LECTURE 11. J-HOLOMORPHIC CURVES

11.1. **Almost complex structure and integrability.**

Let M be a $2n$-dimensional real manifold. An almost complex structure J on M is a tensor field $J : TM \to TM$ such that $J^2 = -I$.

Example 11.1.1. Let $\Sigma \subset \mathbb{R}^3$ be an oriented hypersurface. Then there exists an almost complex structure J over Σ. Let $\nu : \Sigma \to S^2$ be the Gauss map which associates to every point $x \in \Sigma$ the outer normal vector $\nu(x) \perp T_x\Sigma$. Then the almost complex structure is

$$J_x u = \nu(x) \times u.$$

Such a hypersurface carries a J-compatible metric which is the induced metric from the ambient space. The compatible nondegenerate two form ω is given by

$$\omega_x(v, w) = \langle \nu(x), v \times w \rangle.$$

The following example shows that there exist obstructions to the existence of almost complex structure on a manifold.

Example 11.1.2. S^2 and S^6 are the only two spheres that there exist almost complex structure. S^2 is the only symplectic manifold of the spheres and has a unique complex structure. Whether or not S^6 has a complex structure is still a famous conjecture.

For an almost complex manifold (M, J), the complexified tangent space has the decomposition:

$$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M,$$

The space $T^{1,0}M$ consists of the $(1,0)$ vectors Z such that $JZ = iZ$ and $T^{0,1}M$ consists of the $(0,1)$ vectors Z such that $JZ = -iZ$. We have the expression for the $(1,0)$ vector Z and the $(0,1)$ vector W,

$$Z = \frac{1}{2}(I - iJ)X, \ W = \frac{1}{2}(I + iJ)Y, \tag{1}$$

where X and Y are real vector fields.

Denote by $P^{1,0}$ or $P^{0,1}$ as the projections from $TM \otimes C$ to $T^{1,0}M$ or $T^{0,1}M$.

Definition 11.1.3. Let M be a $2n$-dimensional manifold. An almost complex structure J on M is called integrable if there exists an atlas $\{U_\alpha, \alpha : U_\alpha \to \mathbb{R}^{2n}\}$ such that

$$d\alpha \circ J = J_0 \circ d\alpha,$$

and the transition function $d(\beta \circ \alpha^{-1})(z) \in GL(n, \mathbb{C})$. Here J_0 is the standard complex structure on $\mathbb{R}^{2n}$.

For any two vector fields X, Y on M, the Nijenhuis tensor N_J is defined as

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y].$$

One can prove that N_J is actually a tensor. The following theorem of Newlander-Nirenberg shows the relation of almost complex structure and the complex structure:

Theorem 11.1.4 (Integrability theorem). *An almost complex structure J is integrable if and only if the Nijenhuis tensor N_J vanishes.*

Now the vanishing of the Nijenhuis tensor can be viewed as the Frobenius integrability condition:

Lemma 11.1.5. *The set of the type* $(1,0)$ *vector fields is closed under the operation of Lie bracket if and only if* $N_J \equiv 0$.

Proof. Let X and Y be two real vector fields and we have

$$P^{1,0}X = \frac{1}{2}(I - iJ)X, \;\; P^{0,1}Y = \frac{1}{2}(I + iJ)Y.$$

It is easy to show that

$$[P^{1,0}X, P^{1,0}Y] + iJ[P^{1,0}X, P^{1,0}Y] = -N_J(X,Y) - iJN_J(X,Y),$$

which is equivalent to

$$P^{0,1}([P^{1,0}X, P^{1,0}Y]) = P^{0,1}(-N_J(X,Y)). \tag{2}$$

Hence $N_J(X,Y)$ vanishes if and only if $[P^{1,0}X, P^{1,0}Y]$ is a $(1,0)$ vector. □

In $\mathbb{C}^n$, we have the expression

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}), \;\; \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$$

or written in terms of J_0:

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(1 - iJ_0)\frac{\partial}{\partial x_j}, \;\; \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(1 + iJ_0)\frac{\partial}{\partial x_j}$$

Hence in the local coordinates z_j, the type $(1,0)$ vector can be written as the combination $\sum_k a_k \frac{\partial}{\partial z_k}$ where a_k are complex valued functions on M. The Lie bracket operation of two $(1,0)$ type vectors is still of the $(1,0)$ type.

If Σ is a Riemann surface, then the Lie bracket of two $(1,0)$ vectors is of type $(1,0,$ hence we have

Theorem 11.1.6. *Every almost complex structure on a Riemann surface is integrable.*

A nondegenerate 2-form $\omega \in \Omega^2(M)$ is called compatible with J if

$$\langle v, w \rangle = \omega(v, Jw)$$

defines a Riemannian metric on TM. We define by $\mathscr{J}(M,\omega)$ the set of compatible almost complex structures w.r.t. ω.

A Riemannian metric g is called compatible with J if

$$g(v,w) = g(Jv, Jw).$$

In this case, $\omega(v,w) := g(Jv,w)$ is nondegenerate and define a compatible symplectic bilinear form with J.

Proposition 11.1.7. *Let* ω *be nondegenerate 2-form on M and let* $J \in \mathscr{J}(M,\omega)$. *Let* ∇ *be the Levi-civita connection associated to the metric* g_J. *Then the following two conditions are equivalent*

(i) $\nabla J = 0$;
(ii) *J is integrable and* ω *is closed.*

Proof. $(ii) \Rightarrow (i)$. Since

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X,Y],$$

by the formula

$$\nabla_X Y - \nabla_Y X = [X,Y].$$

it is a straightforward computation to show that

$$N_J(X,Y) = (\nabla_{JX} J)Y - (\nabla_{JY} J)X + J(\nabla_Y J)X - J(\nabla_X J)Y. \tag{3}$$

On the other hand, we have the formula:

$$\begin{aligned} d\omega(X,Y,Z) &= X(\omega(Y,Z)) + Y(\omega(Z,X)) + Z(\omega(X,Y)) + \omega([X,Y],Z) + \omega([Y,Z],X) + \omega([Z,X],Y) \\ &= g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y). \end{aligned} \tag{4}$$

So it is easy to prove that

$$g(N_J(X,Y),Z) = d\omega(JX,Y,Z) + d\omega(X,JY,Z) + 2g((\nabla_Z J)X, JY). \tag{5}$$

Thus $N_J \equiv 0$ and $d\omega = 0$ induces that $\nabla J \equiv 0$.

$(i) \Rightarrow (ii)$. By the formula (4), we know that if $\nabla J \equiv 0$, then $d\omega \equiv 0$. From (5), we know that $N_J \equiv 0$. □

11.2. *J*-holomorphic curves.

