COMPUTING CUP-PRODUCTS IN INTEGER COHOMOLOGY OF HILBERT SCHEMES OF POINTS ON K3 SURFACES

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ABSTRACT. We study the images of cup products in integer cohomology of the Hilbert scheme of n points on a K3 surface.

1. PRELIMINARIES

Definition 1.1. Let S be a K3 surface. We fix integral bases 1 of $H^0(S,\mathbb{Z})$, x of $H^4(X,\mathbb{Z})$ and $\alpha_1,\dots,\alpha_{22}$ of $H^2(S,\mathbb{Z})$. The cup product induces a symmetric bilinear form B_{H^2} on $H^2(X,\mathbb{Z})$, written as a symmetric matrix with respect to this basis, looks like

$$B_{H^2} = \begin{pmatrix} U & & & & \\ & U & & & \\ & & U & & \\ & & & E & \\ & & & & E \end{pmatrix},$$

where U stands for the intersection matrix of the hyperbolic lattice and E stands for the negative matrix of the E_8 lattice, *i.e.*

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \end{pmatrix}.$$

We may extend B_{H^2} to a symmetric non-degenerate bilinear form on $H^*(S,\mathbb{Z})$ by setting $B(1,1)=0,\ B(1,\alpha_i)=0,\ B(1,x)=1,\ B(x,x)=0$.

Definition 1.2. B induces a form $B\otimes B$ on $\operatorname{Sym}^2 H^*(S,\mathbb{Z})$. So the cup-product

$$\mu : \operatorname{Sym}^2 H^*(S,\mathbb{Z}) \longrightarrow H^*(S,\mathbb{Z})$$

has an adjoint comultiplication Δ, given by:

$$\Delta : H^*(S,\mathbb{Z}) \longrightarrow \operatorname{Sym}^2 H^*(S,\mathbb{Z}), \quad \Delta = (B\otimes B)^{-1}\mu^T B$$

The image of 1 under the composite map $\mu(\Delta(1)) = B(\Delta(1),\Delta(1)) = 24x$, denoted by e is called the Euler Class.

We denote by $S^{[n]}$ the Hilbert scheme of n points on S. An integral basis for $H^*(S^{[n]},\mathbb{Z})$ in terms of Nakajima's operators was given by Qin–Wang:

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Theorem 1.3. [5, Thm. 5.4.] *The classes*

$$\frac{1}{z_\lambda}\mathfrak{q}_\lambda(1)\mathfrak{q}_\mu(x)\mathfrak{m}_{\nu^1,\alpha_1}\dots\mathfrak{m}_{\nu^{22},\alpha_{22}}|0\rangle, \quad \|\lambda\|+\|\mu\|+\sum_{i=1}^{22}\|\nu^i\|=n$$

form an integral basis for $H^*(S^{[n]},\mathbb{Z})$. Here, λ, μ, ν^i are partitions, $\|\cdot\|$ means the weight of a partition i.e. $\|\lambda\|=\sum_i m_i i$ and $z_\lambda := \prod_i i^{m_i}m_i!$, if $\lambda=(1^{m_1},2^{m_2},\dots)$. The symbol $\mathfrak{q}$ stands for Nakajima's creation operator. The relation of $\mathfrak{m}_{\nu,\alpha}$ to $\mathfrak{q}_{\tilde{\nu}}(\alpha)$ is the same as the monomial symmetric functions m_ν to the power sum symmetric functions $p_{\tilde{\nu}}$.

The ring structure of $H^*(S^{[n]},\mathbb{Q})$ has been studied in [2], where an explicit algebraic model is constructed. Since $H^{\mathrm{odd}}(S^{[n]},\mathbb{Z})=0$ and $H^{\mathrm{even}}(S^{[n]},\mathbb{Z})$ is torsion-free by [3], we can also apply these results to $H^*(S^{[n]},\mathbb{Z})$ to determine the multiplicative structure of cohomology with integer coefficients.

2. Computational results

With the help of a computer, we are able to compute arbitrary products in $H^*(S^{[n]},\mathbb{Z})$. We give some results in low degrees.

