Integral cohomology of IHS varieties

Simon Kapfer

Augsburg University

Summary

We give a description of integral cohomology of the generalised Kummer fourfold. As an application, we describe a new example of a IHS variety with singularities. It is the first example of a Beauville–Bogomolov form which is odd.
This is joint work with Grégoire Menet.

Integral cohomology of the generalised Kummer fourfold

For A an abelian surface, the generalised Kummer fourfold $K_2(A)$ is realised as a subspace of $A^{[3]}$, the Hilbert scheme of three points.

$$\theta^* : K_2(A) \hookrightarrow A^{[3]}$$

Theorem

The pullback homomorphism $\theta^* : H^*(A^{[3]}, \mathbb{Z}) \to H^*(K_2(A), \mathbb{Z})$ of graded rings is surjective in every degree except 4. Moreover, the image of $H^4(A^{[3]}, \mathbb{Z})$ is the primitive overlattice of $\mathrm{Sym}^2(H^2(K_2(A), \mathbb{Z}))$ (of discriminant 3^{22}). The kernel of θ^* is the ideal generated by $H^1(A^{[3]})$.

This is proved using the Nakajima operator algebra to compute the cohomology of the Hilbert scheme, together with an explicit description of θ^*. We get the quotient

$$\frac{H^4(K_2(A), \mathbb{Z})}{\mathrm{Sym}^2(H^2(K_2(A), \mathbb{Z}))} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^7 \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^8 \oplus \mathbb{Z}^{80}.$$

To obtain the remaining 80 classes in $H^4(K_2(A), \mathbb{Z})$, we use an approach of Hassett and Tschinkel: For $\tau \in A[3]$ a point of three-torsion, the Briançon subscheme of $A^{[3]}$ supported at τ yields a class in $H^4(K_2(A), \mathbb{Z})$. These classes give a complementary space to $\mathrm{Sym}^2(H^2(K_2(A), \mathbb{Z}))$. Summing up such classes for τ in an affine plane contained in $A[3]$, one gets a class divisible by three. The orbit of this class under the action of the monodromy group contains enough classes divisible by 3 to obtain an integral basis of middle cohomology.

Involutions on fourfolds of Kummer type

Let $\iota : K_2(A) \to K_2(A)$ be the natural involution induced by $-\mathrm{id}$ on A. Then the fixed locus of ι consists of a $K3$ surface and 36 isolated points. Moreover, for any symplectic involution ι' on a complex fourfold of Kummer type X' we can show:

Theorem

- The induced morphism on $H^2(X, \mathbb{Z})$ acts trivially.
- The couple (X', ι') is deformation equivalent to $(K_2(A), \tau \circ \iota)$, where τ denotes the automorphism induced by a translation by a three-torsion point in A.

This is based on a lattice-theoretic classification of automorphisms, done by Mongardi, Tari and Wandel. Then we conclude from a result by Hassett and Tschinkel, stating that automorphisms which are fixing H^2 are deformation invariant.

A quotient

The fixed locus of ι consists of a $K3$ surface and 36 isolated points, by an observation of Tari. Denote

$$K' \to K_2(A)/\iota$$

the partial resolution of singularities obtained by blowing up the K3 surface. We get an irreducible symplectic variety with singularities of codimension 4.

Theorem

The Beauville–Bogomolov lattice $H^2(K', \mathbb{Z})$ is isomorphic to

$$U(3)^3 \oplus \begin{pmatrix} -5 & -4 \\ -4 & -5 \end{pmatrix}$$

and the Fujiki constant $c_{K'}$ is equal to 8. The Betti numbers are $b_2 = 8$, $b_3 = 0$, $b_4 = 90$. The Euler characteristic is $\chi(K') = 108$.

A general result on the Fujiki relation

Let X be a IHS variety of dimension $2n$ (with singularities or not) with a Beauville–Bogomolov form such that the Fujiki relation holds. Seen as a lattice, $\mathrm{Sym}^n H^2(X, \mathbb{Z})$ is embedded in the unimodular Poincaré lattice $H^{2n}(X, \mathbb{Z})$.

Theorem

Denote $d+1$ the rank of $H^2(X, \mathbb{Z})$ and denote c_X the Fujiki constant. The discriminant of $\mathrm{Sym}^n H^2(X, \mathbb{Z})$ is given by

$$\left(\mathrm{discr}\left(H^2(X, \mathbb{Z})\right)\right)^{\binom{d+n}{d+1}} \cdot c_X^{\binom{d+n}{d}} \cdot \prod_{i=1}^{n} i^{\binom{n-i+d}{d} d} \cdot C,$$

$$\text{with} \quad C = \begin{cases} \displaystyle\prod_{\substack{i=1 \\ i \text{ odd}}}^{2n+d-1} i^{\binom{n-i+d}{d}} & \text{if } d+1 \text{ is odd,} \\ \displaystyle\prod_{i=1}^{n+\frac{d-1}{2}} i^{\binom{n-i+d}{d} - \binom{n-2i+d}{d}} & \text{if } d+1 \text{ is even.} \end{cases}$$

References

- B. Hassett, Y. Tschinkel, *Hodge theory and Lagrangian planes on generalized Kummer fourfolds*, Moscow Math. Journal, 13, no. 1, 2013.
- S. Kapfer, G. Menet, *Integral cohomology of the Generalized Kummer fourfold*, preprint 2016.
- S. Kapfer, *Symmetric Powers of Symmetric Bilinear Forms, Homogeneous Orthogonal Polynomials on the sphere and an application in Compact Hyperkähler Manifolds*, Commun. Contemp. Math., 2016.

Contact information

Email: simon.kapfer@math.uni-augsburg.de

Universität Augsburg
Institut für Mathematik

UNIVERSITAT AUGSBURG · SCIENTIA ET CONSCIENTIA