Let (M, J) be an almost complex manifold and (Σ, j) be a Riemann surface. A smooth map $u : \Sigma \to M$ is called J-holomorphic if the differential du is a complex linear map with respect to j and J:

$$du \circ j = J \circ du.$$

If we write

$$\bar\partial_J(u) := \frac{1}{2}(du + J \circ du \circ j),$$

then $\bar\partial_J$ is a section of the infinitesimal vector bundle $\mathscr{E} \to \mathrm{Map}(\Sigma, M)$, whose fiber at $u \in \mathrm{Map}(\Sigma, M)$ is $\mathscr{E}_u = \Omega^{0,1}(u^*TM)$. This can be seen from the local formula of $\bar\partial_J$.

Let $\{U_\alpha\, \alpha : U_\alpha \to \mathbb{C}\}$ be an atlas of conformal charts of Σ. This means that if we denote i to be the standard complex structure in $\mathbb{C}$, then

$$i d\alpha = d\alpha \circ j,$$

and the transition function $\alpha \circ \beta^{-1}$ is a holomorphic function. Let $z = s + it$ be the conformal coordinate in a chart U_α. Then j has has the action:

$$j(\frac{\partial}{\partial s}) = \frac{\partial}{\partial t},\ j(\frac{\partial}{\partial t}) = -\frac{\partial}{\partial s}. \tag{6}$$

Let $\{u^1, \cdots, u^{2n}\}$ be the local coordinates of the chart in M containing the image of $u(U_\alpha)$. We have

$$\begin{aligned} (\bar\partial_J u)(\frac{\partial}{\partial s}) &= \frac{1}{2}(\frac{\partial u}{\partial s} + J \cdot \frac{\partial u}{\partial t}) \\ (\bar\partial_J u)(\frac{\partial}{\partial t}) &= \frac{1}{2}(\frac{\partial u}{\partial t} - J \cdot \frac{\partial u}{\partial s}) = -\frac{1}{2}J(\frac{\partial u}{\partial s} + J \cdot \frac{\partial u}{\partial t}). \end{aligned}$$

Then we have the local formula

$$\bar\partial_J u = \frac{1}{2}(I + jJ)ds \cdot (\frac{\partial u}{\partial s} + J \cdot \frac{\partial u}{\partial t}). \tag{7}$$

This shows that $\bar{\partial}_J u$ is a u^*TM-valued $(0,1)$-form on Σ. u is a J-curve iff u satisfies the following equation locally:

$$\frac{\partial u}{\partial s} + J(u) \cdot \frac{\partial u}{\partial t} = 0 \tag{8}$$

If (M, J) is $(\mathbb{R}^{2n}, J_0)$ and the $2n$-dimensional vector

$$u = \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f^1 \\ \vdots \\ f^n \\ g^1 \\ \vdots \\ g^n \end{pmatrix}$$

is identified as $f + ig$, then u is J_0-holomorphic if and only if f^i and g^i satisfies the Cauchy-Riemann equation:

$$\frac{\partial f^i}{\partial s} = \frac{\partial g^i}{\partial t}, \frac{\partial f^i}{\partial t} = -\frac{\partial g^i}{\partial s}.$$

Thus a J_0-curve is a holomorphic curve in the usual sense. Moreover, if the almost complex structure on M is integrable, then the J-curve is the holomorphic curve in usual in the complex manifold M.

11.3. **Analytical properties of J-curves.**

Unique continuation. Since the unique continuation is a local property, we consider a J-curve $u : B_\epsilon(0) \to \mathbb{C}^n$, where

$$B_\epsilon(0) = \{z \in \mathbb{C} \| z| < \epsilon\} \subset \mathbb{C}$$

and J is assumed to be a C^1 almost complex structure in $\mathbb{C}^n$.

Definition 11.3.1. An integrable function $w : B_\epsilon(0) \to \mathbb{C}^n$ is said to vanish to infinite order at $z = 0$ if

$$\int_{|z|\leq r} |w(z)| = O(r), \tag{9}$$

for every $k > 0$. If w is smooth, then this means that the ∞-jet of w vanishes at 0.

Theorem 11.3.2. *Let J be a C^1 almost complex structure on $\mathbb{C}^n$. Assume that $u, v \in C^1(B_\epsilon(0), \mathbb{C}^n)$ are two J-curves. If $u - v$ vanishes at 0 to infinity order, then $u \equiv v$.*

Proof. Let u be a solution of the J-curve equation:

$$\partial_s u + J(u)\partial_t u = 0. \tag{10}$$

Take the (distribution) derivative, then

$$\begin{aligned} 0 =& \partial_s^2 u + \partial_s J \cdot \partial_t u + J \cdot \partial_s \partial_t u \\ =& \partial_s^2 u + \partial_s J \cdot \partial_t u + J \cdot \partial_t(-J \cdot \partial_t u) \\ =& \Delta u + \partial_s J \cdot \partial_t u. \end{aligned}$$

Let $w = u - v$, then w satisfies

$$\Delta w = -\partial_s J \cdot \partial_t w. \tag{11}$$

Since the function in the right hand side is a bounded function, by the L^p estimate of the elliptic equation of second order, we know that $w \in W^{2,p}$ for any $1 < p < \infty$. Furthermore, we have

$$|\Delta w| \le C(|w| + |\partial_s w| + |\partial_t w|), \tag{12}$$

by the following Aronszajin theorem, $w \equiv 0$. □

Theorem 11.3.3 (Aronszajin)**.** *Let $\omega \subset \mathbb{C}$ be a connected open set. Suppose the function $w \in W^{2,2}_{loc}(\Omega, \mathbb{R}^m)$ satisfies the estimate (12)(almost everwhere) in Ω and that w vanishes to infinite order at some point $z_0 \in \Omega$. Then $w \equiv 0$.*

Remark 11.3.4. Darboux coordinate

Critical points and intersection points. Let $u : \Sigma \to M$ be a smooth map. A critical point $z \in \Sigma$ is the point such that $du(z) = 0$, and $u(z)$ is called the critical value of u. If M is a complex manifold and u is holomorphic, then we know that the critical points of u are isolated due to the uniqueness theorem of holomorphic functions of one variable. This result holds also for J-curve.

Proposition 11.3.5. *Let J be a C^1 almost complex structure on M, and u is a non-constant J-curve. Then the set of critical points of u is a finite set.*

Proof. We first assume that J is smooth. Since u is nonconstant, the ∞-jet at the critical point $0 \in \Sigma$ is nonzero by the unique continuation theorem. Then we have the expansion,

$$u(z) = O(|z|^l),\ u(z) \neq O(|z|^{l+1}),\ J(u(z)) = J(0) + O(|z|^l), \tag{13}$$

for some $l \in \mathbb{N}$. Let u_l be the expansion of u up to order l. Then u_l satisfies the Cauchy-Riemann equation,

$$\frac{\partial u_l}{\partial s} + J(0) \cdot \frac{\partial u_l}{\partial t} = 0. \tag{14}$$

Without loss of generality, we assume that $J(0) = J_0$ being the standard complex structure. Then u_l is a holomorphic function having the order l, and u can be written as

$$u(z) = a_l z^l + O(|z|^{l+1}),\ \partial_z u(z) = l a_l z^{l-1} + O(|z|^{l-1}),\ \text{for some } a_l \neq 0. \tag{15}$$

This proves that the critical points are isolated. J is C^1? □

Proposition 11.3.6. *Let J be a C^2 almost complex structure on M and $u_0 : \Sigma_0 \to M, u_1 : \Sigma_1 \to M$ be two J-curves such that $u_0(\Sigma_0) \neq u_1(\Sigma_1)$ and u_0 is non constant. Then the set $u_0(u_1(\Sigma_1))$ is at most countable and can accumulate only at the critical points of u_0.*

Proof. ?? □

Simple J-curves.

