On the cohomology of irreducible holomorphically symplectic varieties

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Contents

1 Introduction

1.1 Irreducible holomorphically symplectic manifolds

Irreducible holomorphically symplectic (IHS) manifolds have been introduced by Beauville [1] as simply-connected compact Kähler manifolds admitting an everywhere non-degenerate holomorphic two-form, unique up to a scalar. Alternatively, they can be described in a differential geometric setting, according to Berger's classification as compact Riemannian manifolds with holonomy group isomorphic to the symplectic group $\mathrm{Sp}(n, \mathbb{R})$. This implies the existence of a set of complex structures, parametrized by imaginary quaternions of unit norm, such that the metric is Kähler with respect to all of these [25, Sect. 23]. Another name, compact Hyperkähler manifolds, is therefore common to emphasize this aspect of that class of manifolds. We will use the two names interchangeably.

It can be shown that any such manifold must have even complex dimension. The IHS manifolds in dimension two are the K3 surfaces, and the concept of an IHS manifold can be seen as a generalization of them. The two main example series are given by the deformation classes of Hilbert schemes of points on K3 surfaces and generalized Kummer varieties. Both were identified by Beauville [1]. Apart from that, only two further examples due to O'Grady are known up to now.

To enlarge the short list of known IHS manifolds, it is possible to generalize to irreducible symplectic V-manifolds. A V-manifold is an algebraic variety with at worst finite quotient singularities. It is called symplectic if the nonsingular locus is endowed with an everywhere non-degenerate holomorphic 2-form. A symplectic V-manifold is called irreducible if it is complete, simply connected, and if the holomorphic 2-form is unique up to a scalar. They have been studied by Grégoire Menet, and we include one of his results as an application of our result on the generalized Kummer fourfold.

An important structure of any IHS variety X of complex dimension $2n$ is the so-called Beauville–Bogomolov form, a non-degenerate quadratic form on $H^2(X, \mathbb{Z})$ that can be described with the help of the map of the symmetric power of $H^2(X, \mathbb{Z})$ to the middle cohomology group via the cup product. The cup product relates the Beauville–Bogomolov form with the form given by Poincaré duality. This relation is named after Fujiki. It implies that the map $\mathrm{Sym}^n H^2(X, \mathbb{Z}) \to H^{2n}(X, \mathbb{Z})$ is an embedding of lattices. We shall study the algebraic properties of that situation and give an explicit formula for the discriminant of the embedded lattice. The result depends only on the Fujiki relation and therefore holds whenever such an equation is fulfilled. In particular, it applies also to IHS varieties with singularities.

An automorphism of an IHS manifold X induces a lattice automorphism on $H^2(X, \mathbb{Z})$. This obviously gives some restrictions on the set of possible automorphisms. In recent years lattice theoretic methods have been used by Boissère, Camere, Joumaah, Menet, Mongardi, Nieper-Wißkirchen, Oguiso, Sarti, Tari, Wandel and others to give results on automorphisms of finite order. An important information in this setting is given by the quotient of lattices

$$\frac{H^{2n}(X, \mathbb{Z})}{\operatorname{Sym}^n H^2(X, \mathbb{Z})}$$

where $2n$ is the dimension of X and we determine it explicitly for X deformation equivalent to the Hilbert scheme of n points on a K3 surface, $n = 2, 3$ or the generalized Kummer in dimension four.

Moreover, for these examples, we give a complete description of $H^*(X, \mathbb{Z})$. In the generalized Kummer case, this description, based on preliminary work of Hassett and Tschinkel [24], is new. The method we use is to study in detail the cohomology of Hilbert schemes of points on surfaces. This has been started by Nakajima [46] and was further developed by Ellingsrud, Göttsche, Lehn, Sorger [14, 30] and Li, Qin and Wang [32, 31, 53]. We wrote two computer programs implementing their results. The first one models integral cohomology of Hilbert schemes points on K3 surfaces. The second one computes rational cohomology for Hilbert schemes of points on general surfaces. The source code is included in the appendix.

Cohomology of generalized Kummer manifolds is more subtle. With complex coefficients, a modification of the above mentioned model was developed by Britze [8]. This includes representation theory and prevents the methods from applying to cohomology with rational coeffients, although a general roadmap is contained in [48]. However, in low dimensions it can be done by other means: while in dimension two the resulting manifolds are the well-known Kummer K3 surfaces, the four dimensional case is much less studied. It turns out that the cohomology can by described by pulling back from the surrounding Hilbert scheme of three points on a torus and this description is sufficient for all degrees except 4, where the ideas from [24] complete the picture.

1.2 Overview of the results

This work has partly been published in [26] and [27]. Accordingly, the thesis is divided into several parts:

The first part [27] studies the algebraic properties of the Fujiki relation. For a compact Hyperkähler manifold X of dimension $2n$ this allows to equip the symmetric power $\operatorname{Sym}^n H^2(X)$ with a symmetric bilinear form induced by the Beauville–Bogomolov form. We develop a formula for its discriminant and compare it to the form given by the Poincaré pairing. We obtain:

Theorem 7.1. *Let $d+1$ be the rank of $H^2(X,\mathbb{Z})$ and denote c_X the Fujiki constant. The discriminant of $\operatorname{Sym}^n H^2(X,\mathbb{Z})$ is given by*

$$\left(\operatorname{discr}\left(H^2(X,\mathbb{Z})\right)\right)^{\binom{d+n}{d+1}} \cdot c_X^{\binom{d+n}{d}} \cdot \prod_{i=1}^{n} i^{\binom{n-i+d}{d}d} \cdot C,$$

$$\textit{with}\quad C = \begin{cases} \displaystyle\prod_{\substack{i=1\\ i\ \text{odd}}}^{2n+d-1} i^{\binom{n-i+d}{d}} & \text{if } d+1 \text{ is odd,}\\[2ex] \displaystyle\prod_{i=1}^{n+\frac{d-1}{2}} i^{\binom{n-i+d}{d}-\binom{n-2i+d}{d}} & \text{if } d+1 \text{ is even.}\end{cases}$$

The construction generalizes to a definition for an induced symmetric bilinear form on the symmetric power of any free module equipped with a symmetric bilinear form, yielding Theorem 5.6. We point out in Section 6 how the situation is related to the theory of orthogonal polynomials in several variables. In Definition 6.14 we construct a basis of homogeneous polynomials that are orthogonal when integrated over the unit sphere $\mathbb{S}^d$, or equivalently, over $\mathbb{R}^{d+1}$ with a Gaussian kernel.

In the second part we recall the theory on cohomology of Hilbert schemes of points on surfaces, using the Nakajima operator technique, working a bit on commutator relations (Section 13). We give a description of the Hilbert scheme of two points on a torus via Nakajima operators in Proposition 14.6, but it is clear how to derive the generalization to general surfaces.

We proceed by describing the integral cohomology of the generalized Kummer fourfold giving an explicit basis, using Hilbert scheme cohomology and tools developed by Hassett and Tschinkel. It turns out that Hilbert scheme cohomology is almost sufficient:

Theorem 18.1(i). *Let A be a complex abelian surface and denote $\theta : K_2(A) \hookrightarrow A^{[3]}$ the embedding of the generalized Kummer fourfold into the Hilbert scheme of three points on A. The homomorphism $\theta^* : H^*(A^{[3]},\mathbb{Z}) \to H^*(K_2(A),\mathbb{Z})$ of graded rings is surjective in every degree except 4. Moreover, the image of $H^4(A^{[3]},\mathbb{Z})$ is the* *primitive overlattice of* $\operatorname{Sym}^2(H^2(K_2(A),\mathbb{Z}))$. *The kernel of* θ^* *is the ideal generated by* $H^1(A^{[3]},\mathbb{Z})$.

The remaining discussion of the middle cohomology group is summarized in Theorem 18.2. Roughly, the idea is to start with some extra classes in $H^4(A^{[3]},\mathbb{Z})$ and apply a suitable set of diffeomorphisms of the generalized Kummer, obtained through deformations, to get all missing classes.

As an illustration of the result, we include an application due to Grégoire Menet to an IHS variety with singularities, obtained by a partial resolution of the generalized Kummer quotiented by a symplectic involution. The Beauville–Bogomolov form of this new variety is the first example of such a form that is odd.

The last part [26] is computational. In Section 22 we work out some structural results for integral cohomology of Hilbert schemes on K3 surfaces, using a computer based method. As an example, we get:

Propositions 22.2 and 22.5. *Let* $S^{[3]}$ *be the Hilbert scheme of three points on a projective K3 surface (or a deformation equivalent space). The cup product mappings have the following cokernels:*

$$\frac{H^4(S^{[3]},\mathbb{Z})}{\operatorname{Sym}^2 H^2(S^{[3]},\mathbb{Z})} \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 23}$$
$$\frac{H^6(S^{[3]},\mathbb{Z})}{H^2(S^{[3]},\mathbb{Z}) \smile H^4(S^{[3]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 23}$$

We also obtain the following freeness result for Hilbert schemes of points on K3 surfaces, which is obtained by analyzing the structure of the computational model:

Theorem 21.15. *The quotient*

$$\frac{H^{2k}(S^{[n]},\mathbb{Z})}{\operatorname{Sym}^k H^2(S^{[n]},\mathbb{Z})}$$

is a free $\mathbb{Z}$*-module for* $n \geq k+2$.

The appendix includes the source code for computing the cohomology of Hilbert schemes. In Appendix A an implementation for integral cohomology of $\mathrm{K3}^{[n]}$ is given. Appendix B implements rational cohomology of $A^{[n]}$ for general surfaces A. We use the functional programming language Haskell [52]. The source code is also available under `https://github.com/s--kapfer`.

2 Lattices

Let us first recall some elementary facts about lattices that will be needed throughout the text. A reference for this section is Chapter 8.2.1 of [12].

Definition 2.1. By a *lattice* L we mean a free $\mathbb{Z}$–module of finite rank, equipped with a non-degenerate, integer-valued symmetric bilinear form, denoted by B or $\langle\ ,\ \rangle$. By a homomorphism or *embedding* $L \subset M$ of lattices we mean a map $: L \to M$ that preserves the bilinear forms on L and M respectively. It is automatically injective. We always have the injection of a lattice L into its dual space $L^* := \mathrm{Hom}(L, \mathbb{Z})$, given by $x \mapsto \langle x,\ \rangle$. A lattice is called *unimodular*, if this injection is an isomorphism, *i.e.* if it is surjective. By tensoring with $\mathbb{Q}$, we can interpret L as well as L^* as a discrete subset of the $\mathbb{Q}$–vector space $L \otimes \mathbb{Q}$. Note that this gives a kind of lattice structure to L^*, too, but the symmetric bilinear form on L^* may now take rational coefficients.

If $L \subset M$ is an embedding of lattices of the same rank, then the *index* $|M : L|$ of L in M is defined as the order of the finite group M/L. There is a chain of embeddings $L \subset M \subset M^* \subset L^*$ with $|L^* : M^*| = |M : L|$.

The quotient L^*/L is called the *discriminant group* and denoted A_L. The index of L in L^* is called $\operatorname{discr} L$, the *discriminant* of L. Choosing a basis $(x_i)_i$ of L, we may express $\operatorname{discr} L$ as the absolute value of the determinant of the so-called *Gram matrix* G of L, which is defined by $G_{ij} := \langle x_i, x_j \rangle$. L is unimodular, if and only if $\det G = \pm 1$.

The lattice L is called *odd*, if there exists a $v \in L$, such that $\langle v, v \rangle$ is odd, otherwise it is called *even*. If the map $v \mapsto \langle v, v \rangle$ takes both negative and positive values on L, the lattice is called *indefinite*.

Example 2.2. Up to isomorphism there is a unique even unimodular lattice of rank two. It is called the *hyperbolic lattice* U. Its Gram matrix is given by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Proposition 2.3. *Let M be a lattice. Let $L \subset M$ be a sublattice of the same rank. Then $|M : L|$ equals $\sqrt{\frac{\operatorname{discr} L}{\operatorname{discr} M}}$.*

Proof. We have $|L^* : L| = |L^* : M^*||M^* : M||M : L|$. It follows that $\frac{|L^* : L|}{|M^* : M|} = |L^* : M^*||M : L| = |M : L|^2$. $\square$

An embedding $L \subset M$ is called *primitive*, if the quotient M/L is free. We denote by $L^\perp$ the orthogonal complement of L within M. Since an orthogonal complement is always primitive, the double orthogonal complement $L^{\perp\perp}$ is a primitively embedded overlattice of L. It is clear that $\operatorname{discr}(L^{\perp\perp})$ divides $\operatorname{discr} L$.

Proposition 2.4. *Let $L \subset M$ be an embedding of lattices. Then the order of the torsion part of M/L divides* $\operatorname{discr} L$.

Proof. The torsion part is the index of $M/(L^{\perp\perp})$ in M/L. But this is equal to $|L^{\perp\perp} : L| = |L^* : (L^{\perp\perp})^*|$ and clearly divides $|L^* : L|$. □

Proposition 2.5. *Let M be unimodular. Let $L \subset M$ be a primitive embedding. Then* $\operatorname{discr} L = \operatorname{discr} L^\perp$.

Proof. Consider the orthogonal projection : $M \otimes \mathbb{Q} \to L \otimes \mathbb{Q}$. Its restriction to M has kernel equal to $L^\perp$ and image in L^*. Hence we have an embedding of lattices $M/L^\perp \subset L^*$. Quotienting by L, we get an injective map : $M/(L \oplus L^\perp) \to L^*/L$. Now by Proposition 2.3, $\sqrt{\operatorname{discr}(L)\operatorname{discr}(L^\perp)} = |M : (L \oplus L^\perp)| \leq |L^* : L| = \operatorname{discr} L$. So we get $\operatorname{discr} L^\perp \leq \operatorname{discr} L$. Exchanging the roles of $L = L^{\perp\perp}$ and $L^\perp$ gives the inequality in the opposite direction. □

Corollary 2.6. *Let $L \subset M$ be an embedding of lattices with unimodular M. Let n be the order of the torsion part of M/L. Then* $\operatorname{discr} L^\perp = \operatorname{discr} L^{\perp\perp} = \frac{1}{n^2} \operatorname{discr} L$.

Example 2.7. Given a free $\mathbb{Z}$-module L with the structure of a commutative ring and a linear form $I : V \to \mathbb{Z}$, the setting $\langle v, w\rangle = I(vw)$ defines a bilinear form giving L the structure of a lattice if it is non-degenerate. This is the case in topology: For a compact complex manifold X of dimension d Poincaré duality induces a non-degenerate bilinear form on $H^d(X, \mathbb{Z})$:

$$\langle \alpha, \beta \rangle = \int_X \alpha\beta.$$

This unimodular lattice will be referred to as the *Poincaré lattice*.

Part I

Symmetric Powers of Symmetric Bilinear Forms

3 Polarized Fujiki relation

The Beauville–Bogomolov–Fujiki form q for a compact Hyperkähler manifold X is a quadratic form on the integral cohomology $H^2 := H^2(X, \mathbb{Z})$, defined by an equation of the structure

$$q(x)^k = I(x^{2k}), \tag{1}$$

where x^{2k} means a power in the cohomology ring, and I is a linear form (in fact, a scaled integral).

Now every quadratic form q has an associated symmetric bilinear form $\langle \ , \ \rangle$, obtained by polarization: $2\langle x, y\rangle = q(x+y) - q(x) - q(y)$. This allows us to retrieve some information about I from $\langle \ , \ \rangle$, by comparing coefficients in the equality

$$I\Big((x_1 + \ldots + x_{2k})^{2k}\Big) = q(x_1 + \ldots + x_{2k})^k = \left(\sum_{i=1}^{2k} q(x_i) + \sum_{1 \le i < j \le 2k} 2\langle x_i, x_j\rangle\right)^k .$$

If we look at the summands belonging to the monomial $x_1 \ldots x_{2k}$, we obtain a seemingly more general but in fact equivalent version of (1):

$$(2k)!\, I(x_1 \ldots x_{2k}) = 2^k k! \sum_{\mathcal{P}} \prod_{\{i,j\} \in \mathcal{P}} \langle x_i, x_j\rangle , \tag{2}$$

where the sum is over all partitions $\mathcal{P}$ of $\{1, \ldots, 2k\}$ into pairs. This is a classical observation, see also [50, Eq. 3.2.4]. Let us develop this idea a bit further. The map $(f, g) \mapsto I(fg)$ clearly defines a symmetric bilinear form on the symmetric product $\operatorname{Sym}^k H^2$. Equation (2) gives now a redefinition of this form by means of a bilinear form on H^2. So we liberate ourselves from the initial setting and take the right hand side of (2) as a general recipe to construct a symmetric bilinear form $\langle\!\langle \ , \ \rangle\!\rangle$ on $\operatorname{Sym}^k V$ from a symmetric bilinear form on an appropriate space V. This is carried out in Section 5. Our main result, Theorem 5.6, gives a formula for the determinant of the Gram matrix of $\langle\!\langle \ , \ \rangle\!\rangle$.

If V is a real vector space, then there is a notable description in terms of an analytic integral given in Prop. 5.5: After some simplifications this amounts to integrating homogeneous polynomials over a sphere. Essentially, we have:

$$\langle\!\langle f, g \rangle\!\rangle = \int_{\mathbb{S}^d} f(\omega)g(\omega)d\omega$$

This is very comfortable, since it allows to use the whole bunch of techniques from calculus to investigate the algebraic properties of our construction. Since we are interested in the determinant of the Gram matrix of $\langle\!\langle\ ,\ \rangle\!\rangle$ and for computing determinants it is good to have diagonal matrices, we look for polynomials that are mutually orthogonal on the sphere. The theory of orthogonal polynomials is well developed, and a basis of such polynomials is given by spherical harmonics, see Remark 6.3. But as spherical harmonics are not suitable for our determinant problem, we construct a different (and slightly simpler) basis of homogeneous polynomials that are orthogonal on the sphere in Section 6.

After doing that, we come back to our starting point and apply our results to Hyperkähler manifolds. The bilinear form on $\operatorname{Sym}^k H^2$ allows us to compare $\operatorname{Sym}^k H^2$ with H^{2k}. We give some results on torsion factors of the quotient $\frac{H^{2k}}{\operatorname{Sym}^k H^2}$ in Section 7, similar to those studied in Section 22.

4 Terminology and helper formulas

In this section we give a few standard definitions and recall some facts on elementary calculus. We also mention technical formulas needed for our proofs.

4.1 Combinatorial formulas

Definition 4.1. For a multi-index $\alpha = (\alpha_0, \dots, \alpha_d)$ of length $\operatorname{len}(\alpha) := d + 1$ we define: $x^\alpha := x_0^{\alpha_0} \dots x_d^{\alpha_d}$. The degree is defined by $|\alpha| := \sum \alpha_i$, the factorial is $\alpha! := \prod \alpha_i!$. Further, we set $\alpha' := (\alpha_0, \dots, \alpha_{d-1})$. We introduce the lexicographical ordering on multi-indices: $\alpha < \beta$ iff $\alpha_d < \beta_d$ or $(\alpha_d = \beta_d) \wedge (\alpha' < \beta')$.

Definition 4.2. The binomial coefficient for nonnegative integers k and arbitrary z is defined as: $\binom{z}{k} := \frac{z(z-1)\dots(z-k+1)}{k!}$. Thus we have $\binom{-z}{k} = (-1)^k\binom{z+k-1}{k}$. For negative k we set $\binom{z}{k} := 0$.

We introduce the difference operator $\Delta f(n) := f(n+1) - f(n)$. It has the following properties similar to the differential operator:

$$\sum_{i=0}^{n} \Delta(f) = f\Big|_0^{n+1} = f(n+1) - f(0) \qquad \text{(telescoping sum)} \tag{3}$$

$$\Delta(fg)(n) = f(n+1)\Delta g(n) + g(n)\Delta f(n) \qquad \text{(product rule)} \tag{4}$$

$$\sum_{i=0}^{n} g(i)\Delta f(i) = (fg)\Big|_0^{n+1} - \sum_{i=0}^{n} f(i+1)\Delta g(i) \qquad \text{(summation by parts)} \tag{5}$$

This often applies to the binomial coefficient, since we have:

$$\Delta\tbinom{n}{k} = \tbinom{n+1}{k} - \tbinom{n}{k} = \tbinom{n}{k-1}. \tag{6}$$

Let K be a commutative ring and let $r_{d,k} = \operatorname{rank}\left(\operatorname{Sym}^k K^{d+1}\right)$ be the rank of the symmetric power of a free K-module of rank $d+1$. Because we have the decomposition $\operatorname{Sym}^k K^{d+1} \cong \operatorname{Sym}^k K^d \oplus (\operatorname{Sym}^{k-1} K^{d+1}) \otimes K$, we obtain the recurrence $r_{d,k} = r_{d-1,k} + r_{d,k-1}$. So we deduce:

$$\tbinom{k+d}{d} = \tbinom{k+d}{k} = r_{d,k} = \operatorname{rank}\left(\operatorname{Sym}^k K^{d+1}\right) = \operatorname{card}(\{|\alpha| = k\}). \tag{7}$$

The following identity for integers $d, k \geq 0$ is proven by induction over k:

$$\prod_{j=0}^{k} (k-j)!^{\binom{j+d-1}{d-1}} = \prod_{i=1}^{k} i^{\binom{k-i+d}{d}}, \tag{8}$$

where the induction step $k \to k+1$ produces a factor $\prod_{i=1}^{k+1} i^{\binom{k-i+d}{d-1}}$ on both sides.

We will also need the identity:

$$\sum_{\substack{i=1 \\ i \text{ even}}}^{2k+d+1} \tbinom{k-i+d}{d-1} = \begin{cases} 0 & \text{if } d \text{ is even,} \\ \tbinom{k+d}{d} & \text{if } d \text{ is odd,} \end{cases} \tag{9}$$

which is proven by splitting the sum into:

$$\sum_{\substack{i=1 \\ i \text{ even}}}^{k+1} \tbinom{k-i+d}{d-1} + \sum_{\substack{i=k+d+1 \\ i \text{ even}}}^{2k+d+1} \tbinom{k-i+d}{d-1} = \sum_{\substack{i=1 \\ k-i \text{ even}}}^{k+1} \tbinom{i+d-2}{d-1} + (-1)^{d-1} \sum_{\substack{i=1 \\ k+d+i \text{ even}}}^{k+1} \tbinom{i+d-2}{d-1}.$$

Definition 4.3. We define the double factorial for $n \geq -1$ by

$$n!! := \prod_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (n-2i) = n(n-2)(n-4)\dots$$

Clearly, $(n-1)!!\, n!! = n!$ and $(2n)!! = 2^n n!$.

Proposition 4.4. *The number of partitions of the set $\{1, \dots, 2k\}$ into pairs equals $(2k-1)!! = \frac{(2k)!}{2^k k!}$.*

Proof. Given such a partition, look at the pair that contains the element 1. There are $2k-1$ possible partners for this element; removing the pair leaves a partition of a set of cardinality $(2k-2)$ into pairs. Then proceed by induction. $\square$

Corollary 4.5. *Let $D_1, \dots, D_n$ be disjoint finite sets with $|D_i| = \alpha_i$. Then the number of partitions of the set $D = D_1 \cup \dots \cup D_n$ into pairs, such that the elements of every pair come from the same D_i, is equal to $\prod_i (\alpha_i - 1)!!$ if all α_i are even and 0 otherwise.*

4.2 Formulas from calculus

Denote $\Gamma(t) := \int_0^\infty r^{t-1} e^{-r} dr$ the gamma function. It satisfies:

$$n! = \Gamma(n+1), \qquad (2n-1)!!\sqrt{\pi} = 2^n \Gamma\left(n + \tfrac{1}{2}\right), \tag{10}$$

$$n!\sqrt{\pi} = 2^n \Gamma\left(\tfrac{n}{2} + 1\right) \Gamma\left(\tfrac{n+1}{2}\right), \tag{11}$$

$$\int_0^\infty r^s e^{-\frac{1}{2}r^2} dr = 2^{\frac{s-1}{2}} \Gamma\left(\tfrac{s+1}{2}\right). \tag{12}$$

It follows, that:

$$\begin{aligned} \int_{\mathbb{R}^{d+1}} x^\alpha x^\beta e^{-\frac{1}{2}\|x\|^2} dx &= \prod_{i=0}^{d} \int_{-\infty}^{\infty} x_i^{\alpha_i+\beta_i} e^{-\frac{1}{2}x_i^2} dx_i \qquad (13) \\ &= \begin{cases} (2\pi)^{\frac{d+1}{2}} \prod_{i=0}^{d} (\alpha_i + \beta_i - 1)!! & \text{if all } \alpha_i + \beta_i \text{ are even,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The reader may also consult [17] for that kind of calculus. In particular, [17, Eq. (4)] yields:

Lemma 4.6. *Let $f : \mathbb{R}^{d+1} \to \mathbb{R}$ be a continuous homogeneous function of degree k, that is $f(sx) = s^k f(x)\ \forall s \in \mathbb{R}$. Then, using polar coordinates $(r, \omega) = (\|x\|, \frac{x}{\|x\|})$:*

$$\begin{aligned}\int_{\mathbb{R}^{d+1}} f(x) e^{-\frac{1}{2}\|x\|^2} dx &= \int_{\mathbb{S}^d} \int_0^\infty f(r\omega) r^d e^{-\frac{1}{2}r^2} dr d\omega \\ &= 2^{\frac{k+d-1}{2}} \Gamma\Big(\frac{k+d+1}{2}\Big) \int_{\mathbb{S}^d} f(\omega) d\omega.\end{aligned}$$

5 Symmetric Bilinear Forms on Symmetric Powers

Let V be a vector space (or a free module) over a field (resp. a commutative ring) K of rank $d+1$ with basis $\{x_0, \ldots, x_d\}$, equipped with a symmetric bilinear form $\langle\ ,\ \rangle : V \times V \to K$. We will freely identify the symmetric power $\operatorname{Sym}^k V$ with the space $K[x_0, \ldots, x_d]_k$ of homogeneous polynomials of degree k.

There are at least two possibilities to define an induced bilinear form on $\operatorname{Sym}^k V$. We will use the following

Definition 5.1. On the basis $\{x_{n_1} \ldots x_{n_k} \mid 0 \le n_1 \le \ldots \le n_k \le d\}$ of $\operatorname{Sym}^k V$, we define a symmetric bilinear form $\langle\!\langle\ ,\ \rangle\!\rangle$ by:

$$\langle\!\langle x_{n_1} \ldots x_{n_k}\, ,\, x_{n_{k+1}} \ldots x_{n_{2k}} \rangle\!\rangle := \sum_{\mathcal{P}} \prod_{\{i,j\} \in \mathcal{P}} \Big\langle x_{n_i}, x_{n_j} \Big\rangle\,, \tag{14}$$

where the sum is over all partitions $\mathcal{P}$ of $\{1, \ldots, 2k\}$ into pairs.

We emphasize that this is not the only possibility. One could alternatively define

$$(\!(x_{n_1} \ldots x_{n_k}\, ,\, x_{m_1} \ldots x_{m_k})\!) := \sum_{\sigma} \prod_{i=1}^{k} \Big\langle x_{n_i}, x_{m_{\sigma(i)}} \Big\rangle\,, \tag{15}$$

the sum being over all permutations σ of $\{1, \ldots, k\}$, as studied by McGarraghy in [39]. However, this is a different construction that doesn't match the situation described in the introduction. We will *not* consider it here.

If $U \in O(V)$ is an orthogonal transformation, then the induced diagonal action of $U^{\otimes k}$ on $\operatorname{Sym}^k V$ is orthogonal in both cases. This shows that the values of $\langle\!\langle\ ,\ \rangle\!\rangle$ and $(\!(\ ,\ )\!)$ are independent of the choice of the base of V up to orthogonal transformation.

Example 5.2. To contrast the two definitions, observe that in the case $k = 2$

$$\langle\!\langle ab, cd \rangle\!\rangle = \langle a, c\rangle \langle b, d\rangle + \langle a, d\rangle \langle b, c\rangle + \langle a, b\rangle \langle c, d\rangle , \tag{16}$$
$$(\!(ab, cd)\!) = \langle a, c\rangle \langle b, d\rangle + \langle a, d\rangle \langle b, c\rangle . \tag{17}$$

Remark 5.3. Note that (15) does not require symmetry of the bilinear form $\langle\ ,\ \rangle$ on V. Indeed, the definition would also be valid for an arbitrary bilinear form $V \times W \to K$, yielding a bilinear form $\operatorname{Sym}^k V \times \operatorname{Sym}^k W \to K$. On the other hand, if the form on V is not symmetric, then (14) is not well-defined.

Remark 5.4. The defining equation (14) works equally well, if the two arguments have different degree. So we can easily extend our definition to a symmetric bilinear form $\langle\!\langle\ ,\ \rangle\!\rangle : \operatorname{Sym}^* V \times \operatorname{Sym}^* V \to K$. Then we have: $\langle\!\langle a, bc \rangle\!\rangle = \langle\!\langle ab, c \rangle\!\rangle$. Note that $\operatorname{Sym}^k V$ is in general not orthogonal to $\operatorname{Sym}^l V$ unless $k - l$ is an odd number.

We wish to investigate some properties of this construction. Let G be the Gram matrix of $\langle\ ,\ \rangle$, *i.e.* $G_{ij} = \langle x_i, x_j\rangle$ and let $\mathbb{G}$ be the Gram matrix of $\langle\!\langle\ ,\ \rangle\!\rangle$. We use multi-index notation, cf. Definition 4.1.

Proposition 5.5. *Assume $K = \mathbb{R}$ and G is positive definite, so its inverse G^{-1} exists. Then $\langle\!\langle\ ,\ \rangle\!\rangle$ takes an analytic integral form:*

$$\left\langle\!\!\left\langle x^\alpha, x^\beta \right\rangle\!\!\right\rangle = \frac{1}{c} \int_{\mathbb{R}^{d+1}} x^\alpha x^\beta d\mu(x),$$

where the integration measure is $d\mu(x) = \exp\left(-\frac{1}{2}\sum_{i,j} G^{-1}_{ij} x_i x_j\right) dx$ and the normalization constant is $c = \int_{\mathbb{R}^{n+1}} d\mu(x) = \sqrt{(2\pi)^{d+1} \det G}$.

Proof. Note that we need positive definiteness of G to make the integral converge. We make use of the content in Section 4. First, observe that both sides of the equation are invariant under orthogonal transformations of the base space $\mathbb{R}^{d+1}$. We may therefore assume that $G = \operatorname{diag}(a_0, \dots, a_d)$ is a diagonal matrix. Then the integral splits nicely:

$$\begin{aligned}\frac{1}{c}\int_{\mathbb{R}^{d+1}} x^\alpha x^\beta d\mu(x) &= \frac{1}{c}\prod_{i=0}^{d}\int_{-\infty}^{\infty} x_i^{\alpha_i+\beta_i} e^{-\frac{1}{2a_i}x_i^2} dx_i \\ &= \frac{1}{c}\prod_{i=0}^{d} a_i^{\frac{\alpha_i+\beta_i+1}{2}} \int_{-\infty}^{\infty} x^{\alpha_i+\beta_i} e^{-\frac{1}{2}x^2} dx \\ &\overset{(13)}{=} \begin{cases} \displaystyle\prod_{i=0}^{d} a_i^{\frac{\alpha_i+\beta_i}{2}} (\alpha_i+\beta_i-1)!! & \text{if all } \alpha_i+\beta_i \text{ are even,} \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

On the other hand, if G is diagonal, then every partition into pairs in Equation (14) that contains a pair of two different numbers will not contribute to the sum. Corollary 4.5 shows then, that we get the same formula for $\left\langle\!\left\langle x^\alpha, x^\beta \right\rangle\!\right\rangle$. □

The next theorem gives a formula for the determinant of $\mathbb{G}$. This is of particular interest when $K = \mathbb{Z}$, because in this case we are in the setting of lattice theory, and $|\det \mathbb{G}|$ is the discriminant of the lattice $\operatorname{Sym}^k V$.

Theorem 5.6. *The determinant of the Gram matrix $\mathbb{G}$ of $\langle\!\langle\ ,\ \rangle\!\rangle$, the induced bilinear form on $\operatorname{Sym}^k V$, $\operatorname{rank} V = d+1$, is:*

$$\det(\mathbb{G}) = \det(G)^{\binom{d+k}{d+1}}\, \theta_{d,k} \tag{18}$$

where $\theta_{d,k}$ is a combinatorial factor given by:

$$\theta_{d,k} = \begin{cases} \displaystyle\prod_{i=1}^{k} i^{\binom{k-i+d}{d}d} \prod_{\substack{i=1\\ i \text{ odd}}}^{2k+d-1} i^{\binom{k-i+d}{d}} & \text{if } d \text{ is even,} \\ \displaystyle\prod_{i=1}^{k} i^{\binom{k-i+d}{d}d} \prod_{i=1}^{k+\frac{d-1}{2}} i^{\binom{k-i+d}{d}-\binom{k-2i+d}{d}} & \text{if } d \text{ is odd.}\end{cases} \tag{19}$$

Remark 5.7. If d or k is small, this simplifies as follows:

$$\theta_{d,0} = \theta_{d,1} = 1, \qquad \theta_{d,2} = 2^d(d+3),$$
$$\theta_{0,k} = (2k-1)!!, \qquad \theta_{1,k} = (k!)^{k+1}.$$

Proof. We prove the theorem in three steps. Let us first consider the case when V is a vector space over $\mathbb{R}$ and G is positive definite. We further reduce this to the special case when G is the identity matrix. That is the essential difficulty of the proof, which we will treat in Section 6.3.

