Odd cohomology of $A^{[2]}$

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With your notation, we consider the following exact sequence:

$$0 \longrightarrow \pi_*(H^k(\widetilde{A \times A}, \mathbb{Z})) \longrightarrow H^k(A^{[2]}, \mathbb{Z})/\operatorname{tors} \longrightarrow (\mathbb{Z}/2\mathbb{Z})^{\alpha_k} \longrightarrow 0,$$

with $k \in \{1, ..., 8\}$. We want to prove the following proposition.

Proposition 0.1. *We have:*

$$\alpha_3 = 0 \text{ and } \alpha_5 = 4.$$

1 Preliminary Lemmas

We denote $V = \widetilde{A \times A} \smallsetminus E$ and $U = V/\sigma_2$, where $\mathfrak{S}_2 = \langle \sigma_2 \rangle$.

Lemma 1.1. *We have: $H^k(A \times A, \mathbb{Z}) = H^k(V, \mathbb{Z})$ for all $k \leq 3$.*

Proof. We have $V = A \times A \smallsetminus \Delta$. We have the following natural exact sequence:

$$\cdots \longrightarrow H^k(A \times A, V, \mathbb{Z}) \longrightarrow H^k(A \times A, \mathbb{Z}) \longrightarrow H^k(V, \mathbb{Z}) \longrightarrow \cdots$$

Moreover by Thom isomorphism $H^k(A \times A, V, \mathbb{Z}) = H^{k-4}(\Delta, \mathbb{Z}) = H^{k-4}(A, \mathbb{Z})$. Hence $H^k(A \times A, V, \mathbb{Z}) = 0$ for all $k \leq 3$. Hence $H^k(A \times A, \mathbb{Z}) = H^k(V, \mathbb{Z})$ for all $k \leq 2$. It remains to consider the following exact sequence:

$$0 \longrightarrow H^3(A \times A, \mathbb{Z}) \longrightarrow H^3(V, \mathbb{Z}) \longrightarrow H^4(A \times A, V, \mathbb{Z}) \xrightarrow{\rho} H^4(A \times A, \mathbb{Z}) .$$

The map ρ is given by $\mathbb{Z}[\Delta] \to H^4(A \times A, \mathbb{Z})$. The class $\{x\} \times A$ is also in $H^4(A \times A, \mathbb{Z})$ and intersects Δ in one point. Hence the class of Δ in $H^4(A \times A, \mathbb{Z})$ is not trivial and the map ρ is injective. It follows

$$H^3(A \times A, \mathbb{Z}) = H^3(V, \mathbb{Z}).$$

□

Now we will calculate the invariant $l_{1,-}^2(A \times A)$ and $l_{1,+}^1(A \times A)$ defined in Section 1.2 of [1] (I also recall the definition in the redaction of the application).

Lemma 1.2. *We have: $l_{1,-}^2(A \times A) = l_{1,+}^1(A \times A) = 0$.*

Proof. By Künneth formula we have:

$$H^1(A \times A, \mathbb{Z}) = H^0(A, \mathbb{Z}) \otimes H^1(A, \mathbb{Z}) \oplus H^1(A, \mathbb{Z}) \otimes H^0(A, \mathbb{Z}).$$

The elements of $H^0(A, \mathbb{Z}) \otimes H^1(A, \mathbb{Z})$ and $H^1(A, \mathbb{Z}) \otimes H^0(A, \mathbb{Z})$ are exchanged under the action of σ_2. It follows that $l_2^1(A \times A) = 4$ and necessary $l_{1,-}^1(A \times A) = l_{1,+}^1(A \times A) = 0$.

By Künneth formula we also have:

$$H^2(A \times A, \mathbb{Z}) = H^0(A, \mathbb{Z}) \otimes H^2(A, \mathbb{Z}) \oplus H^1(A, \mathbb{Z}) \otimes H^1(A, \mathbb{Z}) \\ \oplus H^2(A, \mathbb{Z}) \otimes H^0(A, \mathbb{Z}).$$

As before every elements $x\otimes y\in H^2(A\times A,\mathbb{Z})$ are sent to $y\otimes x$ by the action of σ_2. A such element is fixed by the action of σ_2 if $x=y$. It follows:

$$l_2^2(A\times A)=6+6=12,$$

$$l_{1,+}^2(A\times A)=4,$$

and necessary:

$$l_{1,-}^2(A\times A)=0.$$

□

Lemma 1.3. *The group $H^3(U,\mathbb{Z})$ is torsion free.*

Proof. Using the spectral sequence of equivariant cohomology, it follows from Proposition 2.6 of [1], Lemma 1.1 and 1.2. □

2 Proof of $\alpha_3=0$

By Theorem 7.31 of Voisin, we have:

$$H^3(\widetilde{A\times A},\mathbb{Z})=H^3(A\times A,\mathbb{Z})\oplus H^1(\Delta,\mathbb{Z}). \tag{1}$$

