

INTEGRAL COHOMOLOGY OF $K_2(A)$

SIMON KAPFER AND GRÉGOIRE MENET

ABSTRACT. What we know already

1. PRELIMINARIES

Definition 1.1. Let n be a natural number. A partition of n is a decreasing sequence $\lambda = (\lambda_1, \dots, \lambda_k)$, $\lambda_1 \geq \dots \geq \lambda_k > 0$ of natural numbers such that $\sum_i \lambda_i = n$. Sometimes it is convenient to write $\lambda = (\dots, 2^{m_2}, 1^{m_1})$ with multiplicities in the exponent. We define the weight $\|\lambda\| := \sum m_i i = n$ and the length $|\lambda| := \sum_i m_i = k$. We also define $z_\lambda := \prod_i i^{m_i} m_i!$.

Definition 1.2. Let $\Lambda_n := \mathbb{Q}[x_1, \dots, x_n]^{S_n}$ be the graded ring of symmetric polynomials. There are canonical projections $\Lambda_{n+1} \rightarrow \Lambda_n$ which send x_{n+1} to zero. The graded projective limit $\Lambda := \varprojlim \Lambda_n$ is called the ring of symmetric functions. Let m_λ and p_λ denote the monomial and the power sum symmetric functions. They are defined as follows: For a monomial $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \dots x_{i_k}^{\lambda_k}$ of total degree n , the (ordered) sequence of exponents $(\lambda_1, \dots, \lambda_k)$ defines a partition λ of n , which is called the shape of the monomial. Then we define m_λ being the sum of all monomials of shape λ . For the power sums, first define $p_n := x_1^n + x_2^n + \dots$. Then $p_\lambda := p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_k}$. The families $(m_\lambda)_\lambda$ and $(p_\lambda)_\lambda$ form two \mathbb{Q} -bases of Λ , so they are linearly related by $p_\lambda = \sum_\mu \psi_{\lambda\mu} m_\mu$. It turns out that the base change matrix $(\psi_{\lambda\mu})$ has integral entries, but its inverse $(\psi_{\lambda\mu}^{-1})$ has not.

Definition 1.3. We define the double factorial for $n \geq -1$ by

$$n!! := \prod_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (n-2i) = n(n-2)(n-4) \dots$$

Clearly, $(n-1)!! n!! = n!$ and $(2n)!! = 2^n n!$.

Proposition 1.4. The number of partitions of the set $\{1, \dots, 2k\}$ into pairs equals $(2k-1)!! = \frac{(2k)!}{2^k k!}$.

Proof. Given such a partition, look at the pair that contains the element 1. There are $2k-1$ possible partners for this element; removing the pair leaves a partition of a set of cardinality $(2k-2)$ into pairs. Then proceed by induction. \square

2. HILBERT SCHEMES OF POINTS ON SURFACES

Let A be a smooth projective complex surface. Let $A^{[n]}$ the Hilbert scheme of n points on the surface, *i.e.* the moduli space of finite subschemes of A of length n . $A^{[n]}$ is again smooth and projective of dimension $2n$.

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Their rational cohomology can be described in terms of Nakajima's operators. First consider the direct sum

$$\mathbb{H} := \bigoplus_{n=0}^{\infty} H^*(A^{[n]}, \mathbb{Q})$$

This space is bigraded by cohomological *degree* and the *weight*, which is given by the number of points n . The unit element in $H^0(A^{[0]}, \mathbb{Q}) \cong \mathbb{Q}$ is denoted by $|0\rangle$, called the *vacuum*. There are linear operators $\mathfrak{q}_m(a)$, for each $m \geq 1$ and $a \in H^*(A, \mathbb{Q})$, acting on \mathbb{H} which have the following properties: They depend linearly on a , and if $a \in H^k(A, \mathbb{Q})$ is homogeneous, the operator $\mathfrak{q}_m(a)$ is bihomogeneous of degree $k + 2(|m| - 1)$ and weight m :

$$\mathfrak{q}_m(a) : H^l(A^{[n]}) \rightarrow H^{l+k+2(|m|-1)}(A^{[n+m]})$$

To construct them, first define incidence varieties $Z_m \subset A^{[n]} \times A \times A^{[n+m]}$ by

$$Z_m := \{(\xi, x, \xi') \mid \xi \subset \xi', \text{supp}(\xi') - \text{supp}(\xi) = mx\}.$$

Then $\mathfrak{q}_m(a)(\beta)$ is defined as the Poincaré dual of

$$pr_{3*}((pr_2^*(\alpha) \cdot pr_3^*(\beta)) \cap [Z_m]).$$

Every element in \mathbb{H} can be decomposed uniquely as a linear combination of products of operators $\mathfrak{q}_m(a)$, acting on the vacuum.

