INTEGRAL COHOMOLOGY OF $K_2(A)$

SIMON KAPFER AND GRÉGOIRE MENET

Abstract. What we know already

1. Preliminaries

Definition 1.1. Let n be a natural number. A partition of n is a decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda_1 \geq \ldots \geq \lambda_k > 0$ of natural numbers such that $\sum_i \lambda_i = n$. Sometimes it is convenient to write $\lambda = (\ldots, 2^{m_2}, 1^{m_1})$ with multiplicities in the exponent. We define the weight $\|\lambda\| := \sum m_i i = n$ and the length $|\lambda| := \sum_i m_i = k$. We also define $z_\lambda := \prod_i i^{m_i} m_i!$.

Definition 1.2. Let $\Lambda_n := \mathbb{Q}[x_1, \ldots, x_n]^{S_n}$ be the graded ring of symmetric polynomials. There are canonical projections : $\Lambda_{n+1} \to \Lambda_n$ which send x_{n+1} to zero. The graded projective limit $\Lambda := \varprojlim \Lambda_n$ is called the ring of symmetric functions. Let m_λ and p_λ denote the monomial and the power sum symmetric functions. They are defined as follows: For a monomial $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \ldots x_{i_k}^{\lambda_k}$ of total degree n, the (ordered) sequence of exponents $(\lambda_1, \ldots, \lambda_k)$ defines a partition λ of n, which is called the shape of the monomial. Then we define m_λ being the sum of all monomials of shape λ. For the power sums, first define $p_n := x_1^n + x_2^n + \ldots$. Then $p_\lambda := p_{\lambda_1} p_{\lambda_2} \ldots p_{\lambda_k}$. The families $(m_\lambda)_\lambda$ and $(p_\lambda)_\lambda$ form two $\mathbb{Q}$-bases of Λ, so they are linearly related by $p_\lambda = \sum_\mu \psi_{\lambda\mu} m_\mu$. It turns out that the base change matrix $(\psi_{\lambda\mu})$ has integral entries, but its inverse $(\psi^{-1}_{\lambda\mu})$ has not.

2. Hilbert schemes of points on surfaces

Let A be a smooth projective complex surface. Let $A^{[n]}$ the Hilbert scheme of n points on the surface, *i.e.* the moduli space of finite subschemes of A of length n. $A^{[n]}$ is again smooth and projective of dimension $2n$.

Their rational cohomology can be described in terms of Nakajima's operators. First consider the direct sum

$$\mathbb{H} := \bigoplus_{n=0}^{\infty} H^*(A^{[n]}, \mathbb{Q})$$

This space is bigraded by cohomological *degree* and the *weight*, which is given by the number of points n. The unit element in $H^0(A^{[0]}, \mathbb{Q}) \cong \mathbb{Q}$ is denoted by $|0\rangle$, called the *vacuum*. There are linear operators $\mathfrak{q}_m(a)$, for each $m \geq 1$ and $a \in H^*(A, \mathbb{Q})$, acting on $\mathbb{H}$ which have the following properties: They depend linearly on a, and if $a \in H^k(A, \mathbb{Q})$ is homogeneous, the operator $\mathfrak{q}_m(a)$ is bihomogeneous of degree $k + 2(|m| - 1)$ and weight m:

$$\mathfrak{q}_m(a) : H^l(A^{[n]}) \to H^{l+k+2(|m|-1)}(A^{[n+m]})$$

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To construct them, first define incidence varieties $Z_m \subset A^{[n]} \times A \times A^{[n+m]}$ by

$$Z_m := \{(\xi, x, \xi') \,|\, \xi \subset \xi',\, \mathrm{supp}(\xi') - \mathrm{supp}(\xi) = mx\}\,.$$

Then $\mathfrak{q}_m(a)(\beta)$ is defined as the Poincaré dual of

$$pr_{3*}\left((pr_2^*(\alpha) \cdot pr_3^*(\beta)) \cap [Z_m]\right).$$

Every element in $\mathbb{H}$ can be decomposed uniquely as a linear combination of products of operators $\mathfrak{q}_m(a)$, acting on the vacuum.

