Integral cohomology of the Generalized Kummer fourfold

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1 Introduction

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Definition 3.1. A super vector space V over a field k is a vector space with a $\mathbb{Z}/2\mathbb{Z}$-graduation, that is a decomposition

$$V = V^+ \oplus V^-,$$

called the even and the odd part of V. Elements of V^+ are called homogeneous of even degree, elements of V^- are called homogeneous of odd degree. The degree of a homogeneous element v is denoted by $|v| \in \mathbb{Z}/2\mathbb{Z}$. Direct sum and tensor product of two super vector spaces V and W yield again super vector spaces:

$$\begin{aligned} (V \oplus W)^+ &= V^+ \oplus W^+, & (V \oplus W)^- &= V^- \oplus W^-, \\ (V \otimes W)^+ &= (V^+ \otimes W^+) \oplus (V^- \otimes W^-), & (V \otimes W)^- &= (V^+ \otimes W^-) \oplus (V^- \otimes W^+). \end{aligned}$$

Definition 3.2. A superalgebra R is a unital associative k-algebra which carries a super vector space structure. Define the supercommutator by setting for homogeneous elements $u, v \in R$:

$$[u, v] := uv - (-1)^{|u||v|} vu.$$

R is called supercommutative, if $[u, v] = 0$ for all $u, v \in R$. Note that a graded commutative algebra $R = \bigoplus_n R^n$ is supercommutative in a natural way, by setting $R^+ = \bigoplus_{n \text{ even}} R^n$, $R^- = \bigoplus_{n \text{ odd}} R^n$.

For a supercommutative algebra R, the tensor power $R^{\otimes n}$ is again a supercommutative algebra, if we set for the product:

$$(u_1 \otimes \cdots \otimes u_n) \cdot (v_1 \otimes \cdots \otimes v_n) = (-1)^{\sum\limits_{i>j} |u_i||v_j|} u_1 v_1 \otimes \cdots \otimes u_n v_n.$$

Definition 3.3. Let V be a super vector space over k and $n \geq 0$. Then the supersymmetric power $\mathrm{SSym}^n(V)$ of V is a super vector space, given by

$$\mathrm{SSym}^n(V) = \bigoplus_{p+q=n} \mathrm{Sym}^p(V^+) \otimes \Lambda^q(V^-),$$

$$\mathrm{SSym}^n(V)^+ = \bigoplus_{\substack{p+q=n \\ q \text{ even}}} \mathrm{Sym}^p(V^+) \otimes \Lambda^q(V^-), \qquad \mathrm{SSym}^n(V)^- = \bigoplus_{\substack{p+q=n \\ q \text{ odd}}} \mathrm{Sym}^p(V^+) \otimes \Lambda^q(V^-).$$

The supersymmetric algebra $\mathrm{SSym}^*(V) := \bigoplus_n \mathrm{SSym}^n(V)$ on V is a supercommutative algebra over k, where the product of two elements $s \otimes e \in \mathrm{Sym}^p(V^+) \otimes \Lambda^q(V^-)$ and $s' \otimes e' \in \mathrm{Sym}^{p'}(V^+) \otimes \Lambda^{q'}(V^-)$ is given by

$$(s \otimes e) \diamond (s' \otimes e') = (ss') \otimes (e \wedge e') \quad \in \mathrm{Sym}^{p+p'}(V^+) \otimes \Lambda^{q+q'}(V^-).$$

Remark 3.4. The supersymmetric power $\mathrm{SSym}^n(V)$ can be realized as a quotient of $V^{\otimes n}$ by an action of the symmetric group $\mathfrak{S}_n$. This action can be described as follows: If $\tau \in \mathfrak{S}_n$ is a transposition that exchanges two numbers $i < j$, then τ permutes the corresponding tensor factors in $v_1 \otimes \cdots \otimes v_n$ introducing a sign $(-1)^{|v_i||v_j| + (|v_i|+|v_j|)\sum_{i<k<j}|v_k|}$.

Now let U be a vector space over a field k of characteristic 0 and look at the exterior algebra $H := \Lambda^* U$. Since H is a super vector space, we can construct the supersymmetric power $S^n := \mathrm{SSym}^n(H)$. We may identify S^n with the space of $\mathfrak{S}_n$-invariants in $H^{\otimes n}$ by means of the linear projection operator

$$\mathrm{pr} : H^{\otimes n} \longrightarrow S^n, \quad \mathrm{pr} = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \pi.$$

The multiplication in $H^{\otimes n}$ induces a multiplication on the subspace of invariants, which makes S^n a supercommutative algebra. Of course, it is different from the product $\diamond$ discussed above.

Since H is generated as an algebra by $U = \Lambda^1(U) \subset H$, we may define a homomorphism of algebras:

$$s : H \longrightarrow S^n, \quad s(u) = \mathrm{pr}(u \otimes 1 \otimes \cdots \otimes 1) \text{ for } u \in U,$$

so S becomes an algebra over H.

Lemma 3.5. *The morphism s turns S^n into a free module over H, for $n \geq 1$.*

Proof. We start with the tensor power $H^{\otimes n}$ and the ring homomorphism

$$\iota : H \longrightarrow H^{\otimes n}, \quad h \longmapsto h \otimes 1 \otimes \cdots \otimes 1$$

that makes $H^{\otimes n}$ a free H-module. Note that $\mathrm{pr}\,\iota \neq s$, since pr is not a ring homomorphism. (For example, $\mathrm{pr}\,\iota(h) \neq s(h)$ for any nonzero $h \in \Lambda^2(U)$.) We therefore modify the H-module structure of $H^{\otimes n}$:

For some $u \in U$, denote $u^{(i)} := 1^{\otimes i-1} \otimes u \otimes 1^{\otimes n-i+1} \in H^{\otimes n}$. Then $H^{\otimes n}$ is generated as a k-algebra by the elements $\{u^{(i)}, u \in U\}$. Now consider the ring automorphism

$$\sigma : H^{\otimes n} \longrightarrow H^{\otimes n}, \quad u^{(1)} \longmapsto u^{(1)} + u^{(2)} + \ldots + u^{(n)}, \quad u^{(i)} \longmapsto u^{(i)} \text{ for } i > 1.$$

Then we have $\sigma\iota = s$ on S^n. On the other hand, if $\{b_i\}$ is a k-basis of V, then $\{b_i^{(j)}, j > 1\}$ is a ι-basis for $H^{\otimes n}$, and $\{\sigma(b_i^{(j)})\}$ is a $\sigma\iota$-basis for $H^{\otimes n}$. Now if we project the basis elements, we get a set $\{\mathrm{pr}(\sigma(b_i^{(j)}))\}$ that spans S^n. Eliminating linear dependent vectors (this is possible over the rationals), we get a s-basis of S^n. $\square$

4 Actions of the symplectic group over finite fields

Let V be a symplectic vector space of dimension $n \in 2\mathbb{N}$ over a field k with a nondegenerate symplectic form $\omega : \Lambda^2 V \to k$. A line is a one-dimensional subspace ov V, a plane is a two-dimensional subspace of V. A plane $P \subset V$ is called isotropic, if $\omega(x, y) = 0$ for any $x, y \in P$, otherwise non-isotropic. The symplectic group $\mathrm{Sp}\,V$ is the set of all linear maps $\phi : V \to V$ with the property $\omega(\phi(x), \phi(y)) = \omega(x, y)$ for all $x, y \in V$.

Proposition 4.1. *The symplectic group $\mathrm{Sp}\,V$ acts transitively on the set of non-isotropic planes as well as on the set of isotropic planes.*

Proof. Given two planes P_1 and P_2, we may choose vectors v_1, v_2, w_1, w_2 such that v_1, v_2 span P_1, w_1, w_2 span P_2 and $\omega(u_1, u_2) = \omega(w_1, w_2)$. We complete $\{v_1, v_2\}$ as well as $\{w_1, w_2\}$ to a symplectic basis of V. Then define $\phi(v_1) = w_1$ and $\phi(v_2) = w_2$. It is now easy to see that the definition of ϕ can be extended to the remaining basis elements to give a symplectic morphism. $\square$

Remark 4.2. The set of planes in V can be identified with the simple tensors in $\Lambda^2 V$ up to multiples. Indeed, given a simple tensor $v \wedge w \in \Lambda^2 V$, the span of v and w yields the corresponding plane. Conversely, any two spanning vectors v and w of a plane give the same element $v \wedge w$ (up to multiples).

Proposition 4.3. *If $\phi \in \operatorname{Sp} V$ acts through multiplication of a scalar, $\phi(v) = \alpha v$, then $\alpha = \pm 1$ (this is immediate from the definition). Moreover, if $\phi(v) \wedge \phi(w) = \alpha v \wedge w$, then $\alpha = 1$.*

Proof. We may assume that V is two-dimensional, generated by v and w. Our condition on ϕ reads then $\det \phi = \alpha$. But the condition on ϕ being symplectic is $\det \phi = 1$, because on a two-dimensional vector space there is only one symplectic form up to scalar multiple. $\square$

Remark 4.4. If k is the field with two elements, then the set of planes in V can be identified with the set $\{\{x, y, z\} \mid x, y, z \in V \backslash \{0\}, x + y + z = 0\}$. Observe that for such a $\{x, y, z\}$, $\omega(x, y) = \omega(x, y) = \omega(y, x)$ and this value gives the criterion for isotropy.

Proposition 4.5. *Assume that k is finite of cardinality q.*

$$\text{The number of lines in } V \text{ is } \frac{q^n - 1}{q - 1}, \tag{1}$$

$$\text{the number of planes in } V \text{ is } \frac{(q^n - 1)(q^{n-1} - 1)}{(q^2 - 1)(q - 1)}, \tag{2}$$

$$\text{the number of isotropic planes in } V \text{ is } \frac{(q^n - 1)(q^{n-2} - 1)}{(q^2 - 1)(q - 1)}, \tag{3}$$

$$\text{the number of non-isotropic planes in } V \text{ is } \frac{q^{n-2}(q^n - 1)}{q^2 - 1}. \tag{4}$$

Proof. A line ℓ in V is determined by a nonzero vector. There are $q^n - 1$ nonzero vectors in V and $q - 1$ nonzero vectors in ℓ. A plane P is determined by a line $\ell_1 \subset V$ and a unique second line $\ell_2 \in V/\ell_1$. We have $\frac{q^2-1}{q-1}$ choices for ℓ_1 in P. The number of planes is therefore

$$\frac{\frac{q^n-1}{q-1} \cdot \frac{q^{n-1}-1}{q-1}}{\frac{q^2-1}{q-1}} = \frac{(q^n - 1)(q^{n-1} - 1)}{(q^2 - 1)(q - 1)}.$$

For an isotropic plane we have to choose the second line from $\ell_1^\perp/\ell_1$. This is a space of dimension $n - 2$, hence the formula. The number of non-isotropic planes is the difference of the two previous numbers. $\square$

Assume now that V is a four-dimensional vector space over $k = \mathbb{F}_q$. Consider the free k-module $k[V]$ with basis $\{X_i \mid i \in V\}$. It carries a natural k-algebra structure, given by $X_i \cdot X_j := X_{i+j}$ with unit $1 = X_0$. This algebra is local with maximal ideal $\mathfrak{m}$ generated by all elements of the form $(X_i - 1)$.

We introduce an action of $\operatorname{Sp}(4, k)$ on $k[V]$ by setting $\phi(X_i) = X_{\phi(i)}$. Furthermore, the underlying additive group of V acts on $k[V]$ by $v(X_i) = X_{i+v} = X_i X_v$.

Definition 4.6. We define a subset of $k[V]$:

$$N := \left\{ \sum_{i \in P} X_i \mid P \subset V \text{ non-isotropic plane} \right\}.$$

Denote by $\langle N \rangle$ and by (N) the linear span of N and the ideal generated by N, respectively. Note that (N) is the linear span of $\{v \cdot b \mid b \in N, v \in V\}$. Further, let D be the linear span of $\{v(b) - b \mid b \in N, v \in V\}$. Then D is in fact an ideal, namely the product of ideals $\mathfrak{m} \cdot (N)$.

The following table illustrates the dimensions of these objects for some fields k:

k	$\dim_k \langle N \rangle$	$\dim_k (N)$	$\dim_k D$
$\mathbb{F}_2$	10	11	5
$\mathbb{F}_3$	30	50	31
$\mathbb{F}_5$	121	355	270

Remark 4.7. When $k = F_3$, we remark that $\sum_{i \in V} X_i \in D$.

Let us now consider some special orthogonal sums. Set $S := \mathrm{Sym}^2(\Lambda^2 V)$. Take two vectors $v, w \in V$ with $\omega(v, w) = 1$ and set $x := (v \wedge w)^2 \in S$. Denote P the plane spanned by v and w and set $y := \sum_{i \in P} X_i \in k[V]$. We set $Y' := y \cdot \mathfrak{m} = \{\sum_{i \in P} X_{i+j} - X_i \mid j \in V\}$.

We consider now the action of $\mathrm{Sp}\, V$ on $S \oplus k[V]$. Denote O_1 the vector space spanned by the elements $\phi(x) \oplus \phi(z)$, for $\phi \in \mathrm{Sp}\, V$, $z \in (y)$, denote O_2 the vector space spanned by the elements $\phi(x) \oplus \phi(y')$, for $\phi \in \mathrm{Sp}\, V$, $y' \in Y'$ and U the vector space spanned by the elements $\phi(x)$, for $\phi \in \mathrm{Sp}\, V$. Then we have:

k	$\dim_k O_1$	$\dim_k O_2$	$\dim_k U$
$\mathbb{F}_2$	11	10	
$\mathbb{F}_3$	51	50	20
$\mathbb{F}_5$	375	289	

Now, we prove the following lemma.

Lemma 4.8. *We assume that $k = \mathbb{F}_3$. Let $\mathrm{pr}_1 : S \oplus k[V] \to S$ and $\mathrm{pr}_2 : S \oplus k[V] \to k[V]$ the projection. We have:* $\dim \mathrm{Ker}\, \mathrm{pr}_{2|O_1} = 1$.

Proof. We have $\mathrm{pr}_2(O_1) = (N)$, so $\dim \mathrm{pr}_2(O_1) = 50$. Since $\dim O_1 = 51$, it follows $\dim \mathrm{Ker}\, \mathrm{pr}_{2|O_1} = 1$. □

5 Complex abelian surfaces

Denote A a complex abelian surface (a torus of dimension 2). As such, it always can be written as a quotient

$$A = \mathbb{C}^2 / \Lambda,$$

where $\Lambda \subset \mathbb{C}^2$ is a lattice of rank 4, embedded in $\mathbb{C}^2$. Depending on the imbedding, we get different complex manifolds, projective or not. Of course, all of them are equivalent by monodromy.

5.1 Morphisms and special cases

Definition 5.1. An isogeny between abelian surfaces $A = \mathbb{C}^2/\Lambda \to A' = \mathbb{C}^2/\Lambda'$ means a surjective holomorphic map that preserves the group structure. It is given by a complex linear map, that maps Λ to a sublattice of Λ'.

Example 5.2. For a number $n \neq 0$, the multiplication map $n : A \to A$, $x \mapsto n \cdot x$ is an isogeny.

By an automorphism of A we mean a biholomorphism preserving the group structure. It can be represented by a $\mathbb{C}$-linear map $M : \mathbb{C}^2 \to \mathbb{C}^2$ with $M\Lambda = \Lambda$. Have a look in [10] or the appendix of [13] for some reference. Let us now come to the very special case that $A = E \times E$ can be written as the square of an elliptic curve. Note that A is projective, because every elliptic curve is. Now write E as $E = \mathbb{C}/\Lambda_0$. We may assume that Λ_0 is spanned by 1 and a vector $\tau \in \mathbb{C} \backslash \mathbb{R}$. The automorphism group, up to isogeny, is given by ([13]) $\mathrm{GL}(2, \mathrm{End}(\Lambda_0))$, where $\mathrm{End}(\Lambda_0)$ is the set $\{z \in \mathbb{C} \mid z\Lambda_0 \subset \Lambda_0\}$.

Proposition 5.3. *There are two possibilities for* $\mathrm{End}(\Lambda_0)$*, depending on τ:*

1. *Both the real part and the square norm of τ are rational numbers, say $2\Re(\tau) = \frac{p}{r}$ and $\|\tau\|^2 = \frac{q}{r}$ with $r > 0$ as small as possible. Then* $\mathrm{End}(\Lambda_0) = \mathbb{Z} + r\tau\mathbb{Z}$.
2. *At least one of $\Re(\tau), \|\tau\|^2$ is irrational. Then* $\mathrm{End}(\Lambda_0) = \mathbb{Z}$.

Proof. Given $z \in \mathrm{End}(\Lambda_0)$, we have

$$z \cdot 1 = a + b\tau \text{ and } z \cdot \tau = c + d\tau \text{ with } a, b, c, d \in \mathbb{Z}.$$

We get the condition

$$(a + b\tau)\tau = c + d\tau \quad \Leftrightarrow \quad b\tau^2 + (a - d)\tau - c = 0.$$

Up to scalar multiples, there is a unique real quadratic polynomial that annihilates τ, namely $(x - \tau)(x - \bar{\tau}) = x^2 - 2\Re(\tau)x + \|\tau\|^2$. If all coefficients of that polynomial are rational numbers, then $z = a + b\tau$ gives a solution for arbitrary $a \in \mathbb{Z}, b \in r\mathbb{Z}$. Otherwise, the condition must be the zero polynomial, so $b = 0$. □

Definition 5.4. Denote $\xi \in \mathbb{C}$ a primitive sixth root of unity and E_ξ the elliptic curve given by the choice $\Lambda_0 = \langle 1, \xi \rangle$, so $\mathrm{End}(\Lambda_0) = \Lambda_0$ is the ring of Eisenstein integers. Define a group G_ξ of automorphisms of $E_\xi \times E_\xi$ by the following generators in $\mathrm{GL}(2, \mathrm{End}(\Lambda_0))$:

$$g_1 = \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad g_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For $A = E_\xi \times E_\xi$, let $V = A[2]$ be the (fourdimensional) $\mathbb{F}_2$-vector space of 2-torsion points on A and let $\mathfrak{T}$ be the set of planes in V. Note that by Remark 4.4 a plane in V can be identified with an unordered triple $\{x, y, z\}$ with $0 \neq x, y, z \in V$ and $x + y + z = 0$. The action of G_ξ on A induces actions of G_ξ on $A[2]$ and $\mathfrak{T}$.

Proposition 5.5. *There are two orbits of G_ξ on $\mathfrak{T}$, of cardinalities* 5 *and* 30.

