PLANES IN SYMPLECTIC VECTOR SPACES

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1. Symplectic linear algebra

Let V be a symplectic vector space of dimension $n \in 2\mathbb{N}$ over a field F with a nondegenerate symplectic form $\omega : \Lambda^2 V \to F$. A line is a one-dimensional subspace ov V, a plane is a two-dimensional subspace of V. A plane $P \subset V$ is called isotropic, if $\omega(x, y) = 0$ for any $x, y \in P$, otherwise non-isotropic. The symplectic group $\mathrm{Sp}\, V$ is the set of all linear maps $\phi : V \to V$ with the property $\omega(\phi(x), \phi(y)) = \omega(x, y)$ for all $x, y \in V$.

Proposition 1.1. *The symplectic group $\mathrm{Sp}\, V$ acts transitively on the set of non-isotropic planes as well as on the set of isotropic planes.*

Proof. Given two planes P_1 and P_2, we may choose vectors v_1, v_2, w_1, w_2 such that v_1, v_2 span P_1, w_1, w_2 span P_2 and $\omega(u_1, u_2) = \omega(w_1, w_2)$. We complete $\{v_1, v_2\}$ as well as $\{w_1, w_2\}$ to a symplectic basis of V. Then define $\phi(v_1) = w_1$ and $\phi(v_2) = w_2$. It is now easy to see that the definition of ϕ can be extended to the remaining basis elements to give a symplectic morphism. $\qquad\square$

Remark 1.2. The set of planes in V can be identified with the simple tensors in $\Lambda^2 V$ up to multiples. Indeed, given a simple tensor $v \wedge w \in \Lambda^2 V$, the span of v and w yields the corresponding plane. Conversely, any two spanning vectors v and w of a plane give the same element $v \wedge w$ (up to multiples).

Proposition 1.3. *If $\phi \in \mathrm{Sp}\, V$ acts through multiplication of a scalar, $\phi(v) = \alpha v$, then $\alpha = \pm 1$ (this is immediate from the definition). Moreover, if $\phi(v) \wedge \phi(w) = \alpha v \wedge w$, then $\alpha = 1$.*

Proof. We may assume that V is two-dimensional, generated by v and w. Our condition on ϕ reads then $\det \phi = \alpha$. But the condition on ϕ being symplectic is $\det \phi = 1$, because on a two-dimensional vector space there is only one symplectic form up to scalar multiple. $\qquad\square$

Remark 1.4. If F is the field with two elements, then the set of planes in V can be identified with the set $\{\{x, y, z\} \mid x, y, z \in V \backslash \{0\}, x + y + z = 0\}$. Observe that for such a $\{x, y, z\}$, $\omega(x, y) = \omega(x, y) = \omega(y, x)$ and this value gives the criterion for isotropy.

Proposition 1.5. *Assume that F is finite of cardinality q.*

(1)   *The number of lines in V is $\dfrac{q^n - 1}{q - 1}$,*

(2)   *the number of planes in V is $\dfrac{(q^n - 1)(q^{n-1} - 1)}{(q^2 - 1)(q - 1)}$,*

(3)   *the number of isotropic planes in V is $\dfrac{(q^n - 1)(q^{n-2} - 1)}{(q^2 - 1)(q - 1)}$,*

(4)   *the number of non-isotropic planes in V is $\dfrac{q^{n-2}(q^n - 1)}{q^2 - 1}$.*

Proof. A line ℓ in V is determined by a nonzero vector. There are $q^n - 1$ nonzero vectors in V and $q - 1$ nonzero vectors in ℓ. A plane P is determined by a line $\ell_1 \subset V$ and a unique second line $\ell_2 \in V/\ell_1$. We have $\frac{q^2-1}{q-1}$ lines in P. The number of planes is therefore

$$\frac{\frac{q^n - 1}{q - 1} \cdot \frac{q^{n-1} - 1}{q - 1}}{\frac{q^2 - 1}{q - 1}} = \frac{(q^n - 1)(q^{n-1} - 1)}{(q^2 - 1)(q - 1)}.$$

For an isotropic plane we have to choose the second line from $\ell_1^{\perp}/\ell_1$. This is a space of dimension $n - 2$, hence the formula. The number of non-isotropic planes is the difference of the two previous numbers.   □

Conjecture 1.6. *There are $6q$ orbits of the induced action of $\mathrm{Sp}(4, q)$ on $\Lambda^2 \mathbb{F}_q^4$.*

2. Symplectic vector spaces as index sets

Assume now that V is a four-dimensional vector space over $F = \mathbb{F}_q$. Consider the free F-module $F[V]$ with basis $\{X_i \mid i \in V\}$. It carries a natural F-algebra structure, given by $X_i \cdot X_j := X_{i+j}$ with unit $1 = X_0$. Let $\mathfrak{m}$ be the ideal generated by all elements of the form $(X_i - 1)$. Since $F[V]/\mathfrak{m} = F$, it is a maximal ideal.