To give the cup product structure of $\mathbb{H}$, define oprators $\mathfrak{G}(a)$ for $a \in H^*(A)$. Let $\Xi_n \subset A^{[n]} \times A$ be the universal subscheme. Then the action of $\mathfrak{G}(a)$ on $H^*(A^{[n]})$ is multiplication with the class

$$pr_{1*}\left(\mathrm{ch}(\mathcal{O}_{\Xi_n}) \cdot pr_2^*(\mathrm{td}(A) \cdot a)\right) \in H^*(A^{[n]}).$$

For $a \in H^k(A)$, we define $\mathfrak{G}_i(a)$ to be the component of $\mathfrak{G}(a)$ of cohomological degree $k + 2i$. A differential operator $\mathfrak{d}$ is given by $\mathfrak{G}_1(1)$. It means multiplication with the first Chern class of the tautological sheaf $pr_{1*}(\mathcal{O}_{\Xi_n})$.

In [6] and [7] there are various commutation relations between these operators, that allow to determine all multiplications in the cohomology of the Hilbert scheme. First of all, if X and Y are operators of degree d and d', their commutator is defined as

$$[X, Y] := XY - (-1)^{dd'} YX.$$

The integral on $A^{[n]}$ induces a bilinar form on $\mathbb{H}$: for classes α, $\beta \in H^*(A^{[n]})$ it is given by

$$(\alpha, \beta) \longmapsto \int_{A^{[n]}} \alpha \cdot \beta.$$

If X is a homogeneous linear operator of degree d and weight m, acting on $\mathbb{H}$, define its adjont $X^\dagger$ by

$$\int_{A^{[n+m]}} X(\alpha) \cdot \beta = (-1)^{d \deg(\alpha)} \int_{A^{[n]}} \alpha \cdot X^\dagger(\beta).$$

We put $\mathfrak{q}_0(a) := 0$ and for $m < 0$, $\mathfrak{q}_m(a) := (-1)^n \mathfrak{q}_{-m}(a)^\dagger$. Then define

$$\mathfrak{L}_m(a) := \begin{cases} \frac{1}{2} \sum\limits_{k \in \mathbb{Z}} \sum\limits_i \mathfrak{q}_k(a_{(1)}) \mathfrak{q}_{m-k}(a_{(2)}), & \text{if } m \neq 0, \\ \\ \sum\limits_{k > 0} \sum\limits_i \mathfrak{q}_k(a_{(1)}) \mathfrak{q}_{-k}(a_{(2)}), & \text{if } m = 0. \end{cases}$$

where $\sum_i a_{(1)} \otimes a_{(2)}$ is the push-forward of a along the diagonal : $A \to A \times A$ (in Sweedler notation). Then we have ([7, Thm. 2.16]):

$$[\mathfrak{q}_m(a), \mathfrak{q}_l(b)] = m \cdot \delta_{m+l} \cdot \int_A ab \tag{1}$$

$$[\mathfrak{L}_m(a), \mathfrak{q}_l(b)] = -m \cdot \mathfrak{q}_{m+l}(ab) \tag{2}$$

$$[\mathfrak{d}, \mathfrak{q}_m(b)] = m \cdot \mathfrak{L}_m(a) + \tfrac{m(|m|-1)}{2} \mathfrak{q}_m(K\alpha) \tag{3}$$

$$[\mathfrak{G}(a), \mathfrak{q}_1(b)] = \exp(\mathrm{ad}(\mathfrak{d}))(\mathfrak{q}_1(ab)) \tag{4}$$

Note (cf. [6, Thm. 3.8]) that this implies that

$$\mathfrak{q}_{m+1}(a) = \tfrac{(-1)^m}{m!} \,\mathrm{ad}^m([\mathfrak{d}, \mathfrak{q}_1(1)])(\mathfrak{q}_1(a)), \tag{5}$$

so there are two ways of writing an element of $\mathbb{H}$: As a linear combination of products of creation operators $\mathfrak{q}_m(a)$ or as a linear combination of products of the operators $\mathfrak{d}$ and $\mathfrak{q}_1(a)$. While the first one is more intuitive, the second one is more suitable for computing cup-products. Equations (3) and (5) permit now to switch between the two representations, using that

$$\mathfrak{d}|0\rangle = 0, \tag{6}$$

$$\mathfrak{L}_m(a)|0\rangle = \begin{cases} \frac{1}{2}\sum_{k=1}^{m-1}\sum_i \mathfrak{q}_k(b_i)\mathfrak{q}_{m-k}(c_i)|0\rangle, & \text{if } m > 1, \\ 0, & \text{if } m \leq 1. \end{cases} \tag{7}$$

