Integral cohomology of the Generalized Kummer fourfold

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1 Introduction

Part I
Preliminaries

2 Symmetric bilinear forms

3 Super algebras

Definition 3.1. A super vector space V over a field k is a vector space with a $\mathbb{Z}/2\mathbb{Z}$-graduation, that is a decomposition

$$V = V^+ \oplus V^-,$$

called the even and the odd part of V. Elements of V^+ are called homogeneous of even degree, elements of V^- are called homogeneous of odd degree. The degree of a homogeneous element v is denoted by $|v| \in \mathbb{Z}/2\mathbb{Z}$. Direct sum and tensor product of two super vector spaces V and W yield again super vector spaces:

$$\begin{aligned} (V \oplus W)^+ &= V^+ \oplus W^+, & (V \oplus W)^- &= V^- \oplus W^-, \\ (V \otimes W)^+ &= (V^+ \otimes W^+) \oplus (V^- \otimes W^-), & (V \otimes W)^- &= (V^+ \otimes W^-) \oplus (V^- \otimes W^+). \end{aligned}$$

Definition 3.2. A superalgbra R is a unital associative k-algebra which carries a super vector space structure. Define the supercommutator by setting for homogeneous elements $u, v \in R$:

$$[u, v] := uv - (-1)^{|u||v|} vu.$$

R is called supercommutative, if $[u, v] = 0$ for all $u, v \in R$. Note that a graded commutative algebra $R = \bigoplus_n R^n$ is supercommutative in a natural way, by setting $R^+ = \bigoplus_{n \text{ even}} R^n$, $R^- = \bigoplus_{n \text{ odd}} R^n$.

For a supercommutative algebra R, the tensor power $R^{\otimes n}$ is again a supercommutative algebra, if we set for the product:

$$(u_1 \otimes \cdots \otimes u_n) \cdot (v_1 \otimes \cdots \otimes v_n) = (-1)^k u_1 v_1 \otimes \cdots \otimes u_n v_n, \quad \text{where } k = \sum_{j<i} |u_i||v_j|.$$

Definition 3.3. Let V be a super vector space over k and $n \geq 0$. Then the supersymmetric power $\mathrm{SSym}^n(V)$ of V is a super vector space, given by

$$\mathrm{SSym}^n(V) = \bigoplus_{p+q=n} \mathrm{Sym}^p(V^+) \otimes \Lambda^q(V^-),$$

$$\mathrm{SSym}^n(V)^+ = \bigoplus_{\substack{p+q=n \\ q \text{ even}}} \mathrm{Sym}^p(V^+) \otimes \Lambda^q(V^-), \qquad \mathrm{SSym}^n(V)^- = \bigoplus_{\substack{p+q=n \\ q \text{ odd}}} \mathrm{Sym}^p(V^+) \otimes \Lambda^q(V^-).$$

The supersymmetric algebra $\mathrm{SSym}^*(V) := \bigoplus_n \mathrm{SSym}^n(V)$ on V is a supercommutative algebra over k, where the product of two elements $s \otimes e \in \mathrm{Sym}^p(V^+) \otimes \Lambda^q(V^-)$ and $s' \otimes e' \in \mathrm{Sym}^{p'}(V^+) \otimes \Lambda^{q'}(V^-)$ is given by

$$(s \otimes e) \diamond (s' \otimes e') = (ss') \otimes (e \wedge e') \quad \in \mathrm{Sym}^{p+p'}(V^+) \otimes \Lambda^{q+q'}(V^-).$$

Remark 3.4. The supersymmetric power $\mathrm{SSym}^n(V)$ can be realized as a quotient of $V^{\otimes n}$ by an action of the symmetric group $\mathfrak{S}_n$. This action can be described as follows: If $\tau \in \mathfrak{S}_n$ is a transposition that exchanges two numbers $i < j$, then τ permutes the corresponding tensor factors in $v_1 \otimes \cdots \otimes v_n$ introducing a sign $(-1)^{|v_i||v_j| + (|v_i|+|v_j|)\sum_{i<k<j} |v_k|}$.

Now let U be a vector space over a field k of characteristic 0 and look at the exterior algebra $H := \Lambda^* U$. Since H is a super vector space, we can construct the supersymmetric power $S^n := \mathrm{SSym}^n(H)$. We may identify S^n with the space of $\mathfrak{S}_n$-invariants in $H^{\otimes n}$ by means of the linear projection operator

$$\mathrm{pr} : H^{\otimes n} \longrightarrow S^n, \quad \mathrm{pr} = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \pi.$$

The multiplication in $H^{\otimes n}$ induces a multiplication on the subspace of invariants, which makes S^n a supercommutative algebra. Of course, it is different from the product $\diamond$ discussed above.

Since H is generated as an algebra by $U = \Lambda^1(U) \subset H$, we may define a homomorphism of algebras:

$$s : H \longrightarrow S^n, \quad s(u) = \mathrm{pr}(u \otimes 1 \otimes \cdots \otimes 1) \text{ for } u \in U,$$

so S becomes an algebra over H.

Lemma 3.5. *The morphism s turns S^n into a free module over H, for $n \geq 1$.*

Proof. We start with the tensor power $H^{\otimes n}$ and the ring homomorphism

$$\iota : H \longrightarrow H^{\otimes n}, \quad h \longmapsto h \otimes 1 \otimes \cdots \otimes 1$$

that makes $H^{\otimes n}$ a free H-module. Note that $\mathrm{pr}\,\iota \neq s$, since pr is not a ring homomorphism. (For example, $\mathrm{pr}\,\iota(h) \neq s(h)$ for any nonzero $h \in \Lambda^2(U)$.) We therefore modify the H-module structure of $H^{\otimes n}$:

For some $u \in U$, denote $u^{(i)} := 1^{\otimes i-1} \otimes u \otimes 1^{\otimes n-i+1} \in H^{\otimes n}$. Then $H^{\otimes n}$ is generated as a k-algebra by the elements $\{u^{(i)}, u \in U\}$. Now consider the ring automorphism

$$\sigma : H^{\otimes n} \longrightarrow H^{\otimes n}, \quad u^{(1)} \longmapsto u^{(1)} + u^{(2)} + \ldots + u^{(n)}, \quad u^{(i)} \longmapsto u^{(i)} \text{ for } i > 1.$$

Then we have $\sigma\iota = s$ on S^n. On the other hand, if $\{b_i\}$ is a k-basis of V, then $\{b_i^{(j)}, j > 1\}$ is a ι-basis for $H^{\otimes n}$, and $\{\sigma(b_i^{(j)})\}$ is a $\sigma\iota$-basis for $H^{\otimes n}$. Now if we project the basis elements, we get a set $\{\mathrm{pr}(\sigma(b_i^{(j)}))\}$ that spans S^n. Eliminating linear dependent vectors (this is possible over the rationals), we get a s-basis of S^n. $\square$

4 Actions of the symplectic group over finite fields

Let V be a symplectic vector space of dimension $n \in 2\mathbb{N}$ over a field F with a nondegenerate symplectic form $\omega : \Lambda^2 V \to F$. A line is a one-dimensional subspace ov V, a plane is a two-dimensional subspace of V. A plane $P \subset V$ is called isotropic, if $\omega(x, y) = 0$ for any $x, y \in P$, otherwise non-isotropic. The symplectic group $\mathrm{Sp}\,V$ is the set of all linear maps $\phi : V \to V$ with the property $\omega(\phi(x), \phi(y)) = \omega(x, y)$ for all $x, y \in V$.

Proposition 4.1. *The symplectic group $\mathrm{Sp}\,V$ acts transitively on the set of non-isotropic planes as well as on the set of isotropic planes.*

Proof. Given two planes P_1 and P_2, we may choose vectors v_1, v_2, w_1, w_2 such that v_1, v_2 span P_1, w_1, w_2 span P_2 and $\omega(u_1, u_2) = \omega(w_1, w_2)$. We complete $\{v_1, v_2\}$ as well as $\{w_1, w_2\}$ to a symplectic basis of V. Then define $\phi(v_1) = w_1$ and $\phi(v_2) = w_2$. It is now easy to see that the definition of ϕ can be extended to the remaining basis elements to give a symplectic morphism. $\square$

Remark 4.2. The set of planes in V can be identified with the simple tensors in $\Lambda^2 V$ up to multiples. Indeed, given a simple tensor $v \wedge w \in \Lambda^2 V$, the span of v and w yields the corresponding plane. Conversely, any two spanning vectors v and w of a plane give the same element $v \wedge w$ (up to multiples).

