Integral cohomology of the Generalized Kummer fourfold

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Abstract

We provide a basis of the integral cohomology of the Generalized Kummer fourfold, using Hilbert scheme cohomology, Nakajima operators and tools developed by Hassett and Tschinkel in [18]. Then we apply our results to a IHS manifold with singularities, obtained by a partial resolution of the Generalized Kummer quotiented by a symplectic involution. We calculate the Beauville–Bogomolov form of this new variety, presenting the first example of such a form that is odd.

1 Introduction

1.1 Context and main results

In algebraic geometry compact hyperkähler manifolds became an important objects of study, these last years, after the fundamental results from Beauville [1] and Huybrechts [19]. Among all the developments concerning this field, integral cohomology has an inescapable role. First of all due to the Beauville–Bogomolov form which is the unique non-degenerated symmetric integral and primitive bilinear pairing on the second cohomology group. This form endows the second cohomology group with a lattice structure enabling to use all lattice theory as a fundamental tool omnipresent in all the last developments. As examples, we can cite works on classifications of automorphisms [35], [36], [3] or the important survey of Markman [26] with results on the Kähler cone and the monodromy. In a more modest term, the fourth integral cohomology group is also quite useful. As examples, we can underline Theorem 1.2 of [6] providing formulas which apply for the classification of automorphism on hyperkähler manifolds of $K3^{[2]}$-type, in particularly used in [3]; furthermore Theorem 1.10 of [28] provides a description of the monodromy group of the hyperkähler manifolds of $K3^{[n]}$-type; we can also cite [31], where the second author provide the Beauville–Bogomolov lattice of the Markushevich–Tikhomirov varieties constructed in [25]. Taking X a hyperkähler manifolds of $K3^{[2]}$-type, in all these works a description of $\frac{H^4(X,\mathbb{Z})}{\mathrm{Sym}^2(H^2(X,\mathbb{Z}))}$ was essential.

Until now, no complete description of the integral cohomology of the generalized Kummer fourfold was existing. In particular the relation between the fourth cohomology group and the symmetric power of the second cohomology group was not know. For all reasons mentioned above, it appeared to us that it was an interesting lack to fill. It is the main result of this paper (Theorem **??**):

Theorem 1.1. *A basis of $H^*(K_2(A),\mathbb{Z})$ is given by the following classes: TODO*

As an illustration of this result we propose one application which is a generalization of [31] related to irreducible symplectic V-manifolds. A V-manifold is an algebraic variety with at worst finite quotient singularities. A V-manifold will be called symplectic if its nonsingular locus is endowed with an everywhere nondegenerate holomorphic 2-form. A symplectic V-manifold will be called irreducible if it is complete, simply connected, and if the holomorphic 2-form is unique up to $\mathbb{C}^*$. Such varieties are good candidate to generalize the short list of compact hyperkähler manifolds, since some aspects of the theory was already generalized in [40] and [30], for instance the Beauville–Bogomolov form, the local Torelli theorem and the Fujiki formula.

Concretely, let X be an irreducible symplectic fourfold of Kummer type and ι a symplectic involution on X. By results of Mongardi, Tari and Wandel [36] and [48], it can be established that the fixed locus of ι is the union of 36 points and a K3 surface Z_0. Then the singular locus of $K := X/\iota$ is the union of a K3 surface and 36 points. A more interesting variety to consider is the partial resolution K' of K obtained by blowing up the image of Z_0. By Section 2.3 and Lemma 1.2 of [14], this variety is again an irreducible symplectic V-manifold. Something interesting to note, is that the moduli space $\mathcal{M}_{K'}$ of irreducible symplectic V-manifolds deformation equivalent to K' will be of dimension 6 (see Proposition 15.19). However, the space of the V-manifold in $\mathcal{M}_{K'}$ coming from a partial resolution of the quotient X/ι are of dimension 5. It means that $\mathcal{M}_{K'}$ mostly contains V-manifolds which are completely unknown, not related to a quotient of some smooth irreducible symplectic manifold. By the local Torelli theorem of [40], the moduli space $\mathcal{M}_{K'}$ will be related to the Beauville–Bogomolov lattice that we provide in Theorem 15.3.

Theorem 1.2. *Let X be an irreducible symplectic fourfold of Kummer type and ι a symplectic involution on X. Let Z_0 be the K3 surface which is in the fixed locus of ι. We denote $K = X/\iota$ and K' the partial resolution of singularities of K obtained by blowing up the image of Z_0. Then the Beauville–Bogomolov lattice $H^2(K',\mathbb{Z})$ is isomorphic to $U(3)^3 \oplus \begin{pmatrix} -5 & -4 \\ -4 & -5 \end{pmatrix}$, and the Fujiki constant $c_{K'}$ is equal to 8.*

We can remark that it is the first example of Beauville–Bogomolov form which is not even.

Another illustration of Theorem 1.1 is Theorem 14.1 where the knowledge on the third integral cohomology group of a generalized Kummer fourfold X (see Corollary 11.3), allow us to end the classification of symplectic involutions on X as a corollary of the lattice classification by Mongardi Tari and Wandel in [36].

1.2 Overview on the results

The article is divided in 3 parts.

(I) In the first part, we recall some basic tools which will be used in all the paper. There are recalls on basic lattices considerations (Section 2), on super algebra (Section 3), on abelian surface (Section 5), on integral cohomology tools (Section 6) and on Nakajima operators (Section 8). We also provide two new results which will be used to prove Theorem 1.1. Let A be a smooth compact surface with torsion free cohomology, Proposition 7.1 describes $H^{2*+1}(A^{[2]},\mathbb{Z})$. Now, let A be an abelian surface, Proposition 9.6 provides an integral basis of $H^*(A^{[2]},\mathbb{Z})$ in terms of Nakajima operators.

(II) In part II, we prove Theorem 1.1. Let A be an abelian surface and $\theta : K_2(A) \to A^{[3]}$ the natural embedding, in Theorem 13.1, we also provide a description of $\theta^* : H^*(A^{[3]},\mathbb{Z}) \to H^*(K_2(A),\mathbb{Z})$.

(III) In part III, Section 14, we first recall and prove results about symplectic involution on $K_2(A)$ (Theorem 14.1). Then Section 15 is devoted to prove Theorem 1.2.

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Part I
Preliminaries

2 Lattices

A reference for this section is Chapter 8.2.1 of [9].

Definition 2.1. By a lattice L we mean a free $\mathbb{Z}$–module of finite rank, equipped with a non–degenerate, integer–valued symmetric bilinear form, denoted by B or $(\ ,\ )$. By a homomorphism or embedding $L \subset M$ of lattices we mean a map : $L \to M$ that preserves the bilinear forms on L and M respectively. It is automatically injective. We always have the injection of a lattice L into its dual space $L^* := \mathrm{Hom}(L, \mathbb{Z})$, given by $x \mapsto (x,\ )$. A lattice is called unimodular, if this injection is an isomorphism, *i.e.* if it is surjective. By tensoring with $\mathbb{Q}$, we can interpret L as well as L^* as a discrete subset of the $\mathbb{Q}$–vector space $L \otimes \mathbb{Q}$. Note that this gives a kind of lattice structure to L^*, too, but the symmetric bilinear form on L^* may now take rational coefficients.

If $L \subset M$ is an embedding of lattices of the same rank, then the index $|M : L|$ of L in M is defined as the order of the finite group M/L. There is a chain of embeddings $L \subset M \subset M^* \subset L^*$ with $|L^* : M^*| = |M : L|$.

The quotient L^*/L is called the discriminant group and denoted A_L. The index of L in L^* is called $\mathrm{discr}\, L$, the discriminant of L. Choosing a basis $(x_i)_i$ of L, we may express $\mathrm{discr}\, L$ as the absolute value of the determinant of the so–called Gram matrix G of L, which is defined by $G_{ij} := (x_i, x_j)$. L is unimodular, iff $\det G = \pm 1$.

The lattice L is called odd, if there exists a $v \in L$, such that (v, v) is odd, otherwise it is called even. If the map $v \mapsto (v, v)$ takes both negative and positive values on L, the lattice is called indefinite.

Example 2.2. Up to isomorphism there is a unique even unimodular lattice of rank two. It is called the hyperbolic lattice U. Its Gram matrix is given by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Proposition 2.3. *Let M be a unimodular lattice. Let $L \subset M$ be a sublattice of the same rank. Then $|M : L|$ equals $\sqrt{\mathrm{discr}\, L}$.*

Proof. Since M is unimodular, $|L^* : M| = |L^* : M^*| = |M : L|$ and therefore $|L^* : L| = |L^* : M||M : L| = |M : L|^2$. □

An embedding $L \subset M$ is called primitive, if the quotient M/L is free. We denote by $L^\perp$ the orthogonal complement of L within M. Since an orthogonal complement is always primitive, the double orthogonal complement $L^{\perp\perp}$ is a primitively embedded overlattice of L. It is clear that $\mathrm{discr}(L^{\perp\perp})$ divides $\mathrm{discr}\, L$.

Proposition 2.4. *Let $L \subset M$ be an embedding of lattices. Then the order of the torsion part of M/L divides* $\mathrm{discr}\, L$.

Proof. The torsion part is the index of $M/(L^{\perp\perp})$ in M/L. But this is equal to $|L^{\perp\perp} : L| = |L^* : (L^{\perp\perp})^*|$ and clearly divides $|L^* : L|$. □

Proposition 2.5. *Let M be unimodular. Let $L \subset M$ be a primitive embedding. Then* $\mathrm{discr}\, L = \mathrm{discr}\, L^\perp$.

Proof. Consider the orthogonal projection : $M \otimes \mathbb{Q} \to L \otimes \mathbb{Q}$. Its restriction to M has kernel equal to $L^\perp$ and image in L^*. Hence we have an embedding of lattices $M/L^\perp \subset L^*$. Quotienting by L, we get an injective map : $M/(L \oplus L^\perp) \to L^*/L$. Now by Proposition 2.3, $\sqrt{\mathrm{discr}(L)\,\mathrm{discr}(L^\perp)} = |M : (L \oplus L^\perp)| \leq |L^* : L| = \mathrm{discr}\, L$. So we get $\mathrm{discr}\, L^\perp \leq \mathrm{discr}\, L$. Exchanging the roles of $L = L^{\perp\perp}$ and $L^\perp$ gives the inequality in the opposite direction. □

Corollary 2.6. *Let $L \subset M$ be an embedding of lattices with unimodular M. Let n be the order of the torsion part of M/L. Then* $\mathrm{discr}\, L^\perp = \mathrm{discr}\, L^{\perp\perp} = \frac{1}{n^2}\, \mathrm{discr}\, L$.

Example 2.7. Given a free $\mathbb{Z}$-module L with the structure of a commutative ring and a linear form $I : V \to \mathbb{Z}$, the setting $\langle v, w\rangle = I(vw)$ defines a bilinear form giving L the structure of a lattice if it is non-degenerate. This is the case in topology: For a compact complex manifold X of dimension d Poincaré duality induces a non-degenerate bilinear form on $H^d(X, \mathbb{Z})$:

$$\langle \alpha, \beta\rangle = \int_X \alpha\beta.$$

This unimodular lattice will be referred to as the Poincaré lattice.

3 Super algebras

Let us recall some material on super algebras, which will be useful in Sections 8 and 10 to understand the cohomology structure of the Hilbert schemes of points on abelian surfaces.

Definition 3.1. A super vector space V over a field k is a vector space with a $\mathbb{Z}/2\mathbb{Z}$-graduation, that is a decomposition

$$V = V^+ \oplus V^-,$$

called the even and the odd part of V. Elements of V^+ are called homogeneous of even degree, elements of V^- are called homogeneous of odd degree. The degree of a homogeneous element v is denoted by $|v| \in \mathbb{Z}/2\mathbb{Z}$. Direct sum and tensor product of two super vector spaces V and W yield again super vector spaces:

$$\begin{aligned} (V\oplus W)^+ &= V^+ \oplus W^+, & (V\oplus W)^- &= V^- \oplus W^-, \\ (V\otimes W)^+ &= (V^+\otimes W^+)\oplus(V^-\otimes W^-), & (V\otimes W)^- &= (V^+\otimes W^-)\oplus(V^-\otimes W^+). \end{aligned}$$

Definition 3.2. A superalgebra R is a unital associative k-algebra which carries a super vector space structure. Define the supercommutator by setting for homogeneous elements $u, v \in R$:

$$[u,v] := uv - (-1)^{|u||v|}vu.$$

R is called supercommutative, if $[u,v]=0$ for all $u,v\in R$. Note that a graded commutative algebra $R = \bigoplus_n R^n$ is supercommutative in a natural way, by setting $R^+ = \bigoplus_{n \text{ even}} R^n$, $R^- = \bigoplus_{n \text{ odd}} R^n$.

For a supercommutative algebra R, the tensor power $R^{\otimes n}$ is again a supercommutative algebra, if we set for the product:

$$(u_1\otimes\cdots\otimes u_n)\cdot(v_1\otimes\cdots\otimes v_n) = (-1)^{\sum\limits_{i>j}|u_i||v_j|} u_1v_1\otimes\cdots\otimes u_nv_n.$$

Definition 3.3. Let V be a super vector space over k and $n\geq 0$. Then the supersymmetric power $\mathbb{S}\mathrm{ym}^n(V)$ of V is a super vector space, given by

$$\mathbb{S}\mathrm{ym}^n(V) = \bigoplus_{p+q=n} \mathrm{Sym}^p(V^+)\otimes\Lambda^q(V^-),$$

$$\mathbb{S}\mathrm{ym}^n(V)^+ = \bigoplus_{\substack{p+q=n\\ q \text{ even}}} \mathrm{Sym}^p(V^+)\otimes\Lambda^q(V^-), \qquad \mathbb{S}\mathrm{ym}^n(V)^- = \bigoplus_{\substack{p+q=n\\ q \text{ odd}}} \mathrm{Sym}^p(V^+)\otimes\Lambda^q(V^-).$$

The supersymmetric algebra $\mathbb{S}\mathrm{ym}^*(V) := \bigoplus_n \mathbb{S}\mathrm{ym}^n(V)$ on V is a supercommutative algebra over k, where the product of two elements $s\otimes e\in\mathrm{Sym}^p(V^+)\otimes\Lambda^q(V^-)$ and $s'\otimes e'\in\mathrm{Sym}^{p'}(V^+)\otimes\Lambda^{q'}(V^-)$ is given by

$$(s\otimes e)\diamond(s'\otimes e') = (ss')\otimes(e\wedge e') \quad \in \mathrm{Sym}^{p+p'}(V^+)\otimes\Lambda^{q+q'}(V^-).$$

Remark 3.4. The supersymmetric power $\mathbb{S}\mathrm{ym}^n(V)$ can be realized as a quotient of $V^{\otimes n}$ by an action of the symmetric group $\mathfrak{S}_n$. This action can be described as follows: If $\tau\in\mathfrak{S}_n$ is a transposition that exchanges two numbers $i<j$, then τ permutes the corresponding tensor factors in $v_1\otimes\cdots\otimes v_n$ introducing a sign $(-1)^{|v_i||v_j|+(|v_i|+|v_j|)\sum_{i<k<j}|v_k|}$.

Now let U be a vector space over a field k of characteristic 0 and look at the exterior algebra $H:=\Lambda^*U$. Since H is a super vector space, we can construct the supersymmetric power $S^n := \mathbb{S}\mathrm{ym}^n(H)$. We may identify S^n with the space of $\mathfrak{S}_n$-invariants in $H^{\otimes n}$ by means of the linear projection operator

$$\mathrm{pr}: H^{\otimes n}\longrightarrow S^n, \quad \mathrm{pr} = \frac{1}{n!}\sum_{\pi\in\mathfrak{S}_n}\pi.$$

The multiplication in $H^{\otimes n}$ induces a multiplication on the subspace of invariants, which makes S^n a supercommutative algebra. Of course, it is different from the product $\diamond$ discussed above.

Since H is generated as an algebra by $U=\Lambda^1(U)\subset H$, we may define a homomorphism of algebras:

$$s: H\longrightarrow S^n, \quad s(u) = \mathrm{pr}(u\otimes 1\otimes\cdots\otimes 1) \text{ for } u\in U,$$

so S^n becomes an algebra over H.

Lemma 3.5. *The morphism s turns S^n into a free module over H, for $n \geq 1$.*

Proof. We start with the tensor power $H^{\otimes n}$ and the ring homomorphism

$$\iota : H \longrightarrow H^{\otimes n}, \quad h \longmapsto h \otimes 1 \otimes \cdots \otimes 1$$

that makes $H^{\otimes n}$ a free H-module. Note that $\operatorname{pr} \iota \neq s$, since pr is not a ring homomorphism. (For example, $\operatorname{pr} \iota(h) \neq s(h)$ for any nonzero $h \in \Lambda^2(U)$.) We therefore modify the H-module structure of $H^{\otimes n}$:

For some $u \in U$, denote $u^{(i)} := 1^{\otimes i-1} \otimes u \otimes 1^{\otimes n-i+1} \in H^{\otimes n}$. Then $H^{\otimes n}$ is generated as a k-algebra by the elements $\{u^{(i)}, u \in U\}$. Now consider the ring automorphism

$$\sigma : H^{\otimes n} \longrightarrow H^{\otimes n}, \quad u^{(1)} \longmapsto u^{(1)} + u^{(2)} + \ldots + u^{(n)}, \quad u^{(i)} \longmapsto u^{(i)} \text{ for } i > 1.$$

Then we have $\sigma\iota = s$ on S^n. On the other hand, if $\{b_i\}$ is a k-basis of V, then $\{b_i^{(j)}, j > 1\}$ is a ι-basis for $H^{\otimes n}$, and $\{\sigma(b_i^{(j)})\}$ is a $\sigma\iota$-basis for $H^{\otimes n}$. Now if we project the basis elements, we get a set $\{\operatorname{pr}(\sigma(b_i^{(j)}))\}$ that spans S^n. Eliminating linear dependent vectors (this is possible over the rationals), we get a s-basis of S^n. □

4 Actions of the symplectic group over finite fields

The aim of this section is to provide some special computations used in Section 12.

Let V be a symplectic vector space of dimension $n \in 2\mathbb{N}$ over a field k with a nondegenerate symplectic form $\omega : \Lambda^2 V \to k$. A line is a one-dimensional subspace of V, a plane is a two-dimensional subspace of V. A plane $P \subset V$ is called isotropic, if $\omega(x, y) = 0$ for any $x, y \in P$, otherwise non-isotropic. The symplectic group $\operatorname{Sp} V$ is the set of all linear maps $\phi : V \to V$ with the property $\omega(\phi(x), \phi(y)) = \omega(x, y)$ for all $x, y \in V$.

Proposition 4.1. *The symplectic group* $\operatorname{Sp} V$ *acts transitively on the set of non-isotropic planes as well as on the set of isotropic planes.*

Proof. Given two planes P_1 and P_2, we may choose vectors v_1, v_2, w_1, w_2 such that v_1, v_2 span P_1, w_1, w_2 span P_2 and $\omega(u_1, u_2) = \omega(w_1, w_2)$. We complete $\{v_1, v_2\}$ as well as $\{w_1, w_2\}$ to a symplectic basis of V. Then define $\phi(v_1) = w_1$ and $\phi(v_2) = w_2$. It is now easy to see that the definition of ϕ can be extended to the remaining basis elements to give a symplectic morphism. □

Remark 4.2. The set of planes in V can be identified with the simple tensors in $\Lambda^2 V$ up to multiples. Indeed, given a simple tensor $v \wedge w \in \Lambda^2 V$, the span of v and w yields the corresponding plane. Conversely, any two spanning vectors v and w of a plane give the same element $v \wedge w$ (up to multiples).

Remark 4.3. If k is the field with two elements, then the set of planes in V can be identified with the set $\{\{x, y, z\} \mid x, y, z \in V \backslash \{0\}, x + y + z = 0\}$. Observe that for such a $\{x, y, z\}$, $\omega(x, y) = \omega(x, y) = \omega(y, x)$ and this value gives the criterion for isotropy.

Proposition 4.4. *Assume that k is finite of cardinality q.*

$$\text{The number of lines in } V \text{ is } \frac{q^n - 1}{q - 1}, \tag{1}$$

$$\text{the number of planes in } V \text{ is } \frac{(q^n - 1)(q^{n-1} - 1)}{(q^2 - 1)(q - 1)}, \tag{2}$$

$$\text{the number of isotropic planes in } V \text{ is } \frac{(q^n - 1)(q^{n-2} - 1)}{(q^2 - 1)(q - 1)}, \tag{3}$$

$$\text{the number of non-isotropic planes in } V \text{ is } \frac{q^{n-2}(q^n - 1)}{q^2 - 1}. \tag{4}$$

Proof. A line ℓ in V is determined by a nonzero vector. There are q^n-1 nonzero vectors in V and $q-1$ nonzero vectors in ℓ. A plane P is determined by a line $\ell_1 \subset V$ and a unique second line $\ell_2 \in V/\ell_1$. We have $\frac{q^2-1}{q-1}$ choices for ℓ_1 in P. The number of planes is therefore

$$\frac{\frac{q^n-1}{q-1}\cdot\frac{q^{n-1}-1}{q-1}}{\frac{q^2-1}{q-1}} = \frac{(q^n-1)(q^{n-1}-1)}{(q^2-1)(q-1)}.$$

For an isotropic plane we have to choose the second line from $\ell_1^\perp/\ell_1$. This is a space of dimension $n-2$, hence the formula. The number of non-isotropic planes is the difference of the two previous numbers. ☐

Assume now that V is a four-dimensional vector space over $k=\mathbb{F}_q$. Consider the free k-module $k[V]$ with basis $\{X_i \mid i \in V\}$. It carries a natural k-algebra structure, given by $X_i \cdot X_j := X_{i+j}$ with unit $1 = X_0$. This algebra is local with maximal ideal $\mathfrak{m}$ generated by all elements of the form $(X_i - 1)$.

We introduce an action of $\mathrm{Sp}(4,k)$ on $k[V]$ by setting $\phi(X_i) = X_{\phi(i)}$. Furthermore, the underlying additive group of V acts on $k[V]$ by $v(X_i) = X_{i+v} = X_i X_v$.

Definition 4.5. We define a subset of $k[V]$:

$$N := \left\{ \sum_{i\in P} X_i \mid P \subset V \text{ non-isotropic plane} \right\}.$$

Denote by $\langle N\rangle$ and by (N) the linear span of N and the ideal generated by N, respectively. Note that (N) is the linear span of $\{v\cdot b \mid b \in N, v \in V\}$. Further, let D be the linear span of $\{v(b) - b \mid b \in N, v \in V\}$. Then D is in fact an ideal, namely the product of ideals $\mathfrak{m}\cdot(N)$.

Proposition 4.6. *The following table illustrates the dimensions of these objects for some fields k:*

k	$\dim_k \langle N\rangle$	$\dim_k (N)$	$\dim_k D$
$\mathbb{F}_2$	*10*	*11*	*5*
$\mathbb{F}_3$	*30*	*50*	*31*
$\mathbb{F}_5$	*121*	*355*	*270*

Since this is computed numerically using a naive approach, we do not give a formal proof.