Definition 11.3.7. A J-curve $u : (\Sigma, j) \to (M, J)$ is called multiply covered if there exists another surface (Σ', j') and a holomorphic branched covering $\phi : \Sigma \to \Sigma'$ such that

$$u = u' \circ \phi,\ \deg \phi > 1. \tag{16}$$

The curve is called simple, if it is not multiply covered.

We shall see that the simple J-curve in a given homological class forms a smooth finite dimensional manifold for generic J. The proof is based on the fact that every simple J-curve is somewhere injective in the sense that

$$du(z) \neq 0,\ u^{-1}(u(z)) = \{z\}, \tag{17}$$

for some $z \in \Sigma$. Denote by $Z(u)$ the complementary set of the set of injective points of u. Then we have

Theorem 11.3.8. *Let J be a C^2 almost complex structure on M and $u : \Sigma \to M$ be simple J-curve. Then u is somewhere injective. Furthermore, the set $Z(u)$ is at most countable and can only accumulate at the critical points of u.*

Proof. We will show that any curve $u : \Sigma \to M$ can be expressed as a composition $u' \circ \phi : \Sigma \to M$, where $u : \Sigma' \to M$ is somewhere injective. If u is simple, then $\deg \phi = 1$. We construct Σ' from the image $u(\Sigma)$ in M.

Let X be the set of critical points of u in Σ and $X' = u(X)$ is the set of critical values. Because of the isolation of critical points, we know that both sets are of finite sets. Let Q be the set of points in $M \setminus X'$ such that different branches of $u(\Sigma)$ meet at those points. By the isolation of the intersection points of two J-curves, we know that Q is a finite set. We obtain an embedding map $\iota : S := u(\Sigma) \setminus (X' \cup Q) \to M$.

Assume that at the point $p \in Q$ there are r_p branches meeting. We can resolve them at p by adding one point in each branch. We can do this for all the points in Q such that we can get a surface S' with a unique complex structure j' and an immersion $\iota' : S' \to M$. This immersion is a (j', J)-holomorphic.

S' is not a compact surface, because their boundary consists of the critical values in X'. Each end of S' corresponds a distinct branch of $\mathrm{im}(u)$ through a point in X'. The punctured disc has the fixed conformal structure. Hence we can add one point in each end to form a closed Riemann surface Σ'. Since u is well-defined near such points, the map ι' must extend to a J-curve $u' : \Sigma' \to M$. This map u' is somewhere injective and $u = u' \circ \phi$ factors through u' by a holomorphic map $\phi : \Sigma \to \Sigma'$. Thus ϕ is a branched covering and has degree 1 if and only if u is somewhere injective. □

Corollary 11.3.9. *Let J be a C^2 almost complex structure on M. Assume that $u_j : \Sigma_j \to M$, $j = 0, 1$ are two simple J-curves such that $u_0(\Sigma) = u_1(\Sigma)$, then there exists a holomorphic diffeomorphism $\phi : \Sigma_0 \to \Sigma_1$ such that $u_1 = u_0 \circ \phi$.*

Proof. □

11.4. **Moduli space of J-curves.**

<u>*Energy and its critical points.*</u> In the rest part, we assume that M is a symplectic manifold with compatible almost complex structure $J \in \mathscr{J}(\omega)$. Let g_J be the metric determined by ω and J.

Let u be a J-holomorphic curve representing the homology class A. Then we have

$$\begin{aligned}\int_\Sigma u^*\omega &= \int_\Sigma \omega(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}) ds \wedge dt \\ &= \int_\Sigma \omega(\frac{\partial u}{\partial s}, J\frac{\partial u}{\partial s}) dsdt \\ &= \int_\Sigma g_J(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial s}) dsdt \\ &= \frac{1}{2}\int_\Sigma \left(|\frac{\partial u}{\partial s}|^2 + |\frac{\partial u}{\partial t}|^2 \right) \\ &= g_J - \text{area of } \mathrm{im}\, u.\end{aligned}$$

Thus the g_J area of a J-curve is determined entirely by the homology class it represents. Notice that the g_J-area is also the energy of u:

$$E(u) = \frac{1}{2}\int_\Sigma |du|^2.$$

Let h be the Riemannian metric on a surface Σ and g be the Riemannian metric on M, then

$$|du|^2 = \sum_{\alpha,\beta,i,j} h^{\alpha\beta} g_{ij} u^i_\alpha u^j_\beta, \tag{18}$$

and is called the energy density, sometimes denoted by $e(u)$. The critical point of $E(u)$ is the harmonic map determined by the Euler-Lagrangian equation:

$$\Delta_M u^i + \Gamma^i_{jk} u^j_\alpha u^k_\beta h^{\alpha\beta} = 0. \tag{19}$$

Moduli spaces. We define the working spaces:

- $C^\infty(\Sigma, M)$: the set of smooth maps $u : \Sigma \to M$. The tangent space at u is given by the Fréchet space $\Omega^0(u^*TM)$, i.e., the vector field over u.
- $\mathscr{X} = C^\infty(\Sigma, M; A)$: the subspace of $C^\infty(\Sigma, M)$ consisting of the smooth map u representing the homology class $A \in H_2(M, \mathbb{Z})$.
- $\pi : \mathscr{E} \to \mathscr{X}$: the infinite dimensional bundle whose fiber at u is $\Omega^{0,1}(u^*TM)$.
- $\mathscr{X}^{k,p}$: the completeness of $\mathscr{X}$ under the $W^{k,p}$ Sobolev norm.
- $\mathscr{E}^{k,p}$ is completeness of $\mathscr{E}$ under the $W^{k,p}$ norm.

We know that $C^\infty(\Sigma, M)$ is a Fréchet manifold and $\mathscr{E}$ is a Fréchet bundle. By Sobolev embedding theorem, if $kp > 2$, $\mathscr{X}^{k,p}$ is a Banach manifold and $\mathscr{E}^{k,p}$ is a Banach bundle.

The various moduli spaces are defined below:

- $\mathscr{M}(\Sigma, A, J) := \{u \in C^\infty(\Sigma, M) | \bar\partial_J u = 0,\ [u] = A\}$.
- $\mathscr{M}^*(\Sigma, A, J) := \{u \in \mathscr{M}(\Sigma, A, J) | u \text{is simple}\}$.
- $\mathscr{M}(A, J) := \mathscr{M}(\mathbb{CP}^1, A, J)$ and $\mathscr{M}^*(A, J) := \mathscr{M}^*(\mathbb{CP}^1, A, J)$.
- $\mathscr{M}^*(\Sigma, A, \mathscr{J}) = \{(u, J) | J \in \mathscr{J}(M, \omega), u \in \mathscr{M}^*(\Sigma, A, J)\}$: the universal moduli space of simple curves.