Proof. Note that the action of the multiplication with ξ induces a cyclic permutation on $E_\xi[2]$. The orbits can be explicitely computed. Denote x_1, x_2, x_3 the non-zero points in $E_\xi[2]$. The orbit of cardinality five is then given by

$$\{(0, x_1), (0, x_2), (0, x_3)\}, \qquad \{(x_1, 0), (x_2, 0), (x_3, 0)\},$$
$$\{(x_1, x_1), (x_2, x_2), (x_3, x_3)\}, \quad \{(x_1, x_2), (x_2, x_3), (x_3, x_1)\}, \quad \{(x_1, x_3), (x_2, x_1), (x_3, x_2)\}.$$

□

5.2 Homology and Cohomology

The fundamental group $\pi_1(A, \mathbb{Z}) = H_1(A, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 4, which is canonically identified with the lattice Λ. Indeed, the projection of every path in $\mathbb{C}^2$ from 0 to $v \in \Lambda$ gives a unique element of $\pi_1(A, \mathbb{Z})$. Conversely, any closed path in A with basepoint 0 lifts to a unique path in $\mathbb{C}^2$ from 0 to some $v \in \Lambda$. So the first cohomology $H^1(A, \mathbb{Z})$ is freely generated by four elements, too. Moreover, by [32, Sect. I.1], the cohomology ring is isomorphic to the exterior algebra

$$H^*(A, \mathbb{Z}) = \Lambda^*(H^1(A, \mathbb{Z})).$$

Notation 5.6. We denote the generators of $H^1(A, \mathbb{Z})$ by a_i, $1 \leq i \leq 4$ and their respective duals by $a_i^* \in H^3(A, \mathbb{Z})$. If $A = E \times E$ is the product of two elliptic curves, we choose the a_i in a way such that $\{a_1, a_2\}$ and $\{a_3, a_4\}$ give bases of $H^1(E, \mathbb{Z})$ in the decomposition $H^1(A) = H^1(E) \oplus H^1(E)$. We denote the generator of the top cohomology $H^4(A, \mathbb{Z})$ by $x := a_1 a_2 a_3 a_4$.

Let A be a abelian surface. We recall that a *principal polarization* of A is a polarization L such that there exists a basis of $H_1(X, \mathbb{Z})$, with respect to which the bilinear form on $H_1(X, \mathbb{Z})$ induced by $c_1(L)$ is given by the matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We remark that a principal polarization L provide a symplectic bilinear form $\mathcal{E}$ on $H_1(X, \mathbb{Z})$ as follows: $\mathcal{E}(x, y) = x \cdot c_1(L) \cdot y$, for all x and y in $H_1(X, \mathbb{Z})$.

TODO: proof of $\mathrm{Mon}(H_1(X, \mathbb{Z})) = \mathrm{Sp}(H_1(X, \mathbb{Z}))$.

6 Recall on the theory of integral cohomology of quotients

Let $G = \langle \iota \rangle$ be the group generated by an involution ι on a complex manifold X. We denote by $\mathcal{O}_K$ the ring $\mathbb{Z}$ with the following G-module structure: $\iota \cdot x = -x$ for $x \in \mathcal{O}_K$. For $a \in \mathbb{Z}$, we also denote by $(\mathcal{O}_K, a)$ the module $\mathbb{Z} \oplus \mathbb{Z}$ whose G-module structure is defined by $\iota \cdot (x, k) = (-x + ka, k)$. We also denote by N_2 the $\mathbb{F}_2[G]$-module $(\mathcal{O}_K, a) \otimes \mathbb{F}_2$. We recall Definition-Proposition 2.2.2 of [26].

Definition-Proposition 6.1. *Assume that $H^*(X,\mathbb{Z})$ is torsion-free. Then for all $0 \leq k \leq 2 \dim X$, we have an isomorphism of $\mathbb{Z}[G]$-module:*

$$H^k(X,\mathbb{Z}) \simeq \bigoplus_{i=1}^{r} (\mathcal{O}_K, a_i) \oplus \mathcal{O}_K^{\oplus s} \oplus \mathbb{Z}^{\oplus t},$$

for some $a_i \notin 2\mathbb{Z}$ and $(r, s, t) \in \mathbb{N}^3$. It follows the following isomorphism of $\mathbb{F}_2[G]$-module:

$$H^k(X, \mathbb{F}_2) \simeq N_2^{\oplus r} \oplus \mathbb{F}_2^{\oplus (s+t)}.$$

We denote $l_2^k(X) := r$, $l_{1,-}^k(X) := s$, $l_{1,+}^k(X) := t$, $\mathcal{N}_2 := N_2^{\oplus r}$ and $\mathcal{N}_1 := \mathbb{F}_2^{\oplus s+t}$.

Remark 6.2. These invariants are uniquely determined by G, X and k.

We recall an adaptation of Proposition 5.1 and Corollary 5.8 of [5] that can be found in Section 2.2 of [26].

Proposition 6.3. *Let X be a compact complex manifold of dimension n and ι an involution. Assume that $H^*(X,\mathbb{Z})$ is torsion free. We have:*

(i) $\operatorname{rk} H^k(X,\mathbb{Z})^\iota = l_2^k(X) + l_{1,+}^k(X)$.

(ii) We denote $\sigma := \mathrm{id} + \iota^*$ and $S_\iota^k := \operatorname{Ker} \sigma \cap H^k(X,\mathbb{Z})$. We have $H^k(X,\mathbb{Z})^\iota \cap S_\iota^k = 0$ and

$$\frac{H^k(X,\mathbb{Z})}{H^k(X,\mathbb{Z})^\iota \oplus S_\iota^k} = (\mathbb{Z}/2\mathbb{Z})^{l_2^k(X)}.$$

Remark 6.4. Remark that the elements of $(\mathcal{O}_K, a_i)^\iota$ are written $x + \iota^*(x)$ with $x \in (\mathcal{O}_K, a_i)$.

Let $\pi : X \to X/G$ be the quotient map. We denote by π^* and π_* respectively the pull-back and the push-forward of π. We recall that $\pi_* \circ \pi^* = 2\,\mathrm{id}$ and $\pi^* \circ \pi_* = \mathrm{id} + \iota^*$. Assuming that $H^k(X,\mathbb{Z})$ is torsion free, it follows the exact sequence of Proposition 3.3.3 of [26], which will be useful in the next section.

$$0 \longrightarrow \pi_*(H^k(X,\mathbb{Z})) \longrightarrow H^k(X/G,\mathbb{Z})/\mathrm{tors} \longrightarrow (\mathbb{Z}/2\mathbb{Z})^{\alpha_k} \longrightarrow 0, \tag{5}$$

with $\alpha_k \in \mathbb{N}$. We also recall the commutativity behaviour of π_* with the cup product (Lemma 3.3.7 of [26]).

Proposition 6.5. *Let X be a compact complex manifold of dimension n and ι an involution. Assume that $H^*(X,\mathbb{Z})$ is torsion free. Let $0 \leq k \leq 2n$, q an integer such that $kq \leq 2n$, and let $(x_i)_{1 \leq i \leq q}$ be elements of $H^k(X,\mathbb{Z})^\iota$. Then*

$$\pi_*(x_1) \cdot \ldots \cdot \pi_*(x_q) = 2^{q-1} \pi_*(x_1 \cdot \ldots \cdot x_q).$$

We also recall Definition 3.3.4 of [26].

Definition 6.6. Let X be a compact complex manifold and ι be an involution. Let $0 \leq k \leq 2n$, and assume that $H^k(X,\mathbb{Z})$ is torsion free. Then if the map $\pi_* : H^k(X,\mathbb{Z}) \to H^k(X/G,\mathbb{Z})/\mathrm{tors}$ is surjective we say that (X, ι) is H^k*-normal*.

Remark 6.7. H^k-normal property is equivalent to the following property.

For $x \in H^k(X,\mathbb{Z})^\iota$, $\pi_*(x)$ is divisible by 2 if and only if there exists $y \in H^k(X,\mathbb{Z})$ such that $x = y + \iota^*(y)$.

We also need to recall Definition 3.5.1 of [26] about fixed locus.

Definition 6.8. Let X be a compact complex manifold of dimension n and G an automorphism group of prime order p.

1) We will say that $\operatorname{Fix} G$ is negligible if the following conditions are verified:

   - $H^*(\operatorname{Fix} G, \mathbb{Z})$ is torsion-free.
   - $\operatorname{Codim} \operatorname{Fix} G \geq \frac{n}{2} + 1$.

2) We will say that $\operatorname{Fix} G$ is almost negligible if the following conditions are verified:

   - $H^*(\operatorname{Fix} G, \mathbb{Z})$ is torsion-free.
   - n is even and $n \geq 4$.
   - $\operatorname{Codim} \operatorname{Fix} G = \frac{n}{2}$, and the purely $\frac{n}{2}$-dimensional part of $\operatorname{Fix} G$ is connected and simply connected. We denote the $\frac{n}{2}$-dimensional component by Σ.
   - The cocycle $[\Sigma]$ associated to Σ is primitive in $H^n(X, \mathbb{Z})$.

Now, we are ready to provide Corollary 2.65 of [26] that we will our key tool in the next of this article.

Corollary 6.9. *Let $G = \langle \varphi \rangle$ be a group of prime order $p = 2$ acting by automorphisms on a Kähler manifold X of dimension $2n$. We assume:*

i) $H^*(X, \mathbb{Z})$ *is torsion-free,*

ii) $\operatorname{Fix} G$ *is negligible or almost negligible,*

iii) $l_{1,-}^{2k}(X) = 0$ *for all* $1 \leq k \leq n$, *and*

iv) $l_{1,+}^{2k+1}(X) = 0$ *for all* $0 \leq k \leq n-1$, *when* $n > 1$.

v) $l_{1,+}^{2n}(X) + 2\left[\sum_{i=0}^{n-1} l_{1,-}^{2i+1}(X) + \sum_{i=0}^{n-1} l_{1,+}^{2i}(X)\right] = \sum_{k=0}^{\dim \operatorname{Fix} G} h^{2k}(\operatorname{Fix} G, \mathbb{Z})$.

Then (X, G) is H^{2n}-normal.

We will also need a proposition from Section 7 of [5] about Smith theory. Let T be a topological space and let $G = \langle \iota \rangle$ be an involution acting on T. Let $\sigma := 1 + \iota \in \mathbb{F}_2[G]$. We consider the chain complex $C_*(T)$ of T with coefficients in $\mathbb{F}_2$ and its subcomplexes $\sigma C_*(T)$. We denote also X^G the fixed locus of the action of G on T.

Proposition 6.10. *(1) ([6], Theorem 3.1). There is an exact sequence of complexes:*

$$0 \longrightarrow \sigma C_*(T) \oplus C_*(T^G) \xrightarrow{f} C_*(T) \xrightarrow{\sigma} \sigma C_*(T) \longrightarrow 0\ ,$$

where f denotes the sum of the inclusions.

(2) ([6], (3.4) p.124). There is an isomorphism of complexes:

$$\sigma C_*(T) \simeq C_*(T/G, T^G),$$

where T^G is identified with its image in T/G.

7 Odd cohomology of the Hilbert scheme of two points

Let A be a smooth compact surface with torsion free cohomology and $A^{[2]}$ the Hilbert scheme of 2 points. It can be constructed as follows: Consider the direct product $A \times A$. Denote

$$b : \widetilde{A \times A} \to A \times A$$

the blow-up along the diagonal $\Delta \cong A$ with exceptional divisor E. , so we have $j : E \to \widetilde{A \times A}$. The action of $\mathfrak{S}_2$ on $A \times A$ lifts to an action on $\widetilde{A \times A}$. We have the pushforward $j_* : H^*(E, \mathbb{Z}) \to H^*(\widetilde{A \times A}, \mathbb{Z})$.

The quotient by the action of $\mathfrak{S}_2$ is

$$\pi : \widetilde{A \times A} \to A^{[2]}.$$

Now, $A^{[2]}$ is a compact complex manifold with torsion-free cohomology, [43, Theorem 2.2]. By (5) of the last section, there is an exact sequence

$$0 \to \pi_*(H^k(\widetilde{A \times A}, \mathbb{Z})) \to H^k(A^{[2]}, \mathbb{Z}) \to \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\alpha_k} \to 0$$

with $k \in \{1, ..., 8\}$. In this section, we want to prove the following proposition.

Proposition 7.1. *Let A be a smooth compact surface with torsion free cohomology. Then*

(i) $H^3(A^{[2]}, \mathbb{Z}) = \pi_*(b^*(H^3(A \times A, \mathbb{Z}))) \oplus \pi_*(j_*(b^*_{|E}(H^1(\Delta, \mathbb{Z}))))$,

(ii) $H^5(A^{[2]}, \mathbb{Z}) = \pi_*(b^*(H^5(A \times A, \mathbb{Z}))) \oplus \frac{1}{2}\pi_*(j_*(b^*_{|E}(H^3(\Delta, \mathbb{Z}))))$.

The section is dedicated to the proof of this proposition. The proof is organized as follows. Section 7.1 is devoted to calculate the torsion of $H^3(A^{[2]} \smallsetminus E, \mathbb{Z})$ (Lemma 7.4) using equivariant cohomology technique developed in [26]. Then this knowledge allow us to deduce $\alpha_3 = 0$ using the exact sequence (9) and $\alpha_5 = 4$ using the unimodularity of the lattice $H^3(\widetilde{A \times A}, \mathbb{Z}) \oplus H^5(\widetilde{A \times A}, \mathbb{Z})$.

7.1 Preliminary Lemmas

We denote $V = \widetilde{A \times A} \smallsetminus E$ and $U = V/\mathfrak{S}_2 = A^{[2]} \smallsetminus E$, where $\mathfrak{S}_2 = \langle \sigma_2 \rangle$.

Lemma 7.2. *We have: $H^k(A \times A, \mathbb{Z}) = H^k(V, \mathbb{Z})$ for all $k \leq 3$.*

Proof. We have $V = A \times A \smallsetminus \Delta$. We have the following natural exact sequence:

$$\cdots \longrightarrow H^k(A \times A, V, \mathbb{Z}) \longrightarrow H^k(A \times A, \mathbb{Z}) \longrightarrow H^k(V, \mathbb{Z}) \longrightarrow \cdots$$

Moreover, by Thom isomorphism $H^k(A \times A, V, \mathbb{Z}) = H^{k-4}(\Delta, \mathbb{Z}) = H^{k-4}(A, \mathbb{Z})$. Hence $H^k(A \times A, V, \mathbb{Z}) = 0$ for all $k \leq 3$. Hence $H^k(A \times A, \mathbb{Z}) = H^k(V, \mathbb{Z})$ for all $k \leq 2$. It remains to consider the following exact sequence:

$$0 \longrightarrow H^3(A \times A, \mathbb{Z}) \longrightarrow H^3(V, \mathbb{Z}) \longrightarrow H^4(A \times A, V, \mathbb{Z}) \xrightarrow{\rho} H^4(A \times A, \mathbb{Z}) .$$

The map ρ is given by $\mathbb{Z}[\Delta] \to H^4(A \times A, \mathbb{Z})$. The class $\{x\} \times A$ is also in $H^4(A \times A, \mathbb{Z})$ and intersects Δ in one point. Hence the class of Δ in $H^4(A \times A, \mathbb{Z})$ is not trivial and the map ρ is injective. It follows

$$H^3(A \times A, \mathbb{Z}) = H^3(V, \mathbb{Z}).$$

□

Now we will calculate the invariant $l^2_{1,-}(A \times A)$ and $l^1_{1,+}(A \times A)$ defined in Definition-Proposition 6.1.

Lemma 7.3. *We have: $l^2_{1,-}(A \times A) = l^1_{1,+}(A \times A) = 0$.*

Proof. By Künneth formula we have:

$$H^1(A\times A,\mathbb{Z}) = H^0(A,\mathbb{Z})\otimes H^1(A,\mathbb{Z})\oplus H^1(A,\mathbb{Z})\otimes H^0(A,\mathbb{Z}).$$

The elements of $H^0(A,\mathbb{Z})\otimes H^1(A,\mathbb{Z})$ and $H^1(A,\mathbb{Z})\otimes H^0(A,\mathbb{Z})$ are exchanged under the action of σ_2. It follows that $l_2^1(A\times A)=4$ and necessary $l_{1,-}^1(A\times A)=l_{1,+}^1(A\times A)=0$.

By Künneth formula we also have:

$$\begin{aligned}H^2(A\times A,\mathbb{Z}) &= H^0(A,\mathbb{Z})\otimes H^2(A,\mathbb{Z})\oplus H^1(A,\mathbb{Z})\otimes H^1(A,\mathbb{Z})\\ &\oplus H^2(A,\mathbb{Z})\otimes H^0(A,\mathbb{Z}).\end{aligned}$$

As before every elements $x\otimes y\in H^2(A\times A,\mathbb{Z})$ are sent to $y\otimes x$ by the action of σ_2. A such element is fixed by the action of σ_2 if $x=y$. It follows:

$$l_2^2(A\times A)=6+6=12,$$

$$l_{1,+}^2(A\times A)=4,$$

and necessary:

$$l_{1,-}^2(A\times A)=0.$$

□

Lemma 7.4. *The group $H^3(U,\mathbb{Z})$ is torsion free.*

Proof. Using the spectral sequence of equivariant cohomology, it follows from Proposition 3.2.5 of [26], Lemma 7.2 and 7.3. □

7.2 Third cohomology group

By Theorem 7.31 of [44], we have:

$$H^3(\widetilde{A\times A},\mathbb{Z}) = b^*(H^3(A\times A,\mathbb{Z}))\oplus j_*(b_{|E}^*(H^1(\Delta,\mathbb{Z}))). \tag{6}$$

It follows that

$$H^3(A^{[2]},\mathbb{Z})\supset \pi_*(b^*(H^3(A\times A,\mathbb{Z})))\oplus \pi_*(j_*(b_{|E}^*(H^1(\Delta,\mathbb{Z}))).$$

Moreover, by Künneth formula, we have:

$$\begin{aligned}H^3(A\times A,\mathbb{Z}) &= H^0(A,\mathbb{Z})\otimes H^3(A,\mathbb{Z})\oplus H^1(A,\mathbb{Z})\otimes H^2(A,\mathbb{Z})\\ &\oplus H^2(A,\mathbb{Z})\otimes H^1(A,\mathbb{Z})\oplus H^3(A,\mathbb{Z})\otimes H^0(A,\mathbb{Z}).\end{aligned}$$

Hence all elements in $H^3(A\times A,\mathbb{Z})^{\mathfrak{S}_2}$ are written $x+\sigma_2^*(x)$ with $x\in H^3(A\times A,\mathbb{Z})$. Since $\frac{1}{2}\pi_*(x+\sigma_2^*(x))=\pi_*(x)$, it follows that $\pi_*(b^*(H^3(A\times A,\mathbb{Z})))$ is primitive in $H^3(A^{[2]},\mathbb{Z})$. Moreover by (6), we have the following values which will be used in Section 7.3:

$$l_2^3(\widetilde{A\times A}) = \operatorname{rk} H^3(A\times A,\mathbb{Z})^{\mathfrak{S}_2} = 28. \tag{7}$$

and

$$l_{1,+}^3(\widetilde{A\times A}) = \operatorname{rk} H^1(\Delta,\mathbb{Z})^{\mathfrak{S}_2}=4, \quad \text{and} \quad l_{1,-}^3(\widetilde{A\times A})=0.$$

It remains to prove the following lemma.