Remark 4.7. We remark that $X := \sum_{i\in V} X_i \in D$. Indeed, let P, P' be two non-isotropic planes with $P \cap P' = 0$. Then $X = \left(\sum_{i\in P} X_i\right)\left(\sum_{i\in P'} X_i\right)$ and both factors are contained in $(N) \subset \mathfrak{m}$, so $X \in \mathfrak{m}\cdot(N) = D$.

Let us now consider some special orthogonal sums. Set $S := \mathrm{Sym}^2(\Lambda^2 V)$. Take two vectors $v, w \in V$ with $\omega(v,w) = 1$ and set $x := (v\wedge w)^2 \in S$. Denote P the plane spanned by v and w and set $y := \sum_{i\in P} X_i \in k[V]$.

We consider now the action of $\mathrm{Sp}\,V$ on $S \oplus k[V]$. Denote O the vector space spanned by the elements $\phi(x)\oplus\phi(z)$, for $\phi \in \mathrm{Sp}\,V$, $z \in (y)$ and U the vector space spanned by the elements $\phi(x)$, for $\phi \in \mathrm{Sp}\,V$.

Proposition 4.8. *Then we have by numerical computation:*

k	$\dim_k O$	$\dim_k U$
$\mathbb{F}_2$	*11*	*6*
$\mathbb{F}_3$	*51*	*20*
$\mathbb{F}_5$	*375*	*20*

Now we prove the following lemma that we will need for a divisibility argument in Section 12.

Lemma 4.9. *We assume that $k = \mathbb{F}_3$. Let $\mathrm{pr}_1 : S \oplus k[V] \to S$ and $\mathrm{pr}_2 : S \oplus k[V] \to k[V]$ the projection. We have:*

(i) $\dim \operatorname{Ker} \operatorname{pr}_{2|O} = 1$.

(ii) $\dim \operatorname{Ker} \operatorname{pr}_{1|O} = 31$.

Proof. (i) We have $\operatorname{pr}_2(O) = (N)$, so $\dim \operatorname{pr}_2(O) = 50$. Since $\dim O = 51$, it follows $\dim \operatorname{Ker} \operatorname{pr}_{2|O} = 1$.

(ii) We have $\operatorname{pr}_1(O) = U$. So

$$\dim \operatorname{pr}_1(O) = 20.$$

Since $\dim O = 51$, it follows $\dim \operatorname{Ker} \operatorname{pr}_{1|O} = 31$.

□

5 Complex abelian surfaces

Denote A a complex abelian surface (a torus of dimension 2). As such, it always can be written as a quotient

$$A = \mathbb{C}^2/\Lambda,$$

where $\Lambda \subset \mathbb{C}^2$ is a lattice of rank 4, embedded in $\mathbb{C}^2$. Depending on the imbedding, we get different complex manifolds, projective or not. Of course, all of them are equivalent by deformation.

5.1 Morphisms and special cases

Definition 5.1. An isogeny between abelian surfaces $A = \mathbb{C}^2/\Lambda \to A' = \mathbb{C}^2/\Lambda'$ means a surjective holomorphic map that preserves the group structure. It is given by a complex linear map, that maps Λ to a sublattice of Λ'.

Example 5.2. For a number $n \neq 0$, the multiplication map $n : A \to A$, $x \mapsto n \cdot x$ is an isogeny.

By an automorphism of A we mean a biholomorphism preserving the group structure. It can be represented by a $\mathbb{C}$-linear map $M : \mathbb{C}^2 \to \mathbb{C}^2$ with $M\Lambda = \Lambda$. Have a look in [13] or the appendix of [16] for some reference. Let us now come to the very special case that $A = E \times E$ can be written as the square of an elliptic curve. Note that A is projective, because every elliptic curve is. Now write E as $E = \mathbb{C}/\Lambda_0$. We may assume that Λ_0 is spanned by 1 and a vector $\tau \in \mathbb{C}\backslash\mathbb{R}$. The automorphism group, up to isogeny, is given by ([16]) $\mathrm{GL}(2, \operatorname{End}(\Lambda_0))$, where $\operatorname{End}(\Lambda_0)$ is the set $\{z \in \mathbb{C} \mid z\Lambda_0 \subset \Lambda_0\}$.

Proposition 5.3. *There are two possibilities for* $\operatorname{End}(\Lambda_0)$*, depending on* τ*:*

1. *Both the real part and the square norm of* τ *are rational numbers, say* $2\Re(\tau) = \frac{p}{r}$ *and* $\|\tau\|^2 = \frac{q}{r}$ *with* $r > 0$ *as small as possible. Then* $\operatorname{End}(\Lambda_0) = \mathbb{Z} + r\tau\mathbb{Z}$.
2. *At least one of* $\Re(\tau), \|\tau\|^2$ *is irrational. Then* $\operatorname{End}(\Lambda_0) = \mathbb{Z}$.

Proof. Given $z \in \operatorname{End}(\Lambda_0)$, we have

$$z \cdot 1 = a + b\tau \text{ and } z \cdot \tau = c + d\tau \text{ with } a, b, c, d \in \mathbb{Z}.$$

We get the condition

$$(a + b\tau)\tau = c + d\tau \quad \Leftrightarrow \quad b\tau^2 + (a - d)\tau - c = 0.$$

Up to scalar multiples, there is a unique real quadratic polynomial that annihilates τ, namely $(x - \tau)(x - \bar{\tau}) = x^2 - 2\Re(\tau)x + \|\tau\|^2$. If all coefficients of that polynomial are rational numbers, then $z = a + b\tau$ gives a solution for arbitrary $a \in \mathbb{Z}$, $b \in r\mathbb{Z}$. Otherwise, the condition must be the zero polynomial, so $b = 0$. □

Now we study the action of automorphisms on torsion points in a very special case. This will be needed in the technical proof of Theorem 15.9.

Definition 5.4. Denote $\xi \in \mathbb{C}$ a primitive sixth root of unity and E_ξ the elliptic curve given by the choice $\Lambda_0 = \langle 1, \xi \rangle$, so by Proposition 5.3, $\mathrm{End}(\Lambda_0) = \Lambda_0$ is the ring of Eisenstein integers. Define a group G_ξ of automorphisms of $E_\xi \times E_\xi$ by the following generators in $\mathrm{GL}(2, \mathrm{End}(\Lambda_0))$:

$$g_1 = \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad g_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For $A = E_\xi \times E_\xi$, let $V = A[2]$ be the (fourdimensional) $\mathbb{F}_2$-vector space of 2-torsion points on A and let $\mathfrak{T}$ be the set of planes in V. Note that by Remark 4.3 a plane in V can be identified with an unordered triple $\{x, y, z\}$ with $0 \neq x, y, z \in V$ and $x + y + z = 0$. The action of G_ξ on A induces actions of G_ξ on $A[2]$ and $\mathfrak{T}$.

Lemma 5.5. *There are two orbits of G_ξ on $\mathfrak{T}$, of cardinalities 5 and 30.*

Proof. Note that the generators g_2 and g_3 exist because A is of the form $E \times E$, while g_1 exists only in the special case $E = E_\xi$. Indeed, multiplication with ξ induces a cyclic permutation on $E_\xi[2]$. The orbits can be explicitely determined by a suitable computer program. For verification, we give one of the orbits explicitely. Denote x_1, x_2, x_3 the non-zero points in $E_\xi[2]$. The orbit of cardinality five is then given by

$$\{(0, x_1), (0, x_2), (0, x_3)\}, \qquad \{(x_1, 0), (x_2, 0), (x_3, 0)\},$$
$$\{(x_1, x_1), (x_2, x_2), (x_3, x_3)\}, \qquad \{(x_1, x_2), (x_2, x_3), (x_3, x_1)\}, \qquad \{(x_1, x_3), (x_2, x_1), (x_3, x_2)\}.$$

□

5.2 Homology and Cohomology

The fundamental group $\pi_1(A, \mathbb{Z}) = H_1(A, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 4, which is canonically identified with the lattice Λ. Indeed, the projection of every path in $\mathbb{C}^2$ from 0 to $v \in \Lambda$ gives a unique element of $\pi_1(A, \mathbb{Z})$. Conversely, any closed path in A with basepoint 0 lifts to a unique path in $\mathbb{C}^2$ from 0 to some $v \in \Lambda$. So the first cohomology $H^1(A, \mathbb{Z})$ is freely generated by four elements, too. Moreover, by [38, Sect. I.1], the cohomology ring is isomorphic to the exterior algebra

$$H^*(A, \mathbb{Z}) = \Lambda^*(H^1(A, \mathbb{Z})).$$

Notation 5.6. We denote the generators of $H^1(A, \mathbb{Z})$ by a_i, $1 \leq i \leq 4$ and their respective duals by $a_i^* \in H^3(A, \mathbb{Z})$. If $A = E \times E$ is the product of two elliptic curves, we choose the a_i in a way such that $\{a_1, a_2\}$ and $\{a_3, a_4\}$ give bases of $H^1(E, \mathbb{Z})$ in the decomposition $H^1(A) = H^1(E) \oplus H^1(E)$. We denote the generator of the top cohomology $H^4(A, \mathbb{Z})$ by $x := a_1 a_2 a_3 a_4$.

Let A be a abelian surface. We recall that a *principal polarization* of A is a polarization L such that there exists a basis of $H_1(A, \mathbb{Z})$, with respect to which the bilinear form on $H_1(A, \mathbb{Z})$ induced by $c_1(L)$ is given by the matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We remark that a principal polarization L provides a symplectic bilinear form ω_L on $H_1(A, \mathbb{Z})$ as follows:

$$\omega_L(x, y) = x \cdot c_1(L) \cdot y, \tag{5}$$

for all x and y in $H_1(A, \mathbb{Z})$. We recall the following result.

Proposition 5.7. *Let (A, L) be a principally polarized abelian surface. The group $H_1(A, \mathbb{Z})$ is endowed with the symplectic from ω_L defined in (5). Let $\mathrm{Mon}(H_1(A, \mathbb{Z}))$ be the group of monodromy actions on $H_1(A, \mathbb{Z})$. Then $\mathrm{Mon}(H_1(A, \mathbb{Z})) \supset \mathrm{Sp}(H_1(A, \mathbb{Z}))$.*

Proof. It can be seen as follows. Let $\mathcal{M}_2$ be the moduli space of curves of genus 2 and $\mathcal{A}_2$ be the moduli space of principally polarized abelian surfaces. By the Torelli theorem (see for instance [33, Theorem 12.1]), we have an injection $J : \mathcal{M}_2 \hookrightarrow \mathcal{A}_2$ given by taking the Jacobian of the curve endowed with its canonical polarization. Moreover the moduli spaces $\mathcal{M}_2$ and $\mathcal{A}_2$ are both of dimension 3.

Now if $\mathscr{C}_2$ is a curve of genus 2, we have by Theorem 6.4 of [10]:

$$\mathrm{Mon}(H_1(\mathscr{C}_2,\mathbb{Z})) \supset \mathrm{Sp}(H_1(\mathscr{C}_2,\mathbb{Z})),$$

where the symplectic form on $H_1(\mathscr{C}_2,\mathbb{Z})$ is given by the cup product. Then the result follows from the fact that the lattices $H_1(\mathscr{C}_2,\mathbb{Z})$ and $H_1(J(\mathscr{C}_2),\mathbb{Z})$ are isometric. $\square$

Remark 5.8. Let (A, L) be a principally polarized abelian surface and p a prime number. The group $H_1(A,\mathbb{Z})$ tensorized by $\mathbb{F}_p$ can be seen as the group $A[p]$ of points of p-torsion on A and the from $\omega_L \otimes \mathbb{F}_p$ provides a symplectic form on $A[p]$. Then the group $\mathrm{Mon}(A[p])$ of the monodromy action on $A[p]$ contains the group $\mathrm{Sp}(A[p])$.

Now, we are ready to recall Proposition 5.2 of [18] on the monodromy of the generalized Kummer fourfold.

Proposition 5.9. *Let A be an abelian surface and $K_2(A)$ the associated generalized Kummer fourfold. The image of the monodromy representation on $\Pi = \langle Z_\tau |\ \tau \in A[3]\rangle$ contains the semidirect product $\mathrm{Sp}(A[3]) \ltimes A[3]$ which act as follows:*

$$f \cdot Z_\tau = Z_{f(\tau)} \text{ and } \tau' \cdot Z_\tau = Z_{\tau+\tau'},$$

for all $f \in \mathrm{Sp}(A[3])$ and $\tau' \in A[3]$.

6 Recall on the theory of integral cohomology of quotients

The main reference of this section is [32].

Let $G = \langle \iota \rangle$ be the group generated by an involution ι on a complex manifold X. As denoted in [6, Section 5], let $\mathcal{O}_K$ be the ring $\mathbb{Z}$ with the following G-module structure: $\iota \cdot x = -x$ for $x \in \mathcal{O}_K$. For $a \in \mathbb{Z}$, we also denote by $(\mathcal{O}_K, a)$ the module $\mathbb{Z} \oplus \mathbb{Z}$ whose G-module structure is defined by $\iota \cdot (x, k) = (-x + ka, k)$. We also denote by N_2 the $\mathbb{F}_2[G]$-module $(\mathcal{O}_K, a) \otimes \mathbb{F}_2$. We recall Definition-Proposition 2.2.2 of [32].

Definition-Proposition 6.1. *Assume that $H^*(X,\mathbb{Z})$ is torsion-free. Then for all $0 \le k \le 2\dim X$, we have an isomorphism of $\mathbb{Z}[G]$-modules:*

$$H^k(X,\mathbb{Z}) \simeq \bigoplus_{i=1}^{r} (\mathcal{O}_K, a_i) \oplus \mathcal{O}_K^{\oplus s} \oplus \mathbb{Z}^{\oplus t},$$

for some $a_i \notin 2\mathbb{Z}$ and $(r, s, t) \in \mathbb{N}^3$. We get the following isomorphism of $\mathbb{F}_2[G]$-modules:

$$H^k(X,\mathbb{F}_2) \simeq N_2^{\oplus r} \oplus \mathbb{F}_2^{\oplus(s+t)}.$$

We denote $l_2^k(X) := r$, $l_{1,-}^k(X) := s$, $l_{1,+}^k(X) := t$, $\mathcal{N}_2 := N_2^{\oplus r}$ and $\mathcal{N}_1 := \mathbb{F}_2^{\oplus s+t}$.

Remark 6.2. These invariants are uniquely determined by G, X and k.

We recall an adaptation of Proposition 5.1 and Corollary 5.8 of [6] that can be found in Section 2.2 of [32].

Proposition 6.3. *Let X be a compact complex manifold of dimension n and ι an involution. Assume that $H^*(X,\mathbb{Z})$ is torsion free. We have:*

(i) $\mathrm{rk}\, H^k(X,\mathbb{Z})^\iota = l_2^k(X) + l_{1,+}^k(X)$.

(ii) We denote $\sigma := \mathrm{id} + \iota^*$ *and* $S_\iota^k := \operatorname{Ker}\sigma \cap H^k(X,\mathbb{Z})$. *We have* $H^k(X,\mathbb{Z})^\iota \cap S_\iota^k = 0$ *and*

$$\frac{H^k(X,\mathbb{Z})}{H^k(X,\mathbb{Z})^\iota \oplus S_\iota^k} = (\mathbb{Z}/2\mathbb{Z})^{l_2^k(X)}.$$

Remark 6.4. Note that the elements of $(\mathcal{O}_K, a_i)^\iota$ are written $x + \iota^*(x)$ with $x \in (\mathcal{O}_K, a_i)$.

Let $\pi : X \to X/G$ be the quotient map. We denote by π^* and π_* respectively the pull-back and the push-forward along π. We recall that

$$\pi_* \circ \pi^* = 2\,\mathrm{id} \ \text{ and } \ \pi^* \circ \pi_* = \mathrm{id} + \iota^*. \tag{6}$$

Assuming that $H^k(X,\mathbb{Z})$ is torsion free, it follows the exact sequence of Proposition 3.3.3 of [32], which will be useful in the next section.

$$0 \longrightarrow \pi_*(H^k(X,\mathbb{Z})) \longrightarrow H^k(X/G,\mathbb{Z})/\mathrm{tors} \longrightarrow (\mathbb{Z}/2\mathbb{Z})^{\alpha_k} \longrightarrow 0, \tag{7}$$

with $\alpha_k \in \mathbb{N}$. We also recall the commutativity behaviour of π_* with the cup product.

Proposition 6.5. *[32, Lemma 3.3.7] Let* X *be a compact complex manifold of dimension* n *and* ι *an involution. Assume that* $H^*(X,\mathbb{Z})$ *is torsion free. Let* $0 \le k \le 2n$, m *an integer such that* $km \le 2n$, *and let* $(x_i)_{1\le i\le m}$ *be elements of* $H^k(X,\mathbb{Z})^\iota$. *Then*

$$\pi_*(x_1) \cdot \ldots \cdot \pi_*(x_m) = 2^{m-1}\pi_*(x_1 \cdot \ldots \cdot x_m).$$

We also recall Definition 3.3.4 of [32].

Definition 6.6. Let X be a compact complex manifold and ι be an involution. Let $0 \le k \le 2n$, and assume that $H^k(X,\mathbb{Z})$ is torsion free. Then if the map $\pi_* : H^k(X,\mathbb{Z}) \to H^k(X/G,\mathbb{Z})/\mathrm{tors}$ is surjective, we say that (X,ι) is H^k*-normal.*

Remark 6.7. The H^k-normal property is equivalent to the following property.

For $x \in H^k(X,\mathbb{Z})^\iota$, $\pi_*(x)$ is divisible by 2 if and only if there exists $y \in H^k(X,\mathbb{Z})$ such that $x = y + \iota^*(y)$.

We also need to recall Definition 3.5.1 of [32] about fixed loci.

Definition 6.8. Let X be a compact complex manifold of dimension n and G an automorphism group of prime order p.

1) We will say that $\operatorname{Fix} G$ is negligible if the following conditions are verified:

   - $H^*(\operatorname{Fix} G,\mathbb{Z})$ is torsion-free.
   - $\operatorname{Codim} \operatorname{Fix} G \ge \frac{n}{2} + 1$.

2) We will say that $\operatorname{Fix} G$ is almost negligible if the following conditions are verified:

   - $H^*(\operatorname{Fix} G,\mathbb{Z})$ is torsion-free.
   - n is even and $n \ge 4$.
   - $\operatorname{Codim} \operatorname{Fix} G = \frac{n}{2}$, and the purely $\frac{n}{2}$-dimensional part of $\operatorname{Fix} G$ is connected and simply connected. We denote the $\frac{n}{2}$-dimensional component by Z.
   - The cocycle $[Z]$ associated to Z is primitive in $H^n(X,\mathbb{Z})$.

Now, we are ready to provide Theorem 2.65 of [32] which we will be one of the main tools in Part III.

Theorem 6.9. *Let* $G = \langle\varphi\rangle$ *be a group of prime order* $p = 2$ *acting by automorphisms on a Kähler manifold* X *of dimension* $2n$. *We assume:*

i) $H^*(X,\mathbb{Z})$ *is torsion-free,*

ii) $\operatorname{Fix} G$ *is negligible or almost negligible,* *iii)* $l_{1,-}^{2k}(X) = 0$ *for all* $1 \leq k \leq n$, *and*

iv) $l_{1,+}^{2k+1}(X) = 0$ *for all* $0 \leq k \leq n-1$, *when* $n > 1$.

v) $l_{1,+}^{2n}(X) + 2\left[\sum_{i=0}^{n-1} l_{1,-}^{2i+1}(X) + \sum_{i=0}^{n-1} l_{1,+}^{2i}(X)\right] = \sum_{k=0}^{\dim \mathrm{Fix}\, G} h^{2k}(\mathrm{Fix}\, G, \mathbb{Z})$.

Then (X, G) *is* H^{2n}*-normal.*

We will also need a proposition from Section 7 of [6] about Smith theory. Let X be a topological space and let $G = \langle \iota \rangle$ be an involution acting on X. Let $\sigma := 1 + \iota \in \mathbb{F}_2[G]$. We consider the chain complex $C_*(X)$ of X with coefficients in $\mathbb{F}_2$ and its subcomplexes $\sigma C_*(X)$. We denote also X^G the fixed locus of the action of G on X.

Proposition 6.10. *(1) ([7], Theorem 3.1). There is an exact sequence of complexes:*

$$0 \longrightarrow \sigma C_*(X) \oplus C_*(X^G) \xrightarrow{f} C_*(X) \xrightarrow{\sigma} \sigma C_*(X) \longrightarrow 0\ ,$$

where f *denotes the sum of the inclusions.*

(2) ([7], (3.4) p.124). There is an isomorphism of complexes:

$$\sigma C_*(X) \simeq C_*(X/G, X^G),$$

where X^G *is identified with its image in* X/G.

7 Odd cohomology of the Hilbert scheme of two points

Let A be a smooth compact surface with torsion free cohomology and $A^{[2]}$ the Hilbert scheme of two points. It can be constructed as follows: Consider the direct product $A \times A$. Denote

$$b : \widetilde{A \times A} \to A \times A$$

the blow-up along the diagonal $\Delta \cong A$ with exceptional divisor E. Let $j : E \to \widetilde{A \times A}$ be the embedding. The action of $\mathfrak{S}_2$ on $A \times A$ lifts to an action on $\widetilde{A \times A}$. We have the pushforward $j_* : H^*(E, \mathbb{Z}) \to H^*(\widetilde{A \times A}, \mathbb{Z})$.

The quotient by the action of $\mathfrak{S}_2$ is

$$\pi : \widetilde{A \times A} \to A^{[2]}.$$

Now, $A^{[2]}$ is a compact complex manifold with torsion-free cohomology, [49, Theorem 2.2]. By (7), there is an exact sequence

$$0 \to \pi_*(H^k(\widetilde{A \times A}, \mathbb{Z})) \to H^k(A^{[2]}, \mathbb{Z}) \to \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\alpha_k} \to 0$$

with $k \in \{1, ..., 8\}$. In this section, we want to prove the following proposition.

Proposition 7.1. *Let* A *be a smooth compact surface with torsion free cohomology. Then*

(i) $H^3(A^{[2]}, \mathbb{Z}) = \pi_*(b^*(H^3(A \times A, \mathbb{Z}))) \oplus \pi_* j_* b_{|E}^*(H^1(\Delta, \mathbb{Z}))$,

(ii) $H^5(A^{[2]}, \mathbb{Z}) = \pi_*(b^*(H^5(A \times A, \mathbb{Z}))) \oplus \frac{1}{2}\pi_* j_* b_{|E}^*(H^3(\Delta, \mathbb{Z}))$.

The section is dedicated to the proof of this proposition. The proof is organized as follows. Section 7.1 is devoted to calculate the torsion of $H^3(A^{[2]} \smallsetminus E, \mathbb{Z})$ (Lemma 7.4) using equivariant cohomology techniques developed in [32]. Then this knowledge allow us to deduce $\alpha_3 = 0$ using the exact sequence (11) and $\alpha_5 = 4$ using the unimodularity of the lattice $H^3(\widetilde{A \times A}, \mathbb{Z}) \oplus H^5(\widetilde{A \times A}, \mathbb{Z})$.