To give the cup product structure of \mathbb{H} , define operators $\mathfrak{G}(a)$ for $a \in H^*(A)$. Let $\Xi_n \subset A^{[n]} \times A$ be the universal subscheme. Then the action of $\mathfrak{G}(a)$ on $H^*(A^{[n]})$ is multiplication with the class

$$pr_{1*}(\text{ch}(\mathcal{O}_{\Xi_n}) \cdot pr_2^*(\text{td}(A) \cdot a)) \in H^*(A^{[n]}).$$

For $a \in H^k(A)$, we define $\mathfrak{G}_i(a)$ to be the component of $\mathfrak{G}(a)$ of cohomological degree $k + 2i$. A differential operator \mathfrak{d} is given by $\mathfrak{G}_1(1)$. It means multiplication with the first Chern class of the tautological sheaf $pr_{1*}(\mathcal{O}_{\Xi_n})$.

In [6] and [7] there are various commutation relations between these operators, that allow to determine all multiplications in the cohomology of the Hilbert scheme. First of all, if X and Y are operators of degree d and d' , their commutator is defined as

$$[X, Y] := XY - (-1)^{dd'} YX.$$

The integral on $A^{[n]}$ induces a bilinear form on \mathbb{H} : for classes $\alpha, \beta \in H^*(A^{[n]})$ it is given by

$$(\alpha, \beta) \mapsto \int_{A^{[n]}} \alpha \cdot \beta.$$

If X is a homogeneous linear operator of degree d and weight m , acting on \mathbb{H} , define its adjoint X^\dagger by

$$\int_{A^{[n+m]}} X(\alpha) \cdot \beta = (-1)^{d \deg(\alpha)} \int_{A^{[n]}} \alpha \cdot X^\dagger(\beta).$$

We put $\mathfrak{q}_0(a) := 0$ and for $m < 0$, $\mathfrak{q}_m(a) := (-1)^n \mathfrak{q}_{-m}(a)^\dagger$. Then define

$$\mathfrak{L}_m(a) := \begin{cases} \frac{1}{2} \sum_{k \in \mathbb{Z}} \sum_i \mathfrak{q}_k(a_{(1)}) \mathfrak{q}_{m-k}(a_{(2)}), & \text{if } m \neq 0, \\ \sum_{k > 0} \sum_i \mathfrak{q}_k(a_{(1)}) \mathfrak{q}_{-k}(a_{(2)}), & \text{if } m = 0. \end{cases}$$

where $\sum_i a_{(1)} \otimes a_{(2)}$ is the push-forward of a along the diagonal $: A \rightarrow A \times A$ (in Sweedler notation). Then we have ([7, Thm. 2.16]):

$$\begin{aligned}
 (1) \quad & [\mathfrak{q}_m(a), \mathfrak{q}_l(b)] = m \cdot \delta_{m+l} \cdot \int_A ab \\
 (2) \quad & [\mathfrak{L}_m(a), \mathfrak{q}_l(b)] = -m \cdot \mathfrak{q}_{m+l}(ab) \\
 (3) \quad & [\mathfrak{d}, \mathfrak{q}_m(a)] = m \cdot \mathfrak{L}_m(a) + \frac{m(|m|-1)}{2} \mathfrak{q}_m(Ka) \\
 (4) \quad & [\mathfrak{G}(a), \mathfrak{q}_1(b)] = \exp(\text{ad}(\mathfrak{d}))(\mathfrak{q}_1(ab))
 \end{aligned}$$