It follows that

$$H^3(A^{[2]},\mathbb{Z})\supset\pi_*(H^3(A\times A,\mathbb{Z}))\oplus\pi_*(H^1(\Delta,\mathbb{Z})).$$

Moreover by Künneth formula, we have:

$$\begin{aligned}H^3(A\times A,\mathbb{Z})=&\,H^0(A,\mathbb{Z})\otimes H^3(A,\mathbb{Z})\oplus H^1(A,\mathbb{Z})\otimes H^2(A,\mathbb{Z})\\&\oplus H^2(A,\mathbb{Z})\otimes H^1(A\,\mathbb{Z})\oplus H^3(A,\mathbb{Z})\otimes H^0(A,\mathbb{Z}).\end{aligned}$$

Hence all elements in $H^3(A\times A,\mathbb{Z})^{\mathfrak{S}_2}$ are written $x+\sigma_2^*(x)$ with $x\in H^3(A\times A,\mathbb{Z})$. Since $\frac{1}{2}\pi_*(x+\sigma_2^*(x))=\pi_*(x)$, it follows that $\pi_*(H^3(A\times A,\mathbb{Z}))$ is primitive in $H^3(A^{[2]},\mathbb{Z})$. Moreover by (1):

$$l_2^3(\widetilde{A\times A})=\operatorname{rk}H^3(A\times A,\mathbb{Z})^{\mathfrak{S}_2}=28. \tag{2}$$

and

$$l_{1,+}^3(\widetilde{A\times A})=\operatorname{rk}H^1(\Delta,\mathbb{Z})^{\mathfrak{S}_2}=4,\text{ and } l_{1,-}^3(\widetilde{A\times A})=0. \tag{3}$$

It remains to prove the following lemma.

Lemma 2.1. *The group $\pi_*(H^1(\Delta,\mathbb{Z}))$ is primitive in $H^3(A^{[2]},\mathbb{Z})$.*

Proof. We consider the following commutative diagram:

$$\begin{tikzcd} H^3(\mathscr{N}_{A^{[2]}/E},\mathscr{N}_{A^{[2]}/E}-0,\mathbb{Z})=H^3(A^{[2]},U,\mathbb{Z}) \arrow[r, "g"] \arrow[d, "d\widetilde{\pi}^*"] & H^3(A^{[2]},\mathbb{Z}) \arrow[d, "\pi^*"'] \\ H^3(\mathscr{N}_{\widetilde{A\times A}/E},\mathscr{N}_{\widetilde{A\times A}/E}-0,\mathbb{Z})=H^3(\widetilde{A\times A},V,\mathbb{Z}) \arrow[r, "h"] & H^3(\widetilde{A\times A},\mathbb{Z}), \end{tikzcd} \tag{4}$$

By proof of Theorem 7.31 of [2], the map h is injective and its image in $H^3(\widetilde{A\times A},\mathbb{Z})$ is $H^1(\Delta,\mathbb{Z})$. Hence Diagram 4 shows that g is also injective and has image $\pi_*(H^1(\Delta,\mathbb{Z}))$ in $H^3(A^{[2]},\mathbb{Z})$. It follows the exact sequence:

$$0\longrightarrow H^3(A^{[2]},U,\mathbb{Z})\xrightarrow{g}H^3(A^{[2]},\mathbb{Z})\longrightarrow H^3(U,\mathbb{Z}).$$

However, by lemma 1.3, $H^3(U,\mathbb{Z})$ is torsion free; it follows that $\pi_*(H^1(\Delta,\mathbb{Z}))$ is primitive in $H^3(A^{[2]},\mathbb{Z})$. □

3 Proof of $\alpha_5 = 4$

By Theorem 7.31 of Voisin, we have:

$$H^5(\widetilde{A \times A}, \mathbb{Z}) = H^5(A \times A, \mathbb{Z}) \oplus H^3(\Delta, \mathbb{Z}). \tag{5}$$

It follows that

$$H^5(A^{[2]}, \mathbb{Z}) \supset \pi_*(H^5(A \times A, \mathbb{Z})) \oplus \pi_*(H^3(\Delta, \mathbb{Z})).$$

Moreover by Künneth formula, we have:

$$\begin{aligned} H^5(A \times A, \mathbb{Z}) &= H^1(A, \mathbb{Z}) \otimes H^4(A, \mathbb{Z}) \oplus H^2(A, \mathbb{Z}) \otimes H^3(A, \mathbb{Z}) \\ &\oplus H^3(A, \mathbb{Z}) \otimes H^2(A, \mathbb{Z}) \oplus H^4(A, \mathbb{Z}) \otimes H^1(A, \mathbb{Z}). \end{aligned}$$

