INTEGRAL COHOMOLOGY OF $K_2(A)$

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Abstract. What we know already

1. Preliminaries

Definition 1.1. Let n be a natural number. A partition of n is a decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda_1 \geq \ldots \geq \lambda_k > 0$ of natural numbers such that $\sum_i \lambda_i = n$. Sometimes it is convenient to write $\lambda = (\ldots, 2^{m_2}, 1^{m_1})$ with multiplicities in the exponent. We define the weight $\|\lambda\| := \sum m_i i = n$ and the length $|\lambda| := \sum_i m_i = k$. We also define $z_\lambda := \prod_i i^{m_i} m_i!$.

Definition 1.2. Let $\Lambda_n := \mathbb{Q}[x_1, \ldots, x_n]^{S_n}$ be the graded ring of symmetric polynomials. There are canonical projections : $\Lambda_{n+1} \to \Lambda_n$ which send x_{n+1} to zero. The graded projective limit $\Lambda := \varprojlim \Lambda_n$ is called the ring of symmetric functions. Let m_λ and p_λ denote the monomial and the power sum symmetric functions. They are defined as follows: For a monomial $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \ldots x_{i_k}^{\lambda_k}$ of total degree n, the (ordered) sequence of exponents $(\lambda_1, \ldots, \lambda_k)$ defines a partition λ of n, which is called the shape of the monomial. Then we define m_λ being the sum of all monomials of shape λ. For the power sums, first define $p_n := x_1^n + x_2^n + \ldots$. Then $p_\lambda := p_{\lambda_1} p_{\lambda_2} \ldots p_{\lambda_k}$. The families $(m_\lambda)_\lambda$ and $(p_\lambda)_\lambda$ form two $\mathbb{Q}$-bases of Λ, so they are linearly related by $p_\lambda = \sum_\mu \psi_{\lambda\mu} m_\mu$. It turns out that the base change matrix $(\psi_{\lambda\mu})$ has integral entries, but its inverse $(\psi^{-1}_{\lambda\mu})$ has not.

2. Cohomology of Hilbert schemes of points on a torus surface

Let A be a complex projective torus of dimension 2. Its first cohomology $H^1(A, \mathbb{Z})$ is freely generated by four elements a_1, a_2, a_3, a_4, corresponding to the four different circles on the torus. The cohomology ring is isomorphic to the exterior algebra:

$$H^*(A, \mathbb{Z}) = \Lambda^* H^1(A, \mathbb{Z}).$$

We abbreviate for the products $a_i \cdot a_j =: a_{ij}$ and $a_i \cdot a_j \cdot a_k =: a_{ijk}$. We write $a_1 \cdot a_2 \cdot a_3 \cdot a_4 =: x$ for the class corresponding to a point on A. We choose the a_i such that $\int_A x = 1$. The bilinear form, given by $(a_{ij}, a_{kl}) \mapsto \int_A a_{ij} a_{kl}$ gives $H^2(A, \mathbb{Z})$ the structure of a unimodular lattice, isomorphic to $U^{\oplus 3}$, three copies of the hyperbolic lattice.

Let $A^{[n]}$ the Hilbert scheme of n points on the torus, *i.e.* the moduli space of finite subschemes of A of length n. Their rational cohomology can be described in

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terms of Nakajima's operators. First consider the direct sum

$$\mathbb{H} := \bigoplus_{n=0}^{\infty} H^*(A^{[n]}, \mathbb{Q})$$

This space is bigraded by cohomological *degree* and the *weight*, which is given by the number of points n. The unit element in $H^0(A^{[0]}, \mathbb{Q}) \cong \mathbb{Q}$ is denoted by $|0\rangle$, called the *vacuum*.

There are linear operators $\mathfrak{p}_m(\alpha)$, for each $m \in \mathbb{Z}$, $\alpha \in H^*(A, \mathbb{Q})$, acting on $\mathbb{H}$ which have the following properties: They depend linearly on α, and if $\alpha \in H^k(A, \mathbb{Q})$ is homogeneous, the operator $\mathfrak{p}_{-m}(\alpha)$ is bihomogeneous of degree $k + 2(|m| - 1)$ and weight m:

$$\mathfrak{p}_{-m}(\alpha) : H^l(A^{[n]}) \to H^{l+k+2(|m|-1)}(A^{[n+m]})$$

They satisfy the following commutation relations for $\alpha \in H^k(A, \mathbb{Q})$, $\beta \in H^{k'}(A, \mathbb{Q})$:

$$\mathfrak{p}_m(\alpha)\mathfrak{p}_{m'}(\beta) - (-1)^{k \cdot k'}\mathfrak{p}_{m'}(\beta)\mathfrak{p}_m(\alpha) = -m\, \delta_{m,-m'} \int_A \alpha \cdot \beta.$$