Note (cf. [6, Thm. 3.8]) that this implies that

$$(5) \quad \mathfrak{q}_{m+1}(a) = \frac{(-1)^m}{m!} \text{ad}^m([\mathfrak{d}, \mathfrak{q}_1(1)])(\mathfrak{q}_1(a)),$$

so there are two ways of writing an element of \mathbb{H} : As a linear combination of products of creation operators $\mathfrak{q}_m(a)$ or as a linear combination of products of the operators \mathfrak{d} and $\mathfrak{q}_1(a)$. While the first one is more intuitive, the second one is more suitable for computing cup-products. Equations (3) and (5) permit now to switch between the two representations, using that

$$\begin{aligned}
 (6) \quad & \mathfrak{d}|0\rangle = 0, \\
 (7) \quad & \mathfrak{L}_m(a)|0\rangle = \begin{cases} \frac{1}{2} \sum_{k=1}^{m-1} \sum_i \mathfrak{q}_k(b_i) \mathfrak{q}_{m-k}(c_i)|0\rangle, & \text{if } m > 1, \\ 0, & \text{if } m \leq 1. \end{cases}
 \end{aligned}$$

(8)

Remark 2.1. We adopted the notation from [7], which differs from the conventions in [6]. Here is part of a dictionary:

Notation from [7]	Notation from [6]
operator of bidegree (w, d)	operator of bidegree $(w, d - 2w)$
$\mathfrak{q}_m(a)$	$\mathfrak{p}_{-m}(a)$
$\mathfrak{L}_m(a)$	$-L_{-m}(a)$
$\mathfrak{G}(a)$	$a^{[\bullet]}$
\mathfrak{d}	∂
$\tau_{2*}(a)$	$-\Delta(a)$

By sending a subscheme in A to its support, we define a morphism

$$\rho : A^{[n]} \longrightarrow \text{Sym}^n(A),$$

called the Hilbert–Chow morphism. The cohomology of $\text{Sym}^n(A)$ is given by elements of the n -fold tensor power of $H^*(A)$ that are invariant under the action of the group of permutations \mathfrak{S}_n . A class in $H^*(A^{[n]}, \mathbb{Q})$ which can be written using only the operators \mathfrak{q}_1 comes from a pullback along ρ :

$$(9) \quad \mathfrak{q}_1(b_1) \cdots \mathfrak{q}_1(b_n)|0\rangle = \rho^* \left(\sum_{\pi \in \mathfrak{S}_n} b_{\pi(1)} \otimes \cdots \otimes b_{\pi(n)} \right), \quad b_i \in H^*(A, \mathbb{Q}).$$

In particular,

$$(10) \quad \frac{1}{n!} \mathfrak{q}_1(b)^n |0\rangle = \rho^*(b \otimes \cdots \otimes b),$$

$$(11) \quad \frac{1}{(n-1)!} \mathfrak{q}_1(b) (\mathfrak{q}_1(1))^{n-1} |0\rangle = \rho^*(b \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes b).$$

This is sometimes useful for manually computing products:

Proposition 2.2. *Let $b \in H^2(A, \mathbb{Q})$ be of degree 2 and set $B_n := \frac{1}{(n-1)!} \mathfrak{q}_1(b) (\mathfrak{q}_1(1))^{n-1} |0\rangle$, $Q_n := B_n^{2n}$, $P_n := B_n^{2n-2}$. Then:*

$$Q_n = (2n-1)!! \cdot \mathfrak{q}_1(b^2)^n |0\rangle, \quad P_n = \mathfrak{q}_1(1) Q_{n-1} + n \cdot \mathfrak{q}_1(b)^2 Q_{n-2}.$$

Proof. First note that, by (11),

$$B_n^k = \rho^* \left(\sum_{i_1 + \dots + i_n = k} b^{i_1} \otimes \dots \otimes b^{i_n} \right).$$

It is clear by degree reasons that Q_n must be a multiple of $\rho^*(b^2 \otimes \dots \otimes b^2)$. Using Prop. 1.4 one deduces the formula for Q_n . The formula for P_n follows in a similar manner. \square

3. COHOMOLOGY OF HILBERT SCHEMES OF POINTS ON A TORUS SURFACE

For the study of integral cohomology, first note that if $\alpha \in H^*(A, \mathbb{Z})$ is an integral class, then $\mathfrak{p}_{-m}(\alpha)$ maps integral classes to integral classes. Moreover, there is the following theorem:

Theorem 3.1. [11] *An operator is called integral if it maps integral classes in \mathbb{H} to integral classes. The operator $\frac{1}{z_\lambda} \mathfrak{q}_\lambda(1)$ is integral. Let $\alpha \in H^2(A, \mathbb{Z})$ be monodromy equivalent to a divisor. Then the operator $\mathfrak{m}_\lambda(\alpha)$ is integral.*

Remark 3.2. If A is a projective torus, then the sublattice of divisors in $H^2(A, \mathbb{Z})$ contains at least ... By Scattone, etc.