From the definition, we know that $\bar\partial_J$ is a section of the Fréchet bundle $\mathscr{E} \to \mathscr{X}$ (or a section of the Banach bundle $\mathscr{E}^{k-1,p} \to \mathscr{X}^{k,p}$). The zero locus of $\bar\partial_J$ is just

$$\bar\partial_J^{-1}(0) = \mathscr{M}(A, J) \tag{20}$$

We know that the zero locus of a section of a finite dimensional oriented bundle is a cycle and represents the Euler class of this bundle. The main idea of the moduli theory in infinite dimensional case is to make sense the zero locus of a section given by partial differential operators.

If $\bar\partial_J$ is transversal to the zero section, then $\mathscr{M}(A, J)$ is a smooth manifold. Since the differential is

$$d\bar\partial_J(u) : T_u\mathscr{X} \to T_{(u,0)}\mathscr{E} = T_u\mathscr{X} \oplus \mathscr{E}_u.$$

Hence $\bar\partial_J$ is transversal to the zero section if and only if its differential is onto map, and this is equivalent to the requirement that the following linearized operator

$$D_u := D\bar\partial_J(u) = \pi_u \circ d\bar\partial_J(u) : T_u\mathscr{X} \to \mathscr{E}_u$$

is surjective for every $u \in \mathscr{X}$, where $\pi_u : T_u\mathscr{X} \oplus \mathscr{E}_u \to \mathscr{E}_u$ is the projection.

The surjectivity of D_u depends on the almost complex structure J. In general, D_u is not transversal for any (u, J). We want to show that for generic J, and any $u \in \mathscr{M}^*(A, J)$, D_u is onto. We give a definition of such (u, J):

Definition 11.4.1. A J-curve $u : (\Sigma, j) \to (M, J)$ is called regular, if $D_u\bar\partial_J(u)$ is onto.

Regular Criteria. Grothendieck proved that any holomorphic vector bundle E over $\mathbb{CP}^1$ can be split to the direct sum of holomorphic line bundles:

$$E = L_1 \oplus \cdots \oplus L_n.$$

We identify the Chern class $c_1(E)$ with the Chern number $c_1(E) \cdot [\mathbb{CP}^1]$. If u is a J-curve, then we have the above decomposition by replacing $E = u^*TM$. So

$$c_1(u^*TM) = \sum_i c_1(L_i). \tag{21}$$

Lemma 11.4.2. *Assume that J is integrable and u is a J-curve. If each $c_1(L_i) \geq -1$ in (21). Then D_u is onto.*

Proof. If J is integrable, then D_u is complex linear and is exactly the Cauchy-Riemann operator $\bar{\partial}$. The action of $\bar{\partial}$ is decomposed into the direct sum action of $\bar{\partial}$ restricted to each line bundle L_i,

$$\bar{\partial} : \Omega^0(\mathbb{CP}^1, L_i) \to \Omega^{0,1}(\mathbb{CP}^1, L_i)$$

. To prove $\operatorname{coker} \bar{\partial} = \{0\}$, it suffices to prove that

$$H^{0,1}_{\bar{\partial}}(\mathbb{CP}^1, L) = 0. \tag{22}$$

By Dolbeault isomorphism and Serre duality, we have

$$H^{0,1}_{\bar{\partial}}(\mathbb{CP}^1, L) \simeq H^1(\mathbb{CP}^1, \mathscr{O}(L)) \simeq (H^0(\mathbb{CP}^1, K \otimes \mathscr{O}(L^*)))^\vee = \{0\}. \tag{23}$$

The last equality is due to the fact that

$$\deg(K \otimes \mathscr{O}(L^*)) = -2 - c_1(L) \leq -1. \tag{24}$$

$\square$

Proposition 11.4.3. *Let J be an almost complex structure on a 4-manifold and $u : \mathbb{CP}^1 \to M$ be an immersed J-sphere. Then D_u is onto if and only if $c_1(u^*TM) \geq -1$.*

Proof. Since u is a immersed cure, then du is non-degenerate and the tangent space u^*TM has the decomposition,

$$u^*TM = L_0 \oplus L_1, \tag{25}$$

where $L_0 = \operatorname{im} du$ and L_1 is the orthogonal complement of L_0 with respect to any Hermition metric on u^*TM. Note that for any vector field ζ and every J-curve u, there is

$$D_u(du \circ \zeta) = du \circ \bar{\partial}_J \zeta. \tag{26}$$

Hence D_u preserves L_0 and can be decomposed as the matrix form:

$$D_u = \begin{pmatrix} \bar{\partial} & 0 \\ N & \bar{\partial} \end{pmatrix} \tag{27}$$

, where N_0 is a zero order operator.

We have

$$c_1(L_0) = 2, \; c_1(L_1) = c_1(u^*TM) - 2 \geq -1. \tag{28}$$

Hence by Lemma 11.4.2, D_u is surjective. $\square$

Proposition 11.4.4. *Let V be the product space of the sphere $\mathbb{CP}^1$ with a symplectic manifold (M, ω). Let $A = [\mathbb{CP}^1] \times \{pt\} \in H_2(V, \mathbb{Z})$. Then for any $J \in \mathscr{J}(M, \omega)$, the product almost complex structure $J_V = j \times J$ is regular for A.*

Proof. The equation of the J_V-curve is decoupled into two independent equations with the target manifold $\mathbb{CP}^1$ and M respectively. So the A curve u has the form:

$$u(z) = (\phi(z), x_0), \tag{29}$$

where $\phi(z)$ is the fractional linear transformation on $\mathbb{CP}^1$. Hence the pull-back bundle by u is the direct sum of a line bundle L_0 and a trivial bundle L_1. Obviously, we have $c_1(L_0) = 2$ and $c_1(L_1) = 0$ both of which satisfy the hypothesis of Lemma 11.4.2. □

Example 11.4.5. Consider $M = S^2 \times S^2$ with the standard complex structure $J_0 = j \times j$, then any J_0 rational curve has the form $u(z) = (v_1(z), v_2(z))$, where $v_i(z) : S^2 \to S^2$ are holomorphic map with degree $d_i \geq 0$. Thus

$$u^*TM = v_1^*TS^2 \oplus v_2^*TS^2, \tag{30}$$

and

$$c_1(v_i^*TS^2) = 2d_i \geq 0.$$

Hence any J_0 rantional curve u is regular.