Every element in $\mathbb{H}$ can be decomposed uniquely as a linear combination of products of operators $\mathfrak{p}_m(\alpha)$, $m < 0$, acting on the vacuum. We abbreviate for a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$:

$$\mathfrak{q}_\lambda(\alpha) := \prod_{i=1}^{k} \mathfrak{p}_{-\lambda_i}(\alpha) \tag{1}$$

$$\mathfrak{q}_{*\lambda}(\alpha) := \left(\prod_{i=1}^{k} \mathfrak{p}_{-\lambda_i}\right) \left(\Delta_{(k)}(\alpha)\right) \tag{2}$$

For the study of integral cohomology, first note that if $\alpha \in H^*(A, \mathbb{Z})$ is an integral class, then $\mathfrak{p}_{-m}(\alpha)$ maps integral classes to integrall classes. Moreover, there is the following theorem:

Theorem 2.1. [11] *The following operators map integral classes in $\mathbb{H}$ to integral classes:*

- $\frac{1}{z_\lambda}\mathfrak{q}_\lambda(1)$
- $\mathfrak{m}_\lambda(\alpha)$ *for* $\alpha \in H^2(A, \mathbb{Z})$

Here, $\mathfrak{m}_\lambda$ is defined as $\mathfrak{m}_\lambda(\alpha) := \sum_\mu \psi^{-1}_{\lambda\mu}\mathfrak{q}_{-\mu}(\alpha)$ (see Definition 1.2)

To obtain the multiplicative structure of $\mathbb{H}$, given by the cup-products, there is a description in [6] and [7] in terms of multiplication operators $\mathfrak{G}_n(\alpha)$ [7, Def. 5.1], related to chern characters.

Next we focus on the structure of $H^2(A^{[n]}, \mathbb{Z})$ for $n \geq 2$. It has rank 13, and there is a basis consisting of:

- $\frac{1}{(n-1)!}\mathfrak{p}_{-1}(1)^{n-1}\mathfrak{p}_{-1}(a_{ij})|0\rangle$, $1 \leq i < j \leq 4$,
- $\frac{1}{(n-2)!}\mathfrak{p}_{-1}(1)^{n-2}\mathfrak{p}_{-1}(a_i)\mathfrak{p}_{-1}(a_j)|0\rangle$, $1 \leq i < j \leq 4$,
- $\frac{1}{2(n-2)!}\mathfrak{p}_{-1}(1)^{n-2}\mathfrak{p}_{-2}(1)|0\rangle$. We denote this class by δ.

It is clear that these classes form a basis of $H^2(A^{[n]}, \mathbb{Q})$. By [11, Thm. 4.6,Lemma 5.2], they also form a basis for $H^2(A^{[n]}, \mathbb{Z})$. TODO: refine this argument

The first 6 classes give an injection $j : H^2(A, \mathbb{Z}) \to H^2(A^{[n]}, \mathbb{Z})$.

3. Generalized Kummer varieties

Definition 3.1. Let A be a complex projective torus of dimension 2 and $A^{[n]}$, $n \geq 1$, the corresponding Hilbert scheme of points. Denote $\Sigma : A^{[n]} \to A$ the summation morphism, a smooth submersion that factorizes via the Hilbert–Chow morphism : $A^{[n]} \to \mathrm{Sym}^n(A) \to A$. Then the generalized Kummer $K^{n-1}A$ is defined as the fiber over 0:

$$\begin{tikzcd} K^{n-1}A \arrow[r, "\theta"] \arrow[d] & A^{[n]} \arrow[d, "\Sigma"] \\ \{0\} \arrow[r] & A \end{tikzcd} \tag{3}$$

Our first objective is to collect some information about this pullback diagram. Or main reference is [1] where it is shown, that K^{n-1} is an irreducible holomorphically symplectic manifold. So $H^2(K_{n-1}(A), \mathbb{Z})$ admits an integer-valued nondegenerated quadratic form (called Beauville–Bogomolov form) q which gives $H^2(K_{n-1}(A), \mathbb{Z})$ the structure of a lattice isomorphic to $U^{\oplus 3} \oplus \langle -2n \rangle$, for $n \geq 3$. We have the following formula for $\alpha \in H^2(K_{n-1}(A), \mathbb{Z})$:

$$\int_{K_{n-1}(A)} \alpha^{2n-2} = n \frac{(2n-2)!}{2^{n-1}(n-1)!} q(a)^{n-1} \tag{4}$$