Lemma 7.5. *The group $\pi_*(j_*(b_{|E}^*(H^1(\Delta,\mathbb{Z}))))$ is primitive in $H^3(A^{[2]},\mathbb{Z})$.*

Proof. We consider the following commutative diagram:

$$\begin{array}{ccc} H^3(\mathscr{N}_{A^{[2]}/E},\mathscr{N}_{A^{[2]}/E}-0,\mathbb{Z}) = H^3(A^{[2]},U,\mathbb{Z}) & \xrightarrow{g} & H^3(A^{[2]},\mathbb{Z}) \\ \Big\downarrow{\scriptstyle d\widetilde{\pi}^*} & & \Big\downarrow{\scriptstyle \pi^*} \\ H^3(\mathscr{N}_{\widetilde{A\times A}/E},\mathscr{N}_{\widetilde{A\times A}/E}-0,\mathbb{Z}) = H^3(\widetilde{A\times A},V,\mathbb{Z}) & \xrightarrow{h} & H^3(\widetilde{A\times A},\mathbb{Z}), \end{array} \tag{8}$$

By proof of Theorem 7.31 of [44], the map h is injective and its image in $H^3(\widetilde{A\times A},\mathbb{Z})$ is $j_*(b^*_{|E}(H^1(\Delta,\mathbb{Z})))$. Hence Diagram (8) shows that g is also injective and has image $\pi_*(j_*(b^*_{|E}(H^1(\Delta,\mathbb{Z}))))$ in $H^3(A^{[2]},\mathbb{Z})$. It follows the exact sequence:

$$0 \longrightarrow H^3(A^{[2]},U,\mathbb{Z}) \xrightarrow{g} H^3(A^{[2]},\mathbb{Z}) \longrightarrow H^3(U,\mathbb{Z}) \ . \tag{9}$$

However, by Lemma 7.4, $H^3(U,\mathbb{Z})$ is torsion free; it follows that $\pi_*(j_*(b^*_{|E}(H^1(\Delta,\mathbb{Z}))))$ is primitive in $H^3(A^{[2]},\mathbb{Z})$. □

It prove (i) of Proposition 7.1.

7.3 In degree 5

By Theorem 7.31 of Voisin, we have:

$$H^5(\widetilde{A\times A},\mathbb{Z}) = b^*(H^5(A\times A,\mathbb{Z})) \oplus j_*(b^*_{|E}(H^3(\Delta,\mathbb{Z}))). \tag{10}$$

It follows that

$$H^5(A^{[2]},\mathbb{Z}) \supset \pi_*(b^*(H^5(A\times A,\mathbb{Z}))) \oplus \pi_*(j_*(b^*_{|E}(H^3(\Delta,\mathbb{Z})))).$$

Moreover, by Künneth formula, we have:

$$\begin{aligned} H^5(A\times A,\mathbb{Z}) = {} & H^1(A,\mathbb{Z})\otimes H^4(A,\mathbb{Z}) \oplus H^2(A,\mathbb{Z})\otimes H^3(A,\mathbb{Z}) \\ & \oplus H^3(A,\mathbb{Z})\otimes H^2(A,\mathbb{Z}) \oplus H^4(A,\mathbb{Z})\otimes H^1(A,\mathbb{Z}). \end{aligned}$$

As before, $\pi_*(b^*(H^5(A\times A,\mathbb{Z})))$ is primitive in $H^5(A^{[2]},\mathbb{Z})$. Moreover by (10):

$$l_2^5(\widetilde{A\times A}) = \operatorname{rk} H^5(A\times A,\mathbb{Z})^{\mathfrak{S}_2} = 28, \tag{11}$$

and

$$l_{1,+}^5(\widetilde{A\times A}) = \operatorname{rk} H^3(\Delta,\mathbb{Z})^{\mathfrak{S}_2} = 4, \quad \text{and} \quad l_{1,-}^5(\widetilde{A\times A}) = 0.$$

Lemma 7.6. *The lattice* $\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})\oplus H^5(\widetilde{A\times A},\mathbb{Z}))$ *has discriminant* 2^8.

Proof. By Proposition 6.3 (ii):

$$\frac{H^3(\widetilde{A\times A},\mathbb{Z})\oplus H^5(\widetilde{A\times A},\mathbb{Z})}{H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus \left(H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\right)^{\perp}} = (\mathbb{Z}/2\mathbb{Z})^{l_2^3(\widetilde{A\times A})+l_2^5(\widetilde{A\times A})}.$$

Since $H^3(\widetilde{A\times A},\mathbb{Z})\oplus H^5(\widetilde{A\times A},\mathbb{Z})$ is an unimodular lattice, it follows that

$$\operatorname{discr} H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2} = 2^{l_2^3(\widetilde{A\times A})+l_2^5(\widetilde{A\times A})}.$$

Then by Proposition 6.5,

$$\operatorname{discr} \pi_*(H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}) = 2^{l_2^3(\widetilde{A\times A})+l_2^5(\widetilde{A\times A})+\operatorname{rk}\left[H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\right]}.$$

Then by Proposition 6.3 (i):

$$\operatorname{discr} \pi_*(H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}) = 2^{2\left(l_2^3(\widetilde{A\times A})+l_2^5(\widetilde{A\times A})\right)+l_{1,+}^3(\widetilde{A\times A})+l_{1,+}^5(\widetilde{A\times A})}.$$

By Remark 6.4 and since $\pi_*(x+\iota^*(x)) = 2\pi_*(x)$, we have:

$$\frac{\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})\oplus H^5(\widetilde{A\times A},\mathbb{Z}))}{\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})^{\iota}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\iota})} = (\mathbb{Z}/2\mathbb{Z})^{l_2^3(\widetilde{A\times A})+l_2^5(\widetilde{A\times A})}.$$

It follows:

$$\operatorname{discr} \pi_*(H^3(\widetilde{A\times A},\mathbb{Z})\oplus H^5(\widetilde{A\times A},\mathbb{Z})) = 2^{l_{1,+}^3(\widetilde{A\times A})+l_{1,+}^5(\widetilde{A\times A})} = 2^8.$$

□

The lattice $H^3(A^{[2]},\mathbb{Z}) \oplus H^5(A^{[2]},\mathbb{Z})$ is unimodular. Hence:

$$\frac{H^3(A^{[2]},\mathbb{Z}) \oplus H^5(A^{[2]},\mathbb{Z})}{\pi_*(H^3(\widetilde{A\times A},\mathbb{Z}) \oplus H^5(\widetilde{A\times A},\mathbb{Z}))} = (\mathbb{Z}/2\mathbb{Z})^4.$$

However, from the last section, we know that $\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})) = H^3(A^{[2]},\mathbb{Z})$. It follows that

$$\frac{H^5(A^{[2]},\mathbb{Z})}{\pi_*(H^5(\widetilde{A\times A},\mathbb{Z}))} = (\mathbb{Z}/2\mathbb{Z})^4.$$

8 Nakajima operators for Hilbert schemes of points on surfaces

Let A be a smooth projective complex surface. Let $A^{[n]}$ the Hilbert scheme of n points on the surface, *i.e.* the moduli space of finite subschemes of A of length n. $A^{[n]}$ is again smooth and projective of dimension $2n$, cf. [8]. Their rational cohomology can be described in terms of Nakajima's [33] operators. First consider the direct sum

$$\mathbb{H} := \bigoplus_{n=0}^{\infty} H^*(A^{[n]},\mathbb{Q})$$

This space is bigraded by cohomological *degree* and the *weight*, which is given by the number of points n. The unit element in $H^0(A^{[0]},\mathbb{Q}) \cong \mathbb{Q}$ is denoted by $|0\rangle$, called the *vacuum*.

Definition 8.1. There are linear operators $\mathfrak{q}_m(a)$, for each $m \geq 1$ and $a \in H^*(A,\mathbb{Q})$, acting on $\mathbb{H}$ which have the following properties: They depend linearly on a, and if $a \in H^k(A,\mathbb{Q})$ is homogeneous, the operator $\mathfrak{q}_m(a)$ is bihomogeneous of degree $k + 2(m-1)$ and weight m:

$$\mathfrak{q}_m(a) : H^l(A^{[n]}) \to H^{l+k+2(|m|-1)}(A^{[n+m]})$$

To construct them, first define incidence varieties $Z_m \subset A^{[n]} \times A \times A^{[n+m]}$ by

$$Z_m := \{(\xi, x, \xi') \,|\, \xi \subset \xi',\ \mathrm{supp}(\xi') - \mathrm{supp}(\xi) = mx\}.$$

Then $\mathfrak{q}_m(a)(\beta)$ is defined as the Poincaré dual of

$$\mathrm{pr}_{3*}\left((\mathrm{pr}_2^*(\alpha)\cdot \mathrm{pr}_3^*(\beta)) \cap [Z_m]\right).$$

Consider now the superalgebra generated by the $\mathfrak{q}_m(a)$. Every element in $\mathbb{H}$ can be decomposed uniquely as a linear combination of products of operators $\mathfrak{q}_m(a)$, acting on the vacuum. In other words, the $\mathfrak{q}_m(a)$ generate $\mathbb{H}$ and there are no algebraic relations between them (except the linearity in a and the super-commutativity).

Example 8.2. The unit $1_{A^{[n]}} \in H^0(A^{[n]},\mathbb{Q})$ is given by $\frac{1}{n!}\mathfrak{q}_1(1)^n|0\rangle$. The sum of all $1_{A^{[n]}}$ in the formal completion of $\mathbb{H}$ is sometimes denoted by $|1\rangle := \exp(\mathfrak{q}_1(1))|0\rangle$.

Definition 8.3. To give the cup product structure of $\mathbb{H}$, define operators $\mathfrak{G}(a)$ for $a \in H^*(A)$. Let $\Xi_n \subset A^{[n]} \times A$ be the universal subscheme. Then the action of $\mathfrak{G}(a)$ on $H^*(A^{[n]})$ is multiplication with the class

$$\mathrm{pr}_{1*}\left(\mathrm{ch}(\mathcal{O}_{\Xi_n})\cdot \mathrm{pr}_2^*(\mathrm{td}(A)\cdot a)\right) \in H^*(A^{[n]}).$$

For $a \in H^k(A)$, we define $\mathfrak{G}_i(a)$ as the component of $\mathfrak{G}(a)$ of cohomological degree $k+2i$. A differential operator $\mathfrak{d}$ is given by $\mathfrak{G}_1(1)$. It means multiplication with the first Chern class of the tautological sheaf $\mathrm{pr}_{1*}(\mathcal{O}_{\Xi_n})$.

Notation 8.4. We abbreviate $\mathfrak{q} := \mathfrak{q}_1(1)$ and for its derivative $\mathfrak{q}' := [\mathfrak{d}, \mathfrak{q}]$. For any operator X we write $X^{(k)}$ for the k-fold derivative: $X^{(k)} := \mathrm{ad}^k(\mathfrak{d})(X)$.

In [18] and [20] we find various commutation relations between these operators, that allow to determine all multiplications in the cohomology of the Hilbert scheme. First of all, if X and Y are operators of degree d and d', their commutator is defined in the super sense:

$$[X,Y] := XY - (-1)^{dd'}YX.$$

The integral on $A^{[n]}$ induces a non-degenerate bilinar form on $\mathbb{H}$: for classes α, $\beta \in H^*(A^{[n]})$ it is given by

$$(\alpha,\beta)_{A^{[n]}} := \int_{A^{[n]}} \alpha\cdot\beta.$$

If X is a homogeneous linear operator of degree d and weight m, acting on $\mathbb{H}$, define its adjont $X^\dagger$ by

$$(X(\alpha),\beta)_{A^{[n+m]}} = (-1)^{d|\alpha|}(\alpha, X^\dagger(\beta))_{A^{[n]}}.$$

We put $\mathfrak{q}_0(a) := 0$ and for $m<0$, $\mathfrak{q}_m(a) := (-1)^m\mathfrak{q}_{-m}(a)^\dagger$. Note that, for all $m\in\mathbb{Z}$, the bidegree of $\mathfrak{q}_m(a)$ is $(m, |a| + 2(|m|-1))$. Now define

$$\mathfrak{L}_m(a) := \begin{cases} \frac{1}{2}\sum\limits_{k\in\mathbb{Z}}\sum\limits_i \mathfrak{q}_k(a_{(1)})\mathfrak{q}_{m-k}(a_{(2)}), & \text{if } m\neq 0,\\ \sum\limits_{k>0}\sum\limits_i \mathfrak{q}_k(a_{(1)})\mathfrak{q}_{-k}(a_{(2)}), & \text{if } m=0.\end{cases}$$

where $\sum_i a_{(1)}\otimes a_{(2)}$ is the push-forward of a along the diagonal $\tau_2 : A\to A\times A$ (in Sweedler notation).

Remark 8.5. This can be expressed more elegantly using normal ordering: the operator $\mathfrak{q}_m\mathfrak{q}_n(a\otimes b)$ is defined in a way such that the annihilation operator act first. Then we may write $\mathfrak{L}_m(a) = \sum_k {:}\mathfrak{q}_k\mathfrak{q}_{m-k}{:}(\tau_{2*}(a))$.

Remark 8.6. In a similar manner as above, we can use the integral over A to define a bilinear form on $H^*(A,\mathbb{Q})$. The adjoint of the multiplication map gives a coassiocative comultiplication

$$\Delta : H^*(A,\mathbb{Q}) \longrightarrow H^*(A,\mathbb{Q})\otimes H^*(A,\mathbb{Q})$$

that corresponds to τ_{2*}. The sign convention in [18] is such that $-\Delta = \tau_{2*}$. We denote by Δ^k the k-fold composition of Δ.

Lemma 8.7. *[20, Thm. 2.16] Denote $K\in H^2(A,\mathbb{Q})$ the class of the canonical divisor. We have:*

$$[\mathfrak{q}_m(a),\mathfrak{q}_n(b)] = m\cdot\delta_{m+n}\cdot\int_A ab \tag{12}$$

$$[\mathfrak{L}_m(a),\mathfrak{q}_n(b)] = -n\cdot\mathfrak{q}_{m+l}(ab) \tag{13}$$

$$[\mathfrak{d},\mathfrak{q}_m(a)] = m\cdot\mathfrak{L}_m(a) + \tfrac{m(|m|-1)}{2}\mathfrak{q}_m(Ka) \tag{14}$$

$$[\mathfrak{L}_m(a),\mathfrak{L}_n(b)] = (m-n)\mathfrak{L}_{m+n}(ab) - \frac{m^3-m}{12}\delta_{m+n}\int_A abe \tag{15}$$

$$[\mathfrak{G}(a),\mathfrak{q}_1(b)] = \exp(\mathrm{ad}(\mathfrak{d}))(\mathfrak{q}_1(ab)) \tag{16}$$

$$[\mathfrak{G}_i(a),\mathfrak{q}_1(b)] = \tfrac{1}{k!}\,\mathrm{ad}(\mathfrak{d})^k(\mathfrak{q}_1(ab)) \tag{17}$$

Remark 8.8. Note (cf. [18, Thm. 3.8]) that (13) together with (**??**) imply that

$$\mathfrak{q}_{m+1}(a) = \tfrac{(-1)^m}{m!}\,\mathrm{ad}^m\mathfrak{q}'\left(\mathfrak{q}_1(a)\right), \tag{18}$$

so there are two ways of writing an element of $\mathbb{H}$: As a linear combination of products of creation operators $\mathfrak{q}_m(a)$ or as a linear combination of products of the operators $\mathfrak{d}$ and $\mathfrak{q}_1(a)$. This second representation is more suitable for computing cup-products, but not faithful. Equations (14) and

(18) permit now to switch between the two representations, using that

$$\mathfrak{d}|0\rangle = 0, \tag{19}$$

$$\mathfrak{L}_m(a)|0\rangle = \begin{cases} \frac{1}{2}\sum_{k=1}^{m-1}\sum_i \mathfrak{q}_k(a_{(1)})\mathfrak{q}_{m-k}(a_{(2)})|0\rangle, & \text{if } m > 1, \\ 0, & \text{if } m \leq 1. \end{cases} \tag{20}$$

(21)

Lemma 8.9. *Suppose $Ka = 0$. Denote $e = \chi(A)x$ the Euler class of A. For all k, m, the following formulas hold:*

$$\operatorname{ad}\mathfrak{q}\frac{\mathfrak{q}_m^{(k+1)}(a)}{m^{k+1}} = (k+1)\frac{\mathfrak{q}_{m+1}^{(k)}(a)}{(m+1)^k} + \frac{k^3-k}{24}\frac{\mathfrak{q}_{m+1}^{(k-2)}(ae)}{(m+1)^{k-2}}, \tag{22}$$

$$\operatorname{ad}\mathfrak{q}'\frac{\mathfrak{q}_m^{(k)}(a)}{m^k} = (k-m)\frac{\mathfrak{q}_{m+1}^{(k)}(a)}{(m+1)^k} + \frac{k(k-1)(k-3m-2)}{24}\frac{\mathfrak{q}_{m+1}^{(k-2)}(ae)}{(m+1)^{k-2}}. \tag{23}$$

Proof. Let us start with (22). This is a consequence of Theorem 4.2 of [19] which states that

$$\begin{aligned} \frac{\mathfrak{q}_m^{(k)}(a)}{m^k} = &\frac{1}{k+1}\sum_{i_0+\ldots+i_k=m} {:}\mathfrak{q}_{i_0}\cdots\mathfrak{q}_{i_k}{:}(\tau_*(a)) \\ &+ k\sum_{j_0+\ldots+j_{k-2}=m}\frac{j_0^2+\ldots+j_{k-2}^2-1}{24}{:}\mathfrak{q}_{j_0}\cdots\mathfrak{q}_{j_{k-2}}{:}(\tau_*(ae)). \end{aligned}$$