Proposition 2.1. *Studying the image of* $\operatorname{Sym}^2 H^2$ *in* H^4, *we obtain:*

$$\frac{H^4(S^{[2]},\mathbb{Z})}{\operatorname{Sym}^2 H^2(S^{[2]},\mathbb{Z})}\cong\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23}\oplus\frac{\mathbb{Z}}{5\mathbb{Z}}$$

This was already known to Boissière, Nieper-Wißkirchen and Sarti, [1, Prop. 3].

$$\frac{H^4(S^{[3]},\mathbb{Z})}{\operatorname{Sym}^2 H^2(S^{[3]},\mathbb{Z})}\cong\frac{\mathbb{Z}}{3\mathbb{Z}}\oplus\mathbb{Z}^{\oplus 23}$$

The torsion part of the quotient is generated by the integral class $\frac{1}{3}\mathfrak{q}_{(3)}(1)|0\rangle$.

$$\frac{H^4(S^{[n]},\mathbb{Z})}{\operatorname{Sym}^2 H^2(S^{[n]},\mathbb{Z})}\cong\mathbb{Z}^{\oplus 24}, \quad \text{for } n\geq 4.$$

This was already proven by Markman, [4, Thm. 1.10].

Proposition 2.2. *Comparing $H^2\cup H^4$ with H^6, we obtain:*

$$H^2(S^{[2]},\mathbb{Z})\cup H^4(S^{[2]},\mathbb{Z})=H^6(S^{[2]},\mathbb{Z})$$

$$\frac{H^6(S^{[3]},\mathbb{Z})}{H^2(S^{[3]},\mathbb{Z})\cup H^4(S^{[3]},\mathbb{Z})}\cong\left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12}$$

This quotient is generated by the 12 integral classes $\mathfrak{m}_{(1^3),\alpha_i}|0\rangle$, where $i\in\{1,2,3,4,5,6,8,9,11,16,17,19\}$.

$$\frac{H^6(S^{[4]},\mathbb{Z})}{H^2(S^{[4]},\mathbb{Z})\cup H^4(S^{[4]},\mathbb{Z})}\cong\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23}\oplus\left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12}$$

$$\frac{H^6(S^{[5]},\mathbb{Z})}{H^2(S^{[5]},\mathbb{Z})\cup H^4(S^{[5]},\mathbb{Z})}\cong\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23}\oplus\left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12}\oplus\left(\frac{\mathbb{Z}}{5\mathbb{Z}}\right)^{\oplus 3}$$

The 5-torsion part is generated by the 2 integral classes
$\frac{1}{5}\left[\frac{1}{2}\mathfrak{q}_{(1^2)}(1)\mathfrak{m}_{(1^3)}(\alpha_i) + \frac{2}{2}(1)\mathfrak{q}_{(1^2)}\mathfrak{m}_{(2,1)}(\alpha_i) + \frac{3}{2}\mathfrak{q}_{(1^2)}(1)\mathfrak{m}_{(3)}(\alpha_i) + \frac{4}{2}\mathfrak{q}_{(2,1)}(1)\mathfrak{m}_{(1^2)}(\alpha_i) + \frac{2}{4}\mathfrak{q}_{(2,1)}(1)\mathfrak{m}_{(2)}(\alpha_i) + \frac{2}{3}\right.$
$i \in \{13, 21\}$ *and the integral class* $\frac{3}{5}\left[\frac{1}{4}\mathfrak{q}_{(4,1)}(1) + \frac{1}{6}\mathfrak{q}_{(3,2)}(1)\right]|0\rangle$.

$$\frac{H^6(S^{[6]},\mathbb{Z})}{H^2(S^{[6]},\mathbb{Z}) \cup H^4(S^{[6]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 22} \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12} \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}}\right)^{\oplus 2} \oplus \mathbb{Z}$$

The free summand is generated by $\left[\frac{10}{48}\mathfrak{q}_{(2^3)}(1) - \frac{12}{6}\mathfrak{q}_{(3,2,1)}(1) + \frac{3}{8}\mathfrak{q}_{(4,1^2)}(1)\right]|0\rangle$.

Proposition 2.3.

$$\frac{H^6(S^{[2]},\mathbb{Z})}{\mathrm{Sym}^3 H^2(S^{[2]},\mathbb{Z})} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$$

The quotient is generated by the integral class $\frac{1}{2}\mathfrak{q}_{(2)}(1)|0\rangle$.

$$\frac{H^6(S^{[3]},\mathbb{Z})}{\mathrm{Sym}^3 H^2(S^{[3]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 230} \oplus \left(\frac{\mathbb{Z}}{36\mathbb{Z}}\right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{72\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 507}$$

$$\frac{H^6(S^{[4]},\mathbb{Z})}{\mathrm{Sym}^3 H^2(S^{[4]},\mathbb{Z})} \cong$$

$$\frac{H^6(S^{[5]},\mathbb{Z})}{\mathrm{Sym}^3 H^2(S^{[5]},\mathbb{Z})} \cong$$

$$\frac{H^6(S^{[n]},\mathbb{Z})}{\mathrm{Sym}^3 H^2(S^{[n]},\mathbb{Z})} \cong n \geq 6.$$

References

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