Proposition 4.3. *If $\phi \in \mathrm{Sp}\, V$ acts through multiplication of a scalar, $\phi(v) = \alpha v$, then $\alpha = \pm 1$ (this is immediate from the definition). Moreover, if $\phi(v) \wedge \phi(w) = \alpha v \wedge w$, then $\alpha = 1$.*

Proof. We may assume that V is two-dimensional, generated by v and w. Our condition on ϕ reads then $\det \phi = \alpha$. But the condition on ϕ being symplectic is $\det \phi = 1$, because on a two-dimensional vector space there is only one symplectic form up to scalar multiple. □

Remark 4.4. If F is the field with two elements, then the set of planes in V can be identified with the set $\{\{x, y, z\} \mid x, y, z \in V \backslash \{0\},\, x + y + z = 0\}$. Observe that for such a $\{x, y, z\}$, $\omega(x, y) = \omega(x, y) = \omega(y, x)$ and this value gives the criterion for isotropy.

Proposition 4.5. *Assume that F is finite of cardinality q.*

$$\text{The number of lines in } V \text{ is } \frac{q^n - 1}{q - 1}, \tag{1}$$

$$\text{the number of planes in } V \text{ is } \frac{(q^n - 1)(q^{n-1} - 1)}{(q^2 - 1)(q - 1)}, \tag{2}$$

$$\text{the number of isotropic planes in } V \text{ is } \frac{(q^n - 1)(q^{n-2} - 1)}{(q^2 - 1)(q - 1)}, \tag{3}$$

$$\text{the number of non-isotropic planes in } V \text{ is } \frac{q^{n-2}(q^n - 1)}{q^2 - 1}. \tag{4}$$

Proof. A line ℓ in V is determined by a nonzero vector. There are $q^n - 1$ nonzero vectors in V and $q - 1$ nonzero vectors in ℓ. A plane P is determined by a line $\ell_1 \subset V$ and a unique second line $\ell_2 \in V/\ell_1$. We have $\frac{q^2-1}{q-1}$ choices for ℓ_1 in P. The number of planes is therefore

$$\frac{\frac{q^n-1}{q-1} \cdot \frac{q^{n-1}-1}{q-1}}{\frac{q^2-1}{q-1}} = \frac{(q^n - 1)(q^{n-1} - 1)}{(q^2 - 1)(q - 1)}.$$

For an isotropic plane we have to choose the second line from $\ell_1^\perp/\ell_1$. This is a space of dimension $n - 2$, hence the formula. The number of non-isotropic planes is the difference of the two previous numbers. □

Assume now that V is a four-dimensional vector space over $F = \mathbb{F}_q$. Consider the free F-module $F[V]$ with basis $\{X_i \mid i \in V\}$. It carries a natural F-algebra structure, given by $X_i \cdot X_j := X_{i+j}$ with unit $1 = X_0$. This algebra is local with maximal ideal $\mathfrak{m}$ generated by all elements of the form $(X_i - 1)$.

We introduce an action of $\mathrm{Sp}(4, F)$ on $F[V]$ by setting $\phi(X_i) = X_{\phi(i)}$. Furthermore, the underlying additive group of V acts on $F[V]$ by $v(X_i) = X_{i+v} = X_i X_v$.

Definition 4.6. We define a subset of $F[V]$:

$$N := \left\{ \sum_{i \in P} X_i \,\middle|\, P \subset V \text{ non-isotropic plane} \right\}.$$

Denote by $\langle N \rangle$ and by (N) the linear span of N and the ideal generated by N, respectively. Note that (N) is the linear span of $\{v \cdot b \mid b \in N, v \in V\}$. Further, let D be the linear span of $\{v(b) - b \mid b \in N, v \in V\}$. Then D is in fact an ideal, namely the product of ideals $\mathfrak{m} \cdot (N)$.

The following table illustrates the dimensions of these objects for some fields F:

F	$\dim_F \langle N \rangle$	$\dim_F (N)$	$\dim_F D$
$\mathbb{F}_2$	10	11	5
$\mathbb{F}_3$	30	50	31

Let us now consider some special orthogonal sums. Set $S := \mathrm{Sym}^2(\Lambda^2 V)$. Take two vectors $v, w \in V$ with $\omega(v,w) = 1$ and set $x := (v \wedge w)^2 \in S$. Denote P the plane spanned by v and w and set $y := \sum_{i \in P} X_i \in F[V]$. We set $Y' := y \cdot \mathfrak{m} = \{\sum_{i\in P} X_{i+j} - X_i \mid j \in V\}$.

We consider now the action of $\mathrm{Sp}\, V$ on $S \oplus F[V]$.

Proposition 4.7. *The elements $\phi(x) \oplus \phi(z)$, for $\phi \in \mathrm{Sp}\, V$, $z \in (y)$ span a vector space of dimension*

- *11, if $F = \mathbb{F}_2$,*
- *51, if $F = \mathbb{F}_3$,*
- *375, if $F = \mathbb{F}_5$.*

Proposition 4.8. *The elements $\phi(x) \oplus \phi(y')$, for $\phi \in \mathrm{Sp}\, V$, $y' \in Y'$ span a vector space of dimension*

- *10, if $F = \mathbb{F}_2$,*
- *50, if $F = \mathbb{F}_3$,*
- *289, if $F = \mathbb{F}_5$.*

5 Complex abelian surfaces

Denote A a complex abelian surface (a torus of dimension 2). As such, it always can be written as a quotient

$$A = \mathbb{C}^2/\Lambda,$$

where $\Lambda \subset \mathbb{C}^2$ is a lattice of rank 4, embedded in $\mathbb{C}^2$. Depending on the imbedding, we get different complex manifolds, projective or not. Of course, all of them are equivalent by monodromy.

5.1 Morphisms

Definition 5.1. An isogeny between abelian surfaces $A = \mathbb{C}^2/\Lambda \to A' = \mathbb{C}^2/\Lambda'$ means a surjective holomorphic map that preserves the group structure. It is given by a complex linear map, that maps Λ to a sublattice of Λ'.

Example 5.2. For a number $n \neq 0$, the multiplication map $n : A \to A$, $x \mapsto n \cdot x$ is an isogeny.

By an automorphism of A we mean a biholomorphism preserving the group structure. It can be represented by a $\mathbb{C}$-linear map $M : \mathbb{C}^2 \to \mathbb{C}^2$ with $M\Lambda = \Lambda$. Have a look in [7] or the appendix of [8] for some reference. Let us now come to the very special case that $A = E \times E$ can be written as the square of an elliptic curve. Note that A is projective, because every elliptic curve is. Now write E as $E = \mathbb{C}/\Lambda_0$. We may assume that Λ_0 is spanned by 1 and a vector $\tau \in \mathbb{C}\backslash\mathbb{R}$. The automorphism group, up to isogeny, is given by ([8]) $\mathrm{GL}(2, \mathrm{End}(\Lambda_0))$, where $\mathrm{End}(\Lambda_0)$ is the set $\{z \in \mathbb{C} \mid z\Lambda_0 \subset \Lambda_0\}$.

Proposition 5.3. *There are two possibilities for $\mathrm{End}(\Lambda_0)$, depending on τ:*

1. *Both the real part and the square norm of τ are rational numbers, say $2\Re(\tau) = \frac{p}{r}$ and $\|\tau\|^2 = \frac{q}{r}$ with $r > 0$ as small as possible. Then $\mathrm{End}(\Lambda_0) = \mathbb{Z} + r\tau\mathbb{Z}$.*
2. *At least one of $\Re(\tau), \|\tau\|^2$ is irrational. Then $\mathrm{End}(\Lambda_0) = \mathbb{Z}$.*

Proof. Given $z \in \mathrm{End}(\Lambda_0)$, we have

$$z \cdot 1 = a + b\tau \text{ and } z \cdot \tau = c + d\tau \text{ with } a, b, c, d \in \mathbb{Z}.$$

We get the condition

$$(a + b\tau)\tau = c + d\tau \quad \Leftrightarrow \quad b\tau^2 + (a-d)\tau - c = 0.$$

Up to scalar multiples, there is a unique real quadratic polynomial that annihilates τ, namely $(x-\tau)(x-\bar{\tau}) = x^2 - 2\Re(\tau)x + \|\tau\|^2$. If all coefficients of that polynomial are rational numbers, then $z = a + b\tau$ gives a solution for arbitrary $a \in \mathbb{Z}, b \in r\mathbb{Z}$. Otherwise, the condition must be the zero polynomial, so $b = 0$. ∎

Definition 5.4. Denote $\xi \in \mathbb{C}$ a primitive sixth root of unity and E_ξ the elliptic curve given by the choice $\Lambda_0 = \langle 1, \xi \rangle$, so $\mathrm{End}(\Lambda_0) = \Lambda_0$ is the ring of Eisenstein integers. Define a group G_ξ of automorphisms of $E_\xi \times E_\xi$ by the following generators in $\mathrm{GL}(2, \mathrm{End}(\Lambda_0))$:

$$g_1 = \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad g_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For $A = E_\xi \times E_\xi$, let $V = A[2]$ be the (fourdimensional) $\mathbb{F}_2$-vector space of 2-torsion points on A and let $\mathfrak{T}$ be the set of planes in V. Note that by Remark 4.4 a plane in V can be identified with an unordered triple $\{x, y, z\}$ with $0 \neq x, y, z \in V$ and $x + y + z = 0$. The action of G_ξ on A induces actions of G_ξ on $A[2]$ and $\mathfrak{T}$.

