MULTIPLICATIVE STRUCTURE OF INTEGER COHOMOLOGY OF HILBERT SCHEMES OF POINTS ON K3 SURFACES

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Abstract. We study the cokernels of the cup product in integer cohomology of the Hilbert scheme of n points on a K3 surface. The main difference to cohomology with rational coefficients is the presence of torsion, depending on n. We particularly discuss the case $n = 3$.

1. Preliminaries

Let S be a K3 surface. We fix integral bases 1 of $H^0(S,\mathbb{Z})$, x of $H^4(X,\mathbb{Z})$ and $\alpha_1,\ldots,\alpha_{22}$ of $H^2(S,\mathbb{Z})$ such that the cup product pairing on $H^2(X,\mathbb{Z})$, written as a symmetric matrix with respect to this basis, looks like

$$\begin{pmatrix} U & & & & \\ & U & & & \\ & & U & & \\ & & & E & \\ & & & & E \end{pmatrix},$$

where U stands for the intersection matrix of the hyperbolic lattice and E stands for the negative matrix of the E_8 lattice, *i.e.*

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \end{pmatrix}.$$

We denote by $S^{[n]}$ the Hilbert scheme of n points on S. An integral basis for $H^*(S^{[n]},\mathbb{Z})$ in terms of Nakajima's operators was given by Qin–Wang:

Theorem 1.1. [39, Thm. 5.4.] *The classes*

$$\frac{1}{z_\lambda}\mathfrak{q}_\lambda(1)\mathfrak{q}_\mu(x)\mathfrak{m}_{\nu^1,\alpha_1}\ldots\mathfrak{m}_{\nu^{22},\alpha_{22}}|0\rangle, \quad |\lambda|+|\mu|+\sum_{i=1}^{22}|\nu^i| = n$$

form an integral basis for $H^*(S^{[n]},\mathbb{Z})$. Here, λ, μ, ν^i are partitions, $|\cdot|$ means the weight of a partition and $z_\lambda := \prod_i i^{m_i}m_i!$ if $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$. The symbol $\mathfrak{q}$ stands for Nakajima's creation operator. The relation of $\mathfrak{m}_{\nu,\alpha}$ to $\mathfrak{q}_{\tilde{\nu}}(\alpha)$ is the same

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as the monomial symmetric functions m_ν to the power sum symmetric functions $p_{\tilde{\nu}}$.

Using the algebraic model for computing cup-products in rational cohomology described in [24],

2. Computational results

The ring structure of $H^*(S^{[n]}, \mathbb{Q})$ has been studied in [24]. Since $H^{\mathrm{odd}}(S^{[n]}, \mathbb{Z}) = 0$ and $H^{\mathrm{even}}(S^{[n]}, \mathbb{Z})$ is torsion-free by [27], we can also apply these results to $H^*(S^{[n]}, \mathbb{Z})$. A basis for cohomology with integer coefficients was given by Qin–Wang in [39] which allows us to compute explicitly the image of $H^k(S^{[n]}, \mathbb{Z}) \cup H^l(S^{[n]}, \mathbb{Z})$ in $H^{k+l}(S^{[n]}, \mathbb{Z})$. We obtain:

Proposition 2.1. *If X is deformation equivalent to $S^{[3]}$, then:*

$$\frac{H^4(X,\mathbb{Z})}{\mathrm{Sym}^2 H^2(X,\mathbb{Z})} \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 23}$$

The torsion part of the quotient is generated by the integral class $\frac{1}{3}\mathfrak{q}_{(3)}(1)|0\rangle$.

$$\frac{H^6(X,\mathbb{Z})}{H^2(X,\mathbb{Z}) \cup H^4(X,\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12}$$

This quotient is generated by the 12 integral classes $\mathfrak{m}_{(1^3),\alpha_i}|0\rangle$, where $i \in \{1, 2, 3, 4, 5, 6, 8, 9, 11, 16, 17, 19\}$.

$$\frac{H^6(X,\mathbb{Z})}{\mathrm{Sym}^3 H^2(X,\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 2} \oplus \left(\frac{\mathbb{Z}}{4\mathbb{Z}}\right)^{\oplus 21} \oplus \mathbb{Z}^{\oplus 2278}$$

Proposition 2.2.

$$\frac{H^6(S^{[4]},\mathbb{Z})}{H^2(S^{[4]},\mathbb{Z}) \cup H^4(S^{[4]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 11} \oplus \left(\frac{\mathbb{Z}}{6\mathbb{Z}}\right)^{\oplus 12}$$

$$\frac{H^6(S^{[5]},\mathbb{Z})}{H^2(S^{[5]},\mathbb{Z}) \cup H^4(S^{[5]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 11} \oplus \left(\frac{\mathbb{Z}}{6\mathbb{Z}}\right)^{\oplus 9} \oplus \left(\frac{\mathbb{Z}}{30\mathbb{Z}}\right)^{\oplus 3}$$

$$\frac{H^6(S^{[6]},\mathbb{Z})}{H^2(S^{[6]},\mathbb{Z}) \cup H^4(S^{[6]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 11} \oplus \left(\frac{\mathbb{Z}}{6\mathbb{Z}}\right)^{\oplus 10} \oplus \left(\frac{\mathbb{Z}}{30\mathbb{Z}}\right)^{\oplus 2} \oplus \mathbb{Z}$$

The free summand is generated by $\left[\frac{10}{48}\mathfrak{q}_{(2^3)}(1) - \frac{12}{6}\mathfrak{q}_{(3,2,1)}(1) + \frac{3}{8}\mathfrak{q}_{(4,1^2)}(1)\right]|0\rangle$.

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