Any orthogonal transformation $U \in O(V)$ induces a transformation $U^{\otimes k} \in O(\operatorname{Sym}^k V)$ and thus doesn't affect determinants. Since over $\mathbb{R}$, every symmetric matrix can be diagonalized by applying an orthogonal coordinate change, we may assume that $G = \operatorname{diag}(a_0, \dots, a_d)$ is a diagonal matrix. Let us check, what happens if we apply a coordinate transformation $x \mapsto \tilde{x}$ that changes the last coordinate by $\tilde{x}_d = \gamma x_d$ and leaves the other coordinates invariant. Let $\tilde{G}$ and $\tilde{\mathbb{G}}$ be the Gram matrices corresponding to the new coordinates. We clearly have: $\tilde{x}^\alpha = \gamma^{\alpha_d} x^\alpha$. Extracting the factor γ from the Leibniz determinant formula, which is of the form $\det \tilde{\mathbb{G}} = \sum_\sigma \pm \prod_{|\alpha|=k} \left\langle\!\!\left\langle \tilde{x}^\alpha, \tilde{x}^{\sigma(\alpha)} \right\rangle\!\!\right\rangle = \det \mathbb{G} \prod_{|\alpha|=k} \gamma^{2\alpha_d}$, we get:

$$\frac{\det \tilde{\mathbb{G}}}{\det \mathbb{G}} = \prod_{|\alpha|=k} \gamma^{2\alpha_d} = \prod_{i=0}^{k} \prod_{|\alpha'|=k-i} \gamma^{2i} \overset{(7)}{=} \prod_{i=0}^{k} \gamma^{2i\binom{k-i+d-1}{d-1}} \overset{(5)}{=} \gamma^{2\binom{d+k}{d+1}}.$$

Now we apply successively coordinate transformations that map x_i to $\frac{x_i}{\sqrt{a_i}}$. We get a factor $(a_0 \dots a_d)^{\binom{d+k}{d+1}} = \det G^{\binom{d+k}{d+1}}$ and we are left with an identity Gram matrix. The statement follows from Theorem 6.17.

As a second step, still working over $\mathbb{R}$, we show that we can drop the condition that G is positive definite. To see this, let $Q \subset \mathbb{R}^{(d+1)\times(d+1)}$ be the subspace of real symmetric square matrices of size $d+1$. Our formula (18) depends polynomially on the entries of G. The subset $R \subset Q$ of all matrices $G \in Q$ that satisfy (18) is therefore Zariski-closed. But on the other hand, the positive definite matrices form a nonempty subset $P \subset Q$ which is open in the analytic topology. So if $P \subset R$, then necessarily $R = Q$.

Finally, matrices with integer entries form a subset of real matrices. So (18) holds also for free $\mathbb{Z}$-modules V. But (18) is an identity living in $\mathbb{Z}[G_{ij}]$, so it holds true over any commutative ring K, simply by tensoring with K. □

6 Homogeneous Orthogonal Polynomials on the sphere

In this section we will construct a basis for the space of homogeneous polynomials of degree k in $d+1$ variables, $\mathbb{R}[x_0, \dots, x_d]_k$, that is orthogonal with respect to the bilinear form given by

$$\langle\!\langle f, g \rangle\!\rangle = \int_{\mathbb{R}^{d+1}} f(x) g(x) d\mu(x), \tag{20}$$

where the measure is $d\mu(x) = (2\pi)^{-\frac{d+1}{2}} e^{-\frac{1}{2}\|x\|^2} dx$. In order to do this, we wish to apply the Gram-Schmidt process to the (lexicographically ordered) monomial basis $(x^\alpha)_{|\alpha|=k}$. Our result is stated in Subsection 6.3.

Remark 6.1. Although the above definition of $\langle\!\langle\ ,\ \rangle\!\rangle$ doesn't mention the sphere, in view of Lemma 4.6, we could equivalently consider the integral:

$$\langle\!\langle f, g \rangle\!\rangle = c_{d,k} \int_{\mathbb{S}^d} f(\omega) g(\omega) d\omega, \qquad c_{d,k} = 2^{\frac{k}{2}-1} \pi^{-\frac{d+1}{2}} \Gamma\Big(\tfrac{k+d+1}{2}\Big).$$

This is the reason why we speak of polynomials orthogonal on the sphere. However, we prefer to integrate over $\mathbb{R}^{d+1}$, since this avoids the unwanted constant $c_{d,k}$.

Remark 6.2. We stress that this equivalency really depends on the homogeneity. Denote $\langle\!\langle f, g \rangle\!\rangle_{\mathbb{R}^{d+1}} = \langle\!\langle f, g \rangle\!\rangle$ and $\langle\!\langle f, g \rangle\!\rangle_{\mathbb{S}^d} = \int_{\mathbb{S}^d} fg$ for a moment and let us look at what happens if we drop the homogeneity constraint. Since $c_{d,k}$ depends on k, $\langle\!\langle\ ,\ \rangle\!\rangle_{\mathbb{R}^{d+1}}$ and $\langle\!\langle\ ,\ \rangle\!\rangle_{\mathbb{S}^d}$ aren't equivalent anymore. A basis of $\mathbb{R}[x_0, \ldots, x_d]$, consisting of $\langle\!\langle f, g \rangle\!\rangle_{\mathbb{R}^{d+1}}$-orthogonal polynomials is given by products $H_{\alpha_0}(x_0) \ldots H_{\alpha_d}(x_d)$ of Hermite polynomials in one variable, see also [13, Sect. 2.3.4]. On the other hand, the form $\langle\!\langle\ ,\ \rangle\!\rangle_{\mathbb{S}^d}$ becomes degenerate on $\mathbb{R}[x_0, \ldots, x_d]$, because integration on the sphere can't distinguish between 1 and the square radius $\|x\|^2$.

Remark 6.3. A $\langle\!\langle\ ,\ \rangle\!\rangle$-orthogonal basis of homogeneous polynomials which we won't consider here is given by spherical harmonics. Let $\mathcal{H}_k^{d+1} \subset \mathbb{R}[x_0, \ldots, x_d]_k$ be the subspace of harmonic polynomials. By Theorem 1.3 and Proposition 1.4 of [10], there is an orthogonal decomposition

$$\mathbb{R}[x_0, \ldots, x_d]_k = \mathcal{H}_k^{d+1} \oplus r^2 \mathcal{H}_{k-2}^{d+1} \oplus r^4 \mathcal{H}_{k-4}^{d+1} \oplus \ldots$$

where $r^2 = \|x\|^2 = x_0^2 + \ldots + x_d^2$. Orthogonal bases for each of the $\mathcal{H}_k^{d+1}$ in turn are constructed in [13, Sect. 2.2]. However, the basis one obtains this way has nothing to do with monomials. In particular, the transition matrix between them is not triangular, so they are not related by a Gram-Schmidt process.

6.1 Generalities on orthogonal polynomials in one variable

Given a nondegenerate symmetric bilinear form on the space of polynomials $K[x]$, one may ask for a basis of polynomials $(p_n)_n$ that are mutually orthogonal with respect to that form. To find such a basis, one could start with the monomial basis $(x^n)_n$ and apply some version of the Gram–Schmidt algorithm. The result will be an infinite lower triangular matrix T such that $p_n = \sum_j T_{nj} x^j$. We prefer to normalize such that the diagonal elements of T are equal to 1. If our bilinear form now depends only on the product of its two arguments, the procedure simplifies as follows:

Let $\mathcal{L}$ be a linear functional such that the induced bilinear form $(f, g) = \mathcal{L}(fg)$ is nondegenerate when restricted to $K[x]_{\leq n}$, the space of polynomials of bounded degree, for all $n \geq 0$. Let $(p_n)_n$ be the associated sequence of monic orthogonal polynomials, *i.e.* the leading term of $p_n(x)$ is x^n and $(p_k, p_n) = 0$ for $k \neq n$. Then we have

Theorem 6.4. *[9, Thm. 4.1] There are constants c_n, d_n such that*

$$p_0(x) = 1, \qquad p_{n+1}(x) = (x - c_n)p_n(x) - d_n p_{n-1}(x).$$

But also the converse is true:

Theorem 6.5 (Favard's theorem)**.** *[9, Thm. 4.4] Let $(p_n)_n$ be a sequence of polynomials, such that $\deg p_n = n$ and the following three-term recurrence holds:*

$$p_0(x) = 1, \qquad p_{n+1}(x) = (x - c_n)p_n(x) - d_n p_{n-1}(x).$$

Then there exists a unique linear functional $\mathcal{L}$ such that $\mathcal{L}(1) = 1$ and $\mathcal{L}(p_k p_n) = 0$ for $k \neq n$.

Theorem 6.6. *[9, Thm. 4.2] Under the conditions of the above theorems, we have for $n \geq 1$:*

$$\mathcal{L}(p_n^2) = d_n \mathcal{L}(p_{n-1}^2).$$

Remark 6.7. Since we shall deal with finite polynomial families, we need a little modification of Favard's theorem: If $(p_n)_{n \leq N}$ is a finite sequence that satisfies a three-term recurrence as above, then we can always extend it to an infinite sequence by choosing arbitrary constants c_n, d_n for $n \geq N$. But for every such extension, the resulting functional $\mathcal{L}$ from Favard's theorem will satisfy $\mathcal{L}(1) = 1$ and $\mathcal{L}(p_n) = \mathcal{L}(p_n p_0) = 0$ for $n \geq 1$. So $\mathcal{L}$ will always be uniquely determined on $K[x]_{\leq N}$, the space of degree-bounded polynomials.

6.2 A polynomial family

Our construction of homogeneous polynomials orthogonal on the sphere is formally similar to the definition of spherical harmonics, see [13, p. 35]. Those are defined by recursion over the number of variables, as products of Chebychev and Gegenbauer polynomials. Inspired by that procedure, we introduce the following polynomial family in lieu thereof:

Definition 6.8. Let $n,\ m$ be integers with $0 \le 2n \le m+1$, a condition that we always will assume silently. We define monic polynomials p_n^m of degree n with rational coefficients:

$$p_n^m(x) := \sum_{\substack{j=0 \\ n-j \text{ even}}}^{n} (-1)^{\frac{n-j}{2}} \frac{n!\,(m-2n)!!}{j!\,(m-n-j)!!\,(n-j)!!}\, x^j.$$

Remark 6.9. As Yuan Xu pointed out to the author, this can be written in terms of the hypergeometric function, namely $p_n^m(x) = x^n {}_2F_1\left(\begin{smallmatrix} -\frac{n}{2},\, \frac{1-n}{2} \\ \frac{m}{2}-n+1 \end{smallmatrix}; -\frac{1}{x^2}\right)$. To see this, first change summation from j to $n-j$, so that the sum is over $j =$ even, then set $j = 2i$ and rewrite the sum in the notation of the rising Pochhammer symbol $(a)_n = a(a+1)\dots(a+n-1)$. Comparing this formula with [13, Prop. 1.4.11], it follows that the p_n^m are in fact a variant of the Gegenbauer polynomials C_n^λ. More precisely, $p_n^m(x)$ is a multiple of $C_n^{-\frac{m}{2}}(\sqrt{-1}\,x)$, where the factor is chosen such that the polynomial becomes monic.

Lemma 6.10. *For $n \ge 1$, we have a trigonometric differential relation:*

$$\frac{d}{d\omega}\Big[p_{n-1}^{m-2}\big(\tan(\omega)\big)\cos(\omega)^{m-1}\Big] = (n-m)\, p_n^m\big(\tan(\omega)\big)\cos(\omega)^{m-1}.$$

Proof. This is straightforward. Firstly, we calculate $\frac{d}{d\omega}\big[\sin(\omega)^j \cos(\omega)^{m-j-1}\big] = j\sin(\omega)^{j-1}\cos(\omega)^{m-j} - (m-j-1)\sin(\omega)^{j+1}\cos(\omega)^{m-j-2}$, and so

$$\begin{aligned}
&\frac{d}{d\omega}\Big[p_{n-1}^{m-2}\big(\tan(\omega)\big)\cos(\omega)^{m-1}\Big] \\
&\quad= \sum_{\substack{j=0 \\ n-j \text{ odd}}}^{n-1} \frac{(-1)^{\frac{n-j-1}{2}}(n-1)!\,(m-2n)!!}{j!\,(m-n-j-1)!!\,(n-j-1)!!} \frac{d}{d\omega}\big[\sin(\omega)^j\cos(\omega)^{m-j-1}\big] \\
&\quad= \sum_{\substack{j=0 \\ n-j \text{ even}}}^{n-2} \frac{(-1)^{\frac{n-j-2}{2}}(n-1)!\,(m-2n)!!}{j!\,(m-n-j-2)!!\,(n-j-2)!!} \sin(\omega)^j\cos(\omega)^{m-j-1} \\
&\qquad\qquad - \sum_{\substack{j=1 \\ n-j \text{ even}}}^{n} \frac{(-1)^{\frac{n-j}{2}}(n-1)!\,(m-2n)!!\,(m-j)}{(j-1)!\,(m-n-j)!!\,(n-j)!!} \sin(\omega)^j\cos(\omega)^{m-j-1} \\
&\quad= \sum_{\substack{j=0 \\ n-j \text{ even}}}^{n} \frac{(-1)^{\frac{n-j}{2}}(n-1)!\,(m-2n)!!}{j!\,(m-n-j)!!\,(n-j)!!} \underbrace{\big[(j-n)(m-n-j) - j(m-j)\big]}_{=n(n-m)} \tan(\omega)^j\cos(\omega)^{m-1}
\end{aligned}$$

$$= (n-m)\, p_n^m\big(\tan(\omega)\big) \cos(\omega)^{m-1}. \qquad \square$$

Our next goal is to show that the p_n^m, for fixed m, form a set of orthogonal polynomials in the sense of the above subsection. In order to apply Favard's theorem, we claim:

Proposition 6.11. *For $0 \le 2n \le m-1$, we have a three-term recurrence:*

$$p_0^m(x) = 1, \qquad p_1^m(x) = x, \qquad p_{n+1}^m(x) = xp_n^m(x) - d_n^m p_{n-1}^m(x),$$

where $d_n^m := \frac{n(m-n+1)}{(m-2n)(m-2n+2)}$.

Proof. We start from the right: $xp_n^m(x) - d_n^m p_{n-1}^m(x)$ gives

$$\sum_{\substack{j=1 \\ n-j \text{ odd}}}^{n+1} \frac{(-1)^{\frac{n-j+1}{2}} n!\,(m-2n)!!}{(j-1)!\,(m-n-j+1)!!\,(n-j+1)!!}\, x^j \;-\; \sum_{\substack{j=0 \\ n-j \text{ odd}}}^{n-1} d_n^m \frac{(-1)^{\frac{n-j-1}{2}} (n-1)!\,(m-2n+2)!!}{j!\,(m-n-j+1)!!\,(n-j-1)!!}\, x^j$$

$$= \sum_{\substack{j=0 \\ n-j \text{ odd}}}^{n+1} \frac{(-1)^{\frac{n-j+1}{2}} (n+1)!\,(m-2n-2)!!}{j!\,(m-n-j-1)!!\,(n-j+1)!!} \underbrace{\frac{j(m-2n)+(m-n+1)(n-j+1)}{(n+1)(m-n-j+1)}}_{=1} x^j = p_{n+1}^m(x). \qquad \square$$

The next theorem gives a useful analytic form of the corresponding linear functional.

Theorem 6.12. *We define, for $m \ge 1$, a linear functional $\mathcal{L}$ on the vector space of polynomials of degree less than m, by setting*

$$\mathcal{L} : f \longmapsto \int_0^\infty \!\! \int_{-\infty}^\infty z^{m-1} f\left(\frac{y}{z}\right) e^{-\frac{y^2+z^2}{2}}\, dy dz. \tag{21}$$

Then the p_n^m form a set of orthogonal polynomials with respect to the induced bilinear form, i.e. for $k \ne n$, $k+n \le m-1$, we have $\mathcal{L}(p_k^m p_n^m) = 0$ and for $2n \le m-1$:

$$\mathcal{L}(p_n^m p_n^m) = 2^{\frac{3}{2}m-2n-\frac{1}{2}} \frac{n!}{(m-n)!} \Gamma\left(\frac{m}{2}-n\right) \Gamma\left(\frac{m}{2}-n+1\right) \Gamma\left(\frac{m+1}{2}\right). \tag{22}$$

Proof. Since p_n^m satisfy the three-term relation in Prop. 6.11, by Favard's theorem and Remark 6.7, there exists a unique functional $\mathcal{L}'$ with $\mathcal{L}'(1) = 1$, such that the p_n^m form an orthogonal basis with respect to the bilinear form induced by $\mathcal{L}'$. We claim that $\mathcal{L}$ is a scalar multiple of $\mathcal{L}'$. Since $(p_n^m)_n$ is a basis of the space of polynomials, we must show that, for $n \ge 1$, $\mathcal{L}(p_n^m) = \mathcal{L}(p_n^m p_0^m) = 0$. Using polar coordinates $(y, z) = (r\cos\omega, r\sin\omega)$ and Lemma 6.10, we get:

$$\begin{aligned}
\mathcal{L}(p_n^m) &= \int_0^\pi \int_0^\infty p_n^m\big(\tfrac{\cos\omega}{\sin\omega}\big)\sin(\omega)^{m-1} r^m e^{-\frac{r^2}{2}}\,dr\,d\omega \\
&= \int_0^\infty r^m e^{-\frac{r^2}{2}}\,dr \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (-1)^n p_n^m\big(\tan(\omega)\big)\cos(\omega)^{m-1}\,d\omega \\
&= 2^{\frac{m-1}{2}}\Gamma\left(\tfrac{m+1}{2}\right)\left[\tfrac{(-1)^n}{n-m} p_{n-1}^{m-2}\big(\tan(\omega)\big)\cos(\omega)^{m-1}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = 0,
\end{aligned}$$

while $\mathcal{L}(1) = 2^{\frac{m-1}{2}}\sqrt{\pi}\,\Gamma\big(\frac{m}{2}\big) = 2^{\frac{3m-1}{2}}\frac{1}{m!}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{m+1}{2}\right)$ by (13) and (11). To verify that equation (22) holds for $n \geq 1$, too, we must show that the right hand side satisfies the recurrence from Theorem 6.6, but this is immediate:

$$\frac{2^{\frac{3}{2}m-2n-\frac{1}{2}}\frac{n!}{(m-n)!}\Gamma\big(\frac{m}{2}-n\big)\,\Gamma\big(\frac{m}{2}-n+1\big)\,\Gamma\Big(\frac{m+1}{2}\Big)}{2^{\frac{3}{2}m-2n+\frac{3}{2}}\frac{(n-1)!}{(m-n+1)!}\Gamma\big(\frac{m}{2}-n+1\big)\,\Gamma\big(\frac{m}{2}-n+2\big)\,\Gamma\Big(\frac{m+1}{2}\Big)} = \frac{n(m-n+1)}{(m-2n)(m-2n+2)} = d_n^m.$$

□

Corollary 6.13. *$\mathcal{L}(x^k p_n^m) = 0$ for $k < n$ and $\mathcal{L}(x^n p_n^m) = \mathcal{L}(p_n^m p_n^m)$.*

Proof. By the theorem, we have $\mathcal{L}(x^0 p_n^m) = 0$ for $n > 0$, so the case $k = 0$ holds true. Now the three-term recurrence from Proposition 6.11 allows us to inductively conclude that $\mathcal{L}(x^k p_n^m) = \mathcal{L}(x^{k-1}p_{n+1}^m) + d_n^m \mathcal{L}(x^{k-1}p_{n-1}^m) = 0$. The second assertion, $\mathcal{L}(x^n p_n^m) = \mathcal{L}(p_n^m p_n^m)$ is trivial in the case $n \leq 1$. For $n \geq 1$, the three-term recurrence yields now $\mathcal{L}(x^n p_n^m) = \mathcal{L}(x^{n-1}p_{n+1}^m) + d_n^m \mathcal{L}(x^{n-1}p_{n-1}^m) = d_n^m \mathcal{L}(x^{n-1}p_{n-1}^m)$, so $\mathcal{L}(x^n p_n^m)$ and $\mathcal{L}(p_n^m p_n^m)$ (by Theorem 6.6) satisfy the same recurrence relation and therefore must be equal. □

6.3 Homogeneous orthogonal polynomials

We are now ready to give the desired basis $(h_\alpha)_\alpha$ of homogeneous polynomials that are orthogonal on the sphere.

Definition 6.14. For multi-indices $\alpha = (\alpha_0, \ldots, \alpha_d)$ we recursively define homogeneous polynomials h_α of degree $|\alpha|$ by $h_{(\alpha_0)}(x) := x_0^{\alpha_0}$ and, for $d \geq 1$,

$$h_\alpha(x) := p_{\alpha_d}^{2|\alpha|+d}\left(\frac{x_d}{r}\right) r^{\alpha_d} h_{\alpha'}(x'),$$

where we have set $r = \sqrt{x_0^2 + \ldots + x_{d-1}^2} = \|x'\|$. Note that the definition of p_n^m implies that $p_n^m(\frac{1}{y})y^n$ is an even polynomial, so all square roots vanish.

Theorem 6.15. *Let $\langle\!\langle\ ,\ \rangle\!\rangle$ be defined as in (20). For all multi-indices α, β of length $\operatorname{len}(\alpha) = \operatorname{len}(\beta) = d+1$ and degree $|\alpha| = |\beta|$ we have:*

$$\langle\!\langle h_\alpha, h_\alpha \rangle\!\rangle = \alpha_d!\,\frac{(2|\alpha'|+d)!!\ (2|\alpha|+d-1)!!}{(|\alpha'|+|\alpha|+d)!}\,\langle\!\langle h_{\alpha'}, h_{\alpha'} \rangle\!\rangle\,, \tag{23}$$

$$\langle\!\langle h_\alpha, h_\beta \rangle\!\rangle = 0 \quad \textit{for } \alpha \neq \beta, \tag{24}$$

$$\langle\!\langle x^\alpha, h_\alpha \rangle\!\rangle = \langle\!\langle h_\alpha, h_\alpha \rangle\!\rangle\,, \tag{25}$$

$$\langle\!\langle x^\alpha, h_\beta \rangle\!\rangle = 0 \quad \textit{for } \alpha < \beta \textit{ (see Def. 4.1).} \tag{26}$$

Remark 6.16. This means that the $h_\alpha(x)$, $|\alpha| = k$ form an orthogonal basis of $\mathbb{R}[x_0, \ldots, x_d]_k$ that comes from a Gram–Schmidt process applied to the monomials (in lexicographic order). Indeed, equations (23) and (24) imply that the $h_\alpha(x)$ are orthogonal, while equations (25) and (26) say that the transition matrix T^{-1}, defined by $x^\alpha = \sum_\beta T^{-1}_{\alpha\beta} h_\beta$, $T^{-1}_{\alpha\beta} := \frac{\langle\!\langle x^\alpha, h_\beta \rangle\!\rangle}{\langle\!\langle h_\beta, h_\beta \rangle\!\rangle}$ is lower triangular with all diagonal elements equal to 1.

Proof. We begin with equation (23). The term $\langle\!\langle h_{\alpha'}, h_{\alpha'} \rangle\!\rangle$ on the right hand side means of course the form in d instead of $d+1$ variables. We use polar coordinates on $\mathbb{R}^d$ to compute:

$$\begin{aligned}
\sqrt{2\pi}^{d+1}\langle\!\langle h_\alpha, h_\alpha \rangle\!\rangle &= \int_{\mathbb{R}^{d+1}} \left[p_{\alpha_d}^{2|\alpha|+d}\left(\frac{x_d}{r}\right) r^{\alpha_d} h_{\alpha'}(x) \right]^2 e^{-\frac{1}{2}\|x\|^2} dx \\
&= \underbrace{\int_0^\infty \int_{\mathbb{R}} \left[p_{\alpha_d}^{2|\alpha|+d}\left(\frac{x_d}{r}\right) \right]^2 r^{2|\alpha'|+2\alpha_d+d-1} e^{-\frac{r^2+x_d^2}{2}} dx_d dr}_{=\mathcal{L}\left(\left(p_{\alpha_d}^{2|\alpha|+d}\right)^2\right)} \int_{\mathbb{S}^{d-1}} \left[h_{\alpha'}(\omega)\right]^2 d\omega \\
&\overset{(22)}{=} \frac{\alpha_d!\,2^{2|\alpha'|+|\alpha|+\frac{3}{2}d-\frac{1}{2}}}{(|\alpha|+|\alpha'|+d)!} \Gamma\Big(|\alpha'|+\tfrac{d}{2}\Big)\,\Gamma\Big(|\alpha'|+\tfrac{d}{2}+1\Big)\,\Gamma\Big(|\alpha|+\tfrac{d+1}{2}\Big) \int_{\mathbb{S}^{d-1}} [h_{\alpha'}(\omega)]^2\, d\omega \\
&\overset{\text{Lemma 4.6}}{=} \alpha_d!\,2^{|\alpha'|+|\alpha|+d+\frac{1}{2}}\,\frac{\Gamma\Big(|\alpha'|+\frac{d}{2}+1\Big)\,\Gamma\Big(|\alpha|+\frac{d+1}{2}\Big)}{(|\alpha'|+|\alpha|+d)!} \int_{\mathbb{R}^d} \left[h_{\alpha'}(x')\right]^2 e^{-\frac{1}{2}\|x'\|^2} dx' \\
&\overset{(10)}{=} \alpha_d!\,\frac{(2|\alpha'|+d)!!\ (2|\alpha|+d-1)!!}{(|\alpha'|+|\alpha|+d)!}\, \underbrace{\sqrt{2\pi}\int_{\mathbb{R}^d} \left[h_{\alpha'}(x')\right]^2 e^{-\frac{1}{2}\|x'\|^2} dx'}_{=\sqrt{2\pi}^{d+1}\langle\!\langle h_{\alpha'}, h_{\alpha'} \rangle\!\rangle}\,.
\end{aligned}$$

For the proof of (24), let i be the highest index where α and β differ. Due to the recursive nature of (23) and the definition of h_α, we may assume $i = d$, so $\alpha_d \neq \beta_d$. Then we use the calculation above to see that Theorem 6.12 now implies the vanishing of the integral. Equations (25) and (26) follow analogously from Corollary 6.13. $\square$

Theorem 6.17. *Let the bilinear form $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\mathbb{R}[x_0, \ldots x_d]_k$ be defined as in (20). Let $D(d,k) := \det_{|\alpha|,|\beta|=k} \left\langle\!\!\left\langle x^\alpha, x^\beta \right\rangle\!\!\right\rangle$ be the determinant of its Gram matrix. Then:*

$$D(d,k) = \theta_{d,k}$$

where $\theta_{d,k}$ is defined as in (19).

Proof. This is a double induction over k and d. First check that $D(d,0) = \theta_{d,0} = 1$ and $D(0,k) = \theta_{0,k} = (2k-1)!!$. From the above theorem it is clear that $D(d,k) = \prod_{|\alpha|=k} \langle\!\langle h_\alpha, h_\alpha \rangle\!\rangle$. Since $\{|\alpha| = k\} = \bigcup_{j=0}^{k} \{|\alpha'| = j\} \times \{k-j\}$, we have from (23):

$$D(d,k) = \prod_{j=0}^{k} D(d-1,j) \prod_{|\alpha'|=j} (k-j)!\, \frac{(2j+d)!!\,(2k+d-1)!!}{(j+k+d)!},$$

hence we get the ratio

$$\begin{aligned} R(d,k) \ :=\ \frac{D(d,k)}{\prod_{j=0}^{k} D(d-1,j)} \ &=\ \prod_{j=0}^{k} \left[(k-j)!\, \frac{(2j+d)!!\,(2k+d-1)!!}{(j+k+d)!} \right]^{\binom{j+d-1}{d-1}} \\ &\overset{(8)}{=} \prod_{j=0}^{k} \left[\frac{(2j+d)!!}{(j+k+d)!!} \right]^{\binom{j+d-1}{d-1}} (2k+d-1)^{\binom{k+d}{d}} \prod_{i=1}^{k} i^{\binom{k-i+d}{d}}. \end{aligned}$$

We will now show the principal inductive step: $\frac{D(d,k+1)}{D(d,k)D(d-1,k+1)} = \frac{\theta_{d,k+1}}{\theta_{d,k}\theta_{d-1,k+1}}$. The left hand side clearly equals

$$\begin{aligned} \frac{R(d,k+1)}{R(d,k)} &= \frac{(2k+d+2)!!^{\binom{k+d}{d-1}}\,(2k+d+1)^{\binom{k+d+1}{d}}\,(2k+d+1)!!^{\binom{k+d}{d-1}}}{(2k+d+2)!^{\binom{k+d}{d-1}} \prod_{j=0}^{k} (j+k+d+1)^{\binom{j+d-1}{d-1}}} \prod_{i=1}^{k+1} i^{\binom{k-i+d}{d-1}} \\ &= \frac{(2k+d+1)^{\binom{k+d}{d}}}{\prod_{i=k+d+1}^{2k+d+1} i^{\binom{k-i+d}{d-1}}} \prod_{i=1}^{k+1} i^{\binom{k-i+d}{d-1}}. \end{aligned}$$

To simplify the right hand side a little bit, we split $\theta_{d,k} = A(d,k)B(d,k)$ with

$A(d,k) := \prod_{i=1}^{k} i^{\binom{k-i+d}{d}d}$, and $B(d,k)$ the complementary factor of $\theta_{d,k}$ depending on the parity of d. For $A(d,k)$ we have:

$$\frac{A(d,k+1)}{A(d,k)A(d-1,k+1)} = \prod_{i=1}^{k+1} i^{\binom{k-i+d+1}{d}d-\binom{k-i+d}{d}d-\binom{k-i+d}{d-1}(d-1)} \overset{(6)}{=} \prod_{i=1}^{k+1} i^{\binom{k-i+d}{d-1}},$$

while the other factor $B(d,k)$ gives, for even d,

$$\frac{B(d,k+1)}{B(d,k)B(d-1,k+1)} = \frac{(2k+d+1)^{\binom{-k-1}{d}} \prod_{\substack{i=1 \\ i \text{ odd}}}^{2k+d+1} i^{\binom{k-i+d+1}{d}-\binom{k-i+d}{d}}}{\prod_{i=1}^{k+\frac{d}{2}} i^{\binom{k-i+d}{d-1}} \prod_{\substack{i=1 \\ i \text{ even}}}^{2k+d} \left(\frac{i}{2}\right)^{-\binom{k-i+d}{d-1}}}$$

$$\overset{(6)}{=} \frac{(2k+d+1)^{\binom{k+d}{d}} \prod_{i=1}^{2k+d+1} i^{\binom{k-i+d}{d-1}}}{\prod_{i=1}^{k+\frac{d}{2}} i^{\binom{k-i+d}{d-1}} \prod_{\substack{i=1 \\ i \text{ even}}}^{2k+d} 2^{\binom{k-i+d}{d-1}}} \overset{(9)}{=} \frac{(2k+d+1)^{\binom{k+d}{d}}}{\prod_{i=k+d+1}^{2k+d+1} i^{\binom{k-i+d}{d-1}}},$$

but also for odd d,

$$\frac{B(d,k+1)}{B(d,k)B(d-1,k+1)} = \frac{(k+\frac{d+1}{2})^{-\binom{-k-1}{d}} \prod_{i=1}^{k+\frac{d+1}{2}} i^{\binom{k-i+d}{d-1}-\binom{k-2i+d}{d-1}}}{\prod_{\substack{i=1 \\ i \text{ odd}}}^{2k+d+1} i^{\binom{k-i+d}{d-1}}}$$

$$= \frac{(k+\frac{d+1}{2})^{\binom{k+d}{d}} \prod_{i=1}^{k+\frac{d+1}{2}} i^{\binom{k-i+d}{d-1}} \prod_{\substack{i=1 \\ i \text{ even}}}^{2k+d} 2^{\binom{k-i+d}{d-1}}}{\prod_{i=1}^{2k+d+1} i^{\binom{k-i+d}{d-1}}} \overset{(9)}{=} \frac{(2k+d+1)^{\binom{k+d}{d}}}{\prod_{i=k+d+1}^{2k+d+1} i^{\binom{k-i+d}{d-1}}}. \qquad \square$$

7 Application to Hyperkähler cohomology

Let X be a compact Hyperkähler manifold of complex dimension $2k$. The second cohomology group $H^2(X,\mathbb{Z})$ comes with an integral quadratic form, called the Beauville–Bogomolov form q_X, which can be computed by an integration over some cup–product power, see [50, Subsection 2.3]:

$$\int_X \alpha^{2k} = (2k-1)!!\, c_X q_X(\alpha)^k, \qquad \alpha \in H^2(X,\mathbb{Z}). \tag{27}$$

This equation is referred to as the Beauville–Fujiki relation. The constant $c_X \in \mathbb{Q}$ is chosen such that the quadratic form q_X is indivisible and its signum is such that $q_X(\sigma+\bar\sigma) > 0$ for a holomorphic two-form σ with $\int_X \sigma\bar\sigma = 1$. There is an alternative description, as shown in [25, Chap. 23]. Up to a scalar factor $\tilde c$, q_X is equal to:

$$\tilde c\, q_X(\alpha) = \frac{k}{2}\int_X \alpha^2(\sigma\bar\sigma)^{k-1} + (1-k)\left(\int_X \alpha\,\sigma^{n-1}\bar\sigma^n\right)\left(\int_X \alpha\,\sigma^n\bar\sigma^{n-1}\right). \tag{28}$$

Now q_X, by polarization, gives rise to a symmetric bilinear form $\langle\ ,\ \rangle$ on $H^2(X,\mathbb{Z})$, namely $2\langle\alpha,\beta\rangle := q_X(\alpha+\beta) - q_X(\alpha) - q_X(\beta)$. On the other hand, from (27) one deduces again by polarization, as shown in Section 3, that:

$$\int_X \alpha_1 \wedge \ldots \wedge \alpha_{2k} = c_X \langle\!\langle \alpha_1 \ldots \alpha_k\,,\, \alpha_{k+1}\ldots\alpha_{2k}\rangle\!\rangle\,, \tag{29}$$

with the induced form $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\operatorname{Sym}^k H^2(X,\mathbb{Z})$, according to Definition 5.1. The discriminant of $\operatorname{Sym}^k H^2(X,\mathbb{Z})$ can be computed with Theorem 5.6. Since the Poincaré pairing $(\beta_1,\beta_2)_X := \int_X \beta_1\wedge\beta_2$ gives $H^{2k}(X,\mathbb{Z})$ the structure of a unimodular lattice, we have got an embedding of lattices:

$$\Big(\operatorname{Sym}^k H^2(X,\mathbb{Z}),\ c_X\langle\!\langle\ ,\ \rangle\!\rangle\Big) \longrightarrow \Big(H^{2k}(X,\mathbb{Z}),\ (\ ,\ )_X\Big). \tag{30}$$

In general, this embedding is not primitive.