7.1 Preliminary Lemmas

We denote $V = \widetilde{A \times A} \smallsetminus E$ and $U = V/\mathfrak{S}_2 = A^{[2]} \smallsetminus E$, where $\mathfrak{S}_2 = \langle \sigma_2 \rangle$.

Lemma 7.2. *We have: $H^k(A \times A, \mathbb{Z}) = H^k(V, \mathbb{Z})$ for all $k \leq 3$.*

Proof. We have $V = A \times A \smallsetminus \Delta$. We have the following natural exact sequence:

$$\cdots \longrightarrow H^k(A \times A, V, \mathbb{Z}) \longrightarrow H^k(A \times A, \mathbb{Z}) \longrightarrow H^k(V, \mathbb{Z}) \longrightarrow \cdots$$

Moreover, by Thom isomorphism $H^k(A \times A, V, \mathbb{Z}) = H^{k-4}(\Delta, \mathbb{Z}) = H^{k-4}(A, \mathbb{Z})$. Hence $H^k(A \times A, V, \mathbb{Z}) = 0$ for all $k \leq 3$. Hence $H^k(A \times A, \mathbb{Z}) = H^k(V, \mathbb{Z})$ for all $k \leq 2$. It remains to consider the following exact sequence:

$$0 \longrightarrow H^3(A \times A, \mathbb{Z}) \longrightarrow H^3(V, \mathbb{Z}) \longrightarrow H^4(A \times A, V, \mathbb{Z}) \xrightarrow{\rho} H^4(A \times A, \mathbb{Z}) .$$

The map ρ is given by $\mathbb{Z}[\Delta] \to H^4(A \times A, \mathbb{Z})$. Using Notation 5.6, the class $x \otimes 1$ is also in $H^4(A \times A, \mathbb{Z})$ and intersects Δ in one point. Hence the class of Δ in $H^4(A \times A, \mathbb{Z})$ is not trivial and the map ρ is injective. It follows that

$$H^3(A \times A, \mathbb{Z}) = H^3(V, \mathbb{Z}).$$

□

Now we will calculate the invariant $l_{1,-}^2(A \times A)$ and $l_{1,+}^1(A \times A)$ defined in Definition-Proposition 6.1.

Lemma 7.3. *We have: $l_{1,-}^2(A \times A) = l_{1,+}^1(A \times A) = 0$.*

Proof. By Künneth formula we have:

$$H^1(A \times A, \mathbb{Z}) = H^0(A, \mathbb{Z}) \otimes H^1(A, \mathbb{Z}) \oplus H^1(A, \mathbb{Z}) \otimes H^0(A, \mathbb{Z}).$$

The elements of $H^0(A, \mathbb{Z}) \otimes H^1(A, \mathbb{Z})$ and $H^1(A, \mathbb{Z}) \otimes H^0(A, \mathbb{Z})$ are exchanged under the action of σ_2. It follows that $l_2^1(A \times A) = 4$ and necessary $l_{1,-}^1(A \times A) = l_{1,+}^1(A \times A) = 0$.

By Künneth formula we also have:

$$\begin{aligned} H^2(A \times A, \mathbb{Z}) = {} & H^0(A, \mathbb{Z}) \otimes H^2(A, \mathbb{Z}) \oplus H^1(A, \mathbb{Z}) \otimes H^1(A, \mathbb{Z}) \\ & \oplus H^2(A, \mathbb{Z}) \otimes H^0(A, \mathbb{Z}). \end{aligned}$$

As before every elements $x \otimes y \in H^2(A \times A, \mathbb{Z})$ are sent to $y \otimes x$ by the action of σ_2. Such an element is fixed by the action of σ_2 if $x = y$. It follows:

$$l_2^2(A \times A) = 6 + 6 = 12,$$

$$l_{1,+}^2(A \times A) = 4,$$

and thus:

$$l_{1,-}^2(A \times A) = 0.$$

□

Lemma 7.4. *The group $H^3(U, \mathbb{Z})$ is torsion free.*

Proof. Using the spectral sequence of equivariant cohomology, it follows from Proposition 3.2.5 of [32], Lemma 7.2 and 7.3. □

7.2 Third cohomology group

By Theorem 7.31 of [50], we have:

$$H^3(\widetilde{A\times A},\mathbb{Z}) = b^*(H^3(A\times A,\mathbb{Z}))\oplus j_*b_{|E}^*(H^1(\Delta,\mathbb{Z})). \tag{8}$$

It follows that

$$H^3(A^{[2]},\mathbb{Z}) \supset \pi_*b^*(H^3(A\times A,\mathbb{Z}))\oplus \pi_*j_*b_{|E}^*(H^1(\Delta,\mathbb{Z})).$$

We want to show that this inclusion is an equality. We will proceed as follow, we first prove that $\pi_*b^*(H^3(A\times A,\mathbb{Z}))$ is primitive. Then, in Lemma 7.5, it is shown that $\pi_*j_*b_{|E}^*(H^1(\Delta,\mathbb{Z}))$ is primitive and finally we remark that this implies that the sum $\pi_*b^*(H^3(A\times A,\mathbb{Z}))\oplus\pi_*j_*b_{|E}^*(H^1(\Delta,\mathbb{Z}))$ is primitive.

Now, by Künneth formula, we have:

$$\begin{aligned} H^3(A\times A,\mathbb{Z}) = {} & H^0(A,\mathbb{Z})\otimes H^3(A,\mathbb{Z})\oplus H^1(A,\mathbb{Z})\otimes H^2(A,\mathbb{Z})\\ & \oplus H^2(A,\mathbb{Z})\otimes H^1(A,\mathbb{Z})\oplus H^3(A,\mathbb{Z})\otimes H^0(A,\mathbb{Z}). \end{aligned}$$

Hence all elements in $H^3(A\times A,\mathbb{Z})^{\mathfrak{S}_2}$ are written as $x+\sigma_2^*(x)$ with $x\in H^3(A\times A,\mathbb{Z})$. Since $\frac{1}{2}\pi_*(x+\sigma_2^*(x))=\pi_*(x)$, it follows that $\pi_*(b^*(H^3(A\times A,\mathbb{Z})))$ is primitive in $H^3(A^{[2]},\mathbb{Z})$. Moreover by (8), we have the following values which will be used in Section 7.3:

$$l_2^3(\widetilde{A\times A}) = \operatorname{rk} H^3(A\times A,\mathbb{Z})^{\mathfrak{S}_2} = 28. \tag{9}$$

and

$$l_{1,+}^3(\widetilde{A\times A}) = \operatorname{rk} H^1(\Delta,\mathbb{Z})^{\mathfrak{S}_2} = 4, \quad \text{and} \quad l_{1,-}^3(\widetilde{A\times A}) = 0.$$

It remains to prove the following lemma.

Lemma 7.5. *The group $\pi_*(j_*(b_{|E}^*(H^1(\Delta,\mathbb{Z}))))$ is primitive in $H^3(A^{[2]},\mathbb{Z})$.*

Proof. We consider the following commutative diagram:

$$\begin{tikzcd} H^3(\mathscr{N}_{A^{[2]}/\pi(E)},\mathscr{N}_{A^{[2]}/\pi(E)}\smallsetminus 0,\mathbb{Z}) = H^3(A^{[2]},U,\mathbb{Z}) \arrow[r, "g"] \arrow[d, "d\pi^*"] & H^3(A^{[2]},\mathbb{Z}) \arrow[d, "\pi^*"'] \\ H^3(\mathscr{N}_{\widetilde{A\times A}/E},\mathscr{N}_{\widetilde{A\times A}/E}\smallsetminus 0,\mathbb{Z}) = H^3(\widetilde{A\times A},V,\mathbb{Z}) \arrow[r, "h"] & H^3(\widetilde{A\times A},\mathbb{Z}), \end{tikzcd} \tag{10}$$

where $\mathscr{N}_{A^{[2]}/E}$ and $\mathscr{N}_{\widetilde{A\times A}/E}$ are respectively the normal bundles of $\pi(E)$ in $A^{[2]}$ and of E in $\widetilde{A\times A}$. By the proof of Theorem 7.31 of [50], the map h is injective with image in $H^3(\widetilde{A\times A},\mathbb{Z})$ given by $j_*b_{|E}^*(H^1(\Delta,\mathbb{Z}))$. Hence Diagram (10) shows that g is also injective and has image $\pi_*j_*b_{|E}^*(H^1(\Delta,\mathbb{Z}))$ in $H^3(A^{[2]},\mathbb{Z})$. We obtain:

$$0 \longrightarrow H^3(A^{[2]},U,\mathbb{Z}) \xrightarrow{g} H^3(A^{[2]},\mathbb{Z}) \longrightarrow H^3(U,\mathbb{Z})\ . \tag{11}$$

However, by Lemma 7.4, $H^3(U,\mathbb{Z})$ is torsion free; it follows that $\pi_*(j_*(b_{|E}^*(H^1(\Delta,\mathbb{Z}))))$ is primitive in $H^3(A^{[2]},\mathbb{Z})$. □

Now it is remaining to prove that $\pi_*b^*(H^3(A\times A,\mathbb{Z}))\oplus\pi_*j_*b_{|E}^*(H^1(\Delta,\mathbb{Z}))$ is primitive in $H^3(A^{[2]},\mathbb{Z})$. This comes from the fact that all elements in $\pi_*b^*(H^3(A\times A,\mathbb{Z})^{\mathfrak{S}_2})$ are divisible by 2, then the relations (6) on π_* and π^* impose the above sum to be primitive.

In more details, let $x\in\pi_*b^*(H^3(A\times A,\mathbb{Z}))$ and $y\in\pi_*j_*b_{|E}^*(H^1(\Delta,\mathbb{Z}))$. It is enough to show that if $\frac{x+y}{2}\in H^3(A^{[2]},\mathbb{Z})$, then $\frac{x}{2}\in H^3(A^{[2]},\mathbb{Z})$ and $\frac{y}{2}\in H^3(A^{[2]},\mathbb{Z})$. As we have seen, we can write $x=\frac{1}{2}\pi_*(z+\sigma_2^*(z))$, with $z\in b^*(H^3(A\times A,\mathbb{Z}))$ and $y=\pi_*(y')$, with $y'\in j_*b_{|E}^*(H^1(\Delta,\mathbb{Z}))$. If

$$\frac{\frac{1}{2}\pi_*(z+\sigma_2^*(z))+\pi_*(y)}{2}\in H^3(A^{[2]},\mathbb{Z})$$

then taking the image by π^* of this element, we obtain

$$\frac{z+\sigma_2^*(z)}{2} + y' \in H^3(\widetilde{A\times A},\mathbb{Z}).$$

Hence the class $\frac{z+\sigma_2^*(z)}{2} \in b^*(H^3(A\times A,\mathbb{Z}))^{\mathfrak{S}_2}$. Hence there is $z' \in b^*(H^3(A\times A,\mathbb{Z}))$ such that

$$\frac{z+\sigma_2^*(z)}{2} = z' + \sigma_2^*(z').$$

So x is divisible by 2 and then also y.

This finishes the proof of (i) of Proposition 7.1.

7.3 The fifth cohomology group

Now we prove (ii) of Proposition 7.1. By Theorem 7.31 of [50], we have:

$$H^5(\widetilde{A\times A},\mathbb{Z}) = b^*(H^5(A\times A,\mathbb{Z})) \oplus j_*b_{|E}^*(H^3(\Delta,\mathbb{Z})). \tag{12}$$

It follows that

$$H^5(A^{[2]},\mathbb{Z}) \supset \pi_*(b^*(H^5(A\times A,\mathbb{Z}))) \oplus \pi_*j_*b_{|E}^*(H^3(\Delta,\mathbb{Z})).$$

Moreover, by Künneth formula, we have:

$$\begin{aligned} H^5(A\times A,\mathbb{Z}) &= H^1(A,\mathbb{Z})\otimes H^4(A,\mathbb{Z}) \oplus H^2(A,\mathbb{Z})\otimes H^3(A,\mathbb{Z}) \\ &\oplus H^3(A,\mathbb{Z})\otimes H^2(A,\mathbb{Z}) \oplus H^4(A,\mathbb{Z})\otimes H^1(A,\mathbb{Z}). \end{aligned}$$

As before, $\pi_*(b^*(H^5(A\times A,\mathbb{Z})))$ is primitive in $H^5(A^{[2]},\mathbb{Z})$. Moreover by (12):

$$l_2^5(\widetilde{A\times A}) = \operatorname{rk} H^5(A\times A,\mathbb{Z})^{\mathfrak{S}_2} = 28, \tag{13}$$

and

$$l_{1,+}^5(\widetilde{A\times A}) = \operatorname{rk} H^3(\Delta,\mathbb{Z})^{\mathfrak{S}_2} = 4, \quad \text{and} \quad l_{1,-}^5(\widetilde{A\times A}) = 0.$$

Lemma 7.6. *The lattice $\pi_*(H^3(\widetilde{A\times A},\mathbb{Z}) \oplus H^5(\widetilde{A\times A},\mathbb{Z}))$ has discriminant 2^8.*

Proof. By Proposition 6.3 (ii):

$$\frac{H^3(\widetilde{A\times A},\mathbb{Z})\oplus H^5(\widetilde{A\times A},\mathbb{Z})}{H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus\left(H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\right)^{\perp}} = (\mathbb{Z}/2\mathbb{Z})^{l_2^3(\widetilde{A\times A})+l_2^5(\widetilde{A\times A})}.$$

Since $H^3(\widetilde{A\times A},\mathbb{Z})\oplus H^5(\widetilde{A\times A},\mathbb{Z})$ is an unimodular lattice, it follows that

$$\operatorname{discr}\left[H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\right] = 2^{l_2^3(\widetilde{A\times A})+l_2^5(\widetilde{A\times A})}.$$

Then by Proposition 6.5,

$$\operatorname{discr}\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}) = 2^{l_2^3(\widetilde{A\times A})+l_2^5(\widetilde{A\times A})+\operatorname{rk}\left[H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\right]}.$$

Then by Proposition 6.3 (i):

$$\operatorname{discr}\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}) = 2^{2\left(l_2^3(\widetilde{A\times A})+l_2^5(\widetilde{A\times A})\right)+l_{1,+}^3(\widetilde{A\times A})+l_{1,+}^5(\widetilde{A\times A})}.$$

By Remark 6.4 and since $\pi_*(x+\iota^*(x)) = 2\pi_*(x)$, we have:

$$\frac{\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})\oplus H^5(\widetilde{A\times A},\mathbb{Z}))}{\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})^{\iota}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\iota})} = (\mathbb{Z}/2\mathbb{Z})^{l_2^3(\widetilde{A\times A})+l_2^5(\widetilde{A\times A})}.$$

It follows:

$$\operatorname{discr}\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})\oplus H^5(\widetilde{A\times A},\mathbb{Z})) = 2^{l_{1,+}^3(\widetilde{A\times A})+l_{1,+}^5(\widetilde{A\times A})} = 2^8.$$

□

The lattice $H^3(A^{[2]},\mathbb{Z})\oplus H^5(A^{[2]},\mathbb{Z})$ is unimodular. Hence:

$$\frac{H^3(A^{[2]},\mathbb{Z})\oplus H^5(A^{[2]},\mathbb{Z})}{\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})\oplus H^5(\widetilde{A\times A},\mathbb{Z}))}=(\mathbb{Z}/2\mathbb{Z})^4.$$

However, from the last section, we know that $\pi_*(H^3(\widetilde{A\times A},\mathbb{Z}))=H^3(A^{[2]},\mathbb{Z})$. It follows that

$$\frac{H^5(A^{[2]},\mathbb{Z})}{\pi_*(H^5(\widetilde{A\times A},\mathbb{Z}))}=(\mathbb{Z}/2\mathbb{Z})^4.$$

Then by the same argument used in the end of Section 7.2, we can see that the elements in $\frac{H^5(A^{[2]},\mathbb{Z})}{\pi_*(H^5(\widetilde{A\times A},\mathbb{Z}))}$ are provided by $\frac{1}{2}\pi_*j_*b^*_{|E}(H^3(\Delta,\mathbb{Z}))$.

8 Nakajima operators for Hilbert schemes of points on surfaces

Let A be a smooth projective complex surface. Let $A^{[n]}$ the Hilbert scheme of n points on the surface, *i.e.* the moduli space of finite subschemes of A of length n. $A^{[n]}$ is again smooth and projective of dimension $2n$, cf. [11]. Their rational cohomology can be described in terms of Nakajima's [39] operators. First consider the direct sum

$$\mathbb{H}:=\bigoplus_{n=0}^{\infty}H^*(A^{[n]},\mathbb{Q}).$$

This space is bigraded by cohomological *degree* and the *weight*, which is given by the number of points n. The unit element in $H^0(A^{[0]},\mathbb{Q})\cong\mathbb{Q}$ is denoted by $|0\rangle$, called the *vacuum*.

Definition-Proposition 8.1. *There are linear operators $\mathfrak{q}_m(a)$, for each $m\geq 1$ and $a\in H^*(A,\mathbb{Q})$, acting on $\mathbb{H}$ which have the following properties: They depend linearly on a, and if $a\in H^k(A,\mathbb{Q})$ is homogeneous, the operator $\mathfrak{q}_m(a)$ is bihomogeneous of degree $k+2(m-1)$ and weight m:*

$$\mathfrak{q}_m(a):H^l(A^{[n]})\to H^{l+k+2(m-1)}(A^{[n+m]})$$

To construct them, first define incidence varieties $\mathcal{Z}_m\subset A^{[n]}\times A\times A^{[n+m]}$ by

$$\mathcal{Z}_m:=\{(\xi,x,\xi')\,|\,\xi\subset\xi',\,\mathrm{supp}(\xi')-\mathrm{supp}(\xi)=mx\}.$$

Then $\mathfrak{q}_m(a)(\beta)$ is defined as the Poincaré dual of

$$\mathrm{pr}_{3*}\left((\mathrm{pr}_2^*(\alpha)\cdot\mathrm{pr}_3^*(\beta))\cap[\mathcal{Z}_m]\right).$$

Consider now the superalgebra generated by the $\mathfrak{q}_m(a)$. Every element in $\mathbb{H}$ can be decomposed uniquely as a linear combination of products of operators $\mathfrak{q}_m(a)$, acting on the vacuum. In other words, the $\mathfrak{q}_m(a)$ generate $\mathbb{H}$ and there are no algebraic relations between them (except the linearity in a and the super-commutativity).

Example 8.2. The unit $1_{A^{[n]}}\in H^0(A^{[n]},\mathbb{Q})$ is given by $\frac{1}{n!}\mathfrak{q}_1(1)^n|0\rangle$. The sum of all $1_{A^{[n]}}$ in the formal completion of $\mathbb{H}$ is sometimes denoted by $|1\rangle:=\exp(\mathfrak{q}_1(1))|0\rangle$.

Definition 8.3. To give the cup product structure of $\mathbb{H}$, define operators $\mathfrak{G}(a)$ for $a\in H^*(A)$. Let $\Xi_n\subset A^{[n]}\times A$ be the universal subscheme. Then the action of $\mathfrak{G}(a)$ on $H^*(A^{[n]})$ is multiplication with the class

$$\mathrm{pr}_{1*}\left(\mathrm{ch}(\mathcal{O}_{\Xi_n})\cdot\mathrm{pr}_2^*(\mathrm{td}(A)\cdot a)\right)\in H^*(A^{[n]}).$$

For $a\in H^k(A)$, we define $\mathfrak{G}_i(a)$ as the component of $\mathfrak{G}(a)$ of cohomological degree $k+2i$. A differential operator $\mathfrak{d}$ is given by $\mathfrak{G}_1(1)$. It means multiplication with the first Chern class of the tautological sheaf $\mathrm{pr}_{1*}(\mathcal{O}_{\Xi_n})$.

Notation 8.4. We abbreviate $\mathfrak{q} := \mathfrak{q}_1(1)$ and for its derivative $\mathfrak{q}' := [\mathfrak{d}, \mathfrak{q}]$. For any operator X we write $X^{(k)}$ for the k-fold derivative: $X^{(k)} := \mathrm{ad}^k(\mathfrak{d})(X)$.

In [22] and [24] we find various commutation relations between these operators, that allow to determine all multiplications in the cohomology of the Hilbert scheme. First of all, if X and Y are operators of degree d and d', their commutator is defined in the super sense:

$$[X, Y] := XY - (-1)^{dd'} YX.$$

The integral on $A^{[n]}$ induces a non-degenerate bilinar form on $\mathbb{H}$: for classes α, $\beta \in H^*(A^{[n]})$ it is given by

$$(\alpha, \beta)_{A^{[n]}} := \int_{A^{[n]}} \alpha \cdot \beta.$$

If X is a homogeneous linear operator of degree d and weight m, acting on $\mathbb{H}$, define its adjont $X^\dagger$ by

$$(X(\alpha), \beta)_{A^{[n+m]}} = (-1)^{d|\alpha|}(\alpha, X^\dagger(\beta))_{A^{[n]}}.$$

We put $\mathfrak{q}_0(a) := 0$ and for $m < 0$, $\mathfrak{q}_m(a) := (-1)^m \mathfrak{q}_{-m}(a)^\dagger$. Note that, for all $m \in \mathbb{Z}$, the bidegree of $\mathfrak{q}_m(a)$ is $(m, |a| + 2(|m| - 1))$. If m is positive, $\mathfrak{q}_m$ is called a creation operator, otherwise it is called annihiliation operator. Now define

$$\mathfrak{L}_m(a) := \begin{cases} \frac{1}{2} \sum\limits_{k\in\mathbb{Z}} \sum\limits_{i} \mathfrak{q}_k(a_{(1)})\mathfrak{q}_{m-k}(a_{(2)}), & \text{if } m \neq 0, \\ \sum\limits_{k>0} \sum\limits_{i} \mathfrak{q}_k(a_{(1)})\mathfrak{q}_{-k}(a_{(2)}), & \text{if } m = 0. \end{cases}$$

where $\sum_i a_{(1)} \otimes a_{(2)}$ is the push-forward of a along the diagonal $\tau_2 : A \to A \times A$ (in Sweedler notation).

Remark 8.5. This can be expressed more elegantly using normal ordering: the operator $:\mathfrak{q}_m\mathfrak{q}_n:(a\otimes b)$ is defined in a way such that the annihilation operator act first. Then we may write $\mathfrak{L}_m(a) = \sum_k :\mathfrak{q}_k\mathfrak{q}_{m-k}:(\tau_{2*}(a))$.

Remark 8.6. In a similar manner as above, we can use the integral over A to define a bilinear form on $H^*(A, \mathbb{Q})$. The adjoint of the multiplication map gives a coassiocative comultiplication

$$\Delta : H^*(A, \mathbb{Q}) \longrightarrow H^*(A, \mathbb{Q}) \otimes H^*(A, \mathbb{Q})$$

that corresponds to τ_{2*}. The sign convention in [22] is such that $-\Delta = \tau_{2*}$. We denote by Δ^k the k-fold composition of Δ.