Conjecture 2.1. *We define*

$$L := \left\{ \sum_{i \in \ell} X_i =: S_\ell \mid \ell \subset V \ line \right\}$$

Then the ideal generated by L has dimension $q^n - \binom{q+n-2}{n}$.

We introduce an action of $\mathrm{Sp}(4, F)$ on $F[V]$ by setting $\phi(X_i) = X_{\phi(i)}$. Furthermore, the underlying additive group of V acts on $F[V]$ by $v(X_i) = X_{i+v} = X_i X_v$.

Definition 2.2. We define subsets of $F[V]$:

$$B_N := \left\{ \sum_{i \in P} X_i \mid P \subset V \ \text{non-isotropic plane} \right\},$$

$$B_I := \left\{ \sum_{i \in P} X_i \mid P \subset V \ \text{isotropic plane} \right\}.$$

Denote by $\langle B_\alpha \rangle$ and by (B_α) the linear span of B_α and the ideal generated by B_α, respectively. Note that (B_α) is the linear span of $\{v \cdot b \mid b \in B, v \in V\}$. Further, let D_α be the linear span of $\{v(b) - b \mid b \in B, v \in V\}$. Then D_α is in fact an ideal, namely the product of ideals $\mathfrak{m} \cdot (B_\alpha)$.

The following table illustrates the dimensions of these objects:

F	$\dim_F \langle B_N \rangle$	$\dim_F(B_N)$	$\dim_F D_N$	$\dim_F \langle B_I \rangle$	$\dim_F(B_I)$	$\dim_F D_I$
$\mathbb{F}_2$	10	11	5	10	10	10
$\mathbb{F}_3$	30	50	31	?	45	25
$\mathbb{F}_5$	121	355	270	?	?	91

Example 2.3. We give the 31 classes spanning D_N over $\mathbb{F}_3$: First, define for a plane $P \subset V$

$$S_P := \sum_{\tau \in P} X_\tau.$$

Then D_N is spanned by the classes

$$S_{P+\tau} - S_P$$

(5)   for $P = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$ and $0 \neq \tau \in P^\perp = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

(6)   for $P = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$ and $0 \neq \tau \in P^\perp = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \setminus \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

(7)   for $P = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\rangle$ and $\tau \in \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} \right\}$

(8)   for $P = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$ and $\tau \in \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$

(9)   for $P = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$ and $\tau \in \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(10)   for $P = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$ and $\tau \in \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \end{pmatrix} \right\}$

(11)   for $P = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$ and $\tau \in \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \right\}$

(12)   for $P = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} \right\rangle$ and $\tau = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

(13)   for $P = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix} \right\rangle$ and $\tau = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$

So we get $8 + 7 + 4 + 4 + 2 + 2 + 2 + 1 + 1 = 31$ linear independent classes spanning D_N.

Conjecture 2.4. *For $F = \mathbb{F}_q$, $\dim_F D_I = \frac{(q+2)(q^2+1)}{2}$.*

3. ORTHOGONAL SUMS

Set $S := \mathrm{Sym}^2(\Lambda^2 V)$. Take two vectors $v, w \in V$ with $\omega(v, w) = 1$ and set $x := (v \wedge w)^2 \in S$. Denote P the plane spanned by v and w and set $y := \sum_{i \in P} X_i \in F[V]$. We set $Y' := y \cdot \mathfrak{m} = \{ \sum_{i \in P} X_{i+j} - X_i \mid j \in V \}$.

We consider now the action of $\mathrm{Sp}\, V$ on $S \oplus F[V]$.

Proposition 3.1. *The elements $\phi(x) \oplus \phi(z)$, for $\phi \in \mathrm{Sp}\, V$, $z \in (y)$ span a vector space of dimension*

- *11, if $F = \mathbb{F}_2$,*
- *51, if $F = \mathbb{F}_3$,*

- *375, if $F = \mathbb{F}_5$.*

Proposition 3.2. *The elements $\phi(x) \oplus \phi(y')$, for $\phi \in \mathrm{Sp}\, V$, $y' \in Y'$ span a vector space of dimension*

- *10, if $F = \mathbb{F}_2$,*
- *50, if $F = \mathbb{F}_3$,*
- *289, if $F = \mathbb{F}_5$.*

Remark 3.3. If $\omega(v, w) = 0$, we would have the dimensions 10, 25, 105 instead.