INTEGRAL COHOMOLOGY OF $K^2(A)$

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Abstract. What we know already

1. Chomology of Hilbert schemes of points on a torus surface

Let A be a complex projective torus of dimension 2. Its first cohomology $H^1(A,\mathbb{Z})$ is freely generated by four elements a_1, a_2, a_3, a_4, corresponding to the four different circles on the torus. The cohomology ring is isomorphic to the exterior algebra:

$$H^*(A,\mathbb{Z}) = \Lambda^* H^1(A,\mathbb{Z}).$$

We abbreviate for the products $a_i \cdot a_j =: a_{ij}$ and $a_i \cdot a_j \cdot a_k =: a_{ijk}$. We write $a_1 \cdot a_2 \cdot a_3 \cdot a_4 =: x$ for the class corresponding to a point on A. We choose the a_i such that $\int_A x = 1$. The bilinear form, given by $(a_{ij}, a_{kl}) \mapsto \int_A a_{ij} a_{kl}$ gives $H^2(A,\mathbb{Z})$ the structure of a unimodular lattice, isomorphic to $U^{\oplus 3}$, three copies of the hyperbolic lattice.

Let $A^{[n]}$ the Hilbert scheme of n points on the torus, *i.e.* the moduli space of finite subschemes of A of length n. Their rational cohomology can be described in terms of Nakajima's operators. First consider the direct sum

$$\mathbb{H} := \bigoplus_{n=0}^{\infty} H^*(A^{[n]},\mathbb{Q})$$

This space is bigraded by cohomological *degree* and the *weight*, which is given by the number of points n. The unit element in $H^0(A^{[0]},\mathbb{Q}) \cong \mathbb{Q}$ is denoted by $|0\rangle$, called the *vacuum*.

There are linear operators $\mathfrak{p}_m(\alpha)$, for each $m \in \mathbb{Z},\ \alpha \in H^*(A,\mathbb{Q})$, acting on $\mathbb{H}$ which have the following properties: They depend linearly on α, and if $\alpha \in H^k(A,\mathbb{Q})$ is homogeneous, the operator $\mathfrak{p}_{-m}(\alpha)$ is bihomogeneous of degree $k + 2(|m|-1)$ and weight m:

$$\mathfrak{p}_{-m}(\alpha) : H^l(A^{[n]}) \to H^{l+k+2(|m|-1)}(A^{[n+m]})$$

They satisfy the following commutation relations for $\alpha \in H^k(A,\mathbb{Q}),\ \beta \in H^{k'}(A,\mathbb{Q})$:

$$\mathfrak{p}_m(\alpha)\mathfrak{p}_{m'}(\beta) - (-1)^{k\cdot k'}\mathfrak{p}_{m'}(\beta)\mathfrak{p}_m(\alpha) = -m\,\delta_{m,-m'}\int_A \alpha\cdot\beta.$$

Every element in $\mathbb{H}$ can be decomposed uniquely as a linear combination of products of operators $\mathfrak{p}_m(\alpha),\ m < 0$, acting on the vacuum.

For the study of integral cohomology we cite:

Theorem 1.1. [?] *The following operators map integral classes in $\mathbb{H}$ to integral classes:*

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- $\mathfrak{p}_{-\lambda}(\alpha)$ *for* $\alpha \in H^*(A, \mathbb{Z})$
- $\frac{1}{z_\lambda}\mathfrak{p}_{-\lambda}(1)^n$
- $\mathfrak{m}_\lambda(\alpha)$ *for* $\alpha \in H^2(A, \mathbb{Z})$

Here, $\mathfrak{m}_\lambda$ *is defined as* $\mathfrak{m}_\lambda(\alpha) := \sum_\mu c_{\lambda\mu}\mathfrak{p}_{-\mu}(\alpha)$ *and* $c_{\lambda\mu}$ *are the coefficients of the transition matrix between monomial symmetric functions and power sum symmetric functions.*

For a classes $\alpha_1, \dots, \alpha_r \in H^*(A)$ and a partitions $\lambda_1, \dots, \lambda_r$ we write $\alpha_1^{\lambda_1} \dots \alpha_r^{\lambda_r}$ for the class $\mathfrak{p}_{-\lambda_1}(\alpha_1) \dots \mathfrak{p}_{-\lambda_r}(\alpha_r)|0\rangle$.

