MULTIPLICATIVE STRUCTURE OF INTEGER COHOMOLOGY OF HILBERT SCHEMES OF POINTS ON K3 SURFACES

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ABSTRACT. We study the cokernels of the cup product in integer cohomology of the Hilbert scheme of n points on a K3 surface. The main difference to cohomology with rational coefficients is the presence of torsion, depending on n. We particularly discuss the case $n = 3$.

1. PRELIMINARIES

Let S be a K3 surface. We fix integral bases 1 of $H^0(S, \mathbb{Z})$, x of $H^4(X, \mathbb{Z})$ and $\alpha_1, \ldots, \alpha_{22}$ of $H^2(S, \mathbb{Z})$ such that the cup product pairing on $H^2(X, \mathbb{Z})$, written as a symmetric matrix with respect to this basis, looks like

$$
\begin{pmatrix}
U & & & & \\
& U & & & \\
& & U & & \\
& & & E & \\
& & & & E
\end{pmatrix},
$$

where U stands for the intersection matrix of the hyperbolic lattice and E stands for the negative matrix of the E_8 lattice, *i.e.*

$$
U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad
E = \begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -2
\end{pmatrix}.
$$

We denote by $S^{[n]}$ the Hilbert scheme of n points on S. An integral basis for $H^*(S^{[n]}, \mathbb{Z})$ was given by Qin–Wang:

Theorem 1.1. [39, Thm. 5.4.] *The classes*

$$
\frac{1}{z_\lambda} \mathfrak{q}_\lambda(1) \mathfrak{q}_\mu(x) \mathfrak{m}_{\nu^1, \alpha_1} \ldots \mathfrak{m}_{\nu^{22}, \alpha_{22}} |0\rangle, \quad |\lambda| + |\mu| + \sum_{i=1}^{22} |\nu^i| = n
$$

form an integral basis for $H^*(S^{[n]}, \mathbb{Z})$. Here, λ, μ, ν^i are partitions, $|\cdot|$ means the weight of a partition and $z_\lambda := \prod_i i^{m_i} m_i!$ if $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$. The symbol $\mathfrak{q}$ stands for Nakajima's creation operator. The relation of $\mathfrak{m}_{\nu, \alpha}$ to $\mathfrak{q}_{\tilde{\nu}}(\alpha)$ is the same

as the monomial symmetric functions m_ν to the power sum symmetric functions $p_{\tilde\nu}$.

2. Introduction

The ring structure of $H^*(S^{[n]}, \mathbb{Q})$ has been studied in [24]. Since $H^{\mathrm{odd}}(S^{[n]}, \mathbb{Z}) = 0$ and $H^{\mathrm{even}}(S^{[n]}, \mathbb{Z})$ is torsion-free by [27], we can also apply these results to $H^*(S^{[n]}, \mathbb{Z})$. A basis for cohomology with integer coefficients was given by Qin–Wang in [39] which allows us to compute explicitly the image of $H^k(S^{[n]}, \mathbb{Z}) \cup H^l(S^{[n]}, \mathbb{Z})$ in $H^{k+l}(S^{[n]}, \mathbb{Z})$. We obtain:

Proposition 2.1. *If X is deformation equivalent to $S^{[3]}$, then:*

$$\frac{H^4(X, \mathbb{Z})}{\mathrm{Sym}^2 H^2(X, \mathbb{Z})} \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 23}$$

The torsion part of the quotient is generated by the integral class $\frac{1}{3}\mathfrak{q}_{(3)}(1)|0\rangle$.

$$\frac{H^6(X, \mathbb{Z})}{H^2(X, \mathbb{Z}) \cup H^4(X, \mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12}$$

This quotient is generated by the 12 integral classes $\mathfrak{m}_{(1^3),\alpha_i}|0\rangle$, where $i \in \{1, 2, 3, 4, 5, 6, 8, 9, 11, 16, 17, 19\}$.

$$\frac{H^6(X, \mathbb{Z})}{\mathrm{Sym}^3 H^2(X, \mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 2} \oplus \left(\frac{\mathbb{Z}}{4\mathbb{Z}}\right)^{\oplus 21} \oplus \mathbb{Z}^{\oplus 2278}$$

Proposition 2.2. *If X is deformation equivalent to $S^{[4]}$, then:*

$$\frac{H^4(X, \mathbb{Z})}{\mathrm{Sym}^2 H^2(X, \mathbb{Z})} \cong \mathbb{Z}^{\oplus 24}$$

$$\frac{H^6(X, \mathbb{Z})}{H^2(X, \mathbb{Z}) \cup H^4(X, \mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 11} \oplus \left(\frac{\mathbb{Z}}{6\mathbb{Z}}\right)^{\oplus 12}$$

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