Lemma 8.7. *[24, Thm. 2.16] Denote $K_A \in H^2(A, \mathbb{Q})$ the class of the canonical divisor. We have:*

$$[\mathfrak{q}_m(a), \mathfrak{q}_n(b)] = m \cdot \delta_{m+n} \cdot \int_A ab \tag{14}$$

$$[\mathfrak{L}_m(a), \mathfrak{q}_n(b)] = -n \cdot \mathfrak{q}_{m+l}(ab) \tag{15}$$

$$[\mathfrak{d}, \mathfrak{q}_m(a)] = m \cdot \mathfrak{L}_m(a) + \tfrac{m(|m|-1)}{2} \mathfrak{q}_m(K_A a) \tag{16}$$

$$[\mathfrak{L}_m(a), \mathfrak{L}_n(b)] = (m - n)\mathfrak{L}_{m+n}(ab) - \frac{m^3 - m}{12} \delta_{m+n} \int_A abe \tag{17}$$

$$[\mathfrak{G}(a), \mathfrak{q}_1(b)] = \exp(\mathrm{ad}(\mathfrak{d}))(\mathfrak{q}_1(ab)) \tag{18}$$

$$[\mathfrak{G}_i(a), \mathfrak{q}_1(b)] = \tfrac{1}{k!} \,\mathrm{ad}(\mathfrak{d})^k(\mathfrak{q}_1(ab)) \tag{19}$$

Remark 8.8. Note (cf. [22, Thm. 3.8]) that (15) together with (16) imply that

$$\mathfrak{q}_{m+1}(a) = \tfrac{(-1)^m}{m!} (\mathrm{ad}\, \mathfrak{q}')^m (\mathfrak{q}_1(a)), \tag{20}$$

so there are two ways of writing an element of $\mathbb{H}$: As a linear combination of products of creation operators $\mathfrak{q}_m(a)$ or as a linear combination of products of the operators $\mathfrak{d}$ and $\mathfrak{q}_1(a)$. This second representation is more suitable for computing cup-products, but not faithful. Equations (16) and (20) permit now to switch between the two representations, using that

$$\mathfrak{d}|0\rangle = 0, \tag{21}$$

$$\mathfrak{L}_m(a)|0\rangle = \begin{cases} \frac{1}{2}\sum_{k=1}^{m-1}\sum_i \mathfrak{q}_k(a_{(1)})\mathfrak{q}_{m-k}(a_{(2)})|0\rangle, & \text{if } m > 1, \\ 0, & \text{if } m \leq 1. \end{cases} \tag{22}$$

(23)

Next we give some formulas involving higher derivatives of Nakajima operators that can be of use in formal computations.

Proposition 8.9. *Suppose $K_A a = 0$. Denote $\mathbf{e} := -\chi(A)x$ the Euler class of A. Note that if A is a torus, $\mathbf{e} = 0$. For all k, m, the following formulas hold:*

$$\operatorname{ad}\mathfrak{q}\frac{\mathfrak{q}_m^{(k+1)}(a)}{m^{k+1}} = (k+1)\frac{\mathfrak{q}_{m+1}^{(k)}(a)}{(m+1)^k} + \frac{k^3-k}{24}\frac{\mathfrak{q}_{m+1}^{(k-2)}(a\mathbf{e})}{(m+1)^{k-2}}, \tag{24}$$

$$\operatorname{ad}\mathfrak{q}'\frac{\mathfrak{q}_m^{(k)}(a)}{m^k} = (k-m)\frac{\mathfrak{q}_{m+1}^{(k)}(a)}{(m+1)^k} + \frac{k(k-1)(k-3m-2)}{24}\frac{\mathfrak{q}_{m+1}^{(k-2)}(a\mathbf{e})}{(m+1)^{k-2}}. \tag{25}$$

Proof. Let us start with (24). This is a consequence of Theorem 4.2 of [23] which states that

$$\begin{aligned} \frac{\mathfrak{q}_m^{(k)}(a)}{m^k} &= \frac{1}{k+1}\sum_{i_0+\ldots+i_k=m} {:}\mathfrak{q}_{i_0}\cdots\mathfrak{q}_{i_k}{:}(\tau_*(a)) \\ &\quad + k\sum_{j_0+\ldots+j_{k-2}=m}\frac{j_0^2+\ldots+j_{k-2}^2-1}{24}{:}\mathfrak{q}_{j_0}\cdots\mathfrak{q}_{j_{k-2}}{:}(\tau_*(a\mathbf{e})). \end{aligned}$$

Using that $[\mathfrak{q}, {:}\mathfrak{q}_{i_0}\cdots\mathfrak{q}_{i_k}{:}(\Delta^k(a))] = \sum_{r=0}^k \delta_{i_r+1}\, {:}\mathfrak{q}_{i_0}\cdots\widehat{\mathfrak{q}}_{i_r}\cdots\mathfrak{q}_{i_k}{:}(\tau_*(a))$, we calculate:

$$\begin{aligned} \operatorname{ad}\mathfrak{q}\frac{\mathfrak{q}_m^{(k+1)}(a)}{m^{k+1}} &= \frac{1}{k+2}\sum_{i_0+\ldots+i_{k+1}=m}\left[\mathfrak{q}, {:}\mathfrak{q}_{i_0}\cdots\mathfrak{q}_{i_{k+1}}{:}(\tau_*(a))\right] \\ &\quad + (k+1)\sum_{j_0+\ldots+j_{k-1}=m}\frac{j_0^2+\ldots+j_{k-1}^2-1}{24}\left[\mathfrak{q}, {:}\mathfrak{q}_{j_0}\cdots\mathfrak{q}_{j_{k-1}}{:}(\tau_*(a\mathbf{e}))\right] \\ &= \sum_{i_0+\ldots+i_k=m+1} {:}\mathfrak{q}_{i_0}\cdots\mathfrak{q}_{i_k}{:}(\tau_*(a)) \\ &\quad + k(k+1)\sum_{j_0+\ldots+j_{k-2}=m+1}\frac{j_0^2+\ldots+j_{k-2}^2}{24}{:}\mathfrak{q}_{j_0}\cdots\mathfrak{q}_{j_{k-2}}{:}(\tau_*(a\mathbf{e})) \\ &= (k+1)\frac{\mathfrak{q}_{m+1}^{(k)}(a)}{(m+1)^k} + \frac{k^3-k}{24}\frac{\mathfrak{q}_{m+1}^{(k-2)}(a\mathbf{e})}{(m+1)^{k-2}}. \end{aligned}$$

Equation (25) follows from (24) using the Jacobi identity: $\operatorname{ad}\mathfrak{q}' = \operatorname{ad}[\mathfrak{d},\mathfrak{q}] = \operatorname{ad}\mathfrak{d}\operatorname{ad}\mathfrak{q} - \operatorname{ad}\mathfrak{q}\operatorname{ad}\mathfrak{d}$. □

Corollary 8.10. *Suppose $K_A a = 0$. Iterated application of the above lemma gives*

$$\operatorname{ad}(\mathfrak{q})^s\frac{\mathfrak{q}_m^{(k+s)}(a)}{m^{k+s}(k+s)!} = \frac{\mathfrak{q}_{m+s}^{(k)}(a)}{(m+s)^k k!} + \frac{s}{24}\frac{\mathfrak{q}_{m+s}^{(k-2)}(a\mathbf{e})}{(m+s)^{k-2}(k-2)!}. \tag{26}$$

Proposition 8.11. *Suppose $K_A a = 0$. In the formal completion of $\mathbb{H}$ we have:*

$$[\mathfrak{G}(a), \exp(\mathfrak{q})] = \exp(\mathfrak{q})\sum_{\substack{s\geq 1\\ k\geq 0}}\frac{(-1)^{s-1}}{s!}\left(\frac{\mathfrak{q}_s^{(k)}(a)}{s^k k!} + \frac{s-1}{24}\frac{\mathfrak{q}_s^{(k)}(a\mathbf{e})}{s^k k!}\right).$$

Proof. Equation (4.6) of [22] evaluates

$$
\begin{aligned}
[\mathfrak{G}(a), \exp(\mathfrak{q})] &= \exp(\mathfrak{q}) \sum_{\substack{s\geq 1\\ k\geq 0}} \frac{(-\operatorname{ad}\mathfrak{q})^{s-1}}{s!} \left(\frac{(\operatorname{ad}\mathfrak{d})^k}{k!}(\mathfrak{q}_1(a)) \right) \\
&\overset{\text{Cor 8.10}}{=} \exp(\mathfrak{q}) \sum_{s\geq 1} \frac{(-1)^{s-1}}{s!} \left(\sum_{k\geq s-1} \frac{\mathfrak{q}_s^{(k-s+1)}(a)}{s^{k-s+1}(k-s+1)!} + \sum_{k\geq s+1} \frac{s-1}{24} \frac{\mathfrak{q}_s^{(k-s-1)}(a)}{s^{k-s-1}(k-s-1)!} \right) \\
&= \exp(\mathfrak{q}) \sum_{\substack{s\geq 1\\ k\geq 0}} \frac{(-1)^{s-1}}{s!} \left(\frac{\mathfrak{q}_s^{(k)}(a)}{s^k k!} + \frac{s-1}{24} \frac{\mathfrak{q}_s^{(k)}(a\mathbf{e})}{s^k k!} \right).
\end{aligned}
$$

□

Example 8.12.

$$\mathfrak{G}_0(a)\mathfrak{q}^n|0\rangle = n \cdot \mathfrak{q}^{n-1}\mathfrak{q}_1(a)|0\rangle, \tag{27}$$

$$\mathfrak{G}_1(a)\mathfrak{q}^n|0\rangle = -\binom{n}{2}\mathfrak{q}^{n-2}\mathfrak{q}_2(a)|0\rangle, \tag{28}$$

$$\mathfrak{G}_2(a)\mathfrak{q}^n|0\rangle = \binom{n}{3}\mathfrak{q}^{n-3}\mathfrak{q}_3(a)|0\rangle - \binom{n}{2}\mathfrak{q}^{n-2}\mathfrak{L}_2(a)|0\rangle. \tag{29}$$

Remark 8.13. We adopted the notation from [24], which differs from the conventions in [22]. Here is part of a dictionary:

Notation from [24]	Notation from [22]
operator of weight w and degree d	operator of weight w and degree $d-2w$
$\mathfrak{q}_m(a)$	$\mathfrak{p}_{-m}(a)$
$\mathfrak{L}_m(a)$	$-L_{-m}(a)$
$\mathfrak{G}(a)$	$a^{[\bullet]}$
$\mathfrak{d}$	∂
$\tau_{2*}(a)$	$-\Delta(a)$

By sending a subscheme in A to its support, we define a morphism

$$\rho : A^{[n]} \longrightarrow \operatorname{Sym}^n(A),$$

called the Hilbert–Chow morphism. The cohomology of $\operatorname{Sym}^n(A)$ is given by elements of the n-fold tensor power of $H^*(A)$ that are invariant under the action of the group of permutations $\mathfrak{S}_n$. A class in $H^*(A^{[n]}, \mathbb{Q})$ which can be written using only the operators $\mathfrak{q}_1(a)$ of weight 1 comes from a pullback along ρ:

$$\mathfrak{q}_1(b_1)\cdots\mathfrak{q}_1(b_n)|0\rangle = \rho^*\left(\sum_{\pi\in\mathfrak{S}_n} \pm b_{\pi(1)} \otimes \ldots \otimes b_{\pi(n)} \right), \quad b_i \in H^*(A, \mathbb{Q}), \tag{30}$$

where signs arise from permuting factors of odd degrees. In particular,

$$\frac{1}{n!}\mathfrak{q}_1(b)^n|0\rangle = \rho^*\big(b \otimes \ldots \otimes b\big), \tag{31}$$

$$\frac{1}{(n-1)!}\mathfrak{q}_1(b)\mathfrak{q}^{n-1}|0\rangle = \rho^*\Big(b \otimes 1 \otimes \ldots \otimes 1 \;+\; \ldots \;+\; 1 \otimes \ldots \otimes 1 \otimes b\Big). \tag{32}$$

Remark 8.14. With the notation from Section 3, we have that

$$H^*(\operatorname{Sym}^n(A), \mathbb{Q}) \cong \mathbb{S}\mathrm{ym}^n(H^*(A, \mathbb{Q})).$$

Under this isomorphism the ring structure of $\mathbb{S}\mathrm{ym}^n(H^*(A,\mathbb{Q}))$ corresponds to the cup product and the action of the operator $\mathfrak{q}_1(a)$ corresponds to the operation $a\,\diamond$.

9 On integral cohomology of Hilbert schemes

For the study of integral cohomology, first note that if $a \in H^*(A,\mathbb{Z})$ is an integral class, then $\mathfrak{q}_m(a)$ maps integral classes to integral classes. Such operators are called integral. Qin and Wang studied them in [43]. We need the following results:

Lemma 9.1. *[43] The operators $\frac{1}{n!}\mathfrak{q}_1(1)^n$ and $\frac{1}{2}\mathfrak{q}_2(1)$ are integral. Let $a \in H^2(A,\mathbb{Z})$ be monodromy equivalent to a divisor. Then the operator $\frac{1}{2}\mathfrak{q}_1(a)^2 - \frac{1}{2}\mathfrak{q}_2(a)$ is integral.*

Remark 9.2. Qin and Wang [43, Thm 1.1 et seq.] conjecture that their theory works even without the restriction on $a \in H^2(A,\mathbb{Z})$.

Corollary 9.3. *If A is a torus, the operator $\frac{1}{2}\mathfrak{q}_1(a)^2 - \frac{1}{2}\mathfrak{q}_2(a)$ is integral for all $a \in H^2(A,\mathbb{Z})$.*

Proof. The Nakajima operators are preserved under deformations of A. By [46, Thm. II], the group of monodromy actions spans the entire automorphism group of $H^2(A,\mathbb{Z})$. Since the lattice is even and contains two copies of the hyperbolic lattice, a theorem of Eichler [45, Prop. 3.7.3] states that the automorphism group of $H^2(A,\mathbb{Z})$ acts transitively on classes of the same norm.

Suppose now that the Néron-Severi group $\mathrm{NS}(A)$ contains a copy of the hyperbolic lattice U (such A exist). Since $U \subset \mathrm{NS}(A)$, there are divisors of arbitrary even norm, so every class can be mapped to a divisor by the action of a monodromy and Lemma 9.1 establishes proposition for that particular A. Now, since all tori are equivalent by deformation, a general torus can always be deformed to our special A. Since the integrality of an operator is a topolocial invariant, $\frac{1}{2}\mathfrak{q}_1(a)^2 - \frac{1}{2}\mathfrak{q}_2(a)$ remains integral for all a. □

Proposition 9.4. *Assume that $H^*(A,\mathbb{Z})$ is free of torsion. Let $(a_i) \subset H^1(A,\mathbb{Z})$ and $(b_i) \subset H^2(A,\mathbb{Z})$ be bases of integral cohomology. Denote $a_i^* \in H^3(A,\mathbb{Z})$ resp. $b_i^* \in H^2(A,\mathbb{Z})$ the elements of the dual bases. Let x be the generator of $H^4(A,\mathbb{Z})$. Modulo torsion, the following classes form a basis of $H^2(A^{[n]},\mathbb{Z})$:*

- $\frac{1}{(n-1)!}\mathfrak{q}_1(b_i)\mathfrak{q}_1(1)^{n-1}|0\rangle = \mathfrak{G}_0(b_i)1$,
- $\frac{1}{(n-2)!}\mathfrak{q}_1(a_i)\mathfrak{q}_1(a_j)\mathfrak{q}_1(1)^{n-2}|0\rangle = \mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j)1,\ i<j$,
- $\frac{1}{2(n-2)!}\mathfrak{q}_2(1)\mathfrak{q}_1(1)^{n-2}|0\rangle$. *We denote this class by $\delta = \mathfrak{d}1$.*

Their respective duals in $H^{2n-2}(A^{[n]},\mathbb{Z})$ are given by

- $\mathfrak{q}_1(b_i^*)\mathfrak{q}_1(x)^{n-1}|0\rangle$,
- $\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(x)^{n-2}|0\rangle,\ i<j$,
- $\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle$.

Proof. It is clear from the above lemma that these classes are all integral. Göttsche's formula [17, p. 35] gives the Betti numbers of $A^{[n]}$ in terms of the Betti numbers of A: $h^1(A^{[n]}) = h^1(A)$, and $h^2(A^{[n]}) = h^2(A) + \frac{h^1(A)(h^1(A)-1)}{2} + 1$. It follows that the given classes span a lattice of full rank.

Next we have to show that the intersection matrix between these classes is in fact the identity matrix. Most of the entries can be computed easily using the simplification from (30). For products involving δ (this is the action of $\mathfrak{d}$) or its dual, first observe that $\mathfrak{d}\mathfrak{q}_1(x)^m|0\rangle = 0$ and $\mathfrak{L}_1(a)\mathfrak{q}_1(x)^m|0\rangle = 0$ for every class a of degree at least 1. Then compute:

$$\delta \cdot \mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = \mathfrak{d}\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = 2\mathfrak{L}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = \mathfrak{q}_1(x)^n|0\rangle,$$
$$\mathfrak{d}\mathfrak{q}_1(b_i^*)\mathfrak{q}_1(x)^{n-1}|0\rangle = \mathfrak{L}_1(b_i^*)\mathfrak{q}_1(x)^{n-1}|0\rangle = 0,$$
$$\mathfrak{d}\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(x)^{n-2}|0\rangle = \left(\mathfrak{L}_1(a_j^*) + \mathfrak{q}_1(a_j^*)\mathfrak{d}\right)\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(x)^{n-2}|0\rangle =$$
$$= \left(-\mathfrak{q}_1(a_i^*)\mathfrak{L}_1(a_j^*) + \mathfrak{q}_1(a_j^*)\mathfrak{L}_1(a_i^*)\right)\mathfrak{q}_1(x)^{n-2}|0\rangle = 0,$$
$$\mathfrak{G}_0(b_i)\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = 0,$$
$$\mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j)\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = 0.$$

□

Remark 9.5. If A is a complex torus, a theorem of Markman [27] ensures that $H^*(A^{[n]}, \mathbb{Z})$ is free of torsion.

Proposition 9.6. *Let A be a complex abelian surface. Let $(b_i) \subset H^2(A, \mathbb{Z})$ be an integral basis. Then a basis of $H^*(A^{[2]}, \mathbb{Z})$ is given by the following classes.*

degree	*Betti number*	*class*	*multiplication with class*
0	*1*	$\frac{1}{2}\mathfrak{q}_1(1)^2\vert 0\rangle$	id
1	*4*	$\mathfrak{q}_1(1)\mathfrak{q}_1(a_i)\vert 0\rangle$	$\mathfrak{G}_0(a_i)$
2	*13*	$\frac{1}{2}\mathfrak{q}_2(1)\vert 0\rangle$	$\mathfrak{d}$
		$\mathfrak{q}_1(a_i)\mathfrak{q}_1(a_j)\vert 0\rangle$ *for* $i<j$	$\mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j)$
		$\mathfrak{q}_1(1)\mathfrak{q}_1(b_i)\vert 0\rangle$	$\mathfrak{G}_0(b_i)$
3	*32*	$\mathfrak{q}_2(a_i)\vert 0\rangle$	$-2\mathfrak{G}_1(a_i)$
		$\mathfrak{q}_1(a_i)\mathfrak{q}_1(b_j)\vert 0\rangle$	$\mathfrak{G}_0(a_i)\mathfrak{G}_0(b_j)$
		$\mathfrak{q}_1(1)\mathfrak{q}_1(a_i^*)\vert 0\rangle$	$\mathfrak{G}_0(a_i^*)$
4	*44*	$\left(\frac{1}{2}\mathfrak{q}_1(b_i)^2 - \frac{1}{2}\mathfrak{q}_2(b_i)\right)\vert 0\rangle$	$\frac{1}{2}\mathfrak{G}_0(b_i)^2 + \mathfrak{G}_1(b_i)$
		$\mathfrak{q}_1(a_i)\mathfrak{q}_1(a_j^*)\vert 0\rangle$	$\mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j^*)$
		$\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)\vert 0\rangle$ *for* $i\le j$	$\mathfrak{G}_0(b_i)\mathfrak{G}_0(b_j)$
5	*32*	$\frac{1}{2}\mathfrak{q}_2(a_i^*)\vert 0\rangle$	$-\mathfrak{G}_1(a_i^*)$
		$\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(b_j)\vert 0\rangle$	$\mathfrak{G}_0(a_i^*)\mathfrak{G}_0(b_j)$
		$\mathfrak{q}_1(a_i)\mathfrak{q}_1(x)\vert 0\rangle$	$\mathfrak{G}_0(a_i)\mathfrak{G}_0(x)$
6	*13*	$\mathfrak{q}_2(x)\vert 0\rangle$	$-2\mathfrak{G}_1(x)$
		$\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(a_j^*)\vert 0\rangle$ *for* $i<j$	$\mathfrak{G}_0(a_i^*)\mathfrak{G}_0(a_j^*)$
		$\mathfrak{q}_1(b_i)\mathfrak{q}_1(x)\vert 0\rangle$	$\mathfrak{G}_0(b_i)\mathfrak{G}_0(x)$
7	*4*	$\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(x)\vert 0\rangle$	$\mathfrak{G}_0(a_i^*)\mathfrak{G}_0(x)$
8	*1*	$\mathfrak{q}_1(x)^2\vert 0\rangle$	$\mathfrak{G}_0(x)^2$

Proof. The Betti numbers come from Göttsche's formula [17]. One computes the intersection matrix of all classes under the Poincaré duality pairing and finds that it is unimodular. So it remains to show that all these classes are integral. By Lemma 9.1 this is clear for all classes except those of the form $\frac{1}{2}\mathfrak{q}_2(a_i^*)|0\rangle \in H^5(A^{[2]}, \mathbb{Z})$.

