INTEGRAL COHOMOLOGY OF $K^2(A)$

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Abstract. What we know already

Definition 0.1. Let A be a complex projective torus of dimesion 2 and $A^{[n]}$ the corresponding Hilbert scheme of points. Denote $\Sigma : A^{[n]} \to A$ the summation morphism. Then the generalized Kummer $K^{n-1}A$ is defined as the fiber over 0:

$$\begin{array}{ccc} K^{n-1}A & \xrightarrow{\theta} & A^{[n]} \\ \big\downarrow & & \big\downarrow \Sigma \\ \{0\} & \longrightarrow & A \end{array}$$

By [?], $\theta^* : H^2(A^{[n]}) \to H^2(K^{n-1}A)$ is surjective. We have injections $j : H^2(A) \to H^2(A^{[n]})$ and $i = \theta^* j$. The cohomology $H^*(A^{[n]})$ is described in terms of vertex operators in [?] and [?].

We describe now the image of θ^* in the case $n = 3$:

- We know $j(a) = \frac{1}{2}\mathfrak{p}_{-1}(a)\mathfrak{p}_{-1}(1)^2|0\rangle$, because the two are must be linearly dependent and
$$\int_{A^{[3]}} j(a)^6 = 15q(a)^3, \quad \left(\frac{1}{2}\mathfrak{p}_{-1}(a)\mathfrak{p}_{-1}(1)^2|0\rangle\right)^3 = 15q(a)^3\mathfrak{p}_{-1}(x)^3|0\rangle.$$
- By [?, p. 8], we have for $\alpha = j(a) = \frac{1}{2}\mathfrak{p}_{-1}(a)\mathfrak{p}_{-1}(1)^2|0\rangle$:
$$\int_{A^{[3]}} \alpha^6 = \frac{5}{3}q(a)\int_{K^2} \theta^*\alpha^4$$
On the other hand,
$$\alpha^4 = 3q(a)^2\mathfrak{p}_{-1}(x)^2\mathfrak{p}_{-1}(0)|0\rangle + 3q(a)\mathfrak{p}_{-1}(x)\mathfrak{p}_{-1}(a)^2|0\rangle,$$
so if the image of both summands under $\int \theta^*$ is positive, then
$$\int \theta^*\mathfrak{p}_{-1}(x)^2\mathfrak{p}_{-1}(0)|0\rangle = \int \theta^*\tfrac{1}{2}\mathfrak{p}_{-1}(x)\mathfrak{p}_{-1}(a)^2|0\rangle = 1.$$

Proposition 0.2. *Let $(\alpha_i)_{i=1\ldots4}$ be an oriented basis of $H^1(A,\mathbb{Z})$. Then the class of $K^2(A)$ in $H^4(A^{[3]})$ is given by*

$$\prod_{i=1}^{4}\left(\tfrac{1}{2}\mathfrak{p}_{-1}(1)^2\mathfrak{p}_{-1}(\alpha_i)|0\rangle\right).$$

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Proof. We know that for all i and all $\beta \in H^7(A^{[3]})$, we have $\int_{K^2(A)} \theta^*(\alpha_i \cdot \beta) = \int_{A^{[3]}} \alpha_i \cdot \beta \cdot [K^2(A)] = 0$ and for a basis (γ_i) of $H^2(A^{[3]})$,

$$\int_{A^{[3]}} \gamma_i \cdot \gamma_j \cdot \gamma_k \cdot \gamma_l \cdot [K^2(A)] = 3\left(\langle \gamma_i, \gamma_j\rangle \langle \gamma_k, \gamma_l\rangle + \langle \gamma_i, \gamma_k\rangle \langle \gamma_j, \gamma_l\rangle + \langle \gamma_i, \gamma_l\rangle \langle \gamma_j, \gamma_k\rangle\right)$$

These equations admit a unique solution. □

Let $\{a_i\}_{i=1\ldots 6}$ be a hyperbolic basis of $H^2(A, \mathbb{Z})$.