(8)

Remark 2.1. We adopted the notation from [7], which differs from the conventions in [6]. Here is part of a dictionary:

Notation from [7]	Notation from [6]
operator of bidegree (w, d)	operator of bidegree $(w, d-2w)$
$\mathfrak{q}_m(a)$	$\mathfrak{p}_{-m}(a)$
$\mathfrak{L}_m(a)$	$-L_{-m}(a)$
$\mathfrak{G}(a)$	$a^{[\bullet]}$
$\mathfrak{d}$	∂
$\tau_{2*}(a)$	$-\Delta(a)$

By sending a subscheme in A to is support, we define a morphism

$$\rho : A^{[n]} \longrightarrow \mathrm{Sym}^n(A),$$

called the Hilbert–Chow morphism. The cohomology of $\mathrm{Sym}^n(A)$ is given by elements of the n-fold tensor power of $H^*(A)$ that are invariant under the action of the group of permutations $\mathfrak{S}_n$. A class in $H^*(A^{[n]}, \mathbb{Q})$ which can be written using only the operators $\mathfrak{q}_1$ comes from a pullback along ρ:

$$\mathfrak{q}_1(b_1)\cdots\mathfrak{q}_1(b_n)|0\rangle = \rho^*\left(\sum_{\pi\in\mathfrak{S}_n} b_{\pi(1)}\otimes\ldots\otimes b_{\pi(n)}\right), \quad b_i \in H^*(A, \mathbb{Q}). \tag{9}$$

In particular,

$$\frac{1}{n!}\mathfrak{q}_1(b)^n|0\rangle = \rho^*\big(b\otimes\ldots\otimes b\big), \tag{10}$$

$$\frac{1}{(n-1)!}\mathfrak{q}_1(b)\left(\mathfrak{q}_1(1)\right)^{n-1}|0\rangle = \rho^*\Big(b\otimes 1\otimes\ldots\otimes 1 \;+\; \ldots \;+\; 1\otimes\ldots\otimes 1\otimes b\Big). \tag{11}$$

This is sometimes useful for manually computing products. For instance, if $b \in H^2(A, \mathbb{Q})$ is of degree 2, then

$$\begin{aligned} \left(\tfrac{1}{(n-1)!}\mathfrak{q}_1(b)\left(\mathfrak{q}_1(1)\right)^{n-1}|0\rangle\right)^{2n} &= n!\cdot(2n-1)!!\cdot\rho^*\left(b^2\otimes\cdots\otimes b^2\right) \\ &= (2n-1)!!\cdot\mathfrak{q}_1(b^2)^n|0\rangle. \end{aligned} \tag{12}$$

3. Cohomology of Hilbert schemes of points on a torus surface

For the study of integral cohomology, first note that if $\alpha \in H^*(A, \mathbb{Z})$ is an integral class, then $\mathfrak{p}_{-m}(\alpha)$ maps integral classes to integral classes. Moreover, there is the following theorem:

Theorem 3.1. [11] *An operator is called integral if it maps integral classes in $\mathbb{H}$ to integral classes. The operator $\frac{1}{z_\lambda}\mathfrak{q}_\lambda(1)$ is integral. Let $\alpha \in H^2(A, \mathbb{Z})$ be monodromy equivalent to a divisor. Then the operator $\mathfrak{m}_\lambda(\alpha)$ is integral.*

Remark 3.2. If A is a projective torus, then the sublattice of divisors in $H^2(A, \mathbb{Z})$ contains at least ... By Scattone, etc.

