AUTOMORPHISMS OF COMPLEX TORI OF THE FORM $E \times E$

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Let $\Lambda_0 \subset \mathbb{C}$ be a lattice of rank 2. We may assume that $\Lambda_0 = \mathbb{Z} + \mathbb{Z}\tau$ for a complex number $\tau \in \mathbb{C}\backslash\mathbb{R}$. We set $\Lambda := \Lambda_0 \times \Lambda_0$, $E := \mathbb{C}/\Lambda_0$ and $A := E \times E$. By an automorphism of A we mean a biholomorphism preserving the group structure. This is the same as a $\mathbb{C}$-linear map $M : \mathbb{C}^2 \to \mathbb{C}^2$ with $M\Lambda = \Lambda$.

In the appendix of [1] we find a study of automorphisms of 2-dimensional complex tori. In our case, $A = E \times E$, the automorphism group is given by $\mathrm{GL}(2, \mathrm{End}(\Lambda_0))$. Here, $\mathrm{End}(\Lambda_0)$ is the set $\{z \in \mathbb{C} \mid z\Lambda_0 \subset \Lambda_0\}$. Given such a z, then we have

$$z \cdot 1 = a + b\tau \text{ and } z \cdot \tau = c + d\tau \text{ with } a, b, c, d \in \mathbb{Z}.$$

We get the condition

$$(a + b\tau)\tau = c + d\tau \quad \Leftrightarrow \quad b\tau^2 + (d - a)\tau + c = 0.$$

Up to scalar multiples, there is a unique real quadratic polynomial that annihilates τ, namely $(x - \tau)(x - \bar{\tau}) = x^2 - 2\Re(\tau)x + \|\tau\|^2$. So we have 2 possibilities:

(1) Both the real part and the square norm of τ are rational numbers, say $2\Re(\tau) = \frac{p}{r}$ and $\|\tau\|^2 = \frac{q}{r}$ with $r > 0$ as small as possible. Then $z = a + b\tau$ with arbitrary $a \in \mathbb{Z}$ and $b \in r\mathbb{Z}$, so $\mathrm{End}(\Lambda_0) = \mathbb{Z} + r\tau\mathbb{Z}$.

(2) At least one of $\Re(\tau), \|\tau\|^2$ is irrational. Then $z = a$ and $\mathrm{End}(\Lambda_0) = \mathbb{Z}$.

If I understand properly, in [2, proof of Prop. 5.2] it is claimed that the automorphism group of A, when restricted to the three-torsion points $A[3]$, contains the symplectic group $\mathrm{Sp}(A[3]) = \mathrm{Sp}(4, \mathbb{F}_3)$. At least in the second case this is impossible, because then $\mathrm{Aut}(A) = \mathrm{GL}(2, \mathbb{Z})$, so $\mathrm{Aut}(A) \otimes \mathbb{F}_3 = \mathrm{GL}(2, \mathbb{F}_3)$ is a group of order 48 (cf [3]). On the other hand, the order of $\mathrm{Sp}(4, \mathbb{F}_3)$ is 51840 (cf. [4]).

References

1. E. Ghys, A. Verjovsky, *Locally free holomorphic actions of the complex affine group*, `http://perso.ens-lyon.fr/ghys/articles/locallyfreeaffine.pdf`.
2. B. Hassett, Y. Tschinkel, *Hodge theory and Lagrangian planes on generalized Kummer fourfolds.*
3. *Order of general linear group over finite fields*, `https://proofwiki.org/wiki/Order_of_General_Linear_Group_over_Finite_Field`.
4. *Order of symplectic group over finite fields*, `http://groupprops.subwiki.org/wiki/Order_formulas_for_symplectic_groups`.

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