Using that $[\mathfrak{q}, {:}\mathfrak{q}_{i_0}\cdots\mathfrak{q}_{i_k}{:}(\Delta^k(a))] = \sum_{r=0}^k \delta_{i_r+1}\,{:}\mathfrak{q}_{i_0}\cdots\widehat{\mathfrak{q}}_{i_r}\cdots\mathfrak{q}_{i_k}{:}(\tau_*(a))$, we calculate:

$$\begin{aligned} \operatorname{ad}\mathfrak{q}\frac{\mathfrak{q}_m^{(k+1)}(a)}{m^{k+1}} &= \frac{1}{k+2}\sum_{i_0+\ldots+i_{k+1}=m}\left[\mathfrak{q}, {:}\mathfrak{q}_{i_0}\cdots\mathfrak{q}_{i_{k+1}}{:}(\tau_*(a))\right] \\ &\quad + (k+1)\sum_{j_0+\ldots+j_{k-1}=m}\frac{j_0^2+\ldots+j_{k-1}^2-1}{24}\left[\mathfrak{q}, {:}\mathfrak{q}_{j_0}\cdots\mathfrak{q}_{j_{k-1}}{:}(\tau_*(ae))\right] \\ &= \sum_{i_0+\ldots+i_k=m+1}{:}\mathfrak{q}_{i_0}\cdots\mathfrak{q}_{i_k}{:}(\tau_*(a)) \\ &\quad + k(k+1)\sum_{j_0+\ldots+j_{k-2}=m+1}\frac{j_0^2+\ldots+j_{k-2}^2}{24}{:}\mathfrak{q}_{j_0}\cdots\mathfrak{q}_{j_{k-2}}{:}(\tau_*(ae)) \\ &= (k+1)\frac{\mathfrak{q}_{m+1}^{(k)}(a)}{(m+1)^k} + \frac{k^3-k}{24}\frac{\mathfrak{q}_{m+1}^{(k-2)}(ae)}{(m+1)^{k-2}}. \end{aligned}$$

Equation (23) follows from (22) using the Jacobi identity: $\operatorname{ad}\mathfrak{q}' = \operatorname{ad}[\mathfrak{d}, \mathfrak{q}] = \operatorname{ad}\mathfrak{d}\operatorname{ad}\mathfrak{q} - \operatorname{ad}\mathfrak{q}\operatorname{ad}\mathfrak{d}$. □

Corollary 8.10. *Suppose $Ka = 0$. Iterated application of the above lemma gives*

$$\operatorname{ad}(\mathfrak{q})^s\frac{\mathfrak{q}_m^{(k+s)}(a)}{m^{k+s}(k+s)!} = \frac{\mathfrak{q}_{m+s}^{(k)}(a)}{(m+s)^k k!} + \frac{s}{24}\frac{\mathfrak{q}_{m+s}^{(k-2)}(ae)}{(m+s)^{k-2}(k-2)!}. \tag{24}$$

Proposition 8.11. *Suppose $Ka = 0$. In the formal completion of $\mathbb{H}$ we have:*

$$[\mathfrak{G}(a), \exp(\mathfrak{q})] = \exp(\mathfrak{q})\sum_{\substack{s\geq 1\\ k\geq 0}}\frac{(-1)^{s-1}}{s!}\left(\frac{\mathfrak{q}_s^{(k)}(a)}{s^k k!} + \frac{s-1}{24}\frac{\mathfrak{q}_s^{(k)}(ae)}{s^k k!}\right).$$

Proof. Equation (4.6) of [18] evaluates

$$
\begin{aligned}
[\mathfrak{G}(a), \exp(\mathfrak{q})] &= \exp(\mathfrak{q}) \sum_{\substack{s\geq 1\\ k\geq 0}} \frac{(-\operatorname{ad}\mathfrak{q})^{s-1}}{s!}\left(\frac{(\operatorname{ad}\mathfrak{d})^k}{k!}(\mathfrak{q}_1(a))\right)\\
&\overset{\text{Cor 8.10}}{=} \exp(\mathfrak{q}) \sum_{s\geq 1} \frac{(-1)^{s-1}}{s!}\left(\sum_{k\geq s-1} \frac{\mathfrak{q}_s^{(k-s+1)}(a)}{s^{k-s+1}(k-s+1)!} + \sum_{k\geq s+1}\frac{s-1}{24}\frac{\mathfrak{q}_s^{(k-s-1)}(a)}{s^{k-s-1}(k-s-1)!}\right)\\
&= \exp(\mathfrak{q}) \sum_{\substack{s\geq 1\\ k\geq 0}} \frac{(-1)^{s-1}}{s!}\left(\frac{\mathfrak{q}_s^{(k)}(a)}{s^k k!} + \frac{s-1}{24}\frac{\mathfrak{q}_s^{(k)}(ae)}{s^k k!}\right).
\end{aligned}
$$

□

Example 8.12.

$$\mathfrak{G}_0(a)\mathfrak{q})^n|0\rangle = n\cdot \mathfrak{q}_1(1)^{n-1}\mathfrak{q}_1(a)|0\rangle, \tag{25}$$

$$\mathfrak{G}_1(a)\mathfrak{q}^n|0\rangle = -\binom{n}{2}\mathfrak{q}_1(1)^{n-2}\mathfrak{q}_2(a)|0\rangle, \tag{26}$$

$$\mathfrak{G}_2(a)\mathfrak{q}^n|0\rangle = \binom{n}{3}\mathfrak{q}_1(1)^{n-3}\mathfrak{q}_3(a)|0\rangle - \binom{n}{2}\mathfrak{q}_1(1)^{n-2}\mathfrak{L}_2(a)|0\rangle. \tag{27}$$

Remark 8.13. We adopted the notation from [20], which differs from the conventions in [18]. Here is part of a dictionary:

Notation from [20]	Notation from [18]
operator of weight w and degree d	operator of weight w and degree $d-2w$
$\mathfrak{q}_m(a)$	$\mathfrak{p}_{-m}(a)$
$\mathfrak{L}_m(a)$	$-L_{-m}(a)$
$\mathfrak{G}(a)$	$a^{[\bullet]}$
$\mathfrak{d}$	∂
$\tau_{2*}(a)$	$-\Delta(a)$

By sending a subscheme in A to its support, we define a morphism

$$\rho : A^{[n]} \longrightarrow \operatorname{Sym}^n(A),$$

called the Hilbert–Chow morphism. The cohomology of $\operatorname{Sym}^n(A)$ is given by elements of the n-fold tensor power of $H^*(A)$ that are invariant under the action of the group of permutations $\mathfrak{S}_n$. A class in $H^*(A^{[n]},\mathbb{Q})$ which can be written using only the operators $\mathfrak{q}_1$ comes from a pullback along ρ:

$$\mathfrak{q}_1(b_1)\cdots\mathfrak{q}_1(b_n)|0\rangle = \rho^*\left(\sum_{\pi\in\mathfrak{S}_n} \pm b_{\pi(1)}\otimes\ldots\otimes b_{\pi(n)}\right), \quad b_i\in H^*(A,\mathbb{Q}), \tag{28}$$

where signs arise from permuting factors of odd degrees. In particular,

$$\frac{1}{n!}\mathfrak{q}_1(b)^n|0\rangle = \rho^*\big(b\otimes\ldots\otimes b\big), \tag{29}$$

$$\frac{1}{(n-1)!}\mathfrak{q}_1(b)\,(\mathfrak{q})^{n-1}\,|0\rangle = \rho^*\Big(b\otimes 1\otimes\ldots\otimes 1 \;+\; \ldots \;+\; 1\otimes\ldots\otimes 1\otimes b\Big). \tag{30}$$

Remark 8.14. With the notation from Section 3, we have that

$$H^*(\operatorname{Sym}^n(A),\mathbb{Q}) \cong \operatorname{SSym}^n(H^*(A,\mathbb{Q})).$$

Under this isomorphism the ring structure of $\operatorname{SSym}^n(H^*(A,\mathbb{Q}))$ corresponds to the cup product and the action of the operator $\mathfrak{q}_1(a)$ corresponds to the operation $a\diamond$.

9 On integral cohomology of Hilbert schemes

For the study of integral cohomology, first note that if $a \in H^*(A,\mathbb{Z})$ is an integral class, then $\mathfrak{q}_m(a)$ maps integral classes to integral classes. Such operators are called integral. Qin and Wang studied them in [37]. We need the following results:

Lemma 9.1. *[37] The operators $\frac{1}{n!}\mathfrak{q}^n$ and $\frac{1}{2}\mathfrak{q}_2(1)$ are integral. Let $a \in H^2(A,\mathbb{Z})$ be monodromy equivalent to a divisor. Then the operator $\frac{1}{2}\mathfrak{q}_1(a)^2 - \frac{1}{2}\mathfrak{q}_2(a)$ is integral.*

Remark 9.2. If A is a projective torus, then the sublattice of divisors in $H^2(A,\mathbb{Z})$ (the Néron–Severi group) is not trivial. By [40, Thm. II], the group of monodromy actions spans the entire automorphism group of $H^2(A,\mathbb{Z})$. Since the lattice is even and contains two copies of the hyperbolic lattice, a theorem of Eichler [39, Prop. 3.7.3] states that the automorphism group of $H^2(A,\mathbb{Z})$ acts transitively on classes of the same norm. So every class can be mapped to a divisor by the action of a monodromy.

Remark 9.3. Qin and Wang [37, Thm 1.1 et seq.] conjecture that their theory works even without the restriction on $a \in H^2(A,\mathbb{Z})$.

Proposition 9.4. *Assume that $H^*(A,\mathbb{Z})$ is free of torsion. Let $(a_i) \subset H^1(A,\mathbb{Z})$ and $(b_i) \subset H^2(A,\mathbb{Z})$ be bases of integral cohomology. Denote $a_i^* \in H^3(A,\mathbb{Z})$ resp. $b_i^* \in H^2(A,\mathbb{Z})$ the elements of the dual bases. Let x be the generator of $H^4(A,\mathbb{Z})$. Modulo torsion, the following classes form a basis of $H^2(A^{[n]},\mathbb{Z})$:*

- $\frac{1}{(n-1)!}\mathfrak{q}_1(b_i)\mathfrak{q}_1(1)^{n-1}|0\rangle = \mathfrak{G}_0(b_i)1$,
- $\frac{1}{(n-2)!}\mathfrak{q}_1(a_i)\mathfrak{q}_1(a_j)\mathfrak{q}_1(1)^{n-2}|0\rangle = \mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j)1,\ i<j$,
- $\frac{1}{2(n-2)!}\mathfrak{q}_2(1)\mathfrak{q}_1(1)^{n-2}|0\rangle$. *We denote this class by $\delta = \mathfrak{d}1$.*

Their respective duals in $H^{2n-2}(A^{[n]},\mathbb{Z})$ are given by

- $\mathfrak{q}_1(b_i^*)\mathfrak{q}_1(x)^{n-1}|0\rangle$,
- $\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(x)^{n-2}|0\rangle,\ i<j$,
- $\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle$.

Proof. It is clear from the above lemma that these classes are all integral. Göttsche's formula [14, p. 35] gives the Betti numbers of $A^{[n]}$ in terms of the Betti numbers of A: $h^1(A^{[n]}) = h^1(A)$, and $h^2(A^{[n]}) = h^2(A) + \frac{h^1(A)(h^1(A)-1)}{2} + 1$. It follows that the given classes span a lattice of full rank.

Next we have to show that the intersection matrix between these classes is in fact the identity matrix. Most of the entries can be computed easily using (28). For products involving δ (this is the action of $\mathfrak{d}$) or its dual, first observe that $\mathfrak{d}\mathfrak{q}_1(x)^m|0\rangle = 0$ and $\mathfrak{L}_1(a)\mathfrak{q}_1(x)^m|0\rangle = 0$ for every class a of degree at least 1. Then compute:

$$\begin{gathered}
\delta \cdot \mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = \mathfrak{d}\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = 2\mathfrak{L}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = \mathfrak{q}_1(x)^n|0\rangle,\\
\mathfrak{d}\mathfrak{q}_1(b_i^*)\mathfrak{q}_1(x)^{n-1}|0\rangle = \mathfrak{L}_1(b_i^*)\mathfrak{q}_1(x)^{n-1}|0\rangle = 0,\\
\mathfrak{d}\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(x)^{n-2}|0\rangle = \left(\mathfrak{L}_1(a_j^*) + \mathfrak{q}_1(a_j^*)\mathfrak{d}\right)\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(x)^{n-2}|0\rangle =\\
= \left(-\mathfrak{q}_1(a_i^*)\mathfrak{L}_1(a_j^*) + \mathfrak{q}_1(a_j^*)\mathfrak{L}_1(a_i^*)\right)\mathfrak{q}_1(x)^{n-2}|0\rangle = 0,\\
\mathfrak{G}_0(b_i)\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = 0,\\
\mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j)\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = 0.
\end{gathered}$$

□

Proposition 9.5. *Let A be a complex torus of dimension 2. Let $(b_i) \subset H^2(A,\mathbb{Z})$ be an integral basis. By [43], $H^*(A^{[2]},\mathbb{Z})$ is torsion-free and a basis is given by the following classes.*

degree	*Betti number*	*class*	*multiplication with class*
0	*1*	$\frac{1}{2}\mathfrak{q}_1(1)^2\lvert 0\rangle$	id
1	*4*	$\mathfrak{q}_1(1)\mathfrak{q}_1(a_i)\lvert 0\rangle$	$\mathfrak{G}_0(a_i)$
2	*13*	$\frac{1}{2}\mathfrak{q}_2(1)\lvert 0\rangle$ $\mathfrak{q}_1(a_i)\mathfrak{q}_1(a_j)\lvert 0\rangle$ *for* $i<j$ $\mathfrak{q}_1(1)\mathfrak{q}_1(b_i)\lvert 0\rangle$	$\mathfrak{d}$ $\mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j)$ $\mathfrak{G}_0(b_i)$
3	*32*	$\mathfrak{q}_2(a_i)\lvert 0\rangle$ $\mathfrak{q}_1(a_i)\mathfrak{q}_1(b_j)\lvert 0\rangle$ $\mathfrak{q}_1(1)\mathfrak{q}_1(a_i^*)\lvert 0\rangle$	$-2\mathfrak{G}_1(a_i)$ $\mathfrak{G}_0(a_i)\mathfrak{G}_0(b_j)$ $\mathfrak{G}_0(a_i^*)$
4	*44*	$\left(\frac{1}{2}\mathfrak{q}_1(b_i)^2-\frac{1}{2}\mathfrak{q}_2(b_i)\right)\lvert 0\rangle$ $\mathfrak{q}_1(a_i)\mathfrak{q}_1(a_j^*)\lvert 0\rangle$ $\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)\lvert 0\rangle$ *for* $i\le j$	$\frac{1}{2}\mathfrak{G}_0(b_i)^2+\mathfrak{G}_1(b_i)$ $\mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j^*)$ $\mathfrak{G}_0(b_i)\mathfrak{G}_0(b_j)$
5	*32*	$\frac{1}{2}\mathfrak{q}_2(a_i^*)\lvert 0\rangle$ $\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(b_j)\lvert 0\rangle$ $\mathfrak{q}_1(a_i)\mathfrak{q}_1(x)\lvert 0\rangle$	$-\mathfrak{G}_1(a_i^*)$ $\mathfrak{G}_0(a_i^*)\mathfrak{G}_0(b_j)$ $\mathfrak{G}_0(a_i)\mathfrak{G}_0(x)$
6	*13*	$\mathfrak{q}_2(x)\lvert 0\rangle$ $\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(a_j^*)\lvert 0\rangle$ *for* $i<j$ $\mathfrak{q}_1(b_i)\mathfrak{q}_1(x)\lvert 0\rangle$	$-2\mathfrak{G}_1(x)$ $\mathfrak{G}_0(a_i^*)\mathfrak{G}_0(a_j^*)$ $\mathfrak{G}_0(b_i)\mathfrak{G}_0(x)$
7	*4*	$\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(x)\lvert 0\rangle$	$\mathfrak{G}_0(a_i^*)\mathfrak{G}_0(x)$
8	*1*	$\mathfrak{q}_1(x)^2\lvert 0\rangle$	$\mathfrak{G}_0(x)^2$

Proof. The Betti numbers come from Göttsche's formula [14]. One computes the intersection matrix of all classes under the Poincaré duality pairing and finds that it is unimodular. So it remains to show that all these classes are integral. By Proposition 9.1 this is clear for all classes except those of the form $\frac{1}{2}\mathfrak{q}_2(a_i^*)\lvert 0\rangle \in H^5(A^{[2]},\mathbb{Z})$.

Evaluating the Poincaré duality pairing between degrees 3 and 5 gives:

$$\int \mathfrak{q}_2(a_i)\lvert 0\rangle \cdot \mathfrak{q}_2(a_i^*)\lvert 0\rangle = 2,$$
$$\int \mathfrak{q}_1(a_i)\mathfrak{q}_1(b_j)\lvert 0\rangle \cdot \mathfrak{q}_1(a_i^*)\mathfrak{q}_1(b_j^*)\lvert 0\rangle = 1,$$
$$\int \mathfrak{q}_1(1)\mathfrak{q}_1(a_i^*)\lvert 0\rangle \cdot \mathfrak{q}_1(x)\mathfrak{q}_1(a_i)\lvert 0\rangle = 1,$$

while the other pairings vanish. Therefore, one of $\mathfrak{q}_2(a_i)\lvert 0\rangle$ and $\mathfrak{q}_2(a_i^*)\lvert 0\rangle$ must be divisible by 2. We can interpret $\mathfrak{q}_2(a_i)\lvert 0\rangle \in H^3(A^{[2]},\mathbb{Z})$ and $\mathfrak{q}_2(a_i^*)\lvert 0\rangle \in H^5(A^{[2]},\mathbb{Z})$ as classes concentrated on the exceptional divisor, that is, as elements of $\pi_* i_* H^*(E,\mathbb{Z})$. Indeed, the pushforward of a class $a\otimes 1 \in H^k(E,\mathbb{Z})$ is given by

$$\pi_* i_*(a\otimes 1) = \mathfrak{q}_2(a)\lvert 0\rangle \in H^{k+2}(A^{[n]},\mathbb{Z}).$$

When pushing forward to the Hilbert scheme, the only possibility to get a factor 2 is in degree 5, by Proposition 7.1. □

Part II
The Generalized Kummer fourfold

10 Generalized Kummer varieties and the morphism to the Hilbert scheme

Definition 10.1. Let A be a complex projective torus of dimension 2 and $A^{[n]}$, $n \geq 1$, the corresponding Hilbert scheme of points. Denote $\Sigma : A^{[n]} \to A$ the summation morphism, a smooth submersion that factorizes via the Hilbert–Chow morphism : $A^{[n]} \xrightarrow{\rho} \mathrm{Sym}^n(A) \xrightarrow{\sigma} A$. Then the generalized Kummer variety $K_{n-1}(A)$ is defined as the fiber over 0:

$$\begin{array}{ccc} K_{n-1}(A) & \xrightarrow{\theta} & A^{[n]} \\ \Big\downarrow & & \Big\downarrow \Sigma \\ \{0\} & \longrightarrow & A \end{array} \tag{31}$$

Theorem 10.2. *[41, Theorem 2] The cohomology of the generalized Kummer, $H^*(K_{n-1}(A), \mathbb{Z})$, is torsion free.*

Our first objective is to collect some information about this pullback diagram. Recall Notation 5.6.