However, there exists an almost complex structure J on $S^2 \times S^2$ and a J-curve u such that (u, J) is not regular. On the other hand, we can blow up $\mathbb{CP}^1$. Consider the blow-up one point manifold $\mathbb{CP}^1\#\overline{\mathbb{CP}^1}$. The exceptional divisor C has self-intersection number $C \cdot C = -1$. Then D_C is still regular by Lemma 11.4.2. We can blow up one point on C of $\mathbb{CP}^1\#\overline{\mathbb{CP}^1}$ to get a new manifold M. The new exceptional divisor C' has intersection number $C' \cdot C' = -2$ and then $D_{C'}$ is not surjective.

For higher genus curve u, D_u is in general non regular. Assume that J is integrable and u is an embedding J-curve such that $u(\Sigma)$ is a summand of the bundle T_uM. The Cauchy-Riemann operator $\bar\partial$ when restricted to this line bundle may have nonzero cokernel, because $H^{0,1}_{\bar\partial}(\Sigma, T\Sigma)$ does not vanish.

11.5. **Transversality.**

Now we fix a genus g Rieman surface Σ with Riemannian metric h. The conformal class of h gives the complex structure j. Fix a homology class $A \in H_2(M, \mathbb{Z})$.

Definition 11.5.1. An almost complex structure J on M is called regular (for A and Σ) if D_u is onto for any $u \in \mathscr{M}^*(\Sigma, A, J)$. Denote by $\mathscr{J}_{reg}$ the set of all J that are regular for A and Σ.

Theorem 11.5.2. *The following conclusions hold.*

(1) *For any $J \in \mathscr{J}_{reg}$, the moduli space $\mathscr{M}^*(\Sigma, A, J)$ is a smooth manifold of dimension*
$$\dim \mathscr{M}^*(\Sigma, A, J) = 2c_1(A) + 2n(1-g), \tag{31}$$
and it carries a natural orientation.

(2) *The set $\mathscr{J}_{reg}$ is of the second category in $\mathscr{J}$, i.e., this means that it is the intersection of countably many open and dense subsets of $\mathscr{J}$.*

This theorem is based on the description of the universal moduli space as below.

Proposition 11.5.3. *For a fixed $A \in H_2(M, \mathbb{Z})$, $l \in \mathbb{N}, l \geq 2, p > 2$ and $k \in \{1, \cdots, l\}$. The universal moduli space $\mathscr{M}^*(\Sigma, A, \mathscr{J}^l)$ is a separable C^{l-k} Banach submanifold of $\mathscr{X}^{k,p} \times \mathscr{J}^l$. Here $\mathscr{J}^l$ represents the set of C^l-differentiable almost complex structures on M.*

Proof. Consider the Banach bundle

$$\mathscr{E}^{k-1,p} \to \mathscr{X}^{k,p} \times \mathscr{J}^l, \tag{32}$$

whose fiber at (u, J) is the linear space

$$\mathscr{E}^{k-1,p}_{(u,J)} = W^{k-1,p}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM).$$

This bundle is a C^{l-k} Banach bundle.

Define a section of the Banach bundle $\mathscr{E}^{k-1,p} \to \mathscr{X}^{k,p} \times \mathscr{J}^l$:

$$F(u, J) = \bar{\partial}_J u. \tag{33}$$

We shall show that the vertical differential

$$DF(u, J) : W^{k,p}(\Sigma, u^*TM) \times C^l(M, \mathrm{End}(TM, J, \omega)) \to W^{k-1,p}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM) \tag{34}$$

is surjective wherever u is simple and $F(u, J) = 0$. It is easy to see that

$$DF(u, J)(\zeta, Y) = D_u\zeta + \frac{1}{2}Y(u) \circ du \circ j. \tag{35}$$

Since D_u is Fredholm the operator $DF(u, J)$ is also Fredholm when plus a compact perturbation, hence has closed Range. So it suffices to prove that its image is dense whenever $\bar{\partial}_J u = 0$.

We assume that $k = 1$ first. If the image is not dense, then by the Hahn-Banach theorem, there exists a nonzero section $\eta \in L^q(\Lambda^{0,1} \otimes_J u^*TM)$ where $1/p + 1/q = 1$ such that

$$\int_\Sigma (\eta, D_u\xi) dv_\Sigma = 0, \tag{36}$$

for every $\xi \in W^{1,p}(\Sigma, u^*TM)$ and

$$\int_\Sigma (\eta, Y(u) \circ du \circ j) dv_\Sigma = 0, \tag{37}$$

for every $Y \in C^l(M, \mathrm{End}(TM, J, \omega))$. Hence η is a weak solution of $D_u^*\eta = 0$ and by elliptic regularity we have $\eta \in W^{1,q}(\Sigma, u^*TM)$.

Since u is simple, the set of injective points of u is open and dense in Σ. Let $z_0 \in \Sigma$ is such an injective point, i.e.,

$$du(z_0) \neq 0, u^{-1}(u(z_0)) = \{z_0\}.$$

We will show that η must vanish at a small neighborhood of z_0. If it is not the case, since $du(z_0) \neq 0$, then we can choose an endomorphism $Y_0 \in \mathrm{End}(T_{u(z_0)}M, J_{u(z_0)}, \omega_{u(z_0)})$ such that

$$(\eta(z_0), Y_0 \circ du(z_0) \circ j(z_0)) > 0. \tag{38}$$

We can choose a Y with compact support in a small neighborhood of $u(z_0)$ with $Y(u(z_0)) = Y_0$ such that

$$\int_\Sigma (\eta, Y(u) \circ du \circ j) dv_\Sigma > 0, \tag{39}$$

which contradicts with (37). □

Hence η must vanish in a small neighborhood of z_0. By unique continuation theorem, we know that $\eta \equiv 0$.

To prove surjectivity for general k. Let $\eta \in W^{k-1,q}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM)$ be given. Then by the $k = 1$ surjectivity, there exists a pair

$$(\xi, Y) \in W^{1,p}(\Sigma, u^*TM) \times C^l(M, \mathrm{End}(TM, J, \omega))$$

such that

$$DF(u,J)(\xi,Y) = \eta.$$

This is equivalent to

$$D_u\xi = \eta - \frac{1}{2}Y(u)\circ du\circ j \in W^{k-1,p}.$$

Now by elliptic regularity, $\xi \in W^{k,q}$. Hence $DF(u,J)$ is onto for every pair $(u,J) \in \mathscr{M}^*(\Sigma, A, \mathscr{J}^l)$. Since $DF(u,J)$ is Fredholm, there exists a right inverse of $DF(u,J)$. Applying the implicit functional theorem, we know that $\mathscr{M}^*(\Sigma, A, \mathscr{J}^l)$ is a C^{k-l}-Banach submanifold of $\mathscr{X}^{k,p}\times\mathscr{J}^l$. Since $\mathscr{X}^{k,p}\times\mathscr{J}^l$ is separable, so is the submanifold $\mathscr{M}^*(\Sigma, A, \mathscr{J}^l)$.

Proof of Theorem 11.5.2. We divide the proof in some procedures.