Theorem 7.1. *Let $d+1$ be the rank of $H^2(X,\mathbb{Z})$ and denote c_X the Fujiki constant. The discriminant of $\operatorname{Sym}^n H^2(X,\mathbb{Z})$ is given by*

$$\Big(\operatorname{discr}\Big(H^2(X,\mathbb{Z})\Big)\Big)^{\binom{d+n}{d+1}} \cdot c_X^{\binom{d+n}{d}} \cdot \prod_{i=1}^{n} i^{\binom{n-i+d}{d}d} \cdot C,$$

$$\textit{with}\ \ C = \begin{cases} \displaystyle\prod_{\substack{i=1\\ i\ \text{odd}}}^{2n+d-1} i^{\binom{n-i+d}{d}} & \textit{if } d{+}1 \textit{ is odd,}\\[2ex] \displaystyle\prod_{i=1}^{n+\frac{d-1}{2}} i^{\binom{n-i+d}{d}-\binom{n-2i+d}{d}} & \textit{if } d{+}1 \textit{ is even.}\end{cases}$$

This statement is deduced from Theorem 5.6 and Proposition 2.4. We get some interesting corollaries, by looking at the prime factors contained in the formula:

Corollary 7.2. *Let X be a compact Hyperkähler manifold of complex dimension $2k$. Denote b_2 resp. d_2 the rank and the discriminant of $H^2(X,\mathbb{Z})$. Define a set of integers Z by:*

$$\{c_X^{b_2} d_2\} \cup \{1,\dots,k\} \cup \begin{cases} \{i \in \mathbb{Z} \mid k+b_2 \le i \le 2k+b_2-2,\ i \text{ odd}\} & \text{if } b_2 \text{ is odd,} \\ \{i \in \mathbb{Z} \mid \frac{k+b_2}{2} \le i \le k+\frac{b_2}{2}-1\} & \text{if } b_2 \text{ is even.} \end{cases}$$

Then the discriminant of $\operatorname{Sym}^k H^2(X,\mathbb{Z})$ *and hence the torsion part of the quotient*

$$\frac{H^{2k}(X,\mathbb{Z})}{\operatorname{Sym}^k H^2(X,\mathbb{Z})}$$

contain only prime factors that divide at least one of the numbers contained in Z.

For the known examples of compact Hyperkähler manifolds in higher dimensions, we can use the list given in [50, Table 1]: Let $S^{[k]}$ for $k \ge 2$ be the Hilbert scheme of k points on a K3 surface S, let $A^{[[k]]}$ be the generalized Kummer variety of a torus A and let OG_6 and OG_{10} be the 6– resp. 10–dimensional O'Grady manifold. Let X be deformation equivalent to one of these. Then we have:

X	$\dim X$	b_2	d_2	c_X
$S^{[k]}$	$2k$	23	$2(k-1)$	1
$A^{[[k]]}$	$2k$	7	$2(k+1)$	$k+1$
OG_6	6	8	4	4
OG_{10}	10	24	3	1

In particular, the torsion part of $\frac{H^{2k}(X,\mathbb{Z})}{\operatorname{Sym}^k H^2(X,\mathbb{Z})}$ contains no prime factors bigger than

- 3, if $X \sim_{\text{def}} OG_6$,
- 5, if $X \sim_{\text{def}} OG_{10}$.

Remark 7.3. The case $X \sim_{\text{def}} S^{[2]}$ was already studied in [5, Prop. 6.6], using explicit calculations. It is special, because $\operatorname{Sym}^2 H^2(S^{[2]},\mathbb{Z})$ and $H^4(S^{[2]},\mathbb{Z})$ have the same rank. So Theorem 7.1 implies that the cardinality of the quotient is precisely $\sqrt{2^{24}\cdot 2^{22}\cdot (22+3)} = 2^{23}\cdot 5$.

Remark 7.4. If the exact value of the torsion part of $H^{2k}/(\operatorname{Sym}^k H^2)$ is known, we can say more, by applying Corollary 2.6. For instance, in the case $X \sim_{\mathrm{def}} S^{[3]}$, Theorem 7.1 says that the discriminant of $\operatorname{Sym}^3 H^2$ is equal to $2^{1106} \cdot 3^{92}$. But on the other hand, according to [27, Prop. 2.4], the torsion part of $H^6/(\operatorname{Sym}^3 H^2)$ has order $2^{277} \cdot 3^{46}$. So it follows that both the orthogonal complement and the primitive overlattice of $\operatorname{Sym}^3 H^2$ must have discriminant $\frac{2^{1106} \cdot 3^{92}}{(2^{277} \cdot 3^{46})^2} = 2^{552}$.

If the torsion part of $H^{2k}/(\operatorname{Sym}^k H^2)$ is unknown, then Corollary 2.6 still allows the conclusion, that the square-free parts of the discriminants of $\operatorname{Sym}^k H^2$ and its orthogonal complement are equal.

Part II

Hilbert schemes and generalized Kummer varieties

8 Super algebras

Let us recall some material on super algebras, which will be useful in Sections 13 and 15 to understand the cohomology structure of the Hilbert schemes of points on surfaces.

Definition 8.1. A super vector space V over a field k is a vector space with a $\mathbb{Z}/2\mathbb{Z}$-graduation, that is a decomposition

$$V = V^+ \oplus V^-,$$

called the even and the odd part of V. Elements of V^+ are called homogeneous of even degree, elements of V^- are called homogeneous of odd degree. The degree of a homogeneous element v is denoted by $|v| \in \mathbb{Z}/2\mathbb{Z}$. Direct sum and tensor product of two super vector spaces V and W yield again super vector spaces:

$$(V \oplus W)^+ = V^+ \oplus W^+, \qquad (V \oplus W)^- = V^- \oplus W^-,$$
$$(V \otimes W)^+ = (V^+ \otimes W^+) \oplus (V^- \otimes W^-), \quad (V \otimes W)^- = (V^+ \otimes W^-) \oplus (V^- \otimes W^+).$$

Definition 8.2. A superalgebra R is a unital associative k-algebra which carries a super vector space structure. Define the supercommutator by setting for homogeneous elements $u, v \in R$:

$$[u, v] := uv - (-1)^{|u||v|} vu.$$

R is called supercommutative, if $[u, v] = 0$ for all $u, v \in R$. Note that a graded commutative algebra $R = \bigoplus_n R^n$ is supercommutative in a natural way, by setting $R^+ = \bigoplus_{n \text{ even}} R^n$, $R^- = \bigoplus_{n \text{ odd}} R^n$.

For a supercommutative algebra R, the tensor power $R^{\otimes n}$ is again a supercommutative algebra, if we set for the product:

$$(u_1 \otimes \cdots \otimes u_n) \cdot (v_1 \otimes \cdots \otimes v_n) = (-1)^{\sum\limits_{i>j} |u_i||v_j|} u_1 v_1 \otimes \cdots \otimes u_n v_n.$$

Definition 8.3. Let V be a super vector space over k and $n \geq 0$. Then the supersymmetric power $\mathbb{S}\mathrm{ym}^n(V)$ of V is a super vector space, given by

$$\mathbb{S}\mathrm{ym}^n(V) = \bigoplus_{p+q=n} \mathrm{Sym}^p(V^+) \otimes \Lambda^q(V^-),$$

$$\mathbb{S}\mathrm{ym}^n(V)^+ = \bigoplus_{\substack{p+q=n \\ q \text{ even}}} \mathrm{Sym}^p(V^+) \otimes \Lambda^q(V^-), \quad \mathbb{S}\mathrm{ym}^n(V)^- = \bigoplus_{\substack{p+q=n \\ q \text{ odd}}} \mathrm{Sym}^p(V^+) \otimes \Lambda^q(V^-).$$

The supersymmetric algebra $\mathbb{S}\mathrm{ym}^*(V) := \bigoplus_n \mathbb{S}\mathrm{ym}^n(V)$ on V is a supercommutative algebra over k, where the product of two elements $s \otimes e \in \mathrm{Sym}^p(V^+) \otimes \Lambda^q(V^-)$ and $s' \otimes e' \in \mathrm{Sym}^{p'}(V^+) \otimes \Lambda^{q'}(V^-)$ is given by

$$(s \otimes e) \diamond (s' \otimes e') = (ss') \otimes (e \wedge e') \quad \in \mathrm{Sym}^{p+p'}(V^+) \otimes \Lambda^{q+q'}(V^-).$$

Remark 8.4. The supersymmetric power $\mathbb{S}\mathrm{ym}^n(V)$ can be realized as a quotient of $V^{\otimes n}$ by an action of the symmetric group $\mathfrak{S}_n$. This action can be described as follows: If $\tau \in \mathfrak{S}_n$ is a transposition that exchanges two numbers $i < j$, then τ permutes the corresponding tensor factors in $v_1 \otimes \cdots \otimes v_n$ introducing a sign $(-1)^{|v_i||v_j|+(|v_i|+|v_j|)\sum_{i<k<j}|v_k|}$.

Now let U be a vector space over a field k of characteristic 0 and look at the exterior algebra $H := \Lambda^* U$. Since H is a super vector space, we can construct the supersymmetric power $S^n := \mathbb{S}\mathrm{ym}^n(H)$. We may identify S^n with the space of $\mathfrak{S}_n$-invariants in $H^{\otimes n}$ by means of the linear projection operator

$$\mathrm{pr} : H^{\otimes n} \longrightarrow S^n, \quad \mathrm{pr} = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \pi.$$

The multiplication in $H^{\otimes n}$ induces a multiplication on the subspace of invariants, which makes S^n a supercommutative algebra. Of course, it is different from the product $\diamond$ discussed above.

Since H is generated as an algebra by $U = \Lambda^1(U) \subset H$, we may define a homomorphism of algebras:

$$s : H \longrightarrow S^n, \quad s(u) = \mathrm{pr}(u \otimes 1 \otimes \cdots \otimes 1) \text{ for } u \in U,$$

so S^n becomes an algebra over H.

Lemma 8.5. *The morphism s turns S^n into a free module over H, for $n \geq 1$.*

Proof. We start with the tensor power $H^{\otimes n}$ and the ring homomorphism

$$\iota : H \longrightarrow H^{\otimes n}, \quad h \longmapsto h \otimes 1 \otimes \cdots \otimes 1$$

that makes $H^{\otimes n}$ a free H-module. Note that $\operatorname{pr} \iota \neq s$, since pr is not a ring homomorphism. (For example, $\operatorname{pr} \iota(h) \neq s(h)$ for any nonzero $h \in \Lambda^2(U)$.) We therefore modify the H-module structure of $H^{\otimes n}$:

For some $u \in U$, denote $u^{(i)} := 1^{\otimes i-1} \otimes u \otimes 1^{\otimes n-i+1} \in H^{\otimes n}$. Then $H^{\otimes n}$ is generated as a k-algebra by the elements $\{u^{(i)}, u \in U\}$. Now consider the ring automorphism

$$\sigma : H^{\otimes n} \longrightarrow H^{\otimes n}, \quad u^{(1)} \longmapsto u^{(1)} + u^{(2)} + \ldots + u^{(n)}, \quad u^{(i)} \longmapsto u^{(i)} \text{ for } i > 1.$$

Then we have $\sigma\iota = s$ on S^n. On the other hand, if $\{b_i\}$ is a k-basis of V, then $\{b_i^{(j)}, j > 1\}$ is a ι-basis for $H^{\otimes n}$, and $\{\sigma(b_i^{(j)})\}$ is a $\sigma\iota$-basis for $H^{\otimes n}$. Now if we project the basis elements, we get a set $\{\operatorname{pr}(\sigma(b_i^{(j)}))\}$ that spans S^n. Eliminating linear dependent vectors (this is possible over the rationals), we get a s-basis of S^n. □

9 Actions of the symplectic group over finite fields

The aim of this section is to provide some special computations used in Section 17.

Let V be a symplectic vector space of dimension $n \in 2\mathbb{N}$ over a field k with a nondegenerate symplectic form $\omega : \Lambda^2 V \to k$. A line is a one-dimensional subspace of V through the origin, a plane is a two-dimensional subspace of V. A plane $P \subset V$ is called isotropic, if $\omega(x, y) = 0$ for any $x, y \in P$, otherwise non-isotropic. The symplectic group $\operatorname{Sp} V$ is the set of all linear maps $\phi : V \to V$ with the property $\omega(\phi(x), \phi(y)) = \omega(x, y)$ for all $x, y \in V$.

Proposition 9.1. *The symplectic group* $\operatorname{Sp} V$ *acts transitively on the set of non-isotropic planes as well as on the set of isotropic planes.*

Proof. Given two planes P_1 and P_2, we may choose vectors v_1, v_2, w_1, w_2 such that v_1, v_2 span P_1, w_1, w_2 span P_2 and $\omega(u_1, u_2) = \omega(w_1, w_2)$. We complete $\{v_1, v_2\}$ as well as $\{w_1, w_2\}$ to a symplectic basis of V. Then define $\phi(v_1) = w_1$ and $\phi(v_2) = w_2$. It is now easy to see that the definition of ϕ can be extended to the remaining basis elements to give a symplectic morphism. □

Remark 9.2. The set of planes in V can be identified with the simple tensors in $\Lambda^2 V$ up to multiples. Indeed, given a simple tensor $v \wedge w \in \Lambda^2 V$, the span of v and w yields the corresponding plane. Conversely, any two spanning vectors v and w of a plane give the same element $v \wedge w$ (up to multiples).

Remark 9.3. If k is the field with two elements, then the set of planes in V can be identified with the set $\{\{x, y, z\} \mid x, y, z \in V\backslash\{0\},\, x + y + z = 0\}$. Observe that for such a $\{x, y, z\}$, $\omega(x, y) = \omega(x, y) = \omega(y, x)$ and this value gives the criterion for isotropy.

From now on, we assume that k is finite of cardinality q.

Proposition 9.4.

$$\textit{The number of lines in } V \textit{ is } \frac{q^n - 1}{q - 1}, \tag{31}$$

$$\textit{the number of planes in } V \textit{ is } \frac{(q^n - 1)(q^{n-1} - 1)}{(q^2 - 1)(q - 1)}, \tag{32}$$

$$\textit{the number of isotropic planes in } V \textit{ is } \frac{(q^n - 1)(q^{n-2} - 1)}{(q^2 - 1)(q - 1)}, \tag{33}$$

$$\textit{the number of non-isotropic planes in } V \textit{ is } \frac{q^{n-2}(q^n - 1)}{q^2 - 1}. \tag{34}$$

Proof. A line ℓ in V is determined by a nonzero vector. There are $q^n - 1$ nonzero vectors in V and $q - 1$ nonzero vectors in ℓ. A plane P is determined by a line $\ell_1 \subset V$ and a unique second line $\ell_2 \in V/\ell_1$. We have $\frac{q^2-1}{q-1}$ choices for ℓ_1 in P. The number of planes is therefore

$$\frac{\frac{q^n-1}{q-1} \cdot \frac{q^{n-1}-1}{q-1}}{\frac{q^2-1}{q-1}} = \frac{(q^n - 1)(q^{n-1} - 1)}{(q^2 - 1)(q - 1)}.$$

For an isotropic plane we have to choose the second line from $\ell_1^{\perp}/\ell_1$. This is a space of dimension $n - 2$, hence the formula. The number of non-isotropic planes is the difference of the two previous numbers. $\square$

We want to study the free k-module $k[V]$ with basis $\{X_i \,|\, i \in V\}$. It carries a natural k-algebra structure, given by $X_i \cdot X_j := X_{i+j}$ with unit $1 = X_0$. This algebra is local with maximal ideal $\mathfrak{m}$ generated by all elements of the form $(X_i - 1)$.

We introduce an action of $\mathrm{Sp}(4, k)$ on $k[V]$ by setting $\phi(X_i) = X_{\phi(i)}$. Furthermore, the underlying additive group of V acts on $k[V]$ by $v(X_i) = X_{i+v} = X_i X_v$.

Definition 9.5. For a line $\ell \subset V$ define $S_\ell := \sum_{i\in\ell} X_i$. For a vector $0 \neq v \in \ell$ we set $S_v := S_\ell$.

Lemma 9.6. *Let $P \subset V$ be a plane and $\ell_1, \ell_2 \subset P$ two different lines spanning P. Then we have*

$$S_{\ell_1} S_{\ell_2} = \sum_{i\in P} X_i = \sum_{\ell\subset P} S_\ell.$$

Proof. The first equality is clear. For the second equality observe that every point $i \in P$ is contained in one line, if we count modulo q. ☐

Definition 9.7. We define two subsets of $k[V]$:

$$M := \left\{ \sum_{i\in P} X_i \,|\, P \subset V \text{ plane} \right\},$$

$$N := \left\{ \sum_{i\in P} X_i \,|\, P \subset V \text{ non-isotropic plane} \right\}.$$

Let (M) and (N) be the ideals generated by M and N, respectively. Further, let D be the linear span of $\{v(b) - b \,|\, b \in N, v \in V\}$. Then D is in fact an ideal, namely the product of ideals $\mathfrak{m} \cdot (N)$.

Proposition 9.8. *We have $(M) = (N)$.*

Proof. We have to show that $\sum_{i\in P} X_i \in (N)$ for an isotropic plane P. Let v, w be two spanning vectors of P and u a vector with $\omega(u, v) \neq 0$. Denote P' the non-isotropic plane spanned by u and v. By Lemma 9.6, we have

$$S_u S_v S_w = \sum_{\ell\subset P'} S_\ell S_w = \left(S_v + \sum_{\lambda\in k} S_{u+\lambda v} \right) S_w.$$

Now w spans a non-isotropic plane with every line in P', except one, namely the line that contains v. So it follows that

$$\sum_{i\in P} X_i = S_v S_w = S_u S_v S_w - \sum_{\lambda\in k} S_{u+\lambda v} S_w,$$

and we see that the right hand side is an element of (N). ☐

For the rest of this section, we assume $\dim_k V = 4$.

Proposition 9.9. *The following table illustrates the dimensions of* (N) *and* D *for some* k*.*

k	$\dim_k(N)$	$\dim_k D$
$\mathbb{F}_2$	*11*	*5*
$\mathbb{F}_3$	*50*	*31*
$\mathbb{F}_5$	*355*	*270*

Since this is computed numerically using a naive approach, we do not give a formal proof.

Remark 9.10. We remark that $X := \sum_{i \in V} X_i \in D$. Indeed, let P, P' be two non-isotropic planes with $P \cap P' = 0$. Then $X = (\sum_{i \in P} X_i)(\sum_{i \in P'} X_i)$ and both factors are contained in $(N) \subset \mathfrak{m}$, so $X \in \mathfrak{m} \cdot (N) = D$.

Let us now consider some special orthogonal sums. Set $S := \mathrm{Sym}^2(\Lambda^2 V)$. Take two vectors $v, w \in V$ with $\omega(v, w) = 1$ and set $x := (v \wedge w)^2 \in S$. Denote P the plane spanned by v and w and set $y := \sum_{i \in P} X_i \in k[V]$.

We consider now the action of $\mathrm{Sp}\, V$ on $S \oplus k[V]$. Denote O the vector space spanned by the elements $\phi(x) \oplus \phi(z)$, for $\phi \in \mathrm{Sp}\, V$, $z \in (y)$ and U the vector space spanned by the elements $\phi(x)$, for $\phi \in \mathrm{Sp}\, V$.

Proposition 9.11. *Then we have by numerical computation:*

k	$\dim_k O$	$\dim_k U$
$\mathbb{F}_2$	*11*	*6*
$\mathbb{F}_3$	*51*	*20*
$\mathbb{F}_5$	*375*	*20*

Now we prove the following lemma that we will need for a divisibility argument in Section 17.

Lemma 9.12. *We assume that* $k = \mathbb{F}_3$*. Let* $\mathrm{pr}_1 : S \oplus k[V] \to S$ *and* $\mathrm{pr}_2 : S \oplus k[V] \to k[V]$ *the projection. We have:*

(i) $\dim \ker \mathrm{pr}_{2|O} = 1$.

(ii) $\dim \ker \mathrm{pr}_{1|O} = 31$.

Proof. We have $\mathrm{pr}_1(O) = U$ and $\mathrm{pr}_2(O) = (N)$. Using the dimension tables from Propositions 9.11 and 9.12, we get

$$\dim \ker \mathrm{pr}_{1|O} = \dim O - \dim U = 31,$$
$$\dim \ker \mathrm{pr}_{2|O} = \dim O - \dim(N) = 1. \qquad \square$$

10 Complex abelian surfaces

Denote A a complex abelian surface (a torus of dimension 2). As such, it always can be written as a quotient

$$A = \mathbb{C}^2/\Lambda,$$

where $\Lambda \subset \mathbb{C}^2$ is a lattice of rank 4, embedded in $\mathbb{C}^2$. Depending on the imbedding, we get different complex manifolds, projective or not. Of course, all of them are equivalent by deformation.

10.1 Morphisms and special cases

Definition 10.1. An isogeny between abelian surfaces $A = \mathbb{C}^2/\Lambda \to A' = \mathbb{C}^2/\Lambda'$ means a surjective holomorphic map that preserves the group structure. It is given by a complex linear map, that maps Λ to a sublattice of Λ'.

Example 10.2. For a number $n \neq 0$, the multiplication map $n : A \to A$, $x \mapsto n \cdot x$ is an isogeny.

By an automorphism of A we mean a biholomorphism preserving the group structure. It can be represented by a $\mathbb{C}$-linear map $M : \mathbb{C}^2 \to \mathbb{C}^2$ with $M\Lambda = \Lambda$. Have a look in [19] or the appendix of [22] for some reference. Let us now come to the very special case that $A = E \times E$ can be written as the square of an elliptic curve. Note that A is projective, because every elliptic curve is. Now write E as $E = \mathbb{C}/\Lambda_0$. We may assume that Λ_0 is spanned by 1 and a vector $\tau \in \mathbb{C}\backslash\mathbb{R}$. The automorphism group, up to isogeny, is given by ([22]) $\mathrm{GL}(2, \mathrm{End}(\Lambda_0))$, where $\mathrm{End}(\Lambda_0)$ is the set $\{z \in \mathbb{C} \mid z\Lambda_0 \subset \Lambda_0\}$.

Proposition 10.3. *There are two possibilities for* $\mathrm{End}(\Lambda_0)$*, depending on* τ*:*

(i) Both the real part and the square norm of τ *are rational numbers, say* $2\Re(\tau) = \frac{p}{r}$ *and* $\|\tau\|^2 = \frac{q}{r}$ *with* $r > 0$ *as small as possible. Then* $\mathrm{End}(\Lambda_0) = \mathbb{Z} + r\tau\mathbb{Z}$*.*

(ii) At least one of $\Re(\tau), \|\tau\|^2$ *is irrational. Then* $\mathrm{End}(\Lambda_0) = \mathbb{Z}$*.*

Proof. Given $z \in \mathrm{End}(\Lambda_0)$, we have

$$z \cdot 1 = a + b\tau \text{ and } z \cdot \tau = c + d\tau \text{ with } a, b, c, d \in \mathbb{Z}.$$

We get the condition

$$(a + b\tau)\tau = c + d\tau \quad \Leftrightarrow \quad b\tau^2 + (a - d)\tau - c = 0.$$

Up to scalar multiples, there is a unique real quadratic polynomial that annihilates τ, namely $(x-\tau)(x-\bar{\tau}) = x^2 - 2\Re(\tau)x + \|\tau\|^2$. If all coefficients of that polynomial are rational numbers, then $z = a + b\tau$ gives a solution for arbitrary $a \in \mathbb{Z}$, $b \in r\mathbb{Z}$. Otherwise, the condition must be the zero polynomial, so $b = 0$. $\square$

Now we study the action of automorphisms on torsion points in a very special case. This will be needed in the technical proof of Theorem 20.9.

Definition 10.4. Denote $\xi \in \mathbb{C}$ a primitive sixth root of unity and E_ξ the elliptic curve given by the choice $\Lambda_0 = \langle 1, \xi \rangle$, so by Proposition 10.3, $\mathrm{End}(\Lambda_0) = \Lambda_0$ is the ring of Eisenstein integers. Define a group G_ξ of automorphisms of $E_\xi \times E_\xi$ by the following generators in $\mathrm{GL}(2, \mathrm{End}(\Lambda_0))$:

$$g_1 = \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad g_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For $A = E_\xi \times E_\xi$, let $V = A[2]$ be the (fourdimensional) $\mathbb{F}_2$-vector space of 2-torsion points on A and let $\mathfrak{T}$ be the set of planes in V. Note that by Remark 9.3 a plane in V can be identified with an unordered triple $\{x, y, z\}$ with $0 \neq x, y, z \in V$ and $x + y + z = 0$. The action of G_ξ on A induces actions of G_ξ on $A[2]$ and $\mathfrak{T}$.

Lemma 10.5. *There are two orbits of G_ξ on $\mathfrak{T}$, of cardinalities 5 and 30.*

Proof. Note that the generators g_2 and g_3 exist because A is of the form $E \times E$, while g_1 exists only in the special case $E = E_\xi$. Indeed, multiplication with ξ induces a cyclic permutation on $E_\xi[2]$. The orbits can be explicitely determined by a suitable computer program. For verification, we give one of the orbits explicitely. Denote x_1, x_2, x_3 the non-zero points in $E_\xi[2]$. The orbit of cardinality five is then given by

$$\begin{gathered}\{(0,x_1),(0,x_2),(0,x_3)\}, \quad \{(x_1,0),(x_2,0),(x_3,0)\}, \quad \{(x_1,x_1),(x_2,x_2),(x_3,x_3)\} \\ \{(x_1,x_2),(x_2,x_3),(x_3,x_1)\}, \quad \{(x_1,x_3),(x_2,x_1),(x_3,x_2)\}. \end{gathered}$$ $\square$

10.2 Homology and Cohomology

The fundamental group $\pi_1(A, \mathbb{Z}) = H_1(A, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 4, which is canonically identified with the lattice Λ. Indeed, the projection of every path in $\mathbb{C}^2$ from 0 to $v \in \Lambda$ gives a unique element of $\pi_1(A, \mathbb{Z})$. Conversely, any closed path in A with basepoint 0 lifts to a unique path in $\mathbb{C}^2$ from 0 to some $v \in \Lambda$. So the first cohomology $H^1(A,\mathbb{Z})$ is freely generated by four elements, too. Moreover, by [45, Sect. I.1], the cohomology ring is isomorphic to the exterior algebra

$$H^*(A,\mathbb{Z}) = \Lambda^*(H^1(A,\mathbb{Z})).$$

Notation 10.6. We denote the generators of $H^1(A,\mathbb{Z})$ by a_i, $1 \leq i \leq 4$ and their respective duals by $a_i^* \in H^3(A,\mathbb{Z})$. If $A = E \times E$ is the product of two elliptic curves, we choose the a_i in a way such that $\{a_1, a_2\}$ and $\{a_3, a_4\}$ give bases of $H^1(E,\mathbb{Z})$ in the decomposition $H^1(A) = H^1(E) \oplus H^1(E)$. We denote the generator of the top cohomology $H^4(A,\mathbb{Z})$ by $x := a_1 a_2 a_3 a_4$. A basis of $H^2(A,\mathbb{Z})$ will be denoted by $(b_i)_{1 \leq i \leq 6}$.

Let A be a abelian surface. We recall that a *principal polarization* of A is a polarization L such that there exists a basis of $H_1(A,\mathbb{Z})$, with respect to which the symplectic bilinear form on $H_1(A,\mathbb{Z})$ induced by $c_1(L)$:

$$\omega_L(x,y) = x \cdot c_1(L) \cdot y, \tag{35}$$

is given by the matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We recall the following result.

Proposition 10.7. *Let (A, L) be a principally polarized abelian surface. The group $H_1(A,\mathbb{Z})$ is endowed with the symplectic from ω_L defined in (35). Let $\mathrm{Mon}(H_1(A,\mathbb{Z}))$ be the image of monodromy representations on $H_1(A,\mathbb{Z})$. Then $\mathrm{Mon}(H_1(A,\mathbb{Z})) \supset \mathrm{Sp}(H_1(A,\mathbb{Z}))$.*

Proof. It can be seen as follows. Let $\mathcal{M}_2$ be the moduli space of curves of genus 2 and $\mathcal{A}_2$ be the moduli space of principally polarized abelian surfaces. By the Torelli theorem (see for instance [40, Theorem 12.1]), we have an injection $J : \mathcal{M}_2 \hookrightarrow \mathcal{A}_2$ given by taking the Jacobian of the curve endowed with its canonical polarization. Moreover, the moduli spaces $\mathcal{M}_2$ and $\mathcal{A}_2$ are both of dimension 3.

Now if $\mathscr{C}_2$ is a curve of genus 2, we have by Theorem 6.4 of [15]:

$$\mathrm{Mon}(H_1(\mathscr{C}_2,\mathbb{Z})) \supset \mathrm{Sp}(H_1(\mathscr{C}_2,\mathbb{Z})),$$

where the symplectic form on $H_1(\mathscr{C}_2, \mathbb{Z})$ is given by the cup product. Then the result follows from the fact that the lattices $H_1(\mathscr{C}_2, \mathbb{Z})$ and $H_1(J(\mathscr{C}_2), \mathbb{Z})$ are isometric. $\square$

Remark 10.8. Let (A, L) be a principally polarized abelian surface and p a prime number. The group $H_1(A, \mathbb{Z})$ tensorized by $\mathbb{F}_p$ can be seen as the group $A[p]$ of points of p-torsion on A and the form $\omega_L \otimes \mathbb{F}_p$ provides a symplectic form on $A[p]$. Then $\mathrm{Mon}(A[p])$, the image of the monodromy representation on $A[p]$ contains the group $\mathrm{Sp}(A[p])$.

Now, we are ready to recall Proposition 5.2 of [24] on the monodromy of the generalized Kummer fourfold.

Proposition 10.9. *Let A be an abelian surface and $K_2(A)$ the associated generalized Kummer fourfold. The image of the monodromy representation on $\Pi = \langle Z_\tau |\ \tau \in A[3]\rangle$ contains the semidirect product $\mathrm{Sp}(A[3]) \ltimes A[3]$ which acts as follows:*

$$f \cdot Z_\tau = Z_{f(\tau)} \text{ and } \tau' \cdot Z_\tau = Z_{\tau+\tau'},$$

for all $f \in \mathrm{Sp}(A[3])$ and $\tau' \in A[3]$.

11 Recall on the theory of integral cohomology of quotients

The main references of this section are [38] and [5].

Let $G = \langle \iota \rangle$ be the group generated by an involution ι on a complex manifold X. As denoted in [5, Section 5], let $\mathcal{O}_K$ be the ring $\mathbb{Z}$ with the following G-module structure: $\iota \cdot x = -x$ for $x \in \mathcal{O}_K$. For $a \in \mathbb{Z}$, we also denote by $(\mathcal{O}_K, a)$ the module $\mathbb{Z} \oplus \mathbb{Z}$ whose G-module structure is defined by $\iota \cdot (x, k) = (-x + ka, k)$. We also denote by N_2 the $\mathbb{F}_2[G]$-module $(\mathcal{O}_K, a) \otimes \mathbb{F}_2$. We recall Definition-Proposition 2.2.2 of [38].

Definition-Proposition 11.1. *Assume that $H^*(X, \mathbb{Z})$ is torsion-free. Then for all $0 \le k \le 2\dim X$, we have an isomorphism of $\mathbb{Z}[G]$-modules:*

$$H^k(X, \mathbb{Z}) \simeq \bigoplus_{i=1}^{r} (\mathcal{O}_K, a_i) \oplus \mathcal{O}_K^{\oplus s} \oplus \mathbb{Z}^{\oplus t},$$

for some $a_i \notin 2\mathbb{Z}$ and $(r, s, t) \in \mathbb{N}^3$. We get the following isomorphism of $\mathbb{F}_2[G]$-modules:

$$H^k(X, \mathbb{F}_2) \simeq N_2^{\oplus r} \oplus \mathbb{F}_2^{\oplus (s+t)}.$$

We denote $l_2^k(X) := r$, $l_{1,-}^k(X) := s$, $l_{1,+}^k(X) := t$, $\mathcal{N}_2 := N_2^{\oplus r}$ *and* $\mathcal{N}_1 := \mathbb{F}_2^{\oplus s+t}$.

Remark 11.2. These invariants are uniquely determined by G, X and k.

We recall an adaptation of Proposition 5.1 and Corollary 5.8 of [5] that can be found in Section 2.2 of [38].

Proposition 11.3. *Let X be a compact complex manifold of dimension n and ι an involution. Assume that $H^*(X,\mathbb{Z})$ is torsion free. We have:*

(i) $\operatorname{rk} H^k(X,\mathbb{Z})^\iota = l_2^k(X) + l_{1,+}^k(X)$.

(ii) We denote $\sigma := \mathrm{id} + \iota^*$ *and* $S_\iota^k := \operatorname{Ker}\sigma \cap H^k(X,\mathbb{Z})$. *We have* $H^k(X,\mathbb{Z})^\iota \cap S_\iota^k = 0$ *and*
$$\frac{H^k(X,\mathbb{Z})}{H^k(X,\mathbb{Z})^\iota \oplus S_\iota^k} = (\mathbb{Z}/2\mathbb{Z})^{\oplus l_2^k(X)}.$$

Remark 11.4. Note that the elements of $(\mathcal{O}_K, a_i)^\iota$ are of the form $x + \iota^*(x)$ with $x \in (\mathcal{O}_K, a_i)$.

Let $\pi : X \to X/G$ be the quotient map. We denote by π^* and π_* respectively the pull-back and the push-forward along π. We recall that

$$\pi_* \circ \pi^* = 2\,\mathrm{id} \text{ and } \pi^* \circ \pi_* = \mathrm{id} + \iota^*. \tag{36}$$

Assuming that $H^k(X,\mathbb{Z})$ is torsion free, we obtain the following exact sequence (Proposition 3.3.3 of [38]), which will be useful in the next section:

$$0 \longrightarrow \pi_*(H^k(X,\mathbb{Z})) \longrightarrow H^k(X/G,\mathbb{Z})/\mathrm{tors} \longrightarrow (\mathbb{Z}/2\mathbb{Z})^{\oplus \alpha_k} \longrightarrow 0, \tag{37}$$

with $\alpha_k \in \mathbb{N}$. We also recall the commutativity behaviour of π_* with the cup product.

Proposition 11.5. *[38, Lemma 3.3.7] Let X be a compact complex manifold of dimension n and ι an involution. Assume that $H^*(X,\mathbb{Z})$ is torsion free. Let $0 \leq k \leq 2n$, m be integers such that $km \leq 2n$, and let $(x_i)_{1\leq i\leq m}$ be elements of $H^k(X,\mathbb{Z})^\iota$. Then*

$$\pi_*(x_1) \cdot \ldots \cdot \pi_*(x_m) = 2^{m-1}\pi_*(x_1 \cdot \ldots \cdot x_m).$$

We also recall Definition 3.3.4 of [38].

