2) = \omega(w_1, w_2)$. We complete $\{v_1, v_2\}$ as well as $\{w_1, w_2\}$ to a symplectic basis of V. Then define $\phi(v_1) = w_1$ and $\phi(v_2) = w_2$. It is now easy to see that the definition of ϕ can be extended to the remaining basis elements to give a symplectic morphism. $\square$

Remark 1.2. The set of planes in V can be identified with the simple tensors in $\Lambda^2 V$ up to multiples. Indeed, given a simple tensor $v \wedge w \in \Lambda^2 V$, the span of v and w yields the corresponding plane. Conversely, any two spanning vectors v and w of a plane give the same element $v \wedge w$ (up to multiples).

Proposition 1.3. *If* $\phi \in \operatorname{Sp} V$ *acts through multiplication of a scalar,* $\phi(v) = \alpha v$, *then* $\alpha = \pm 1$ *(this is immediate from the definition). Moreover, if* $\phi(v) \wedge \phi(w) = \alpha v \wedge w$, *then* $\alpha = 1$.

Proof. We may assume that V is two-dimensional, generated by v and w. Our condition on ϕ reads then $\det \phi = \alpha$. But the condition on ϕ being symplectic is $\det \phi = 1$, because on a two-dimensional vector space there is only one symplectic form up to scalar multiple. $\square$

Remark 1.4. If F is the field with two elements, then the set of planes in V can be identified with the set $\{\{x, y, z\} \mid x, y, z \in V \backslash \{0\},\ x + y + z = 0\}$. Observe that for such a $\{x, y, z\}$, $\omega(x, y) = \omega(x, y) = \omega(y, x)$ and this value gives the criterion for isotropy.

Date: February 14, 2016.

Proposition 1.5. *Assume that F is finite of cardinality q.*

$$\text{(1)} \qquad \textit{The number of lines in } V \textit{ is } \frac{q^n-1}{q-1},$$

$$\text{(2)} \qquad \textit{the number of planes in } V \textit{ is } \frac{(q^n-1)(q^{n-1}-1)}{(q^2-1)(q-1)},$$

$$\text{(3)} \qquad \textit{the number of isotropic planes in } V \textit{ is } \frac{(q^n-1)(q^{n-2}-1)}{(q^2-1)(q-1)},$$

$$\text{(4)} \qquad \textit{the number of non-isotropic planes in } V \textit{ is } \frac{q^{n-2}(q^n-1)}{q^2-1}.$$

Proof. A line ℓ in V is determined by a nonzero vector. There are q^n-1 nonzero vectors in V and $q-1$ nonzero vectors in ℓ. A plane P is determined by a line $\ell_1 \subset V$ and a unique second line $\ell_2 \in V/\ell_1$. We have $\frac{q^2-1}{q-1}$ lines in P. The number of planes is therefore

$$\frac{\frac{q^n-1}{q-1} \cdot \frac{q^{n-1}-1}{q-1}}{\frac{q^2-1}{q-1}} = \frac{(q^n-1)(q^{n-1}-1)}{(q^2-1)(q-1)}.$$

For an isotropic plane we have to choose the second line from $\ell_1^\perp/\ell_1$. This is a space of dimension $n-2$, hence the formula. The number of non-isotropic planes is the difference of the two previous numbers. □

Conjecture 1.6. *There are $6q$ orbits of the induced action of $\mathrm{Sp}(4,q)$ on $\Lambda^2\mathbb{F}_q^4$.*

2. Symplectic vector spaces as index sets

Assume now that V is a four-dimensional vector space over $F = \mathbb{F}_q$. Consider the free F-module $F[V]$ with basis $\{X_i \mid i \in V\}$. It carries a natural F-algebra structure, given by $X_i \cdot X_j := X_{i+j}$ with unit $1 = X_0$. Let J be the ideal generated by all elements of the form $(X_i - 1)$.

We introduce an action of $\mathrm{Sp}(4,F)$ on $F[V]$ by setting $\phi(X_i) = X_{\phi(i)}$. Furthermore, the underlying additive group of V acts on $F[V]$ by $v(X_i) = X_{i+v} = X_v X_i$.

Definition 2.1. We define subsets of $F[V]$:

$$B_N := \left\{ \sum_{i\in P} X_i \mid P \subset V \text{ non-isotropic plane} \right\},$$

$$B_I := \left\{ \sum_{i\in P} X_i \mid P \subset V \text{ isotropic plane} \right\}.$$

Denote by $\langle B_\alpha \rangle$ and by (B_α) the linear span of B_α and the ideal generated by B_α, respectively. Note that (B_α) is the linear span of $\{v \cdot b \mid b \in B, v \in V\}$. Further, let D_α be the linear span of $\{v(b) - b \mid b \in B, v \in V\}$. Then D_α is in fact an ideal, namely the product of ideals $J \cdot (B_\alpha)$.

The following table illustrates the dimensions of these objects:

F	$\dim_F \langle B_N \rangle$	$\dim_F (B_N)$	$\dim_F D_N$	$\dim_F \langle B_I \rangle$	$\dim_F (B_I)$	$\dim_F D_I$
$\mathbb{F}_2$	10	11	5	10	10	10
$\mathbb{F}_3$	30	50	31	25	25	25
$\mathbb{F}_5$	121	355	270	91	91	91

Conjecture 2.2. *For* $F = \mathbb{F}_q$, $\dim_F \langle B_I \rangle = \dim_F (B_I) = \dim_F D_I = \frac{(q+2)(q^2+1)}{2}$.