Proposition 5.5. *There are two orbits of G_ξ on $\mathfrak{T}$, of cardinalities 5 and 30.*

Proof. Note that the action of the multiplication with ξ induces a cyclic permutation on $E_\xi[2]$. The orbits can be explicitely computed. □

5.2 Homology and Cohomology

The fundamental group $\pi_1(A, \mathbb{Z}) = H_1(A, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 4, which is canonically identified with the lattice Λ. Indeed, the projection of every path in $\mathbb{C}^2$ from 0 to $v \in \Lambda$ gives a unique element of $\pi_1(A, \mathbb{Z})$. Conversely, any closed path in A with basepoint 0 lifts to a unique path in $\mathbb{C}^2$ from 0 to some $v \in \Lambda$. So the first cohomology $H^1(A, \mathbb{Z})$ is freely generated by four elements, too. Moreover, by [19, Sect. I.1], the cohomology ring is isomorphic to the exterior algebra

$$H^*(A, \mathbb{Z}) = \Lambda^*(H^1(A, \mathbb{Z})).$$

Notation 5.6. We denote the generators of $H^1(A, \mathbb{Z})$ by a_i, $1 \leq i \leq 4$. If $A = E \times E$ is the product of two elliptic curves, we choose the a_i in a way such that $\{a_1, a_2\}$ and $\{a_3, a_4\}$ give bases of $H^1(E, \mathbb{Z})$ in the decomposition $H^1(A) = H^1(E) \oplus H^1(E)$. We denote the generator of the top cohomology $H^4(A, \mathbb{Z})$ by $x := a_1 a_2 a_3 a_4$.

TODO: Principal polarization/Jacobians
TODO: Weil pairing for tori

6 Recall on Irreducible holomorphically symplectic manifolds

7 Integral cohomology of quotients

8 Odd cohomology of the Hilbert scheme of two points

Let A be a complex torus **TODO: Does it work if A is a general (projective) surface?** of dimension 2 and $A^{[2]}$ the Hilbert scheme of 2 points. It can be constructed as follows: Consider the direct product $A \times A$. Denote

$$b : \widetilde{A \times A} \to A \times A$$

the blow-up along the diagonal $\Delta \cong A$ with exceptional divisor E, so we have $i : E \to \widetilde{A \times A}$. Since the normal bundle of Δ in $A \times A$ is trivial, we have:

$$E \cong \Delta \times \mathbb{P}^1.$$

The action of $\mathfrak{S}_2$ on $A \times A$ lifts to an action on $\widetilde{A \times A}$. We have the pushforward $i_* : H^*(E, \mathbb{Z}) \to H^*(\widetilde{A \times A}, \mathbb{Z})$.

The quotient by the action of $\mathfrak{S}_2$ is

$$\pi : \widetilde{A \times A} \to A^{[2]}.$$

Now, $A^{[2]}$ is a compact complex manifold with torsion-free cohomology. By [16], we have an exact sequence
$$0 \to \pi_*(H^k(\widetilde{A\times A},\mathbb{Z})) \to H^k(A^{[2]},\mathbb{Z}) \to \left(\frac{\mathbb{Z}}{2Z}\right)^{\alpha_k} \to 0$$
with $k \in \{1,...,8\}$.

Proposition 8.1. *We have:*
$$\alpha_3 = 0 \quad and \quad \alpha_5 = 4.$$

8.1 Preliminary Lemmas

We denote $V = \widetilde{A\times A} \smallsetminus E$ and $U = V/\sigma_2$, where $\mathfrak{S}_2 = \langle \sigma_2 \rangle$.

Lemma 8.2. *We have: $H^k(A\times A,\mathbb{Z}) = H^k(V,\mathbb{Z})$ for all $k \leq 3$.*

Proof. We have $V = A\times A \smallsetminus \Delta$. We have the following natural exact sequence:
$$\cdots \longrightarrow H^k(A\times A, V,\mathbb{Z}) \longrightarrow H^k(A\times A,\mathbb{Z}) \longrightarrow H^k(V,\mathbb{Z}) \longrightarrow \cdots$$
Moreover by Thom isomorphism $H^k(A\times A, V,\mathbb{Z}) = H^{k-4}(\Delta,\mathbb{Z}) = H^{k-4}(A,\mathbb{Z})$. Hence $H^k(A \times A, V,\mathbb{Z}) = 0$ for all $k \leq 3$. Hence $H^k(A\times A,\mathbb{Z}) = H^k(V,\mathbb{Z})$ for all $k \leq 2$. It remains to consider the following exact sequence:
$$0 \longrightarrow H^3(A\times A,\mathbb{Z}) \longrightarrow H^3(V,\mathbb{Z}) \longrightarrow H^4(A\times A, V,\mathbb{Z}) \xrightarrow{\rho} H^4(A\times A,\mathbb{Z})\ .$$
The map ρ is given by $\mathbb{Z}[\Delta] \to H^4(A\times A,\mathbb{Z})$. The class $\{x\}\times A$ is also in $H^4(A\times A,\mathbb{Z})$ and intersects Δ in one point. Hence the class of Δ in $H^4(A\times A,\mathbb{Z})$ is not trivial and the map ρ is injective. It follows
$$H^3(A\times A,\mathbb{Z}) = H^3(V,\mathbb{Z}).$$
□

Now we will calculate the invariant $l_{1,-}^2(A\times A)$ and $l_{1,+}^1(A\times A)$ defined in Section 1.2 of [16] **TODO : recall the definition in the redaction of the application.**

Lemma 8.3. *We have: $l_{1,-}^2(A\times A) = l_{1,+}^1(A\times A) = 0$.*

Proof. By Künneth formula we have:
$$H^1(A\times A,\mathbb{Z}) = H^0(A,\mathbb{Z})\otimes H^1(A,\mathbb{Z}) \oplus H^1(A,\mathbb{Z})\otimes H^0(A,\mathbb{Z}).$$
The elements of $H^0(A,\mathbb{Z})\otimes H^1(A,\mathbb{Z})$ and $H^1(A,\mathbb{Z})\otimes H^0(A,\mathbb{Z})$ are exchanged under the action of σ_2. It follows that $l_2^1(A\times A) = 4$ and necessary $l_{1,-}^1(A\times A) = l_{1,+}^1(A\times A) = 0$.

By Künneth formula we also have:
$$\begin{aligned} H^2(A\times A,\mathbb{Z}) = H^0(A,\mathbb{Z})\otimes H^2(A,\mathbb{Z}) &\oplus H^1(A,\mathbb{Z})\otimes H^1(A,\mathbb{Z}) \\ \oplus H^2(A,\mathbb{Z})\otimes H^0(A,\mathbb{Z}).& \end{aligned}$$
As before every elements $x\otimes y \in H^2(A\times A,\mathbb{Z})$ are sent to $y\otimes x$ by the action of σ_2. A such element is fixed by the action of σ_2 if $x = y$. It follows:
$$l_2^2(A\times A) = 6+6 = 12,$$
$$l_{1,+}^2(A\times A) = 4,$$
and necessary:
$$l_{1,-}^2(A\times A) = 0.$$
□

Lemma 8.4. *The group $H^3(U,\mathbb{Z})$ is torsion free.*

Proof. Using the spectral sequence of equivariant cohomology, it follows from Proposition 2.6 of [16], Lemma 8.2 and 8.3. □

8.2 In degree 3

By Theorem 7.31 of Voisin, we have:

$$H^3(\widetilde{A\times A},\mathbb{Z}) = H^3(A\times A,\mathbb{Z})\oplus H^1(\Delta,\mathbb{Z}). \tag{5}$$