Proposition 10.3. *Let $\alpha_i := \frac{1}{(n-1)!}\mathfrak{q}_1(1)^{n-1}\mathfrak{q}_1(a_i)|0\rangle = \mathfrak{G}_0(a_i)1$. The class of the Poincaré dual of $K_{n-1}(A)$ in $H^4(A^{[n]}, \mathbb{Z})$ is given by*

$$[K_{n-1}(A)] = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4.$$

Proof. Since the generalized Kummer variety is the fiber over a point, its Poincaré dual must be the pullback of $x \in H^4(A)$ under Σ. But $\Sigma^*(x) = \Sigma^*(a_1) \cdot \Sigma^*(a_2) \cdot \Sigma^*(a_3) \cdot \Sigma^*(a_4)$, so we have to verify that $\Sigma^*(a_i) = \alpha_i$. To do this, we want to use the decomposition $\Sigma = \sigma\rho$. The pullback along σ of a class $a \in H^1(A, \mathbb{Q})$ on $H^1(\mathrm{Sym}^n(A), \mathbb{Q})$ is given by $a \otimes 1 \otimes \cdots \otimes 1 + \ldots + 1 \otimes \cdots \otimes 1 \otimes a$. It follows from (30) that $\Sigma^*(a_i) = \frac{1}{(n-1)!}\mathfrak{q}_1(1)^{n-1}\mathfrak{q}_1(a_i)|0\rangle$. □

The morphism θ induces a homomorphism of graded rings

$$\theta^* : H^*(A^{[n]}) \longrightarrow H^*(K_{n-1}(A)) \tag{32}$$

and on the image of θ^*, the integral over $K_{n-1}(A)$ can be written as follows:

$$\int_{K_{n-1}(A)} \theta^*(\alpha) = \int_{A^{[n]}} [K_{n-1}(A)] \cdot \alpha. \tag{33}$$

Lemma 10.4. *Let $\alpha \in H^*(A^{[n]}, \mathbb{Q})$ be such that $\theta^*(\alpha) \in H^*(K_{n-1}(A), \mathbb{Z})$ is an integral class. Then $[K_{n-1}(A)] \cdot \alpha \in H^*(A^{[n]}, \mathbb{Z})$.*

Proof. The integrality of $\theta^*(\alpha)$ implies that, for all $\beta \in H^*(A^{[n]}, \mathbb{Z})$,

$$\int_{K_{n-1}(A)} \theta^*(\alpha) \cdot \theta^*(\beta) = \int_{A^{[n]}} [K_{n-1}(A)] \cdot \alpha \cdot \beta$$

is integral. Since the Poincaré pairing on $A^{[n]}$ is unimodular, $[K_{n-1}(A)] \cdot \alpha$ must be integral. □

Proposition 10.5. *The kernel of θ^* is equal to the annihilator of $[K_{n-1}(A)]$.*

Proof. Assume $\alpha \in \ker(\theta^*)$. Then, for all $\beta \in H^*(A^{[n]})$, we have

$$\int_{A^{[n]}} [K_{n-1}(A)] \cdot \alpha \cdot \beta = \int_{K_{n-1}(A)} \theta^*(\alpha) \cdot \theta^*(\beta) = 0.$$

Since the Poincaré pairing on $A^{[n]}$ is non-degenerate, this implies $[K_{n-1}(A)] \cdot \alpha = 0$.

Conversely, suppose that $[K_{n-1}(A)] \cdot \alpha = 0$. Then, for all $\beta \in H^*(A^{[n]})$, we have

$$\int_{K_{n-1}(A)} \theta^*(\alpha) \cdot \theta^*(\beta) = 0,$$

so $\theta^*(\alpha) \in \mathrm{Im}(\theta^*) \cap \mathrm{Im}(\theta^*)^{\perp}$. The Poincaré pairing on $K_{n-1}(A)$ is non-degenerate when restricted to $\mathrm{Im}(\theta^*)$. To see this, observe that the dual element of $\theta^*(\alpha)$ in $\mathrm{Im}(\theta^*)$ is given by $\theta^*(\alpha^*)$, where α^* is the Poincaré dual of $[K_{n-1}(A)] \cdot \alpha$. So it follows that $\mathrm{Im}(\theta^*) \cap \mathrm{Im}(\theta^*)^{\perp} = 0$. □

Corollary 10.6. *$\theta^*(\alpha) = \theta^*(\beta)$ if and only if $[K_{n-1}(A)] \cdot \alpha = [K_{n-1}(A)] \cdot \beta$.* □

Proposition 10.7. *The annihilator of $[K_{n-1}(A)]$ in $H^*(A^{[n]}, \mathbb{Q})$ is the ideal generated by $H^1(A^{[n]})$.*

Proof. Set $H = H^*(A, \mathbb{Q})$ and consider the exact sequence of H-modules

$$0 \longrightarrow J \longrightarrow H \xrightarrow{x\cdot} H.$$

It is clear that J is the ideal in H generated by $H^1(A, \mathbb{Q})$. Now denote $J^{(n)}$ the ideal generated by $H^1(\mathrm{Sym}^n(A), \mathbb{Q})$ in $H^*(\mathrm{Sym}^n(A), \mathbb{Q}) \cong \mathrm{SSym}^n(H)$. By the freeness result of Lemma 3.5, tensoring with $\mathrm{SSym}^n(H)$ yields another exact sequence of H-modules

$$0 \longrightarrow J^{(n)} \longrightarrow \mathrm{SSym}^n(H) \xrightarrow{\sigma(x)\cdot} \mathrm{SSym}^n(H).$$

Now let $\mathfrak{H}$ be the operator algebra spanned by products of $\mathfrak{d}$ and $\mathfrak{q}_1(a)$ for $a \in H^*(A)$. Let $\mathfrak{C}$ be the graded commutative subalgebra of $\mathfrak{H}$ generated by $\mathfrak{q}_1(a)$ for $a \in H^*(A)$. The action of $\mathfrak{H}$ on $|0\rangle$ gives $\mathbb{H}$ and the action of $\mathfrak{C}$ on $|0\rangle$ gives $\rho^*(H^*(\mathrm{Sym}^n(A), \mathbb{Q})) \cong \mathrm{SSym}^n(H)$. By sending $\mathfrak{d}$ to the identity, we define a linear map $c : \mathfrak{H} \to \mathfrak{C}$. Denote $J^{[n]}$ the ideal generated by $H^1(A^{[n]}, \mathbb{Q})$ in $H^*(A^{[n]}, \mathbb{Q})$. We claim that for every $\mathfrak{y} \in \mathfrak{H}$:

$$\mathfrak{y}|0\rangle \in J^{[n]} \Leftrightarrow c(\mathfrak{y})|0\rangle \in J^{[n]}.$$

To see this, we remark that $H^1(A^{[n]}, \mathbb{Q}) \cong H^1(A, \mathbb{Q})$ and the multiplication with a class in $H^1(A^{[n]}, \mathbb{Q})$ is given by the operator $\mathfrak{G}_0(a)$ for some $a \in H^1(A, \mathbb{Q})$. Due to the fact that $\mathfrak{d}$ is also a multiplication operator (of degree 2), $\mathfrak{G}_0(a)$ commutes with $\mathfrak{d}$. It follows that for $\mathfrak{y} = \mathfrak{G}_0(a)\mathfrak{r}$ we have $c(\mathfrak{y}) = \mathfrak{G}_0(a)c(\mathfrak{r})$.

Now denote $\mathfrak{k}$ the multiplication operator with the class $[K_{n-1}(A)]$. We have: $[\mathfrak{k}, \mathfrak{d}] = 0$. Now let $y \in H^*(A^{[n]}, \mathbb{Q})$ be a class in the annihilator of $[K_{n-1}(A)]$. We can write $y = \mathfrak{y}|0\rangle$ for a $\mathfrak{y} \in \mathfrak{H}$. Choose $\tilde{y} \in \mathrm{SSym}^n(H)$ in a way that $\rho^*(\tilde{y}) = c(\mathfrak{y})|0\rangle$. Then we have:

$$0 = [K_{n-1}(A)] \cdot y = \mathfrak{k}\,\mathfrak{y}|0\rangle = \mathfrak{k}\,c(\mathfrak{y})|0\rangle = \rho^*(\sigma^*(x) \cdot \tilde{y}).$$

Since ρ^* is injective, $\tilde{y}$ is in the annihilator of $\sigma^*(x)$, so $\tilde{y} \in J^{(n)}$. It follows that $c(\mathfrak{y})|0\rangle$ and y are in the ideal generated by $H^1(A^{[n]}, \mathbb{Q})$. □

Theorem 10.8. *[1, Théorème 4] $K_{n-1}(A)$ is a irreducible holomorphically symplectic manifold. In particular, it is simply connected and the canonical bundle is trivial.*

This implies that $H^2(K_{n-1}(A), \mathbb{Z})$ admits an integer-valued nondegenerated quadratic form (called Beauville–Bogomolov form) q which gives $H^2(K_{n-1}(A), \mathbb{Z})$ the structure of a lattice. This lattice is isomorphic to $U^{\oplus 3} \oplus \langle -2n \rangle$, for $n \geq 3$.

TODO: find a reference for Beauville-Bogomolov form

We have the following formula for $\alpha \in H^2(K_{n-1}(A), \mathbb{Z})$:

$$\int_{K_{n-1}(A)} \alpha^{2n-2} = n \cdot (2n-3)!! \cdot q(\alpha)^{n-1} \tag{34}$$

Proposition 10.9. *Assume $n \geq 3$. Then θ^* is surjective on $H^2(A^{[n]}, \mathbb{Z})$.*

Proof. By [1, Sect. 7], $\theta^* : H^2(A^{[n]}, \mathbb{C}) \to H^2(K_{n-1}(A), \mathbb{C})$ is surjective. Then the idea is to prove that the lattice structure of $\operatorname{Im} \theta^*$ is the same as of $H^2(K_{n-1}(A))$. We use two formulas in [7, pp. 8–11]. Let $b \in H^2(A, \mathbb{Z})$ and set $\alpha = \frac{1}{(n-1)!}\mathfrak{q}_1(1)^{n-1}\mathfrak{q}_1(b)|0\rangle \in H^2(A^{[n]}, \mathbb{Z})$. Then

$$\int_{A^{[n]}} \alpha^{2n} = \binom{2n}{2} \frac{\int_A b^2}{n^2} \int_{K_{n-1}(A)} \theta^* \alpha^{2n-2} \tag{35}$$

By Proposition (TODO), the left hand side of this equation equals $(2n-1)!! \cdot \left(\int_a b^2\right)^n$. By (34), the right hand side gives $(2n-1)!! \cdot \left(\int_A b^2\right) \cdot q(\alpha)^{n-1}$. So we get $\int_A b^2 = q(\alpha)$, giving the set of all α a lattice structure isomorphic to $H^2(A, \mathbb{Z})$. Secondly, we must show that for $\delta =$ half of the exceptional divisor: $q(\theta^*\delta) = -2n$. But this follows now from Proposition 1 in [7]. □

Notation 10.10. We have seen that, for $n \geq 3$,

$$H^2(K_{n-1}(A), \mathbb{Z}) \cong H^2(A, \mathbb{Z}) \oplus \langle \theta^*(\delta) \rangle .$$

We denote the injection : $H^2(A, \mathbb{Z}) \to H^2(K_{n-1}(A), \mathbb{Z})$ by j. Further, we set $e := \theta^*(\delta)$. We give the following names for classes in $H^2(K_{n-1}(A), \mathbb{Z})$:

$$\begin{aligned} u_1 &:= j(a_1a_2), & v_1 &:= j(a_1a_3), & w_1 &:= j(a_1a_4), \\ u_2 &:= j(a_3a_4), & v_2 &:= j(a_4a_2), & w_2 &:= j(a_2a_3), \end{aligned}$$

These elements form a basis of $H^2(K_{n-1}(A), \mathbb{Z})$ with the following intersection relations under the Beauville-Bogomolov form:

$$q(u_1, u_2) = 1, \qquad q(v_1, v_2) = 1, \qquad q(w_1, w_2) = 1, \qquad q(e, e) = -2n,$$

and all other pairs of basis elements are orthogonal.

11 Odd Cohomology of the Generalized Kummer fourfold

Now we come to the special case $n = 3$, so we study $K_2(A)$, the Generalized Kummer fourfolds.

Proposition 11.1. *The Betti numbers of $K_2(A)$ are:* 1, 0, 7, 8, 108, 8, 7, 0, 1.

Proof. This follows from Göttsche's formula [14, page 49]. □

By means of the morphism θ^*, we may express part of the cohomology of $K_2(A)$ in terms of Hilbert scheme cohomology. We have seen in Proposition 10.9 that θ^* is surjective for degree 2 and (by duality) also in degree 6. The next proposition shows that this also holds true for odd degrees.

Proposition 11.2. *A basis of $H^3(K_2(A), \mathbb{Z})$ is given by:*

$$\frac{1}{2}\theta^*\Big(\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(1)^2|0\rangle\Big), \tag{36}$$

$$\theta^*\Big(\mathfrak{q}_2(a_i)\mathfrak{q}_1(1)|0\rangle\Big). \tag{37}$$

and a basis of $H^5(K_2(A), \mathbb{Z})$ is given by:

$$\frac{2}{3}\theta^*\Big(\mathfrak{G}_2(a_i)\Big) = \theta^*\Big(-\frac{1}{3}\mathfrak{q}_3(a_i)|0\rangle - \mathfrak{L}_2(a_i)\mathfrak{q}|0\rangle\Big), \tag{38}$$

$$\frac{1}{2}\theta^*\Big(\mathfrak{q}_2(a_i^*)\mathfrak{q}_1(1)|0\rangle\Big). \tag{39}$$

Proof. The classes (36) are Poincaré dual to (38) and the classes (37) are Poincaré dual to (39) by computation, so it remains to show that all of them are integral.

It is clear from Theorem 9.1 that (36) and (37) are integral. By Proposition 9.5, $\frac{1}{2}\mathfrak{q}_2(a_i^*)|0\rangle$ is integral as well. If the operator $\mathfrak{q}_1(1)$ is applied, we get again an integral class.

Further, $\frac{2}{3}\mathfrak{G}_2(a_i)[K_2(A)]$ is an integral class and $[K_2(A)] \cdot \mathfrak{q}_1(a_i^*)\mathfrak{q}_1(1)^2|0\rangle$ is non-divisible. Because of $\int_{A^{[3]}} [K_2(A)] \cdot \mathfrak{G}_2(a_i)\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(1)^2|0\rangle = 3$, one of $\frac{1}{2}\theta^*\left(\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(1)^2|0\rangle\right)$ and $2\theta^*\left(\mathfrak{G}_2(a_i)\right)$ is divisible by three. In view of Lemma 10.4, it must be the latter one. □

It follows the following corollary which will be used in Part III.

Corollary 11.3. *Let A be an abelian surface and g be an automorphisms on A. Let $g^{[[3]]}$ be the automorphisms induced by g on $K_2(A)$. We have $H^3(K_2(A), \mathbb{Z}) \simeq H^1(A, \mathbb{Z}) \oplus H^3(A, \mathbb{Z})$ and the action of $g^{[[3]]}$ on $H^3(K_2(A), \mathbb{Z})$ is given by the action of g on $H^1(A, \mathbb{Z}) \oplus H^3(A, \mathbb{Z})$.*

Proof. Let $g^{[3]}$ be the involution on $A^{[3]}$ induced by g. We have $g^{[3]*}(a_i^{(1)}) = (g^*a_i)^{(1)}$ and $g^{[3]*}(a_{\bar{i}}^{(0)}) = (g^*a_{\bar{i}})^{(0)}$. Moreover, we have by definition, $g^{[[3]]*} \circ \theta^* = \theta^* \circ g^{[3]*}$. The result follows from Proposition 11.2. □

12 Middle cohomology

The middle cohomology $H^4(K_2(A), \mathbb{Z})$ has been studied by Hassett and Tschinkel in [15]. We first recall some of their results, then we proceed by using θ^* to give a partial description of $H^4(K_2(A), \mathbb{Z})$ in terms of the well-understood cohomology of $A^{[3]}$. Finally, we find a basis of $H^4(K_2(A), \mathbb{Z})$ using the action of the monodromy group.

Notation 12.1. For each $\tau \in A$, denote W_τ the Briançon subscheme of $A^{[3]}$ supported enitrely at the point τ. If $\tau \in A[3]$ is a point of three-torsion, W_τ is actually a subscheme of $K_2(A)$. We will also use the symbol W_τ for the corresponding class in $H^4(K_2(A), \mathbb{Z})$. Further, set

$$W := \sum_{\tau \in A[3]} W_\tau.$$

For $p \in A$, denote Y_p the locus of all $\{x, y, p\}$ in $K_2(A)$. The corresponding class $Y_p \in H^4(K_2(A), \mathbb{Z})$ is independent of the choice of the point p. Then set $Z_\tau := Y_p - W_\tau$ and denote Π the lattice generated by all Z_τ, $\tau \in A[3]$.

Proposition 12.2. *Denote by* $\mathrm{Sym} := \mathrm{Sym}^2\left(H^2\left(K_2(A), \mathbb{Z}\right)\right) \subset H^4\left(K_2(A), \mathbb{Z}\right)$ *the span of products of integral classes in degree* 2. *Then*

$$\mathrm{Sym} + \Pi \ \subset \ H^4\left(K_2(A), \mathbb{Z}\right)$$

is a sublattice of full rank.

Proof. This follows from [15, Proposition 4.3]. □

Proposition 12.3. *The class W can be written with the help of the square of half the diagonal as*

$$W = \theta^*\Big(\mathfrak{q}_3(1)|0\rangle\Big) \tag{40}$$

$$= 9Y_p + e^2. \tag{41}$$

The second Chern class is non-divisible and given by

$$c_2(K_2(A)) = \frac{1}{3} \sum_{\tau \in A[3]} Z_\tau \tag{42}$$

$$= \frac{1}{3}\Big(72Y_p - e^2\Big). \tag{43}$$

Proof. In Section 4 of [15] one finds the equations

$$W = \frac{3}{8}\Big(c_2(K_2(A)) + 3e^2\Big), \tag{44}$$

$$Y_p = \frac{1}{72}\Big(c_2(K_2(A)) + e^2\Big), \tag{45}$$

from which we deduce (41) and (43). Equation (42) and the non-divisibility are from [15, Proposition 5.1]. □

Proposition 12.4. *We have:*

$$c_2(K_2(A)) = 4u_1u_2 + 4v_1v_2 + 4w_1w_2 - \frac{1}{3}e^2. \tag{46}$$

In particular, $c_2(K_2(A)) \in \mathrm{Sym}$.

Proof. First note that the defining diagram (31) of the Kummer manifold is the pullback of the inclusion of a point, so the normal bundle of $K_2(A)$ in $A^{[3]}$ is trivial. The Chern class of the tangent bundle of $K_2(A)$ is therefore given by the pullback from $A^{[3]}$: $c(K_2(A)) = \theta^*\left(c(A^{[3]})\right)$. Theorem **??** allows now to conclude that $c_2(K_2(A)) \in \mathrm{Sym}$.