Evaluating the Poincaré duality pairing between degrees 3 and 5 gives:

$$\mathfrak{q}_2(a_i)|0\rangle \cdot \mathfrak{q}_2(a_i^*)|0\rangle = 2,$$
$$\mathfrak{q}_1(a_i)\mathfrak{q}_1(b_j)|0\rangle \cdot \mathfrak{q}_1(a_i^*)\mathfrak{q}_1(b_j^*)|0\rangle = 1,$$
$$\mathfrak{q}_1(1)\mathfrak{q}_1(a_i^*)|0\rangle \cdot \mathfrak{q}_1(x)\mathfrak{q}_1(a_i)|0\rangle = 1,$$

while the other pairings vanish. Therefore, one of $\mathfrak{q}_2(a_i)|0\rangle$ and $\mathfrak{q}_2(a_i^*)|0\rangle$ must be divisible by 2. We can interpret $\mathfrak{q}_2(a_i)|0\rangle \in H^3(A^{[2]}, \mathbb{Z})$ and $\mathfrak{q}_2(a_i^*)|0\rangle \in H^5(A^{[2]}, \mathbb{Z})$ as classes concentrated on the exceptional divisor, that is, as elements of $\pi_* j_* H^*(E, \mathbb{Z})$. Indeed, the pushforward of a class $a \otimes 1 \in H^k(E, \mathbb{Z})$ is given by

$$\pi_* j_*(a \otimes 1) = \mathfrak{q}_2(a)|0\rangle \in H^{k+2}(A^{[n]}, \mathbb{Z}).$$

When pushing forward to the Hilbert scheme, the only possibility to get a factor 2 is in degree 5, by Proposition 7.1. $\square$

Part II
The Generalized Kummer fourfold

10 Generalized Kummer varieties and the morphism to the Hilbert scheme

Definition 10.1. Let A be a complex projective torus of dimension 2 and $A^{[n]}$, $n \geq 1$, the corresponding Hilbert scheme of points. Denote $\Sigma : A^{[n]} \to A$ the summation morphism, a smooth submersion that factorizes via the Hilbert–Chow morphism : $A^{[n]} \xrightarrow{\rho} \mathrm{Sym}^n(A) \xrightarrow{\sigma} A$. Then the generalized Kummer variety $K_{n-1}(A)$ is defined as the fiber over 0:

$$\begin{array}{ccc} K_{n-1}(A) & \xrightarrow{\theta} & A^{[n]} \\ \downarrow & & \downarrow{\scriptstyle \Sigma} \\ \{0\} & \longrightarrow & A \end{array} \tag{33}$$

Theorem 10.2. *[47, Theorem 2] The cohomology of the generalized Kummer, $H^*(K_{n-1}(A), \mathbb{Z})$, is torsion free.*

Our first objective is to collect some information about this pullback diagram. Recall Notation 5.6.

Proposition 10.3. *Let $\alpha_i := \frac{1}{(n-1)!}\mathfrak{q}_1(1)^{n-1}\mathfrak{q}_1(a_i)|0\rangle = \mathfrak{G}_0(a_i)1$. The class of the Poincaré dual of $K_{n-1}(A)$ in $H^4(A^{[n]}, \mathbb{Z})$ is given by*

$$[K_{n-1}(A)] = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4.$$

Proof. Since the generalized Kummer variety is the fiber over a point, its Poincaré dual must be the pullback of $x \in H^4(A)$ under Σ. But $\Sigma^*(x) = \Sigma^*(a_1) \cdot \Sigma^*(a_2) \cdot \Sigma^*(a_3) \cdot \Sigma^*(a_4)$, so we have to verify that $\Sigma^*(a_i) = \alpha_i$. To do this, we want to use the decomposition $\Sigma = \sigma\rho$. The pullback along σ of a class $a \in H^1(A, \mathbb{Q})$ on $H^1(\mathrm{Sym}^n(A), \mathbb{Q})$ is given by $a \otimes 1 \otimes \cdots \otimes 1 + \ldots + 1 \otimes \cdots \otimes 1 \otimes a$. It follows from (32) that $\Sigma^*(a_i) = \frac{1}{(n-1)!}\mathfrak{q}_1(1)^{n-1}\mathfrak{q}_1(a_i)|0\rangle$. □

The morphism θ induces a homomorphism of graded rings

$$\theta^* : H^*(A^{[n]}) \longrightarrow H^*(K_{n-1}(A)) \tag{34}$$

and by the projection formula, we have

$$\theta_*\theta^*(\alpha) = [K_{n-1}(A)] \cdot \alpha. \tag{35}$$

Proposition 10.4. *The kernel of θ^* is equal to the annihilator of $[K_{n-1}(A)]$.*

Proof. Assume $\alpha \in \ker(\theta^*)$. Then we have $[K_{n-1}(A)] \cdot \alpha = \theta_*\theta^*(\alpha) = 0$. Conversely, suppose that $[K_{n-1}(A)] \cdot \alpha = 0$. Then, for all $\beta \in H^*(A^{[n]})$, we have

$$\int_{K_{n-1}(A)} \theta^*(\alpha) \cdot \theta^*(\beta) = 0,$$

so $\theta^*(\alpha) \in \mathrm{Im}(\theta^*) \cap \mathrm{Im}(\theta^*)^{\perp}$. The Poincaré pairing on $K_{n-1}(A)$ is non-degenerate when restricted to $\mathrm{Im}(\theta^*)$. To see this, observe that the dual element of $\theta^*(\alpha)$ in $\mathrm{Im}(\theta^*)$ is given by $\theta^*(\alpha^*)$, where α^* is the Poincaré dual of $[K_{n-1}(A)] \cdot \alpha$. So it follows that $\mathrm{Im}(\theta^*) \cap \mathrm{Im}(\theta^*)^{\perp} = 0$. □

Corollary 10.5. *$\theta^*(\alpha) = \theta^*(\beta)$ if and only if $[K_{n-1}(A)] \cdot \alpha = [K_{n-1}(A)] \cdot \beta$.* □

Proposition 10.6. *The annihilator of $[K_{n-1}(A)]$ in $H^*(A^{[n]}, \mathbb{Q})$ is the ideal generated by $H^1(A^{[n]})$.*

Proof. Set $H = H^*(A, \mathbb{Q})$ and consider the exact sequence of H-modules

$$0 \longrightarrow J \longrightarrow H \xrightarrow{x\cdot} H.$$

It is clear that J is the ideal in H generated by $H^1(A, \mathbb{Q})$. Now denote $J^{(n)}$ the ideal generated by $H^1(\mathrm{Sym}^n(A), \mathbb{Q})$ in $H^*(\mathrm{Sym}^n(A), \mathbb{Q}) \cong \mathbb{S}\mathrm{ym}^n(H)$. By the freeness result of Lemma 3.5, tensoring with $\mathbb{S}\mathrm{ym}^n(H)$ yields another exact sequence of H-modules

$$0 \longrightarrow J^{(n)} \longrightarrow \mathbb{S}\mathrm{ym}^n(H) \xrightarrow{\sigma(x)\cdot} \mathbb{S}\mathrm{ym}^n(H).$$

Now let $\mathfrak{H}$ be the operator algebra spanned by products of $\mathfrak{d}$ and $\mathfrak{q}_1(a)$ for $a \in H^*(A)$. Let $\mathfrak{C}$ be the graded commutative subalgebra of $\mathfrak{H}$ generated by $\mathfrak{q}_1(a)$ for $a \in H^*(A)$. The action of $\mathfrak{H}$ on $|0\rangle$ gives $\mathbb{H}$ and the action of $\mathfrak{C}$ on $|0\rangle$ gives $\rho^*(H^*(\mathrm{Sym}^n(A), \mathbb{Q})) \cong \mathbb{S}\mathrm{ym}^n(H)$. By sending $\mathfrak{d}$ to the identity, we define a linear map $c : \mathfrak{H} \to \mathfrak{C}$. Denote $J^{[n]}$ the ideal generated by $H^1(A^{[n]}, \mathbb{Q})$ in $H^*(A^{[n]}, \mathbb{Q})$. We claim that for every $\mathfrak{y} \in \mathfrak{H}$:

$$\mathfrak{y}|0\rangle \in J^{[n]} \Leftrightarrow c(\mathfrak{y})|0\rangle \in J^{[n]}.$$

To see this, we remark that $H^1(A^{[n]}, \mathbb{Q}) \cong H^1(A, \mathbb{Q})$ and the multiplication with a class in $H^1(A^{[n]}, \mathbb{Q})$ is given by the operator $\mathfrak{G}_0(a)$ for some $a \in H^1(A, \mathbb{Q})$. Due to the fact that $\mathfrak{d}$ is also a multiplication operator (of degree 2), $\mathfrak{G}_0(a)$ commutes with $\mathfrak{d}$. It follows that for $\mathfrak{y} = \mathfrak{G}_0(a)\mathfrak{r}$ we have $c(\mathfrak{y}) = \mathfrak{G}_0(a)c(\mathfrak{r})$.

Now denote $\mathfrak{k}$ the multiplication operator with the class $[K_{n-1}(A)]$. We have: $[\mathfrak{k}, \mathfrak{d}] = 0$. Now let $y \in H^*(A^{[n]}, \mathbb{Q})$ be a class in the annihilator of $[K_{n-1}(A)]$. We can write $y = \mathfrak{y}|0\rangle$ for a $\mathfrak{y} \in \mathfrak{H}$. Choose $\tilde{y} \in \mathbb{S}\mathrm{ym}^n(H)$ in a way that $\rho^*(\tilde{y}) = c(\mathfrak{y})|0\rangle$. Then we have:

$$0 = [K_{n-1}(A)] \cdot y = \mathfrak{k}\,\mathfrak{y}|0\rangle = \mathfrak{k}\,c(\mathfrak{y})|0\rangle = \rho^*(\sigma^*(x) \cdot \tilde{y}).$$

Since ρ^* is injective, $\tilde{y}$ is in the annihilator of $\sigma^*(x)$, so $\tilde{y} \in J^{(n)}$. It follows that $c(\mathfrak{y})|0\rangle$ and y are in the ideal generated by $H^1(A^{[n]}, \mathbb{Q})$. ◻

Theorem 10.7. *[1, Théorème 4] $K_{n-1}(A)$ is a irreducible holomorphically symplectic manifold. In particular, it is simply connected and the canonical bundle is trivial.*

This implies that $H^2(K_{n-1}(A), \mathbb{Z})$ admits an integer-valued nondegenerated quadratic form (called Beauville–Bogomolov form) q which gives $H^2(K_{n-1}(A), \mathbb{Z})$ the structure of a lattice. Looking for instance, the useful table in the introduction of [44], we know that this lattice is isomorphic to $U^{\oplus 3} \oplus \langle -2n \rangle$, for $n \geq 3$. We have the Fujiki formula for $\alpha \in H^2(K_{n-1}(A), \mathbb{Z})$:

$$\int_{K_{n-1}(A)} \alpha^{2n-2} = n \cdot (2n-3)!! \cdot B(\alpha, \alpha)^{n-1} \tag{36}$$

Proposition 10.8. *Assume $n \geq 3$. Then θ^* is surjective on $H^2(A^{[n]}, \mathbb{Z})$.*

Proof. By [1, Sect. 7], $\theta^* : H^2(A^{[n]}, \mathbb{C}) \to H^2(K_{n-1}(A), \mathbb{C})$ is surjective. But by Proposition 1 of [8], the lattice structure of $\mathrm{Im}\,\theta^*$ is the same as of $H^2(K_{n-1}(A))$, so the image of $H^2(A^{[n]}, \mathbb{Z})$ must be primitive. The result follows. ◻

Notation 10.9. We have seen that, for $n \geq 3$,

$$H^2(K_{n-1}(A), \mathbb{Z}) \cong H^2(A, \mathbb{Z}) \oplus \langle \theta^*(\delta) \rangle.$$

We denote the injection : $H^2(A, \mathbb{Z}) \to H^2(K_{n-1}(A), \mathbb{Z})$ by j. It can be described by

$$j : a \longmapsto \frac{1}{(n-1)!}\mathfrak{q}_1(a)\mathfrak{q}_1(1)^{n-1}|0\rangle.$$

Further, we set $e := \theta^*(\delta)$. We give the following names for classes in $H^2(K_{n-1}(A), \mathbb{Z})$:

$$\begin{aligned} u_1 &:= j(a_1a_2), & v_1 &:= j(a_1a_3), & w_1 &:= j(a_1a_4), \\ u_2 &:= j(a_3a_4), & v_2 &:= j(a_4a_2), & w_2 &:= j(a_2a_3), \end{aligned}$$

These elements form a basis of $H^2(K_{n-1}(A), \mathbb{Z})$ with the following intersection relations under the Beauville-Bogomolov form:

$$B(u_1, u_2) = 1, \qquad B(v_1, v_2) = 1, \qquad B(w_1, w_2) = 1, \qquad B(e, e) = -2n,$$

and all other pairs of basis elements are orthogonal.

11 Odd Cohomology of the Generalized Kummer fourfold

Now we come to the special case $n = 3$, so we study $K_2(A)$, the Generalized Kummer fourfolds.

Proposition 11.1. *The Betti numbers of $K_2(A)$ are:* 1, 0, 7, 8, 108, 8, 7, 0, 1.

Proof. This follows from Göttsche's formula [17, page 49]. $\square$

By means of the morphism θ^*, we may express part of the cohomology of $K_2(A)$ in terms of Hilbert scheme cohomology. We have seen in Proposition 10.8 that θ^* is surjective for degree 2 and (by duality) also in degree 6. The next proposition shows that this also holds true for odd degrees.

Proposition 11.2. *A basis of $H^3(K_2(A), \mathbb{Z})$ is given by:*

$$\frac{1}{2}\theta^*\Big(\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(1)^2|0\rangle\Big), \tag{37}$$

$$\theta^*\Big(\mathfrak{q}_2(a_i)\mathfrak{q}_1(1)|0\rangle\Big). \tag{38}$$

and a dual basis of $H^5(K_2(A), \mathbb{Z})$ is given by:

$$\theta^*\Big(\mathfrak{q}_1(a_ia_j)\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(1)|0\rangle\Big) \text{ for any } j \neq i, \tag{39}$$

$$\frac{1}{2}\theta^*\Big(\mathfrak{q}_2(a_i^*)\mathfrak{q}_1(1)|0\rangle\Big). \tag{40}$$

Proof. The classes (37) are Poincaré dual to (39) and the classes (38) are Poincaré dual to (40) by direct computation:

$$\begin{aligned}\frac{1}{2}\theta^*\Big(\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(1)^2|0\rangle\Big)\cdot\theta^*\Big(\mathfrak{q}_1(a_ia_j)\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(1)|0\rangle\Big) &= \frac{1}{2}\theta^*\Big(\mathfrak{G}_0(a_i^*)\mathfrak{q}_1(a_ia_j)\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(1)|0\rangle\Big)\\ &= \frac{1}{2}[K_2(A)]\cdot\mathfrak{q}_1(a_ia_j)\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(a_i^*) = 1,\\ \frac{1}{2}\theta^*\Big(\mathfrak{q}_2(a_i)\mathfrak{q}_1(1)|0\rangle\Big)\cdot\theta^*\Big(\mathfrak{q}_2(a_i^*)\mathfrak{q}_1(1)|0\rangle\Big) &= \theta^*\Big(\mathfrak{G}_1(a_i)\mathfrak{q}_2(a_i^*)\mathfrak{q}_1(1)|0\rangle\Big)\\ &= [K_2(A)]\cdot\big(2\mathfrak{q}_3(x)-\mathfrak{q}_1(x)^2\mathfrak{q}_1(1)\big)\,|0\rangle = 0-1=-1.\end{aligned}$$

It remains to show that all classes are integral. It is clear from Lemma 9.1 that (37) is integral, while the integrality of (38) and (39) is obvious. By Proposition 9.6, $\frac{1}{2}\mathfrak{q}_2(a_i^*)|0\rangle$ is integral as well. If the operator $\mathfrak{q}_1(1)$ is applied, we get again an integral class. $\square$

The following corollary will be used in Part III.

Corollary 11.3. *Let A be an abelian surface and g be an automorphisms on A. Let $g^{[[3]]}$ be the automorphisms induced by g on $K_2(A)$. We have $H^3(K_2(A), \mathbb{Z}) \simeq H^1(A, \mathbb{Z}) \oplus H^3(A, \mathbb{Z})$ and the action of $g^{[[3]]}$ on $H^3(K_2(A), \mathbb{Z})$ is given by the action of g on $H^1(A, \mathbb{Z}) \oplus H^3(A, \mathbb{Z})$.*

Proof. Let $g^{[3]}$ be the involution on $A^{[3]}$ induced by g. We have $g^{[3]*}(\mathfrak{q}_2(a_i)\mathfrak{q}_1(1)|0\rangle) = \mathfrak{q}_2(g^*a_i)\mathfrak{q}_1(1)|0\rangle$ and $g^{[3]*}(\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(1)^2|0\rangle) = \mathfrak{q}_1(g^*a_i^*)\mathfrak{q}_1(1)^2|0\rangle$. Moreover, we have by definition, $g^{[[3]]*} \circ \theta^* = \theta^* \circ g^{[3]*}$. The result follows from Proposition 11.2. $\square$

12 Middle cohomology

The middle cohomology $H^4(K_2(A), \mathbb{Z})$ has been studied by Hassett and Tschinkel in [18]. We first recall some of their results, then we proceed by using θ^* to give a partial description of $H^4(K_2(A), \mathbb{Z})$ in terms of the well-understood cohomology of $A^{[3]}$. Finally, we find a basis of $H^4(K_2(A), \mathbb{Z})$ using the action of the monodromy group.

Notation 12.1. For each $\tau \in A$, denote W_τ the Briançon subscheme of $A^{[3]}$ supported enitrely at the point τ. If $\tau \in A[3]$ is a point of three-torsion, W_τ is actually a subscheme of $K_2(A)$. We will also use the symbol W_τ for the corresponding class in $H^4(K_2(A), \mathbb{Z})$. Further, set

$$W := \sum_{\tau \in A[3]} W_\tau.$$

For $p \in A$, denote Y_p the locus of all $\{x, y, p\}$ in $K_2(A)$. The corresponding class $Y_p \in H^4(K_2(A), \mathbb{Z})$ is independent of the choice of the point p. Then set $Z_\tau := Y_p - W_\tau$ and denote Π the lattice generated by all Z_τ, $\tau \in A[3]$.

Proposition 12.2. *Denote by* $\mathrm{Sym} := \mathrm{Sym}^2\left(H^2\left(K_2(A), \mathbb{Z}\right)\right) \subset H^4\left(K_2(A), \mathbb{Z}\right)$ *the span of products of integral classes in degree* 2. *Then*

$$\mathrm{Sym} + \Pi \;\subset\; H^4\left(K_2(A), \mathbb{Z}\right)$$

is a sublattice of full rank.

Proof. This follows from [18, Proposition 4.3]. □

In Section 4 of [18], one finds the following formula:

$$Z_\tau \cdot D_1 \cdot D_2 = 2 \cdot B(D_1, D_2), \tag{41}$$

for all D_1, D_2 in $H^2(K_2(A), \mathbb{Z})$, $\tau \in A[3]$ and B the Beauville-Bogomolov form on $K_2(A)$.

Definition 12.3. We define $\Pi' := \Pi \cap \mathrm{Sym}^\perp$. It follows from (41) that Π' can be described as the span of all classes of the form $Z_\tau - Z_0$ or alternatively as the set of all $\sum_\tau \alpha_\tau Z_\tau$, such that $\sum_\tau \alpha_\tau = 0$. Note that in [18] the lattice Π' denotes something different.

Remark 12.4. Since $\mathrm{rk}\,\mathrm{Sym} = 28$ and $\mathrm{rk}\,\Pi' = 80$, the lattice $\mathrm{Sym} \oplus \Pi' \subset H^4\left(K_2(A), \mathbb{Z}\right)$ has full rank.

Proposition 12.5. *The class W can be written with the help of the square of half the diagonal as*

$$W = \theta^*\Big(\mathfrak{q}_3(1)|0\rangle\Big) \tag{42}$$

$$= 9Y_p + e^2. \tag{43}$$

The second Chern class is non-divisible and given by

$$c_2(K_2(A)) = \frac{1}{3} \sum_{\tau \in A[3]} Z_\tau \tag{44}$$

$$= \frac{1}{3}\Big(72Y_p - e^2\Big). \tag{45}$$

Proof. In Section 4 of [18] one finds the equations

$$W = \frac{3}{8}\Big(c_2(K_2(A)) + 3e^2\Big), \tag{46}$$

$$Y_p = \frac{1}{72}\Big(3c_2(K_2(A)) + e^2\Big), \tag{47}$$

from which we deduce (43) and (45). Equation (44) and the non-divisibility are from [18, Proposition 5.1]. □

Proposition 12.6. *The image of* $H^4(A^{[3]}, \mathbb{Q})$ *under the morphism* θ^* *is equal to* $\mathrm{Sym}^2 H^2(K_2(A), \mathbb{Q})$.

Proof. We start by giving set of generators of $H^4(A^{[n]}, \mathbb{Q})$. Theorem 5.30 of [24] ensures that it is possible to do this in terms of multiplication operators. To enumerate elements of $H^*(A, \mathbb{Q})$, we follow Notation 5.6. Bais elements of $H^2(A, \mathbb{Q})$ will be denoted by b_i for $1 \leq i \leq 6$. Then our set of generators is given by:

multiplication operator	number of classes
$\mathfrak{G}_0(a_1)\mathfrak{G}_0(a_2)\mathfrak{G}_0(a_3)\mathfrak{G}_0(a_4)$	1
$\mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j)\mathfrak{G}_0(b_k)$ for $i<j$	$\binom{4}{2}\cdot 6 = 36$
$\mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j^*)$	$4\cdot 4 = 16$
$\mathfrak{G}_0(b_i)\mathfrak{G}_0(b_j)$ for $i\le j$	$\binom{6+1}{2} = 21$
$\mathfrak{G}_0(x)$	1
$\mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j)\mathfrak{G}_1(1)$ for $i<j$	$\binom{4}{2} = 6$
$\mathfrak{G}_0(a_i)\mathfrak{G}_1(a_j)$	$4\cdot 4 = 16$
$\mathfrak{G}_0(b_i)\mathfrak{G}_1(1)$	6
$\mathfrak{G}_1(b_i)$	6
$\mathfrak{G}_1(1)^2$	1
$\mathfrak{G}_2(1)$	1

Any multiplication operator of degree 4 can be written as a linear combination of these 111 classes. Likewise, the dimension of $H^4(A^{[n]},\mathbb{Q})$ is 111 for all $n\ge 4$, according to Göttsche's formula [17]. However, for $n=3$, the 8 classes $\mathfrak{G}_0(x)$, $\mathfrak{G}_1(b_i)$ and $\mathfrak{G}_2(1)$ can be expressed as linear combinations of the others, so we are left with 103 linearly independent classes that form a basis of $H^4(A^{[3]},\mathbb{Q})$. Multiplication with the class $[K_2(A)]$ is given by the operator $\mathfrak{G}_0(a_1)\mathfrak{G}_0(a_2)\mathfrak{G}_0(a_3)\mathfrak{G}_0(a_4)$ and annihilates every class that contains an operator of the form $\mathfrak{G}_0(a_i)$. There are 75 such classes, so by Proposition 10.4, $\ker\theta^*\subset H^4(A^{[3]},\mathbb{Q})$ has dimension at least 75 and $\operatorname{Im}\theta^*$ has dimension at most $103-75=28$. However, since the image of θ^* must contain $\operatorname{Sym}^2 H^2(K_2(A),\mathbb{Q})$, which is 28-dimensional, equality follows. □

Proposition 12.7. *We have:*

$$c_2(K_2(A)) = 4u_1u_2 + 4v_1v_2 + 4w_1w_2 - \frac{1}{3}e^2. \tag{48}$$

In particular, $c_2(K_2(A))\in\operatorname{Sym}$.

We shall give two different proofs. The first one uses Nakajima operators, the second one is based on results of [18].

Proof 1. First note that the defining diagram (33) of the Kummer manifold is the pullback of the inclusion of a point, so the normal bundle of $K_2(A)$ in $A^{[3]}$ is trivial. The Chern class of the tangent bundle of $K_2(A)$ is therefore given by the pullback from $A^{[3]}$: $c(K_2(A)) = \theta^*\big(c(A^{[3]})\big)$. Proposition 12.6 allows now to conclude that $c_2(K_2(A))\in\operatorname{Sym}$.