It follows that

$$H^3(A^{[2]},\mathbb{Z})\supset \pi_*(H^3(A\times A,\mathbb{Z}))\oplus\pi_*(H^1(\Delta,\mathbb{Z})).$$

Moreover by Künneth formula, we have:

$$\begin{aligned}H^3(A\times A,\mathbb{Z}) &= H^0(A,\mathbb{Z})\otimes H^3(A,\mathbb{Z})\oplus H^1(A,\mathbb{Z})\otimes H^2(A,\mathbb{Z})\\ &\oplus H^2(A,\mathbb{Z})\otimes H^1(A\mathbb{Z})\oplus H^3(A,\mathbb{Z})\otimes H^0(A,\mathbb{Z}).\end{aligned}$$

Hence all elements in $H^3(A\times A,\mathbb{Z})^{\mathfrak{S}_2}$ are written $x+\sigma_2^*(x)$ with $x\in H^3(A\times A,\mathbb{Z})$. Since $\frac{1}{2}\pi_*(x+\sigma_2^*(x))=\pi_*(x)$, it follows that $\pi_*(H^3(A\times A,\mathbb{Z}))$ is primitive in $H^3(A^{[2]},\mathbb{Z})$. Moreover by (5):

$$l_2^3(\widetilde{A\times A}) = \operatorname{rk} H^3(A\times A,\mathbb{Z})^{\mathfrak{S}_2} = 28. \tag{6}$$

and

$$l_{1,+}^3(\widetilde{A\times A}) = \operatorname{rk} H^1(\Delta,\mathbb{Z})^{\mathfrak{S}_2} = 4,\quad \text{and}\quad l_{1,-}^3(\widetilde{A\times A}) = 0. \tag{7}$$

It remains to prove the following lemma.

Lemma 8.5. *The group $\pi_*(H^1(\Delta,\mathbb{Z}))$ is primitive in $H^3(A^{[2]},\mathbb{Z})$.*

Proof. We consider the following commutative diagram:

$$\begin{tikzcd} H^3(\mathscr{N}_{A^{[2]}/E},\mathscr{N}_{A^{[2]}/E}-0,\mathbb{Z}) = H^3(A^{[2]},U,\mathbb{Z}) \arrow[r, "g"] \arrow[d, "d\widetilde{\pi}^*"] & H^3(A^{[2]},\mathbb{Z}) \arrow[d, "\pi^*"'] \\ H^3(\mathscr{N}_{\widetilde{A\times A}/E},\mathscr{N}_{\widetilde{A\times A}/E}-0,\mathbb{Z}) = H^3(\widetilde{A\times A},V,\mathbb{Z}) \arrow[r, "h"] & H^3(\widetilde{A\times A},\mathbb{Z}), \end{tikzcd} \tag{8}$$

By proof of Theorem 7.31 of [25], the map h is injective and its image in $H^3(\widetilde{A\times A},\mathbb{Z})$ is $H^1(\Delta,\mathbb{Z})$. Hence Diagram 8 shows that g is also injective and has image $\pi_*(H^1(\Delta,\mathbb{Z}))$ in $H^3(A^{[2]},\mathbb{Z})$. It follows the exact sequence:

$$0\longrightarrow H^3(A^{[2]},U,\mathbb{Z})\xrightarrow{g} H^3(A^{[2]},\mathbb{Z})\longrightarrow H^3(U,\mathbb{Z}).$$

However, by lemma 8.4, $H^3(U,\mathbb{Z})$ is torsion free; it follows that $\pi_*(H^1(\Delta,\mathbb{Z}))$ is primitive in $H^3(A^{[2]},\mathbb{Z})$. $\square$

8.3 In degree 5

By Theorem 7.31 of Voisin, we have:

$$H^5(\widetilde{A\times A},\mathbb{Z}) = H^5(A\times A,\mathbb{Z})\oplus H^3(\Delta,\mathbb{Z}). \tag{9}$$

It follows that

$$H^5(A^{[2]},\mathbb{Z})\supset \pi_*(H^5(A\times A,\mathbb{Z}))\oplus\pi_*(H^3(\Delta,\mathbb{Z})).$$

Moreover by Künneth formula, we have:

$$\begin{aligned}H^5(A\times A,\mathbb{Z}) &= H^1(A,\mathbb{Z})\otimes H^4(A,\mathbb{Z})\oplus H^2(A,\mathbb{Z})\otimes H^3(A,\mathbb{Z})\\ &\oplus H^3(A,\mathbb{Z})\otimes H^2(A,\mathbb{Z})\oplus H^4(A,\mathbb{Z})\otimes H^1(A,\mathbb{Z}).\end{aligned}$$

As before, $\pi_*(H^5(A\times A,\mathbb{Z}))$ is primitive in $H^5(A^{[2]},\mathbb{Z})$. Moreover by (9):

$$l_2^5(\widetilde{A\times A}) = \operatorname{rk} H^5(A\times A,\mathbb{Z})^{\mathfrak{S}_2} = 28, \tag{10}$$

and

$$l_{1,+}^5(\widetilde{A\times A}) = \operatorname{rk} H^3(\Delta,\mathbb{Z})^{\mathfrak{S}_2} = 4,\quad \text{and}\quad l_{1,-}^5(\widetilde{A\times A}) = 0. \tag{11}$$

Lemma 8.6. *The lattice* $\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})\oplus H^5(\widetilde{A\times A},\mathbb{Z}))$ *has discriminant* 2^8.

Proof. By Definition-Proposition 1.7 2) and 3) of [16], (6) and (10):

$$\frac{H^3(\widetilde{A\times A},\mathbb{Z})\oplus H^5(\widetilde{A\times A},\mathbb{Z})}{H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus\left(H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\right)^{\perp}}=(\mathbb{Z}/2\mathbb{Z})^{l_2^3(\widetilde{A\times A})+l_2^5(\widetilde{A\times A})}.$$

Since $H^3(\widetilde{A\times A},\mathbb{Z})\oplus H^5(\widetilde{A\times A},\mathbb{Z})$ is an unimodular lattice, it follows that

$$\operatorname{discr} H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}=2^{l_2^3(\widetilde{A\times A})+l_2^5(\widetilde{A\times A})}.$$

Then by Lemma 2.18 3) of [16],

$$\operatorname{discr}\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2})=2^{l_2^3(\widetilde{A\times A})+l_2^5(\widetilde{A\times A})+\operatorname{rk}\left[H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\right]}.$$

Then by Proposition 1.6 of [16]:

$$\operatorname{discr}\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2})=2^{2\left(l_2^3(\widetilde{A\times A})+l_2^5(\widetilde{A\times A})\right)+l_{1,+}^3(\widetilde{A\times A})+l_{1,+}^5(\widetilde{A\times A})}.$$

Then by Lemma 2.17 and 2.3 of [16],

$$\operatorname{discr}\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})\oplus H^5(\widetilde{A\times A},\mathbb{Z}))=2^{l_{1,+}^3(\widetilde{A\times A})+l_{1,+}^5(\widetilde{A\times A})}=2^8.$$

□

The lattice $H^3(A^{[2]},\mathbb{Z})\oplus H^5(A^{[2]},\mathbb{Z})$ is unimodular. Hence:

$$\frac{H^3(A^{[2]},\mathbb{Z})\oplus H^5(A^{[2]},\mathbb{Z})}{\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})\oplus H^5(\widetilde{A\times A},\mathbb{Z}))}=(\mathbb{Z}/2\mathbb{Z})^4.$$

However, by Section 8.2, we know that $\pi_*(H^3(\widetilde{A\times A},\mathbb{Z}))=H^3(A^{[2]},\mathbb{Z})$. It follows

$$\frac{H^5(A^{[2]},\mathbb{Z})}{\pi_*(H^5(\widetilde{A\times A},\mathbb{Z}))}=(\mathbb{Z}/2\mathbb{Z})^4.$$

9 Nakajima operators for Hilbert schemes of points on surfaces

Let A be a smooth projective complex surface. Set $H:=H^*(A,\mathbb{Q})$. Let $A^{[n]}$ the Hilbert scheme of n points on the surface, *i.e.* the moduli space of finite subschemes of A of length n. $A^{[n]}$ is again smooth and projective of dimension $2n$, cf. [6].