To obtain the multiplicative structure of $\mathbb{H}$, given by the cup-products, there is a description in [6] and [7] in terms of multiplication operators $\mathfrak{G}_k(a),\ a \in H^*(A)$ [7, Def. 5.1], related to chern characters. There is the following commutation relation:

$$[\mathfrak{G}_k(a), \mathfrak{q}_1(b)] = \frac{1}{k!}\,\mathrm{ad}(\mathfrak{d})^k(\mathfrak{q}_1(ab)),$$

where the operator $\mathfrak{d}$ means multiplication with the first Chern class of the tautological sheaf. We set $a^{(k)} := \mathfrak{G}_k(a)(1) \in H^{\deg a+2k}(A^{[n]}, \mathbb{Q})$.

Next we focus on the structure of $H^2(A^{[n]}, \mathbb{Z})$ for $n \geq 2$. It has rank 13, and there is a basis consisting of:

- $\frac{1}{(n-1)!}\mathfrak{p}_{-1}(1)^{n-1}\mathfrak{p}_{-1}(a_{ij})|0\rangle,\ 1 \leq i < j \leq 4,$
- $\frac{1}{(n-2)!}\mathfrak{p}_{-1}(1)^{n-2}\mathfrak{p}_{-1}(a_i)\mathfrak{p}_{-1}(a_j)|0\rangle,\ 1 \leq i < j \leq 4,$
- $\frac{1}{2(n-2)!}\mathfrak{p}_{-1}(1)^{n-2}\mathfrak{p}_{-2}(1)|0\rangle$. We denote this class by δ.

It is clear that these classes form a basis of $H^2(A^{[n]}, \mathbb{Q})$. By [11, Thm. 4.6,Lemma 5.2], they also form a basis for $H^2(A^{[n]}, \mathbb{Z})$. TODO: refine this argument

The first 6 classes give an injection $j : H^2(A, \mathbb{Z}) \to H^2(A^{[n]}, \mathbb{Z})$.

4. Generalized Kummer varieties

Let A be a complex projective torus of dimension 2. Its first cohomology $H^1(A, \mathbb{Z})$ is freely generated by four elements a_1, a_2, a_3, a_4, corresponding to the four different circles on the torus. The cohomology ring is isomorphic to the exterior algebra:

$$H^*(A, \mathbb{Z}) = \Lambda^* H^1(A, \mathbb{Z}).$$

We abbreviate for the products $a_i \cdot a_j =: a_{ij}$ and $a_i \cdot a_j \cdot a_k =: a_{ijk}$. We write $a_1 \cdot a_2 \cdot a_3 \cdot a_4 =: x$ for the class corresponding to a point on A. We choose the a_i such that $\int_A x = 1$. We set $a_{\bar{i}}$ for the dual class of a_i, *i.e.* $a_i \cdot a_{\bar{i}} = x$. The bilinear form, given by $(a_{ij}, a_{kl}) \mapsto \int_A a_{ij}a_{kl}$ gives $H^2(A, \mathbb{Z})$ the structure of a unimodular lattice, isomorphic to $U^{\oplus 3}$, three copies of the hyperbolic lattice.

Definition 4.1. Let A be a complex projective torus of dimension 2 and $A^{[n]}$, $n \geq 1$, the corresponding Hilbert scheme of points. Denote $\Sigma : A^{[n]} \to A$ the summation morphism, a smooth submersion that factorizes via the Hilbert–Chow morphism : $A^{[n]} \xrightarrow{\rho} \mathrm{Sym}^n(A) \xrightarrow{\sigma} A$. Then the generalized Kummer $K_{n-1}(A)$ is defined as the fiber over 0:

$$\begin{array}{ccc} K_{n-1}(A) & \xrightarrow{\theta} & A^{[n]} \\ \downarrow & & \downarrow \Sigma \\ \{0\} & \longrightarrow & A \end{array} \tag{13}$$

Our first objective is to collect some information about this pullback diagram.