To obtain the precise formula, we use a result of Boissière, [2, Lemma 3.12], giving a commutation relation for the multiplication operator with the Chern class on the Hilbert scheme. We get:

$$c_2(A^{[3]}) = 3\mathfrak{q}_1(1)\mathfrak{L}_2(1)|0\rangle - \frac{8}{3}\mathfrak{q}_3(1)|0\rangle.$$

With Corollary 10.6 one shows now, that $c_2(K_2(A))$ is given as stated. □

Corollary 12.5. *The intersection* $\mathrm{Sym} \cap \Pi$ *is one-dimensional and spanned by* $c_2(K_2(A))$.

Proof. By Proposition 12.4 and (42), $c_2(K_2(A)) \in \mathrm{Sym} \cap \Pi$. Since the ranks of Sym, Π and $H^4(K_2(A), \mathbb{Z})$ are 28, 81 and 108, respectively, the intersection cannot contain more. □

Corollary 12.6.

$$Y_p = \frac{1}{6}\Big(u_1u_2 + v_1v_2 + w_1w_2\Big). \tag{47}$$

Remark 12.7. Using Nakajima operators, we may write

$$Y_p = \frac{1}{9}\theta^*\Big(\mathfrak{q}_1(1)\mathfrak{L}_2(1)|0\rangle\Big). \tag{48}$$

Proposition 12.8. *The image of* $H^4(A^{[3]}, \mathbb{Z})$ *under* θ^* *is a (non-primitive?) overlattice of* Sym *of the same rank.*

Definition 12.9. We set $\Pi' := \mathrm{Sym}^\perp \subset \Pi$. This lattice can be described as the span of all classes of the form $Z_\tau - Z_0$ or alternatively as the set of all $\sum_\tau \alpha_\tau Z_\tau$, such that $\sum_\tau \alpha_\tau = 0$.

From the intersection properties $Z_\tau \cdot Z_{\tau'} = 1$ for $\tau \neq \tau'$ and $Z_\tau^2 = 4$ from Section 4 of [15], we compute

$$\operatorname{discr} \Pi' = 3^{84}. \tag{49}$$

Since Sym has discriminant $2^{14} \cdot 3^{38}$, the lattices Sym and Π' cannot be primitive. To obtain a basis of $H^4(K_2(A), \mathbb{Z})$, we are now going to find

- 7 classes in Sym divisible by 2,
- 8 classes in Sym divisible by 3,
- 31 classes in Π' divisible by 3 and
- 22 classes in $\mathrm{Sym} \oplus \Pi'$ divisible by 3.

Proposition 12.10. *For* $y \in \{u_1, u_2, v_1, v_2, w_1, w_2\}$, *the class* $e \cdot y$ *is divisible by* 3 *and* $y^2 - \frac{1}{3}e \cdot y$ *is divisible by* 2.

Proof. We have $y = \theta^*\left(\mathfrak{q}_1(a)\mathfrak{q}_1(1)^2|0\rangle\right)$ for some $a \in H^2(A, \mathbb{Z})$. A computation yields:

$$e \cdot y = 3 \cdot \theta^*\Big(\mathfrak{q}_2(a)\mathfrak{q}_1(1)|0\rangle\Big) \quad \text{and} \quad y^2 = \theta^*\Big(\mathfrak{q}_1(a)^2\mathfrak{q}_1(1)|0\rangle\Big)$$

so $e \cdot y$ is divisible by 3. Furthermore, by Lemma 9.1, the class $\frac{1}{2}\mathfrak{q}_1(a)^2\mathfrak{q}_1(1)|0\rangle - \frac{1}{2}\mathfrak{q}_2(a)\mathfrak{q}_1(1)|0\rangle$ is contained in $H^4(A^{[3]}, \mathbb{Z})$, so its pullback $\frac{1}{2}y^2 - \frac{1}{6}e \cdot y$ is an integral class, too. □

Looking at Proposition 12.4, we see that e^2 is divisible by 3 and by (47), the class $u_1u_2 + v_1v_2 + w_1w_2$ is divisible by 6.

Now we come to Π'. For a non-isotropic plane $\Lambda \subset A[3]$ and any $\tau_0 \in A[3]$, the classes

$$\frac{1}{3}\sum_{\tau \in \Lambda}\Big(Z_\tau - Z_{\tau+\tau_0}\Big) \tag{50}$$

are integral (cf. (12) of [15]). By the considerations after Definition 4.6, these give a space of rank 31 of classes in Π' divisible by 3.

There is one missing class provided by Z_0 which is not in Sym nor in Π'. The class Z_0 can be written as follows:

$$Z_0 = \frac{(\sum_{\tau\in A[3]} Z_\tau) + (\sum_{\tau\in A[3]} Z_\tau - Z_0)}{81}.$$

From Remark 4.7, we have already counted the class $\sum_{\tau\in A[3]}(Z_\tau - Z_0)$ which is divisible by 3. Then we can write:

$$Z_0 = \frac{c_2(K_2(A)) + \frac{1}{3}(\sum_{\tau\in A[3]} Z_\tau - Z_0)}{27}.$$

It provides the equivalent of 3 classes divisible by 3.

Hence it is remaining 19 classes divisible by 3 to be found. To find these 19 remaining classes, we start by assuming that A is the product of two elliptic curves $A = E_1 \times E_2$. Hassett and Tschinkel in Proposition 7.1 of [15], provide the class of a Lagrangian plan $P \subset K_2(A)$ which can be expressed as follows:

$$[P] = \frac{1}{216}(6u_1 - 3e)^2 + \frac{1}{8}c_2(K_2(A)) - \frac{1}{3}\sum_{\tau\in\Lambda'} Z_\tau,$$

where $\Lambda' = E_1[3] \times 0 \subset A[3]$. We rearrange a bit this expression using (44):

$$\begin{aligned}
[P] &= \frac{1}{216}(6u_1 - 3e)^2 + \frac{1}{8}c_2(K_2(A)) - \frac{1}{3}\sum_{\tau\in\Lambda'} Z_\tau \\
&= \frac{36u_1^2 + 9e^2 - 36u_1\cdot e}{216} + \frac{W}{3} - \frac{3}{8}e^2 - \frac{1}{3}\sum_{\tau\in\Lambda'} Z_\tau \\
&= \frac{36u_1^2 - 72e^2 - 36u_1\cdot e}{216} + \frac{W}{3} - \frac{1}{3}\sum_{\tau\in\Lambda'} Z_\tau \\
&= \frac{u_1^2 - 2e^2 - u_1\cdot e}{6} + \frac{W}{3} - \frac{1}{3}\sum_{\tau\in\Lambda'} Z_\tau.
\end{aligned}$$

However, by Proposition 11.4, e^2 is divisible by 3, by Proposition 11.9, $u_1\cdot e$ is divisible by 3 and by (44), W is divisible by 3. Hence, we obtain the following integral class in $H^4(K_2(A),\mathbb{Z})$:

$$\mathfrak{N}_0 := \frac{u_1^2 + \sum_{\tau\in\Lambda'} Z_\tau}{3}.$$

Now let A be a principally polarized abelian surface. The surface A can be deform in $E_1 \times E_2$. Then by an appropriated choice of the basis of $H^1(A,\mathbb{Z})$, the class $\mathfrak{N}_0$ deforms in a class of in $H^4(K_2(A),\mathbb{Z})$ that can be expressed similary:

$$\frac{u_1^2 + \sum_{\tau\in\Lambda'} Z_\tau}{3},$$

where Λ' is the plane of $A[3]$ generated by $\frac{a_1}{3}$ and $\frac{a_2}{3}$. It provides this new class:

$$\mathfrak{N} := \frac{u_1^2 + \sum_{\tau\in\Lambda'\smallsetminus\{0\}} Z_\tau - Z_0}{3}.$$

Now we will conclude using the action of the monodromy group $\mathrm{Sp}(A[3]) \ltimes A[3]$ on the element $\mathfrak{N}$ and the considerations after Definition 4.6.

By Lemma 4.8, the orbit of $\mathfrak{N}$ under the action of $\mathrm{Sp}(A[3]) \ltimes A[3]$ provides 19 linearly independent elements of the form: $\frac{x+y}{3}$ with $x \in \mathrm{Sym} \smallsetminus \{0\}$, and $y \in \Pi' \smallsetminus \{0\}$. It is the 19 classes which was missing.

Part III
A quotient

13 Symplectic involution on $K_2(A)$

Let X be an irreducible symplectic manifold. Let

$$\nu : \mathrm{Aut}(X) \to H^2(X, \mathbb{Z})$$

the natural morphism. Hassett and Tschinkel (Theorem 2.1 in [15]) have shown that $\operatorname{Ker}\nu$ is a deformation invariant. Let X be an irreducible symplectic fourfold of Kummer type. Then Oguiso in [36] has shown that $\operatorname{Ker}\nu = (\mathbb{Z}/3\mathbb{Z})^3 \rtimes \mathbb{Z}/2\mathbb{Z}$.

Let A be an abelian variety and g an automorphism of A. Let denote by $T_{A[3]}$ the group of translation of A by elements of $A[3]$. If $g \in T_{A[3]} \rtimes \mathrm{Aut}_{\mathbb{Z}}(X)$ then g induces a natural automorphism on $K_2(A)$. We denote the induced automorphism by $g^{[[3]]}$. If there is no ambiguity, we also denote the *induced automorphism* by the same letter g.

When $X = K_2(A)$, we have more precisely, by Corollary 3.3 of [4],

$$\operatorname{Ker}\nu = T_{A[3]} \rtimes (-\mathrm{id}_A)^{[[3]]}.$$

13.1 Uniqueness and fixed locus

Theorem 13.1. *Let X be an irreducible symplectic fourfold of Kummer type and ι a symplectic involution on X then:*

(1) We have $\iota \in \operatorname{Ker}\nu$.

(2) Let A be an abelian surface then the couple (X, ι) is deformation equivalent to $(K_2(A), t_\tau \circ (-\mathrm{id}_A)^{[[3]]})$, where t_τ is the morphism induced on $K_2(A)$ by the translation by $\tau \in A[3]$.

(3) The fixed locus of ι is given by a K3 surface and 36 isolated points.

Proof. (1) If $\iota \notin \operatorname{Ker}\nu$, by Section 5 of [30], the unique possible action of ι on $H^2(X,\mathbb{Z})$ is given by $H^2(X,\mathbb{Z})^\iota = U \oplus A_1^2 \oplus (-6)$. We will show that it is impossible. Let assume that $H^2(X,\mathbb{Z})^\iota = U \oplus A_1^2 \oplus (-6)$, we will find a contradiction.

As done in Section 3 of [29], consider a local universal deformation space of X:

$$\Phi : \mathcal{X} \to \Delta,$$

where Δ is a small polydisk and $\mathcal{X}_0 = X$. By eventually restricting Δ, we can assume that ι extends to an automorphisms M on $\mathcal{X}$ and m on Δ such that, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "M"] \arrow[d] & \mathcal{X} \arrow[d] \\ \Delta \arrow[r, "m"] & \Delta \end{tikzcd}$$

Moreover, the differential of m at 0 is given by the action of ι on $H^1(T_X)$ which is the same as the action on $H^{1,1}(X)$ since the symplectic holomorphic 2-form induces an isomorphism between those two and the symplectic holomorphic 2-form is preserved by the action of ι. The morphism m is linearizable, then Δ^m is smooth and $\dim \Delta^m = \operatorname{rk} H^2(X,\mathbb{Z})^\iota - 2 = 3$. Moreover, by [23] that we can find $x \in \Delta^m$ such that $\mathcal{X}_x$ is bimeromorphic to a Kummer fourfold $K_2(A)$. Since $H^2(X,\mathbb{Z})^\iota = U \oplus A_1^2 \oplus (-6)$, $\iota_x := M_{\mathcal{X}_x}$ induces a bimeromorphic involution i on $K_2(A)$ with $H^2(K_2(A),\mathbb{Z})^i = U \oplus A_1^2 \oplus (-6)$.

Necessary, we have $\mathrm{NS}(K_2(A)) \supset A_1(-1)^2 \oplus (-6)$. It follows that $\mathrm{NS}(A) \supset A_1(-1)^2$. Let consider the involution g defined by $-\mathrm{id}$ on $A_1(-1)^2$ and id on $(A_1(-1)^2)^\perp$. By Corollary

1.5.2 of [35], g can be extend to an involution on $H^2(A,\mathbb{Z})$. Then by Theorem 1 of [40], g provides a symplectic automorphism on A with: $H^2(A,\mathbb{Z})^g=(A_1(-1)^2)^{\perp}=U\oplus A_1^2$. It follows from classification of Section 4 of [31], that $A=\mathbb{C}/\Lambda$ with $\Lambda=\langle(1,0),(0,1),(x,-y),(y,x)\rangle$, $(x,y)\in\mathbb{C}^2\smallsetminus\mathbb{R}^2$ and $g=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$.

Let also denote g the automorphism on $K_2(A)$ induces by g. By construction, $g\circ i$ acts trivially on $H^2(K_2(A),\mathbb{Z})$. Hence by Corollary 3.3 and Lemma 3.4 of [9], $g\circ\iota$ extends to an automorphism of $K_2(A)$. In particular, i extends to a symplectic involution on $K_2(A)$. Then $g\circ i\in\operatorname{Ker}\nu$.

By Corollary 11.3, t_τ acts trivially on $H^3(K_2(A),\mathbb{Z})$. Hence by Corollary 3.3 of [4], we have necessary:

$$g^*_{|H^3(K_2(A),\mathbb{Z})}=i^*_{|H^3(K_2(A),\mathbb{Z})}\circ(-\operatorname{id}_A)^*_{|H^3(K_2(A),\mathbb{Z})}\ \text{ or }\ g^*_{|H^3(K_2(A),\mathbb{Z})}=i^*_{|H^3(K_2(A),\mathbb{Z})}.$$

But $g^*_{|H^3(K_2(A),\mathbb{Z})}$ has order 4 and $i^*_{|H^3(K_2(A),\mathbb{Z})}\circ(-\operatorname{id}_A)^*_{|H^3(K_2(A),\mathbb{Z})}$ and $i^*_{|H^3(K_2(A),\mathbb{Z})}$ have order 2, which is a contradiction.

(2) It follows from (1), Theorem 2.1 of [15] and Corollary 3.3 of [4].

(3) It follows from (2) and Section 1.2.1 of [42].

□

Remark 13.2. (1) We also remark that the K3 surface fixed by $(t_\tau\circ(-\operatorname{id}_A))$ is given by the sub-manifold Z_τ defined in Section 4 of [15].

(2) Considering the involution $-\operatorname{id}_A$, the set

$$\mathcal{P}:=\{\xi\in K_2(A)|\ \operatorname{Supp}\xi=\{a_1,a_2,a_3\},\ a_i\in A[2]\smallsetminus\{0\},1\leq i\leq 3\}$$

provides 35 fixed points and the vertex of

$$W_0:=\{\xi\in K_2(A)|\ \operatorname{Supp}\xi=\{0\}\}$$

supplies the 36th point. We denote by $p_1,...,p_{35}$ the points of $\mathcal{P}$ and by p_{36} the vertex of W_0.

13.2 Action on the cohomology

From Theorem 13.1, we can assume that $X=K_2(A)$ and $\iota=-\operatorname{id}_A$. To consider $t_\tau\circ(-\operatorname{id}_A)$ instead of $-\operatorname{id}_A$ will only has the effect to exchange the role of $[Z_0]$ and $[Z_\tau]$. Hence we do not lose any generality assuming that $\iota=-\operatorname{id}_A$.

Proposition 13.3. *(1) The involution ι acts trivially on $H^2(K_2(A),\mathbb{Q})$. It follows $l_2^2(K_2(A))=l_{1,-}^2(K_2(A))=0$ and $l_{1,+}^2(K_2(A))=7$.*

(2) The involution ι acts as $-\operatorname{id}$ on $H^3(K_2(A),\mathbb{Q})$. It follows $l_2^3(K_2(A))=l_{1,+}^3(K_2(A))=0$ and $l_{1,-}^3(K_2(A))=8$.

(3) By Corollary 12.5, we have:

$$H^4(K_2(A),\mathbb{Q})=\operatorname{Sym}^2H^2(K_2(A),\mathbb{Q})\oplus^{\perp}\Pi'\otimes\mathbb{Q},$$

where $\Pi'=\langle Z_\tau-Z_0,\ \tau\in A[3]\smallsetminus\{0\}\rangle$. The involution ι^* fixes $\operatorname{Sym}^2H^2(K_2(A),\mathbb{Z})$ and $\iota^*(Z_\tau-Z_0)=Z_{-\tau}-Z_0$. It follows that $l_{1,-}^4(K_2(A))=0$, $l_{1,+}^4(K_2(A))=28$ and $l_2^4(K_2(A))=40$.

Proof. (1) It follows from Corollary 3.3 of [4].

(2) It follows from Corollary 11.3.

(3) Let $\mathcal{S}$ be the over-lattice of $\operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z})$ where we add all the classes divisible by 2 in $H^4(K_2(A),\mathbb{Z})$. We can calculate that the discriminant of Π' is not divisible by 2. Since $H^4(K_2(A),\mathbb{Z})$ is unimodular, it follows that the discriminant of $\mathcal{S}$ is also not divisible by 2. Hence, we have:

$$H^4(K_2(A),\mathbb{F}_2)=\mathcal{S}\otimes\mathbb{F}_2\oplus\Pi'\otimes\mathbb{F}_2.$$

Moreover, we have:

$$\iota^*(Z_\tau-Z_0)=Z_{-\tau}-Z_0,$$

for all $\tau\in A[3]\smallsetminus\{0\}$. Hence $\operatorname{Vect}_{\mathbb{F}_2}(Z_\tau-Z_0,Z_{-\tau}-Z_0)$ is isomorphic to N_2 as a $\mathbb{F}_2[G]$-module. Moreover $H^2(K_2(A),\mathbb{Z})$ is invariant by the action of ι, hence $\operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z})$ and $\mathcal{S}$ is also invariant by the action of ι. It follows that $\mathcal{S}\otimes\mathbb{F}_2=\mathcal{N}_1$ and $\Pi'\otimes\mathbb{F}_2=\mathcal{N}_2$. Since $\operatorname{rk}\mathcal{S}=28$, we have $l_{1,+}^4+l_{1,-}^4=28$. However, $\mathcal{S}$ is invariant by the action of ι, it follows that $l_{1,-}^4=0$ and $l_{1,+}^4=28$. On the other hand $\operatorname{rk}\Pi'=80$, it follows that $l_2^4=40$. □

14 Application to Beauville–Bogomolov form

14.1 Statement of the main theorem

In [34], Namikawa propose a definition of the Beauville-Bogomolov form for some singular irreducible symplectic varieties. He Assumes that the singularities are only $\mathbb{Q}$-factorial singularities with a singular locus of codimension of ≥ 4. Under these assumptions, he proves a local Torelli theorem. This result was completed by a generalization of the Fujiki formula by Matsushita in [24].