9.1 The rational cohomology

Their rational cohomology can be described in terms of Nakajima's [20] operators. First consider the direct sum

$$\mathbb{H}:=\bigoplus_{n=0}^{\infty}H^*(A^{[n]},\mathbb{Q})$$

This space is bigraded by cohomological *degree* and the *weight*, which is given by the number of points n. The unit element in $H^0(A^{[0]},\mathbb{Q})\cong\mathbb{Q}$ is denoted by $|0\rangle$, called the *vacuum*. There are linear operators $\mathfrak{q}_m(a)$, for each $m\geq 1$ and $a\in H^*(A,\mathbb{Q})$, acting on $\mathbb{H}$ which have the following properties: They depend linearly on a, and if $a\in H^k(A,\mathbb{Q})$ is homogeneous, the operator $\mathfrak{q}_m(a)$ is bihomogeneous of degree $k+2(|m|-1)$ and weight m:

$$\mathfrak{q}_m(a):H^l(A^{[n]})\to H^{l+k+2(|m|-1)}(A^{[n+m]})$$

To construct them, first define incidence varieties $Z_m \subset A^{[n]} \times A \times A^{[n+m]}$ by

$$Z_m := \{(\xi, x, \xi') \,|\, \xi \subset \xi',\, \mathrm{supp}(\xi') - \mathrm{supp}(\xi) = mx\}\,.$$

Then $\mathfrak{q}_m(a)(\beta)$ is defined as the Poincaré dual of

$$pr_{3*}\left((pr_2^*(\alpha) \cdot pr_3^*(\beta)) \cap [Z_m]\right).$$

Every element in $\mathbb{H}$ can be decomposed uniquely as a linear combination of products of operators $\mathfrak{q}_m(a)$, acting on the vacuum.

To give the cup product structure of $\mathbb{H}$, define oprators $\mathfrak{G}(a)$ for $a \in H^*(A)$. Let $\Xi_n \subset A^{[n]} \times A$ be the universal subscheme. Then the action of $\mathfrak{G}(a)$ on $H^*(A^{[n]})$ is multiplication with the class

$$pr_{1*}\left(\mathrm{ch}(\mathcal{O}_{\Xi_n}) \cdot pr_2^*(\mathrm{td}(A) \cdot a)\right) \in H^*(A^{[n]}).$$

For $a \in H^k(A)$, we define $\mathfrak{G}_i(a)$ to be the component of $\mathfrak{G}(a)$ of cohomological degree $k + 2i$. A differential operator $\mathfrak{d}$ is given by $\mathfrak{G}_1(1)$. It means multiplication with the first Chern class of the tautological sheaf $pr_{1*}(\mathcal{O}_{\Xi_n})$.

In [12] and [13] there are various commutation relations between these operators, that allow to determine all multiplications in the cohomology of the Hilbert scheme. First of all, if X and Y are operators of degree d and d', their commutator is defined as

$$[X, Y] := XY - (-1)^{dd'} YX.$$

The integral on $A^{[n]}$ induces a bilinar form on $\mathbb{H}$: for classes α, $\beta \in H^*(A^{[n]})$ it is given by

$$(\alpha, \beta) \longmapsto \int_{A^{[n]}} \alpha \cdot \beta.$$

If X is a homogeneous linear operator of degree d and weight m, acting on $\mathbb{H}$, define its adjont $X^\dagger$ by

$$\int_{A^{[n+m]}} X(\alpha) \cdot \beta = (-1)^{d \deg(\alpha)} \int_{A^{[n]}} \alpha \cdot X^\dagger(\beta).$$

We put $\mathfrak{q}_0(a) := 0$ and for $m < 0$, $\mathfrak{q}_m(a) := (-1)^m \mathfrak{q}_{-m}(a)^\dagger$. Then define

$$\mathfrak{L}_m(a) := \begin{cases} \frac{1}{2} \sum\limits_{k \in \mathbb{Z}} \sum\limits_i \mathfrak{q}_k(a_{(1)}) \mathfrak{q}_{m-k}(a_{(2)}), & \text{if } m \neq 0, \\[2ex] \sum\limits_{k>0} \sum\limits_i \mathfrak{q}_k(a_{(1)}) \mathfrak{q}_{-k}(a_{(2)}), & \text{if } m = 0. \end{cases}$$

where $\sum_i a_{(1)} \otimes a_{(2)}$ is the push-forward of a along the diagonal $\tau_2 : A \to A \times A$ (in Sweedler notation). Then we have ([13, Thm. 2.16]):

$$[\mathfrak{q}_m(a), \mathfrak{q}_l(b)] = m \cdot \delta_{m+l} \cdot \int_A ab \tag{12}$$

$$[\mathfrak{L}_m(a), \mathfrak{q}_l(b)] = -m \cdot \mathfrak{q}_{m+l}(ab) \tag{13}$$

$$[\mathfrak{d}, \mathfrak{q}_m(a)] = m \cdot \mathfrak{L}_m(a) + \tfrac{m(|m|-1)}{2} \mathfrak{q}_m(Ka) \tag{14}$$

$$[\mathfrak{G}(a), \mathfrak{q}_1(b)] = \exp(\mathrm{ad}(\mathfrak{d}))(\mathfrak{q}_1(ab)) \tag{15}$$

Note (cf. [12, Thm. 3.8]) that this implies that

$$\mathfrak{q}_{m+1}(a) = \tfrac{(-1)^m}{m!} \,\mathrm{ad}^m([\mathfrak{d}, \mathfrak{q}_1(1)])(\mathfrak{q}_1(a)), \tag{16}$$

so there are two ways of writing an element of $\mathbb{H}$: As a linear combination of products of creation operators $\mathfrak{q}_m(a)$ or as a linear combination of products of the operators $\mathfrak{d}$ and $\mathfrak{q}_1(a)$. While the first one is more intuitive, the second one is more suitable for computing cup-products. Equations (14) and (16) permit now to switch between the two representations, using that

$$\mathfrak{d}|0\rangle = 0, \tag{17}$$

$$\mathfrak{L}_m(a)|0\rangle = \begin{cases} \frac{1}{2}\sum_{k=1}^{m-1}\sum_i \mathfrak{q}_k(a_{(1)})\mathfrak{q}_{m-k}(a_{(2)})|0\rangle, & \text{if } m > 1, \\ 0, & \text{if } m \leq 1. \end{cases} \tag{18}$$

(19)

Remark 9.1. We adopted the notation from [13], which differs from the conventions in [12]. Here is part of a dictionary:

Notation from [13]	Notation from [12]
operator of bidegree (w, d)	operator of bidegree $(w, d-2w)$
$\mathfrak{q}_m(a)$	$\mathfrak{p}_{-m}(a)$
$\mathfrak{L}_m(a)$	$-L_{-m}(a)$
$\mathfrak{G}(a)$	$a^{[\bullet]}$
$\mathfrak{d}$	∂
$\tau_{2*}(a)$	$-\Delta(a)$

By sending a subscheme in A to is support, we define a morphism

$$\rho : A^{[n]} \longrightarrow \mathrm{Sym}^n(A),$$

called the Hilbert–Chow morphism. The cohomology of $\mathrm{Sym}^n(A)$ is given by elements of the n-fold tensor power of $H^*(A)$ that are invariant under the action of the group of permutations $\mathfrak{S}_n$. A class in $H^*(A^{[n]}, \mathbb{Q})$ which can be written using only the operators $\mathfrak{q}_1$ comes from a pullback along ρ:

$$\mathfrak{q}_1(b_1)\cdots\mathfrak{q}_1(b_n)|0\rangle = \rho^*\left(\sum_{\pi\in\mathfrak{S}_n} \pm b_{\pi(1)}\otimes\ldots\otimes b_{\pi(n)}\right), \quad b_i \in H^*(A,\mathbb{Q}), \tag{20}$$

where signs arise from permuting factors of odd degrees. In particular,

$$\frac{1}{n!}\mathfrak{q}_1(b)^n|0\rangle = \rho^*\big(b\otimes\ldots\otimes b\big), \tag{21}$$

$$\frac{1}{(n-1)!}\mathfrak{q}_1(b)\,(\mathfrak{q}_1(1))^{n-1}\,|0\rangle = \rho^*\Big(b\otimes 1\otimes\ldots\otimes 1 \;+\; \ldots \;+\; 1\otimes\ldots\otimes 1\otimes b\Big). \tag{22}$$

Remark 9.2. With the notation from Section 3, we have that

$$H^*(\mathrm{Sym}^n(A), \mathbb{Q}) \cong \mathrm{SSym}^n(H^*(A,\mathbb{Q})).$$

Under this isomorphism the ring structure of $\mathrm{SSym}^n(H^*(A,\mathbb{Q}))$ corresponds to the cup product and the action of the operator $\mathfrak{q}_1(a)$ corresponds to the operation $a\diamond$.