<u>*Chart construction*</u>. Let $J \in \mathscr{J}_{reg}(\Sigma, A)$ and u be a J-curve. Then u is smooth and D_u is onto by definition of J. We have the map

$$F_u : W^{k,p}(\Sigma, u^*TM) \to W^{k-1,p}(\Sigma, \Lambda^{0,1}\otimes_J u^*TM)$$

defined by

$$F_u = P_u^{-1}(\xi)\circ \bar\partial_J(\exp_u(\xi)). \tag{40}$$

Remember that P_u is the parallel transition from u to $\exp_u(\xi)$ and we proved that $dF_u(0) = D_u$. Since D_u is surjective, by transversality theorem between Banach spaces, the local zero locus $F_u^{-1}(0)$ at 0 is a submanifold of $W^{k,p}(\Sigma, u^*TM)$ with dimension $2c_1(A) + 2n(1-g)$. The image of such submanifold under $\exp_u(\xi)$ is a submanifold of $W^{k,p}(\Sigma, M)$ and provides a chart near $u \in \mathscr{M}^*(\Sigma, A, J)$. This chart is independent of parameters $k \geq 1, p > 1$, since all nearby J-curves are smooth.

<u>*Orientation*</u>. The tangent space of $\mathscr{M}^*(\Sigma, A, J)$ at u is $T_u\mathscr{M}^*(\Sigma, A, J)$ which is $\ker D_u$. D_u can be expressed by the sum

$$D_u\xi = (\hat\nabla\xi)^{0,1} + T(\xi, \partial_J(u)). \tag{41}$$

This first term is a first order operator commuting with the almost complex structure J and the second term is a compact operator of zero order. The determinant is

$$\det(D_u) = \Lambda^{\max}(\ker D_u)\otimes\Lambda^{\max}(\operatorname{coker} D_u). \tag{42}$$

Now by continuity of Fredholm operator, the determinant bundle $\det(D_u)$ is isomorphic to $\det((\hat\nabla\xi)^{0,1})$. But the later is orientable since the operator $(\hat\nabla\xi)^{0,1}$ is complex linear (commuting with J). This argument shows that $\mathscr{M}^*(\Sigma, A, J)$ is orientable.

<u>*The set of $\mathscr{J}_{reg}(\Sigma, A)$*</u>. We want to show that $\mathscr{J}_{reg}(\Sigma, A)$ is of the second category of $\mathscr{J}$.

Consider the projection

$$\pi : \mathscr{M}^*(\Sigma, A, \mathscr{J}^l) \to \mathscr{J}^l \tag{43}$$

π is a C^{l-1} map between two Banach manifolds. The tangent space is

$$T_{(u,J)}\mathscr{M}^*(\Sigma, A, \mathscr{J}^l) \subset W^{1,p}(\Sigma, u^*TM)\times C^l(M, \operatorname{End}(TM, J, \omega)) \tag{44}$$

and consists of all pairing (ξ, Y) such that

$$D_u\xi + \frac{1}{2}Y(u)\circ du\circ j = 0. \tag{45}$$

The linearized operator of π is

$$d\pi(u,J) : T_{(u,J)}\mathscr{M}^*(\Sigma, A, J) \to T_J\mathscr{J}^l \tag{46}$$

given by $(\xi, Y) \to Y$. Therefore

$$\ker d\pi(u,J) = \ker D_u$$

. On the other hand, by considering the adjoint operator of $d\pi(u, J)$ and using the same argument as in Proposition 11.5.3, we know that

$$\operatorname{coker} d\pi(u, J) = \operatorname{coker} D_u.$$

So π is a Fredholm operator and $d\pi(u, J)$ is onto if and only if D_u is onto. Hence the regular value J of π is an almost complex structure such that D_u is onto for every $u \in \mathscr{M}^*(\Sigma, A, J)$. This means that $\mathscr{J}^l_{reg}(\Sigma, A)$ is just the set of regular values of π. The projection π is of C^{l-1}, by Sard-Smale theorem, if

$$l - 2 \geq \operatorname{index} \pi = \operatorname{index} D_u = 2c_1(A) + 2n(1 - g),$$

the set of regular values $\mathscr{J}^l_{reg}(\Sigma, A)$ is of the second category in $\mathscr{J}^l$. We can improve the smoothness to get a second category set $\mathscr{J}_{reg}$ in $\mathscr{J}$ by Taubes' technique (see[?]).

□

11.6. **Cobordism.**

In this part, we will discuss the dependence of the moduli space $\mathscr{M}^*(\Sigma, A, J)$ on the choice of $J \in \mathscr{J}_{reg}(\Sigma, A)$. A smooth path $\lambda \in [0, 1] \to J_\lambda$ of the almost complex structures in $\mathscr{J}$ is called a (smooth) homotopy between J_0 and J_1. Denote by $\mathscr{J}(J_0, J_1)$ the space of all smooth homotopies between J_0 and J_1. In general, even if $\mathscr{J}$ is path-connected, there does not exist a homotopy such that $J_\lambda \in \mathscr{J}_{reg}(\Sigma, A)$ for every λ.

For each homotopy J_λ, we can define the moduli space:

$$\mathscr{W}^*(\Sigma, A, J_\lambda) := \{(\lambda, u) | \lambda \in [0, 1], u \in \mathscr{M}^*(\Sigma, A, J_\lambda)\}.$$

Definition 11.6.1. Let $J_0, J_1 \in \mathscr{J}_{reg}(\Sigma, A)$. A homotopy J_λ from J_0 to J_1 is called regular(for fixed Σ, A) , if

$$\Omega^{0,1}(\Sigma, u^*TM) = \operatorname{im} D_{J_\lambda, u} + \mathbb{R}v_\lambda,$$

for every $(\lambda, u) \in \mathscr{W}^*(\Sigma, A, J_\lambda)$ where $v_\lambda := (\partial_\lambda J_\lambda) du \circ j_\Sigma$ is the tangent vector to the path $\lambda \to J_\lambda$. The space of regular homotopies will be denoted by $\mathscr{J}_{reg}(\Sigma, A, J_0, J_1)$.

From the discussion of the transversality, we know that a homotopy J_λ is regular if and only if the projection

$$\pi : \mathscr{M}^*(\Sigma, A, \mathscr{J}^l) \to \mathscr{J}^l$$

is transversal to the path J_λ in $\mathscr{J}^l$. Hence using the same argument as in the transversality discussion, we can obtain the following cobordant theorem under the hypothesis there is no compactness loss phenomenon. We will discuss the compactness in the next part.

We fix the Riemannian surface (Σ, h) with the conformal structure $j = [h]$ and fix $A \in H_2(M, \mathbb{Z})$.