Theorem 14.1. *Let Z be a projective irreducible symplectic variety of dimension $2n$ with only $\mathbb{Q}$-factorial singularities, and* $\operatorname{Codim}\operatorname{Sing} Z\geq 4$. *There exists a unique indivisible integral symmetric bilinear form $B_Z\in S^2(H^2(Z,\mathbb{Z}))^*$ and a unique positive constant $c_Z\in\mathbb{Q}$, such that for any $\alpha\in H^2(Z,\mathbb{C})$,*

$$\alpha^{2n}=c_ZB_Z(\alpha,\alpha)^n. \qquad (1)$$

For $0\neq\omega\in H^0(\Omega_U^2)$

$$B_Z(\omega+\overline{\omega},\omega+\overline{\omega})>0. \qquad (2)$$

Moreover the signature of B_Z is $(3,h^2(Z,\mathbb{C})-3)$.

The form B_Z is proportional to q_Z and is called the Beauville–Bogomolov form of Z.

Proof. The statement of the theorem in [24] does not say that the form is integral. However, let Z_s be a fiber of the Kuranishi family of Z, with the same idea as Matsushita's proof, we can see that q_Z and q_{Z_s} are proportional. Then, it follows using the proof of Theorem 5 a), c) of [1]. □

We can also consider its polarizeed form.

Proposition 14.2. *Let X be a projective irreducible symplectic variety of dimension $2n$ with* $\operatorname{Codim}\operatorname{Sing} X\geq 4$. *The equality (1) of Theorem 14.1 implies that*

$$\alpha_1\cdot\ldots\cdot\alpha_{2n}=\frac{c_X}{(2n)!}\sum_{\sigma\in S_{2n}}B_X(\alpha_{\sigma(1)},\alpha_{\sigma(2)})\ldots B_X(\alpha_{\sigma(2n-1)},\alpha_{\sigma(2n)}).$$

for all $\alpha_i\in H^2(X,\mathbb{Z})$.

These results were then generalized by Kirschner for complex spaces in [16]. In [25] (Theorem 2.5) was propose the first concrete example of Beauville-Bogomolov lattice for a singular irreducible symplectic variety. The variety studied in [25] is a partial resolution of the quotient of the Hilbert scheme of 2 points on a K3 surface quotiented by a symplectic involution. The objective of this section is to provide a new example of Beauville-Bogomolov lattice replacing Hilbert schemes of 2 points on a K3 surface by generalized Kummer fourfolds. Knowing the integral basis of the cohomology group of the generalized Kummer provided in Part II, this calculation becomes possible. Moreover the calculation will be much more simple as in [25] because of the general techniques for calculating integral cohomology of quotients developed in [26] and the new technique using monodromy developed in Lemma 14.14. The other techniques developed in [25] are also in [26], so to simplify the reading, we will only cite [25] in the rest of the section.

Concretely, let X be an irreducible symplectic fourfold of Kummer type and ι a symplectic involution on X. By Theorem 13.1 the fixed locus of ι is the union of 36 points and a K3 surface Z_0. Then the singular locus of $K := X/\iota$ is the union of a K3 and 36 points. The singular locus is not of codimension four. We will lift to a partial resolution of singularities, K' of K, obtained by blowing up the image of Z_0. By Section 2.3 and Lemma 1.2 of [11], the variety K' is an irreducible symplectic V-manifold which has singular locus of codimension four.

All Section 14 is devoted to prove the following theorem.

Theorem 14.3. *Let X be an irreducible symplectic fourfold of Kummer type and ι a symplectic involution on X. Let Z_0 be the K3 surface which is in the fixed locus of ι. We denote $K = X/\iota$ and K' the partial resolution of singularities of K obtained by blowing up the image of Z_0. Then the Beauville–Bogomolov lattice $H^2(K',\mathbb{Z})$ is isomorphic to $U(3)^3 \oplus \begin{pmatrix} -5 & -4 \\ -4 & -5 \end{pmatrix}$, and the Fujiki constant $C_{K'}$ is equal to* 8.

The Beauville-Bogomolov form is a topological invariant, hence from Theorem 13.1 we can assume that X is a generalized Kummer fourfold and $\iota = -\mathrm{id}_A$. As it will be useful to prove Lemma 14.14, we can assume even more. All generalized Kummer fourfolds are deformation equivalent, hence, we can assume that $A = E \times E$ with

$$E := \frac{\mathbb{C}}{\left\langle 1, e^{\frac{2i\pi}{6}} \right\rangle}.$$

14.2 Overview on the proof of Theorem 14.3

We first provide all the notation that we will need during the proof in Section 14.3. Then the proof is divided in the following steps:

(1) First Proposition 13.3 and Corollary 6.9 will provide the H^4-normality in Section 14.4.

(2) The knowledge of the element divisible by 2 in $\mathrm{Sym}^2 H^2(K_2(A),\mathbb{Z})$ from Section 12 and the H^4-normality allow us to prove the H^2-normality in Section 14.5.

(3) An adaptation of the H^2-normality (Lemma 14.8) and several lemmas in Section 14.6 will provide an integral basis of $H^2(K',\mathbb{Z})$ (Theorem 14.9).

(4) Knowing an integral basis of $H^2(K',\mathbb{Z})$, we end the calculation of the Beauville–Bogomolov form in Section 14.7 using intersection theory and the generalized Fujiki formula (Theorem 14.1).

14.3 Notation

Let $K_2(A)$ be a generalized Kummer fourfold endowed with the symplectic involution ι induced by $-\mathrm{id}_A$. We denote by π the quotient map $K_2(A) \to K_2(A)/\iota$. From Theorem 13.1, we know that the singular locus of the quotient $K_2(A)/\iota$ is the K3 surface, image by π of Z_0, and 36 isolated points. We denote $\overline{Z_0} := \pi(Z_0)$. We consider $r' : K' \to K_2(A)/\iota$ the blow-up of $K_2(A)/\iota$ in $\overline{Z_0}$ and we denote by $\overline{Z_0}'$ the exceptional divisor. We also denote by $s_1 : N_1 \to K_2(A)$ the blowup of $K_2(A)$ in Z_0; and denote by Z_0' the exceptional divisor in N_1. Denote by ι_1 the involution on N_1 induced by ι. We have $K' \simeq N_1/\iota_1$, and we denote $\pi_1 : N_1 \to K'$ the quotient map.

Consider the blowup $s_2 : N_2 \to N_1$ of N_1 in the 36 points $p_1, ..., p_{36}$ fixed by ι_1 and the blowup $\widetilde{r} : \widetilde{K} \to K'$ of K' in its 36 singulars points. We denote the exceptional divisors by $E_1, ..., E_{36}$ and $D_1, ..., D_{36}$ respectively. We also denote $\widetilde{\overline{Z_0}} = \widetilde{r}^*(\overline{Z_0}')$ and $\widetilde{Z_0} = s_2^*(Z_0')$. Denote ι_2 the involution induced by ι on N_2 and $\pi_2 : N_2 \to N_2/\iota_2$ the quotient map. We have $N_2/\iota_2 \simeq \widetilde{K}$. We collect this notation in commutative diagram

$$\begin{tikzcd}
\widetilde{K} \arrow[r, "\widetilde{r}"] & K' \arrow[r, "r'"] & K_2(A)/\iota \\
N_2 \arrow[u, "\pi_2"] \arrow[r, "s_2"] \arrow[loop below, "\iota_2"] & N_1 \arrow[u, "\pi_1"] \arrow[r, "s_1"] \arrow[loop below, "\iota_1"] & K_2(A) \arrow[u, "\pi"] \arrow[loop below, "\iota"]
\end{tikzcd}$$

To finish, we denote $V = K_2(A) \smallsetminus \operatorname{Fix}\iota$ and $U = V/\iota$. Also, we set $s = s_2 \circ s_1$ and $r = \widetilde{r} \circ r'$. We denote also δ the half of the class of the diagonal in $H^2(K_2(A), \mathbb{Z})$.

We can commute the push-forward maps and the blow-up maps as proved in Lemma 3.3.21 of [26].

Remark 14.4. Let $x \in H^2(N_1, \mathbb{Z})$, $y \in H^2(K_2(A), \mathbb{Z})$, we have:

$$\pi_{2*}(s_2^*(x)) = \widetilde{r}^*(\pi_{1*}(x)),$$

$$\pi_{1*}(s_1^*(y)) = r'^*(\pi_*(y)),$$

14.4 The couple $(K_2(A), \iota)$ is H^4-normal

Proposition 14.5. *The couple $(K_2(A), \iota)$ is H^4-normal.*

Proof. We apply Corollary 6.9.

i) By Theorem 2 of [41], $H^*(K_2(A), \mathbb{Z})$ is torsion-free.

ii) From Remark 13.2 (1), we know that the connected component of dimension 2 of $\operatorname{Fix}\iota$ is given by Z_0. We know that Z_0 is a K3 surface, hence is simply connected. Moreover by Proposition 4.3 of [15] $Z_0 \cdot Z_\tau = 1$ for all $\tau \in A[3] \smallsetminus \{0\}$. Hence the class of Z_0 in $H^4(K_2(A), \mathbb{Z})$ is primitive. It follows that $\operatorname{Fix}\iota$ is almost negligible (Definition 6.8).

iii) By Proposition 13.3, we have $l_{1,-}^2(K_2(A)) = l_{1,-}^4(K_2(A)) = 0$.

iv) By Proposition 13.3, we have $l_{1,+}^3(K_2(A)) = 0$. Moreover $H^1(K_2(A)) = 0$, so $l_{1,+}^1(K_2(A)) = 0$.

v) We have to check the following equality:

$$\begin{aligned} &l_{1,+}^4(K_2(A)) + 2\left[l_{1,-}^1(X) + l_{1,-}^3(X) + l_{1,+}^0(X) + l_{1,+}^2(X)\right] \\ &= 36h^0(pt) + h^0(Z_0, \mathbb{Z}) + h^2(Z_0, \mathbb{Z}) + h^4(Z_0, \mathbb{Z}). \end{aligned}$$

By Proposition 13.3:

$$l_{1,+}^4(K_2(A)) + 2\left[l_{1,-}^1(X) + l_{1,-}^3(X) + l_{1,+}^0(X) + l_{1,+}^2(X)\right] = 28 + 2(8 + 1 + 7) = 60.$$

Moreover since Z_0 is a K3 surface, we have:

$$36h^0(pt) + h^0(Z_0, \mathbb{Z}) + h^2(Z_0, \mathbb{Z}) + h^4(Z_0, \mathbb{Z}) = 36 + 2 + 22 = 60.$$

It follows from Corollary 6.9 that $(K_2(A), \iota)$ is H^4-normal. □

As explained in Proposition 3.5.20 of [26], the proof of Corollary 6.9 provide first that $\pi_{2*}(s^*(H^4(K_2(A), \mathbb{Z})))$ is primitive in $H^4(\widetilde{K}, \mathbb{Z})$ and then the H^4 normality.

Remark 14.6. The lattice $\pi_{2*}(s^*(H^4(K_2(A), \mathbb{Z})))$ is primitive in $H^4(\widetilde{K}, \mathbb{Z})$.

14.5 The couple $(K_2(A), \iota)$ is H^2-normal

Proposition 14.7. *The couple $(K_2(A), \iota)$ is H^2-normal.*

Proof. We want to prove that the pushforward $\pi_* : H^2(K_2(A),\mathbb{Z}) \to H^2(K_2(A)/\iota,\mathbb{Z})/\text{tors}$ is surjective. By Remark 6.7, it is equivalent to prove that for all $x \in H^2(K_2(A),\mathbb{Z})^\iota$ $\pi_*(x)$ is divisible by 2 if and only if there exist $y \in H^2(K_2(A),\mathbb{Z})$ such that $x = y + \iota^*(y)$.

Let $x \in H^2(K_2(A),\mathbb{Z})^\iota = H^2(K_2(A),\mathbb{Z})$ such that $\pi_*(x)$ is divisible by 2, we will show that there exists $y \in H^2(K_2(A),\mathbb{Z})$ such that $x = y+\iota^*(y)$. Then, by Proposition 6.5, $\pi_*(x^2)$ is divisible by 2. However, $x^2 \in H^4(K_2(A),\mathbb{Z})^\iota$; since $(K_2(A),\iota)$ is H^4-normal by Proposition 14.5, it means that there is $z \in H^4(K_2(A),\mathbb{Z})$ such that $x^2 = z + \iota^*(z)$.

Let $\mathcal{S}$ be, as before, the over-lattice of $\operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z})$ where we add all the classes divisible by 2 in $H^4(K_2(A),\mathbb{Z})$. By Corollary 12.5, there exist $z_s \in \mathcal{S}$, $z_p \in \Pi'$ and $\alpha \in \mathbb{N}$ such that: $5 \cdot 3^\alpha \cdot z = z_s + z_p$. Hence, we have:

$$5 \cdot 3^\alpha \cdot x^2 = 2z_s + z_p + \iota^*(z_p).$$

Since $x^2 \in \operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z})$, by Corollary 12.5, $z_p + \iota^*(z_p) = 0$. It follows:

$$5 \cdot 3^\alpha \cdot x^2 = 2z_s. \tag{51}$$

let $(u_1, u_2, v_1, v_2, w_1, w_2, \delta)$ be the integral basis of $H^2(K_2(A),\mathbb{Z})$ introduced in Notation10.10. The couples (u_1,u_2), (v_1,v_2) and (w_1,w_2) are basis of the hyperbolic plans U and δ is half the diagonal of $K_2(A)$. We can write:

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \beta_1 v_1 + \beta_2 v_2 + \gamma_1 w_1 + \gamma_2 w_2 + d\delta.$$

Then

$$x^2 = \alpha_1^2 u_1^2 + \alpha_2^2 u_2^2 + \beta_1^2 v_1^2 + \beta_2^2 v_2^2 + \gamma_1^2 w_1^2 + \gamma_2^2 w_2^2 + d^2\delta^2 \mod 2H^4(K_2(A),\mathbb{Z}).$$

We also have:

$$5 \cdot 3^\alpha \cdot x^2 = \alpha_1^2 u_1^2 + \alpha_2^2 u_2^2 + \beta_1^2 v_1^2 + \beta_2^2 v_2^2 + \gamma_1^2 w_1^2 + \gamma_2^2 w_2^2 + d^2\delta^2 \mod 2H^4(K_2(A),\mathbb{Z}).$$

It follows by (51) that $\alpha_1^2 u_1^2 + \alpha_2^2 u_2^2 + \beta_1^2 v_1^2 + \beta_2^2 v_2^2 + \gamma_1^2 w_1^2 + \gamma_2^2 w_2^2 + d^2\delta^2$ is divisible by 2. However by Corollary 12.6 and Proposition 12.10, we have:

$$\mathcal{S} = \left\langle \operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z}); \frac{u_1 \cdot u_2 + v_1 \cdot v_2 + w_1 \cdot w_2}{2}; \frac{u_i^2 - \frac{1}{3}u_i \cdot \delta}{2}; \frac{v_i^2 - \frac{1}{3}v_i \cdot \delta}{2}; \frac{w_i^2 - \frac{1}{3}w_i \cdot \delta}{2}, i \in \{1,2\} \right\rangle.$$

It follows that all the coefficients of $\alpha_1^2 u_1^2 + \alpha_2^2 u_2^2 + \beta_1^2 v_1^2 + \beta_2^2 v_2^2 + \gamma_1^2 w_1^2 + \gamma_2^2 w_2^2 + d^2\delta^2$ are divisible by 2. It means that x is divisible by 2. It is what we wanted to prove. □

With exactly the same proof working in $H^4(\widetilde{K},\mathbb{Z})$ and using Remark 14.6, we provide the following lemma.

Lemma 14.8. *The lattice $\pi_{2*}(s^*(H^2(K_2(A),\mathbb{Z})))$ is primitive in $H^2(\widetilde{K},\mathbb{Z})$.*

14.6 Calculation of $H^2(K',\mathbb{Z})$

This section is devoted to prove the following theorem.

Theorem 14.9. *Let K', π_1, s_1 and $\overline{Z_0}'$ be respectively the variety, the maps and the class defined in Section 14.3. We have*

$$H^2(K',\mathbb{Z}) = \pi_{1*}(s_1^*(H^2(K_2(A),\mathbb{Z}))) \oplus \mathbb{Z}\left(\frac{\pi_{1*}(s_1^*(\delta)) + \overline{Z_0}'}{2}\right) \oplus \mathbb{Z}\left(\frac{\pi_{1*}(s_1^*(\delta)) - \overline{Z_0}'}{2}\right).$$

First we need to calculate some intersections.

Lemma 14.10. *(1) We have $E_l \cdot E_k = 0$ if $l \neq k$, $E_l^4 = -1$ and $E_l \cdot z = 0$ for all $(l,k) \in \{1,...,28\}^2$ and for all $z \in s^*(H^2(K_2(A),\mathbb{Z}))$.*

(2) We have $\delta^4 = 324$.

Proof. (1) Same proof as Proposition 4.6.16 1) of [26].

(2) It follows directly from the Fujiki formula.

□

We already have properties of primitivity.

Lemma 14.11. *(1) $\pi_{1*}(s_1^*(H^2(K_2(A),\mathbb{Z})))$ is primitive in $H^2(K',\mathbb{Z})$,*

(2) The group $\widetilde{\mathcal{D}} = \left\langle \widetilde{\overline{Z_0}}, D_1, ..., D_{36}, \frac{\widetilde{\overline{Z_0}}+D_1+...+D_{36}}{2} \right\rangle$ is primitive in $H^2(\widetilde{K},\mathbb{Z})$.

(3) $\overline{Z_0}'$ is primitive in $H^2(K',\mathbb{Z})$,

Proof. (1) By Lemma 14.8, $\pi_{2*}(s^*(H^2(K_2(A),\mathbb{Z})))$ is primitive in $H^2(\widetilde{K},\mathbb{Z})$. Then by Remark 14.4, $r'^*(\pi_*(H^2(K_2(A),\mathbb{Z})))$ is primitive in $H^2(K',\mathbb{Z})$. Using again Remark 14.4, we get the result.

(2), (3) See proof of Lemma 4.6.14 of [26].

□

With Lemma 14.11 (1) and (3), it only remains to prove that $\pi_{1*}(s_1^*(\delta)) + \overline{Z_0}'$ is divisible by 2 which will be gone in Lemma 14.15. To prove this lemma, we first prove that $\pi_{2*}(s^*(\delta)) + \widetilde{\overline{Z_0}}$ is divisible by 2. Knowing that $\widetilde{\overline{Z_0}} + D_1 + ... + D_{36}$ is divisible by 2, we have only to show that $\pi_{2*}(s^*(\delta)) + D_1 + ... + D_{36}$ is divisible by 2 which is done by Lemma 14.13 and 14.14.