Definition 11.6. Let X be a compact complex manifold and ι be an involution. Let $0 \leq k \leq 2n$, and assume that $H^k(X,\mathbb{Z})$ is torsion free. Then if the map $\pi_*: H^k(X,\mathbb{Z}) \to H^k(X/G,\mathbb{Z})/\mathrm{tors}$ is surjective, we say that (X,ι) is H^k-*normal.*

Remark 11.7. The H^k-normality is equivalent to the following property:

For $x \in H^k(X,\mathbb{Z})^\iota$, $\pi_*(x)$ is divisible by 2 if and only if there exists a $y \in H^k(X,\mathbb{Z})$ such that $x = y + \iota^*(y)$.

We also need to recall Definition 3.5.1 of [38] about fixed loci.

Definition 11.8. Let X be a compact complex manifold of dimension n and G an automorphism group of prime order p.

(i) We will say that $\operatorname{Fix} G$ is negligible if the following conditions are verified:

- $H^*(\operatorname{Fix} G,\mathbb{Z})$ is torsion-free.
- $\operatorname{Codim} \operatorname{Fix} G \geq \frac{n}{2} + 1$.

(ii) We will say that $\operatorname{Fix} G$ is almost negligible if the following conditions are verified:

- $H^*(\operatorname{Fix} G,\mathbb{Z})$ is torsion-free.
- n is even and $n \geq 4$.
- $\operatorname{Codim} \operatorname{Fix} G = \frac{n}{2}$, and the purely $\frac{n}{2}$-dimensional part of $\operatorname{Fix} G$ is connected and simply connected. We denote the $\frac{n}{2}$-dimensional component by Z.
- The cocycle $[Z]$ associated to Z is primitive in $H^n(X,\mathbb{Z})$.

Now we are ready to provide Theorem 2.65 of [38].

Theorem 11.9. *Let $G = \langle \varphi \rangle$ be a group of prime order $p = 2$ acting by automorphisms on a Kähler manifold X of dimension $2n$. We assume:*

(i) $H^*(X,\mathbb{Z})$ *is torsion-free,*

(ii) $\operatorname{Fix} G$ *is negligible or almost negligible,*

(iii) $l_{1,-}^{2k}(X) = 0$ *for all* $1 \leq k \leq n$, *and*

(iv) $l_{1,+}^{2k+1}(X) = 0$ *for all* $0 \leq k \leq n-1$, *when* $n > 1$.

(v) $l_{1,+}^{2n}(X) + 2\left[\sum_{i=0}^{n-1} l_{1,-}^{2i+1}(X) + \sum_{i=0}^{n-1} l_{1,+}^{2i}(X)\right] = \sum_{k=0}^{\dim \operatorname{Fix} G} h^{2k}(\operatorname{Fix} G,\mathbb{Z})$.

Then (X, G) is H^{2n}-normal.

We will also need a proposition from Section 7 of [5] about Smith theory. Let X be a topological space and let $G = \langle \iota \rangle$ be an involution acting on X. Let $\sigma := 1 + \iota \in \mathbb{F}_2[G]$. We consider the chain complex $C_*(X)$ of X with coefficients in $\mathbb{F}_2$ and its subcomplex $\sigma C_*(X)$. We denote by X^G the fixed locus of the action of G on X.

Proposition 11.10. *(i) ([7], Theorem 3.1). There is an exact sequence of complexes:*

$$0 \longrightarrow \sigma C_*(X) \oplus C_*(X^G) \xrightarrow{f} C_*(X) \xrightarrow{\sigma} \sigma C_*(X) \longrightarrow 0 ,$$

where f denotes the sum of the inclusions.

(ii) ([7], (3.4) p.124). There is an isomorphism of complexes:

$$\sigma C_*(X) \simeq C_*(X/G, X^G),$$

where X^G is identified with its image in X/G.

12 Odd cohomology of $A^{[2]}$

Let A be a smooth compact surface with torsion free cohomology and $A^{[2]}$ the Hilbert scheme of two points. It can be constructed as follows: Consider the direct product $A \times A$. Denote

$$b : \widetilde{A \times A} \to A \times A$$

the blow-up along the diagonal $\Delta \cong A$ with exceptional divisor E. Let $j : E \to \widetilde{A \times A}$ be the embedding. The action of $\mathfrak{S}_2$ on $A \times A$ lifts to an action on $\widetilde{A \times A}$. We have the pushforward $j_* : H^*(E, \mathbb{Z}) \to H^*(\widetilde{A \times A}, \mathbb{Z})$.

The quotient by the action of $\mathfrak{S}_2$ is

$$\pi : \widetilde{A \times A} \to A^{[2]}.$$

Now, $A^{[2]}$ is a compact complex manifold with torsion-free cohomology, [59, Theorem 2.2]. By (37), there is an exact sequence

$$0 \to \pi_*(H^k(\widetilde{A \times A}, \mathbb{Z})) \to H^k(A^{[2]}, \mathbb{Z}) \to \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus \alpha_k} \to 0$$

with $k \in \{1, ..., 8\}$. In this section, we want to prove the following proposition.

Proposition 12.1. *Let A be a smooth compact surface with torsion free cohomology. Then*

(i) $H^3(A^{[2]}, \mathbb{Z}) = \pi_*(b^*(H^3(A \times A, \mathbb{Z}))) \oplus \pi_* j_* b^*_{|E}(H^1(\Delta, \mathbb{Z}))$,

(ii) $H^5(A^{[2]}, \mathbb{Z}) = \pi_*(b^*(H^5(A \times A, \mathbb{Z}))) \oplus \frac{1}{2}\pi_* j_* b^*_{|E}(H^3(\Delta, \mathbb{Z}))$.

The section is dedicated to the proof of this proposition. The proof is organized as follows. Subsection 12.1 is devoted to calculate the torsion of $H^3(A^{[2]} \smallsetminus E, \mathbb{Z})$ (Lemma 12.4) using equivariant cohomology techniques developed in [38]. Then this knowledge allow us to deduce $\alpha_3 = 0$ using the exact sequence (41) and $\alpha_5 = 4$ using the unimodularity of the lattice $H^3\left(\widetilde{A \times A}, \mathbb{Z}\right) \oplus H^5\left(\widetilde{A \times A}, \mathbb{Z}\right)$.

12.1 Preliminary lemmas

We denote $V = \widetilde{A \times A} \smallsetminus E$ and $U = V/\mathfrak{S}_2 = A^{[2]} \smallsetminus E$, where $\mathfrak{S}_2 = \langle \sigma_2 \rangle$.

Lemma 12.2. *We have: $H^k(A \times A, \mathbb{Z}) = H^k(V, \mathbb{Z})$ for all $k \leq 3$.*

Proof. We have $V = A \times A \smallsetminus \Delta$. We have the following natural exact sequence:

$$\cdots \longrightarrow H^k(A \times A, V, \mathbb{Z}) \longrightarrow H^k(A \times A, \mathbb{Z}) \longrightarrow H^k(V, \mathbb{Z}) \longrightarrow \cdots$$

Moreover, by Thom isomorphism $H^k(A \times A, V, \mathbb{Z}) = H^{k-4}(\Delta, \mathbb{Z}) = H^{k-4}(A, \mathbb{Z})$. Hence $H^k(A \times A, V, \mathbb{Z}) = 0$ for all $k \leq 3$. Hence $H^k(A \times A, \mathbb{Z}) = H^k(V, \mathbb{Z})$ for all $k \leq 2$. It remains to consider the following exact sequence:

$$0 \longrightarrow H^3(A \times A, \mathbb{Z}) \longrightarrow H^3(V, \mathbb{Z}) \longrightarrow H^4(A \times A, V, \mathbb{Z}) \xrightarrow{\rho} H^4(A \times A, \mathbb{Z}) .$$

The map ρ is given by $\mathbb{Z}[\Delta] \to H^4(A \times A, \mathbb{Z})$. Using Notation 10.6, the class $x \otimes 1$ is also in $H^4(A \times A, \mathbb{Z})$ and intersects Δ in one point. Hence the class of Δ in $H^4(A \times A, \mathbb{Z})$ is not trivial and the map ρ is injective. It follows that

$$H^3(A \times A, \mathbb{Z}) = H^3(V, \mathbb{Z}). \qquad \square$$

Now we will calculate the invariant $l^2_{1,-}(A \times A)$ and $l^1_{1,+}(A \times A)$ from Definition-Proposition 11.1.

Lemma 12.3. *We have: $l^2_{1,-}(A \times A) = l^1_{1,+}(A \times A) = 0$.*

Proof. By Künneth formula we have:

$$H^1(A \times A, \mathbb{Z}) = H^0(A, \mathbb{Z}) \otimes H^1(A, \mathbb{Z}) \oplus H^1(A, \mathbb{Z}) \otimes H^0(A, \mathbb{Z}).$$

The elements of $H^0(A, \mathbb{Z}) \otimes H^1(A, \mathbb{Z})$ and $H^1(A, \mathbb{Z}) \otimes H^0(A, \mathbb{Z})$ are exchanged under the action of σ_2. It follows that $l_2^1(A \times A) = 4$ and necessary $l_{1,-}^1(A \times A) = l_{1,+}^1(A \times A) = 0$. Using Künneth again, we get:

$$\begin{aligned} H^2(A \times A, \mathbb{Z}) = {} & H^0(A, \mathbb{Z}) \otimes H^2(A, \mathbb{Z}) \oplus H^1(A, \mathbb{Z}) \otimes H^1(A, \mathbb{Z}) \\ & \oplus H^2(A, \mathbb{Z}) \otimes H^0(A, \mathbb{Z}). \end{aligned}$$

As before, elements $x \otimes y \in H^2(A \times A, \mathbb{Z})$ are sent to $y \otimes x$ by the action of σ_2. Such an element is fixed by the action of σ_2 if $x = y$. It follows:

$$l_2^2(A \times A) = 6 + 6 = 12,$$

$$l_{1,+}^2(A \times A) = 4,$$

and thus:

$$l_{1,-}^2(A \times A) = 0.$$

□

Lemma 12.4. *The group $H^3(U, \mathbb{Z})$ is torsion free.*

Proof. Using the spectral sequence of equivariant cohomology, it follows from Proposition 3.2.5 of [38], Lemma 12.2 and 12.3. □

12.2 Third cohomology group

By Theorem 7.31 of [61], we have:

$$H^3(\widetilde{A \times A}, \mathbb{Z}) = b^*(H^3(A \times A, \mathbb{Z})) \oplus j_* b_{|E}^*(H^1(\Delta, \mathbb{Z})). \tag{38}$$

It follows that

$$H^3(A^{[2]}, \mathbb{Z}) \supset \pi_* b^*(H^3(A \times A, \mathbb{Z})) \oplus \pi_* j_* b_{|E}^*(H^1(\Delta, \mathbb{Z})).$$

We want to show that this inclusion is an equality. We will proceed as follows: We first prove that $\pi_* b^*(H^3(A \times A, \mathbb{Z}))$ is primitive. Then, in Lemma 12.5, we show that $\pi_* j_* b_{|E}^*(H^1(\Delta, \mathbb{Z}))$ is primitive and finally we remark that this implies that the direct sum $\pi_* b^*(H^3(A \times A, \mathbb{Z})) \oplus \pi_* j_* b_{|E}^*(H^1(\Delta, \mathbb{Z}))$ is primitive.

By Künneth formula, we have:

$$\begin{aligned} H^3(A\times A,\mathbb{Z}) = {} & H^0(A,\mathbb{Z})\otimes H^3(A,\mathbb{Z})\oplus H^1(A,\mathbb{Z})\otimes H^2(A,\mathbb{Z}) \\ & \oplus H^2(A,\mathbb{Z})\otimes H^1(A,\mathbb{Z})\oplus H^3(A,\mathbb{Z})\otimes H^0(A,\mathbb{Z}). \end{aligned}$$

Hence all elements in $H^3(A\times A,\mathbb{Z})^{\mathfrak{S}_2}$ are written as $x+\sigma_2^*(x)$ with $x\in H^3(A\times A,\mathbb{Z})$. Since $\frac{1}{2}\pi_*(x+\sigma_2^*(x))=\pi_*(x)$, it follows that $\pi_*(b^*(H^3(A\times A,\mathbb{Z})))$ is primitive in $H^3(A^{[2]},\mathbb{Z})$. Moreover by (38), we have the following values which will be used in Section 12.3:

$$l_2^3(\widetilde{A\times A}) = \operatorname{rk} H^3(A\times A,\mathbb{Z})^{\mathfrak{S}_2} = 28. \tag{39}$$

and

$$l_{1,+}^3(\widetilde{A\times A}) = \operatorname{rk} H^1(\Delta,\mathbb{Z})^{\mathfrak{S}_2} = 4, \quad \text{and} \quad l_{1,-}^3(\widetilde{A\times A}) = 0.$$

Lemma 12.5. *The group $\pi_*j_*b_{|E}^*(H^1(\Delta,\mathbb{Z}))$ is primitive in $H^3(A^{[2]},\mathbb{Z})$.*

Proof. We consider the following commutative diagram:

$$\begin{tikzcd} H^3(\mathscr{N}_{A^{[2]}/\pi(E)},\mathscr{N}_{A^{[2]}/\pi(E)}\smallsetminus 0,\mathbb{Z}) = H^3(A^{[2]},U,\mathbb{Z}) \arrow[d, "d\pi^*"] \arrow[r, "g"] & H^3(A^{[2]},\mathbb{Z}) \arrow[d, "\pi^*"] \\ H^3(\mathscr{N}_{\widetilde{A\times A}/E},\mathscr{N}_{\widetilde{A\times A}/E}\smallsetminus 0,\mathbb{Z}) = H^3(\widetilde{A\times A},V,\mathbb{Z}) \arrow[r, "h"] & H^3(\widetilde{A\times A},\mathbb{Z}), \end{tikzcd} \tag{40}$$

where $\mathscr{N}_{A^{[2]}/E}$ and $\mathscr{N}_{\widetilde{A\times A}/E}$ are respectively the normal bundles of $\pi(E)$ in $A^{[2]}$ and of E in $\widetilde{A\times A}$. By the proof of Theorem 7.31 of [61], the map h is injective with image in $H^3(\widetilde{A\times A},\mathbb{Z})$ given by $j_*b_{|E}^*(H^1(\Delta,\mathbb{Z}))$. Hence Diagram (40) shows that g is also injective and has image $\pi_*j_*b_{|E}^*(H^1(\Delta,\mathbb{Z}))$ in $H^3(A^{[2]},\mathbb{Z})$. We obtain:

$$0 \longrightarrow H^3(A^{[2]},U,\mathbb{Z}) \xrightarrow{g} H^3(A^{[2]},\mathbb{Z}) \longrightarrow H^3(U,\mathbb{Z}) . \tag{41}$$

However, by Lemma 12.4, $H^3(U,\mathbb{Z})$ is torsion free; it follows that $\pi_*j_*b_{|E}^*(H^1(\Delta,\mathbb{Z}))$ is primitive in $H^3(A^{[2]},\mathbb{Z})$. □

Now it remains to prove that $\pi_*b^*(H^3(A\times A,\mathbb{Z}))\oplus\pi_*j_*b_{|E}^*(H^1(\Delta,\mathbb{Z}))$ is primitive in $H^3(A^{[2]},\mathbb{Z})$. This comes from the fact that all elements in $\pi_*b^*(H^3(A\times A,\mathbb{Z})^{\mathfrak{S}_2})$ are divisible by 2, so the relations (36) on π_* and π^* impose the above sum to be primitive.

In more details, let $x \in \pi_* b^*(H^3(A\times A,\mathbb{Z}))$ and $y \in \pi_* j_* b_{|E}^*(H^1(\Delta,\mathbb{Z}))$. It is enough to show that if $\frac{x+y}{2} \in H^3(A^{[2]},\mathbb{Z})$, then $\frac{x}{2} \in H^3(A^{[2]},\mathbb{Z})$ and $\frac{y}{2} \in H^3(A^{[2]},\mathbb{Z})$. As we have seen, we can write $x = \frac{1}{2}\pi_*(z+\sigma_2^*(z))$, with $z \in b^*(H^3(A\times A,\mathbb{Z}))$ and $y = \pi_*(y')$, with $y' \in j_* b_{|E}^*(H^1(\Delta,\mathbb{Z}))$. If

$$\frac{\frac{1}{2}\pi_*(z+\sigma_2^*(z)) + \pi_*(y)}{2} \in H^3(A^{[2]},\mathbb{Z})$$

then taking the image by π^* of this element, we obtain

$$\frac{z+\sigma_2^*(z)}{2} + y' \in H^3(\widetilde{A\times A},\mathbb{Z}).$$

Hence $\frac{z+\sigma_2^*(z)}{2} \in b^*(H^3(A\times A,\mathbb{Z}))^{\mathfrak{S}_2}$. Hence there is $z' \in b^*(H^3(A\times A,\mathbb{Z}))$ such that

$$\frac{z+\sigma_2^*(z)}{2} = z' + \sigma_2^*(z').$$

So x is divisible by 2 and then also y.

This finishes the proof of (i) of Proposition 12.1.

12.3 The fifth cohomology group

Now we prove (ii) of Proposition 12.1. By Theorem 7.31 of [61], we have:

$$H^5(\widetilde{A\times A},\mathbb{Z}) = b^*(H^5(A\times A,\mathbb{Z})) \oplus j_* b_{|E}^*(H^3(\Delta,\mathbb{Z})). \tag{42}$$

It follows that

$$H^5(A^{[2]},\mathbb{Z}) \supset \pi_*(b^*(H^5(A\times A,\mathbb{Z}))) \oplus \pi_* j_* b_{|E}^*(H^3(\Delta,\mathbb{Z})).$$

Moreover, by Künneth formula, we have:

$$\begin{aligned} H^5(A\times A,\mathbb{Z}) &= H^1(A,\mathbb{Z})\otimes H^4(A,\mathbb{Z}) \oplus H^2(A,\mathbb{Z})\otimes H^3(A,\mathbb{Z}) \\ &\quad \oplus H^3(A,\mathbb{Z})\otimes H^2(A,\mathbb{Z}) \oplus H^4(A,\mathbb{Z})\otimes H^1(A,\mathbb{Z}). \end{aligned}$$

As before, $\pi_*(b^*(H^5(A\times A,\mathbb{Z})))$ is primitive in $H^5(A^{[2]},\mathbb{Z})$. Moreover by (42):

$$l_2^5(\widetilde{A\times A}) = \operatorname{rk} H^5(A\times A,\mathbb{Z})^{\mathfrak{S}_2} = 28, \tag{43}$$

and

$$l^5_{1,+}(\widetilde{A\times A}) = \operatorname{rk} H^3(\Delta,\mathbb{Z})^{\mathfrak{S}_2} = 4, \quad \text{and} \quad l^5_{1,-}(\widetilde{A\times A}) = 0.$$

Lemma 12.6. *The lattice $\pi_*(H^3(\widetilde{A\times A},\mathbb{Z}) \oplus H^5(\widetilde{A\times A},\mathbb{Z}))$ has discriminant 2^8.*

Proof. By Proposition 11.3 (ii):

$$\frac{H^3(\widetilde{A\times A},\mathbb{Z}) \oplus H^5(\widetilde{A\times A},\mathbb{Z})}{H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2} \oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2} \oplus \left(H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2} \oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\right)^{\perp}}$$
$$= (\mathbb{Z}/2\mathbb{Z})^{l^3_2(\widetilde{A\times A})+l^5_2(\widetilde{A\times A})}.$$

Since $H^3(\widetilde{A\times A},\mathbb{Z}) \oplus H^5(\widetilde{A\times A},\mathbb{Z})$ is a unimodular lattice, it follows from Propositions 2.5 and 2.3 that

$$\operatorname{discr}\left[H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2} \oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\right] = 2^{l^3_2(\widetilde{A\times A})+l^5_2(\widetilde{A\times A})}.$$

Then by Proposition 11.5,

$$\operatorname{discr}\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2} \oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2})$$
$$= 2^{l^3_2(\widetilde{A\times A})+l^5_2(\widetilde{A\times A})+\operatorname{rk}\left[H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}\right]}.$$

Then by Proposition 11.3 (i):

$$\operatorname{discr}\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2} \oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2})$$
$$= 2^{2\left(l^3_2(\widetilde{A\times A})+l^5_2(\widetilde{A\times A})\right)+l^3_{1,+}(\widetilde{A\times A})+l^5_{1,+}(\widetilde{A\times A})}.$$

By Remark 11.4 and since $\pi_*(x+\iota^*(x)) = 2\pi_*(x)$, we have:

$$\frac{\pi_*(H^3(\widetilde{A\times A},\mathbb{Z}) \oplus H^5(\widetilde{A\times A},\mathbb{Z}))}{\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2} \oplus H^5(\widetilde{A\times A},\mathbb{Z})^{\mathfrak{S}_2}} = \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus l^3_2(\widetilde{A\times A})+l^5_2(\widetilde{A\times A})}.$$

It follows from Proposition 2.3:

$$\operatorname{discr}\pi_*(H^3(\widetilde{A\times A},\mathbb{Z}) \oplus H^5(\widetilde{A\times A},\mathbb{Z})) = 2^{l^3_{1,+}(\widetilde{A\times A})+l^5_{1,+}(\widetilde{A\times A})} = 2^8. \qquad \square$$

The lattice $H^3(A^{[2]},\mathbb{Z}) \oplus H^5(A^{[2]},\mathbb{Z})$ is unimodular. Hence by Proposition 2.3:

$$\frac{H^3(A^{[2]},\mathbb{Z}) \oplus H^5(A^{[2]},\mathbb{Z})}{\pi_*(H^3(\widetilde{A\times A},\mathbb{Z}) \oplus H^5(\widetilde{A\times A},\mathbb{Z}))} = \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 4}.$$

However, from the last section, we know that $\pi_*(H^3(\widetilde{A\times A},\mathbb{Z})) = H^3(A^{[2]},\mathbb{Z})$. It follows that

$$\frac{H^5(A^{[2]},\mathbb{Z})}{\pi_*(H^5(\widetilde{A\times A},\mathbb{Z}))} = \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 4}.$$

Then by the same argument as used in the end of Section 12.2, we can see that the elements in $\frac{H^5(A^{[2]},\mathbb{Z})}{\pi_*(H^5(\widetilde{A\times A},\mathbb{Z}))}$ are given by $\frac{1}{2}\pi_* j_* b^*_{|E}(H^3(\Delta,\mathbb{Z}))$.

13 Nakajima operators for Hilbert schemes of points on surfaces

Let A be a smooth projective complex surface. Let $A^{[n]}$ the Hilbert scheme of n points on the surface, *i.e.* the moduli space of finite subschemes of A of length n. $A^{[n]}$ is again smooth and projective of dimension $2n$, cf. [16]. Their rational cohomology can be described in terms of Nakajima's [46] operators. First consider the direct sum

$$\mathbb{H} := \bigoplus_{n=0}^{\infty} H^*(A^{[n]},\mathbb{Q}).$$

This space is bigraded by cohomological *degree* and the *weight*, which is given by the number of points n. The unit element in $H^0(A^{[0]},\mathbb{Q}) \cong \mathbb{Q}$ is denoted by $|0\rangle$, called the *vacuum*.

Definition-Proposition 13.1. *There are linear operators $\mathfrak{q}_m(a)$, for each $m \geq 1$ and $a \in H^*(A,\mathbb{Q})$, acting on $\mathbb{H}$, which have the following properties: They depend linearly on a, and if $a \in H^k(A,\mathbb{Q})$ is homogeneous, the operator $\mathfrak{q}_m(a)$ is bihomogeneous of degree $k+2(m-1)$ and weight m:*

$$\mathfrak{q}_m(a) : H^l(A^{[n]}) \to H^{l+k+2(m-1)}(A^{[n+m]})$$

To construct them, first define incidence varieties $\mathcal{Z}_m \subset A^{[n]} \times A \times A^{[n+m]}$ by

$$\mathcal{Z}_m := \left\{(\xi, x, \xi') \,|\, \xi \subset \xi',\, \mathrm{supp}(\xi') - \mathrm{supp}(\xi) = mx\right\}.$$

Then $\mathfrak{q}_m(a)(\beta)$ is defined as the Poincaré dual of

$$\mathrm{pr}_{3*}\left((\mathrm{pr}_2^*(\alpha)\cdot \mathrm{pr}_3^*(\beta)) \cap [\mathcal{Z}_m]\right).$$

Consider now the superalgebra generated by the $\mathfrak{q}_m(a)$. Every element in $\mathbb{H}$ can be decomposed uniquely as a linear combination of products of operators

$\mathfrak{q}_m(a)$, acting on the vacuum. In other words, the $\mathfrak{q}_m(a)$ generate $\mathbb{H}$ and there are no algebraic relations between them (except the linearity in a and the super-commutativity).

Example 13.2. The unit $1_{A^{[n]}} \in H^0(A^{[n]}, \mathbb{Q})$ is given by $\frac{1}{n!}\mathfrak{q}_1(1)^n|0\rangle$. The sum of all $1_{A^{[n]}}$ in the formal completion of $\mathbb{H}$ is sometimes denoted by $|1\rangle := \exp(\mathfrak{q}_1(1))|0\rangle$.

Definition 13.3. To give the cup product structure of $\mathbb{H}$, define operators $\mathfrak{G}(a)$ for $a \in H^*(A)$. Let $\Xi_n \subset A^{[n]} \times A$ be the universal subscheme. Then the action of $\mathfrak{G}(a)$ on $H^*(A^{[n]})$ is multiplication with the class

$$\mathrm{pr}_{1*}\left(\mathrm{ch}(\mathcal{O}_{\Xi_n}) \cdot \mathrm{pr}_2^*(\mathrm{td}(A) \cdot a)\right) \in H^*(A^{[n]}).$$

For $a \in H^k(A)$, we define $\mathfrak{G}_i(a)$ as the component of $\mathfrak{G}(a)$ of cohomological degree $k + 2i$. A differential operator $\mathfrak{d}$ is given by $\mathfrak{G}_1(1)$. It means multiplication with the first Chern class of the tautological sheaf $\mathrm{pr}_{1*}(\mathcal{O}_{\Xi_n})$.

Notation 13.4. We abbreviate $\mathfrak{q} := \mathfrak{q}_1(1)$ and for its derivative $\mathfrak{q}' := [\mathfrak{d}, \mathfrak{q}]$. For any operator X we write $X^{(k)}$ for the k-fold derivative: $X^{(k)} := \mathrm{ad}^k(\mathfrak{d})(X)$.

In [30] and [32] we find various commutation relations between these operators, that allow to determine all multiplications in the cohomology of the Hilbert scheme. First of all, if X and Y are operators of degree d and d', their commutator is defined in the super sense:

$$[X, Y] := XY - (-1)^{dd'}YX.$$

The integral on $A^{[n]}$ induces a non-degenerate bilinar form on $\mathbb{H}$: for classes α, $\beta \in H^*(A^{[n]})$ it is given by

$$(\alpha, \beta)_{A^{[n]}} := \int_{A^{[n]}} \alpha \cdot \beta.$$

If X is a homogeneous linear operator of degree d and weight m, acting on $\mathbb{H}$, define its adjont $X^\dagger$ by

$$(X(\alpha), \beta)_{A^{[n+m]}} = (-1)^{d|\alpha|}(\alpha, X^\dagger(\beta))_{A^{[n]}}.$$

We put $\mathfrak{q}_0(a) := 0$ and for $m < 0$, $\mathfrak{q}_m(a) := (-1)^m\mathfrak{q}_{-m}(a)^\dagger$. Note that, for all $m \in \mathbb{Z}$, the bidegree of $\mathfrak{q}_m(a)$ is $(m, |a| + 2(|m| - 1))$. If m is positive, $\mathfrak{q}_m$ is called a creation operator, otherwise it is called annihiliation operator. Now define

$$\mathfrak{L}_m(a) := \begin{cases} \frac{1}{2}\sum\limits_{k\in\mathbb{Z}}\sum\limits_i \mathfrak{q}_k(a_{(1)})\mathfrak{q}_{m-k}(a_{(2)}), & \text{if } m \neq 0, \\ \sum\limits_{k>0}\sum\limits_i \mathfrak{q}_k(a_{(1)})\mathfrak{q}_{-k}(a_{(2)}), & \text{if } m = 0. \end{cases}$$

where $\sum_i a_{(1)} \otimes a_{(2)}$ is the push-forward of a along the diagonal $\tau_2 : A \to A \times A$ (in Sweedler notation).

Remark 13.5. This can be expressed more elegantly using normal ordering: the operator $:\mathfrak{q}_m\mathfrak{q}_n:(a \otimes b)$ is defined in a way such that the annihilation operator act first. Then we may write $\mathfrak{L}_m(a) = \sum_k :\mathfrak{q}_k\mathfrak{q}_{m-k}:(\tau_{2*}(a))$.

Remark 13.6. In a similar manner as above, we can use the integral over A to define a bilinear form on $H^*(A, \mathbb{Q})$. The adjoint of the multiplication map gives a coassiocative comultiplication

$$\Delta : H^*(A, \mathbb{Q}) \longrightarrow H^*(A, \mathbb{Q}) \otimes H^*(A, \mathbb{Q})$$

that corresponds to τ_{2*}. The sign convention in [30] is such that $-\Delta = \tau_{2*}$. We denote by Δ^k the k-fold composition of Δ.

Lemma 13.7. *[32, Thm. 2.16] Denote $K_A \in H^2(A, \mathbb{Q})$ the class of the canonical divisor. We have:*

$$[\mathfrak{q}_m(a), \mathfrak{q}_n(b)] = m \cdot \delta_{m+n} \cdot \int_A ab \tag{44}$$

$$[\mathfrak{L}_m(a), \mathfrak{q}_n(b)] = -n \cdot \mathfrak{q}_{m+n}(ab) \tag{45}$$

$$[\mathfrak{d}, \mathfrak{q}_m(a)] = m \cdot \mathfrak{L}_m(a) + \tfrac{m(|m|-1)}{2}\mathfrak{q}_m(K_A a) \tag{46}$$

$$[\mathfrak{L}_m(a), \mathfrak{L}_n(b)] = (m-n)\mathfrak{L}_{m+n}(ab) - \frac{m^3 - m}{12}\delta_{m+n}\int_A abe \tag{47}$$

$$[\mathfrak{G}(a), \mathfrak{q}_1(b)] = \exp(\mathrm{ad}(\mathfrak{d}))(\mathfrak{q}_1(ab)) \tag{48}$$

$$[\mathfrak{G}_k(a), \mathfrak{q}_1(b)] = \tfrac{1}{k!}\,\mathrm{ad}(\mathfrak{d})^k(\mathfrak{q}_1(ab)) \tag{49}$$

Remark 13.8. Note (cf. [30, Thm. 3.8]) that (45) together with (46) imply that

$$\mathfrak{q}_{m+1}(a) = \tfrac{(-1)^m}{m!}(\mathrm{ad}\,\mathfrak{q}')^m\left(\mathfrak{q}_1(a)\right), \tag{50}$$

so there are two ways of writing an element of $\mathbb{H}$: As a linear combination of products of creation operators $\mathfrak{q}_m(a)$ or as a linear combination of products of the operators $\mathfrak{d}$ and $\mathfrak{q}_1(a)$. This second representation is more suitable for computing cup-products, but not faithful. Equations (46) and (50) permit now to switch between the two representations, using that

$$\mathfrak{d}|0\rangle = 0, \tag{51}$$

$$\mathfrak{L}_m(a)|0\rangle = \begin{cases} \frac{1}{2}\sum_{k=1}^{m-1}\sum_i \mathfrak{q}_k(a_{(1)})\mathfrak{q}_{m-k}(a_{(2)})|0\rangle, & \text{if } m > 1, \\ 0, & \text{if } m \leq 1. \end{cases} \tag{52}$$

(53)

Next we give some formulas involving higher derivatives of Nakajima operators that can be of use in formal computations.