2. Generalized Kummer varieties

Definition 2.1. Let A be a complex projective torus of dimesion 2 and $A^{[n]}$ the corresponding Hilbert scheme of points. Denote $\Sigma : A^{[n]} \to A$ the summation morphism. Then the generalized Kummer $K^{n-1}A$ is defined as the fiber over 0:

$$\begin{array}{ccc} K^{n-1}A & \xrightarrow{\theta} & A^{[n]} \\ \downarrow & & \downarrow \Sigma \\ \{0\} & \longrightarrow & A \end{array} \tag{1}$$

By [?], $\theta^* : H^2(A^{[n]}) \to H^2(K^{n-1}A)$ is surjective. We have injections $j : H^2(A) \to H^2(A^{[n]})$ and $i = \theta^* j$. The cohomology $H^*(A^{[n]})$ is described in terms of vertex operators in [?] and [?].

We describe now the image of θ^* in the case $n = 3$:

- We know $j(a) = \frac{1}{2}\mathfrak{p}_{-1}(a)\mathfrak{p}_{-1}(1)^2|0\rangle$, because the two must be linearly dependent and
$$\int_{A^{[3]}} j(a)^6 = 15q(a)^3, \quad \left(\frac{1}{2}\mathfrak{p}_{-1}(a)\mathfrak{p}_{-1}(1)^2|0\rangle\right)^3 = 15q(a)^3\mathfrak{p}_{-1}(x)^3|0\rangle.$$
- By [?, p. 8], we have for $\alpha = j(a) = \frac{1}{2}\mathfrak{p}_{-1}(a)\mathfrak{p}_{-1}(1)^2|0\rangle$:
$$\int_{A^{[3]}} \alpha^6 = \frac{5}{3}q(a)\int_{K^2} \theta^*\alpha^4$$
On the other hand,
$$\alpha^4 = 3q(a)^2\mathfrak{p}_{-1}(x)^2\mathfrak{p}_{-1}(0)|0\rangle + 3q(a)\mathfrak{p}_{-1}(x)\mathfrak{p}_{-1}(a)^2|0\rangle,$$
so if the image of both summands under $\int \theta^*$ is positive, then
$$\int \theta^*\mathfrak{p}_{-1}(x)^2\mathfrak{p}_{-1}(0)|0\rangle = \int \theta^* \tfrac{1}{2}\mathfrak{p}_{-1}(x)\mathfrak{p}_{-1}(a)^2|0\rangle = 1.$$

Proposition 2.2. *The class of* $K^2(A)$ *in* $H^4(A^{[3]}, \mathbb{Q})$ *is given by*
$$a_1^{(1)} \cdot a_2^{(1)} \cdot a_3^{(1)} \cdot a_4^{(1)}.$$

Proof. We know that for all i and all $\beta \in H^7(A^{[3]})$, we have $\int_{K^2(A)} \theta^*(\alpha_i \cdot \beta) = \int_{A^{[3]}} \alpha_i \cdot \beta \cdot [K^2(A)] = 0$ and for a basis (γ_i) of $H^2(A^{[3]})$,
$$\int_{A^{[3]}} \gamma_i \cdot \gamma_j \cdot \gamma_k \cdot \gamma_l \cdot [K^2(A)] = 3\left(\langle\gamma_i, \gamma_j\rangle\langle\gamma_k, \gamma_l\rangle + \langle\gamma_i, \gamma_k\rangle\langle\gamma_j, \gamma_l\rangle + \langle\gamma_i, \gamma_l\rangle\langle\gamma_j, \gamma_k\rangle\right)$$
These equations admit a unique solution. $\square$

Let $\{a_i\}_{i=1\ldots 6}$ be a hyperbolic basis of $H^2(A,\mathbb{Z})$.

Proposition 2.3. *The classes* $\theta^*\left(\mathfrak{p}_{-2}(a_i)\mathfrak{p}_{-1}(1)|0\rangle\right)$ *and* $\theta^*\left(\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_i)|0\rangle\right)$ *are linearly dependent.*

Proposition 2.4. $\theta^*\left(\mathfrak{p}_{-3}(x)|0\rangle\right) = 0$

Corollary 2.5. $\theta^*\left(\mathfrak{p}_{-2}(a_i)\mathfrak{p}_{-1}(1)|0\rangle\right) = \frac{1}{4}\theta^*\left(\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_i)|0\rangle\right)$