The morphism θ induces a homomorphism of graded rings

$$\theta^* : H^*(A^{[n]}, \mathbb{Z}) \longrightarrow H^*(K_{n-1}(A), \mathbb{Z}). \tag{5}$$

Proposition 3.2. *Let $n \geq 3$.*

1. *θ^* maps $H^1(A^{[n]}, \mathbb{Z})$ to zero.*
2. *θ^* is surjective on $H^2(A^{[n]}, \mathbb{Z})$ with kernel $\Lambda^2 H^1(A^{[n]}, \mathbb{Z})$.*

Proof. The first statement is clear since $H^1(K_{n-1}(A))$ is always zero [1, Thm. 3]. Furthermore, by [1, Sect. 7], $\theta^* : H^2(A^{[n]}, \mathbb{C}) \to H^2(K_{n-1}(A), \mathbb{C})$ is surjective. The second Betti numbers of $A^{[n]}$ and $K_{n-1}(A)$ are 13 and 7, respectively. It is clear that $\Lambda^2 H^1(A^{[n]}, \mathbb{Z})$ is contained in the kernel, and since the dimension of the kernel has to be 6, it must be all.

It remains to show that θ^* is surjective for integral coefficients, too. We do it only for $n = 3$. We use a formula in [4, p. 8], namely:

$$\int_{A^{[3]}} j(a)^6 = \frac{5}{3} \int_A a^2 \int_{K_2(A)} \theta^* j(a)^4 \tag{6}$$

for all $a \in H^2(A)$. One computes $\int_{A^{[3]}} j(a)^6 = 15 \left(\int_A a^2 \right)^3$. Comparing this with (4), we see that the sublattice given by the image of $\theta^* \circ j$ is unimodular. Secondly, we must show that $q(\theta^* \delta) = -6$. TODO: show this! □

Proposition 3.3. *Set $a_i^{(1)} := \frac{1}{2} \mathfrak{q}_1(1)^2 \mathfrak{q}_1(a_i)|0\rangle$. The class of $K_2(A)$ in $H^4(A^{[3]}, \mathbb{Q})$ is given by*

$$a_1^{(1)} \cdot a_2^{(1)} \cdot a_3^{(1)} \cdot a_4^{(1)}.$$

Conjecture: This is true for all n, not only $n = 3$.

Proof. We know that for all i and all $\beta \in H^7(A^{[3]})$, we have $\int_{K_2(A)} \theta^*(\alpha_i \cdot \beta) = \int_{A^{[3]}} \alpha_i \cdot \beta \cdot [K_2(A)] = 0$ and for a basis (γ_i) of $H^2(A^{[3]})$,

$$\int_{A^{[3]}} \gamma_i \cdot \gamma_j \cdot \gamma_k \cdot \gamma_l \cdot [K_2(A)] = 3\left(\langle\gamma_i,\gamma_j\rangle\langle\gamma_k,\gamma_l\rangle + \langle\gamma_i,\gamma_k\rangle\langle\gamma_j,\gamma_l\rangle + \langle\gamma_i,\gamma_l\rangle\langle\gamma_j,\gamma_k\rangle\right)$$

These equations admit a unique solution. □

Now we focus on classes of cohomological degree 4.

Proposition 3.4. *The classes* $\theta^*\left(\mathfrak{p}_{-2}(a_{ij})\mathfrak{p}_{-1}(1)|0\rangle\right)$ *and* $\theta^*\left(\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_{ij})|0\rangle\right)$ *are linearly dependent.*

Proof. We can compute the product of these two classes with $[K_2(A)]$ in $H^*(A^{[3]})$. The two results are linearly dependent. Is this sufficient? If not, is there a direct geometric proof? □

Proposition 3.5. $\theta^*\left(\mathfrak{p}_{-3}(x)|0\rangle\right) = 0$

Corollary 3.6. $\theta^*\left(\mathfrak{p}_{-2}(a_{ij})\mathfrak{p}_{-1}(1)|0\rangle\right) = \frac{1}{4}\theta^*\left(\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_{ij})|0\rangle\right)$