Proposition 4.2. *Recall that* $a_i^{(0)} = \frac{1}{(n-1)!}\mathfrak{q}_1(1)^{n-1}\mathfrak{q}_1(a_i)|0\rangle$*. The class of the Poincaré dual of* $K_{n-1}(A)$ *in* $H^4(A^{[n]},\mathbb{Z})$ *is given by*

$$a_1^{(0)} \cdot a_2^{(0)} \cdot a_3^{(0)} \cdot a_4^{(0)}.$$

Proof. Since the generalized Kummer variety is the fiber over a point, its Poincaré dual must be the pullback of $x \in H^4(A)$ under Σ. But $\Sigma^*(x) = \Sigma^*(a_1) \cdot \Sigma^*(a_2) \cdot \Sigma^*(a_3) \cdot \Sigma^*(a_4)$, so we have to verify that $\Sigma^*(a_i) = a_i^{(0)}$. To do this, we want to use the decomposition $\Sigma = \sigma\rho$. The pullback along σ of a class $a \in H^1(A,\mathbb{Q})$ on $H^1(\mathrm{Sym}^n(A),\mathbb{Q}) \cong H^*(A^n,\mathbb{Q})^{\mathfrak{S}_n}$ is given by $a \otimes 1 \otimes \cdots \otimes 1 + \ldots + 1 \otimes \cdots \otimes 1 \otimes a$. It follows from (11) that $\Sigma^*(a_i) = \frac{1}{(n-1)!}\mathfrak{q}_1(1)^{n-1}\mathfrak{q}_1(a_i)|0\rangle$. □

Remark 4.3. This allows us to better understand the morphism θ^*. Since the Poincaré pairing is nondegenerated, $[K_{n-1}(A)] \cdot \alpha = 0$ implies $\theta^*\alpha = 0$.

We recall Theorem 2 of [12].

Theorem 4.4. *The cohomology of* $K_{n-1}(A)$ *is torsion free.*

Theorem 4.5. [1] $K_{n-1}(A)$ *is a irreducible holomorphically symplectic manifold. In particular, it is simply connected and the canonical bundle is trivial.*

So $H^2(K_{n-1}(A),\mathbb{Z})$ admits an integer-valued nondegenerated quadratic form (called Beauville–Bogomolov form) q which gives $H^2(K_{n-1}(A),\mathbb{Z})$ the structure of a lattice isomorphic to $U^{\oplus 3} \oplus \langle -2n\rangle$, for $n \geq 3$. We have the following formula for $\alpha \in H^2(K_{n-1}(A),\mathbb{Z})$:

$$\int_{K_{n-1}(A)} \alpha^{2n-2} = n \cdot (2n-3)!! \cdot q(\alpha)^{n-1} \tag{14}$$

The morphism θ induces a homomorphism of graded rings

$$\theta^* : H^*(A^{[n]},\mathbb{Z}) \longrightarrow H^*(K_{n-1}(A),\mathbb{Z}). \tag{15}$$

Proposition 4.6. *Assume* $n \geq 3$*. Then* θ^* *is surjective on* $H^2(A^{[n]},\mathbb{Z})$ *with kernel* $\Lambda^2 H^1(A^{[n]},\mathbb{Z})$*.*

Proof. By [1, Sect. 7], $\theta^* : H^2(A^{[n]},\mathbb{C}) \to H^2(K_{n-1}(A),\mathbb{C})$ is surjective. Because of $H^1(K_{n-1}(A),\mathbb{Z}) = 0$, it is clear that $\Lambda^2 H^1(A^{[n]},\mathbb{Z})$ is contained in the kernel. This is a free $\mathbb{Z}$-module of rank 6. But the second Betti numbers of $A^{[n]}$ and $K_{n-1}(A)$ are 13 and 7, respectively, so it must be all.

It remains to show that θ^* is surjective for cohomology with integral coefficients, too. The idea is to prove that the lattice structure of $\operatorname{Im}\theta^*$ is the same as of $H^2(K_{n-1}(A))$. We use two formulas in [4, pp. 8–11]. Let $b \in H^2(A,\mathbb{Z})$ and set $\alpha = \frac{1}{(n-1)!}\mathfrak{q}_1(1)^{n-1}\mathfrak{q}_1(b)|0\rangle \in H^2(A^{[n]},\mathbb{Z})$. Then

$$\int_{A^{[n]}} \alpha^{2n} = \binom{2n}{2}\frac{\int_A b^2}{n^2}\int_{K_{n-1}(A)} \theta^*\alpha^{2n-2} \tag{16}$$

Combining this formula with (14) and (11), we get $\int_A b^2 = q(\alpha)$.