To obtain the multiplicative structure of \mathbb{H} , given by the cup-products, there is a description in [6] and [7] in terms of multiplication operators $\mathfrak{G}_k(a)$, $a \in H^*(A)$ [7, Def. 5.1], related to chern characters. There is the following commutation relation:

$$[\mathfrak{G}_k(a), \mathfrak{q}_1(b)] = \frac{1}{k!} \text{ad}(\mathfrak{d})^k(\mathfrak{q}_1(ab)),$$

where the operator \mathfrak{d} means multiplication with the first Chern class of the tautological sheaf. We set $a^{(k)} := \mathfrak{G}_k(a)(1) \in H^{\deg a + 2k}(A^{[n]}, \mathbb{Q})$.

Next we focus on the structure of $H^2(A^{[n]}, \mathbb{Z})$ for $n \geq 2$. It has rank 13, and there is a basis consisting of:

- $\frac{1}{(n-1)!} \mathfrak{p}_{-1}(1)^{n-1} \mathfrak{p}_{-1}(a_{ij}) |0\rangle$, $1 \leq i < j \leq 4$,
- $\frac{1}{(n-2)!} \mathfrak{p}_{-1}(1)^{n-2} \mathfrak{p}_{-1}(a_i) \mathfrak{p}_{-1}(a_j) |0\rangle$, $1 \leq i < j \leq 4$,
- $\frac{1}{2(n-2)!} \mathfrak{p}_{-1}(1)^{n-2} \mathfrak{p}_{-2}(1) |0\rangle$. We denote this class by δ .

It is clear that these classes form a basis of $H^2(A^{[n]}, \mathbb{Q})$. By [11, Thm. 4.6, Lemma 5.2], they also form a basis for $H^2(A^{[n]}, \mathbb{Z})$. TODO: refine this argument

The first 6 classes give an injection $j : H^2(A, \mathbb{Z}) \rightarrow H^2(A^{[n]}, \mathbb{Z})$.

4. GENERALIZED KUMMER VARIETIES

Let A be a complex projective torus of dimension 2. Its first cohomology $H^1(A, \mathbb{Z})$ is freely generated by four elements a_1, a_2, a_3, a_4 , corresponding to the four different circles on the torus. The cohomology ring is isomorphic to the exterior algebra:

$$H^*(A, \mathbb{Z}) = \Lambda^* H^1(A, \mathbb{Z}).$$

We abbreviate for the products $a_i \cdot a_j =: a_{ij}$ and $a_i \cdot a_j \cdot a_k =: a_{ijk}$. We write $a_1 \cdot a_2 \cdot a_3 \cdot a_4 =: x$ for the class corresponding to a point on A . We choose the a_i such that $\int_A x = 1$. We set $a_{\bar{i}}$ for the dual class of a_i , i.e. $a_i \cdot a_{\bar{i}} = x$. The bilinear form, given by $(a_{ij}, a_{kl}) \mapsto \int_A a_{ij} a_{kl}$ gives $H^2(A, \mathbb{Z})$ the structure of a unimodular lattice, isomorphic to $U^{\oplus 3}$, three copies of the hyperbolic lattice.

Definition 4.1. Let A be a complex projective torus of dimension 2 and $A^{[n]}$, $n \geq 1$, the corresponding Hilbert scheme of points. Denote $\Sigma : A^{[n]} \rightarrow A$ the summation morphism, a smooth submersion that factorizes via the Hilbert–Chow morphism $A^{[n]} \xrightarrow{\rho} \text{Sym}^n(A) \xrightarrow{\sigma} A$. Then the generalized Kummer $K_{n-1}(A)$ is defined as the fiber over 0:

$$(12) \quad \begin{array}{ccc} K_{n-1}(A) & \xrightarrow{\theta} & A^{[n]} \\ \downarrow & & \downarrow \Sigma \\ \{0\} & \longrightarrow & A \end{array}$$

Our first objective is to collect some information about this pullback diagram.