Theorem 11.6.2. *Let $J_0, J_1 \in \mathscr{J}_{reg}(\Sigma, A)$ and J_λ is a homotopy connecting J_0 and J_1. Then we have the following conclusions:*

(1) *If $J_\lambda \in \mathscr{J}_{reg}(\Sigma, A, J_0, J_1)$, then $\mathscr{W}^*(\Sigma, A, J_\lambda)$ is a smooth oriented manifold with boundary*

$$\partial \mathscr{W}^*(\Sigma, A, J_\lambda) = \mathscr{M}^*(\Sigma, A, J_0) \cup \mathscr{M}^*(\Sigma, A, J_1).$$

The boundary orientation agrees with the orientation of $\mathscr{M}^*(\Sigma, A, J_1)$ and is opposite to the orientation of $\mathscr{M}^*(\Sigma, A, J_0)$.

(2) *The set $\mathscr{J}_{reg}(\Sigma, A, J_0, J_1)$ is of the second category in the space of all smooth homotopies in $\mathscr{J}$ from J_0 to J_1.*

Remark 11.6.3. To get the above cobordism results, one needs to consider the compactness problem, one must show there is not blow-up phenomenon in the limit course of a family of J_{λ_i}-curves $u_{\lambda_i,n}$.

11.7. **Compactness.**

From this part, we always assume that $\Sigma = \mathbb{CP}^1$ and with the unique complex structure j.

Because of the action of the noncompact group $G = PGL(2, \mathbb{C})$ on the moduli space $\mathscr{M}(A, J)$, this moduli space can never be compact. For each $u \in \mathscr{M}(A, J)$, we know that

$$E(u) = \frac{1}{2}\int_{\mathbb{C}P^1} |du|^2 = \omega(A).$$

This shows that the $W^{1,2}$-norms of all the J-curves in $\mathscr{M}(A, J)$ are equal and bounded. Since the domain of J-curve is 2-dimensional, the energy is conformal invariant. This case is on the borderline of Sobolev embedding theorem. This means that if $u \in W^{1,p}$ for $p > 1$, then by the sobolev embedding theory and the regularity argument any sequence of curves u_ν has uniform C^k norm and will converge to a smooth J-curve.

However for a sequence of J-curves, the uniform bound of $W^{1,2}$-norm can't give the uniform L^∞-norm estimate. The loss of compactness phenomenon arises. The reason is that though the energy is uniformly bounded, the energy density of a subsequence u_ν may tends to infinity at one point. The usual convergence loses its meaning. But under Gromov-Hausdorff convergence, the sequence will converge to a stable curve (cusp-curve). This is the content of Gromov's compactness theorem.

In this part, we only care about the compactness theorem related to the proof of Gromov's nonsqueezing theorem. Since the J-curve u is a harmonic map, we first discuss some analytic properties of harmonic maps.

11.7.1. <u>*Harmonic maps.*</u> Let (Σ, h) be a Riemann surface with metric h and M be a Riemannian manifold with metric g. A harmonic map $u : \Sigma \to M$ is the critical point of the energy functional $E(u)$ and satisfies the Euler-Lagrangian equation which has the local form:

$$\Delta_M u^i + h^{\alpha\beta}\Gamma^i_{jk}\frac{\partial u^j}{\partial s_\alpha}\frac{\partial u^k}{\partial s_\beta} = 0, \tag{47}$$

where $s_\alpha, \alpha = 1, 2$ are the local coordinates of Σ.

Theorem 11.7.1 (Convergence)**.** *Let u_ν be a sequence of harmonic maps from a domain $\Omega \subset \Sigma$ to M. Assume that for any compact subset $K \subset \Omega$, there is*

$$\sup_\nu \|du_\nu\|_{L^\infty(K)} < \infty, \tag{48}$$

Then for any compact subset $K \subset \Omega$ and any $k \in \mathbb{N}$, there is a subsequence of u_ν such that it converges in $C^k(K)$ topology to a harmonic map u.

Proof. It is a conclusion of L^p estimate for elliptic equations and the Sobolev compact embedding theorem. □

Theorem 11.7.2 (Removal of singularity)**.** *Let $u : B_1(0) \setminus \{0\} \to M$ be a smooth harmonic map with energy $E(u, B_1) < \infty$, then u can be extended smoothly to $B_1(0)$ and become a smooth harmonic map defined on $B_1(0)$.*

11.7.2. Blow-up analysis for holomorphic curves.

Theorem 11.7.3. *Assume that there is no spherical homology class $B \in H_2(M)$ such that $0 < \omega(B) < \omega(A)$. Then the moduli space $\mathscr{M}(A, J)/G$ is compact.*

Proof. Let $u_\nu : \mathbb{C} \to M$ be a sequence of J-curves which represents the class A. We must show that there exist $\phi_\nu \in PSL(2, \mathbb{C})$ such that $u_\nu \circ \phi_\nu$ converges in $\mathbb{C} \cup \{\infty\}$.

It is easy to show that $|du_\nu(z)|$ attains its maximum at some point $a_\nu \in \mathbb{C}$. Denote

$$c_\nu = |du_\nu(a_\nu)| = \|du_\nu\|_{L^\infty}.$$

We can define a reparametrized curve $v_\nu : \mathbb{C} \to M$:

$$v_\nu(z) = u_\nu(a_\nu + c_\nu^{-1} z).$$

Then the curves satisfy

$$|dv_\nu(0)| = 1, \ \|dv_\nu\|_{L^\infty} \leq 1, \ E(v_\nu) = E(u_\nu) = \omega(A).$$

By Theorem 11.7.1, there exists a subsequence, still denoted by v_ν, which converges uniformly with all derivatives on compact sets. The limit function $v : \mathbb{C} \to M$ is still a J-curve such that

$$dv(0) = 1, \ 0 < E(v) = \int_{\mathbb{C}} v^* \omega \leq \omega(A).$$

Now by Theorem 11.7.2, the map $\mathbb{C} - \{0\} \to M : z \to v(1/z)$ is a J-curve.

In order to prove the $\mathscr{M}(A, J)/G$ is compact, we must prove that the functions $v'_\nu(z) := v_\nu(1/z)$ converge to $v(1/z)$ uniformly on compact neighborhood of 0.

Assume that this is not true. Then

$$v'_\nu = \|dv'_\nu\|_{L^\infty} \to \infty$$

for some subsequence. Let $c'_\nu \in \mathbb{C}$ be the maximum point of $|dv'_\nu(z)|$. Since $v'_\nu(z)$ converges to $v(1/z)$ uniformly away from 0, we must have $a'_\nu \to 0$. Define the reparametrized curve

$$w_\nu(z) = v'_z(a'_\nu + (c'_\nu)^{-1} z).$$

These maps again satisfy

$$dw_\nu(0) = 1, \|dw_\nu\|_{L^\infty} \leq 1, \ E(w_\nu) = \omega(A).$$

Hence passing to a subsequence, we may assume that w_ν converges uniformly in any C^k-topology on a compact set to a J-curve $w : \mathbb{C} \to M$ such that

$$|dw(0)| = 1, \ 0 < E(w) = \int_{\mathbb{C}} w^* \omega \leq \omega(A).$$

By the removable singularity theorem, $w(1/z)$ extends to a smooth J-curve at 0.