Secondly, we must show that $q(\theta^*\delta) = -2n$. TODO: show this! □

5. Study of $K_2(A)$

Let us summarize our results on θ^* for the case $n = 3$:

Theorem 5.1. *The homomorphism $\theta^* : H^*(A^{[3]}, \mathbb{Q}) \to H^*(K_2(A), \mathbb{Q})$ of graded rings is surjective in every degree except 4. Moreover, the image of $H^4(A^{[3]}, \mathbb{Q})$ is equal to $\mathrm{Sym}^2(H^2(K_2(A), \mathbb{Q}))$. The kernel of θ^* is the ideal generated by $H^1(A^{[3]}, \mathbb{Q})$.*

Now we focus on classes of cohomological degree 4.

Proposition 5.2. *The classes $\theta^*\left(\mathfrak{p}_{-2}(a_{ij})\mathfrak{p}_{-1}(1)|0\rangle\right)$ and $\theta^*\left(\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_{ij})|0\rangle\right)$ are linearly dependent.*

Proof. We can compute the product of these two classes with $[K_2(A)]$ in $H^*(A^{[3]})$. The two results are linearly dependent. Is there a direct geometric proof? $\square$

Proposition 5.3. $\theta^*\left(\mathfrak{p}_{-3}(x)|0\rangle\right) = 0$

Corollary 5.4. $\theta^*\left(\mathfrak{p}_{-2}(a_{ij})\mathfrak{p}_{-1}(1)|0\rangle\right) = \frac{1}{4}\theta^*\left(\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_{ij})|0\rangle\right)$

Proof. Let a_{ij} and a_{kl} be complementary, *i.e.* $a_{ij}a_{kl} = 1$. We have $a_{kl}^{(1)} = -\frac{1}{2}\mathfrak{p}_{-2}(a_{kl})\mathfrak{p}_{-1}(1)|0\rangle$. Then:

$$\theta^*\left(a_{ij}^{(1)} \cdot a_{kl}^{(1)}\right) = \theta^*\left(\mathfrak{p}_{-3}(1)|0\rangle + \frac{1}{2}\mathfrak{p}_{-1}(x)^2\mathfrak{p}_{-1}(1)|0\rangle\right)$$

But on the other hand, $\delta \cdot j(a) = \frac{1}{2}\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_{ij})|0\rangle + \mathfrak{p}_{-2}(a_{ij})\mathfrak{p}_{-1}(1)|0\rangle$, and

$$\theta^*\left(a_{kl}^{(1)} \cdot \delta \cdot j(a)\right) = \theta^*\left(-3\mathfrak{p}_{-3}(1)|0\rangle - 3\mathfrak{p}_{-1}(x)^2\mathfrak{p}_{-1}(1)|0\rangle\right).$$

$\square$

Corollary 5.5. $\theta^*\left(\delta \cdot j(a_{ij})\right) = \theta^*\left(\frac{3}{4}\mathfrak{p}_{-2}(1)\mathfrak{p}_{-1}(a_{ij})|0\rangle\right)$ *is divisible by 3.* $\square$

Proposition 5.6. *The classes $\theta^*\left(j(a_{ij})^2 - \frac{1}{3}j(a_{ij}) \cdot \delta\right)$ are divisible by 2.*

Proof. By [11], the classes $\frac{1}{2}\mathfrak{p}_{-1}(a_{ij})^2\mathfrak{p}_{-1}(1)|0\rangle - \frac{1}{2}\mathfrak{p}_{-2}(a_{ij})\mathfrak{p}_{-1}(1)|0\rangle$ are integral in $H^4(A^{[n]})$. But $j(a_{ij})^2 = \mathfrak{p}_{-1}(a_{ij})^2\mathfrak{p}_{-1}(1)|0\rangle$ and $\theta^*\left(\frac{1}{3}j(a_{ij}) \cdot \delta\right) = \theta^*\left(\mathfrak{p}_{-2}(a_{ij})\mathfrak{p}_{-1}(1)|0\rangle\right)$. $\square$