To obtain the precise formula, we use a result of Boissière, [2, Lemma 3.12], giving a commutation relation for the Chern character multiplication operator on the Hilbert scheme. We get:

$$\begin{aligned} c_2(A^{[3]}) &= 3\mathfrak{q}_1(1)\mathfrak{L}_2(1)|0\rangle - \frac{1}{3}\mathfrak{q}_3(1)|0\rangle \\ &= \frac{8}{3}\mathfrak{q}_1(1)\mathfrak{L}_2(1)|0\rangle - \frac{1}{3}\delta^2. \end{aligned}$$

With Corollary 10.5 one shows now, that $c_2(K_2(A))$ is given as stated. □

Proof 2. It follows from (44) that $c_2(K_2(A))\in\Pi'^{\perp}$, so $c_2(K_2(A))\in\operatorname{Sym}\otimes\mathbb{Q}$. Moreover, together with (41) we get that

$$c_2(K_2(A))\cdot D_1\cdot D_2 = 54\cdot B(D_1,D_2)$$

for all D_1, D_2 in $H^2(K_2(A),\mathbb{Z})$. Using the non-degeneracy of the Poincaré pairing and our knowledge about the Beauville–Bogomolov form on $K_2(A)$, we can calculate that

$$c_2(K_2(A)) = 4u_1u_2 + 4v_1v_2 + 4w_1w_2 - \frac{1}{3}e^2. \quad \square$$

Corollary 12.8. *The intersection* $\operatorname{Sym}\cap\Pi$ *is one-dimensional and spanned by* $c_2(K_2(A))$.

Proof. By Proposition 12.7 and (44), $c_2(K_2(A))\in\operatorname{Sym}\cap\Pi$. Since the ranks of Sym, Π and $H^4(K_2(A),\mathbb{Z})$ are 28, 81 and 108, respectively, the intersection cannot contain more. □

Corollary 12.9.

$$Y_p = \frac{1}{6}\Big(u_1u_2 + v_1v_2 + w_1w_2\Big). \tag{49}$$

Remark 12.10. Using Nakajima operators, we may write

$$Y_p = \frac{1}{9}\theta^*\Big(\mathfrak{q}_1(1)\mathfrak{L}_2(1)|0\rangle\Big) = \frac{1}{2}\theta^*\Big(\mathfrak{q}_1(x)\mathfrak{q}_1(1)^2|0\rangle\Big). \tag{50}$$

From the intersection properties $Z_\tau \cdot Z_{\tau'} = 1$ for $\tau \neq \tau'$ and $Z_\tau^2 = 4$ from Section 4 of [18], we compute

$$\operatorname{discr}\Pi' = 3^{84}. \tag{51}$$

On the other hand, a formula developed in [21] evaluates

$$\operatorname{discr}\mathrm{Sym} = 2^{14} \cdot 3^{38}, \tag{52}$$

so the lattices Sym and Π' cannot be primitive. Denote Sym^{sat} and Π'^{sat} the respective primitive overlattices. $\mathrm{Sym} \oplus \Pi'$ is a sublattice of $H^4(K_2(A),\mathbb{Z})$ of index $2^7 \cdot 3^{61}$ and we claim that $\mathrm{Sym}^{sat} \oplus \Pi'^{sat}$ has index 3^{22}. To obtain a basis of $H^4(K_2(A),\mathbb{Z})$, we are now going to find

- 7 classes in Sym divisible by 2,
- 8 classes in Sym divisible by 3,
- 31 classes in Π' divisible by 3 and
- 20 classes in $\mathrm{Sym}^{sat} \oplus \Pi'^{sat}$, one divisible by 3^3 and 19 divisible by 3.

Proposition 12.11. *For $y \in \{u_1, u_2, v_1, v_2, w_1, w_2\}$, the class $e \cdot y$ is divisible by* 3 *and $y^2 - \frac{1}{3}e \cdot y$ is divisible by* 2*.*

Proof. We have $y = \theta^*\left(\mathfrak{q}_1(a)\mathfrak{q}_1(1)^2|0\rangle\right)$ for some $a \in H^2(A,\mathbb{Z})$. A computation yields:

$$e \cdot y = 3 \cdot \theta^*\Big(\mathfrak{q}_2(a)\mathfrak{q}_1(1)|0\rangle\Big) \quad \text{and} \quad y^2 = \theta^*\Big(\mathfrak{q}_1(a)^2\mathfrak{q}_1(1)|0\rangle\Big)$$

so $e \cdot y$ is divisible by 3. Furthermore, by Corollary 9.3, the class $\frac{1}{2}\mathfrak{q}_1(a)^2\mathfrak{q}_1(1)|0\rangle - \frac{1}{2}\mathfrak{q}_2(a)\mathfrak{q}_1(1)|0\rangle$ is contained in $H^4(A^{[3]},\mathbb{Z})$, so its pullback $\frac{1}{2}y^2 - \frac{1}{6}e \cdot y$ is an integral class, too. □

From Proposition 12.7 we see that e^2 is divisible by 3 and by Corollary 12.9 the class $u_1u_2 + v_1v_2 + w_1w_2$ is divisible by 6.

Corollary 12.12. *The image of $H^4(A^{[3]},\mathbb{Z})$ under θ^* is equal to* Sym^{sat}*.* □

Now we come to Π'. By applying a suitable deformation, we may achieve that A is the product of two elliptic curves $A = E_1 \times E_2$. After [18, Eq. (12)], for a non-isotropic plane $\Lambda \subset A[3]$ and any $\tau_0 \in A[3]$, the classes

$$\frac{1}{3}\sum_{\tau \in \Lambda}\Big(Z_\tau - Z_{\tau+\tau_0}\Big) \tag{53}$$

are integral. The monodromy representation acts on the Z_τ via the symplectic group $\mathrm{Sp}(4,\mathbb{F}_3)$. Modulo Π', the orbit of these classes is a $\mathbb{F}_3$-vector space naturally isomorphic to D as introduced in Definition 4.5. By Proposition 4.6, we get a subspace of Π' of rank 31 of classes divisible by 3.

The class Z_0 is not contained in Sym nor in Π'. It can be written as follows:

$$Z_0 = \frac{\sum_{\tau \in A[3]} Z_\tau - \sum_{\tau \in A[3]}(Z_\tau - Z_0)}{81} \overset{(44)}{=} \frac{c_2(K_2(A)) - \frac{1}{3}\sum_{\tau \in A[3]}(Z_\tau - Z_0)}{27}.$$

This is the class in $\mathrm{Sym}^{sat} \oplus \Pi'^{sat}$ divisible by 27. Let us now find the remaining 19 classes divisible by 3.

Hassett and Tschinkel in Proposition 7.1 of [18], provide the class of a Lagrangian plane $P \subset K_2(A)$ which can be expressed as follows:

$$[P] = \frac{1}{216}(6u_1 - 3e)^2 + \frac{1}{8}c_2(K_2(A)) - \frac{1}{3}\sum_{\tau \in \Lambda'} Z_\tau,$$

where $\Lambda' = E_1[3] \times 0 \subset A[3]$. We rearrange this expression a bit using (46):

$$\begin{aligned}[P] &= \frac{1}{216}(6u_1 - 3e)^2 + \frac{1}{8}c_2(K_2(A)) - \frac{1}{3}\sum_{\tau \in \Lambda'} Z_\tau \\ &= \frac{36u_1^2 + 9e^2 - 36u_1 \cdot e}{216} + \frac{W}{3} - \frac{3}{8}e^2 - \frac{1}{3}\sum_{\tau \in \Lambda'} Z_\tau \\ &= \frac{u_1^2 - 2e^2 - u_1 \cdot e}{6} + \frac{W}{3} - \frac{1}{3}\sum_{\tau \in \Lambda'} Z_\tau.\end{aligned}$$

The classes e^2 and $u_1 \cdot e$ are both divisible by 3 and by (46), W is divisible by 3, so the following class is integral:

$$\mathfrak{N} := \frac{u_1^2 + \sum_{\tau \in \Lambda'}(Z_\tau - Z_0)}{3}.$$

Now we will conclude using the action of the monodromy group $\mathrm{Sp}(A[3]) \ltimes A[3]$ on the element $\mathfrak{N}$ and the considerations from Section 4. Proposition 4.8 states now that the orbit of $\mathfrak{N}$ under the action of $\mathrm{Sp}(A[3]) \ltimes A[3]$ gives a space of rank 51 modulo $\mathrm{Sym}^{sat} \oplus \Pi'^{sat}$. However, by Lemma 4.9, the intersection of this orbit with Sym^{sat} is one-dimensional and the intersection with Π'^{sat} has dimension 31, so we are left with 19 linearly independent elements of the form: $\frac{x+y}{3}$ with $x \in \mathrm{Sym} \setminus \{0\}$, and $y \in \Pi' \setminus \{0\}$. These are the 19 classes which were missing.

13 Conclusion

Let us summarize our results on θ^*:

Theorem 13.1. *The homomorphism $\theta^* : H^*(A^{[3]}, \mathbb{Z}) \to H^*(K_2(A), \mathbb{Z})$ of graded rings is surjective in every degree except 4. Moreover, the image of $H^4(A^{[3]}, \mathbb{Z})$ is the primitive overlattice of $\mathrm{Sym}^2(H^2(K_2(A), \mathbb{Z}))$. The kernel of θ^* is the ideal generated by $H^1(A^{[3]})$. The following integral classes give a basis of the image of θ^*:*

degree	*preimage of class*	*alternative name*
0	$\frac{1}{6}\mathfrak{q}_1(1)^3\lvert 0\rangle$	*1*
2	$\frac{1}{2}\mathfrak{q}_1(b_i)\mathfrak{q}_1(1)^2\lvert 0\rangle$ *for* $1 \le i \le 6$	$j(b_i)$
	$\frac{1}{2}\mathfrak{q}_2(1)\mathfrak{q}_1(1)\lvert 0\rangle$	e
3	$\frac{1}{2}\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(1)^2\lvert 0\rangle$	
	$\mathfrak{q}_2(a_i)\mathfrak{q}_1(1)\lvert 0\rangle$	
4	$\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)\mathfrak{q}_1(1)\lvert 0\rangle$ *for* $1 \le i \le j \le 6$*, but* $(b_i, b_j) \neq (a_1a_2, a_3a_4)$	
	$\frac{1}{2}\mathfrak{q}_1(x)\mathfrak{q}_1(1)^2\lvert 0\rangle$ *(instead of the missing case above)*	Y_p
	$\frac{1}{2}\left(\mathfrak{q}_1(b_i)^2 - \mathfrak{q}_2(b_i)\right)\mathfrak{q}_1(1)\lvert 0\rangle$	
	$\frac{1}{3}\mathfrak{q}_3(1)\lvert 0\rangle$	W
5	$\mathfrak{q}_1(a_ia_j)\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(1)\lvert 0\rangle$ *for any choice of* $j \neq i$	
	$\frac{1}{2}\mathfrak{q}_2(a_i^*)\mathfrak{q}_1(1)\lvert 0\rangle$	
6	$\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(1)\lvert 0\rangle$ *for* $1 \le i < j \le 4$	
	$\mathfrak{q}_2(x)\mathfrak{q}_1(1)\lvert 0\rangle$	
8	$\mathfrak{q}_1(x)^3\lvert 0\rangle$	*top class*

Proof. The table is established by the following results: For degree 2, see Proposition 10.8. The dual classes of degree 6 can be computed using Proposition 10.3 and the methods from Section 8. The odd degrees are treated by Proposition 11.2. Classes of degree 4 are studied in Section 12. The classes are chosen in a way that they give a basis of Sym^{sat}, which is possible by Corollary

12.12. The condition $(b_i, b_j) \neq (a_1a_2, a_3a_4)$ is more or less arbitrary, but we had to remove one class to avoid a relation of linear dependence.

The kernel of θ^* is described by the Propositions 10.4 and 10.6. $\square$

Part III
A quotient

14 Symplectic involution on $K_2(A)$

Let X be an irreducible symplectic manifold. Let

$$\nu : \operatorname{Aut}(X) \to \operatorname{Aut} H^2(X, \mathbb{Z})$$

the natural morphism. Hassett and Tschinkel (Theorem 2.1 in [18]) have shown that $\operatorname{Ker}\nu$ is a deformation invariant. Let X be an irreducible symplectic fourfold of Kummer type. Then Oguiso in [42] has shown that $\operatorname{Ker}\nu = (\mathbb{Z}/3\mathbb{Z})^3 \rtimes \mathbb{Z}/2\mathbb{Z}$.

Let A be an abelian variety and g an automorphism of A. Let denote by $T_{A[3]}$ the group of translation of A by elements of $A[3]$. If $g \in T_{A[3]} \rtimes \operatorname{Aut}_{\mathbb{Z}}(A)$ then g induces a natural automorphism on $K_2(A)$. We denote the induced automorphism by $g^{[[3]]}$. If there is no ambiguity, we also denote the *induced automorphism* by the same letter g.

When $X = K_2(A)$, we have more precisely, by Corollary 3.3 of [5],

$$\operatorname{Ker}\nu = T_{A[3]} \rtimes (-\operatorname{id}_A)^{[[3]]}.$$

14.1 Uniqueness and fixed locus

Theorem 14.1. *Let X be an irreducible symplectic fourfold of Kummer type and ι a symplectic involution on X. Then:*

(1) We have $\iota \in \operatorname{Ker}\nu$.

(2) Let A be an abelian surface. Then the couple (X, ι) is deformation equivalent to $(K_2(A), t_\tau \circ (-\operatorname{id}_A)^{[[3]]})$, where t_τ is the morphism induced on $K_2(A)$ by the translation by $\tau \in A[3]$.

(3) The fixed locus of ι is given by a K3 surface and 36 isolated points.

Proof. (1) If $\iota \notin \operatorname{Ker}\nu$, by the classification of Section 5 of [36], the unique possible action of ι on $H^2(X, \mathbb{Z})$ is given by $H^2(X, \mathbb{Z})^\iota = U \oplus (2)^2 \oplus (-6)$. We will show that it is impossible. Let us assume that $H^2(X, \mathbb{Z})^\iota = U \oplus (2)^2 \oplus (-6)$, we will find a contradiction.

As done in Section 3 of [35], consider a local universal deformation space of X:

$$\Phi : \mathcal{X} \to \Delta,$$

where Δ is a small polydisk and $\mathcal{X}_0 = X$. By restricting Δ, we can assume that ι extends to an automorphism M on $\mathcal{X}$ and μ on Δ such that, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "M"] \arrow[d] & \mathcal{X} \arrow[d] \\ \Delta \arrow[r, "\mu"] & \Delta \end{tikzcd}$$

Moreover, the differential of μ at 0 is given by the action of ι on $H^1(T_X)$ which is the same as the action on $H^{1,1}(X)$, since the symplectic holomorphic 2-form induces an isomorphism between the two and the symplectic holomorphic 2-form is preserved by the action of ι. We may assume that μ is a linear map. So Δ^μ is smooth and $\dim \Delta^\mu = \operatorname{rk} H^2(X, \mathbb{Z})^\iota - 2 = 3$. Moreover, by [29] we can find $x \in \Delta^\mu$ such that $\mathcal{X}_x$ is bimeromorphic to a Kummer fourfold

$K_2(A)$. Since $H^2(X,\mathbb{Z})^\iota = U \oplus (2)^2 \oplus (-6)$, $\iota_x := M_{\mathcal{X}_x}$ induces a bimeromorphic involution i on $K_2(A)$ with $H^2(K_2(A),\mathbb{Z})^i = U \oplus (2)^2 \oplus (-6)$.

Since i preserve the holomorphic 2-from, we have $\mathrm{NS}(K_2(A)) \supset \left[H^2(K_2(A),\mathbb{Z})^i\right]^\perp = (-2)^2$. The involution i also induces a trivial involution on $A_{H^2(X,\mathbb{Z})}$, so the half class of the diagonal e is in $H^2(K_2(A),\mathbb{Z})^i \cap \mathrm{NS}(K_2(A))$. It follows $\mathrm{NS}(K_2(A)) \supset (-2)^2 \oplus \mathbb{Z}e$. Moreover the morphism j defined in Notation 10.9 respects the Hodge structure so $\mathrm{NS}(K_2(A)) = j(\mathrm{NS}(A)) \oplus \mathbb{Z}e$. It follows that $\mathrm{NS}(A) \supset (-2)^2$. Now we construct an involution g on $H^2(A,\mathbb{Z})$ given by $-\mathrm{id}$ on $(-2)^2$ and id on $((-2)^2)^\perp$ and extended to an involution on $H^2(A,\mathbb{Z})$ by Corollary 1.5.2 of [41]. Then by Theorem 1 of [46], g provides a symplectic automorphism on A with: $H^2(A,\mathbb{Z})^g = ((-2)^2)^\perp = U \oplus (2)^2$. It follows from the classification of Section 4 of [37], that $A = \mathbb{C}/\Lambda$ with $\Lambda = \langle (1,0),(0,1),(x,-y),(y,x)\rangle$, $(x,y) \in \mathbb{C}^2 \smallsetminus \mathbb{R}^2$ and $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Let also denote by g the automorphism on $K_2(A)$ induced by g. By construction, $g \circ i$ acts trivially on $H^2(K_2(A),\mathbb{Z})$. Hence by Corollary 3.3 and Lemma 3.4 of [12], $g \circ \iota$ extends to an automorphism of $K_2(A)$. In particular, i extends to a symplectic involution on $K_2(A)$. Then $g \circ i \in \operatorname{Ker} \nu$.

By Corollary 11.3, t_τ acts trivially on $H^3(K_2(A),\mathbb{Z})$. Hence by Corollary 3.3 of [5], we have necessarily:

$$g^*_{|H^3(K_2(A),\mathbb{Z})} = i^*_{|H^3(K_2(A),\mathbb{Z})} \circ (-\mathrm{id}_A)^*_{|H^3(K_2(A),\mathbb{Z})} \text{ or } g^*_{|H^3(K_2(A),\mathbb{Z})} = i^*_{|H^3(K_2(A),\mathbb{Z})}.$$

But $g^*_{|H^3(K_2(A),\mathbb{Z})}$ has order 4 and $i^*_{|H^3(K_2(A),\mathbb{Z})} \circ (-\mathrm{id}_A)^*_{|H^3(K_2(A),\mathbb{Z})}$ and $i^*_{|H^3(K_2(A),\mathbb{Z})}$ have order 2, which is a contradiction.

(2) Let X be a irreducible symplectic fourfold of Kummer type and ι a symplectic involution on X. By (1) of the above theorem, we have $\iota \in \operatorname{Ker} \nu$. Then by Theorem 2.1 of [18], the couple (X,ι) deform to a couple $(K_2(A),\iota')$ with A an abelian surface and $\iota' \in \operatorname{Ker} \nu$ a symplectic involution on $K_2(A)$. Then we conclud with Corollary 3.3 of [5].

(3) Let A be an abelian surface, by Section 1.2.1 of [48], the fixed locus of $t_\tau \circ (-\mathrm{id}_A)^{[[3]]}$ on $K_2(A)$ is given by a K3 surface an 36 isolated points. Now let X be a irreducible symplectic fourfold of Kummer type and ι a symplectic involution on X. By (2) of the above theorem, $\operatorname{Fix} \iota$ deforms to the disjoint union of a K3 surface and 36 isolated points. Moreover ι is a symplectic involution, so the holomorphic 2-form of X restricts to a non-degenerated holomorphic 2-form on $\operatorname{Fix} \iota$. Then necessarily, $\operatorname{Fix} \iota$ consists of a K3 surface and 36 isolated points.

□

Remark 14.2. (1) We also remark that the K3 surface fixed by $(t_\tau \circ (-\mathrm{id}_A))$ is given by the sub-manifold

$$Z_{-\tau} = \overline{\{(a_1,a_2,a_3) \mid a_1 = -\tau,\ a_2 = -a_3 + \tau,\ a_2 \neq -\tau\}}$$

defined in Section 4 of [18].

(2) Considering the involution $-\mathrm{id}_A$, the set

$$\mathcal{P} := \{\xi \in K_2(A) |\ \operatorname{Supp} \xi = \{a_1,a_2,a_3\},\ a_i \in A[2] \smallsetminus \{0\}, 1 \leq i \leq 3\}$$

provides 35 fixed points and the vertex of

$$W_0 := \{\xi \in K_2(A) |\ \operatorname{Supp} \xi = \{0\}\}$$

supplies the 36th point. We denote by $p_1, ..., p_{35}$ the points of $\mathcal{P}$ and by p_{36} the vertex of W_0.

14.2 Action on the cohomology

From Theorem 14.1, we can assume that $X = K_2(A)$ and $\iota = -\operatorname{id}_A$. To consider $t_\tau \circ (-\operatorname{id}_A)$ instead of $-\operatorname{id}_A$ only has the effect of exchanging the role of $[Z_0]$ and $[Z_\tau]$. Hence we do not lose any generality assuming that $\iota = -\operatorname{id}_A$.

From Theorem 14.1 (1), the involution ι acts trivially on $H^2(K_2(A),\mathbb{Z})$. It follows

$$l_2^2(K_2(A)) = l_{1,-}^2(K_2(A)) = 0 \text{ and } l_{1,+}^2(K_2(A)) = 7. \tag{54}$$

From Corollary 11.3, the involution ι acts as $-\operatorname{id}$ on $H^3(K_2(A),\mathbb{Z})$. It follows

$$l_2^3(K_2(A)) = l_{1,+}^3(K_2(A)) = 0 \text{ and } l_{1,-}^3(K_2(A)) = 8. \tag{55}$$

By Definition 12.3, we have:

$$H^4(K_2(A),\mathbb{Q}) = \operatorname{Sym}^2 H^2(K_2(A),\mathbb{Q}) \oplus^\perp \Pi' \otimes \mathbb{Q},$$

where $\Pi' = \langle Z_\tau - Z_0,\ \tau \in A[3] \smallsetminus \{0\}\rangle$. The involution ι^* fixes $\operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z})$ and $\iota^*(Z_\tau - Z_0) = Z_{-\tau} - Z_0$. It provides the following proposition.

