PLANES IN SYMPLECTIC VECTOR SPACES

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Let V be a symplectic vector space of dimension $n \in 2\mathbb{N}$ over a field F with a nondegenerate symplectic form $\omega : \Lambda^2 V \to F$. A line is a one-dimensional subspace ov V, a plane is a two-dimensional subspace of V. A plane $P \subset V$ is called isotropic, if $\omega(x, y) = 0$ for any $x, y \in P$, otherwise non-isotropic. The symplectic group $\operatorname{Sp} V$ is the set of all linear maps $\phi : V \to V$ with the property $\omega(\phi(x), \phi(y)) = \omega(x, y)$ for all $x, y \in V$.

Proposition 0.1. *The symplectic group* $\operatorname{Sp} V$ *acts transitively on the set of non-isotropic planes as well as on the set of isotropic planes.*

Proof. Given two planes P_1 and P_2, we may choose vectors v_1, v_2, w_1, w_2 such that v_1, v_2 span P_1, w_1, w_2 span P_2 and $\omega(u_1, u_2) = \omega(w_1, w_2)$. We complete $\{v_1, v_2\}$ as well as $\{w_1, w_2\}$ to a symplectic basis of V. Then define $\phi(v_1) = w_1$ and $\phi(v_2) = w_2$. It is now easy to see that the definition of ϕ can be extended to the remaining basis elements to give a symplectic morphism. $\square$

Remark 0.2. If F is the field with two elements, then the set of planes in V can be identified with the set $\{\{x, y, z\} \mid x, y, z \in V\backslash\{0\},\ x + y + z = 0\}$. Observe that for such a $\{x, y, z\}$, $\omega(x, y) = \omega(x, y) = \omega(y, x)$ and this value gives the criterion for isotropy.

From now on we assume that F is finite of cardinality q.

Proposition 0.3.

$$\text{(1)} \qquad \textit{The number of lines in } V \textit{ is } \frac{q^n - 1}{q - 1},$$

$$\text{(2)} \qquad \textit{the number of planes in } V \textit{ is } \frac{(q^n - 1)(q^{n-1} - 1)}{(q^2 - 1)(q - 1)},$$

$$\text{(3)} \qquad \textit{the number of isotropic planes in } V \textit{ is } \frac{(q^n - 1)(q^{n-2} - 1)}{(q^2 - 1)(q - 1)},$$

$$\text{(4)} \qquad \textit{the number of non-isotropic planes in } V \textit{ is } \frac{q^{n-2}(q^n - 1)}{q^2 - 1}.$$

Proof. A line ℓ in V is determined by a nonzero vector. There are $q^n - 1$ nonzero vectors in V and $q - 1$ nonzero vectors in ℓ. A plane P is determined by a line $\ell_1 \subset V$ and a unique second line $\ell_2 \in V/\ell$. We have $\frac{q^2-1}{q-1}$ lines in P. The number of planes is therefore

$$\frac{\frac{q^n-1}{q-1} \cdot \frac{q^{n-1}-1}{q-1}}{\frac{q^2-1}{q-1}} = \frac{(q^n - 1)(q^{n-1} - 1)}{(q^2 - 1)(q - 1)}.$$

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For an isotropic plane we have to choose the second line from $\ell_1^\perp$. This is a space of dimension $n-2$, hence the formula. The number of non-isotropic planes is the difference of the two previous numbers. $\square$