Proof. Let a_{ij} and a_{kl} be complementary, *i.e.* $a_{ij}a_{kl} = 1$. Let $\mathrm{ch}_1(a_{kl}) = -\frac{1}{2}\mathfrak{p}_{-2}(a_{kl})\mathfrak{p}_{-1}(1)|0\rangle$ be the chern character in the vertex algebra description of $H^*(A^{[3]})$. Then:

$$\theta^*\left(-\frac{1}{2}\,\mathrm{ch}_1(a_{kl}) \cdot \mathfrak{p}_{-2}(a_{ij})\mathfrak{p}_{-1}(1)|0\rangle\right) = \theta^*\left(\mathfrak{p}_{-3}(1)|0\rangle + \frac{1}{2}\mathfrak{p}_{-1}(x)^2\mathfrak{p}_{-1}(1)|0\rangle\right)$$

But on the other hand, $\delta \cdot j(a) = \frac{1}{2}\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_{ij})|0\rangle + \mathfrak{p}_{-2}(a_{ij})\mathfrak{p}_{-1}(1)|0\rangle$, and

$$\theta^*\left(\mathrm{ch}_1(a_{kl}) \cdot \delta \cdot j(a)\right) = \theta^*\left(-3\mathfrak{p}_{-3}(1)|0\rangle - 3\mathfrak{p}_{-1}(x)^2\mathfrak{p}_{-1}(1)|0\rangle\right).$$

□

Corollary 3.7. $\theta^*\left(\delta \cdot j(a_{ij})\right) = \theta^*\left(\frac{3}{4}\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_{ij})|0\rangle\right)$ *is divisible by 3.* □

Proposition 3.8. *The classes* $\theta^*\left(j(a_{ij})^2 - \frac{1}{3}j(a_{ij}) \cdot \delta\right)$ *are divisible by 2.*

Proof. By [11], the classes $\frac{1}{2}\mathfrak{p}_{-1}(a_{ij})^2\mathfrak{p}_{-1}(1)|0\rangle - \frac{1}{2}\mathfrak{p}_{-2}(a_{ij})\mathfrak{p}_{-1}(1)|0\rangle$ are integral in $H^4(A^{[n]})$. But $j(a_{ij})^2 = \mathfrak{p}_{-1}(a_{ij})^2\mathfrak{p}_{-1}(1)|0\rangle$ and $\theta^*\left(\frac{1}{3}j(a_{ij}) \cdot \delta\right) = \theta^*\left(\mathfrak{p}_{-2}(a_{ij})\mathfrak{p}_{-1}(1)|0\rangle\right)$. □

Proposition 3.9. *The class* δ^2 *is divisible by 2.*

Proof. By [5, Prop. 4.1], $\mathrm{Sym}^2 H^2 \oplus \left(\mathrm{Sym}^2 H^2\right)^\perp = H^4$. We want to show that $\delta^2 \cdot \mathrm{Sym}^2 H^2 = 2\mathbb{Z}$. We know a $\mathbb{Q}$–basis of $\mathrm{Sym}^2 H^2$ with at most one class divisible by 2, given by $j(a_{ij})j(a_{kl})$, δ^2 and the above proposition. By computation, $\int \delta^4$ is divisible by 4 and $\int \delta^2 j(a_{ij})j(a_{kl})$ and $\int \delta^3 j(a_{ij})$ are all divisible by 2. So $\delta^2 \cdot H^4 = 2\mathbb{Z}$ and therefore δ^2 is divisible by 2, since H^4 is unimodular. □

Proposition 3.10. *The class* $\theta^*\left(\delta^2 - j(a_1) \cdot j(a_2) - j(a_3) \cdot j(a_4) - j(a_5) \cdot j(a_6)\right)$ *is divisible by 3.*

Proof. It is equal to $\theta^*\left(\mathfrak{p}_{-3}(1)|0\rangle + \frac{3}{2}\mathfrak{p}_{-1}(x)\mathfrak{p}_{-1}(1)^2|0\rangle\right)$. □

Next we look at the Chern classes of the tangent sheaves. Since the morphism Σ from the defining pullback diagram (3) is a submersion, the normal bundle of $K_{n-1}(A)$ in $A^{[n]}$ is trivial. Hence $c(K_2(A)) = \theta^* c(A^{[3]})$. Looking in [2, Sect. 8], we find a general formula for Chern classes of Hilbert schemes of points on surfaces. So we deduce

$$c_2(A^{[3]}) = \left(\tfrac{3}{2}\mathfrak{q}_{*(1,1)}(1)\mathfrak{q}_1(1) - \tfrac{1}{3}\mathfrak{q}_3\right)|0\rangle$$
$$= 10(1^{[\bullet]}_{(4)}) - 2(1^{[\bullet]}_{(2)})^2$$
$$c_4(A^{[3]}) = \tfrac{4}{3}\mathfrak{q}_{*(1,1,1)}(1)|0\rangle = 4(1^{[\bullet]}_{(4)})^2.$$

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