Proposition 5.7. *The class $\theta^*\left(\delta^2 + j(a_{12}) \cdot j(a_{34}) - j(a_{13}) \cdot j(a_{24}) + j(a_{14}) \cdot j(a_{23})\right)$ is divisible by 3.*

Proof. It is equal to $\theta^*\left(\mathfrak{p}_{-3}(1)|0\rangle - \frac{3}{2}\mathfrak{p}_{-1}(x)\mathfrak{p}_{-1}(1)^2|0\rangle\right)$. $\square$

Proposition 5.8. *We have:*

$$H^4(K_2(A), \mathbb{Q}) = \mathrm{Sym}^2 H^2(K_2(A), \mathbb{Q}) \oplus^{\perp} \Pi' \otimes \mathbb{Q}.$$

Proof. In Section 4 of [5], we can find the following formula:

$$Z_\tau \cdot D_1 \cdot D_2 = 2 \cdot q(D_1, D_2), \tag{17}$$

for all D_1, D_2 in $H^2(K_2(A), \mathbb{Z})$, $\tau \in A[3]$ and q the Beauville-Bogomolov form. It follows that $\Pi' \subset \mathrm{Sym}^2 H^2(K_2(A), \mathbb{Z})^{\perp}$. Since the cup-product is non-degenerated and by Proposition 4.3 of [5], we have:

$$\begin{aligned}
\mathrm{rk}\left(\mathrm{Sym}^2 H^2(K_2(A), \mathbb{Z}) \oplus \Pi'\right) &= \mathrm{rk}\,\mathrm{Sym}^2 H^2(K_2(A), \mathbb{Z}) + \mathrm{rk}\,\Pi' \\
&= 28 + 80 \\
&= \mathrm{rk}\, H^4(K_2(A), \mathbb{Z}).
\end{aligned}$$

It follows that

$$H^4(K_2(A), \mathbb{Q}) = \mathrm{Sym}^2 H^2(K_2(A), \mathbb{Q}) \oplus^{\perp} \Pi' \otimes \mathbb{Q}.$$

□

Next we look at the Chern classes of the tangent sheaves. Since the morphism Σ from the defining pullback diagram (13) is a submersion, the normal bundle of $K_{n-1}(A)$ in $A^{[n]}$ is trivial. Hence $c(K_2(A)) = \theta^* c(A^{[3]})$. Looking in [2, Sect. 8], we find a general formula for Chern classes of Hilbert schemes of points on surfaces. So we deduce

$$\begin{aligned} c_2(A^{[3]}) &= \left(\tfrac{3}{2}\mathfrak{q}_{*(1,1)}(1)\mathfrak{q}_1(1) - \tfrac{1}{3}\mathfrak{q}_3\right)|0\rangle \\ &= 10(1_{(4)}^{[\bullet]}) - 2(1_{(2)}^{[\bullet]})^2 \\ c_4(A^{[3]}) &= \tfrac{4}{3}\mathfrak{q}_{*(1,1,1)}(1)|0\rangle = 4(1_{(4)}^{[\bullet]})^2. \end{aligned}$$

Proposition 5.9. *We have:*

$$c_2(K_2(A)) = \theta^* \left(4j(a_{12}) \cdot j(a_{34}) - 4j(a_{13}) \cdot j(a_{24}) + 4j(a_{14}) \cdot j(a_{23}) - \frac{1}{3}\delta^2\right).$$

In particular $c_2(K_2(A)) \in \mathrm{Sym}^2 H^2(K_2(A), \mathbb{Z})$.

Proof. We can write:

$$c_2(K_2(A)) = a + b,$$

with $a \in \mathrm{Sym}^2 H^2(K_2(A), \mathbb{Q})$ and $b \in \Pi'$. First, we prove that $b = 0$. We have $c_2(K_2(A)) \in \Pi'^{\perp}$ and also $a \in \Pi'^{\perp}$, it follows that $b \in \Pi'^{\perp}$. Since the cup-product is non-degenerated, it follows that b is of torsion. Then by Theorem 4.4, $b = 0$.