Proposition 4.2. Recall that $a_i^{(0)} = \frac{1}{(n-1)!} \mathbf{q}_1(1)^{n-1} \mathbf{q}_1(a_i) |0\rangle$. The class of the Poincaré dual of $K_{n-1}(A)$ in $H^4(A^{[n]}, \mathbb{Z})$ is given by

$$a_1^{(0)} \cdot a_2^{(0)} \cdot a_3^{(0)} \cdot a_4^{(0)}.$$

Proof. Since the generalized Kummer variety is the fiber over a point, its Poincaré dual must be the pullback of $x \in H^4(A)$ under Σ . But $\Sigma^*(x) = \Sigma^*(a_1) \cdot \Sigma^*(a_2) \cdot \Sigma^*(a_3) \cdot \Sigma^*(a_4)$, so we have to verify that $\Sigma^*(a_i) = a_i^{(0)}$. To do this, we want to use the decomposition $\Sigma = \sigma\rho$. The pullback along σ of a class $a \in H^1(A, \mathbb{Q})$ on $H^1(\text{Sym}^n(A), \mathbb{Q}) \cong H^*(A^n, \mathbb{Q})^{\oplus n}$ is given by $a \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes a$. It follows from (11) that $\Sigma^*(a_i) = \frac{1}{(n-1)!} \mathbf{q}_1(1)^{n-1} \mathbf{q}_1(a_i) |0\rangle$. \square

Remark 4.3. This allows us to better understand the morphism θ^* . Since the Poincaré pairing is nondegenerated, $[K_{n-1}(A)] \cdot \alpha = 0$ implies $\theta^* \alpha = 0$.

We recall Theorem 2 of [12].

Theorem 4.4. The cohomology of $K_{n-1}(A)$ is torsion free.

Theorem 4.5. [1] $K_{n-1}(A)$ is a irreducible holomorphically symplectic manifold. In particular, it is simply connected and the canonical bundle is trivial.

So $H^2(K_{n-1}(A), \mathbb{Z})$ admits an integer-valued nondegenerated quadratic form (called Beauville–Bogomolov form) q which gives $H^2(K_{n-1}(A), \mathbb{Z})$ the structure of a lattice isomorphic to $U^{\oplus 3} \oplus \langle -2n \rangle$, for $n \geq 3$. We have the following formula for $\alpha \in H^2(K_{n-1}(A), \mathbb{Z})$:

$$(13) \quad \int_{K_{n-1}(A)} \alpha^{2n-2} = n \cdot (2n-3)!! \cdot q(\alpha)^{n-1}$$

The morphism θ induces a homomorphism of graded rings

$$(14) \quad \theta^* : H^*(A^{[n]}, \mathbb{Z}) \longrightarrow H^*(K_{n-1}(A), \mathbb{Z}).$$

Proposition 4.6. Assume $n \geq 3$. Then θ^* is surjective on $H^2(A^{[n]}, \mathbb{Z})$ with kernel $\Lambda^2 H^1(A^{[n]}, \mathbb{Z})$.

Proof. By [1, Sect. 7], $\theta^* : H^2(A^{[n]}, \mathbb{C}) \rightarrow H^2(K_{n-1}(A), \mathbb{C})$ is surjective. Because of $H^1(K_{n-1}(A), \mathbb{Z}) = 0$, it is clear that $\Lambda^2 H^1(A^{[n]}, \mathbb{Z})$ is contained in the kernel. This is a free \mathbb{Z} -module of rank 6. But the second Betti numbers of $A^{[n]}$ and $K_{n-1}(A)$ are 13 and 7, respectively, so it must be all.