Proof. Let a_j be complementary, *i.e.* $a_i a_j = 1$. Let $\mathrm{ch}_1(a_j) = -\frac{1}{2}\mathfrak{p}_{-2}(a_j)\mathfrak{p}_{-1}(1)|0\rangle$ be the chern character in the vertex algebra description of $H^*(A^{[3]})$. Then:

$$\theta^*\left(-\frac{1}{2}\,\mathrm{ch}_1(a_j)\cdot\mathfrak{p}_{-2}(a_i)\mathfrak{p}_{-1}(1)|0\rangle\right) = \theta^*\left(\mathfrak{p}_{-3}(1)|0\rangle + \frac{1}{2}\mathfrak{p}_{-1}(x)^2\mathfrak{p}_{-1}(1)|0\rangle\right)$$

But on the other hand, $\delta\cdot j(a) = \frac{1}{2}\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_i)|0\rangle + \mathfrak{p}_{-2}(a_i)\mathfrak{p}_{-1}(1)|0\rangle$, and

$$\theta^*\left(\mathrm{ch}_1(a_j)\cdot\delta\cdot j(a)\right) = \theta^*\left(-3\mathfrak{p}_{-3}(1)|0\rangle - 3\mathfrak{p}_{-1}(x)^2\mathfrak{p}_{-1}(1)|0\rangle\right).$$

□

Corollary 2.6. $\theta^*\left(\delta\cdot j(a_i)\right) = \theta^*\left(\frac{3}{4}\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_i)|0\rangle\right)$ *is divisible by 3.* □

Proposition 2.7. *The classes* $\theta^*\left(j(a_i)^2 - \frac{1}{3}j(a_i)\cdot\delta\right)$ *are divisible by 2.*

Proof. By [**?**], the classes $\frac{1}{2}\mathfrak{p}_{-1}(a_i)^2\mathfrak{p}_{-1}(1)|0\rangle - \frac{1}{2}\mathfrak{p}_{-2}(a_i)\mathfrak{p}_{-1}(1)|0\rangle$ are integral in $H^4(A^{[n]})$. But $j(a_i)^2 = \mathfrak{p}_{-1}(a_i)^2\mathfrak{p}_{-1}(1)|0\rangle$ and $\theta^*\left(\frac{1}{3}j(a_i)\cdot\delta\right) = \theta^*\left(\mathfrak{p}_{-2}(a_i)\mathfrak{p}_{-1}(1)|0\rangle\right)$. □

Proposition 2.8. *The class* δ^2 *is divisible by 2.*

Proof. By [**?**, Prop. 4.1], $\mathrm{Sym}^2 H^2 \oplus \left(\mathrm{Sym}^2 H^2\right)^\perp = H^4$. We want to show that $\delta^2\cdot\mathrm{Sym}^2 H^2 = 2\mathbb{Z}$. We know a $\mathbb{Q}$–basis of $\mathrm{Sym}^2 H^2$ with at most one class divisible by 2, given by $j(a_i)j(a_j)$, δ^2 and the above proposition. By computation, $\int\delta^4$ is divisible by 4 and $\int\delta^2 j(a_i)j(a_j)$ and $\int\delta^3 j(a_i)$ are all divisible by 2. So $\delta^2\cdot H^4 = 2\mathbb{Z}$ and therefore δ^2 is divisible by 2, since H^4 is unimodular. □

Proposition 2.9. *The class* $\theta^*\left(\delta^2 - j(a_1)\cdot j(a_2) - j(a_3)\cdot j(a_4) - j(a_5)\cdot j(a_6)\right)$ *is divisible by 3.*

Proof. It is equal to $\theta^*\left(\mathfrak{p}_{-3}(1)|0\rangle + \frac{3}{2}\mathfrak{p}_{-1}(x)\mathfrak{p}_{-1}(1)^2|0\rangle\right)$. □

Next we look at the Chern classes of the tangent sheaves. Since the morphism Σ from the defining pullback diagram (1) is a submersion, the normal bundle of $K_{n-1}(A)$ in $A^{[n]}$ is trivial. Hence $c(K_2(A)) = \theta^* c(A^{[3]})$. Looking in [**?**, Sect. 8], we find a general formula for Chern classes of Hilbert schemes of points on surfaces. So we deduce

$$\begin{aligned} c_2(A^{[3]}) &= \left(\tfrac{3}{2}\mathfrak{q}_{(1,1)}(\Delta 1)\mathfrak{q}_1(1) - \tfrac{1}{3}\mathfrak{q}_3\right)|0\rangle \\ &= 10(1^{[\bullet]}_{(4)}) - 2(1^{[\bullet]}_{(2)})^2 \\ c_4(A^{[3]}) &= \tfrac{4}{3}\mathfrak{q}_{(1,1,1)}(\Delta 1)|0\rangle = 4(1^{[\bullet]}_{(4)})^2. \end{aligned}$$

References

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