By (17) and Proposition 5.1 of [5], we can see that for all D_1 and D_2 in $H^2(K_2(A), \mathbb{Z})$, we have:

$$c_2(K_2(A)) \cdot D_1 \cdot D_2 = 54 \cdot q(D_1, D_2),$$

where q is the Beauville-Bogomolov form. Then we can calculate that:

$$c_2(K_2(A)) = \theta^* \left(4j(a_{12}) \cdot j(a_{34}) - 4j(a_{13}) \cdot j(a_{24}) + 4j(a_{14}) \cdot j(a_{23}) - \frac{1}{3}\delta^2\right).$$

□

Corollary 5.10. *The class $\theta^*\delta^2$ is divisible by 3.*

Proposition 5.11. *The element*

$$\theta^* \left(j(a_{12}) \cdot j(a_{34}) - j(a_{13}) \cdot j(a_{24}) + j(a_{14}) \cdot j(a_{23})\right)$$

is divisible by 6. More precisely, it is equal to $6\,Y_p$ (see [5]).

Proof. Again by Section 4 of [5], we have:

$$W = \frac{3}{8}(c_2(K_2(A)) + 3\theta^*(\delta)^2).$$

It follows:

$$W = \frac{3}{8}\theta^* \left(4j(a_{12}) \cdot j(a_{34}) - 4j(a_{13}) \cdot j(a_{24}) + 4j(a_{14}) \cdot j(a_{23}) + \frac{8}{3}\delta^2\right). \tag{18}$$

It follows that

$$\theta^*(j(a_{12}) \cdot j(a_{34}) - j(a_{13}) \cdot j(a_{24}) + j(a_{14}) \cdot j(a_{23})).$$

is divisible by 2. For the divisibility by 3, combine Proposition 5.7 with Corollary 5.10. ☐

Remark 5.12. We also have the following formulas:

$$W = \theta^* \left(\mathfrak{p}_{-3}(1)|0\rangle\right) \tag{19}$$

$$Y_p = -\tfrac{1}{9}\,\theta^* \left(\mathfrak{p}_{-1}(1)L_{-2}(1)|0\rangle\right) \tag{20}$$

Let us now look at cohomology classes of odd degree. Since $H^1(K_2(A)) = H^7(K_2(A)) = 0$, we only need to consider the degrees 3 and 5.

Proposition 5.13. *The map $\theta^* : H^*(A^{[3]}, \mathbb{Q}) \to H^*(K_2(A), \mathbb{Q})$ is surjective in degrees 3 and 5. If we set*

$$B_3 := \{a_{\bar{i}}^{(0)},\ 1 \leq i \leq 4\} \cup \{a_i^{(1)},\ 1 \leq i \leq 4\} \tag{21}$$

$$B_5 := \{a_{\bar{i}}^{(1)},\ 1 \leq i \leq 4\} \cup \{a_i^{(2)},\ 1 \leq i \leq 4\}, \tag{22}$$

then the images of B_3 and B_5 give bases of $H^3(K_2(A), \mathbb{Q})$ and $H^5(K_2(A), \mathbb{Q})$ that are orthogonal under the intersection pairing. We have

$$\int \theta^* \left(a_{\bar{i}}^{(0)} \cdot a_i^{(2)}\right) = \pm\frac{3}{2} \tag{23}$$

$$\int \theta^* \left(a_i^{(1)} \cdot a_{\bar{i}}^{(1)}\right) = \pm\frac{1}{2}. \tag{24}$$

Moreover, $a_{\bar{i}}^{(0)} \cdot [K_2(A)]$ and $\frac{2}{3} a_i^{(2)} \cdot [K_2(A)]$ are integral classes. This implies (by Poincaré duality) that $\theta^* a_{\bar{i}}^{(0)}$ and $\frac{2}{3}\theta^* a_i^{(2)}$ are integral.

Question: Which of $\theta^* a_i^{(1)}$ and $\theta^* a_{\bar{i}}^{(1)}$ is not integral?

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Simon Kapfer, Laboratoire de Mathématiques et Applications, UMR CNRS 6086, Université de Poitiers, Téléport 2, Boulevard Marie et Pierre Curie, F-86962 Futuroscope Chasseneuil

E-mail address: simon.kapfer@math.univ-poitiers.fr