Proposition 13.9. *Suppose $K_A a = 0$. Denote $\mathbf{e} := -\chi(A)x$ the Euler class of A. Note that if A is a torus, $\mathbf{e} = 0$. For all k, m, the following formulas hold:*

$$\operatorname{ad}\mathfrak{q}\frac{\mathfrak{q}_m^{(k+1)}(a)}{m^{k+1}} = (k+1)\frac{\mathfrak{q}_{m+1}^{(k)}(a)}{(m+1)^k} + \frac{k^3-k}{24}\frac{\mathfrak{q}_{m+1}^{(k-2)}(a\mathbf{e})}{(m+1)^{k-2}}, \tag{54}$$

$$\operatorname{ad}\mathfrak{q}'\frac{\mathfrak{q}_m^{(k)}(a)}{m^k} = (k-m)\frac{\mathfrak{q}_{m+1}^{(k)}(a)}{(m+1)^k} + \frac{k(k-1)(k-3m-2)}{24}\frac{\mathfrak{q}_{m+1}^{(k-2)}(a\mathbf{e})}{(m+1)^{k-2}}. \tag{55}$$

Proof. Let us start with (54). This is a consequence of Theorem 4.2 of [31] which states that

$$\begin{aligned}\frac{\mathfrak{q}_m^{(k)}(a)}{m^k} &= \frac{1}{k+1}\sum_{i_0+\ldots+i_k=m} {:}\mathfrak{q}_{i_0}\cdots\mathfrak{q}_{i_k}{:}(\tau_*(a)) \\ &\quad + k\sum_{j_0+\ldots+j_{k-2}=m}\frac{j_0^2+\ldots+j_{k-2}^2-1}{24}\,{:}\mathfrak{q}_{j_0}\cdots\mathfrak{q}_{j_{k-2}}{:}(\tau_*(a\mathbf{e})).\end{aligned}$$

Using that $[\mathfrak{q}, {:}\mathfrak{q}_{i_0}\cdots\mathfrak{q}_{i_k}{:}(\Delta^k(a))] = \sum_{r=0}^k \delta_{i_r+1}\,{:}\mathfrak{q}_{i_0}\cdots\widehat{\mathfrak{q}}_{i_r}\cdots\mathfrak{q}_{i_k}{:}(\tau_*(a))$, we calculate:

$$\begin{aligned}\operatorname{ad}\mathfrak{q}\frac{\mathfrak{q}_m^{(k+1)}(a)}{m^{k+1}} &= \frac{1}{k+2}\sum_{i_0+\ldots+i_{k+1}=m}[\mathfrak{q}, {:}\mathfrak{q}_{i_0}\cdots\mathfrak{q}_{i_{k+1}}{:}(\tau_*(a))] \\ &\quad + (k+1)\sum_{j_0+\ldots+j_{k-1}=m}\frac{j_0^2+\ldots+j_{k-1}^2-1}{24}\,[\mathfrak{q}, {:}\mathfrak{q}_{j_0}\cdots\mathfrak{q}_{j_{k-1}}{:}(\tau_*(a\mathbf{e}))]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i_0+\ldots+i_k=m+1} {:}\mathfrak{q}_{i_0}\cdots\mathfrak{q}_{i_k}{:}(\tau_*(a)) \\
&\quad + k(k+1) \sum_{j_0+\ldots+j_{k-2}=m+1} \frac{j_0^2+\ldots+j_{k-2}^2}{24} {:}\mathfrak{q}_{j_0}\cdots\mathfrak{q}_{j_{k-2}}{:}(\tau_*(a\mathbf{e})) \\
&= (k+1)\frac{\mathfrak{q}^{(k)}_{m+1}(a)}{(m+1)^k} + \frac{k^3-k}{24}\frac{\mathfrak{q}^{(k-2)}_{m+1}(a\mathbf{e})}{(m+1)^{k-2}}.
\end{aligned}
$$

Equation (55) follows from (54) using the Jacobi identity: $\operatorname{ad}\mathfrak{q}' = \operatorname{ad}[\mathfrak{d},\mathfrak{q}] = \operatorname{ad}\mathfrak{d}\operatorname{ad}\mathfrak{q} - \operatorname{ad}\mathfrak{q}\operatorname{ad}\mathfrak{d}$. □

Corollary 13.10. *Suppose $K_A a = 0$. Iterated application of the above proposition gives*

$$
\operatorname{ad}(\mathfrak{q})^s \frac{\mathfrak{q}^{(k+s)}_m(a)}{m^{k+s}(k+s)!} = \frac{\mathfrak{q}^{(k)}_{m+s}(a)}{(m+s)^k k!} + \frac{s}{24}\frac{\mathfrak{q}^{(k-2)}_{m+s}(a\mathbf{e})}{(m+s)^{k-2}(k-2)!}. \tag{56}
$$

Proposition 13.11. *Suppose $K_A a = 0$. In the formal completion of $\mathbb{H}$ we have:*

$$
[\mathfrak{G}(a), \exp(\mathfrak{q})] = \exp(\mathfrak{q}) \sum_{\substack{s\geq 1\\ k\geq 0}} \frac{(-1)^{s-1}}{s!}\left(\frac{\mathfrak{q}^{(k)}_s(a)}{s^k k!} + \frac{s-1}{24}\frac{\mathfrak{q}^{(k)}_s(a\mathbf{e})}{s^k k!}\right).
$$

Proof. Equation (4.6) of [30] evaluates

$$
\begin{aligned}
[\mathfrak{G}(a), \exp(\mathfrak{q})] &= \exp(\mathfrak{q}) \sum_{\substack{s\geq 1\\ k\geq 0}} \frac{(-\operatorname{ad}\mathfrak{q})^{s-1}}{s!}\left(\frac{(\operatorname{ad}\mathfrak{d})^k}{k!}(\mathfrak{q}_1(a))\right) \\
&\overset{\text{Cor 13.10}}{=} \exp(\mathfrak{q}) \sum_{s\geq 1} \frac{(-1)^{s-1}}{s!}\left(\sum_{k\geq s-1} \frac{\mathfrak{q}^{(k-s+1)}_s(a)}{s^{k-s+1}(k-s+1)!}\right. \\
&\qquad\qquad \left. + \sum_{k\geq s+1} \frac{s-1}{24}\frac{\mathfrak{q}^{(k-s-1)}_s(a)}{s^{k-s-1}(k-s-1)!}\right) \\
&= \exp(\mathfrak{q}) \sum_{\substack{s\geq 1\\ k\geq 0}} \frac{(-1)^{s-1}}{s!}\left(\frac{\mathfrak{q}^{(k)}_s(a)}{s^k k!} + \frac{s-1}{24}\frac{\mathfrak{q}^{(k)}_s(a\mathbf{e})}{s^k k!}\right).
\end{aligned}
$$

□

Example 13.12.

$$\mathfrak{G}_0(a)\mathfrak{q}^n|0\rangle = n \cdot \mathfrak{q}^{n-1}\mathfrak{q}_1(a)|0\rangle, \tag{57}$$

$$\mathfrak{G}_1(a)\mathfrak{q}^n|0\rangle = -\binom{n}{2}\mathfrak{q}^{n-2}\mathfrak{q}_2(a)|0\rangle, \tag{58}$$

$$\mathfrak{G}_2(a)\mathfrak{q}^n|0\rangle = \binom{n}{3}\mathfrak{q}^{n-3}\mathfrak{q}_3(a)|0\rangle - \binom{n}{2}\mathfrak{q}^{n-2}\mathfrak{L}_2(a)|0\rangle. \tag{59}$$

Remark 13.13. We adopted the notation from [32], which differs from the conventions in [30]. Here is part of a dictionary:

Notation from [32]	Notation from [30]
operator of weight w and degree d	operator of weight w and degree $d-2w$
$\mathfrak{q}_m(a)$	$\mathfrak{p}_{-m}(a)$
$\mathfrak{L}_m(a)$	$-L_{-m}(a)$
$\mathfrak{G}(a)$	$a^{[\bullet]}$
$\mathfrak{d}$	∂
$\tau_{2*}(a)$	$-\Delta(a)$

By sending a subscheme in A to its support, we define a morphism

$$\rho : A^{[n]} \longrightarrow \mathrm{Sym}^n(A), \tag{60}$$

called the Hilbert–Chow morphism. The cohomology of $\mathrm{Sym}^n(A)$ is given by elements of the n-fold tensor power of $H^*(A)$ that are invariant under the action of the group of permutations $\mathfrak{S}_n$. A class in $H^*(A^{[n]}, \mathbb{Q})$ which can be written using only the operators $\mathfrak{q}_1(a)$ of weight 1 comes from a pullback along ρ:

$$\mathfrak{q}_1(b_1)\cdots\mathfrak{q}_1(b_n)|0\rangle = \rho^*\left(\sum_{\pi\in\mathfrak{S}_n} \pm b_{\pi(1)} \otimes \ldots \otimes b_{\pi(n)}\right), \quad b_i \in H^*(A, \mathbb{Q}), \tag{61}$$

where signs arise from permuting factors of odd degrees. In particular,

$$\frac{1}{n!}\mathfrak{q}_1(b)^n|0\rangle = \rho^*\big(b \otimes \ldots \otimes b\big), \tag{62}$$

$$\frac{1}{(n-1)!}\mathfrak{q}_1(b)\mathfrak{q}^{n-1}|0\rangle = \rho^*\Big(b \otimes 1 \otimes \ldots \otimes 1 \;+\; \ldots \;+\; 1 \otimes \ldots \otimes 1 \otimes b\Big). \tag{63}$$

Remark 13.14. With the notation from Section 8, we have that

$$H^*(\mathrm{Sym}^n(A), \mathbb{Q}) \cong \mathbb{S}\mathrm{ym}^n(H^*(A, \mathbb{Q})).$$

Under this isomorphism the ring structure of $\mathbb{S}\mathrm{ym}^n(H^*(A, \mathbb{Q}))$ corresponds to the cup product and the action of the operator $\mathfrak{q}_1(a)$ corresponds to the operation $a \diamond$.

14 On integral cohomology of Hilbert schemes

For the study of integral cohomology, first note that if $a \in H^*(A, \mathbb{Z})$ is an integral class, then $\mathfrak{q}_m(a)$ maps integral classes to integral classes. Such operators are called integral. Qin and Wang studied them in [53]. We need the following results:

Lemma 14.1. *([53], see also Thm. 21.5). The operators $\frac{1}{n!}\mathfrak{q}_1(1)^n$ and $\frac{1}{2}\mathfrak{q}_2(1)$ are integral. Let $b \in H^2(A, \mathbb{Z})$ be monodromy equivalent to a divisor. Then the operator $\frac{1}{2}\mathfrak{q}_1(b)^2 - \frac{1}{2}\mathfrak{q}_2(b)$ is integral.*

Remark 14.2. Qin and Wang [53, Thm 1.1 et seq.] conjecture that their theory works even without the restriction on $b \in H^2(A, \mathbb{Z})$.

Corollary 14.3. *If A is a torus, the operator $\frac{1}{2}\mathfrak{q}_1(b)^2 - \frac{1}{2}\mathfrak{q}_2(b)$ is integral for all $b \in H^2(A, \mathbb{Z})$.*

Proof. The Nakajima operators are preserved under deformations of A. Moreover, by [6], the image of the monodromy representation on $H^2(A, \mathbb{Z})$ is given by $O^{+,+}(H^2(A, \mathbb{Z}))$, the group of isometry on $H^2(A, \mathbb{Z})$ which preserve the orientation of the negative and positive definite part of $H^2(A, \mathbb{R})$.

Suppose now that the Néron-Severi group $\mathrm{NS}(A)$ contains a copy of the hyperbolic lattice U (such A exist). Let us denote $H^2(A, \mathbb{Z}) = U_1 \oplus U_2 \oplus U_3$ with $\mathrm{NS}(A) = U_1$. We consider two isometries in $O^{+,+}(H^2(A, \mathbb{Z}))$, φ_2 and φ_3, defined in the following way: φ_2 exchanges U_1 and U_2 and acts as $-\mathrm{id}$ on U_3 and φ_3 exchanges U_1 and U_3 and acts as $-\mathrm{id}$ on U_2. Using these two isometries, all elements of U_2 and U_3 are monodromy equivalent to a divisor. Then Lemma 14.1 establishes the corollary for that particular A. Now, since all tori are equivalent by deformation, a general torus can always be deformed to our special A. Since the integrality of an operator is a topological invariant, $\frac{1}{2}\mathfrak{q}_1(b)^2 - \frac{1}{2}\mathfrak{q}_2(b)$ remains integral for all $b \in H^2(A, \mathbb{Z})$. □

Proposition 14.4. *Assume that $H^*(A, \mathbb{Z})$ is free of torsion. Let $(a_i) \subset H^1(A, \mathbb{Z})$ and $(b_i) \subset H^2(A, \mathbb{Z})$ be bases of integral cohomology as in Notation 10.6. Denote*

$a_i^* \in H^3(A,\mathbb{Z})$ resp. $b_i^* \in H^2(A,\mathbb{Z})$ the elements of the dual bases. Let x be the generator of $H^4(A,\mathbb{Z})$. Modulo torsion, the following classes form a basis of $H^2(A^{[n]},\mathbb{Z})$:

- $\frac{1}{(n-1)!}\mathfrak{q}_1(b_i)\mathfrak{q}_1(1)^{n-1}|0\rangle = \mathfrak{G}_0(b_i)1$,
- $\frac{1}{(n-2)!}\mathfrak{q}_1(a_i)\mathfrak{q}_1(a_j)\mathfrak{q}_1(1)^{n-2}|0\rangle = \mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j)1,\ i<j$,
- $\frac{1}{2(n-2)!}\mathfrak{q}_2(1)\mathfrak{q}_1(1)^{n-2}|0\rangle$. *We denote this class by $\delta = \mathfrak{d}1$.*

Their respective duals in $H^{2n-2}(A^{[n]},\mathbb{Z})$ are given by

- $\mathfrak{q}_1(b_i^*)\mathfrak{q}_1(x)^{n-1}|0\rangle$,
- $\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(x)^{n-2}|0\rangle,\ i<j$,
- $\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle$.

Proof. It is clear from the above lemma that these classes are all integral. Göttsche's formula [23, p. 35] gives the Betti numbers of $A^{[n]}$ in terms of the Betti numbers of A: $h^1(A^{[n]}) = h^1(A)$, and $h^2(A^{[n]}) = h^2(A) + \frac{h^1(A)(h^1(A)-1)}{2} + 1$. It follows that the given classes span a lattice of full rank.

Next we have to show that the intersection matrix between these classes is in fact the identity matrix. Most of the entries can be computed easily using the simplification from (61). For products involving δ (this is the action of $\mathfrak{d}$) or its dual, first observe that $\mathfrak{d}\mathfrak{q}_1(x)^m|0\rangle = 0$ and $\mathfrak{L}_1(a)\mathfrak{q}_1(x)^m|0\rangle = 0$ for every class a of degree at least 1. Then compute:

$$\begin{gathered}
\delta\cdot\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = \mathfrak{d}\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = 2\mathfrak{L}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = \mathfrak{q}_1(x)^n|0\rangle,\\
\mathfrak{d}\mathfrak{q}_1(b_i^*)\mathfrak{q}_1(x)^{n-1}|0\rangle = \mathfrak{L}_1(b_i^*)\mathfrak{q}_1(x)^{n-1}|0\rangle = 0,\\
\mathfrak{d}\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(x)^{n-2}|0\rangle = \left(\mathfrak{L}_1(a_j^*) + \mathfrak{q}_1(a_j^*)\mathfrak{d}\right)\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(x)^{n-2}|0\rangle =\\
= \left(-\mathfrak{q}_1(a_i^*)\mathfrak{L}_1(a_j^*) + \mathfrak{q}_1(a_j^*)\mathfrak{L}_1(a_i^*)\right)\mathfrak{q}_1(x)^{n-2}|0\rangle = 0,\\
\mathfrak{G}_0(b_i)\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = 0,\\
\mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j)\mathfrak{q}_2(x)\mathfrak{q}_1(x)^{n-2}|0\rangle = 0.
\end{gathered}$$ $\square$

Remark 14.5. If A is a complex torus, a theorem of Markman [33] ensures that $H^*(A^{[n]},\mathbb{Z})$ is free of torsion.

Proposition 14.6. *Let A be a complex abelian surface. Using Notation 10.6, a basis of $H^*(A^{[2]},\mathbb{Z})$ is given by the following classes.*

degree	*Betti number*	*class*	*multiplication with class*
0	*1*	$\frac{1}{2}\mathfrak{q}_1(1)^2\|0\rangle$	id
1	*4*	$\mathfrak{q}_1(1)\mathfrak{q}_1(a_i)\|0\rangle$	$\mathfrak{G}_0(a_i)$
2	*13*	$\frac{1}{2}\mathfrak{q}_2(1)\|0\rangle$	$\mathfrak{d}$
		$\mathfrak{q}_1(a_i)\mathfrak{q}_1(a_j)\|0\rangle$ *for* $i<j$	$\mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j)$
		$\mathfrak{q}_1(1)\mathfrak{q}_1(b_i)\|0\rangle$	$\mathfrak{G}_0(b_i)$
3	*32*	$\mathfrak{q}_2(a_i)\|0\rangle$	$-2\mathfrak{G}_1(a_i)$
		$\mathfrak{q}_1(a_i)\mathfrak{q}_1(b_j)\|0\rangle$	$\mathfrak{G}_0(a_i)\mathfrak{G}_0(b_j)$
		$\mathfrak{q}_1(1)\mathfrak{q}_1(a_i^*)\|0\rangle$	$\mathfrak{G}_0(a_i^*)$
4	*44*	$\left(\frac{1}{2}\mathfrak{q}_1(b_i)^2-\frac{1}{2}\mathfrak{q}_2(b_i)\right)\|0\rangle$	$\frac{1}{2}\mathfrak{G}_0(b_i)^2+\mathfrak{G}_1(b_i)$
		$\mathfrak{q}_1(a_i)\mathfrak{q}_1(a_j^*)\|0\rangle$	$\mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j^*)$
		$\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)\|0\rangle$ *for* $i\le j$	$\mathfrak{G}_0(b_i)\mathfrak{G}_0(b_j)$
5	*32*	$\frac{1}{2}\mathfrak{q}_2(a_i^*)\|0\rangle$	$-\mathfrak{G}_1(a_i^*)$
		$\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(b_j)\|0\rangle$	$\mathfrak{G}_0(a_i^*)\mathfrak{G}_0(b_j)$
		$\mathfrak{q}_1(a_i)\mathfrak{q}_1(x)\|0\rangle$	$\mathfrak{G}_0(a_i)\mathfrak{G}_0(x)$
6	*13*	$\mathfrak{q}_2(x)\|0\rangle$	$-2\mathfrak{G}_1(x)$
		$\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(a_j^*)\|0\rangle$ *for* $i<j$	$\mathfrak{G}_0(a_i^*)\mathfrak{G}_0(a_j^*)$
		$\mathfrak{q}_1(b_i)\mathfrak{q}_1(x)\|0\rangle$	$\mathfrak{G}_0(b_i)\mathfrak{G}_0(x)$
7	*4*	$\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(x)\|0\rangle$	$\mathfrak{G}_0(a_i^*)\mathfrak{G}_0(x)$
8	*1*	$\mathfrak{q}_1(x)^2\|0\rangle$	$\mathfrak{G}_0(x)^2$

Proof. The Betti numbers come from Göttsche's formula [23]. One computes the intersection matrix of all classes under the Poincaré duality pairing and finds that it is unimodular. So it remains to show that all these classes are integral. By Lemma 14.1 this is clear for all classes except those of the form $\frac{1}{2}\mathfrak{q}_2(a_i^*)|0\rangle \in H^5(A^{[2]},\mathbb{Z})$.

Evaluating the Poincaré duality pairing between degrees 3 and 5 gives:

$$\begin{aligned}
\mathfrak{q}_2(a_i)|0\rangle\cdot\mathfrak{q}_2(a_i^*)|0\rangle &= 2,\\
\mathfrak{q}_1(a_i)\mathfrak{q}_1(b_j)|0\rangle\cdot\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(b_j^*)|0\rangle &= 1,\\
\mathfrak{q}_1(1)\mathfrak{q}_1(a_i^*)|0\rangle\cdot\mathfrak{q}_1(x)\mathfrak{q}_1(a_i)|0\rangle &= 1,
\end{aligned}$$

while the other pairings vanish. Therefore, one of $\mathfrak{q}_2(a_i)|0\rangle$ and $\mathfrak{q}_2(a_i^*)|0\rangle$ must be divisible by 2. With the considerations from Section 12 in mind, we can interpret $\mathfrak{q}_2(a_i)|0\rangle \in H^3(A^{[2]},\mathbb{Z})$ and $\mathfrak{q}_2(a_i^*)|0\rangle \in H^5(A^{[2]},\mathbb{Z})$ as classes concentrated on the exceptional divisor, that is, as elements of $\pi_*j_*H^*(E,\mathbb{Z})$. Indeed, the pushforward of a class $a \otimes 1 \in H^k(E, \mathbb{Z})$ is given by

$$\pi_* j_*(a \otimes 1) = \mathfrak{q}_2(a)|0\rangle \in H^{k+2}(A^{[n]}, \mathbb{Z}).$$

When pushing forward to the Hilbert scheme, the only possibility to get a factor 2 is in degree 5, by Proposition 12.1. $\square$

15 Generalized Kummer varieties and the morphism to the Hilbert scheme

Definition 15.1. Let A be a complex projective torus of dimension 2 and $A^{[n]}$, $n \geq 1$, the corresponding Hilbert scheme of points. Denote $\Sigma : A^{[n]} \to A$ the summation morphism, a smooth submersion that factorizes via (60) the Hilbert–Chow morphism $A^{[n]} \xrightarrow{\rho} \mathrm{Sym}^n(A) \xrightarrow{\sigma} A$. Then the generalized Kummer variety $K_{n-1}(A)$ is defined as the fiber over 0:

$$\begin{array}{ccc} K_{n-1}(A) & \xrightarrow{\theta} & A^{[n]} \\ \big\downarrow & & \big\downarrow \Sigma \\ \{0\} & \longrightarrow & A \end{array} \tag{64}$$

Theorem 15.2. *[57, Theorem 2] The cohomology of the generalized Kummer, $H^*(K_{n-1}(A), \mathbb{Z})$, is torsion free.*

Our first objective is to collect some information about the pullback diagram (64). We make use of Notation 10.6.

Proposition 15.3. *Set $\alpha_i := \frac{1}{(n-1)!}\mathfrak{q}_1(1)^{n-1}\mathfrak{q}_1(a_i)|0\rangle = \mathfrak{G}_0(a_i)1$. The class of $K_{n-1}(A)$ in $H^4(A^{[n]}, \mathbb{Z})$ is given by*

$$[K_{n-1}(A)] = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4.$$

Proof. Since the generalized Kummer variety is the fiber over a point, its class must be the pullback of $x \in H^4(A)$ under Σ. But $\Sigma^*(x) = \Sigma^*(a_1) \cdot \Sigma^*(a_2) \cdot \Sigma^*(a_3) \cdot \Sigma^*(a_4)$, so we have to verify that $\Sigma^*(a_i) = \alpha_i$. To do this, we want to use the decomposition $\Sigma = \sigma\rho$. The pullback along σ of a class $a \in H^1(A, \mathbb{Q})$ on $H^1(\mathrm{Sym}^n(A), \mathbb{Q})$ is given by $a \otimes 1 \otimes \cdots \otimes 1 + \ldots + 1 \otimes \cdots \otimes 1 \otimes a$. It follows from (63) that $\Sigma^*(a_i) = \frac{1}{(n-1)!}\mathfrak{q}_1(1)^{n-1}\mathfrak{q}_1(a_i)|0\rangle$. $\square$

The morphism θ induces a homomorphism of graded rings

$$\theta^* : H^*(A^{[n]}) \longrightarrow H^*(K_{n-1}(A)) \tag{65}$$

and by the projection formula, we have

$$\theta_*\theta^*(\alpha) = [K_{n-1}(A)] \cdot \alpha. \tag{66}$$

Lemma 15.4. *Let $\beta \in H^*(K_{n-1}(A), \mathbb{Q})$. Then there is a class $B \in H^*(A^{[n]}, \mathbb{Q})$ such that*

$$\theta_*(\beta) = \frac{1}{n^4} B \cdot [K_{n-1}(A)].$$

Proof. For a point $a \in A$, we denote by t_a the morphism on $A^{[n]}$ induced by the translation by a. Then we consider the morphism $\Theta : K_{n-1}(A) \times A \longrightarrow A^{[n]}$ defined by $\Theta(\xi, a) = t_a(\theta(\xi))$. It fits in a pullback diagram

$$\begin{tikzcd} K_{n-1}(A) \times A \arrow[r, "\Theta"] \arrow[d, "\mathrm{pr}_2"] & A^{[n]} \arrow[d, "\Sigma"] \\ A \arrow[r, "n\cdot"] & A \end{tikzcd} \tag{67}$$

that realizes $K_{n-1}(A) \times A$ as a n^4-fold covering of $A^{[n]}$ over A. Now, for $\beta \in H^*(K_{n-1}(A), \mathbb{Q})$ set

$$B := \Theta_*(\beta \otimes 1).$$

Then the projection formula gives

$$\begin{aligned} B \cdot [K_{n-1}(A)] &= \Theta_* \left(\beta \otimes 1 \cdot \Theta^*[K_{n-1}(A)]\right) \\ &= n^4 \Theta_* \left((\beta \otimes 1) \cdot (1 \otimes x)\right) \\ &= n^4 \Theta_*(\beta \otimes x) \\ &= n^4 \theta_*(\beta). \end{aligned}$$

□

Proposition 15.5. *The kernel of θ^* is equal to the annihilator of $[K_{n-1}(A)]$.*

Proof. Assume $\alpha \in \ker(\theta^*)$. Then we have $[K_{n-1}(A)] \cdot \alpha = \theta_*\theta^*(\alpha) = 0$. Conversely, if $\alpha \notin \ker(\theta^*)$, let $\beta \in H^*(K_{n-1}(A), \mathbb{Q})$ be the Poincaré dual of $\theta^*(\alpha)$, so $\beta \cdot \theta^*(\alpha) \neq 0$. Then by projection formula: $\theta_*(\beta) \cdot \alpha \neq 0$. By Lemma 15.4, there exists $B \in H^*(A^{[n]}, \mathbb{Q})$ such that $B \cdot [K_{n-1}(A)] \cdot \alpha \neq 0$. It follows that $[K_{n-1}(A)] \cdot \alpha \neq 0$. □

Corollary 15.6. $\theta^*(\alpha) = \theta^*(\beta)$ *if and only if* $[K_{n-1}(A)] \cdot \alpha = [K_{n-1}(A)] \cdot \beta$. □

Proposition 15.7. *The annihilator of $[K_{n-1}(A)]$ in $H^*(A^{[n]}, \mathbb{Q})$ is the ideal generated by $H^1(A^{[n]})$.*

Proof. Set $H = H^*(A, \mathbb{Q})$ and consider the exact sequence of H-modules

$$0 \longrightarrow J \longrightarrow H \xrightarrow{x\cdot} H.$$

It is clear that J is the ideal in H generated by $H^1(A, \mathbb{Q})$. Now denote $J^{(n)}$ the ideal generated by $H^1(\mathrm{Sym}^n(A), \mathbb{Q})$ in $H^*(\mathrm{Sym}^n(A), \mathbb{Q}) \cong \mathbb{S}\mathrm{ym}^n(H)$. By the freeness result of Lemma 8.5, tensoring with $\mathbb{S}\mathrm{ym}^n(H)$ yields another exact sequence of H-modules

$$0 \longrightarrow J^{(n)} \longrightarrow \mathbb{S}\mathrm{ym}^n(H) \xrightarrow{\sigma(x)\cdot} \mathbb{S}\mathrm{ym}^n(H).$$

Now let $\mathfrak{H}$ be the operator algebra spanned by products of $\mathfrak{d}$ and $\mathfrak{q}_1(a)$ for $a \in H^*(A)$. Let $\mathfrak{C}$ be the graded commutative subalgebra of $\mathfrak{H}$ generated by $\mathfrak{q}_1(a)$ for $a \in H^*(A)$. The action of $\mathfrak{H}$ on $|0\rangle$ gives $\mathbb{H}$ and the action of $\mathfrak{C}$ on $|0\rangle$ gives $\rho^*(H^*(\mathrm{Sym}^n(A), \mathbb{Q})) \cong \mathbb{S}\mathrm{ym}^n(H)$. By sending $\mathfrak{d}$ to the identity, we define a linear map $c : \mathfrak{H} \to \mathfrak{C}$. Denote $J^{[n]}$ the ideal generated by $H^1(A^{[n]}, \mathbb{Q})$ in $H^*(A^{[n]}, \mathbb{Q})$. We claim that for every $\mathfrak{y} \in \mathfrak{H}$:

$$\mathfrak{y}|0\rangle \in J^{[n]} \Leftrightarrow c(\mathfrak{y})|0\rangle \in J^{[n]}.$$

To see this, we remark that $H^1(A^{[n]}, \mathbb{Q}) \cong H^1(A, \mathbb{Q})$ and the multiplication with a class in $H^1(A^{[n]}, \mathbb{Q})$ is given by the operator $\mathfrak{G}_0(a)$ for some $a \in H^1(A, \mathbb{Q})$. Due to the fact that $\mathfrak{d}$ is also a multiplication operator (of degree 2), $\mathfrak{G}_0(a)$ commutes with $\mathfrak{d}$. It follows that for $\mathfrak{y} = \mathfrak{G}_0(a)\mathfrak{r}$ we have $c(\mathfrak{y}) = \mathfrak{G}_0(a)c(\mathfrak{r})$.

Now denote $\mathfrak{k}$ the multiplication operator with the class $[K_{n-1}(A)]$. We have: $[\mathfrak{k}, \mathfrak{d}] = 0$. Now let $y \in H^*(A^{[n]}, \mathbb{Q})$ be a class in the annihilator of $[K_{n-1}(A)]$. We can write $y = \mathfrak{y}|0\rangle$ for a $\mathfrak{y} \in \mathfrak{H}$. Choose $\tilde{y} \in \mathbb{S}\mathrm{ym}^n(H)$ in a way that $\rho^*(\tilde{y}) = c(\mathfrak{y})|0\rangle$. Then we have:

$$0 = [K_{n-1}(A)] \cdot y = \mathfrak{k}\,\mathfrak{y}|0\rangle = \mathfrak{k}\,c(\mathfrak{y})|0\rangle = \rho^*(\sigma^*(x) \cdot \tilde{y}).$$

Since ρ^* is injective, $\tilde{y}$ is in the annihilator of $\sigma^*(x)$, so $\tilde{y} \in J^{(n)}$. It follows that $c(\mathfrak{y})|0\rangle$ and y are in the ideal generated by $H^1(A^{[n]}, \mathbb{Q})$. $\square$

Theorem 15.8. *[1, Théorème 4] $K_{n-1}(A)$ is a irreducible holomorphically symplectic manifold. In particular, it is simply connected and the canonical bundle is trivial.*

This implies that $H^2(K_{n-1}(A),\mathbb{Z})$ admits an integer-valued nondegenerated symmetric bilinear form (the Beauville–Bogomolov form) B which gives $H^2(K_{n-1}(A),\mathbb{Z})$ the structure of a lattice. Looking, for instance, in the useful table from the introduction of [54], we know that this lattice is isomorphic to $U^{\oplus 3} \oplus \langle -2n \rangle$, for $n \geq 3$. We have the Fujiki formula for $\alpha \in H^2(K_{n-1}(A),\mathbb{Z})$:

$$\int_{K_{n-1}(A)} \alpha^{2n-2} = n \cdot (2n-3)!! \cdot B(\alpha,\alpha)^{n-1} \tag{68}$$

Proposition 15.9. *Assume $n \geq 3$. Then θ^* is surjective on $H^2(A^{[n]},\mathbb{Z})$.*

Proof. By [1, Sect. 7], $\theta^* : H^2(A^{[n]},\mathbb{C}) \to H^2(K_{n-1}(A),\mathbb{C})$ is surjective. But by Proposition 1 of [8], the lattice structure of $\operatorname{Im}\theta^*$ is the same as of $H^2(K_{n-1}(A))$, so the image of $H^2(A^{[n]},\mathbb{Z})$ must be primitive. The result follows. □

Notation 15.10. We have seen that, for $n \geq 3$,

$$H^2(K_{n-1}(A),\mathbb{Z}) \cong H^2(A,\mathbb{Z}) \oplus \langle \theta^*(\delta) \rangle .$$

We denote the injection : $H^2(A,\mathbb{Z}) \to H^2(K_{n-1}(A),\mathbb{Z})$ by j. It can be described by

$$j : a \longmapsto \frac{1}{(n-1)!}\theta^* \left(\mathfrak{q}_1(a)\mathfrak{q}_1(1)^{n-1}|0\rangle\right).$$

Further, we set $e := \theta^*(\delta)$. Using Notation 10.6, we give the following names for classes in $H^2(K_{n-1}(A),\mathbb{Z})$:

$$\begin{aligned} u_1 &:= j(a_1a_2), & v_1 &:= j(a_1a_3), & w_1 &:= j(a_1a_4),\\ u_2 &:= j(a_3a_4), & v_2 &:= j(a_4a_2), & w_2 &:= j(a_2a_3), \end{aligned}$$

These elements form a basis of $H^2(K_{n-1}(A),\mathbb{Z})$ with the following intersection relations under the Beauville-Bogomolov form:

$$B(u_1,u_2) = 1, \qquad B(v_1,v_2) = 1, \qquad B(w_1,w_2) = 1, \qquad B(e,e) = -2n,$$

and all other pairs of basis elements are orthogonal.

16 Odd Cohomology of the generalized Kummer fourfold

Now we come to the special case $n = 3$, so we study $K_2(A)$, the generalized Kummer fourfolds.

Proposition 16.1. *The Betti numbers of $K_2(A)$ are:* 1, 0, 7, 8, 108, 8, 7, 0, 1.

Proof. This follows from Göttsche's formula [23, page 49]. □

By means of the morphism θ^*, we may express part of the cohomology of $K_2(A)$ in terms of Hilbert scheme cohomology. We have seen in Proposition 15.9 that θ^* is surjective for degree 2 and (by duality) also in degree 6. The next proposition shows that this also holds true for odd degrees.

Proposition 16.2. *A basis of $H^3(K_2(A), \mathbb{Z})$ is given by:*

$$\frac{1}{2}\theta^*\Big(\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(1)^2|0\rangle\Big), \tag{69}$$

$$\theta^*\Big(\mathfrak{q}_2(a_i)\mathfrak{q}_1(1)|0\rangle\Big). \tag{70}$$

and a dual basis of $H^5(K_2(A), \mathbb{Z})$ is given by:

$$\theta^*\Big(\mathfrak{q}_1(a_ia_j)\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(1)|0\rangle\Big) \text{ for any } j \neq i, \tag{71}$$

$$\frac{1}{2}\theta^*\Big(\mathfrak{q}_2(a_i^*)\mathfrak{q}_1(1)|0\rangle\Big). \tag{72}$$

Proof. The classes (69) are Poincaré dual to (71) and the classes (70) are Poincaré dual to (72) by direct computation:

$$\begin{aligned}
&\frac{1}{2}\theta^*\Big(\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(1)^2|0\rangle\Big)\cdot\theta^*\Big(\mathfrak{q}_1(a_ia_j)\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(1)|0\rangle\Big)\\
&\qquad = \frac{1}{2}\theta^*\Big(\mathfrak{G}_0(a_i^*)\mathfrak{q}_1(a_ia_j)\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(1)|0\rangle\Big)\\
&\qquad = \frac{1}{2}[K_2(A)]\cdot\mathfrak{q}_1(a_ia_j)\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(a_i^*) = 1,\\
&\frac{1}{2}\theta^*\Big(\mathfrak{q}_2(a_i)\mathfrak{q}_1(1)|0\rangle\Big)\cdot\theta^*\Big(\mathfrak{q}_2(a_i^*)\mathfrak{q}_1(1)|0\rangle\Big) = \theta^*\Big(\mathfrak{G}_1(a_i)\mathfrak{q}_2(a_i^*)\mathfrak{q}_1(1)|0\rangle\Big)\\
&\qquad = [K_2(A)]\cdot\Big(2\mathfrak{q}_3(x) - \mathfrak{q}_1(x)^2\mathfrak{q}_1(1)\Big)\,|0\rangle = 0 - 1 = -1.
\end{aligned}$$

It remains to show that all classes are integral. It is clear from Lemma 14.1 that (69) is integral, while the integrality of (70) and (71) is obvious. By Proposition 14.6,

$\frac{1}{2}\mathfrak{q}_2(a_i^*)|0\rangle$ is integral as well. If the operator $\mathfrak{q}_1(1)$ is applied, we get again an integral class. $\square$

Corollary 16.3. *Let A be an abelian surface and g be an automorphism of A. Let $g^{[[3]]}$ be the automorphism induced by g on $K_2(A)$. We have $H^3(K_2(A),\mathbb{Z}) \simeq H^1(A,\mathbb{Z}) \oplus H^3(A,\mathbb{Z})$ and the action of $g^{[[3]]}$ on $H^3(K_2(A),\mathbb{Z})$ is given by the action of g on $H^1(A,\mathbb{Z}) \oplus H^3(A,\mathbb{Z})$.*

Proof. Let $g^{[3]}$ be the automorphism on $A^{[3]}$ induced by g. We have for the pullbacks $g^{[3]*}(\mathfrak{q}_2(a_i)\mathfrak{q}_1(1)|0\rangle) = \mathfrak{q}_2(g^*a_i)\mathfrak{q}_1(1)|0\rangle$ and $g^{[3]*}(\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(1)^2|0\rangle) = \mathfrak{q}_1(g^*a_i^*)\mathfrak{q}_1(1)^2|0\rangle$. Moreover, we have by definition $g^{[[3]]*} \circ \theta^* = \theta^* \circ g^{[3]*}$. The result follows from Proposition 16.2. $\square$

17 Middle cohomology

The middle cohomology $H^4(K_2(A),\mathbb{Z})$ has been studied by Hassett and Tschinkel in [24]. We first recall some of their results, then we proceed by using θ^* to give a partial description of $H^4(K_2(A),\mathbb{Z})$ in terms of the well-understood cohomology of $A^{[3]}$. Finally, we find a basis of $H^4(K_2(A),\mathbb{Z})$ using the action of the monodromy group.