First we need to know the group $H^3(\widetilde{K},\mathbb{Z})$.

Lemma 14.12. *We have $H^3(\widetilde{K},\mathbb{Z}) = 0$.*

Proof. We have the following exact sequence:

$$H^3(K_2(A),V,\mathbb{Z}) \to H^3(K_2(A),\mathbb{Z}) \xrightarrow{f} H^3(V,\mathbb{Z}) \to H^4(K_2(A),V,\mathbb{Z}) \xrightarrow{\rho} H^4(K_2(A),\mathbb{Z}).$$

By Thom isomorphism, $H^3(K_2(A),V,\mathbb{Z}) = 0$ and $H^4(K_2(A),V,\mathbb{Z}) = H^0(Z_0,\mathbb{Z})$. Moreover ρ is injective, so $H^3(V,\mathbb{Z}) = H^3(K_2(A),\mathbb{Z})$.

Hence by Proposition 13.3 (2) and Proposition 3.2.8 of [26], we find that $H^3(U,\mathbb{Z}) = 0$. Since $H^3(K_2(A),\mathbb{Z})^\iota = 0$, $H^3(\widetilde{K},\mathbb{Z})$ is a torsion group. Hence the result follows from the exact sequence

$$H^3(\widetilde{K},U,\mathbb{Z}) \to H^3(\widetilde{K},\mathbb{Z}) \to H^3(U,\mathbb{Z})$$

and from the fact that $H^3(\widetilde{K},U,\mathbb{Z}) = 0$ by Thom isomorphism. □

Lemma 14.13. *There exists D_δ which is a linear combination of the D_i with coefficient 0 or 1 such that $\pi_{2*}(s^*(\delta)) + D_\delta$ is divisible by 2.*

Proof. First, we have to use Smith theory as in Section 4.6.4 of [26].

Look at the following exact sequence:

$$0 \to H^2(\widetilde{K},\widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k),\mathbb{F}_2)) \to H^2(\widetilde{K},\mathbb{F}_2) \to H^2(\widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k,\mathbb{F}_2))$$
$$\longrightarrow H^3(\widetilde{K},\widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k),\mathbb{F}_2) \longrightarrow 0.$$

First, we will calculate the dimension of the vector spaces $H^2(\widetilde{K}, \widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2)$ and $H^3(\widetilde{K}, \widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2)$. By (2) of Proposition 6.10, we have

$$H^*(\widetilde{K}, \widetilde{Z_0} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) \simeq H^*_\sigma(N_2).$$

The previous exact sequence gives us the following equation:

$$h^2_\sigma(N_2) - h^2(\widetilde{K}, \mathbb{F}_2) + h^2(\widetilde{Z_0} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) - h^3_\sigma(N_2) = 0.$$

As $h^2(\widetilde{K}, \mathbb{F}_2) = 8 + 36 = 44$ and $h^2(\widetilde{Z_0} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) = 23 + 36 = 59$, we obtain:

$$h^2_\sigma(N_2) - h^3_\sigma(N_2) = -15.$$

Moreover by 2) of Proposition 6.10, we have the exact sequence

$$\begin{aligned} 0 \to H^1_\sigma(N_2) \to H^2_\sigma(N_2) \to H^2(N_2, \mathbb{F}_2) \to H^2_\sigma(N_2) \oplus H^2(\widetilde{Z_0} \cup (\cup_{k=1}^{36} E_k), \mathbb{F}_2) \\ \to H^3_\sigma(N_2) \longrightarrow \operatorname{coker} \longrightarrow 0. \end{aligned}$$

By Lemma 7.4 of [5], $h^1_\sigma(N_2) = h^0(\widetilde{Z_0} \cup (\cup_{k=1}^{36} E_k), \mathbb{F}_2) - 1$. Then we get the equation

$$\begin{aligned} & h^0(\widetilde{Z_0} \cup (\cup_{k=1}^{36} E_k), \mathbb{F}_2) - 1 - h^2_\sigma(N_2) + h^2(N_2, \mathbb{F}_2) \\ & - h^2_\sigma(N_2) - h^2(\widetilde{Z_0} \cup (\cup_{k=1}^{36} E_k), \mathbb{F}_2) + h^3_\sigma(N_2) - \alpha = 0, \end{aligned}$$

where $\alpha = \dim \operatorname{coker}$. So

$$21 - \alpha - 2h^2_\sigma(N_2) + h^3_\sigma(N_2) = 0.$$

From the two equations, we deduce that

$$h^2_\sigma(N_2) = 36 - \alpha, \qquad h^3_\sigma(N_2) = 51 - \alpha.$$

Come back to the exact sequence

$$0 \longrightarrow H^2(\widetilde{K}, \widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) \longrightarrow H^2(\widetilde{K}, \mathbb{F}_2) \xrightarrow{\varsigma^*} H^2(\widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2),$$

where $\varsigma : \widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k) \hookrightarrow \widetilde{K}$ is the inclusion. Since $h^2(\widetilde{K}, \widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) = h^2_\sigma(N_2) = 36 - \alpha$, we have $\dim_{\mathbb{F}_2} \varsigma^*(H^2(\widetilde{K}, \mathbb{F}_2)) = (8+36) - 36 + \alpha = 8 + \alpha$. We can interpret this as follows. Consider the homomorphism

$$\begin{aligned} \varsigma^*_{\mathbb{Z}} : H^2(\widetilde{K}, \mathbb{Z}) &\to H^2(\widetilde{\overline{Z_0}}, \mathbb{Z}) \oplus (\oplus_{k=1}^{36} H^2(D_k, \mathbb{Z})) \\ u &\to (u \cdot \widetilde{\overline{Z_0}}, u \cdot D_1, ..., u \cdot D_{36}). \end{aligned}$$

Since this is a map of torsion free $\mathbb{Z}$-modules (by Lemma 14.12 and universal coefficient formula), we can tensor by $\mathbb{F}_2$,

$$\varsigma^* = \varsigma^*_{\mathbb{Z}} \otimes \mathrm{id}_{\mathbb{F}_2} : H^2(\widetilde{K}, \mathbb{Z}) \otimes \mathbb{F}_2 \to H^2(\widetilde{\overline{Z_0}}, \mathbb{Z}) \oplus (\oplus_{k=1}^{36} H^2(D_k, \mathbb{Z})) \otimes \mathbb{F}_2,$$

and we have $8+\alpha$ independent elements such that the intersection with the D_k $k \in \{1, ..., 36\}$ and $\widetilde{\overline{Z_0}}$ are not all zero. But, $\varsigma^*(\pi_{2*}(H^2(N_2, \mathbb{Z}))) = 0$ and $\varsigma^*(\widetilde{\overline{Z_0}}, \langle D_1, ..., D_{36} \rangle)$, (it follows from Proposition 6.5). By Lemma 14.11 (2), the element $\widetilde{\overline{Z_0}} + D_1 + ... + D_{36}$ is divisible by 2. Hence necessary, it remains $7 + \alpha$ independent elements in $H^2(\widetilde{K}, \mathbb{Z})$ of the form $\frac{u+d}{2}$ with $u \in \pi_{2*}(s^*(H^2(K_2(A), \mathbb{Z})))$ and $d \in \langle D_1, ..., D_{36} \rangle$.

Let denote by $u_1, ..., u_{7+\alpha}$ the $7 + \alpha$ elements in $\pi_{2*}(s^*(H^2(K_2(A), \mathbb{Z})))$ provided above. By Lemma 14.11 (2) $\langle D_1, ..., D_{36} \rangle$ is primitive in $H^2(\widetilde{K}, \mathbb{Z})$. Hence necessary, the element $u_1, ..., u_{7+\alpha}$ view as element in $\pi_{2*}(s^*(H^2(K_2(A), \mathbb{F}_2)))$ are linearly independent. Since $\dim_{\mathbb{F}_2} \pi_{2*}(s^*(H^2(K_2(A), \mathbb{F}_2))) = 7$, it follows that $\alpha = 0$ and $\mathrm{Vect}_{\mathbb{F}_2}(u_1, ..., u_7) = \pi_{2*}(s^*(H^2(K_2(A), \mathbb{F}_2)))$. Hence there exists D_δ which is a linear combination of the D_i with coefficient 0 or 1 such that $\pi_{2*}(s^*(\delta)) + D_\delta$ is divisible by 2. $\square$

Lemma 14.14. *We have:*

$$D_\delta = D_1 + ... + D_{36}.$$

Proof. The know that the monodromy acts on $A[2]$ as the symplectic group $\operatorname{Sp} A[2]$. Hence the monodromy action extends naturally to an action on the divisors $D_1, ..., D_{35}$. Also this monodromy action represented by $SpA[3]$ acts trivially on D_{36} and on $\pi_{2*}(s^*(\delta))$. As explaining in Remark 1.4 4.4 the 2 orbits of the action of $\operatorname{Sp} A[2]$ on the set $\mathfrak{D} := \{D_1, ..., D_{35}\}$ correspond to the two sets of isotropic and non-isotropic planes in $A[2]$. Hence by Proposition 1.5 (3), (4) 4.5 the action of $\operatorname{Sp} A[2]$ on the set $\mathfrak{D}$ has 2 orbits: one of 15 elements and another of 20 elements.

On the other hand, as we mentioned in the end of Section 14.1, we can assume that $A = E \times E$ where $E = \frac{\mathbb{C}}{\left\langle 1, e^{\frac{2i\pi}{6}} \right\rangle}$. Hence there is the following automorphism group acting on A:

$$G := \left\langle \begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle,$$

where $\rho = e^{\frac{2i\pi}{6}}$. The group G extends naturally to an automorphism group of N_2 which we denote also G. Moreover the action of G restricts to the set $\mathfrak{D}$. Then by Proposition 5.5 the action of G on $\mathfrak{D}$ has 2 orbits: one of 5 elements and one of 30 elements. Also the group G acts trivially on D_{36} and on $\pi_{2*}(s^*(\delta))$.

Hence the composition of the action of G and the action of $\operatorname{Sp} A[2]$ acts transitively on $\mathfrak{D}$. Since $\pi_{2*}(s^*(\delta))$ is fixed by the action of G and $\operatorname{Sp} A[2]$, D_δ has also to be fixed by the action of G and $\operatorname{Sp} A[2]$ or it will contradict Lemma 14.11 (2). It follows that there are only 3 possibilities for D_δ:

(1) $D_\delta = D_{36}$,

(2) $D_\delta = D_1 + ... + D_{35}$,

(3) or $D_\delta = D_1 + ... + D_{36}$.

Let d be the number of D_i with coefficient equal to 1 in the linear decomposition of D_δ. The number d can be 1, 35 or 36.

Then from Lemma 14.10 (1), (2) and Proposition 6.5

$$\left(\frac{\pi_{2*}(s^*(\delta)) + D_\delta}{2} \right)^4 = \frac{324 - d}{2}.$$

Hence d has to be divisible by 2. It follows that $D_\delta = D_1 + ... + D_{36}$. □

Lemma 14.15. *The class $\pi_{1*}(s_1^*(\delta)) + \overline{Z_0}'$ is divisible by 2.*

Proof. We know that $\pi_{2,*}(s^*(\delta)) + \widetilde{\overline{Z_0}}$ is divisible by 2. Indeed by Lemma 14.11 (2), $\widetilde{\overline{Z_0}} + D_1 + ... + D_{36}$ is divisible by 2 and by Lemma 14.13 and 14.14, $\pi_{2,*}(s^*(\delta)) + D_1 + ... + D_{36}$ is divisible by 2.

We can find a Cartier divisor on $\widetilde{K}$ which corresponds to $\frac{\pi_{2*}(s^*(\delta)) + \widetilde{\overline{Z_0}}}{2}$ and which does not meet $\cup_{k=1}^{36} D_k$. Then this Cartier divisor induces a Cartier divisor on K' which necessarily corresponds to half the cocycle $\pi_{1*}(s_1^*(\delta)) + \overline{Z_0}'$. □

14.7 Calculation of $B_{K'}$

We finish the proof of Theorem 14.3, calculating $B_{K'}$. We use the notation provided in Section 14.3.

Lemma 14.16. *We have*

$$\overline{Z_0}'^2 = -2r^*(\overline{Z_0}).$$

Proof. We use the same technique as Lemma 4.6.12 of [26]. Consider the following diagram:

$$\begin{tikzcd} Z_0' \arrow[r, hook, "l_1"] \arrow[d, "g"] & N_1 \arrow[d, "s_1"] \\ Z_0 \arrow[r, hook, "l_0"] & K_2(A), \end{tikzcd}$$

where l_0 and l_1 are the inclusions and $g := s_{1|Z_0'}$. By Proposition 6.7 of [12], we have:

$$s_1^* l_{0*}(Z_0) = l_{1*}(c_1(E)),$$

where $E := g^*(\mathscr{N}_{Z_0/K_2(A)})/\mathscr{N}_{Z_0'/N_1}$. Hence

$$s_1^* l_{0*}(Z_0) = c_1(g^*(\mathscr{N}_{Z_0/K_2(A)})) - Z_0'^2.$$

Since $K_2(A)$ is hyperkähler and Z_0 is a K3 surface, we have $c_1(\mathscr{N}_{Z_0/K_2(A)}) = 0$. So

$$Z_0'^2 = -s_1^* l_{0*}(Z_0).$$

Then the result follows from Proposition 6.5. □

Proposition 14.17. *We have the formula*

$$B_{K'}(\pi_{1*}(s_1^*(\alpha), \pi_{1*}(s_1^*(\beta))) = 6\sqrt{\frac{2}{C_{K'}}} B_{K_2(A)}(\alpha, \beta),$$

where $C_{K'}$ is the Fujiki constant of K' and α, β are in $H^2(K_2(A), \mathbb{Z})^\iota$ and $B_{K_2(A)}$ is the Beauville–Bogomolov form of $K_2(A)$.

Proof. An easy use of the Fujiki formula provide the following proposition (same proof as Proposition 4.6.10 of [26]).

By (1) of Theorem 14.1, we have

$$(\pi_{1*}(s_1^*(\alpha)))^4 = C_{K'} B_{K'}(\pi_{1*}(s_1^*(\alpha), \pi_{1*}(s_1^*(\alpha)))^2.$$

And

$$\alpha^4 = 9B_{K_2(A)}(\alpha, \alpha)^2.$$

Moreover, by Proposition 6.5,

$$(\pi_{1*}(s^*(\alpha)))^4 = 8s^*(\alpha)^4 = 8\alpha^4.$$

By Point (2) of Theorem 14.1, we get the result. □

In particular, it follows:

$$B_{K'}(\pi_{1*}(s_1^*(\delta), \pi_{1*}(s_1^*(\delta))) = -36\sqrt{\frac{2}{C_{K'}}} \tag{52}$$

Lemma 14.18.

$$B_{K'}(\pi_{1*}(s_1^*(\alpha)), \overline{Z_0}') = 0,$$

for all $\alpha \in H^2(S^{[2]}, \mathbb{Z})^\iota$.

Proof. We have $\pi_{1*}(s_1^*(\alpha))^3 \cdot \overline{Z_0}' = 8s_1^*(\alpha)^3 \cdot \Sigma_1$ by Proposition 6.5, and $s_{1*}(s_1^*(\alpha^3) \cdot Z_0') = \alpha^3 \cdot s_{1*}(Z_0') = 0$ by the projection formula. We conclude by Proposition 14.2. □

Lemma 14.19. *We have:*

$$B_{K'}(\overline{Z_0}', \overline{Z_0}') = -4\sqrt{\frac{2}{C_{K'}}}.$$

Proof. We have:

$$\begin{aligned}
\overline{Z_0}'^2 \cdot \pi_{1*}(s_1^*(\delta))^2 &= \frac{C_{K'}}{3} B_{M'}(\overline{Z_0}', \overline{Z_0}') \times B_{K'}(\pi_{1*}(s_1^*(\delta)), \pi_{1*}(s_1^*(\delta))) \\
&= \frac{C_{M'}}{3} B_{K'}(\overline{Z_0}', \overline{Z_0}') \times \left(-36\sqrt{\frac{2}{C_{K'}}}\right) \\
&= -12\sqrt{2C_{K'}} B_{K'}(\overline{Z_0}', \overline{Z_0}')
\end{aligned} \tag{53}$$

By Proposition 6.5, we have

$$\overline{Z_0}'^2 \cdot \pi_{1*}(s_1^*(\delta))^2 = 8Z_0'^2 \cdot (s_1^*(\delta))^2. \tag{54}$$

By the projection formula, $Z_0'^2 \cdot (s_1^*(\delta))^2 = s_{1*}(Z_0'^2) \cdot \delta^2$. Moreover by lemma 14.16, $s_{1*}(Z_0'^2) = -Z_0$. Hence

$$Z_0'^2 \cdot (s_1^*(\delta))^2 = -Z_0 \cdot \delta^2. \tag{55}$$

It follows from (53), (54) and (55) that

$$-8Z_0 \cdot \delta^2 = -12\sqrt{2C_{K'}}B_{K'}(\overline{Z_0}', \overline{Z_0}'). \tag{56}$$

Moreover from Section 4 of [15], we have:

$$Z_0 \cdot \delta^2 = -12. \tag{57}$$

So by (56) and (57):

$$B_{K'}(\overline{Z_0}', \overline{Z_0}') = -8\sqrt{\frac{1}{2C_{K'}}}.$$

□

Now we are able to finish the calculation of the Beauville–Bogomolov form on $H^2(K', \mathbb{Z})$. By (52), Propositions 14.17, Lemma 14.18, 14.19 and Theorem 14.9, the Beauville–Bogomolov form on $H^2(K', \mathbb{Z})$ gives the lattice:

$$\begin{aligned} & U^3\left(6\sqrt{\frac{2}{C_{K'}}}\right) \oplus -\frac{1}{4}\sqrt{\frac{2}{C_{K'}}}\begin{pmatrix} 40 & 32 \\ 32 & 40 \end{pmatrix} \\ = & U^3\left(6\sqrt{\frac{2}{C_{K'}}}\right) \oplus -\sqrt{\frac{2}{C_{K'}}}\begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix} \end{aligned}$$

It follows that $C_{K'} = 8$, and we get Theorem 14.3.

14.8 Betti numbers and Euler characteristic of K'

Proposition 14.20. *We have:*

- $b_2(K') = 8$,
- $b_3(K') = 0$,
- $b_4(K') = 90$,
- $\chi(K') = 108$.

Proof. It is the same proof as Proposition 4.7.2 of [26]. From Theorem 7.31 of [44] and Proposition 13.3, we get the betti numbers. Then $\chi(K') = 1 - 0 + 8 - 0 + 90 - 0 + 8 - 0 + 1 = 108$. □

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