Proposition 0.3. *The classes $\theta^*\left(\mathfrak{p}_{-2}(a_i)\mathfrak{p}_{-1}(1)|0\rangle\right)$ and $\theta^*\left(\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_i)|0\rangle\right)$ are linearly dependent.*

Proposition 0.4. $\theta^*\left(\mathfrak{p}_{-3}(x)|0\rangle\right) = 0$

Corollary 0.5. $\theta^*\left(\mathfrak{p}_{-2}(a_i)\mathfrak{p}_{-1}(1)|0\rangle\right) = \frac{1}{4}\theta^*\left(\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_i)|0\rangle\right)$

Proof. Let a_j be complementary, *i.e.* $a_i a_j = 1$. Let $\mathrm{ch}_1(a_j) = -\frac{1}{2}\mathfrak{p}_{-2}(a_j)\mathfrak{p}_{-1}(1)|0\rangle$ be the chern character in the vertex algebra description of $H^*(A^{[3]})$. Then:

$$\theta^*\left(-\frac{1}{2}\,\mathrm{ch}_1(a_j)\cdot \mathfrak{p}_{-2}(a_i)\mathfrak{p}_{-1}(1)|0\rangle\right) = \theta^*\left(\mathfrak{p}_{-3}(1)|0\rangle + \frac{1}{2}\mathfrak{p}_{-1}(x)^2\mathfrak{p}_{-1}(1)|0\rangle\right)$$

But on the other hand, $\delta \cdot j(a) = \frac{1}{2}\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_i)|0\rangle + \mathfrak{p}_{-2}(a_i)\mathfrak{p}_{-1}(1)|0\rangle$, and

$$\theta^*\left(\mathrm{ch}_1(a_j)\cdot \delta \cdot j(a)\right) = \theta^*\left(-3\mathfrak{p}_{-3}(1)|0\rangle - 3\mathfrak{p}_{-1}(x)^2\mathfrak{p}_{-1}(1)|0\rangle\right).$$

□

Corollary 0.6. $\theta^*\left(\delta \cdot j(a_i)\right) = \theta^*\left(\frac{3}{4}\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_i)|0\rangle\right)$ *is divisible by 3.* □

Proposition 0.7. *The classes $\theta^*\left(j(a_i)^2 - \frac{1}{3}j(a_i)\cdot\delta\right)$ are divisible by 2.*

Proof. By [**?**], the classes $\frac{1}{2}\mathfrak{p}_{-1}(a_i)^2\mathfrak{p}_{-1}(1)|0\rangle - \frac{1}{2}\mathfrak{p}_{-2}(a_i)\mathfrak{p}_{-1}(1)|0\rangle$ are integral in $H^4(A^{[n]})$. But $j(a_i)^2 = \mathfrak{p}_{-1}(a_i)^2\mathfrak{p}_{-1}(1)|0\rangle$ and $\theta^*\left(\frac{1}{3}j(a_i)\cdot\delta\right) = \theta^*\left(\mathfrak{p}_{-2}(a_i)\mathfrak{p}_{-1}(1)|0\rangle\right)$. □

Proposition 0.8. *The class δ^2 is divisible by 2.*

Proof. By [**?**, Prop. 4.1], $\mathrm{Sym}^2 H^2 \oplus \left(\mathrm{Sym}^2 H^2\right)^\perp = H^4$. We want to show that $\delta^2 \cdot \mathrm{Sym}^2 H^2 = 2\mathbb{Z}$. We know a $\mathbb{Q}$–basis of $\mathrm{Sym}^2 H^2$ with at most one class divisible by 2, given by $j(a_i)j(a_j)$, δ^2 and the above proposition. By computation, $\int \delta^4$ is divisible by 4 and $\int \delta^2 j(a_i)j(a_j)$ and $\int \delta^3 j(a_i)$ are all divisible by 2. So $\delta^2 \cdot H^4 = 2\mathbb{Z}$ and therefore δ^2 is divisible by 2, since H^4 is unimodular. □

Proposition 0.9. *The class $\theta^*\left(\delta^2 - j(a_1)\cdot j(a_2) - j(a_3)\cdot j(a_4) - j(a_5)\cdot j(a_6)\right)$ is divisible by 3.*

Proof. It is equal to $\theta^*\left(\mathfrak{p}_{-3}(1)|0\rangle + \frac{3}{2}\mathfrak{p}_{-1}(x)\mathfrak{p}_{-1}(1)^2|0\rangle\right)$. □

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