P-adic arctan

Simon Kapfer

August 14, 2014

Abstract

We define the function arctan in the p-adic fields $\mathbb{Q}_p$.

For p a prime, $x \in \mathbb{Q}_p$, the series

$$\arctan(x) := x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + - \dots$$

converges, if $\|x\|_p < 1$. We would like to extend this definition to all $x \in \mathbb{Q}_p$. Therefore, we must use some functional equation for arctan, namely

$$\begin{aligned}
\arctan(x) &= \operatorname{Im}\log(1+ix) \\
&= \frac{1}{m}\operatorname{Im}\log((1+ix)^m) \\
&= \frac{1}{m}\operatorname{Im}\log\left(\sum_k (-1)^k \binom{m}{2k} x^{2k} + i\sum_k (-1)^k \binom{m}{2k+1} x^{2k+1}\right) \\
&= \frac{1}{m}\operatorname{Im}\log\left(1 + i\frac{\sum_k (-1)^k \binom{m}{2k+1} x^{2k+1}}{\sum_k (-1)^k \binom{m}{2k} x^{2k}}\right) \\
&= \frac{1}{m}\arctan\left(\frac{\sum_k (-1)^k \binom{m}{2k+1} x^{2k+1}}{\sum_k (-1)^k \binom{m}{2k} x^{2k}}\right)
\end{aligned}$$

which holds for all $m \geq 1$ and all $x \in \mathbb{R}$. We can use this identity to decrease the p-adic norm of the argument of arctan for an appropriate choice of m:

- If $p = 2$, take $m = 4$.
- If $p = 4n - 1$, take $m = p + 1$.
- If $p = 4n + 1$, take $m = p - 1$. In this case, $i := \sqrt{-1} \in \mathbb{Q}_p$ and we could also have used the identity: $\arctan(x) = \frac{1}{2i}\log\left(\frac{x+i}{x-i}\right)$ (Iwasawa logarithm).

Now we can check some formulas for π. It surprisingly turns out, that all of them become 0, e. g.

$$\begin{aligned}
\pi &= 4\arctan(1) \\
&= 4\arctan\left(\tfrac{1}{2}\right) + \arctan\left(\tfrac{1}{3}\right) \\
&= 16\arctan\left(\tfrac{1}{5}\right) - 4\arctan\left(\tfrac{1}{239}\right) \\
&= 2\arctan(x) + 2\arctan\left(x^{-1}\right) \quad \forall x \in \mathbb{Q}_p \\
&= 0.
\end{aligned}$$