Let B and C be the homology classes represented by v and w. Then $\omega(B)$ and $\omega(C)$ are positive, and we want to prove that

$$\omega(B) + \omega(C) \leq \omega(A).$$

Denote

$$E(w, \Omega) = \int_{\Omega} w^* \omega.$$

Then $\forall \epsilon > 0$, we have

$$\begin{aligned}\omega(C) &= \lim_{R\to\infty} E(w, B_R) = \lim_{R\to\infty}\lim_{\nu\to\infty} E(w_\nu, B_R)\\ &= \lim_{R\to\infty}\lim_{\nu\to\infty} E(v'_\nu, B_{R(c'_\nu)^{-1}}(a'_\nu))\\ &\leq \lim_{R\to\infty}\lim_{\nu\to\infty} E(v'_\nu, B_\epsilon) = \lim_{\nu\to\infty}(E(v_\nu) - E(v_\nu, B_{1/\epsilon}))\\ &= \omega(A) - E(v, B_{1/\epsilon})\end{aligned}$$

Take the limit $\epsilon \to 0$, we have the inequality

$$\omega(C) \leq \omega(A) - \omega(B).$$

This inequality contradicts with our hypothesis, hence this shows that $\mathscr{M}(A, J)/G$ is compact. □

Corollary 11.7.4. *Assume that there is no spherical homology class $B \in H_2(M)$ such that $0 < \omega(B) < \omega(A)$. Assume that J_ν is a sequence of almost complex structure which converges in C^∞-topology to J. Let $u_\nu : \mathbb{C} \cup \{\infty\} \to M$ be a sequence of J_ν holomorphic A-spheres. Then there exist matrices $A_\nu \in SL(2, \mathbb{C})$ such that $u_\nu \circ \phi_{A_\nu}$ has a convergent subsequence.*

We say that the class is indecomposable if it does not decompose as a sum $A = A_1 + \cdots + A_k$ of classes which are spherical and satisfy $\omega(A_i) > 0$ for all i.

Theorem 11.7.5. *If A is indecomposable then the moduli space $\mathscr{M}(A, J)/G$ of unparametrized J-holomorphic A-spheres is compact for all ω-compatible J.*

The conclusion of this theorem comes from the Gromov's compactness theorem.

To get the information of the symplectic manifold, we define the evaluation map:

$$ev : \mathscr{M}(A, J) \times_G S^2 \to M : ev(u, z) = u(z).$$

Here $\mathscr{M}(A, J) \times_G S^2$ is the quotient of the product $\mathscr{M}(A, J) \times S^2$ by the diagonal action of G:

$$\phi^*(u, z) = (u \circ \phi, \phi^{-1}(z)).$$

Theorem 11.7.6. *Suppose that $\mathscr{M}(A, J)/G$ is compact for $J \in \mathscr{J}_{reg}$. Then the map $ev : \mathscr{M}(A, J) \times_G S^2 \to M$ represents a cycle of dimension $2n + 2c_1(A) - 4$ in M and the homology class is independent of $J_{reg}(A)$. If $\mathscr{M}(A, J)/G$ is empty for any $J \in \mathscr{J}(M, \omega)$ then the homology class is zero.*

Proof. The first two conclusions comes from Theorem 11.5.2, 11.6.2 and the later is obvious. □

11.8. Applications.

Suppose that $M = \mathbb{C}P^1 \times V$ be a product of symplectic manifolds and let $A = [\mathbb{C}P^1 \times pt]$. Suppose that $\pi_2(V) = 0$, then A generates the group of spherical 2-classes in M. So $\omega(A)$ is the smallest value that ω takes on the set of spherical classes. So by Theorem 11.7.3, the moduli space $\mathscr{M}(A, J)/G$ is compact and by Theorem 11.7.6, the evaluation map

$$ev_J : \mathscr{M}(A, J) \times_G \mathbb{C}P^1 \to M$$

defines a homology class on M. Notice that

$$\dim \mathscr{M}(A, J) \times_G \mathbb{C}P^1 = 2n + 2c_1(A) - 4 = 2n = \dim M.$$

Therefore, the homology class is uniquely determined by the degree of the evaluation map. For any two almost complex structure $J_0, J_1 \in \mathscr{J}_{reg}(A)$, since the corresponding moduli spaces are cobordant, hence the degree of the two maps ev_{J_0} and ev_{J_1} are the same. By Proposition 11.4.4, the product almost complex structures J of $\mathbb{C}P^1$ and V is regular. Then it is easy to see that the elements of $\mathscr{M}(A, J)$ have the form

$$u(z) = (\phi(z), v_0),$$

where $\phi \in G$ and $v_0 \in V$. Therefore the map ev_J has degree 1.

Use the above facts, we can now prove Gromov's nonsqueezing theorem by using the technique from J-holomorphic curves.

Theorem 11.8.1 (Gromov)**.** *If Φ is a symplectic embedding of the ball $B^{2n}(r)$ of radius r in $(\mathbb{R}^{2n}, \omega_0)$ into the cylinder $Z^{2n}(R) = B^2(R) \times \mathbb{R}^{2n-2}$, then $r \leq R$.*

Proof. Since $\phi : B^{2n}(r) \to Z^{2n}(R)$ is a symplectic embedding. Its image lies in a compact set $B^2(R)\times K$ which can be viewed as a subset in $B^2(R)\times T^{2n-2}$ with the product symplectic form $\Omega = \omega_0 \times \kappa\omega_1$, where ω_0 and ω_1 are the standard symplectic structures on $\mathbb{R}^2$ and T^{2n-2} and κ are chosen large enough. We can embed $B^2(R)$ into a 2-sphere of area $\pi R^2 + \epsilon$. Note that $\pi_2(T^{2n-2}) = 0$ and we have the standard almost complex structure $J = j \times J_T$ where j is the unique complex structure on $\mathbb{CP}^1$ and J_T is the standard complex structure on T^{2n-2}.

Let $A = [S^2 \times pt]$, then according to the above argument, the moduli space $\mathscr{M}(A, J)/G$ is compact and the evaluation map ev_J has degree 1. This shows that for any point in $\mathbb{C}P^1 \times T^{2n-2}$ there exists at least one J-curve going through it. Let C be such a curve through $\phi(0)$ and let S be the component of the inverse image $\phi^{-1}(C)$ that goes through the origin. We can choose J such that when restricted to the image of ϕ it equals the push-forward $\phi_*(J_0)$. Hence S is holomorphic in the standard complex structure J_0. Now S is a g_0 minimal surface (where g_0 is the usual metric in $\mathbb{R}^{2n}$). But it is well-known that the proper surface of smallest area through the center of the ball of radius r is a flat disc with area πr^2. Hence

$$\pi r^2 \leq g_0 - \text{area of } S = \int_S \omega_0 = \int_{\phi(S)} \Omega < \int_C \Omega = \int_{S^2\times pt} \Omega = \pi R^2 + \epsilon.$$

Since this holds for any $\epsilon > 0$, we must have $R \geq r$. This finishes the proof of the nonsqueezing theorem. $\square$