Notation 17.1. For each $\tau \in A$, denote W_τ the Briançon subscheme of $A^{[3]}$ supported enitrely at the point τ. If $\tau \in A[3]$ is a point of three-torsion, W_τ is actually a subscheme of $K_2(A)$. We will also use the symbol W_τ for the corresponding class in $H^4(K_2(A),\mathbb{Z})$. Further, set

$$W := \sum_{\tau \in A[3]} W_\tau.$$

For $p \in A$, denote Y_p the locus of all $\{x,y,p\}$ in $K_2(A)$. The corresponding class $Y_p \in H^4(K_2(A),\mathbb{Z})$ is independent of the choice of the point p. Then set $Z_\tau := Y_p - W_\tau$ and denote Π the lattice generated by all Z_τ, $\tau \in A[3]$.

Proposition 17.2. *Denote by* $\mathrm{Sym} := \mathrm{Sym}^2\left(H^2\left(K_2(A),\mathbb{Z}\right)\right) \subset H^4\left(K_2(A),\mathbb{Z}\right)$ *the span of products of integral classes in degree* 2*. Then*

$$\mathrm{Sym} + \Pi \ \subset \ H^4\left(K_2(A),\mathbb{Z}\right)$$

is a sublattice of full rank.

Proof. This follows from [24, Proposition 4.3]. □

In Section 4 of [24], one finds the following formula:

$$Z_\tau \cdot D_1 \cdot D_2 = 2 \cdot B(D_1, D_2), \tag{73}$$

for all D_1, D_2 in $H^2(K_2(A), \mathbb{Z})$, $\tau \in A[3]$ and B the Beauville-Bogomolov form on $K_2(A)$.

Definition 17.3. We define $\Pi' := \Pi \cap \mathrm{Sym}^\perp$. It follows from (73) that Π' can be described as the span of all classes of the form $Z_\tau - Z_0$ or alternatively as the set of all $\sum_\tau \alpha_\tau Z_\tau$, such that $\sum_\tau \alpha_\tau = 0$. Note that in [24] the symbol Π' denotes something different.

Remark 17.4. Since $\operatorname{rk} \mathrm{Sym} = 28$ and $\operatorname{rk} \Pi' = 80$, the lattice $\mathrm{Sym} \oplus \Pi'$ as a subset of $H^4\left(K_2(A), \mathbb{Z}\right)$ has full rank.

Proposition 17.5. *The class W can be written with the help of the square of half the diagonal as*

$$W = \theta^*\Big(\mathfrak{q}_3(1)|0\rangle\Big) \tag{74}$$

$$= 9Y_p + e^2. \tag{75}$$

The second Chern class is non-divisible and given by

$$c_2(K_2(A)) = \frac{1}{3} \sum_{\tau \in A[3]} Z_\tau \tag{76}$$

$$= \frac{1}{3}\Big(72Y_p - e^2\Big). \tag{77}$$

Proof. In Section 4 of [24] one finds the equations

$$W = \frac{3}{8}\Big(c_2(K_2(A)) + 3e^2\Big), \tag{78}$$

$$Y_p = \frac{1}{72}\Big(3c_2(K_2(A)) + e^2\Big), \tag{79}$$

from which we deduce (75) and (77). Equation (76) and the non-divisibility are from [24, Proposition 5.1]. □

Proposition 17.6. *The image of $H^4(A^{[3]}, \mathbb{Q})$ under the morphism θ^* is equal to* $\mathrm{Sym}^2 H^2(K_2(A), \mathbb{Q})$.

Proof. We start by giving a set of generators of $H^4(A^{[n]}, \mathbb{Q})$. Theorem 5.30 of [32] ensures that it is possible to do this in terms of multiplication operators. To enumerate elements of $H^*(A, \mathbb{Q})$, we follow Notation 10.6. Then our set of generators is given by:

multiplication operator	number of classes
$\mathfrak{G}_0(a_1)\mathfrak{G}_0(a_2)\mathfrak{G}_0(a_3)\mathfrak{G}_0(a_4)$	1
$\mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j)\mathfrak{G}_0(b_k)$ for $i < j$	$\binom{4}{2} \cdot 6 = 36$
$\mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j^*)$	$4 \cdot 4 = 16$
$\mathfrak{G}_0(b_i)\mathfrak{G}_0(b_j)$ for $i \leq j$	$\binom{6+1}{2} = 21$
$\mathfrak{G}_0(x)$	1
$\mathfrak{G}_0(a_i)\mathfrak{G}_0(a_j)\mathfrak{G}_1(1)$ for $i < j$	$\binom{4}{2} = 6$
$\mathfrak{G}_0(a_i)\mathfrak{G}_1(a_j)$	$4 \cdot 4 = 16$
$\mathfrak{G}_0(b_i)\mathfrak{G}_1(1)$	6
$\mathfrak{G}_1(b_i)$	6
$\mathfrak{G}_1(1)^2$	1
$\mathfrak{G}_2(1)$	1

Any multiplication operator of degree 4 can be written as a linear combination of these 111 classes. Likewise, the dimension of $H^4(A^{[n]}, \mathbb{Q})$ is 111 for all $n \geq 4$, according to Göttsche's formula [23]. However, for $n = 3$, the 8 classes $\mathfrak{G}_0(x)$, $\mathfrak{G}_1(b_i)$ and $\mathfrak{G}_2(1)$ can be expressed as linear combinations of the others, so we are left with 103 linearly independent classes that form a basis of $H^4(A^{[3]}, \mathbb{Q})$. Multiplication with the class $[K_2(A)]$ is given by the operator $\mathfrak{G}_0(a_1)\mathfrak{G}_0(a_2)\mathfrak{G}_0(a_3)\mathfrak{G}_0(a_4)$ and annihilates every class that contains an operator of the form $\mathfrak{G}_0(a_i)$. There are 75 such classes, so by Proposition 15.5, $\ker \theta^* \subset H^4(A^{[3]}, \mathbb{Q})$ has dimension at least 75 and $\operatorname{Im} \theta^*$ has dimension at most $103 - 75 = 28$. However, since the image of θ^* must contain $\operatorname{Sym}^2 H^2(K_2(A), \mathbb{Q})$, which is 28-dimensional, equality follows. □

Proposition 17.7. *We have:*

$$c_2(K_2(A)) = 4u_1u_2 + 4v_1v_2 + 4w_1w_2 - \frac{1}{3}e^2. \tag{80}$$

In particular, $c_2(K_2(A)) \in \operatorname{Sym} \otimes \mathbb{Q}$.

We shall give two different proofs. The first one uses Nakajima operators, the second one is based on results of [24].

Proof 1. First note that the defining diagram (64) of the Kummer manifold is the pullback of the inclusion of a point, so the normal bundle of $K_2(A)$ in $A^{[3]}$ is trivial.

The Chern class of the tangent bundle of $K_2(A)$ is therefore given by the pullback from $A^{[3]}$: $c(K_2(A)) = \theta^*\Big(c(A^{[3]})\Big)$. Proposition 17.6 allows now to conclude that $c_2(K_2(A)) \in \mathrm{Sym}\otimes\mathbb{Q}$.

To obtain the precise formula, we use a result of Boissière, [2, Lemma 3.12], giving a commutation relation for the Chern character multiplication operator on the Hilbert scheme. We get:

$$\begin{aligned} c_2(A^{[3]}) &= 3\mathfrak{q}_1(1)\mathfrak{L}_2(1)|0\rangle - \frac{1}{3}\mathfrak{q}_3(1)|0\rangle \\ &= \frac{8}{3}\mathfrak{q}_1(1)\mathfrak{L}_2(1)|0\rangle - \frac{1}{3}\delta^2. \end{aligned}$$

With Corollary 15.6 one shows now, that $c_2(K_2(A))$ is given as stated. □

Proof 2. It follows from (76) that $c_2(K_2(A)) \in \Pi'^{\perp}$, so $c_2(K_2(A)) \in \mathrm{Sym}\otimes\mathbb{Q}$. Moreover, together with (73) we get that

$$c_2(K_2(A)) \cdot D_1 \cdot D_2 = 54 \cdot B(D_1, D_2)$$

for all D_1, D_2 in $H^2(K_2(A),\mathbb{Z})$. Using the non-degeneracy of the Poincaré pairing and our knowledge about the Beauville–Bogomolov form on $K_2(A)$, we can calculate that

$$c_2(K_2(A)) = 4u_1u_2 + 4v_1v_2 + 4w_1w_2 - \frac{1}{3}e^2.$$ □

Corollary 17.8. *The intersection* $\mathrm{Sym}\cap\Pi$ *is one-dimensional and spanned by* $3c_2(K_2(A))$.

Proof. By Proposition 17.7 and (76), $3c_2(K_2(A)) \in \mathrm{Sym}\cap\Pi$. Since the ranks of Sym, Π and $H^4(K_2(A),\mathbb{Z})$ are 28, 81 and 108, respectively, the intersection cannot contain more. □

Corollary 17.9.

$$Y_p = \frac{1}{6}\Big(u_1u_2 + v_1v_2 + w_1w_2\Big). \tag{81}$$

Remark 17.10. Using Nakajima operators, we may write

$$Y_p = \frac{1}{9}\theta^*\Big(\mathfrak{q}_1(1)\mathfrak{L}_2(1)|0\rangle\Big) = \frac{1}{2}\theta^*\Big(\mathfrak{q}_1(x)\mathfrak{q}_1(1)^2|0\rangle\Big). \tag{82}$$

From the intersection properties $Z_\tau \cdot Z_{\tau'} = 1$ for $\tau \neq \tau'$ and $Z_\tau^2 = 4$ from Section 4 of [24], we compute

$$\mathrm{discr}\,\Pi' = 3^{84}. \tag{83}$$

On the other hand, a formula developed in [27] evaluates

$$\operatorname{discr} \mathrm{Sym} = 2^{14} \cdot 3^{38}, \tag{84}$$

so the lattices Sym and Π' cannot be primitive. Denote Sym^{sat} and Π'^{sat} the respective primitive overlattices. $\mathrm{Sym} \oplus \Pi'$ is a sublattice of $H^4(K_2(A), \mathbb{Z})$ of index $2^7 \cdot 3^{61}$ and we claim that $\mathrm{Sym}^{sat} \oplus \Pi'^{sat}$ has index 3^{22}. To obtain a basis of $H^4(K_2(A), \mathbb{Z})$, we are now going to find

- 7 classes in Sym divisible by 2,
- 8 classes in Sym divisible by 3,
- 31 classes in Π' divisible by 3 and
- 20 classes in $\mathrm{Sym}^{sat} \oplus \Pi'^{sat}$, one divisible by 3^3 and 19 divisible by 3.

Proposition 17.11. *For $y \in \{u_1, u_2, v_1, v_2, w_1, w_2\}$, the class $e \cdot y$ is divisible by 3 and $y^2 - \frac{1}{3}e \cdot y$ is divisible by 2.*

Proof. We have $y = \theta^* \left(\mathfrak{q}_1(a)\mathfrak{q}_1(1)^2|0\rangle\right)$ for some $a \in H^2(A, \mathbb{Z})$. A computation yields:

$$e \cdot y = 3 \cdot \theta^*\Big(\mathfrak{q}_2(a)\mathfrak{q}_1(1)|0\rangle\Big) \quad \text{and} \quad y^2 = \theta^*\Big(\mathfrak{q}_1(a)^2\mathfrak{q}_1(1)|0\rangle\Big)$$

so $e \cdot y$ is divisible by 3. Furthermore, by Corollary 14.3, the class $\frac{1}{2}\mathfrak{q}_1(a)^2\mathfrak{q}_1(1)|0\rangle - \frac{1}{2}\mathfrak{q}_2(a)\mathfrak{q}_1(1)|0\rangle$ is contained in $H^4(A^{[3]}, \mathbb{Z})$, so its pullback $\frac{1}{2}y^2 - \frac{1}{6}e \cdot y$ is an integral class, too. □

From Proposition 17.7 we see that e^2 is divisible by 3 and by Corollary 17.9 the class $u_1u_2 + v_1v_2 + w_1w_2$ is divisible by 6.

Corollary 17.12. *The image of $H^4(A^{[3]}, \mathbb{Z})$ under θ^* is equal to* Sym^{sat}. □

Now we come to Π'. The first thing to note is that Π' is defined topologically for all deformations of $K_2(A)$ and the primitive overlattice of Π' is a topological invariant, too. By applying a suitable deformation, we may therefore assume without loss of generality that A is the product of two elliptic curves $A = E_1 \times E_2$. After [24, Eq. (12)], for a non-isotropic plane $\Lambda \subset A[3]$ and any $\tau_0 \in A[3]$, the classes

$$\frac{1}{3} \sum_{\tau \in \Lambda} \Big(Z_\tau - Z_{\tau+\tau_0}\Big) \tag{85}$$

are integral. The monodromy representation acts on the Z_τ via the symplectic group $\mathrm{Sp}(4, \mathbb{F}_3)$. Modulo Π', the orbit of these classes is a $\mathbb{F}_3$-vector space naturally isomorphic to D as introduced in Definition 9.7, so by Proposition 9.9, we get a subspace of Π' of rank 31 of classes divisible by 3. They are explicitely given in Proposition C.1.

The class Z_0 is not contained in Sym nor in Π'. It can be written as follows:

$$Z_0 = \frac{\sum_{\tau \in A[3]} Z_\tau - \sum_{\tau \in A[3]}(Z_\tau - Z_0)}{81} \overset{(76)}{=} \frac{c_2(K_2(A)) - \frac{1}{3}\sum_{\tau \in A[3]}(Z_\tau - Z_0)}{27},$$

This is the class in $\mathrm{Sym}^{sat} \oplus \Pi'^{sat}$ divisible by 27. Let us now find the remaining 19 classes divisible by 3.

Hassett and Tschinkel in Proposition 7.1 of [24] provide the class of a Lagrangian plane $P \subset K_2(A)$, which can be expressed as follows:

$$[P] = \frac{1}{216}(6u_1 - 3e)^2 + \frac{1}{8}c_2(K_2(A)) - \frac{1}{3}\sum_{\tau \in \Lambda'} Z_\tau,$$

where $\Lambda' = E_1[3] \times 0 \subset A[3]$. We rearrange this expression a bit using (78):

$$\begin{aligned}
[P] &= \frac{1}{216}(6u_1 - 3e)^2 + \frac{1}{8}c_2(K_2(A)) - \frac{1}{3}\sum_{\tau \in \Lambda'} Z_\tau \\
&= \frac{36u_1^2 + 9e^2 - 36u_1 \cdot e}{216} + \frac{W}{3} - \frac{3}{8}e^2 - \frac{1}{3}\sum_{\tau \in \Lambda'} Z_\tau \\
&= \frac{u_1^2 - 2e^2 - u_1 \cdot e}{6} + \frac{W}{3} - \frac{1}{3}\sum_{\tau \in \Lambda'} Z_\tau.
\end{aligned}$$

The classes e^2 and $u_1 \cdot e$ are both divisible by 3 and by (78), W is divisible by 3, so the following class is integral:

$$\mathfrak{N} := \frac{u_1^2 + \sum_{\tau \in \Lambda'}(Z_\tau - Z_0)}{3}.$$

Now we will conclude using the action of the monodromy group $\mathrm{Sp}(A[3]) \ltimes A[3]$ on the element $\mathfrak{N}$ and the considerations from Section 9. Proposition 9.11 states now that the orbit of $\mathfrak{N}$ under the action of $\mathrm{Sp}(A[3]) \ltimes A[3]$ is spanning a space of rank 51 modulo $\mathrm{Sym} \oplus \Pi'$. However, by Lemma 9.12, the intersection of that space with Sym^{sat} is one-dimensional and the intersection with Π'^{sat} has dimension 31, so we are left with 19 linearly independent elements of the form: $\frac{x+y}{3}$ with $x \in \mathrm{Sym} \setminus \{0\}$, and $y \in \Pi' \smallsetminus \{0\}$. These are the 19 classes which were missing. They are listed in Proposition C.2.

18 Conclusion on cohomology of generalized Kummer fourfolds

Let A be a complex abelian surface with integral basis of $H^2(A, \mathbb{Z})$ given as in Notation 10.6. Let $\theta : K_2(A) \hookrightarrow A^{[3]}$ be the embedding. Let us summarize our results:

Theorem 18.1. *The homomorphism $\theta^* : H^*(A^{[3]}, \mathbb{Z}) \to H^*(K_2(A), \mathbb{Z})$ of graded rings is surjective in every degree except 4. Moreover, the image of $H^4(A^{[3]}, \mathbb{Z})$ is the primitive overlattice of $\mathrm{Sym}^2(H^2(K_2(A), \mathbb{Z}))$. The kernel of θ^* is the ideal generated by $H^1(A^{[3]}, \mathbb{Z})$. The image of the following integral classes under θ^* gives a basis of $\mathrm{Im}\,\theta^*$:*

degree	*preimage of class*	
0	$\frac{1}{6}\mathfrak{q}_1(1)^3\|0\rangle$	*1*
2	$\frac{1}{2}\mathfrak{q}_1(b_i)\mathfrak{q}_1(1)^2\|0\rangle$ *for* $1 \le i \le 6$	$j(b_i)$
	$\frac{1}{2}\mathfrak{q}_2(1)\mathfrak{q}_1(1)\|0\rangle$	e
3	$\frac{1}{2}\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(1)^2\|0\rangle$	
	$\mathfrak{q}_2(a_i)\mathfrak{q}_1(1)\|0\rangle$	
4	$\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)\mathfrak{q}_1(1)\|0\rangle$ *for* $1 \le i \le j \le 6$, *but* $(b_i, b_j) \neq (a_1a_2, a_3a_4)$	
	$\frac{1}{2}\mathfrak{q}_1(x)\mathfrak{q}_1(1)^2\|0\rangle$ *(instead of the missing case above)*	Y_p
	$\frac{1}{2}\left(\mathfrak{q}_1(b_i)^2 - \mathfrak{q}_2(b_i)\right)\mathfrak{q}_1(1)\|0\rangle$	
	$\frac{1}{3}\mathfrak{q}_3(1)\|0\rangle$	W
5	$\mathfrak{q}_1(a_ia_j)\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(1)\|0\rangle$ *for any choice of* $j \neq i$	
	$\frac{1}{2}\mathfrak{q}_2(a_i^*)\mathfrak{q}_1(1)\|0\rangle$	
6	$\mathfrak{q}_1(a_i^*)\mathfrak{q}_1(a_j^*)\mathfrak{q}_1(1)\|0\rangle$ *for* $1 \le i < j \le 4$	
	$\mathfrak{q}_2(x)\mathfrak{q}_1(1)\|0\rangle$	
8	$\mathfrak{q}_1(x)^3\|0\rangle$	

Proof. The table is established by the following results: For degree 2, see Proposition 15.9. The dual classes of degree 6 can be computed using Proposition 15.3 and the methods from Section 13. The odd degrees are treated by Proposition 16.2. Classes of degree 4 are studied in Section 17. The classes are chosen in a way that they give a basis of Sym^{sat}, which is possible by Corollary 17.12. The condition

$(b_i, b_j) \neq (a_1a_2, a_3a_4)$ is more or less arbitrary, but we had to remove one class to avoid a relation of linear dependence.

The kernel of θ^* is described by the Propositions 15.5 and 15.7. □

Theorem 18.2. *Using Notation 15.10 and 17.1, we have:*

(i) Let $\mathrm{Sym}^{sat} \subset H^4(K_2(A), \mathbb{Z})$ *be the primitive overlattice of the symmetric product* $\mathrm{Sym}^2(H^2(K_2(A), \mathbb{Z}))$. *The group* $\frac{\mathrm{Sym}^{sat}}{\mathrm{Sym}^2(H^2(K_2(A),\mathbb{Z}))} = (\mathbb{Z}/2\mathbb{Z})^{\oplus 7} \oplus (\mathbb{Z}/3\mathbb{Z})^{\oplus 8}$ *is generated by the following elements:*

$$\frac{e \cdot y}{3},\ \frac{y^2 - \frac{1}{3}e \cdot y}{2} \text{ for } y \in \{u_1, u_2, v_1, v_2, w_1, w_2\},$$
$$\frac{e^2}{3} \text{ and } \frac{u_1 \cdot u_2 + v_1 \cdot v_2 + w_1 \cdot w_2}{6}.$$

(ii) Let Π' *be the lattice from Definition 17.3 and let* Π'^{sat} *be the primitive over lattice of* Π' *in* $H^4(K_2(A), \mathbb{Z})$. *The group* $\frac{\Pi'^{sat}}{\Pi'} = (\mathbb{Z}/3\mathbb{Z})^{\oplus 31}$ *is generated by the classes:*

$$\frac{1}{3}\sum_{\tau \in \Lambda} \left(Z_\tau - Z_{\tau+\tau'}\right),$$

with Λ *a non-isotropic group and* $\tau' \in A[3]$. *Moreover a basis of* $\frac{\Pi'^{sat}}{\Pi'}$ *is provided by the 31 classes described in Proposition C.1.*

(iii)

$$\frac{H^4(K_2(A), \mathbb{Z})}{\mathrm{Sym}^{sat} \oplus \Pi'^{sat}} = \left(\frac{\mathbb{Z}}{27\mathbb{Z}}\right) \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 19}.$$

Moreover, this group is generated by the class Z_0 *and the 19 classes described in Proposition C.2.* □

19 Symplectic involution on $K_2(A)$

This section and the following one is contributed by Grégoire Menet.

Let X be an irreducible symplectic manifold. Denote

$$\nu : \mathrm{Aut}(X) \to \mathrm{Aut}\, H^2(X, \mathbb{Z})$$

the natural morphism. Hassett and Tschinkel (Theorem 2.1 in [24]) have shown that $\mathrm{Ker}\,\nu$ is a deformation invariant. Let X be an irreducible symplectic fourfold of Kummer type. Then Oguiso in [51] has shown that $\mathrm{Ker}\,\nu = (\mathbb{Z}/3\mathbb{Z})^3 \rtimes \mathbb{Z}/2\mathbb{Z}$.

Let A be an abelian variety and g an automorphism of A. Let us denote by $T_{A[3]}$ the group of translations of A by elements of $A[3]$. If $g \in T_{A[3]} \rtimes \mathrm{Aut}_{\mathbb{Z}}(A)$, then g induces a natural automorphism on $K_2(A)$. We denote the induced automorphism by $g^{[[3]]}$. If there is no ambiguity, we also denote the *induced automorphism* by the same letter g to avoid too complicated formulas.

When $X = K_2(A)$, we have more precisely, by Corollary 3.3 of [4],

$$\operatorname{Ker}\nu = T_{A[3]} \rtimes (-\operatorname{id}_A)^{[[3]]}.$$

19.1 Uniqueness and fixed locus

Theorem 19.1. *Let X be an irreducible symplectic fourfold of Kummer type and ι a symplectic involution on X. Then:*

(i) We have $\iota \in \operatorname{Ker}\nu$.

(ii) Let A be an abelian surface. Then the couple (X, ι) is deformation equivalent to $(K_2(A), t_\tau \circ (-\operatorname{id}_A)^{[[3]]})$, where t_τ is the morphism induced on $K_2(A)$ by the translation by $\tau \in A[3]$.

(iii) The fixed locus of ι is given by a K3 surface and 36 isolated points.

Proof. (i) If $\iota \notin \operatorname{Ker}\nu$, by the classification of Section 5 of [43], the unique possible action of ι on $H^2(X,\mathbb{Z})$ is given by $H^2(X,\mathbb{Z})^\iota = U \oplus (2)^2 \oplus (-6)$. We will show that it is impossible. Let us assume that $H^2(X,\mathbb{Z})^\iota = U \oplus (2)^2 \oplus (-6)$, we will find a contradiction.

As done in Section 3 of [42], consider a local universal deformation space of X:

$$\Phi : \mathcal{X} \to \Delta,$$

where Δ is a small polydisk and $\mathcal{X}_0 = X$. By restricting Δ, we can assume that ι extends to an automorphism M on $\mathcal{X}$ and μ on Δ, such that we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "M"] \arrow[d] & \mathcal{X} \arrow[d] \\ \Delta \arrow[r, "\mu"] & \Delta \end{tikzcd}$$

Moreover, the differential of μ at 0 is given by the action of ι on $H^1(T_X)$ which is the same as the action on $H^{1,1}(X)$, since the symplectic holomorphic 2-form induces an isomorphism between the two and the symplectic holomorphic 2-form is preserved by the action of ι. We may assume that μ is a linear map. So Δ^μ is smooth and $\dim \Delta^\mu = \operatorname{rk} H^2(X,\mathbb{Z})^\iota - 2 = 3$. Moreover, by [35] we can find $x \in \Delta^\mu$ such that $\mathcal{X}_x$ is bimeromorphic to a Kummer fourfold $K_2(A)$. Since $H^2(X,\mathbb{Z})^\iota = U \oplus (2)^2 \oplus (-6)$, $\iota_x := M_{\mathcal{X}_x}$ induces a bimeromorphic involution i on $K_2(A)$ with $H^2(K_2(A),\mathbb{Z})^i = U \oplus (2)^2 \oplus (-6)$.

Since i preserves the holomorphic 2-from, $\operatorname{NS}(K_2(A)) \supset \left[H^2(K_2(A),\mathbb{Z})^i\right]^\perp = (-2)^2$. The involution i also induces a trivial involution on $A_{H^2(X,\mathbb{Z})}$, so the half class of the diagonal e is contained in $H^2(K_2(A),\mathbb{Z})^i \cap \operatorname{NS}(K_2(A))$. It follows that $\operatorname{NS}(K_2(A)) \supset (-2)^2 \oplus \mathbb{Z}e$. Moreover, the morphism j defined in Notation 15.10 respects the Hodge structure so $\operatorname{NS}(K_2(A)) = j(\operatorname{NS}(A)) \oplus \mathbb{Z}e$. It follows that $\operatorname{NS}(A) \supset (-2)^2$. Now we construct an involution g on $H^2(A,\mathbb{Z})$ given by $-\operatorname{id}$ on $(-2)^2$ and id on $((-2)^2)^\perp$ and extended to an involution on $H^2(A,\mathbb{Z})$ by Corollary 1.5.2 of [49]. Then by Theorem 1 of [56], g provides a symplectic automorphism on A with: $H^2(A,\mathbb{Z})^g = ((-2)^2)^\perp = U \oplus (2)^2$. It follows from the classification of Section 4 of [44], that $A = \mathbb{C}/\Lambda$ with $\Lambda = \langle (1,0),(0,1),(x,-y),(y,x)\rangle$, $(x,y) \in \mathbb{C}^2 \smallsetminus \mathbb{R}^2$ and $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Let also denote by g the automorphism on $K_2(A)$ induced by g. By construction, $g \circ i$ acts trivially on $H^2(K_2(A),\mathbb{Z})$. Hence by Corollary 3.3 and Lemma 3.4 of [18], $g \circ \iota$ extends to an automorphism of $K_2(A)$. In particular, i extends to a symplectic involution on $K_2(A)$. Then $g \circ i \in \operatorname{Ker} \nu$.

By Corollary 16.3, t_τ acts trivially on $H^3(K_2(A),\mathbb{Z})$. Hence by Corollary 3.3 of [4], we have necessarily:

$$\begin{gathered} g^*_{|H^3(K_2(A),\mathbb{Z})} = i^*_{|H^3(K_2(A),\mathbb{Z})} \circ (-\operatorname{id}_A)^*_{|H^3(K_2(A),\mathbb{Z})} \\ \textit{or} \quad g^*_{|H^3(K_2(A),\mathbb{Z})} = i^*_{|H^3(K_2(A),\mathbb{Z})}. \end{gathered}$$

By Corollary 16.3, $g^*_{|H^3(K_2(A),\mathbb{Z})}$ has order 4. But on the other hand, both $i^*_{|H^3(K_2(A),\mathbb{Z})} \circ (-\operatorname{id}_A)^*_{|H^3(K_2(A),\mathbb{Z})}$ and $i^*_{|H^3(K_2(A),\mathbb{Z})}$ have order 2, which is a contradiction.

(ii) Let X be a irreducible symplectic fourfold of Kummer type and ι a symplectic involution on X. By (1) of the above theorem, we have $\iota \in \operatorname{Ker} \nu$. Then by Theorem 2.1 of [24], the couple (X,ι) deform to a couple $(K_2(A),\iota')$ with A an abelian surface and $\iota' \in \operatorname{Ker} \nu$ a symplectic involution on $K_2(A)$. Then we conclude with Corollary 3.3 of [4].

(iii) Let A be an abelian surface. By Section 1.2.1 of [58], the fixed locus of $t_\tau \circ (-\mathrm{id}_A)^{[[3]]}$ on $K_2(A)$ is given by a K3 surface an 36 isolated points. Now let X be a irreducible symplectic fourfold of Kummer type and ι a symplectic involution on X. By (2) of the above theorem, $\operatorname{Fix}\iota$ deforms to the disjoint union of a K3 surface and 36 isolated points. Moreover, ι is a symplectic involution, so the holomorphic 2-form of X restricts to a non-degenerated holomorphic 2-form on $\operatorname{Fix}\iota$. Then necessarily, $\operatorname{Fix}\iota$ consists of a K3 surface and 36 isolated points. □

Remark 19.2. (i) We also remark that the K3 surface fixed by $(t_\tau \circ (-\mathrm{id}_A))$ is given by the sub-manifold

$$Z_{-\tau} = \overline{\{(a_1, a_2, a_3) \mid a_1 = -\tau,\ a_2 = -a_3 + \tau,\ a_2 \neq -\tau\}}$$

defined in Section 4 of [24].

(ii) Considering the involution $-\mathrm{id}_A$, the set

$$\mathcal{P} := \{\xi \in K_2(A) |\ \operatorname{Supp}\xi = \{a_1, a_2, a_3\},\ a_i \in A[2] \smallsetminus \{0\}, 1 \leq i \leq 3\}$$

provides 35 fixed points and the vertex of

$$W_0 := \{\xi \in K_2(A) |\ \operatorname{Supp}\xi = \{0\}\}$$

supplies the 36th point. We denote by $p_1, ..., p_{35}$ the points of $\mathcal{P}$ and by p_{36} the vertex of W_0.

19.2 Action on the cohomology

From Theorem 19.1, we can assume that $X = K_2(A)$ and $\iota = -\mathrm{id}_A$. To consider $t_\tau \circ (-\mathrm{id}_A)$ instead of $-\mathrm{id}_A$ only has the effect of exchanging the role of $[Z_0]$ and $[Z_{-\tau}]$. Hence we do not lose any generality assuming that $\iota = -\mathrm{id}_A$. Now, we calculate the invariants $l_i^j(K_2(A))$ defined in Definition-Proposition 11.1. It will be used in Section 20.

From Theorem 19.1 (1), the involution ι acts trivially on $H^2(K_2(A), \mathbb{Z})$. It follows

$$l_2^2(K_2(A)) = l_{1,-}^2(K_2(A)) = 0 \text{ and } l_{1,+}^2(K_2(A)) = 7. \tag{86}$$

From Corollary 16.3, the involution ι acts as $-\operatorname{id}$ on $H^3(K_2(A),\mathbb{Z})$. It follows

$$l_2^3(K_2(A)) = l_{1,+}^3(K_2(A)) = 0 \text{ and } l_{1,-}^3(K_2(A)) = 8. \tag{87}$$

By Definition 17.3, we have:

$$H^4(K_2(A),\mathbb{Q}) = \operatorname{Sym}^2 H^2(K_2(A),\mathbb{Q}) \oplus^{\perp} \Pi' \otimes \mathbb{Q},$$

where $\Pi' = \langle Z_\tau - Z_0,\ \tau \in A[3] \smallsetminus \{0\}\rangle$. The involution ι^* fixes $\operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z})$ and $\iota^*(Z_\tau - Z_0) = Z_{-\tau} - Z_0$. It provides the following proposition.