It remains to show that θ^* is surjective for cohomology with integral coefficients, too. The idea is to prove that the lattice structure of $\text{Im } \theta^*$ is the same as of $H^2(K_{n-1}(A))$. We use two formulas in [4, pp. 8–11]. Let $b \in H^2(A, \mathbb{Z})$ and set $\alpha = \frac{1}{(n-1)!} \mathbf{q}_1(1)^{n-1} \mathbf{q}_1(b)|0\rangle \in H^2(A^{[n]}, \mathbb{Z})$. Then

$$(15) \quad \int_{A^{[n]}} \alpha^{2n} = \binom{2n}{2} \frac{\int_A b^2}{n^2} \int_{K_{n-1}(A)} \theta^* \alpha^{2n-2}$$

By Proposition 2.2, the left hand side of this equation equals $(2n-1)!! \cdot \left(\int_A b^2\right)^n$. By (13), the right hand side gives $(2n-1)!! \cdot \left(\int_A b^2\right) \cdot q(\alpha)^{n-1}$. So we get $\int_A b^2 = q(\alpha)$, giving the set of all α a lattice structure isomorphic to $H^2(A, \mathbb{Z})$. Secondly, we must show that for $\delta =$ half of the exceptional divisor: $q(\theta^* \delta) = -2n$. But this follows now from Proposition 1 in [4]. \square

5. STUDY OF $K_2(A)$

Let us summarize our results on θ^* for the case $n = 3$:

Theorem 5.1. *The homomorphism $\theta^* : H^*(A^{[3]}, \mathbb{Q}) \rightarrow H^*(K_2(A), \mathbb{Q})$ of graded rings is surjective in every degree except 4. Moreover, the image of $H^4(A^{[3]}, \mathbb{Q})$ is equal to $\text{Sym}^2(H^2(K_2(A), \mathbb{Q}))$. The kernel of θ^* is the ideal generated by $H^1(A^{[3]}, \mathbb{Q})$.*

Now we focus on classes of cohomological degree 4.

Proposition 5.2. *The classes $\theta^*(\mathbf{p}_{-2}(a_{ij})\mathbf{p}_{-1}(1)|0\rangle)$ and $\theta^*(\mathbf{p}_{-2}(1)\mathbf{p}_{-1}(a_{ij})|0\rangle)$ are linearly dependent.*

Proof. We can compute the product of these two classes with $[K_2(A)]$ in $H^*(A^{[3]})$. The two results are linearly dependent. Is there a direct geometric proof? \square

Proposition 5.3. $\theta^*(\mathbf{p}_{-3}(x)|0\rangle) = 0$

Corollary 5.4. $\theta^*(\mathbf{p}_{-2}(a_{ij})\mathbf{p}_{-1}(1)|0\rangle) = \frac{1}{4}\theta^*(\mathbf{p}_{-2}(1)\mathbf{p}_{-1}(a_{ij})|0\rangle)$

Proof. Let a_{ij} and a_{kl} be complementary, i.e. $a_{ij}a_{kl} = 1$. We have $a_{kl}^{(1)} = -\frac{1}{2}\mathbf{p}_{-2}(a_{kl})\mathbf{p}_{-1}(1)|0\rangle$. Then:

$$\theta^*\left(a_{ij}^{(1)} \cdot a_{kl}^{(1)}\right) = \theta^*\left(\mathbf{p}_{-3}(1)|0\rangle + \frac{1}{2}\mathbf{p}_{-1}(x)^2\mathbf{p}_{-1}(1)|0\rangle\right)$$

But on the other hand, $\delta \cdot j(a) = \frac{1}{2}\mathbf{p}_{-2}(1)\mathbf{p}_{-1}(a_{ij})|0\rangle + \mathbf{p}_{-2}(a_{ij})\mathbf{p}_{-1}(1)|0\rangle$, and

$$\theta^*\left(a_{kl}^{(1)} \cdot \delta \cdot j(a)\right) = \theta^*\left(-3\mathbf{p}_{-3}(1)|0\rangle - 3\mathbf{p}_{-1}(x)^2\mathbf{p}_{-1}(1)|0\rangle\right).$$

\square

Corollary 5.5. $\theta^*(\delta \cdot j(a_{ij})) = \theta^*\left(\frac{3}{4}\mathbf{p}_{-2}(1)\mathbf{p}_{-1}(a_{ij})|0\rangle\right)$ is divisible by 3. \square

Proposition 5.6. *The classes $\theta^*(j(a_{ij})^2 - \frac{1}{3}j(a_{ij}) \cdot \delta)$ are divisible by 2.*