9.2 On integral cohomology

For the study of integral cohomology, first note that if $\alpha \in H^*(A,\mathbb{Z})$ is an integral class, then $\mathfrak{q}_m(\alpha)$ maps integral classes to integral classes. Moreover, there is the following theorem:

Theorem 9.3. *[21] An operator is called integral if it maps integral classes in $\mathbb{H}$ to integral classes. The operator $\frac{1}{z_\lambda}\mathfrak{q}_\lambda(1)$ is integral. Let $\alpha \in H^2(A,\mathbb{Z})$ be monodromy equivalent to a divisor. Then the operator $\mathfrak{m}_\lambda(\alpha)$ is integral.*

Remark 9.4. If A is a projective torus, then the sublattice of divisors in $H^2(A,\mathbb{Z})$ (the Néron–Severi group) is not trivial. By [23, Thm. II], the group of monodromy actions spans the entire automorphism group of $H^2(A,\mathbb{Z})$. Since the lattice is even and contains two copies of the hyperbolic lattice, a theorem of Eichler [22, Prop. 3.7.3] states that the automorphism group of $H^2(A,\mathbb{Z})$ acts transitively on classes of the same norm. So every class can be mapped to a divisor by the action of a monodromy.

We set $a^{(k)} := \mathfrak{G}_k(a)(1) \in H^{\deg a+2k}(A^{[n]},\mathbb{Q})$.

Remark 9.5. Qin and Wang [21] conjecture that the above theorem is valid even without any restriction on $\alpha \in H^2(A,\mathbb{Z})$. The following proposition explicits this out for low degrees.

Proposition 9.6. *Assume that $H^*(A,\mathbb{Z})$ is free of torsion. Let $(a_i) \subset H^1(A,\mathbb{Z})$ and $(b_i) \subset H^2(A,\mathbb{Z})$ be bases of integral cohomology. Denote $a_i^* \in H^3(A,\mathbb{Z})$ resp. $b_i^* \in H^2(A,\mathbb{Z})$ the elements of the dual bases. Let x be the generator of $H^4(A,\mathbb{Z})$. Then the classes $\mathfrak{G}_0(a_i)1 = \frac{1}{(n-1)!}\mathfrak{q}_1(1)^{n-1}\mathfrak{q}_1(a_i)|0\rangle$ form a basis of $H^1(A^{[n]},\mathbb{Z})$ with dual basis $\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(x)^{n-1}|0\rangle$. Moreover, the following classes form a basis of $H^2(A^{[n]},\mathbb{Z})$:*

- $\mathfrak{G}_0(b_i)1 = \frac{1}{(n-1)!}\mathfrak{q}_1(1)^{n-1}\mathfrak{q}_1(b_i)|0\rangle$,
- $\mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j)1 = \frac{1}{(n-2)!}\mathfrak{q}_1(1)^{n-2}\mathfrak{q}_1(a_i)\mathfrak{q}_1(a_j)|0\rangle,\ i<j$,
- $\mathfrak{d}1 = \frac{1}{2(n-2)!}\mathfrak{q}_1(1)^{n-2}\mathfrak{q}_2(1)|0\rangle$. *We denote this class by δ.*

Their respective duals in $H^{2n-2}(A^{[n]},\mathbb{Z})$ are given by

- $\mathfrak{q}_1(b_i^*)\mathfrak{q}_1(x)^{n-1}|0\rangle$,
- $\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(x)^{n-2}|0\rangle,\ i<j$,
- $\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle$.

Proof. It is clear from the above theorem that these classes are all integral. Göttsche's formula [9, p. 35] gives the Betti numbers of $A^{[n]}$ in terms of the Betti numbers of A: $h^1(A^{[n]}) = h^1(A)$, and $h^2(A^{[n]}) = h^2(A) + \frac{h^1(A)(h^1(A)-1)}{2} + 1$. It follows that we have not forgotten any classes.

Next we have to show that the intersection matrix between these classes is in fact the identity matrix. Most of the entries can be computed easily using (20). For products involving δ (this is the action of $\mathfrak{d}$) or its dual, first observe that $\mathfrak{d}\mathfrak{q}_1(x)^m|0\rangle = 0$ and $\mathfrak{L}_1(a)\mathfrak{q}_1(x)^m|0\rangle = 0$ for every class a of degree at least 1. Then compute:

$$\begin{gathered}
\delta\cdot\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = \mathfrak{d}\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = 2\mathfrak{L}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = \mathfrak{q}_1(x)^n|0\rangle,\\
\mathfrak{d}\mathfrak{q}_1(b_i^*)\mathfrak{q}_1(x)^{n-1}|0\rangle = \mathfrak{L}_1(b_i^*)\mathfrak{q}_1(x)^{n-1}|0\rangle = 0,\\
\mathfrak{d}\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(x)^{n-2}|0\rangle = \left(\mathfrak{L}_1(a_j^*) + \mathfrak{q}_1(a_j^*)\mathfrak{d}\right)\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(x)^{n-2}|0\rangle =\\
= \left(-\mathfrak{q}_1(a_i^*)\mathfrak{L}_1(a_j^*) + \mathfrak{q}_1(a_j^*)\mathfrak{L}_1(a_i^*)\right)\mathfrak{q}_1(x)^{n-2}|0\rangle = 0,\\
\mathfrak{G}_0(b_i)\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = 0,\\
\mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j)\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = 0.
\end{gathered}$$

□

Proposition 9.7. *Let A be a complex torus of dimension 2. A basis of $H^*(A^{[2]},\mathbb{Z})$ is given by the classes*

0. $\frac{1}{2}\mathfrak{q}_1(1)^2|0\rangle$,
1. $\mathfrak{q}_1(1)\mathfrak{q}_1(a_i)|0\rangle$, *for a basis $(a_i) \subset H^1(A,\mathbb{Z})$,*
2. $\frac{1}{2}\mathfrak{q}_2(1)|0\rangle,\ \mathfrak{q}_1(1)\mathfrak{q}_1(u_i)|0\rangle$ *for a basis $(u_i) \subset H^2(A,\mathbb{Z})$,*
3. $\mathfrak{q}_2(a_i)|0\rangle,\ \mathfrak{q}_1(a_i)\mathfrak{q}_1(u_j)|0\rangle\ \mathfrak{q}_1(1)\mathfrak{q}_1(a_i^*)|0\rangle$,
4. $\mathfrak{q}_2(u_i)|0\rangle,\ \mathfrak{q}_1(a_i)\mathfrak{q}_1(a_j^*)|0\rangle,\ \mathfrak{q}_1(u_i)\mathfrak{q}_1(u_k)|0\rangle$ *for $i\le k$,*
5. $\frac{1}{2}\mathfrak{q}_2(a_i^*)|0\rangle,\ \mathfrak{q}_1(a_i^*)\mathfrak{q}_1(u_j)|0\rangle,\ \mathfrak{q}_1(x)\mathfrak{q}_1(a_i)|0\rangle$,
6. $\mathfrak{q}_2(x)|0\rangle,\ \mathfrak{q}_1(x)\mathfrak{q}_1(u_i)|0\rangle$,
7. $\mathfrak{q}_1(x)\mathfrak{q}_1(a_i^*)|0\rangle$,

8. $\mathfrak{q}_1(x)^2|0\rangle$.

Proof. One computes the intersection matrix of these objects under the Poincaré duality pairing and finds that the determinant is one. So it remains to show that all these classes are integral. By the analysis in [21] this is clear for all classes except those of the form $\frac{1}{2}\mathfrak{q}_2(a_i^*)|0\rangle \in H^5(A^{[2]},\mathbb{Z})$.