Proposition 19.3. *We have $l_{1,-}^4(K_2(A)) = 0$, $l_{1,+}^4(K_2(A)) = 28$ and $l_2^4(K_2(A)) = 40$.*

Proof. Let $\mathcal{S}$ be the over-lattice of $\operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z})$ where we add all the classes divisible by 2 in $H^4(K_2(A),\mathbb{Z})$. From Section 17, we know that the discriminant of $\mathcal{S}$ is not divisible by 2. Hence, we have:

$$H^4(K_2(A),\mathbb{F}_2) = \mathcal{S} \otimes \mathbb{F}_2 \oplus \Pi' \otimes \mathbb{F}_2.$$

Moreover, we have:

$$\iota^*(Z_\tau - Z_0) = Z_{-\tau} - Z_0,$$

for all $\tau \in A[3] \smallsetminus \{0\}$. Hence $\operatorname{Vect}_{\mathbb{F}_2}(Z_\tau - Z_0, Z_{-\tau} - Z_0)$ is isomorphic to N_2 as a $\mathbb{F}_2[G]$-module (see the notation in Definition-Proposition 11.1). Moreover $H^2(K_2(A),\mathbb{Z})$ is invariant by the action of ι, hence $\operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z})$ and $\mathcal{S}$ is also invariant by the action of ι. It follows that $\mathcal{S} \otimes \mathbb{F}_2 = \mathcal{N}_1$ and $\Pi' \otimes \mathbb{F}_2 = \mathcal{N}_2$. Since $\operatorname{rk} \mathcal{S} = 28$, we have $l_{1,+}^4 + l_{1,-}^4 = 28$. However, $\mathcal{S}$ is invariant by the action of ι, it follows that $l_{1,-}^4 = 0$ and $l_{1,+}^4 = 28$. On the other hand $\operatorname{rk} \Pi' = 80$, it follows that $l_2^4 = 40$. $\square$

20 Application to singular irreducible symplectic varieties

20.1 Statement of the main theorem

In [47], Namikawa proposes a definition of the Beauville-Bogomolov form for some singular irreducible symplectic varieties. He assumes that the singularities are only $\mathbb{Q}$-factorial with a singular locus of codimension ≥ 4. Under these assumptions, he proves a local Torelli theorem. This result was completed by a generalization of the Fujiki formula by Matsushita in [36].

Theorem 20.1. *Let X be a projective irreducible symplectic variety of dimension $2n$ with only $\mathbb{Q}$-factorial singularities, and* $\operatorname{Codim}\operatorname{Sing} X \geq 4$*. There exists a unique indivisible integral symmetric non-degenerated bilinear form B_X on $H^2(X,\mathbb{Z})$ and a unique positive constant $c_X \in \mathbb{Q}$, such that for any $\alpha \in H^2(X,\mathbb{C})$,*

$$\alpha^{2n} = c_X B_X(\alpha,\alpha)^n \tag{88}$$

and such that for $0 \neq \omega \in H^0(\Omega^2_U)$ a holomorphic 2-from on the smooth locus U of X:

$$B_X(\omega+\overline{\omega},\omega+\overline{\omega}) > 0. \tag{89}$$

Moreover, the signature of B_X is $(3, h^2(X,\mathbb{C})-3)$.

The form B_X is called the Beauville–Bogomolov form of X.

Proof. The statement of the theorem in [36] does not say that the form is integral. However, let X_s be a fiber of the Kuranishi family of X, with the same idea as Matsushita's proof, we can see that B_X and B_{X_s} are proportional. Then, it follows using the proof of Theorem 5 a), c) of [1]. □

We can also consider its polarized form.

Proposition 20.2. *Let X be a projective irreducible symplectic variety of dimension $2n$ with* $\operatorname{Codim}\operatorname{Sing} X \geq 4$*. The equality (88) of Theorem 20.1 implies that*

$$\alpha_1 \cdot \ldots \cdot \alpha_{2n} = \frac{c_X}{(2n)!} \sum_{\sigma\in S_{2n}} B_X(\alpha_{\sigma(1)},\alpha_{\sigma(2)})\ldots B_X(\alpha_{\sigma(2n-1)},\alpha_{\sigma(2n)}).$$

for all $\alpha_i \in H^2(X,\mathbb{Z})$.

These results were then generalized by Kirschner for symplectic complex spaces in [28]. In [37, Theorem 2.5] was appeared the first concrete example of Beauville-Bogomolov lattice for a singular irreducible symplectic variety. The variety studied in [37] is a partial resolution of an irreducible symplectic manifold of $K3^{[2]}$-type quotiented by a symplectic involution. The objective of this section is to provide a new example of a Beauville-Bogomolov lattice replacing the manifold of $K3^{[2]}$-type by a fourfold of Kummer type. Knowing the integral basis of the cohomology group of the generalized Kummer provided in Part 14, this calculation becomes possible. Moreover the calculation will be much simpler as in [37] because of the general techniques for calculating integral cohomology of quotients developed in [38] and the new technique using monodromy developed in Lemma 20.13. The other techniques developed in [37] are also in [38], so to simplify the reading, we will only cite [37] in the rest of the section.

Concretely, let X be an irreducible symplectic fourfold of Kummer type and ι a symplectic involution on X. By Theorem 19.1 the fixed locus of ι is the union of 36 points and a K3 surface Z_0. Then the singular locus of $K := X/\iota$ is the union of a K3 surface and 36 points. The singular locus is not of codimension four. We will lift to a partial resolution of singularities, K' of K, obtained by blowing up the image of Z_0. By Section 2.3 and Lemma 1.2 of [20], the variety K' is an irreducible symplectic V-manifold which has singular locus of codimension four.

All Section 20 is devoted to prove the following theorem.

Theorem 20.3. *Let X be an irreducible symplectic fourfold of Kummer type and ι a symplectic involution on X. Let Z_0 be the K3 surface which is in the fixed locus of ι. We denote $K = X/\iota$ and K' the partial resolution of singularities of K obtained by blowing up the image of Z_0. Then the Beauville–Bogomolov lattice $H^2(K',\mathbb{Z})$ is isomorphic to $U(3)^3 \oplus \begin{pmatrix} -5 & -4 \\ -4 & -5 \end{pmatrix}$, and the Fujiki constant $c_{K'}$ is equal to* 8.

The Beauville-Bogomolov form is a topological invariant, hence from Theorem 19.1 we can assume that X is a generalized Kummer fourfold and $\iota = -\mathrm{id}_A$. As it will be useful to prove Lemma 20.13, we can assume even more. All generalized Kummer fourfolds are deformation equivalent, hence we can assume that $A = E_\xi \times E_\xi$, where E_ξ is the elliptic curve provided in Definition 10.4:

$$E_\xi := \frac{\mathbb{C}}{\left\langle 1, e^{\frac{2i\pi}{6}} \right\rangle}.$$

20.2 Overview on the proof of Theorem 20.3

We first provide all the notation that we will need during the proof in Section 20.3. Then the proof is divided into the following steps:

(i) First (86), (87), Proposition 19.3 and Corollary 11.9 will prove the H^4-normality of $(K_2(A), \iota)$ in Section 20.4.

(ii) The knowledge of the elements divisible by 2 in $\mathrm{Sym}^2 H^2(K_2(A),\mathbb{Z})$ from Section 17 and the H^4-normality allow us to prove the H^2-normality of $(K_2(A), \iota)$ in Section 20.5.

(iii) An adaptation of the H^2-normality (Lemma 20.8) and several lemmas in Section 20.6 will provide an integral basis of $H^2(K',\mathbb{Z})$ (Theorem 20.9).

(iv) Knowing an integral basis of $H^2(K',\mathbb{Z})$, we end the calculation of the Beauville–Bogomolov form in Section 20.7 using intersection theory and the generalized Fujiki formula (Theorem 20.1).

20.3 Notation

Let $K_2(A)$ be a generalized Kummer fourfold endowed with the symplectic involution ι induced by $-\operatorname{id}_A$. We denote by π the quotient map $K_2(A) \to K_2(A)/\iota$. From Theorem 19.1, we know that the singular locus of the quotient $K_2(A)/\iota$ is the K3 surface, image by π of Z_0, and 36 isolated points. We denote $\overline{Z_0} := \pi(Z_0)$. We consider $r' : K' \to K_2(A)/\iota$ the blow-up of $K_2(A)/\iota$ in $\overline{Z_0}$ and we denote by $\overline{Z_0}'$ the exceptional divisor. We also denote by $s_1 : N_1 \to K_2(A)$ the blowup of $K_2(A)$ in Z_0; and denote by Z_0' the exceptional divisor in N_1. Denote by ι_1 the involution on N_1 induced by ι. We have $K' \simeq N_1/\iota_1$, and we denote $\pi_1 : N_1 \to K'$ the quotient map.

Consider the blowup $s_2 : N_2 \to N_1$ of N_1 in the 36 points $p_1, ..., p_{36}$ fixed by ι_1 and the blowup $\widetilde{r} : \widetilde{K} \to K'$ of K' in its 36 singulars points. We denote the exceptional divisors by $E_1, ..., E_{36}$ and $D_1, ..., D_{36}$ respectively. We also denote $\widetilde{\overline{Z_0}} = \widetilde{r}^*(\overline{Z_0}')$ and $\widetilde{Z_0} = s_2^*(Z_0')$. Denote ι_2 the involution induced by ι on N_2 and $\pi_2 : N_2 \to N_2/\iota_2$ the quotient map. We have $N_2/\iota_2 \simeq \widetilde{K}$. To finish, we denote $V = K_2(A) \smallsetminus \operatorname{Fix}\iota$ and $U = V/\iota$. We collect this notation in a commutative diagram

$$\begin{tikzcd}
\widetilde{K} \arrow[r, "\widetilde{r}"] & K' \arrow[r, "r'"] & K_2(A)/\iota & U \arrow[l, hook'] \\
N_2 \arrow[u, "\pi_2"] \arrow[r, "s_2"] \arrow[loop below, "\iota_2"] & N_1 \arrow[u, "\pi_1"] \arrow[r, "s_1"] \arrow[loop below, "\iota_1"] & K_2(A) \arrow[u, "\pi"] \arrow[loop below, "\iota"] & V \arrow[l, hook'] \arrow[u]
\end{tikzcd} \tag{90}$$

Also, we set $s = s_2 \circ s_1$ and $r = \widetilde{r} \circ r'$. We denote also e the half of the class of the diagonal in $H^2(K_2(A),\mathbb{Z})$ as states in Notation 15.10.

Remark 20.4. We can commute the push-forward maps and the blow-up maps as proved in Lemma 3.3.21 of [38]. Let $x \in H^2(N_1,\mathbb{Z})$, $y \in H^2(K_2(A),\mathbb{Z})$, we have:

$$\pi_{2*}(s_2^*(x)) = \widetilde{r}^*(\pi_{1*}(x)),$$

$$\pi_{1*}(s_1^*(y)) = r'^*(\pi_*(y)),$$

Moreover, we will also use the notation provided in Notation 15.10 and in Section 17.

20.4 The couple $(K_2(A), \iota)$ is H^4-normal

Proposition 20.5. *The couple $(K_2(A), \iota)$ is H^4-normal.*

Proof. We apply Theorem 11.9.

(i) By Theorem 15.2, $H^*(K_2(A), \mathbb{Z})$ is torsion-free.

(ii) From Remark 19.2 (1), we know that the connected component of dimension 2 of $\operatorname{Fix}\iota$ is given by Z_0 which is a K3 surface, hence is simply connected. Moreover by Proposition 4.3 of [24] $Z_0 \cdot Z_\tau = 1$ for all $\tau \in A[3] \smallsetminus \{0\}$. Hence the class of Z_0 in $H^4(K_2(A), \mathbb{Z})$ is primitive. It follows that $\operatorname{Fix}\iota$ is almost negligible (Definition 11.8).

(iii) By (86) and Proposition 19.3, we have $l_{1,-}^2(K_2(A)) = l_{1,-}^4(K_2(A)) = 0$.

(iv) By (87) and Proposition 19.3, we have $l_{1,+}^3(K_2(A)) = 0$. Moreover, we have $H^1(K_2(A)) = 0$, so $l_{1,+}^1(K_2(A)) = 0$.

(v) We have to check the following equality:

$$\begin{aligned} &l_{1,+}^4(K_2(A)) + 2\left[l_{1,-}^1(X) + l_{1,-}^3(X) + l_{1,+}^0(X) + l_{1,+}^2(X)\right] \\ &= 36h^0(pt) + h^0(Z_0) + h^2(Z_0) + h^4(Z_0). \end{aligned}$$

By (86), (87) and Proposition 19.3:

$$\begin{aligned} l_{1,+}^4(K_2(A)) + 2\left[l_{1,-}^1(X) + l_{1,-}^3(X) + l_{1,+}^0(X) + l_{1,+}^2(X)\right] \\ = 28 + 2(8 + 1 + 7) = 60. \end{aligned}$$

Moreover, since Z_0 is a K3 surface, we have:

$$36h^0(pt) + h^0(Z_0) + h^2(Z_0) + h^4(Z_0) = 36 + 1 + 22 + 1 = 60.$$

It follows from Corollary 11.9 that $(K_2(A), \iota)$ is H^4-normal. □

Remark 20.6. As explained in Proposition 3.5.20 of [38], the proof of Theorem 11.9 provide first that $\pi_{2*}(s^*(H^4(K_2(A),\mathbb{Z})))$ is primitive in $H^4(\widetilde{K},\mathbb{Z})$ and then the H^4 normality. So, the lattice $\pi_{2*}(s^*(H^4(K_2(A),\mathbb{Z})))$ is primitive in $H^4(\widetilde{K},\mathbb{Z})$.

20.5 The couple $(K_2(A),\iota)$ is H^2-normal

Proposition 20.7. *The couple $(K_2(A),\iota)$ is H^2-normal.*

Proof. We prove that the pushforward $\pi_*: H^2(K_2(A),\mathbb{Z}) \to H^2(K_2(A)/\iota,\mathbb{Z})/\text{tors}$ is surjective. By Remark 11.7, it is equivalent to prove that for all $x \in H^2(K_2(A),\mathbb{Z})^\iota$, $\pi_*(x)$ is divisible by 2 if and only if there exists $y \in H^2(K_2(A),\mathbb{Z})$ such that $x = y + \iota^*(y)$.

Let $x \in H^2(K_2(A),\mathbb{Z})^\iota = H^2(K_2(A),\mathbb{Z})$ such that $\pi_*(x)$ is divisible by 2, we will show that there exists $y \in H^2(K_2(A),\mathbb{Z})$ such that $x = y + \iota^*(y)$. By Proposition 11.5, $\pi_*(x^2)$ is divisible by 2. However, $x^2 \in H^4(K_2(A),\mathbb{Z})^\iota$; since $(K_2(A),\iota)$ is H^4-normal by Proposition 20.5, it means that there is $z \in H^4(K_2(A),\mathbb{Z})$ such that $x^2 = z + \iota^*(z)$.

Let $\mathcal{S}$ be, as before, the over-lattice of $\operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z})$ where we add all the classes divisible by 2 in $H^4(K_2(A),\mathbb{Z})$. By Definition 17.3 and (83), there exist $z_s \in \mathcal{S}$, $z_p \in \Pi'$ and $\alpha \in \mathbb{N}$ such that: $3^\alpha \cdot z = z_s + z_p$. Hence, we have:

$$3^\alpha \cdot x^2 = 2z_s + z_p + \iota^*(z_p).$$

Since $x^2 \in \text{Sym}$, by Corollary 17.8, $z_p + \iota^*(z_p) = 0$. It follows:

$$3^\alpha \cdot x^2 = 2z_s. \tag{91}$$

let $(u_1,u_2,v_1,v_2,w_1,w_2,e)$ be the integral basis of $H^2(K_2(A),\mathbb{Z})$ introduced in Notation15.10. We can write:

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \beta_1 v_1 + \beta_2 v_2 + \gamma_1 w_1 + \gamma_2 w_2 + de.$$

Then

$$3^\alpha \cdot x^2 = \alpha_1^2 u_1^2 + \alpha_2^2 u_2^2 + \beta_1^2 v_1^2 + \beta_2^2 v_2^2 + \gamma_1^2 w_1^2 + \gamma_2^2 w_2^2 + d^2 e^2 \mod 2H^4(K_2(A),\mathbb{Z}).$$

It follows by (91) that $\alpha_1^2 u_1^2 + \alpha_2^2 u_2^2 + \beta_1^2 v_1^2 + \beta_2^2 v_2^2 + \gamma_1^2 w_1^2 + \gamma_2^2 w_2^2 + d^2 e^2$ is divisible by 2. However by Corollary 17.9 and Proposition 17.11, we have:

$$\mathcal{S} = \operatorname{Sym}^2 H^2(K_2(A),\mathbb{Z}) \ + \\ \left\langle \frac{u_1\cdot u_2+v_1\cdot v_2+w_1\cdot w_2}{2};\frac{u_i^2-\frac{1}{3}u_i\cdot e}{2};\frac{v_i^2-\frac{1}{3}v_i\cdot e}{2};\frac{w_i^2-\frac{1}{3}w_i\cdot e}{2}, i\in\{1,2\}\right\rangle. \tag{92}$$

The $\frac{1}{2}(\alpha_1^2u_1^2+\alpha_2^2u_2^2+\beta_1^2v_1^2+\beta_2^2v_2^2+\gamma_1^2w_1^2+\gamma_2^2w_2^2+d^2e^2)$ is in $\mathcal{S}$ and so can be expressed as a linear combination of the generators of $\mathcal{S}$. Then, it follows from (92) that all the coefficients of $\alpha_1^2u_1^2+\alpha_2^2u_2^2+\beta_1^2v_1^2+\beta_2^2v_2^2+\gamma_1^2w_1^2+\gamma_2^2w_2^2+d^2e^2$ are divisible by 2. It means that x is divisible by 2. This is what we wanted to prove. □

With exactly the same proof working in $H^4(\widetilde{K},\mathbb{Z})$ and using Remark 20.6, we provide the following lemma.

Lemma 20.8. *The lattice $\pi_{2*}(s^*(H^2(K_2(A),\mathbb{Z})))$ is primitive in $H^2(\widetilde{K},\mathbb{Z})$.*

20.6 Calculation of $H^2(K',\mathbb{Z})$

This section is devoted to prove the following theorem.

Theorem 20.9. *Let K', π_1, s_1 and $\overline{Z_0}'$ be respectively the variety, the maps and the class defined in Section 20.3. We have*

$$H^2(K',\mathbb{Z})=\pi_{1*}s_1^*H^2(K_2(A),\mathbb{Z})\oplus\mathbb{Z}\left(\frac{\pi_{1*}(s_1^*(e))+\overline{Z_0}'}{2}\right)\oplus\mathbb{Z}\left(\frac{\pi_{1*}(s_1^*(e))-\overline{Z_0}'}{2}\right).$$

First we need to calculate some intersections.

Lemma 20.10. *(i) We have $E_l\cdot E_k=0$ if $l\neq k$, $E_l^4=-1$ and $E_l\cdot z=0$ for all $(l,k)\in\{1,...,28\}^2$ and for all $z\in s^*(H^2(K_2(A),\mathbb{Z}))$.*

(ii) We have $e^4=324$.
We already have some properties of primitivity:

(i) $\pi_{1*}(s_1^*(H^2(K_2(A),\mathbb{Z})))$ is primitive in $H^2(K',\mathbb{Z})$,

(ii) The group $\widetilde{\mathcal{D}}=\left\langle\widetilde{\overline{Z_0}},D_1,...,D_{36},\frac{\widetilde{\overline{Z_0}}+D_1+...+D_{36}}{2}\right\rangle$ is primitive in $H^2(\widetilde{K},\mathbb{Z})$.

(iii) $\overline{Z_0}'$ is primitive in $H^2(K',\mathbb{Z})$,

Proof. (i) It is proven using adjunction formula. It is the same statement as Proposition 4.6.16 1) of [38].

(ii) It follows directly from the Fujiki formula (68).

(iii) By Lemma 20.8, $\pi_{2*}(s^*(H^2(K_2(A),\mathbb{Z})))$ is primitive in $H^2(\widetilde{K},\mathbb{Z})$. Then by Remark 20.4, $r'^*(\pi_*(H^2(K_2(A),\mathbb{Z})))$ is primitive in $H^2(K',\mathbb{Z})$. Using again Remark 20.4, we get the result.

The proof of the last two points is the same as Lemma 4.6.14 of [38] and will be omitted.

□

With Lemma 20.10 (iii) and (v), it only remains to prove that $\pi_{1*}(s_1^*(e))+\overline{Z_0}'$ is divisible by 2 which will be done in Lemma 20.14. To prove this lemma, we first prove that $\pi_{2*}(s^*(e))+\widetilde{\overline{Z_0}}$ is divisible by 2. Knowing that $\widetilde{\overline{Z_0}}+D_1+...+D_{36}$ is divisible by 2, we only have to show that $\pi_{2*}(s^*(e))+D_1+...+D_{36}$ is divisible by 2 which is done by Lemma 20.12 and 20.13.

First we need to know the group $H^3(\widetilde{K},\mathbb{Z})$.

Lemma 20.11. *We have* $H^3(\widetilde{K},\mathbb{Z})=0$.

Proof. We have the following exact sequence:

$$H^3(K_2(A),V,\mathbb{Z})\to H^3(K_2(A),\mathbb{Z})\xrightarrow{f} H^3(V,\mathbb{Z})\to H^4(K_2(A),V,\mathbb{Z})\xrightarrow{\rho} H^4(K_2(A),\mathbb{Z}).$$

By Thom isomorphism, $H^3(K_2(A),V,\mathbb{Z})=0$ and $H^4(K_2(A),V,\mathbb{Z})=H^0(Z_0,\mathbb{Z})$. Moreover ρ is injective, so $H^3(V,\mathbb{Z})=H^3(K_2(A),\mathbb{Z})$.

Hence by (86), (87) and Proposition 3.2.8 of [38], we find that $H^3(U,\mathbb{Z})=0$. Since $H^3(K_2(A),\mathbb{Z})^{\iota}=0$, $H^3(\widetilde{K},\mathbb{Z})$ is a torsion group. Hence the result follows from the exact sequence

$$H^3(\widetilde{K},U,\mathbb{Z})\to H^3(\widetilde{K},\mathbb{Z})\to H^3(U,\mathbb{Z})$$

and from the fact that $H^3(\widetilde{K},U,\mathbb{Z})=0$ by Thom isomorphism. □

Lemma 20.12. *There exists D_e which is a linear combination of the D_i with coefficient 0 or 1 such that $\pi_{2*}(s^*(e))+D_e$ is divisible by 2.*

Proof. First, we have to use Smith theory as in Section 4.6.4 of [38].

Look at the following exact sequence:

$$0 \to H^2(\widetilde{K}, \widetilde{\widetilde{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2)) \to H^2(\widetilde{K}, \mathbb{F}_2) \to H^2(\widetilde{\widetilde{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2))$$
$$\longrightarrow H^3(\widetilde{K}, \widetilde{\widetilde{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) \longrightarrow 0.$$

First, we will calculate the dimension of the vector spaces $H^2(\widetilde{K}, \widetilde{\widetilde{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2)$ and $H^3(\widetilde{K}, \widetilde{\widetilde{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2)$. By (2) of Proposition 11.10, we have

$$H^*(\widetilde{K}, \widetilde{Z_0} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) \simeq H^*_{\sigma}(N_2).$$

The previous exact sequence gives us the following equation:

$$h^2_{\sigma}(N_2) - h^2(\widetilde{K}, \mathbb{F}_2) + h^2(\widetilde{Z_0} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) - h^3_{\sigma}(N_2) = 0.$$

As $h^2(\widetilde{K}, \mathbb{F}_2) = 8 + 36 = 44$ and $h^2(\widetilde{Z_0} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) = 23 + 36 = 59$, we obtain:

$$h^2_{\sigma}(N_2) - h^3_{\sigma}(N_2) = -15.$$

Moreover by 2) of Proposition 11.10, we have the exact sequence

$$0 \to H^1_{\sigma}(N_2) \to H^2_{\sigma}(N_2) \to H^2(N_2, \mathbb{F}_2) \to H^2_{\sigma}(N_2) \oplus H^2(\widetilde{Z_0} \cup (\cup_{k=1}^{36} E_k), \mathbb{F}_2)$$
$$\to H^3_{\sigma}(N_2) \longrightarrow \text{coker} \longrightarrow 0.$$

By Lemma 7.4 of [5], $h^1_{\sigma}(N_2) = h^0(\widetilde{Z_0} \cup (\cup_{k=1}^{36} E_k), \mathbb{F}_2) - 1$. Then we get the equation

$$\begin{aligned} &h^0(\widetilde{Z_0} \cup (\cup_{k=1}^{36} E_k), \mathbb{F}_2) - 1 - h^2_{\sigma}(N_2) + h^2(N_2, \mathbb{F}_2) \\ &- h^2_{\sigma}(N_2) - h^2(\widetilde{Z_0} \cup (\cup_{k=1}^{36} E_k), \mathbb{F}_2) + h^3_{\sigma}(N_2) - \alpha = 0, \end{aligned}$$

where $\alpha = \dim \text{coker}$. So

$$21 - \alpha - 2h^2_{\sigma}(N_2) + h^3_{\sigma}(N_2) = 0.$$

From the two equations, we deduce that

$$h^2_{\sigma}(N_2) = 36 - \alpha, \qquad h^3_{\sigma}(N_2) = 51 - \alpha.$$

Come back to the exact sequence

$$0 \longrightarrow H^2(\widetilde{K}, \widetilde{\widetilde{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) \longrightarrow H^2(\widetilde{K}, \mathbb{F}_2) \xrightarrow{\varsigma^*} H^2(\widetilde{\widetilde{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2),$$

where $\varsigma : \widetilde{\widetilde{Z_0}} \cup (\cup_{k=1}^{36} D_k) \hookrightarrow \widetilde{K}$ is the inclusion. Since $h^2(\widetilde{K}, \widetilde{\widetilde{Z_0}} \cup (\cup_{k=1}^{36} D_k), \mathbb{F}_2) = h^2_\sigma(N_2) = 36 - \alpha$, we have $\dim_{\mathbb{F}_2} \varsigma^*(H^2(\widetilde{K}, \mathbb{F}_2)) = (8+36) - 36 + \alpha = 8 + \alpha$. We can interpret this as follows. Consider the homomorphism

$$\begin{aligned} \varsigma^*_{\mathbb{Z}} : H^2(\widetilde{K}, \mathbb{Z}) &\to H^2(\widetilde{\widetilde{Z_0}}, \mathbb{Z}) \oplus (\oplus_{k=1}^{36} H^2(D_k, \mathbb{Z})) \\ u &\to (u \cdot \widetilde{\widetilde{Z_0}}, u \cdot D_1, ..., u \cdot D_{36}). \end{aligned}$$

Since this is a map of torsion free $\mathbb{Z}$-modules (by Lemma 20.11 and universal coefficient formula), we can tensor by $\mathbb{F}_2$,

$$\varsigma^* = \varsigma^*_{\mathbb{Z}} \otimes \mathrm{id}_{\mathbb{F}_2} : H^2(\widetilde{K}, \mathbb{Z}) \otimes \mathbb{F}_2 \to H^2(\widetilde{\widetilde{Z_0}}, \mathbb{Z}) \oplus (\oplus_{k=1}^{36} H^2(D_k, \mathbb{Z})) \otimes \mathbb{F}_2,$$

and we have $8 + \alpha$ independent elements such that the intersection with the D_k $k \in \{1, ..., 36\}$ and $\widetilde{\widetilde{Z_0}}$ are not all zero. But, $\varsigma^*(\pi_{2*}(H^2(N_2, \mathbb{Z}))) = 0$ and $\varsigma^*(\widetilde{\widetilde{Z_0}}, \langle D_1, ..., D_{36}\rangle)$, (it follows from Proposition 11.5). By Lemma 20.10 (iv), the element $\widetilde{\widetilde{Z_0}} + D_1 + ... + D_{36}$ is divisible by 2. Hence necessary, it remains $7 + \alpha$ independent elements in $H^2(\widetilde{K}, \mathbb{Z})$ of the form $\frac{u+d}{2}$ with $u \in \pi_{2*}(s^*(H^2(K_2(A), \mathbb{Z})))$ and $d \in \langle D_1, ..., D_{36}\rangle$.

Let denote by $u_1, ..., u_{7+\alpha}$ the $7+\alpha$ elements in $\pi_{2*}(s^*(H^2(K_2(A), \mathbb{Z})))$ provided above. By Lemma 20.10 (iv) $\langle D_1, ..., D_{36}\rangle$ is primitive in $H^2(\widetilde{K}, \mathbb{Z})$. Hence necessary, the element $u_1, ..., u_{7+\alpha}$ view as element in $\pi_{2*}(s^*(H^2(K_2(A), \mathbb{F}_2)))$ are linearly independent. Since $\dim_{\mathbb{F}_2} \pi_{2*}(s^*(H^2(K_2(A), \mathbb{F}_2))) = 7$, it follows that $\alpha = 0$ and $\mathrm{Vect}_{\mathbb{F}_2}(u_1, ..., u_7) = \pi_{2*}(s^*(H^2(K_2(A), \mathbb{F}_2)))$. Hence there exists D_e which is a linear combination of the D_i with coefficient 0 or 1 such that $\pi_{2*}(s^*(e)) + D_e$ is divisible by 2. $\square$

Lemma 20.13. *We have:*

$$D_e = D_1 + ... + D_{36}.$$

Proof. We know from Remark 10.8 that the image of the monodromy representation on $A[2]$ contains the symplectic group $\mathrm{Sp}\, A[2]$. We recall from Remark 19.2 (2), that the $D_1, ..., D_{35}$ are given by $\pi_2(s^{-1}(\mathcal{P})$. It follows that the image of the monodromy representation on $H^2(\widetilde{K},\mathbb{Z})$ contains the isometries which act on $D_1,...,D_{35}$ as the elements f of $A[2]$:

$$f\cdot\pi_2(s^{-1}(\{a_1,a_2,a_3\})=\pi_2(s^{-1}(\{f(a_1),f(a_2),f(a_3)\}),$$

and act trivially on D_{36} and $\pi_{2*}(s^*(e))$. As explained in Remark 9.3 the 2 orbits of the action of $\mathrm{Sp}\,A[2]$ on the set $\mathfrak{D}:=\{D_1,...,D_{35}\}$ correspond to the two sets of isotropic and non-isotropic planes in $A[2]$. Hence by Proposition 9.4 (3), (4) the action of $\mathrm{Sp}\,A[2]$ on the set $\mathfrak{D}$ has 2 orbits: one of 15 elements and another of 20 elements.

On the other hand, as we mentioned in the end of Section 20.1, we can assume that $A=E_\xi\times E_\xi$ where E_ξ is the elliptic curve introduced in Definition 10.4. Hence there is the following automorphism group acting on A:

$$G:=\left\langle\begin{pmatrix}\rho & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right\rangle,$$

where $\rho=e^{\frac{2i\pi}{6}}$. The group G extends naturally to an automorphism group of N_2 which we denote also G. Moreover, the action of G restricts to the set $\mathfrak{D}$. Then by Lemma 10.5 the action of G on $\mathfrak{D}$ has 2 orbits: one of 5 elements and one of 30 elements. Also the group G acts trivially on D_{36} and on $\pi_{2*}(s^*(e))$.

Hence the combined action of G and $\mathrm{Sp}\,A[2]$ acts transitively on $\mathfrak{D}$. Since $\pi_{2*}(s^*(e))$ is fixed by the action of G and $\mathrm{Sp}\,A[2]$, D_e has also to be fixed by the action of G and $\mathrm{Sp}\,A[2]$ else it will contradict Lemma 20.10 (iv). It follows that there are only 3 possibilities for D_e:

(i) $D_e=D_{36}$,

(ii) $D_e=D_1+...+D_{35}$,

(iii) or $D_e=D_1+...+D_{36}$.

Let d be the number of D_i with coefficient equal to 1 in the linear decomposition of D_e. The number d can be 1, 35 or 36.

Then from Lemma 20.10 (i), (ii) and Proposition 11.5

$$\left(\frac{\pi_{2*}(s^*(e))+D_e}{2}\right)^4=\frac{324-d}{2}.$$

Hence d has to be divisible by 2. It follows that $D_e=D_1+...+D_{36}$. □

Lemma 20.14. *The class* $\pi_{1*}(s_1^*(e)) + \overline{Z_0}'$ *is divisible by 2.*

Proof. We know that $\pi_{2,*}(s^*(e)) + \widetilde{\overline{Z_0}}$ is divisible by 2. Indeed by Lemma 20.10 (iv), $\widetilde{\overline{Z_0}} + D_1 + ... + D_{36}$ is divisible by 2 and by Lemma 20.12 and 20.13, $\pi_{2,*}(s^*(e)) + D_1 + ... + D_{36}$ is divisible by 2.

We can find a Cartier divisor on $\widetilde{K}$ which corresponds to $\frac{\pi_{2*}(s^*(e))+\widetilde{\overline{Z_0}}}{2}$ and which does not meet $\cup_{k=1}^{36} D_k$. Then this Cartier divisor induces a Cartier divisor on K' which necessarily corresponds to half the cocycle $\pi_{1*}(s_1^*(e)) + \overline{Z_0}'$. □

20.7 Calculation of $B_{K'}$

We finish the proof of Theorem 20.3, calculating $B_{K'}$. We continue using the notation provided in Section 20.3.

Lemma 20.15. *We have*

$$\overline{Z_0}'^2 = -2r^*(\overline{Z_0}).$$

Proof. We use the same technique as in Lemma 4.6.12 of [38]. Consider the following diagram:

$$\begin{tikzcd} Z_0' \arrow[r, hook, "l_1"] \arrow[d, "g"] & N_1 \arrow[d, "s_1"] \\ Z_0 \arrow[r, hook, "l_0"] & K_2(A), \end{tikzcd}$$

where l_0 and l_1 are the inclusions and $g := s_{1|Z_0'}$. By Proposition 6.7 of [21], we have:

$$s_1^* l_{0*}(Z_0) = l_{1*}(c_1(E)),$$

where $E := g^*(\mathscr{N}_{Z_0/K_2(A)})/\mathscr{N}_{Z_0'/N_1}$. Hence

$$s_1^* l_{0*}(Z_0) = c_1(g^*(\mathscr{N}_{Z_0/K_2(A)})) - Z_0'^2.$$

Since $K_2(A)$ is hyperkähler and Z_0 is a K3 surface, we have $c_1(\mathscr{N}_{Z_0/K_2(A)}) = 0$. So

$$Z_0'^2 = -s_1^* l_{0*}(Z_0).$$

Then the result follows from Proposition 11.5. □

Proposition 20.16. *We have the formula*

$$B_{K'}(\pi_{1*}(s_1^*(\alpha), \pi_{1*}(s_1^*(\beta))) = 6\sqrt{\frac{2}{c_{K'}}} B_{K_2(A)}(\alpha, \beta),$$

where $c_{K'}$ is the Fujiki constant of K' and α, β are in $H^2(K_2(A),\mathbb{Z})^\iota$ and $B_{K_2(A)}$ is the Beauville–Bogomolov form of $K_2(A)$.