Proof. By [11], the classes $\frac{1}{2}\mathbf{p}_{-1}(a_{ij})^2\mathbf{p}_{-1}(1)|0\rangle - \frac{1}{2}\mathbf{p}_{-2}(a_{ij})\mathbf{p}_{-1}(1)|0\rangle$ are integral in $H^4(A^{[n]})$. But $j(a_{ij})^2 = \mathbf{p}_{-1}(a_{ij})^2\mathbf{p}_{-1}(1)|0\rangle$ and $\theta^*\left(\frac{1}{3}j(a_{ij}) \cdot \delta\right) = \theta^*(\mathbf{p}_{-2}(a_{ij})\mathbf{p}_{-1}(1)|0\rangle)$. \square

Proposition 5.7. *The class $\theta^*(\delta^2 + j(a_{12}) \cdot j(a_{34}) - j(a_{13}) \cdot j(a_{24}) + j(a_{14}) \cdot j(a_{23}))$ is divisible by 3.*

Proof. It is equal to $\theta^*(\mathfrak{p}_{-3}(1)|0\rangle - \frac{3}{2}\mathfrak{p}_{-1}(x)\mathfrak{p}_{-1}(1)^2|0\rangle)$. \square

Proposition 5.8. *We have:*

$$H^4(K_2(A), \mathbb{Q}) = \text{Sym}^2 H^2(K_2(A), \mathbb{Q}) \oplus^\perp \Pi' \otimes \mathbb{Q}.$$

Proof. In Section 4 of [5], we can find the following formula:

$$(16) \quad Z_\tau \cdot D_1 \cdot D_2 = 2 \cdot q(D_1, D_2),$$

for all D_1, D_2 in $H^2(K_2(A), \mathbb{Z})$, $\tau \in A[3]$ and q the Beauville-Bogomolov form. It follows that $\Pi' \subset \text{Sym}^2 H^2(K_2(A), \mathbb{Z})^\perp$. Since the cup-product is non-degenerated and by Proposition 4.3 of [5], we have:

$$\begin{aligned} \text{rk}(\text{Sym}^2 H^2(K_2(A), \mathbb{Z}) \oplus \Pi') &= \text{rk} \text{Sym}^2 H^2(K_2(A), \mathbb{Z}) + \text{rk} \Pi' \\ &= 28 + 80 \\ &= \text{rk} H^4(K_2(A), \mathbb{Z}). \end{aligned}$$

It follows that

$$H^4(K_2(A), \mathbb{Q}) = \text{Sym}^2 H^2(K_2(A), \mathbb{Q}) \oplus^\perp \Pi' \otimes \mathbb{Q}.$$

\square

Next we look at the Chern classes of the tangent sheaves. Since the morphism Σ from the defining pullback diagram (12) is a submersion, the normal bundle of $K_{n-1}(A)$ in $A^{[n]}$ is trivial. Hence $c(K_2(A)) = \theta^*c(A^{[3]})$. Looking in [2, Sect. 8], we find a general formula for Chern classes of Hilbert schemes of points on surfaces. So we deduce

$$\begin{aligned} c_2(A^{[3]}) &= \left(\frac{3}{2}\mathfrak{q}_{*(1,1)}(1)\mathfrak{q}_1(1) - \frac{1}{3}\mathfrak{q}_3\right)|0\rangle \\ &= 10(1_{(4)}^{[\bullet]}) - 2(1_{(2)}^{[\bullet]})^2 \\ c_4(A^{[3]}) &= \frac{4}{3}\mathfrak{q}_{*(1,1,1)}(1)|0\rangle = 4(1_{(4)}^{[\bullet]})^2. \end{aligned}$$

Proposition 5.9. *We have:*

$$c_2(K_2(A)) = \theta^* \left(4j(a_{12}) \cdot j(a_{34}) - 4j(a_{13}) \cdot j(a_{24}) + 4j(a_{14}) \cdot j(a_{23}) - \frac{1}{3}\delta^2 \right).$$

In particular $c_2(K_2(A)) \in \text{Sym}^2 H^2(K_2(A), \mathbb{Z})$.

Proof. We can write:

$$c_2(K_2(A)) = a + b,$$

with $a \in \text{Sym}^2 H^2(K_2(A), \mathbb{Q})$ and $b \in \Pi'$. First, we prove that $b = 0$. We have $c_2(K_2(A)) \in \Pi'^\perp$ and also $a \in \Pi'^\perp$, it follows that $b \in \Pi'^\perp$. Since the cup-product is non-degenerated, it follows that b is of torsion. Then by Theorem 4.4, $b = 0$.