Proposition 14.3. *We have $l_{1,-}^4(K_2(A)) = 0$, $l_{1,+}^4(K_2(A)) = 28$ and $l_2^4(K_2(A)) = 40$.*

Proof. Let $\mathcal{S}$ be the over-lattice of $\operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z})$ where we add all the classes divisible by 2 in $H^4(K_2(A),\mathbb{Z})$. By (51), the discriminant of Π' is not divisible by 2. Since $H^4(K_2(A),\mathbb{Z})$ is unimodular, it follows that the discriminant of $\mathcal{S}$ is also not divisible by 2. Hence, we have:

$$H^4(K_2(A),\mathbb{F}_2) = \mathcal{S} \otimes \mathbb{F}_2 \oplus \Pi' \otimes \mathbb{F}_2.$$

Moreover, we have:

$$\iota^*(Z_\tau - Z_0) = Z_{-\tau} - Z_0,$$

for all $\tau \in A[3] \smallsetminus \{0\}$. Hence $\operatorname{Vect}_{\mathbb{F}_2}(Z_\tau - Z_0, Z_{-\tau} - Z_0)$ is isomorphic to N_2 as a $\mathbb{F}_2[G]$-module (see the notation in Definition-Proposition 6.1). Moreover $H^2(K_2(A),\mathbb{Z})$ is invariant by the action of ι, hence $\operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z})$ and $\mathcal{S}$ is also invariant by the action of ι. It follows that $\mathcal{S} \otimes \mathbb{F}_2 = \mathcal{N}_1$ and $\Pi' \otimes \mathbb{F}_2 = \mathcal{N}_2$. Since $\operatorname{rk} \mathcal{S} = 28$, we have $l_{1,+}^4 + l_{1,-}^4 = 28$. However, $\mathcal{S}$ is invariant by the action of ι, it follows that $l_{1,-}^4 = 0$ and $l_{1,+}^4 = 28$. On the other hand $\operatorname{rk} \Pi' = 80$, it follows that $l_2^4 = 40$. □

15 Application to singular irreducible symplectic varieties

15.1 Statement of the main theorem

In [40], Namikawa proposes a definition of the Beauville-Bogomolov form for some singular irreducible symplectic varieties. He assumes that the singularities are only $\mathbb{Q}$-factorial with a singular locus of codimension ≥ 4. Under these assumptions, he proves a local Torelli theorem. This result was completed by a generalization of the Fujiki formula by Matsushita in [30].

Theorem 15.1. *Let X be a projective irreducible symplectic variety of dimension $2n$ with only $\mathbb{Q}$-factorial singularities, and $\operatorname{Codim} \operatorname{Sing} X \geq 4$. There exists a unique indivisible integral symmetric non-degenerated bilinear form B_X on $H^2(X,\mathbb{Z})$ and a unique positive constant $c_X \in \mathbb{Q}$, such that for any $\alpha \in H^2(X,\mathbb{C})$,*

$$\alpha^{2n} = c_X B_X(\alpha,\alpha)^n \tag{56}$$

and such that for $0 \neq \omega \in H^0(\Omega_U^2)$ a holomorphic 2-from on the smooth locus U of X:

$$B_X(\omega + \overline{\omega}, \omega + \overline{\omega}) > 0. \tag{57}$$

Moreover, the signature of B_X is $(3, h^2(X,\mathbb{C}) - 3)$.

The form B_X is called the Beauville–Bogomolov form of X.

Proof. The statement of the theorem in [30] does not say that the form is integral. However, let Z_s be a fiber of the Kuranishi family of Z, with the same idea as Matsushita's proof, we can see that q_Z and q_{Z_s} are proportional. Then, it follows using the proof of Theorem 5 a), c) of [1]. $\square$

We can also consider its polarized form.

Proposition 15.2. *Let X be a projective irreducible symplectic variety of dimension $2n$ with $\operatorname{Codim}\operatorname{Sing} X \geq 4$. The equality (56) of Theorem 15.1 implies that*

$$\alpha_1 \cdot ... \cdot \alpha_{2n} = \frac{c_X}{(2n)!} \sum_{\sigma \in S_{2n}} B_X(\alpha_{\sigma(1)}, \alpha_{\sigma(2)})...B_X(\alpha_{\sigma(2n-1)}, \alpha_{\sigma(2n)}).$$

for all $\alpha_i \in H^2(X, \mathbb{Z})$.

These results were then generalized by Kirschner for symplectic complex spaces in [20]. In [31, Theorem 2.5] was appeared the first concrete example of Beauville-Bogomolov lattice for a singular irreducible symplectic variety. The variety studied in [31] is a partial resolution of the Hilbert scheme of 2 points on a K3 surface quotiented by a symplectic involution. The objective of this section is to provide a new example of a Beauville-Bogomolov lattice replacing Hilbert schemes of 2 points on a K3 surface by generalized Kummer fourfolds. Knowing the integral basis of the cohomology group of the generalized Kummer provided in Part II, this calculation becomes possible. Moreover the calculation will be much simpler as in [31] because of the general techniques for calculating integral cohomology of quotients developed in [32] and the new technique using monodromy developed in Lemma 15.13. The other techniques developed in [31] are also in [32], so to simplify the reading, we will only cite [31] in the rest of the section.

Concretely, let X be an irreducible symplectic fourfold of Kummer type and ι a symplectic involution on X. By Theorem 14.1 the fixed locus of ι is the union of 36 points and a K3 surface Z_0. Then the singular locus of $K := X/\iota$ is the union of a K3 surface and 36 points. The singular locus is not of codimension four. We will lift to a partial resolution of singularities, K' of K, obtained by blowing up the image of Z_0. By Section 2.3 and Lemma 1.2 of [14], the variety K' is an irreducible symplectic V-manifold which has singular locus of codimension four.

All Section 15 is devoted to prove the following theorem.

Theorem 15.3. *Let X be an irreducible symplectic fourfold of Kummer type and ι a symplectic involution on X. Let Z_0 be the K3 surface which is in the fixed locus of ι. We denote $K = X/\iota$ and K' the partial resolution of singularities of K obtained by blowing up the image of Z_0. Then the Beauville–Bogomolov lattice $H^2(K', \mathbb{Z})$ is isomorphic to $U(3)^3 \oplus \begin{pmatrix} -5 & -4 \\ -4 & -5 \end{pmatrix}$, and the Fujiki constant $c_{K'}$ is equal to 8.*

The Beauville-Bogomolov form is a topological invariant, hence from Theorem 14.1 we can assume that X is a generalized Kummer fourfold and $\iota = -\operatorname{id}_A$. As it will be useful to prove Lemma 15.13, we can assume even more. All generalized Kummer fourfolds are deformation equivalent, hence we can assume that $A = E_\xi \times E_\xi$, where E_ξ the elliptic curve provided in Definition 5.4:

$$E_\xi := \frac{\mathbb{C}}{\left\langle 1, e^{\frac{2i\pi}{6}} \right\rangle}.$$

15.2 Overview on the proof of Theorem 15.3

We first provide all the notation that we will need during the proof in Section 15.3. Then the proof is divided into the following steps:

(1) First (54), (55), Proposition 14.3 and Corollary 6.9 will provide the H^4-normality in Section 15.4.

(2) The knowledge of the elements divisible by 2 in $\operatorname{Sym}^2 H^2(K_2(A), \mathbb{Z})$ from Section 12 and the H^4-normality allow us to prove the H^2-normality in Section 15.5.

(3) An adaptation of the H^2-normality (Lemma 15.8) and several lemmas in Section 15.6 will provide an integral basis of $H^2(K',\mathbb{Z})$ (Theorem 15.9).

(4) Knowing an integral basis of $H^2(K',\mathbb{Z})$, we end the calculation of the Beauville–Bogomolov form in Section 15.7 using intersection theory and the generalized Fujiki formula (Theorem 15.1).

15.3 Notation

Let $K_2(A)$ be a generalized Kummer fourfold endowed with the symplectic involution ι induced by $-\operatorname{id}_A$. We denote by π the quotient map $K_2(A) \to K_2(A)/\iota$. From Theorem 14.1, we know that the singular locus of the quotient $K_2(A)/\iota$ is the K3 surface, image by π of Z_0, and 36 isolated points. We denote $\overline{Z_0} := \pi(Z_0)$. We consider $r' : K' \to K_2(A)/\iota$ the blow-up of $K_2(A)/\iota$ in $\overline{Z_0}$ and we denote by $\overline{Z_0}'$ the exceptional divisor. We also denote by $s_1 : N_1 \to K_2(A)$ the blowup of $K_2(A)$ in Z_0; and denote by Z_0' the exceptional divisor in N_1. Denote by ι_1 the involution on N_1 induced by ι. We have $K' \simeq N_1/\iota_1$, and we denote $\pi_1 : N_1 \to K'$ the quotient map.

Consider the blowup $s_2 : N_2 \to N_1$ of N_1 in the 36 points $p_1, ..., p_{36}$ fixed by ι_1 and the blowup $\widetilde{r} : \widetilde{K} \to K'$ of K' in its 36 singulars points. We denote the exceptional divisors by $E_1, ..., E_{36}$ and $D_1, ..., D_{36}$ respectively. We also denote $\widetilde{\overline{Z_0}} = \widetilde{r}^*(\overline{Z_0}')$ and $\widetilde{Z_0} = s_2^*(Z_0')$. Denote ι_2 the involution induced by ι on N_2 and $\pi_2 : N_2 \to N_2/\iota_2$ the quotient map. We have $N_2/\iota_2 \simeq \widetilde{K}$. We collect this notation in a commutative diagram

$$\begin{tikzcd}
\widetilde{K} \arrow[r, "\widetilde{r}"] & K' \arrow[r, "r'"] & K_2(A)/\iota & U \arrow[l, hook'] \\
N_2 \arrow[u, "\pi_2"] \arrow[r, "s_2"] \arrow[loop below, "\iota_2"] & N_1 \arrow[u, "\pi_1"] \arrow[r, "s_1"] \arrow[loop below, "\iota_1"] & K_2(A) \arrow[u, "\pi"] \arrow[loop below, "\iota"] & V \arrow[l, hook'] \arrow[u]
\end{tikzcd} \tag{58}$$

To finish, we denote $V = K_2(A) \smallsetminus \operatorname{Fix}\iota$ and $U = V/\iota$. Also, we set $s = s_2 \circ s_1$ and $r = \widetilde{r} \circ r'$. We denote also e the half of the class of the diagonal in $H^2(K_2(A),\mathbb{Z})$ as states in Notation 10.9.

Remark 15.4. We can commute the push-forward maps and the blow-up maps as proved in Lemma 3.3.21 of [32]. Let $x \in H^2(N_1,\mathbb{Z})$, $y \in H^2(K_2(A),\mathbb{Z})$, we have:

$$\pi_{2*}(s_2^*(x)) = \widetilde{r}^*(\pi_{1*}(x)),$$

$$\pi_{1*}(s_1^*(y)) = r'^*(\pi_*(y)),$$

Moreover, we will also use the notation provided in Notation 10.9 and in Section 12.

15.4 The couple $(K_2(A),\iota)$ is H^4-normal

Proposition 15.5. *The couple $(K_2(A),\iota)$ is H^4-normal.*

Proof. We apply Theorem 6.9.

i) By Theorem 10.2, $H^*(K_2(A),\mathbb{Z})$ is torsion-free.

ii) From Remark 14.2 (1), we know that the connected component of dimension 2 of $\operatorname{Fix}\iota$ is given by Z_0. We know that Z_0 is a K3 surface, hence is simply connected. Moreover by Proposition 4.3 of [18] $Z_0 \cdot Z_\tau = 1$ for all $\tau \in A[3] \smallsetminus \{0\}$. Hence the class of Z_0 in $H^4(K_2(A),\mathbb{Z})$ is primitive. It follows that $\operatorname{Fix}\iota$ is almost negligible (Definition 6.8).

iii) By (54) and Proposition 14.3, we have $l_{1,-}^2(K_2(A)) = l_{1,-}^4(K_2(A)) = 0$.

iv) By (55) and Proposition 14.3, we have $l_{1,+}^3(K_2(A)) = 0$. Moreover $H^1(K_2(A)) = 0$, so $l_{1,+}^1(K_2(A)) = 0$.

v) We have to check the following equality:

$$\begin{aligned} & l^4_{1,+}(K_2(A)) + 2\left[l^1_{1,-}(X) + l^3_{1,-}(X) + l^0_{1,+}(X) + l^2_{1,+}(X)\right] \\ & = 36h^0(pt) + h^0(Z_0) + h^2(Z_0) + h^4(Z_0). \end{aligned}$$

By (54), (55) and Proposition 14.3:

$$l^4_{1,+}(K_2(A)) + 2\left[l^1_{1,-}(X) + l^3_{1,-}(X) + l^0_{1,+}(X) + l^2_{1,+}(X)\right] = 28 + 2(8+1+7) = 60.$$

Moreover since Z_0 is a K3 surface, we have:

$$36h^0(pt) + h^0(Z_0) + h^2(Z_0) + h^4(Z_0) = 36 + 1 + 22 + 1 = 60.$$

It follows from Corollary 6.9 that $(K_2(A), \iota)$ is H^4-normal. □

Remark 15.6. As explained in Proposition 3.5.20 of [32], the proof of Theorem 6.9 provide first that $\pi_{2*}(s^*(H^4(K_2(A),\mathbb{Z})))$ is primitive in $H^4(\widetilde{K},\mathbb{Z})$ and then the H^4 normality. So, the lattice $\pi_{2*}(s^*(H^4(K_2(A),\mathbb{Z})))$ is primitive in $H^4(\widetilde{K},\mathbb{Z})$.

15.5 The couple $(K_2(A), \iota)$ is H^2-normal

Proposition 15.7. *The couple $(K_2(A), \iota)$ is H^2-normal.*

Proof. We want to prove that the pushforward $\pi_* : H^2(K_2(A),\mathbb{Z}) \to H^2(K_2(A)/\iota,\mathbb{Z})/\mathrm{tors}$ is surjective. By Remark 6.7, it is equivalent to prove that for all $x \in H^2(K_2(A),\mathbb{Z})^\iota$, $\pi_*(x)$ is divisible by 2 if and only if there exists $y \in H^2(K_2(A),\mathbb{Z})$ such that $x = y + \iota^*(y)$.

Let $x \in H^2(K_2(A),\mathbb{Z})^\iota = H^2(K_2(A),\mathbb{Z})$ such that $\pi_*(x)$ is divisible by 2, we will show that there exists $y \in H^2(K_2(A),\mathbb{Z})$ such that $x = y + \iota^*(y)$. Then, by Proposition 6.5, $\pi_*(x^2)$ is divisible by 2. However, $x^2 \in H^4(K_2(A),\mathbb{Z})^\iota$; since $(K_2(A),\iota)$ is H^4-normal by Proposition 15.5, it means that there is $z \in H^4(K_2(A),\mathbb{Z})$ such that $x^2 = z + \iota^*(z)$.

Let $\mathcal{S}$ be, as before, the over-lattice of $\mathrm{Sym}^2 H^2(K_2(A),\mathbb{Z})$ where we add all the classes divisible by 2 in $H^4(K_2(A),\mathbb{Z})$. By Definition 12.3 and (51), there exist $z_s \in \mathcal{S}$, $z_p \in \Pi'$ and $\alpha \in \mathbb{N}$ such that: $3^\alpha \cdot z = z_s + z_p$. Hence, we have:

$$3^\alpha \cdot x^2 = 2z_s + z_p + \iota^*(z_p).$$

Since $x^2 \in \mathrm{Sym}$, by Corollary 12.8, $z_p + \iota^*(z_p) = 0$. It follows:

$$3^\alpha \cdot x^2 = 2z_s. \tag{59}$$

let $(u_1, u_2, v_1, v_2, w_1, w_2, e)$ be the integral basis of $H^2(K_2(A),\mathbb{Z})$ introduced in Notation10.9. We can write:

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \beta_1 v_1 + \beta_2 v_2 + \gamma_1 w_1 + \gamma_2 w_2 + de.$$

Then

$$3^\alpha \cdot x^2 = \alpha_1^2 u_1^2 + \alpha_2^2 u_2^2 + \beta_1^2 v_1^2 + \beta_2^2 v_2^2 + \gamma_1^2 w_1^2 + \gamma_2^2 w_2^2 + d^2 e^2 \mod 2H^4(K_2(A),\mathbb{Z}).$$

It follows by (59) that $\alpha_1^2 u_1^2 + \alpha_2^2 u_2^2 + \beta_1^2 v_1^2 + \beta_2^2 v_2^2 + \gamma_1^2 w_1^2 + \gamma_2^2 w_2^2 + d^2 e^2$ is divisible by 2. However by Corollary 12.9 and Proposition 12.11, we have:

$$\mathcal{S} = \left\langle \mathrm{Sym}^2 H^2(K_2(A),\mathbb{Z}); \frac{u_1\cdot u_2 + v_1 \cdot v_2 + w_1 \cdot w_2}{2}; \frac{u_i^2 - \frac{1}{3}u_i\cdot e}{2}; \frac{v_i^2 - \frac{1}{3}v_i\cdot e}{2}; \frac{w_i^2 - \frac{1}{3}w_i\cdot e}{2}, i \in \{1,2\} \right\rangle. \tag{60}$$

The $\frac{1}{2}(\alpha_1^2 u_1^2 + \alpha_2^2 u_2^2 + \beta_1^2 v_1^2 + \beta_2^2 v_2^2 + \gamma_1^2 w_1^2 + \gamma_2^2 w_2^2 + d^2 e^2)$ is in $\mathcal{S}$ and so can be expressed as a linear combination of the generators of $\mathcal{S}$. Then, it follows from (60) that all the coefficients of $\alpha_1^2 u_1^2 + \alpha_2^2 u_2^2 + \beta_1^2 v_1^2 + \beta_2^2 v_2^2 + \gamma_1^2 w_1^2 + \gamma_2^2 w_2^2 + d^2 e^2$ are divisible by 2. It means that x is divisible by 2. This is what we wanted to prove. □

With exactly the same proof working in $H^4(\widetilde{K},\mathbb{Z})$ and using Remark 15.6, we provide the following lemma.

Lemma 15.8. *The lattice $\pi_{2*}(s^*(H^2(K_2(A),\mathbb{Z})))$ is primitive in $H^2(\widetilde{K},\mathbb{Z})$.*

15.6 Calculation of $H^2(K',\mathbb{Z})$

This section is devoted to prove the following theorem.

Theorem 15.9. *Let K', π_1, s_1 and $\overline{Z_0}'$ be respectively the variety, the maps and the class defined in Section 15.3. We have*

$$H^2(K',\mathbb{Z}) = \pi_{1*}(s_1^*(H^2(K_2(A),\mathbb{Z}))) \oplus \mathbb{Z}\left(\frac{\pi_{1*}(s_1^*(e)) + \overline{Z_0}'}{2}\right) \oplus \mathbb{Z}\left(\frac{\pi_{1*}(s_1^*(e)) - \overline{Z_0}'}{2}\right).$$

First we need to calculate some intersections.

Lemma 15.10. *(i) We have $E_l \cdot E_k = 0$ if $l \neq k$, $E_l^4 = -1$ and $E_l \cdot z = 0$ for all $(l,k) \in \{1,...,28\}^2$ and for all $z \in s^*(H^2(K_2(A),\mathbb{Z}))$.*

(ii) We have $e^4 = 324$.

We already have some properties of primitivity:

(iii) $\pi_{1*}(s_1^*(H^2(K_2(A),\mathbb{Z})))$ is primitive in $H^2(K',\mathbb{Z})$,

(iv) The group $\widetilde{\mathcal{D}} = \left\langle \widetilde{\overline{Z_0}}, D_1, ..., D_{36}, \frac{\widetilde{\overline{Z_0}}+D_1+...+D_{36}}{2}\right\rangle$ is primitive in $H^2(\widetilde{K},\mathbb{Z})$.

(v) $\overline{Z_0}'$ is primitive in $H^2(K',\mathbb{Z})$,

Proof. (i) It is proven using adjunction formula. It is the same statement as Proposition 4.6.16 1) of [32].

(ii) It follows directly from the Fujiki formula (36).

(iii) By Lemma 15.8, $\pi_{2*}(s^*(H^2(K_2(A),\mathbb{Z})))$ is primitive in $H^2(\widetilde{K},\mathbb{Z})$. Then by Remark 15.4, $r'^*(\pi_*(H^2(K_2(A),\mathbb{Z})))$ is primitive in $H^2(K',\mathbb{Z})$. Using again Remark 15.4, we get the result.

The proof of (iv) and (v) is the same as Lemma 4.6.14 of [32] and will be omitted. □

With Lemma 15.10 (iii) and (v), it only remains to prove that $\pi_{1*}(s_1^*(e)) + \overline{Z_0}'$ is divisible by 2 which will be done in Lemma 15.14. To prove this lemma, we first prove that $\pi_{2*}(s^*(e)) + \widetilde{\overline{Z_0}}$ is divisible by 2. Knowing that $\widetilde{\overline{Z_0}} + D_1 + ... + D_{36}$ is divisible by 2, we only have to show that $\pi_{2*}(s^*(e)) + D_1 + ... + D_{36}$ is divisible by 2 which is done by Lemma 15.12 and 15.13.
First we need to know the group $H^3(\widetilde{K},\mathbb{Z})$.

Lemma 15.11. *We have $H^3(\widetilde{K},\mathbb{Z}) = 0$.*

Proof. We have the following exact sequence:

$$H^3(K_2(A),V,\mathbb{Z}) \to H^3(K_2(A),\mathbb{Z}) \xrightarrow{f} H^3(V,\mathbb{Z}) \to H^4(K_2(A),V,\mathbb{Z}) \xrightarrow{\rho} H^4(K_2(A),\mathbb{Z}).$$

By Thom isomorphism, $H^3(K_2(A),V,\mathbb{Z}) = 0$ and $H^4(K_2(A),V,\mathbb{Z}) = H^0(Z_0,\mathbb{Z})$. Moreover ρ is injective, so $H^3(V,\mathbb{Z}) = H^3(K_2(A),\mathbb{Z})$.

Hence by (54), (55) and Proposition 3.2.8 of [32], we find that $H^3(U,\mathbb{Z}) = 0$. Since $H^3(K_2(A),\mathbb{Z})^\iota = 0$, $H^3(\widetilde{K},\mathbb{Z})$ is a torsion group. Hence the result follows from the exact sequence

$$H^3(\widetilde{K},U,\mathbb{Z}) \to H^3(\widetilde{K},\mathbb{Z}) \to H^3(U,\mathbb{Z})$$

and from the fact that $H^3(\widetilde{K},U,\mathbb{Z}) = 0$ by Thom isomorphism. □

Lemma 15.12. *There exists D_e which is a linear combination of the D_i with coefficient 0 or 1 such that $\pi_{2*}(s^*(e)) + D_e$ is divisible by 2.*

Proof. First, we have to use Smith theory as in Section 4.6.4 of [32].