Evaluating the Poincaré duality pairing between degrees 3 and 5 gives:

$$\int \mathfrak{q}_2(a_i)|0\rangle \cdot \mathfrak{q}_2(a_i^*)|0\rangle = 2,$$

$$\int \mathfrak{q}_1(a_i)\mathfrak{q}_1(u_j)|0\rangle \cdot \mathfrak{q}_1(a_i^*)\mathfrak{q}_1(u_j^*)|0\rangle = 1,$$

$$\int \mathfrak{q}_1(1)\mathfrak{q}_1(a_i^*)|0\rangle \cdot \mathfrak{q}_1(x)\mathfrak{q}_1(a_i)|0\rangle = 1,$$

while the other pairings vanish. Therefore, one of $\mathfrak{q}_2(a_i)|0\rangle$ and $\mathfrak{q}_2(a_i^*)|0\rangle$ must be divisible by 2. We can interpret $\mathfrak{q}_2(a_i)|0\rangle \in H^3(A^{[2]},\mathbb{Z})$ and $\mathfrak{q}_2(a_i^*)|0\rangle \in H^5(A^{[2]},\mathbb{Z})$ as classes concentrated on the exceptional divisor, that is, as elements of $\pi_* i_* H^*(E,\mathbb{Z})$. Indeed, the pushforward of a class $a \otimes 1 \in H^k(E,\mathbb{Z})$ is given by

$$\pi_* i_*(a \otimes 1) = \mathfrak{q}_2(a)|0\rangle \in H^{k+2}(A^{[n]},\mathbb{Z}).$$

When pushing forward to the Hilbert scheme, the only possibility to get a factor 2 is in degree 5, by Proposition 8.1. □

Part II

The Generalized Kummer fourfold

10 Generalized Kummer varieties and the morphism to the Hilbert scheme

Definition 10.1. Let A be a complex projective torus of dimension 2 and $A^{[n]}$, $n \geq 1$, the corresponding Hilbert scheme of points. Denote $\Sigma : A^{[n]} \to A$ the summation morphism, a smooth submersion that factorizes via the Hilbert–Chow morphism : $A^{[n]} \xrightarrow{\rho} \mathrm{Sym}^n(A) \xrightarrow{\sigma} A$. Then the generalized Kummer variety $K_{n-1}(A)$ is defined as the fiber over 0:

$$\begin{array}{ccc} K_{n-1}(A) & \xrightarrow{\theta} & A^{[n]} \\ \big\downarrow & & \big\downarrow \Sigma \\ \{0\} & \longrightarrow & A \end{array} \tag{23}$$

Our first objective is to collect some information about this pullback diagram. Recall Notation 5.6.

Proposition 10.2. *Let* $\alpha_i := \frac{1}{(n-1)!}\mathfrak{q}_1(1)^{n-1}\mathfrak{q}_1(a_i)|0\rangle = \mathfrak{G}_0(a_i)1$. *The class of the Poincaré dual of* $K_{n-1}(A)$ *in* $H^4(A^{[n]},\mathbb{Z})$ *is given by*

$$[K_{n-1}(A)] = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4.$$

Proof. Since the generalized Kummer variety is the fiber over a point, its Poincaré dual must be the pullback of $x \in H^4(A)$ under Σ. But $\Sigma^*(x) = \Sigma^*(a_1) \cdot \Sigma^*(a_2) \cdot \Sigma^*(a_3) \cdot \Sigma^*(a_4)$, so we have to verify that $\Sigma^*(a_i) = \alpha_i$. To do this, we want to use the decomposition $\Sigma = \sigma\rho$. The pullback along σ of a class $a \in H^1(A,\mathbb{Q})$ on $H^1(\mathrm{Sym}^n(A),\mathbb{Q})$ is given by $a \otimes 1 \otimes \cdots \otimes 1 + \ldots + 1 \otimes \cdots \otimes 1 \otimes a$. It follows from (22) that $\Sigma^*(a_i) = \frac{1}{(n-1)!}\mathfrak{q}_1(1)^{n-1}\mathfrak{q}_1(a_i)|0\rangle$. □

The morphism θ induces a homomorphism of graded rings

$$\theta^* : H^*(A^{[n]}) \longrightarrow H^*(K_{n-1}(A)) \tag{24}$$

and on the image of θ^*, the integral over $K_{n-1}(A)$ can be written as follows:

$$\int_{K_{n-1}(A)} \theta^*(\alpha) \;=\; \int_{A^{[n]}} [K_{n-1}(A)] \cdot \alpha. \tag{25}$$

Proposition 10.3. *The kernel of θ^* is equal to the annihilator of $[K_{n-1}(A)]$.*

Proof. Assume $\alpha \in \ker(\theta^*)$. Then, for all $\beta \in H^*(A^{[n]})$, we have

$$\int_{A^{[n]}} [K_{n-1}(A)] \cdot \alpha \cdot \beta = \int_{K_{n-1}(A)} \theta^*(\alpha) \cdot \theta^*(\beta) = 0.$$

Since the Poincaré pairing on $A^{[n]}$ is non-degenerate, this implies $[K_{n-1}(A)] \cdot \alpha = 0$.

Conversely, suppose that $[K_{n-1}(A)] \cdot \alpha = 0$. Then, for all $\beta \in H^*(A^{[n]})$, we have

$$\int_{K_{n-1}(A)} \theta^*(\alpha) \cdot \theta^*(\beta) = 0,$$

so $\theta^*(\alpha) \in \mathrm{Im}(\theta^*) \cap \mathrm{Im}(\theta^*)^{\perp}$. The Poincaré pairing on $K_{n-1}(A)$ is non-degenerate when restricted to $\mathrm{Im}(\theta^*)$. To see this, observe that the dual element of $\theta^*(\alpha)$ in $\mathrm{Im}(\theta^*)$ is given by $\theta^*(\alpha^*)$, where α^* is the Poincaré dual of $[K_{n-1}(A)] \cdot \alpha$. So it follows that $\mathrm{Im}(\theta^*) \cap \mathrm{Im}(\theta^*)^{\perp} = 0$. $\square$

Proposition 10.4. *The annihilator of $[K_{n-1}(A)]$ in $H^*(A^{[n]}, \mathbb{Q})$ is the ideal generated by $H^1(A^{[n]})$.*

Proof. Set $H = H^*(A, \mathbb{Q})$ and consider the exact sequence of H-modules

$$0 \longrightarrow H^{\geq 1}(A, \mathbb{Q}) \longrightarrow H \xrightarrow{x\cdot} H.$$

It is clear that $H^{\geq 1}(A, \mathbb{Q})$ is the ideal in H generated by $H^1(A, \mathbb{Q})$. Now denote $J^{(n)}$ the ideal generated by $H^1(\mathrm{Sym}^n(A), \mathbb{Q})$ in $H^*(\mathrm{Sym}^n(A), \mathbb{Q}) \cong \mathrm{SSym}^n(H)$. By the freeness result of Lemma 3.5, tensoring with $\mathrm{SSym}^n(H)$ yields another exact sequence of H-modules

$$0 \longrightarrow J^{(n)} \longrightarrow \mathrm{SSym}^n(H) \xrightarrow{\sigma(x)\cdot} \mathrm{SSym}^n(H).$$

Now let $\mathfrak{H}$ be the operator algebra spanned by products of $\mathfrak{d}$ and $\mathfrak{q}_1(a)$ for $a \in H^*(A)$. Let $\mathfrak{C}$ be the graded commutative subalgebra of $\mathfrak{H}$ generated by $\mathfrak{q}_1(a)$ for $a \in H^*(A)$. The action of $\mathfrak{H}$ on $|0\rangle$ gives $\mathbb{H}$ and the action of $\mathfrak{C}$ on $|0\rangle$ gives $\rho^*(H^*(\mathrm{Sym}^n(A), \mathbb{Q})) \cong \mathrm{SSym}^n(H)$. By sending $\mathfrak{d}$ to the identity, we define a linear map $c : \mathfrak{H} \to \mathfrak{C}$. Denote $J^{[n]}$ the ideal generated by $H^1(A^{[n]}, \mathbb{Q})$ in $H^*(A^{[n]}, \mathbb{Q})$. We claim that for every $\mathfrak{y} \in \mathfrak{H}$:

$$\mathfrak{y}|0\rangle \in J^{[n]} \Leftrightarrow c(\mathfrak{y})|0\rangle \in J^{[n]}.$$

To see this, we remark that $H^1(A^{[n]}, \mathbb{Q}) \cong H^1(A, \mathbb{Q})$ and the multiplication with a class in $H^1(A^{[n]}, \mathbb{Q})$ is given by the operator $\mathfrak{G}_0(a)$ for some $a \in H^1(A, \mathbb{Q})$. Due to the fact that $\mathfrak{d}$ is also a multiplication operator (of degree 2), $\mathfrak{G}_0(a)$ commutes with $\mathfrak{d}$. It follows that for $\mathfrak{y} = \mathfrak{G}_0(a)\mathfrak{r}$ we have $c(\mathfrak{y}) = \mathfrak{G}_0(a)c(\mathfrak{r})$.