By (16) and Proposition 5.1 of [5], we can see that for all D_1 and D_2 in $H^2(K_2(A), \mathbb{Z})$, we have:

$$c_2(K_2(A)) \cdot D_1 \cdot D_2 = 54 \cdot q(D_1, D_2),$$

where q is the Beauville-Bogomolov form. Then we can calculate that:

$$c_2(K_2(A)) = \theta^* \left(4j(a_{12}) \cdot j(a_{34}) - 4j(a_{13}) \cdot j(a_{24}) + 4j(a_{14}) \cdot j(a_{23}) - \frac{1}{3}\delta^2 \right).$$

□

Corollary 5.10. *The class $\theta^* \delta^2$ is divisible by 3.*

Proposition 5.11. *The element*

$$\theta^* (j(a_{12}) \cdot j(a_{34}) - j(a_{13}) \cdot j(a_{24}) + j(a_{14}) \cdot j(a_{23}))$$

is divisible by 6. More precisely, it is equal to $6Y_p$ (see [5]).

Proof. Again by Section 4 of [5], we have:

$$W = \frac{3}{8}(c_2(K_2(A)) + 3\theta^*(\delta)^2).$$

It follows:

$$(17) \quad W = \frac{3}{8}\theta^* \left(4j(a_{12}) \cdot j(a_{34}) - 4j(a_{13}) \cdot j(a_{24}) + 4j(a_{14}) \cdot j(a_{23}) + \frac{8}{3}\delta^2 \right).$$

It follows that

$$\theta^* (j(a_{12}) \cdot j(a_{34}) - j(a_{13}) \cdot j(a_{24}) + j(a_{14}) \cdot j(a_{23})).$$

is divisible by 2. For the divisibility by 3, combine Proposition 5.7 with Corollary 5.10. □

Remark 5.12. We also have the following formulas:

$$(18) \quad W = \theta^* (\mathbf{p}_{-3}(1)|0))$$

$$(19) \quad Y_p = -\frac{1}{9}\theta^* (\mathbf{p}_{-1}(1)L_{-2}(1)|0))$$

Let us now look at cohomology classes of odd degree. Since $H^1(K_2(A)) = H^7(K_2(A)) = 0$, we only need to consider the degrees 3 and 5.

Proposition 5.13. *The map $\theta^* : H^*(A^{[3]}, \mathbb{Q}) \rightarrow H^*(K_2(A), \mathbb{Q})$ is surjective in degrees 3 and 5. If we set*

$$(20) \quad B_3 := \{a_i^{(0)}, 1 \leq i \leq 4\} \cup \{a_i^{(1)}, 1 \leq i \leq 4\}$$

$$(21) \quad B_5 := \{a_i^{(1)}, 1 \leq i \leq 4\} \cup \{a_i^{(2)}, 1 \leq i \leq 4\},$$

then the images of B_3 and B_5 give bases of $H^3(K_2(A), \mathbb{Q})$ and $H^5(K_2(A), \mathbb{Q})$ that are orthogonal under the intersection pairing. We have

$$(22) \quad \int \theta^* (a_i^{(0)} \cdot a_i^{(2)}) = \pm \frac{3}{2}$$

$$(23) \quad \int \theta^* (a_i^{(1)} \cdot a_i^{(1)}) = \pm \frac{1}{2}.$$

Moreover, $a_i^{(0)} \cdot [K_2(A)]$ and $\frac{2}{3}a_i^{(2)} \cdot [K_2(A)]$ are integral classes. This implies (by Poincaré duality) that $\theta^ a_i^{(0)}$ and $\frac{2}{3}\theta^* a_i^{(2)}$ are integral.*

Question: Which of $\theta^ a_i^{(1)}$ and $\theta^* a_i^{(1)}$ is not integral?*

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SIMON KAPFER, LABORATOIRE DE MATHÉMATIQUES ET APPLICATIONS, UMR CNRS 6086, UNIVERSITÉ DE POITIERS, TÉLÉPORT 2, BOULEVARD MARIE ET PIERRE CURIE, F-86962 FUTUROSCOPE CHASSENEUIL

E-mail address: `simon.kapfer@math.univ-poitiers.fr`