As before, $\pi_*(H^5(A \times A, \mathbb{Z}))$ is primitive in $H^5(A^{[2]}, \mathbb{Z})$. Moreover by (5):

$$l_2^5(\widetilde{A \times A}) = \operatorname{rk} H^5(A \times A, \mathbb{Z})^{\mathfrak{S}_2} = 28, \tag{6}$$

and

$$l_{1,+}^5(\widetilde{A \times A}) = \operatorname{rk} H^3(\Delta, \mathbb{Z})^{\mathfrak{S}_2} = 4, \text{ and } l_{1,-}^5(\widetilde{A \times A}) = 0. \tag{7}$$

Lemma 3.1. *The lattice $\pi_*(H^3(\widetilde{A \times A}, \mathbb{Z}) \oplus H^5(\widetilde{A \times A}, \mathbb{Z}))$ has discriminant 2^8.*

Proof. By Definition-Proposition 1.7 2) and 3) of [1], (2) and (6):

$$\frac{H^3(\widetilde{A \times A}, \mathbb{Z}) \oplus H^5(\widetilde{A \times A}, \mathbb{Z})}{H^3(\widetilde{A \times A}, \mathbb{Z})^{\mathfrak{S}_2} \oplus H^5(\widetilde{A \times A}, \mathbb{Z})^{\mathfrak{S}_2} \oplus \left(H^3(\widetilde{A \times A}, \mathbb{Z})^{\mathfrak{S}_2} \oplus H^5(\widetilde{A \times A}, \mathbb{Z})^{\mathfrak{S}_2}\right)^{\perp}} = (\mathbb{Z}/2\mathbb{Z})^{l_2^3(\widetilde{A \times A}) + l_2^5(\widetilde{A \times A})}.$$

Since $H^3(\widetilde{A \times A}, \mathbb{Z}) \oplus H^5(\widetilde{A \times A}, \mathbb{Z})$ is an unimodular lattice, it follows that

$$\operatorname{discr} H^3(\widetilde{A \times A}, \mathbb{Z})^{\mathfrak{S}_2} \oplus H^5(\widetilde{A \times A}, \mathbb{Z})^{\mathfrak{S}_2} = 2^{l_2^3(\widetilde{A \times A}) + l_2^5(\widetilde{A \times A})}.$$

Then by Lemma 2.18 3) of [1],

$$\operatorname{discr} \pi_*(H^3(\widetilde{A \times A}, \mathbb{Z})^{\mathfrak{S}_2} \oplus H^5(\widetilde{A \times A}, \mathbb{Z})^{\mathfrak{S}_2}) = 2^{l_2^3(\widetilde{A \times A}) + l_2^5(\widetilde{A \times A}) + \operatorname{rk}\left[H^3(\widetilde{A \times A}, \mathbb{Z})^{\mathfrak{S}_2} \oplus H^5(\widetilde{A \times A}, \mathbb{Z})^{\mathfrak{S}_2}\right]}.$$

Then by Proposition 1.6 of [1]:

$$\operatorname{discr} \pi_*(H^3(\widetilde{A \times A}, \mathbb{Z})^{\mathfrak{S}_2} \oplus H^5(\widetilde{A \times A}, \mathbb{Z})^{\mathfrak{S}_2}) = 2^{2\left(l_2^3(\widetilde{A \times A}) + l_2^5(\widetilde{A \times A})\right) + l_{1,+}^3(\widetilde{A \times A}) + l_{1,+}^5(\widetilde{A \times A})}.$$

Then by Lemma 2.17 and 2.3 of [1],

$$\operatorname{discr} \pi_*(H^3(\widetilde{A \times A}, \mathbb{Z}) \oplus H^5(\widetilde{A \times A}, \mathbb{Z})) = 2^{l_{1,+}^3(\widetilde{A \times A}) + l_{1,+}^5(\widetilde{A \times A})} = 2^8.$$

□

The lattice $H^3(A^{[2]}, \mathbb{Z}) \oplus H^5(A^{[2]}, \mathbb{Z})$ is unimodular. Hence:

$$\frac{H^3(A^{[2]}, \mathbb{Z}) \oplus H^5(A^{[2]}, \mathbb{Z})}{\pi_*(H^3(\widetilde{A \times A}, \mathbb{Z}) \oplus H^5(\widetilde{A \times A}, \mathbb{Z}))} = (\mathbb{Z}/2\mathbb{Z})^4.$$

However, by Section 2, we know that $\pi_*(H^3(\widetilde{A \times A}, \mathbb{Z})) = H^3(A^{[2]}, \mathbb{Z})$. It follows

$$\frac{H^5(A^{[2]}, \mathbb{Z})}{\pi_*(H^5(\widetilde{A \times A}, \mathbb{Z}))} = (\mathbb{Z}/2\mathbb{Z})^4.$$

References

[1] G. Menet *On the integer cohomology of quotients of Kähler manifolds*, arXiv:1312.1584v3[math.AG] 4 Mar 2015.

[2] C. Voisin, *Hodge Theory and Complex Algebraic Geometry. I,II,* Cambridge Stud. Adv. Math., 76, 77, Cambridge Univ. Press, 2003.