Now denote $\mathfrak{k}$ the multiplication operator with the class $[K_{n-1}(A)]$. We have: $[\mathfrak{k}, \mathfrak{d}] = 0$. Now let $y \in H^*(A^{[n]}, \mathbb{Q})$ be a class in the annihilator of $[K_{n-1}(A)]$. We can write $y = \mathfrak{y}|0\rangle$ for a $\mathfrak{y} \in \mathfrak{H}$. Choose $\tilde{y} \in \mathrm{SSym}^n(H)$ in a way that $\rho^*(\tilde{y}) = c(\mathfrak{y})|0\rangle$. Then we have:

$$0 = [K_{n-1}(A)] \cdot y = \mathfrak{k}\,\mathfrak{y}|0\rangle = \mathfrak{k}\,c(\mathfrak{y})|0\rangle = \rho^*(\sigma^*(x) \cdot \tilde{y}).$$

Since ρ^* is injective, $\tilde{y}$ is in the annihilator of $\sigma^*(x)$, so $\tilde{y} \in J^{(n)}$. It follows that $c(\mathfrak{y})|0\rangle$ and y are in the ideal generated by $H^1(A^{[n]}, \mathbb{Q})$. $\square$

We recall Theorem 2 of [24].

Theorem 10.5. *The cohomology of $K_{n-1}(A)$ is torsion free.*

Theorem 10.6. *[1] $K_{n-1}(A)$ is a irreducible holomorphically symplectic manifold. In particular, it is simply connected and the canonical bundle is trivial.*

So $H^2(K_{n-1}(A), \mathbb{Z})$ admits an integer-valued nondegenerated quadratic form (called Beauville–Bogomolov form) q which gives $H^2(K_{n-1}(A), \mathbb{Z})$ the structure of a lattice isomorphic to $U^{\oplus 3} \oplus \langle -2n \rangle$, for $n \geq 3$. We have the following formula for $\alpha \in H^2(K_{n-1}(A), \mathbb{Z})$:

$$\int_{K_{n-1}(A)} \alpha^{2n-2} = n \cdot (2n-3)!! \cdot q(\alpha)^{n-1} \tag{26}$$

Proposition 10.7. *Assume $n \geq 3$. Then θ^* is surjective on $H^2(A^{[n]}, \mathbb{Z})$.*

Proof. By [1, Sect. 7], $\theta^* : H^2(A^{[n]}, \mathbb{C}) \to H^2(K_{n-1}(A), \mathbb{C})$ is surjective. Then the idea is to prove that the lattice structure of $\operatorname{Im} \theta^*$ is the same as of $H^2(K_{n-1}(A))$. We use two formulas in [5, pp. 8–11]. Let $b \in H^2(A, \mathbb{Z})$ and set $\alpha = \frac{1}{(n-1)!} \mathfrak{q}_1(1)^{n-1} \mathfrak{q}_1(b)|0\rangle \in H^2(A^{[n]}, \mathbb{Z})$. Then

$$\int_{A^{[n]}} \alpha^{2n} = \binom{2n}{2} \frac{\int_A b^2}{n^2} \int_{K_{n-1}(A)} \theta^* \alpha^{2n-2} \tag{27}$$

By Proposition (TODO), the left hand side of this equation equals $(2n-1)!! \cdot \left(\int_a b^2\right)^n$. By (26), the right hand side gives $(2n-1)!! \cdot \left(\int_A b^2\right) \cdot q(\alpha)^{n-1}$. So we get $\int_A b^2 = q(\alpha)$, giving the set of all α a lattice structure isomorphic to $H^2(A, \mathbb{Z})$. Secondly, we must show that for $\delta =$ half of the exceptional divisor: $q(\theta^* \delta) = -2n$. But this follows now from Proposition 1 in [5]. □

Proposition 10.8. *Let $\gamma \in H^*(A^{[n]}, \mathbb{Q})$ be a class with $[K_{n-1}(A)] \cdot \gamma \in H^*(A^{[n]}, \mathbb{Z})$. Then $\theta^*(\gamma) \in H^*(K_{n-1}(A), \mathbb{Z})$.*

11 Cohomology of the Generalized Kummer fourfold

Now we come to the special case $n = 3$, so we study $K_2(A)$, the Generalized Kummer fourfolds.

Proposition 11.1. *The Betti numbers of $K_2(A)$ are:* 1, 0, 7, 8, 108, 8, 7, 0, 1.

Proof. This follows from Göttsche's formula [9, page 49]. □

Notation 11.2. We give the following names for classes in $H^2(K_2(A), \mathbb{Z})$:

$$\begin{aligned} u_1 &:= j(a_1 a_2), & v_1 &:= j(a_1 a_3), & w_1 &:= j(a_1 a_4), \\ u_2 &:= j(a_3 a_4), & v_2 &:= j(a_4 a_2), & w_2 &:= j(a_2 a_3), \end{aligned}$$

Further, we set $e := \theta^*(\delta)$. These elements form a basis of $H^2(K_2(A), \mathbb{Z})$ with the following intersection relations under the Beauville-Bogomolov form:

$$q(u_1, u_2) = 1, \qquad q(v_1, v_2) = 1, \qquad q(w_1, w_2) = 1, \qquad q(e, e) = -6,$$

and all other pairs of basis elements are orthogonal.

By means of the morphism θ^*, we may express part of the cohomology of $K_2(A)$ in terms of Hilbert scheme cohomology. We have seen in Proposition 10.7 that θ^* is surjective for degree 2 and (by duality) also in degree 6. The next proposition shows that this also holds true for odd degrees.

Proposition 11.3. *A basis of $H^3(A^{[3]}, \mathbb{Z})$ is given by:*

$$\frac{1}{2}\theta^* \left(\mathfrak{q}_1(1)^2\mathfrak{q}_1(a_i^*)|0\rangle\right), \tag{28}$$

$$\theta^* \left(\mathfrak{q}_2(a_i)|0\rangle\right). \tag{29}$$

and a basis of $H^5(A^{[3]}, \mathbb{Z})$ is given by:

$$\frac{1}{2}\theta^* \left(\mathfrak{q}_1(1)\mathfrak{q}_2(a_i^*)|0\rangle\right), \tag{30}$$

$$\frac{2}{3}\theta^* \left(\mathfrak{G}_2(a_i)1\right). \tag{31}$$

Proof. We claim that The classes (28) are Poincaré dual to (30) and the classes (29) are Poincaré dual to (31), so it remains to show, that all of them are integral. By [21], (28) and (29) are integral. By Proposition 9.7, $\frac{1}{2}\mathfrak{q}_2(a_i^*)|0\rangle$ is integral. If the operator $\mathfrak{q}_1(1)$ is applied, we get again an integral class, by [21, Lemma 3.3].

Further, $2\mathfrak{G}_2(a_i)1$ is integral and $[K_2(A)] \cdot 2\mathfrak{G}_2(a_i)1$ is divisible by 3. By Proposition 10.8, $\frac{2}{3}\theta^* \left(\mathfrak{G}_2(a_i)1\right)$ is integral. □

Let us summarize our results on θ^*:

Theorem 11.4. *The homomorphism $\theta^* : H^*(A^{[3]}, \mathbb{Q}) \to H^*(K_2(A), \mathbb{Q})$ of graded rings is surjective in every degree except 4. Moreover, the image of $H^4(A^{[3]}, \mathbb{Q})$ is equal to $\operatorname{Sym}^2(H^2(K_2(A), \mathbb{Q}))$. The kernel of θ^* is the ideal generated by $H^1(A^{[3]}, \mathbb{Q})$.*

12 Middle cohomology

The middle cohomology $H^4(K_2(A), \mathbb{Z})$ has been studied by Hassett and Tschinkel in [10]. Let us first recall some of their results.

Notation 12.1. For each $\tau \in A$, denote W_τ the Briançon subscheme of $A^{[3]}$ supported enitrely at the point τ. If $\tau \in A[3]$ is a point of three-torsion, W_τ is actually a subscheme of $K_2(A)$. We will also use the symbol W_τ for the corresponding class in $H^4(K_2(A), \mathbb{Z})$. Further, set

$$W := \sum_{\tau \in A[3]} W_\tau.$$

For $p \in A$, denote Y_p the locus of all $\{x, y, p\}$ in $K_2(A)$. The corresponding class $Y_p \in H^4(K_2(A), \mathbb{Z})$ is independent of the choice of the point p. Then set $Z_\tau := Y_p - W_\tau$ and denote Π the lattice generated by all Z_τ, $\tau \in A[3]$.

Proposition 12.2. *The class W can be written with the help of the square of half the diagonal as*

$$W = 9Y_p + e^2.$$

The second Chern class is given by

$$c_2(K_2(A)) = \frac{1}{3} \sum_{\tau \in A[3]} Z_\tau = \frac{1}{3}(72Y_p - e^2).$$

Part III

A quotient

13 Calculation of the Beauville-Bogomolov form

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