Look at the following exact sequence:

$$\begin{aligned} 0 \to H^2(\widetilde{K}, \widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2)) &\to H^2(\widetilde{K}, \mathbb{F}_2) \to H^2(\widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2)) \\ &\longrightarrow H^3(\widetilde{K}, \widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) \longrightarrow 0. \end{aligned}$$

First, we will calculate the dimension of the vector spaces $H^2(\widetilde{K}, \widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2)$ and $H^3(\widetilde{K}, \widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2)$. By (2) of Proposition 6.10, we have

$$H^*(\widetilde{K}, \widetilde{Z_0} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) \simeq H^*_\sigma(N_2).$$

The previous exact sequence gives us the following equation:

$$h^2_\sigma(N_2) - h^2(\widetilde{K}, \mathbb{F}_2) + h^2(\widetilde{Z_0} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) - h^3_\sigma(N_2) = 0.$$

As $h^2(\widetilde{K}, \mathbb{F}_2) = 8 + 36 = 44$ and $h^2(\widetilde{Z_0} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) = 23 + 36 = 59$, we obtain:

$$h^2_\sigma(N_2) - h^3_\sigma(N_2) = -15.$$

Moreover by 2) of Proposition 6.10, we have the exact sequence

$$\begin{aligned} 0 \to H^1_\sigma(N_2) \to H^2_\sigma(N_2) \to H^2(N_2, \mathbb{F}_2) &\to H^2_\sigma(N_2) \oplus H^2(\widetilde{Z_0} \cup (\cup_{k=1}^{36} E_k), \mathbb{F}_2) \\ \to H^3_\sigma(N_2) &\longrightarrow \text{coker} \longrightarrow 0. \end{aligned}$$

By Lemma 7.4 of [6], $h^1_\sigma(N_2) = h^0(\widetilde{Z_0} \cup (\cup_{k=1}^{36} E_k), \mathbb{F}_2) - 1$. Then we get the equation

$$\begin{aligned} &h^0(\widetilde{Z_0} \cup (\cup_{k=1}^{36} E_k), \mathbb{F}_2) - 1 - h^2_\sigma(N_2) + h^2(N_2, \mathbb{F}_2) \\ &- h^2_\sigma(N_2) - h^2(\widetilde{Z_0} \cup (\cup_{k=1}^{36} E_k), \mathbb{F}_2) + h^3_\sigma(N_2) - \alpha = 0, \end{aligned}$$

where $\alpha = \dim \text{coker}$. So

$$21 - \alpha - 2h^2_\sigma(N_2) + h^3_\sigma(N_2) = 0.$$

From the two equations, we deduce that

$$h^2_\sigma(N_2) = 36 - \alpha, \qquad h^3_\sigma(N_2) = 51 - \alpha.$$

Come back to the exact sequence

$$0 \longrightarrow H^2(\widetilde{K}, \widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) \longrightarrow H^2(\widetilde{K}, \mathbb{F}_2) \xrightarrow{\varsigma^*} H^2(\widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2),$$

where $\varsigma : \widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k) \hookrightarrow \widetilde{K}$ is the inclusion. Since $h^2(\widetilde{K}, \widetilde{\overline{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) = h^2_\sigma(N_2) = 36 - \alpha$, we have $\dim_{\mathbb{F}_2} \varsigma^*(H^2(\widetilde{K}, \mathbb{F}_2)) = (8+36) - 36 + \alpha = 8 + \alpha$. We can interpret this as follows. Consider the homomorphism

$$\begin{aligned} \varsigma^*_{\mathbb{Z}} : H^2(\widetilde{K}, \mathbb{Z}) &\to H^2(\widetilde{\overline{Z_0}}, \mathbb{Z}) \oplus (\oplus_{k=1}^{36} H^2(D_k, \mathbb{Z})) \\ u &\to (u \cdot \widetilde{\overline{Z_0}}, u \cdot D_1, ..., u \cdot D_{36}). \end{aligned}$$

Since this is a map of torsion free $\mathbb{Z}$-modules (by Lemma 15.11 and universal coefficient formula), we can tensor by $\mathbb{F}_2$,

$$\varsigma^* = \varsigma^*_{\mathbb{Z}} \otimes \mathrm{id}_{\mathbb{F}_2} : H^2(\widetilde{K}, \mathbb{Z}) \otimes \mathbb{F}_2 \to H^2(\widetilde{\overline{Z_0}}, \mathbb{Z}) \oplus (\oplus_{k=1}^{36} H^2(D_k, \mathbb{Z})) \otimes \mathbb{F}_2,$$

and we have $8+\alpha$ independent elements such that the intersection with the D_k $k \in \{1, ..., 36\}$ and $\widetilde{\overline{Z_0}}$ are not all zero. But, $\varsigma^*(\pi_{2*}(H^2(N_2, \mathbb{Z}))) = 0$ and $\varsigma^*(\widetilde{\overline{Z_0}}, \langle D_1, ..., D_{36} \rangle)$, (it follows from Proposition

6.5). By Lemma 15.10 (iv), the element $\widetilde{\overline{Z_0}}+D_1+...+D_{36}$ is divisible by 2. Hence necessary, it remains $7+\alpha$ independent elements in $H^2(\widetilde{K},\mathbb{Z})$ of the form $\frac{u+d}{2}$ with $u\in\pi_{2*}(s^*(H^2(K_2(A),\mathbb{Z})))$ and $d\in\langle D_1,...,D_{36}\rangle$.

Let denote by $u_1,...,u_{7+\alpha}$ the $7+\alpha$ elements in $\pi_{2*}(s^*(H^2(K_2(A),\mathbb{Z})))$ provided above. By Lemma 15.10 (iv) $\langle D_1,...,D_{36}\rangle$ is primitive in $H^2(\widetilde{K},\mathbb{Z})$. Hence necessary, the element $u_1,...,u_{7+\alpha}$ view as element in $\pi_{2*}(s^*(H^2(K_2(A),\mathbb{F}_2)))$ are linearly independent. Since $\dim_{\mathbb{F}_2}\pi_{2*}(s^*(H^2(K_2(A),\mathbb{F}_2)))=7$, it follows that $\alpha=0$ and $\mathrm{Vect}_{\mathbb{F}_2}(u_1,...,u_7)=\pi_{2*}(s^*(H^2(K_2(A),\mathbb{F}_2)))$. Hence there exists D_e which is a linear combination of the D_i with coefficient 0 or 1 such that $\pi_{2*}(s^*(e))+D_e$ is divisible by 2. □

Lemma 15.13. *We have:*

$$D_e=D_1+...+D_{36}.$$

Proof. The know that the monodromy acts on $A[2]$ as the symplectic group $\mathrm{Sp}\,A[2]$. Hence the monodromy action extends naturally to an action on the divisors $D_1,...,D_{35}$. Also this monodromy action represented by $\mathrm{Sp}\,A[2]$ acts trivially on D_{36} and on $\pi_{2*}(s^*(e))$. As explained in Remark 4.3 the 2 orbits of the action of $\mathrm{Sp}\,A[2]$ on the set $\mathfrak{D}:=\{D_1,...,D_{35}\}$ correspond to the two sets of isotropic and non-isotropic planes in $A[2]$. Hence by Proposition 4.4 (3), (4) the action of $\mathrm{Sp}\,A[2]$ on the set $\mathfrak{D}$ has 2 orbits: one of 15 elements and another of 20 elements.

On the other hand, as we mentioned in the end of Section 15.1, we can assume that $A=E_\xi\times E_\xi$ where E_ξ is the elliptic curve introduced in Definition 5.4. Hence there is the following automorphism group acting on A:

$$G:=\left\langle\begin{pmatrix}\rho & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right\rangle,$$

where $\rho=e^{\frac{2i\pi}{6}}$. The group G extends naturally to an automorphism group of N_2 which we denote also G. Moreover, the action of G restricts to the set $\mathfrak{D}$. Then by Lemma 5.5 the action of G on $\mathfrak{D}$ has 2 orbits: one of 5 elements and one of 30 elements. Also the group G acts trivially on D_{36} and on $\pi_{2*}(s^*(e))$.

Hence the combined action of G and $\mathrm{Sp}\,A[2]$ acts transitively on $\mathfrak{D}$. Since $\pi_{2*}(s^*(e))$ is fixed by the action of G and $\mathrm{Sp}\,A[2]$, D_e has also to be fixed by the action of G and $\mathrm{Sp}\,A[2]$ or it will contradict Lemma 15.10 (iv). It follows that there are only 3 possibilities for D_e:

(1) $D_e=D_{36}$,

(2) $D_e=D_1+...+D_{35}$,

(3) or $D_e=D_1+...+D_{36}$.

Let d be the number of D_i with coefficient equal to 1 in the linear decomposition of D_e. The number d can be 1, 35 or 36.

Then from Lemma 15.10 (i), (ii) and Proposition 6.5

$$\left(\frac{\pi_{2*}(s^*(e))+D_e}{2}\right)^4=\frac{324-d}{2}.$$

Hence d has to be divisible by 2. It follows that $D_e=D_1+...+D_{36}$. □

Lemma 15.14. *The class $\pi_{1*}(s_1^*(e))+\overline{Z_0}'$ is divisible by 2.*

Proof. We know that $\pi_{2,*}(s^*(e))+\widetilde{\overline{Z_0}}$ is divisible by 2. Indeed by Lemma 15.10 (iv), $\widetilde{\overline{Z_0}}+D_1+...+D_{36}$ is divisible by 2 and by Lemma 15.12 and 15.13, $\pi_{2,*}(s^*(e))+D_1+...+D_{36}$ is divisible by 2.

We can find a Cartier divisor on $\widetilde{K}$ which corresponds to $\frac{\pi_{2*}(s^*(e))+\widetilde{\overline{Z_0}}}{2}$ and which does not meet $\cup_{k=1}^{36}D_k$. Then this Cartier divisor induces a Cartier divisor on K' which necessarily corresponds to half the cocycle $\pi_{1*}(s_1^*(e))+\overline{Z_0}'$. □

15.7 Calculation of $B_{K'}$

We finish the proof of Theorem 15.3, calculating $B_{K'}$. We continue using the notation provided in Section 15.3.

Lemma 15.15. *We have*

$$\overline{Z_0}'^2 = -2r^*(\overline{Z_0}).$$

Proof. We use the same technique as in Lemma 4.6.12 of [32]. Consider the following diagram:

$$\begin{tikzcd} Z_0' \arrow[r, hook, "l_1"] \arrow[d, "g"] & N_1 \arrow[d, "s_1"] \\ Z_0 \arrow[r, hook, "l_0"] & K_2(A), \end{tikzcd}$$

where l_0 and l_1 are the inclusions and $g := s_{1|Z_0'}$. By Proposition 6.7 of [15], we have:

$$s_1^* l_{0*}(Z_0) = l_{1*}(c_1(E)),$$

where $E := g^*(\mathscr{N}_{Z_0/K_2(A)})/\mathscr{N}_{Z_0'/N_1}$. Hence

$$s_1^* l_{0*}(Z_0) = c_1(g^*(\mathscr{N}_{Z_0/K_2(A)})) - Z_0'^2.$$

Since $K_2(A)$ is hyperkähler and Z_0 is a K3 surface, we have $c_1(\mathscr{N}_{Z_0/K_2(A)}) = 0$. So

$$Z_0'^2 = -s_1^* l_{0*}(Z_0).$$

Then the result follows from Proposition 6.5. □

Proposition 15.16. *We have the formula*

$$B_{K'}(\pi_{1*}(s_1^*(\alpha), \pi_{1*}(s_1^*(\beta))) = 6\sqrt{\frac{2}{c_{K'}}} B_{K_2(A)}(\alpha, \beta),$$

where $c_{K'}$ is the Fujiki constant of K' and α, β are in $H^2(K_2(A), \mathbb{Z})^\iota$ and $B_{K_2(A)}$ is the Beauville–Bogomolov form of $K_2(A)$.

Proof. The ingredient for the proof is the Fujiki formula.
By (56) of Theorem 15.1, we have

$$(\pi_{1*}(s_1^*(\alpha)))^4 = c_{K'} B_{K'}(\pi_{1*}(s_1^*(\alpha), \pi_{1*}(s_1^*(\alpha)))^2.$$

$$\alpha^4 = 9B_{K_2(A)}(\alpha, \alpha)^2.$$

Moreover, by Proposition 6.5,

$$(\pi_{1*}(s^*(\alpha)))^4 = 8s^*(\alpha)^4 = 8\alpha^4.$$

By statement (57) of Theorem 15.1, we get the result. □

In particular, it follows:

$$B_{K'}(\pi_{1*}(s_1^*(e), \pi_{1*}(s_1^*(e))) = -36\sqrt{\frac{2}{c_{K'}}} \tag{61}$$

Lemma 15.17.

$$B_{K'}(\pi_{1*}(s_1^*(\alpha)), \overline{Z_0}') = 0,$$

for all $\alpha \in H^2(S^{[2]}, \mathbb{Z})^\iota$.

Proof. We have $\pi_{1*}(s_1^*(\alpha))^3 \cdot \overline{Z_0}' = 8s_1^*(\alpha)^3 \cdot \Sigma_1$ by Proposition 6.5, and $s_{1*}(s_1^*(\alpha^3) \cdot Z_0') = \alpha^3 \cdot s_{1*}(Z_0') = 0$ by the projection formula. We conclude by Proposition 15.2. □

Lemma 15.18. *We have:*

$$B_{K'}(\overline{Z_0}', \overline{Z_0}') = -4\sqrt{\frac{2}{c_{K'}}}.$$

Proof. We have:

$$\begin{aligned}
\overline{Z_0}'^2 \cdot \pi_{1*}(s_1^*(e))^2 &= \frac{c_{K'}}{3} B_{M'}(\overline{Z_0}', \overline{Z_0}') \times B_{K'}(\pi_{1*}(s_1^*(e)), \pi_{1*}(s_1^*(e))) \\
&= \frac{c_{K'}}{3} B_{K'}(\overline{Z_0}', \overline{Z_0}') \times \left(-36\sqrt{\frac{2}{c_{K'}}}\right) \\
&= -12\sqrt{2c_{K'}} B_{K'}(\overline{Z_0}', \overline{Z_0}') \qquad (62)
\end{aligned}$$

By Proposition 6.5, we have

$$\overline{Z_0}'^2 \cdot \pi_{1*}(s_1^*(e))^2 = 8Z_0'^2 \cdot (s_1^*(e))^2. \qquad (63)$$

By the projection formula, $Z_0'^2 \cdot (s_1^*(e))^2 = s_{1*}(Z_0'^2) \cdot e^2$. Moreover by lemma 15.15, $s_{1*}(Z_0'^2) = -Z_0$. Hence

$$Z_0'^2 \cdot (s_1^*(e))^2 = -Z_0 \cdot e^2. \qquad (64)$$

It follows from (62), (63) and (64) that

$$-8Z_0 \cdot e^2 = -12\sqrt{2c_{K'}} B_{K'}(\overline{Z_0}', \overline{Z_0}'). \qquad (65)$$

Moreover from Section 4 of [18], we have:

$$Z_0 \cdot e^2 = -12. \qquad (66)$$

So by (65) and (66):

$$B_{K'}(\overline{Z_0}', \overline{Z_0}') = -8\sqrt{\frac{1}{2c_{K'}}}.$$

□

Now we are able to finish the calculation of the Beauville–Bogomolov form on $H^2(K', \mathbb{Z})$. By (61), Propositions 15.16, Lemma 15.17, 15.18 and Theorem 15.9, the Beauville–Bogomolov form on $H^2(K', \mathbb{Z})$ gives the lattice:

$$\begin{aligned}
&U^3\left(6\sqrt{\frac{2}{c_{K'}}}\right) \oplus -\frac{1}{4}\sqrt{\frac{2}{c_{K'}}}\begin{pmatrix} 40 & 32 \\ 32 & 40 \end{pmatrix} \\
&= U^3\left(6\sqrt{\frac{2}{c_{K'}}}\right) \oplus -\sqrt{\frac{2}{c_{K'}}}\begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix}
\end{aligned}$$

Then it follows from the integrality and the indivisibility of the Beauville–Bogomolov form that $c_{K'} = 8$, and we get Theorem 15.3.

15.8 Betti numbers and Euler characteristic of K'

Proposition 15.19. *We have:*

- $b_2(K') = 8$,
- $b_3(K') = 0$,
- $b_4(K') = 90$,
- $\chi(K') = 108$.

Proof. It is the same proof as Proposition 4.7.2 of [32]. From Theorem 7.31 of [50], (54), (55) and Proposition 14.3, we get the betti numbers. Then $\chi(K') = 1-0+8-0+90-0+8-0+1 = 108$. □

References

[1] A. Beauville, *Variétés kähleriennes dont la première classe de Chern est nulle*, J. Differential geometry 18 (1983) 755–782.

[2] S. Boissière, *Chern classes of the tangent bundle on the Hilbert scheme of points on the affine plane*, J. Algebraic Geom. **14** (2005), no. 14, 761–787.

[3] S. Boissière, C. Camere and A. Sarti, *Classification of automorphisms on a deformation family of hyperkähler fourfolds by p-elementary lattices*, arXiv:1402.5154v1.

[4] S. Boissière and M. Nieper-Wißkirchen, *Generating series in the cohomology of Hilbert schemes of points on surfaces*, LMS J. of Computation and Math. 10 (2007), 254–270 .

[5] S. Boissière, M. Nieper-Wisskirchen and A. Sarti, *Higher dimensional Enriques varieties and automorphisms of generalized Kummer varieties*, Journal de Mathématiques Pures et Appliquées, vol. 95 (2011), no. 5 553–563.

[6] S. Boissière, M. Nieper-Wißkirchen, and A. Sarti, *Smith theory and Irreducible Holomorphic Symplectic Manifolds*, Journal of Topology 6 (2013), no. 2, 361–390.

[7] G.E. Bredon, *Intoduction to compact transformation groups*, Academic Press, New York, (1972), Pure and Applied Mathematics, Vol. 46.

[8] M. Britze, *On the cohomology of generalized Kummer varieties*, (2003)

[9] Igor V. Dolgachev, *Classical Algebraic Geometry*, Cambridge University Press (2012).

[10] B. Farb and D. Margalit *A primer on mapping class groups*, Princeton University Press.

[11] J. Fogarty, *Algebraic Families on an Algebraic Surface*, Am. J. Math. **10** (1968), 511–521.

[12] A. Fujiki *A theorem on bimeromorphic maps of Kähler manifolds and its applications*, Publ. Res. Inst. Math. Sci., 17(2): 735–754, 1981.

[13] A. Fujiki, *Finite automorphism groups of complex tori of dimension two*, Publ. RIMS, Kyoto Univ., **24** (1988), 1-97.

[14] A. Fujiki *On Primitively Symplectic Compact Kähler V-manifolds of Dimension Four*, Classification of algebraic and analytic manifolds, (Katata, 1982), 71-250.

[15] W. Fulton *Intersection theory*, Second edition, Springer.

[16] E. Ghys and A. Verjovsky, *Locally free holomorphic actions of the complex affine group*, Geometric Study of Foliations, World Scientific (1994), 201–217.

[17] L. Göttsche, *Hilbert Schemes of Zero-Dimensional Subschemes of Smooth Varieties*, Lecture Notes in Mathematics 1572, Springer (1994).

[18] B. Hassett and Y. Tschinkel, *Hodge theory and Lagrangian planes on generalized Kummer fourfolds*, Moscow Math. Journal, 13, no. 1, 33–56, (2013)

[19] D. Huybrechts, *Compact hyperkähler manifolds: basic results*, Invent. math. 135, 63-113 (1999).

[20] T. Kirschner, *Irreducible symplectic complex spaces*, PhD thesis, arXiv:1210.4197.

[21] S. Kapfer, *Symmetric Powers of Symmetric Bilinear Forms, Homogeneous Orthogonal Polynomials on the Sphere and an Application to Compact Hyperkähler Manifolds* (2015).

[22] M. Lehn and C. Sorger, *The cup product of Hilbert schemes for $K3$ surfaces*, Invent. Math. **152** (2003), no. 2, 305–329.

[23] W. Li, Z. Qin and W. Wang, *Hilbert schemes and W algebras* Int. Math. res. Not. (2002), no. 27, 1427–1456.

[24] —, *Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces* (2002)

[25] D. Markushevich, A.S. Tikhomirov, *New symplectic V-manifolds of dimension four via the relative compactified Prymian*, International Journal of Mathematics, Vol. 18, No. 10 (2007) 1187-1224, World Scientific Publishing Company.

[26] E. Markman, *A survey of Torelli and monodromy results for holomorphic-symplectic varieties*, Complex and differential geometry, 257-322, Springer Proc. Math., 8, Springer, Heidelberg, (2011).

[27] E. Markman, *Integral generators for the cohomology ring of moduli spaces of sheaves over Poisson surfaces*, Adv. Math. **208** (2007), no. 2, 622–646.

[28] E. Markman, *Integral constraints on the monodromy group of the hyperKähler resolution of a symmetric product of a $K3$ surface*, Internat. J. Math. **21** (2010), no. 2, 169–223.

[29] E. Markman and S. Mehrtra *Hilbert schemes of K3 surfaces are dense in moduli*, arXiv:1201.0031v1.

[30] D. Matsushita, *On base manifolds of Lagrangian fibrations*, Science China Mathematics (2014), 531-542.

[31] G. Menet, *Beauville–Bogomolov lattice for a singular symplectic variety of dimension 4*, Journal of pure and apply algebra (2014), 1455-1495.

[32] G. Menet *Cohomologie entière et fibration lagrangiennes sur certaines variétés holomorphiquement symplectiques singulières*, PhD thesis of Lille 1 University, (2014).

[33] J. S. Milne, *Abelian Varieties (v2.00)*, Available at www.jmilne.org/math/ (2008)

[34] J. Milnor, D. Husemöller, *Symmetric bilinear forms*, Ergebnisse der Mathematik und ihrer Grenzgebiete 73, Springer (1973).

[35] G. Mongardi *Symplectic involutions on deformations of $K3^{[2]}$*, Cent. Eur. J. Math. 10 (2012), no.4, 1472-1485.

[36] G. Mongardi, Kévin Tari and Malte Wandel, *Automorphisms of generalised Kummer fourfolds*, arXiv:1512.00225v2.

[37] G. Mongardi, Kévin Tari and Malte Wandel *Prime order automorphisms of abelian surfaces: a lattice-theoretic point of view*, arXiv:1506.05679v1.

[38] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5 (1970).

[39] H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Ann. of Math. (2) **145** (1997), no. 2, 379–388.

[40] Y. Namikawa, *Extension of 2-forms and symplectic varieties*, J. Reine Angew. Math. 539 (2001), 123-147.

[41] V.V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Math. USSR Izv. 14 (1980), 103-167.

[42] K. Oguiso *No cohomologically trivial non-trivial automorphism of generalized Kummer manifolds*, arXiv:1208.3750.

[43] Z. Qin and W. Wang, *Integral operators and integral cohomology classes of Hilbert schemes*, Math. Ann. **331** (2005), no. 3, 669–692.

[44] A. Rapagnetta *On the Beauville form of the known irreducible symplectic varieties*, Math. Ann. 321 (2008), 77-95.

[45] F. Scattone, *On the compactification of moduli spaces for algebraic K3 surfaces*, Memoirs of the American Mathematical Society (1987), Volume 70, Number 374.

[46] T. Shioda, *The period map of abelian surfaces*, J. Fac. S. Univ.Tokyo 25 (1978), 47–59.

[47] E. Spanier, *The homology of Kummer manifolds*, Proc. Amer. Math. Soc. 7, (1956), 155-160.

[48] K. Tari, *Automorphismes des variétés de Kummer généralisées*, PhD Thesis, Université de Poitiers (2015).

[49] B. Totaro, *The integral cohomology of the Hilbert scheme of two points* (2015).

[50] C. Voisin, *Hodge Theory and Complex Algebraic Geometry, I*, Cambridge studies in advanced mathematics 76, Cambridge University Press (2002).