Proof. The ingredient for the proof is the Fujiki formula.

By (88) of Theorem 20.1, we have

$$(\pi_{1*}(s_1^*(\alpha)))^4 = c_{K'} B_{K'}(\pi_{1*}(s_1^*(\alpha), \pi_{1*}(s_1^*(\alpha)))^2.$$

$$\alpha^4 = 9B_{K_2(A)}(\alpha,\alpha)^2.$$

Moreover, by Proposition 11.5,

$$(\pi_{1*}(s^*(\alpha)))^4 = 8s^*(\alpha)^4 = 8\alpha^4.$$

By statement (89) of Theorem 20.1, we get the result. □

In particular, it follows:

$$B_{K'}(\pi_{1*}(s_1^*(e), \pi_{1*}(s_1^*(e))) = -36\sqrt{\frac{2}{c_{K'}}} \tag{93}$$

Lemma 20.17.

$$B_{K'}(\pi_{1*}(s_1^*(\alpha)), \overline{Z_0}') = 0,$$

for all $\alpha \in H^2(S^{[2]},\mathbb{Z})^\iota$.

Proof. We have $\pi_{1*}(s_1^*(\alpha))^3 \cdot \overline{Z_0}' = 8s_1^*(\alpha)^3 \cdot \Sigma_1$ by Proposition 11.5, and $s_{1*}(s_1^*(\alpha^3) \cdot Z_0') = \alpha^3 \cdot s_{1*}(Z_0') = 0$ by the projection formula. We conclude by Proposition 20.2. □

Lemma 20.18. *We have:*

$$B_{K'}(\overline{Z_0}', \overline{Z_0}') = -4\sqrt{\frac{2}{c_{K'}}}.$$

Proof. We have:

$$\begin{aligned}
\overline{Z_0}'^2 \cdot \pi_{1*}(s_1^*(e))^2 &= \frac{c_{K'}}{3} B_{M'}(\overline{Z_0}', \overline{Z_0}') \times B_{K'}(\pi_{1*}(s_1^*(e)), \pi_{1*}(s_1^*(e))) \\
&= \frac{c_{K'}}{3} B_{K'}(\overline{Z_0}', \overline{Z_0}') \times \left(-36\sqrt{\frac{2}{c_{K'}}}\right)
\end{aligned}$$

$$= -12\sqrt{2c_{K'}}B_{K'}(\overline{Z_0}', \overline{Z_0}') \tag{94}$$

By Proposition 11.5, we have

$$\overline{Z_0}'^2 \cdot \pi_{1*}(s_1^*(e))^2 = 8Z_0'^2 \cdot (s_1^*(e))^2. \tag{95}$$

By the projection formula, $Z_0'^2 \cdot (s_1^*(e))^2 = s_{1*}(Z_0'^2) \cdot e^2$. Moreover by lemma 20.15, $s_{1*}(Z_0'^2) = -Z_0$. Hence

$$Z_0'^2 \cdot (s_1^*(e))^2 = -Z_0 \cdot e^2. \tag{96}$$

It follows from (94), (95) and (96) that

$$-8Z_0 \cdot e^2 = -12\sqrt{2c_{K'}}B_{K'}(\overline{Z_0}', \overline{Z_0}'). \tag{97}$$

Moreover from Section 4 of [24], we have:

$$Z_0 \cdot e^2 = -12. \tag{98}$$

So by (97) and (98):

$$B_{K'}(\overline{Z_0}', \overline{Z_0}') = -8\sqrt{\frac{1}{2c_{K'}}}.$$

□

Now we are able to finish the calculation of the Beauville–Bogomolov form on $H^2(K', \mathbb{Z})$. By (93), Propositions 20.16, Lemma 20.17, 20.18 and Theorem 20.9, the Beauville–Bogomolov form on $H^2(K', \mathbb{Z})$ gives the lattice:

$$U^3\left(6\sqrt{\frac{2}{c_{K'}}}\right) \oplus -\frac{1}{4}\sqrt{\frac{2}{c_{K'}}}\begin{pmatrix} 40 & 32 \\ 32 & 40 \end{pmatrix}$$

$$= U^3\left(6\sqrt{\frac{2}{c_{K'}}}\right) \oplus -\sqrt{\frac{2}{c_{K'}}}\begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix}$$

Then it follows from the integrality and the indivisibility of the Beauville–Bogomolov form that $c_{K'} = 8$, and we get Theorem 20.3.

20.8 Betti numbers and Euler characteristic of K'

Proposition 20.19. *We have:*

- $b_2(K') = 8$,
- $b_3(K') = 0$,

- $b_4(K') = 90$,
- $\chi(K') = 108$.

Proof. It is the same proof as Proposition 4.7.2 of [38]. From Theorem 7.31 of [61], (86), (87) and Proposition 19.3, we get the betti numbers. Then $\chi(K') = 1 - 0 + 8 - 0 + 90 - 0 + 8 - 0 + 1 = 108$. □

Part III

Computing Cup-Products in integral cohomology of Hilbert schemes of points on K3 surfaces

Lehn and Sorger in [30] developed an algebraic model to describe the cohomological ring structure of Hilbert schemes of points on a K3 surface. On the other hand, Qin and Wang [53] found a base for integral cohomology in the projective case. By combining these results, we are able to compute all cup-products in the cohomology rings of Hilbert schemes of n points on a projective K3 surface with integral coefficients. For $n = 2$, this was done by Boissière, Nieper-Wißkirchen and Sarti [5], who applied their results to automorphism groups of prime order. When n is increasing, the ranks of the cohomology rings become very large, so we need the help of a computer. Our goal here is to give some properties for low degrees. Although the case $n = 3$ is the most interesting for us, our computer program allows computations for arbitrary n. We give some numerical results in Section 22.

21 The combinatorial model

Definition 21.1. Let n be a natural number. A partition of n is a decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda_1 \geq \ldots \geq \lambda_k > 0$ of natural numbers such that $\sum_i \lambda_i = n$. Sometimes it is convenient to write $\lambda = (\ldots, 2^{m_2}, 1^{m_1})$ with multiplicities in the exponent. No confusion should be possible since numerical exponentiation is never meant in this context. We define the weight $\|\lambda\| := \sum m_i i = n$ and the length $|\lambda| := \sum_i m_i = k$. We also define $z_\lambda := \prod_i i^{m_i} m_i!$.

Definition 21.2. Let $\Lambda_n := \mathbb{Q}[x_1, \ldots, x_n]^{S_n}$ be the graded ring of symmetric polynomials. There are canonical projections : $\Lambda_{n+1} \to \Lambda_n$ which send x_{n+1} to zero. The graded projective limit $\Lambda := \varprojlim \Lambda_n$ is called the ring of symmetric functions. Let m_λ and p_λ denote the monomial and the power sum symmetric functions. They are defined as follows: For a monomial $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \ldots x_{i_k}^{\lambda_k}$ of total degree n, the (ordered) sequence of exponents $(\lambda_1, \ldots, \lambda_k)$ defines a partition λ of n, which is called the shape of the monomial. Then we define m_λ being the sum of all monomials of shape λ. For the power sums, first define $p_n := x_1^n + x_2^n + \ldots$. Then $p_\lambda := p_{\lambda_1} p_{\lambda_2} \ldots p_{\lambda_k}$.

The families $(m_\lambda)_\lambda$ and $(p_\lambda)_\lambda$ form two $\mathbb{Q}$-bases of Λ, so they are linearly related by $p_\lambda = \sum_\mu \psi_{\lambda\mu} m_\mu$. It turns out that the base change matrix $(\psi_{\lambda\mu})$ has integral entries, but its inverse $(\psi^{-1}_{\mu\lambda})$ has not. A method to determine the $(\psi_{\lambda\mu})$ is given by Lascoux in [29, Sect. 3.7].

Definition 21.3. Let S be a projective K3 surface. We fix integral bases 1 of $H^0(S,\mathbb{Z})$, x of $H^4(S,\mathbb{Z})$ and $\alpha_1,\ldots,\alpha_{22}$ of $H^2(S,\mathbb{Z})$. The cup product induces a symmetric bilinear form B_{H^2} on $H^2(S,\mathbb{Z})$ and thus the structure of a unimodular lattice. We may extend B_{H^2} to a symmetric non-degenerate bilinear form B on $H^*(S,\mathbb{Z})$ by setting $B(1,1)=0,\ B(1,\alpha_i)=0,\ B(1,x)=1,\ B(x,x)=0.$

By the Hirzebruch index theorem, we know that $H^2(S,\mathbb{Z})$ has signature -16 and, by the classification theorem for indefinite unimodular lattices, is isomorphic to $U^{\oplus 3}\oplus(-E_8)^{\oplus 2}$.

Definition 21.4. B induces a form $B\otimes B$ on $\operatorname{Sym}^2 H^*(S,\mathbb{Z})$. So the cup-product

$$\mu : \operatorname{Sym}^2 H^*(S,\mathbb{Z}) \longrightarrow H^*(S,\mathbb{Z})$$

induces an adjoint comultiplication Δ that is coassociative, given by:

$$\Delta : H^*(S,\mathbb{Z}) \longrightarrow \operatorname{Sym}^2 H^*(S,\mathbb{Z}), \quad \Delta = -(B\otimes B)^{-1}\mu^T B$$

with the property $(B\otimes B)\left(\Delta(a), b\otimes c\right) = -B\left(a, b\smile c\right)$. Note that this does not define a bialgebra structure. The image of 1 under the composite map $\mu\circ\Delta$, denoted by $e=24x$ is called the Euler Class.

More generally, every linear map $f : A^{\otimes k}\to A^{\otimes m}$ induces an adjoint map g in the other direction that satisfies $(-1)^m B^{\otimes m}(f(x),y) = (-1)^k B^{\otimes k}(x,g(y))$.

We denote by $S^{[n]}$ the Hilbert scheme of n points on S, *i.e.* the classifying space of all zero-dimensional closed subschemes of length n. $S^{[0]}$ consists of a single point and $S^{[1]}=S$. Fogarty [16, Thm. 2.4] proved that the Hilbert scheme is a smooth variety. A theorem by Nakajima [46] gives an explicit description of the vector space structure of $H^*(S^{[n]},\mathbb{Q})$ in terms of creation operators

$$\mathfrak{q}_l(\beta) : H^*(S^{[n]},\mathbb{Q}) \longrightarrow H^{*+k+2(l-1)}(S^{[n+l]},\mathbb{Q}),$$

where $\beta\in H^k(S,\mathbb{Q})$, acting on the direct sum $\mathbb{H} := \bigoplus_n H^*(S^{[n]},\mathbb{Q})$. The operators $\mathfrak{q}_l(\beta)$ are linear and commute with each other. The vacuum vector $|0\rangle$ is defined as the generator of $H^0(S^{[0]},\mathbb{Q})\cong\mathbb{Q}$. The images of $|0\rangle$ under the polynomial algebra generated by the creation operators span $\mathbb{H}$ as a vector space. Following [53], we abbreviate $\mathfrak{q}_{l_1}(\beta)\dots\mathfrak{q}_{l_k}(\beta) =: \mathfrak{q}_\lambda(\beta)$, where the partition λ is composed by the l_i.

An integral basis for $H^*(S^{[n]},\mathbb{Z})$ in terms of Nakajima's operators was given by Qin–Wang:

Theorem 21.5. *[53, Thm. 5.4.] Let $\mathfrak{m}_{\nu,\alpha} := \sum_\rho \psi^{-1}_{\nu\rho}\,\mathfrak{q}_\rho(\alpha)$, with coefficients $\psi^{-1}_{\nu\rho}$ as in Definition 21.2. The classes*

$$\frac{1}{z_\lambda}\mathfrak{q}_\lambda(1)\mathfrak{q}_\mu(x)\mathfrak{m}_{\nu^1,\alpha_1}\dots\mathfrak{m}_{\nu^{22},\alpha_{22}}|0\rangle, \quad \|\lambda\|+\|\mu\|+\sum_{i=1}^{22}\|\nu^i\| = n$$

form an integral basis for $H^*(S^{[n]},\mathbb{Z})$. Here, λ, μ, ν^i are partitions.

Notation 21.6. To enumerate the basis of $H^*(S^{[n]},\mathbb{Z})$, we introduce the following abbreviation:

$$\boldsymbol{\alpha}^{\boldsymbol{\lambda}} := 1^{\lambda^0}\alpha_1^{\lambda^1}\dots\alpha_{22}^{\lambda^{22}}x^{\lambda^{23}} := \frac{1}{z_{\widetilde{\lambda^0}}}\mathfrak{q}_{\widetilde{\lambda^0}}(1)\mathfrak{q}_{\lambda^{23}}(x)\mathfrak{m}_{\lambda^1,\alpha_1}\dots\mathfrak{m}_{\lambda^{22},\alpha_{22}}|0\rangle$$

where the partition $\widetilde{\lambda^0}$ is built from λ^0 by appending sufficiently many ones, such that $\left\|\widetilde{\lambda^0}\right\| + \sum_{i\geq 1}\|\lambda^i\| = n$. If $\sum_{i\geq 0}\|\lambda^i\| > n$, we put $\boldsymbol{\alpha}^{\boldsymbol{\lambda}} = 0$. Thus we can interpret $\boldsymbol{\alpha}^{\boldsymbol{\lambda}}$ as an element of $H^*(S^{[n]},\mathbb{Z})$ for arbitrary n. We say that the symbol $\boldsymbol{\alpha}^{\boldsymbol{\lambda}}$ is reduced, if λ^0 contains no ones. We define also $\|\boldsymbol{\lambda}\| := \sum_{i\geq 0}\|\lambda^i\|$.

Lemma 21.7. *Let $\boldsymbol{\alpha}^{\boldsymbol{\lambda}}$ represent a class of cohomological degree $2k$. If $\boldsymbol{\alpha}^{\boldsymbol{\lambda}}$ is reduced, then $\frac{k}{2} \leq \|\boldsymbol{\lambda}\| \leq 2k$.*

Proof. This is a simple combinatorial observation. We give the two extremal cases. The lowest ratio between $\|\boldsymbol{\lambda}\|$ and $\deg\boldsymbol{\alpha}^{\boldsymbol{\lambda}}$ is achieved by the classes $x^{(1^m)}$, where the degree is $4m$ and the weight of $\boldsymbol{\lambda}$ is m. The highest ratio is achieved by the classes $1^{(2^m)}$, where both degree and weight equal $2m$. So $\frac{1}{4} \leq \frac{\|\boldsymbol{\lambda}\|}{\deg\boldsymbol{\alpha}^{\boldsymbol{\lambda}}} \leq 1$. $\square$

The ring structure of $H^*(S^{[n]},\mathbb{Q})$ has been studied by Lehn and Sorger in [30], where an explicit algebraic model is constructed, which we recall briefly:

Definition 21.8. [30, Sect. 2] Let π be a permutation of n letters, written as a product of disjoint cycles. To each cycle we may associate an element of $A := H^*(S,\mathbb{Q})$. This defines an element in $A^{\otimes m}$, m being the number of cycles. For example, a term like $(1\,2\,3)_{\alpha_1}(4\,5)_{\alpha_2}$ may describe a permutation consisting of two cycles with associated classes $\alpha_1,\alpha_2 \in A$. We interpret the cycles as the orbits of the subgroup $\langle\pi\rangle \subset S_n$ generated by π. We denote the set of orbits by $\langle\pi\rangle\backslash[n]$. Thus we construct a vector space $A\{S_n\} := \bigoplus_{\pi\in S_n} A^{\otimes\langle\pi\rangle\backslash[n]}$.

To define a ring structure, take two permutations π, $\tau \in S_n$ and the subgroup $\langle\pi,\tau\rangle$ generated by them. The natural map of orbit spaces $p_\pi : \langle\pi\rangle\backslash[n] \to \langle\pi,\tau\rangle\backslash[n]$ induces a map $f^{\pi,\langle\pi,\tau\rangle} : A^{\otimes\langle\pi\rangle\backslash[n]} \to A^{\otimes\langle\pi,\tau\rangle\backslash[n]}$, which multiplies the factors of an elementary tensor if the corresponding orbits are glued together. Denote $f_{\langle\pi,\tau\rangle,\pi}$ the adjoint to this map in the sense of Definition 21.4. Then the map

$$\begin{aligned} m_{\pi,\tau} : A^{\otimes\langle\pi\rangle\backslash[n]} \otimes A^{\otimes\langle\tau\rangle\backslash[n]} &\longrightarrow A^{\otimes\langle\pi\tau\rangle\backslash[n]}, \\ a\otimes b &\longmapsto f_{\langle\pi,\tau\rangle,\pi\tau}(f^{\pi,\langle\pi,\tau\rangle}(a)\cdot f^{\tau,\langle\pi,\tau\rangle}(b)\cdot e^{g(\pi,\tau)}) \end{aligned}$$

defines a multiplication on $A\{S_n\}$. Here the dot means the cup product on each tensor factor and $e^{g(\pi,\tau)} \in A^{\otimes\langle\pi,\tau\rangle\backslash[n]}$ is an elementary tensor that is composed by powers of the Euler class e: for each orbit $B \in \otimes\langle\pi,\tau\rangle\backslash[n]$ the exponent $g(\pi,\tau)(B)$ (so-called "graph defect", see [30, 2.6]) is given by:

$$g(\pi,\tau)(B) = \frac{1}{2}\Big(|B| + 2 - |p_\pi^{-1}(\{B\})| - |p_\tau^{-1}(\{B\})| - |p_{\pi\tau}^{-1}(\{B\})|\Big).$$

The symmetric group S_n acts on $A\{S_n\}$ by conjugation, permuting the direct summands: conjugation by $\sigma \in S_n$ maps $A^{\otimes\langle\pi\rangle\backslash[n]}$ to $A^{\otimes\langle\sigma\pi\sigma^{-1}\rangle\backslash[n]}$. This action preserves the ring structure. Therefore the space of invariants $A^{[n]} := (A\{S_n\})^{S_n}$ becomes a subring. The main theorem of [30] can now be stated:

Theorem 21.9. *[30, Thm. 3.2.] The following map is an isomorphism of rings:*

$$\begin{aligned} H^*(S^{[n]},\mathbb{Q}) &\longrightarrow A^{[n]} \\ \mathfrak{q}_{n_1}(\beta_1)\dots\mathfrak{q}_{n_k}(\beta_k)|0\rangle &\longmapsto \sum_{\sigma\in S_n}\sigma a\sigma^{-1} \end{aligned}$$

with $\sum_i n_i = n$ *and* $a = (1\,2\dots n_1)_{\beta_1}(n_1{+}1\dots n_1{+}n_2)_{\beta_2}\cdots(n{-}n_k\dots n)_{\beta_k} \in A\{S_n\}$.

Since $H^{\mathrm{odd}}(S^{[n]},\mathbb{Z}) = 0$ and $H^{\mathrm{even}}(S^{[n]},\mathbb{Z})$ is torsion-free by [33], we can apply these results to $H^*(S^{[n]},\mathbb{Z})$ to determine the multiplicative structure of cohomology with integer coefficients. It turns out, that it is somehow independent of n. More precisely, we have the following stability theorem, by Li, Qin and Wang:

Theorem 21.10. (Derived from [53, Thm. 2.1]). *Let $Q_1,\dots,Q_s$ be products of creation operators, i.e. $Q_i = \prod_j \mathfrak{q}_{\lambda_{i,j}}(\beta_{i,j})$ for some partitions $\lambda_{i,j}$ and classes $\beta_{i,j} \in H^*(S,\mathbb{Z})$. Set $n_i := \sum_j \|\lambda_{i,j}\|$. Then the cup product $\prod_{i=1}^s \left(\frac{1}{(n-n_i)!}\mathfrak{q}_{1^{n-n_i}}(1)\,Q_i\,|0\rangle\right)$*

equals a finite linear combination of classes of the form $\frac{1}{(n-m)!}\mathfrak{q}_{1^{n-m}}(1)\prod_j \mathfrak{q}_{\mu_j}(\gamma_j)|0\rangle$, *with* $\gamma \in H^*(S,\mathbb{Z})$, $m = \sum_j \|\mu_j\|$, *whose coefficients are independent of* n. *We have the upper bound* $m \leq \sum_i n_i$. *Moreover,* $m = \sum_i n_i$ *if and only if the corresponding class is* $\frac{1}{(n-m)!}\mathfrak{q}_{1^{n-m}}(1)Q_1Q_2\dots Q_s|0\rangle$ *with coefficient* 1.

Corollary 21.11. *Let* $\boldsymbol{\alpha}^{\boldsymbol{\lambda}}, \boldsymbol{\alpha}^{\boldsymbol{\mu}}, \boldsymbol{\alpha}^{\boldsymbol{\nu}}$ *be reduced. Assume* $n \geq \|\boldsymbol{\lambda}\|, \|\boldsymbol{\mu}\|$. *Then the coefficients* $c_{\boldsymbol{\nu}}^{\boldsymbol{\lambda}\boldsymbol{\mu}}$ *of the cup product in* $H^*(S^{[n]},\mathbb{Z})$

$$\boldsymbol{\alpha}^{\boldsymbol{\lambda}} \smile \boldsymbol{\alpha}^{\boldsymbol{\mu}} = \sum_{\boldsymbol{\nu}} c_{\boldsymbol{\nu}}^{\boldsymbol{\lambda}\boldsymbol{\mu}} \boldsymbol{\alpha}^{\boldsymbol{\nu}}$$

are polynomials in n *of degree at most* $\|\boldsymbol{\lambda}\| + \|\boldsymbol{\mu}\| - \|\boldsymbol{\nu}\|$.

Proof. Set $Q_{\boldsymbol{\lambda}} := \mathfrak{q}_{\lambda^0}(1)\mathfrak{q}_{\lambda^{23}}(x)\prod_{1\leq j\leq 22}\mathfrak{q}_{\lambda^j}(\alpha_j)$ and $n_{\boldsymbol{\lambda}} := \|\boldsymbol{\lambda}\|$. Then we have: $\boldsymbol{\alpha}^{\boldsymbol{\lambda}} = \frac{1}{(n-n_{\boldsymbol{\lambda}})!\,z_{\lambda^0}}\mathfrak{q}_{1^{n-n_{\boldsymbol{\lambda}}}}(1)Q_{\boldsymbol{\lambda}}|0\rangle$ and $\boldsymbol{\alpha}^{\boldsymbol{\mu}} = \frac{1}{(n-n_{\boldsymbol{\mu}})!\,z_{\mu^0}}\mathfrak{q}_{1^{n-n_{\boldsymbol{\mu}}}}(1)Q_{\boldsymbol{\mu}}|0\rangle$. Thus the coefficient $c_{\boldsymbol{\nu}}^{\boldsymbol{\lambda}\boldsymbol{\mu}}$ in the product expansion is a constant, which depends on $\|\boldsymbol{\lambda}\|$, $\|\boldsymbol{\mu}\|$, $\|\boldsymbol{\nu}\|$, but not on n, multiplied with $\frac{(n-n_{\boldsymbol{\nu}})!}{(n-m)!}$ for a certain $m \leq n_{\boldsymbol{\lambda}} + n_{\boldsymbol{\mu}}$. This is a polynomial of degree $m - n_{\boldsymbol{\nu}} \leq n_{\boldsymbol{\lambda}} + n_{\boldsymbol{\mu}} - n_{\boldsymbol{\nu}} = \|\boldsymbol{\lambda}\| + \|\boldsymbol{\mu}\| - \|\boldsymbol{\nu}\|$. □

Remark 21.12. If $n < \|\boldsymbol{\lambda}\|$ or $n < \|\boldsymbol{\mu}\|$, one has $\boldsymbol{\alpha}^{\boldsymbol{\lambda}} = 0$, resp. $\boldsymbol{\alpha}^{\boldsymbol{\mu}} = 0$. But it is still possible that $\boldsymbol{\alpha}^{\boldsymbol{\nu}} \neq 0$ in $H^*(S^{[n]})$. It seems that in this case the polynomial $c_{\boldsymbol{\nu}}^{\boldsymbol{\lambda}\boldsymbol{\mu}}$ always becomes zero when evaluated at n. So the $c_{\boldsymbol{\nu}}^{\boldsymbol{\lambda}\boldsymbol{\mu}}$ seem to be universal in the sense that the above corollary holds true even without the condition $n \geq \|\boldsymbol{\lambda}\|, \|\boldsymbol{\mu}\|$.

Example 21.13. Here are some explicit examples for illustration. See A.1 for how to compute them.

(i) $1^{(2,2)} \smile \alpha_i^{(2)} = -2\cdot 1^{(2)}\alpha_i^{(1)}x^{(1)} + 1^{(2,2)}\alpha_i^{(2)} + 2\cdot 1^{(2)}\alpha_i^{(3)} + \alpha_i^{(4)}$ for $i \in \{1..22\}$.

(ii) Let $i,j \in \{1\dots 22\}$. If $i \neq j$, then $\alpha_i^{(2)} \smile \alpha_j^{(1)} = \alpha_i^{(2)}\alpha_j^{(1)} + 2B(\alpha_i,\alpha_j)\cdot x^{(1)}$. Otherwise, $\alpha_i^{(2)} \smile \alpha_i^{(1)} = \alpha_i^{(3)} + \alpha_i^{(2,1)} + 2B(\alpha_i,\alpha_i)\cdot x^{(1)}$.

(iii) Set $\boldsymbol{\alpha}^{\boldsymbol{\lambda}} = 1^{(2)}$ and $\boldsymbol{\alpha}^{\boldsymbol{\nu}} = x^{(1)}$. Then $c_{\boldsymbol{\nu}}^{\boldsymbol{\lambda}\boldsymbol{\lambda}} = -(n-1)$.

(iv) Set $\boldsymbol{\alpha}^{\boldsymbol{\lambda}} = 1^{(2,2)}$ and $\boldsymbol{\alpha}^{\boldsymbol{\nu}} = x^{(1,1)}$. Then $c_{\boldsymbol{\nu}}^{\boldsymbol{\lambda}\boldsymbol{\lambda}} = \frac{(n-3)(n-2)}{2}$.

Example 21.14. Let i,j be indices, such that $B(\alpha_i,\alpha_j) = 1$, $B(\alpha_i,\alpha_i) = 0 = B(\alpha_j,\alpha_j)$ and let $k \geq 0$. Set $\boldsymbol{\alpha}^{\boldsymbol{\lambda}} = \alpha_i^{(1)}\alpha_j^{(1)}x^{(1^k)}$ and $\boldsymbol{\alpha}^{\boldsymbol{\nu}} = x^{(1^{2k+2})}$. Then $c_{\boldsymbol{\nu}}^{\boldsymbol{\lambda}\boldsymbol{\lambda}} = 1$.

Proof. It is not hard to see from the definition, that for $\beta_j,\ \gamma_j \in H^*(S)$:

$$\mathfrak{q}_1(\beta_1)\dots\mathfrak{q}_1(\beta_n)|0\rangle \smile \mathfrak{q}_1(\gamma_1)\dots\mathfrak{q}_1(\gamma_n)|0\rangle = \sum_{\sigma\in S_n}\mathfrak{q}_1(\beta_1\cdot\gamma_{\sigma(1)})\dots\mathfrak{q}_1(\beta_n\cdot\gamma_{\sigma(n)})|0\rangle.$$

A combinatorial investigation yields now:

$$\Big(\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)\mathfrak{q}_1(x)^k\mathfrak{q}_1(1)^{k+m}|0\rangle\Big)^2 = \frac{(k+m)!^2}{m!}\mathfrak{q}_1(x)^{2k+2}\mathfrak{q}_1(1)^m|0\rangle + \text{other terms.}$$

Looking at 21.6, the result follows. □

Theorem 21.15. *The quotient*

$$\frac{H^{2k}(S^{[n]},\mathbb{Z})}{\operatorname{Sym}^k H^2(S^{[n]},\mathbb{Z})}$$

is a free $\mathbb{Z}$-module for $n \geq k+2$.

Proof. The idea of the proof is to modify the basis of $H^{2k}(S^{[n]},\mathbb{Z})$, given in Theorem 21.5, in a way that $\operatorname{Sym}^k H^2(S^{[n]},\mathbb{Z})$ splits as a direct summand.

Given a free $\mathbb{Z}$-module M with basis $(b_i)_{i=1\ldots m}$ and a vector $v = a_1b_1+\ldots+a_mb_m$. Then there is another basis of M which contains v, iff $\gcd\{a_1,\ldots,a_m\} = 1$. More generally, given a set of vectors $(v_i)_{i=1\ldots r}$, $v_i = a_{i1}b_1+\ldots+a_{im}b_m$, we can complete it to a basis of M, iff the $r\times r$-minors of the matrix $(a_{ij})_{ij}$ share no common divisor. We want to show that the canonical basis of $\operatorname{Sym}^k H^2(S^{[n]},\mathbb{Z})$ is such a set.

A basis of $H^2(S^{[n]},\mathbb{Z})$ is given by the classes $\alpha_i^{(1)} = \frac{1}{(n-1)!}\mathfrak{q}_{1^{n-1}}(1)\mathfrak{q}_1(\alpha_i)|0\rangle$, $i = 1,\ldots,22$ and $1^{(2)} = \frac{1}{2(n-2)!}\mathfrak{q}_{(2,1^{n-2})}(1)|0\rangle$. A power of $\alpha_i^{(1)}$ looks like (Thm. 21.10):

$$\left(\alpha_i^{(1)}\right)^k = \frac{1}{(n-k)!}\mathfrak{q}_{1^{n-k}}(1)\mathfrak{q}_{1^k}(\alpha_i)|0\rangle + \text{other terms containing } \mathfrak{q}_\lambda(x).$$

Now, by the definition of $\psi_{\lambda\mu}$, $\mathfrak{q}_{1^k}(\alpha_i) = \mathfrak{m}_{(k),\alpha_i} + \ldots + k!\cdot\mathfrak{m}_{(1^k),\alpha_i}$, so

$$\left(\alpha_i^{(1)}\right)^k = \alpha_i^{(k)} + \text{other terms.} \tag{99}$$

Next, we determine the coefficients of $1^{(k+1)}$ and $1^{(k,2)}$ in the expansion of $\left(1^{(2)}\right)^k$. Considering Definition 21.8, we observe that here the graph defect is zero and the adjoint map is trivial, so the problem reduces to combinatorics of the symmetric group: the coefficient of $1^{(k+1)}$ is the number of ways to write a $(k+1)$-cycle as a product of k transpositions. A result of Dénes [11] states that this is $(k+1)^{k-1}$. For the $1^{(k,2)}$-coefficient, we have to choose one transposition, and write a k-cyle as a product of the remaining $k-1$ transpositions. The number of possibilities is therefore $k \cdot k^{k-2} = k^{k-1}$. So

$$\left(1^{(2)}\right)^k = (k+1)^{k-1} \cdot 1^{(k+1)} \; + \; k^{k-1} \cdot 1^{(k,2)} \; + \; \text{other terms}. \tag{100}$$

Note that these two coefficients are coprime. Putting the two cases together, one gets for a general element of $\operatorname{Sym}^k H^2(S^{[n]}, \mathbb{Z})$, $k = k_0 + \ldots + k_{22}$:

$$\begin{aligned}\left(1^{(2)}\right)^{k_0} \prod_{i=1}^{22} \left(\alpha_i^{(1)}\right)^{k_i} = {} & (k_0+1)^{k_0-1} \cdot 1^{(k_0+1)} \alpha_1^{(k_1)} \ldots \alpha_{22}^{(k_{22})} \\ & + k_0^{k_0-1} \cdot 1^{(k_0,2)} \alpha_1^{(k_1)} \ldots \alpha_{22}^{(k_{22})} + \text{other terms}.\end{aligned}$$

One checks, that this is the only element of $\operatorname{Sym}^k H^2(S^{[n]}, \mathbb{Z})$ having a nonzero coefficient at $1^{(k_0+1)} \alpha_1^{(k_1)} \ldots \alpha_{22}^{(k_{22})}$ and $1^{(k_0,2)} \alpha_1^{(k_1)} \ldots \alpha_{22}^{(k_{22})}$. Now it is easy to show the existence of a complementary basis. $\square$

22 Computational results

We now give some results in low degrees, obtained by computing multiplication matrices with respect to the integral basis of $H^*(S^{[n]}, \mathbb{Z})$. To get their cokernels, one has to reduce them to Smith normal form. Both results have been obtained using a computer.

Remark 22.1. Denote $h^k(S^{[n]})$ the rank of $H^k(S^{[n]}, \mathbb{Z})$. We have:

- $h^2(S^{[n]}) = 23$ for $n \geq 2$.
- $h^4(S^{[n]}) = 276, \; 299, \; 300$ for $n = 2, 3, \geq 4$ resp.
- $h^6(S^{[n]}) = 23, \; 2554, \; 2852, \; 2875, \; 2876$ for $n = 2, 3, 4, 5, \geq 6$ resp.

The algebra generated by classes of degree 2 is an interesting object to study. For cohomology with complex coefficients, Verbitsky has proven in [60] that the cup product mapping from $\operatorname{Sym}^k H^2(S^{[n]}, \mathbb{C})$ to $H^{2k}(S^{[n]}, \mathbb{C})$ is injective for $k \leq n$. Since there is no torsion, one concludes that this also holds for integral coefficients.

Proposition 22.2. *We identify* $\operatorname{Sym}^2 H^2(S^{[n]}, \mathbb{Z})$ *with its image in* $H^4(S^{[n]}, \mathbb{Z})$ *under the cup product mapping. Then:*

$$\frac{H^4(S^{[2]}, \mathbb{Z})}{\operatorname{Sym}^2 H^2(S^{[2]}, \mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23} \oplus \frac{\mathbb{Z}}{5\mathbb{Z}}, \tag{101}$$

$$\frac{H^4(S^{[3]},\mathbb{Z})}{\operatorname{Sym}^2 H^2(S^{[3]},\mathbb{Z})} \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 23}, \tag{102}$$

$$\frac{H^4(S^{[n]},\mathbb{Z})}{\operatorname{Sym}^2 H^2(S^{[n]},\mathbb{Z})} \cong \mathbb{Z}^{\oplus 24}, \quad \textit{for } n \geq 4. \tag{103}$$

The 3-torsion part in (102) is generated by the integral class $1^{(3)}$.

Remark 22.3. The torsion in the case $n = 2$ was also computed by Boissière, Nieper-Wißkirchen and Sarti, [5, Prop. 3] using similar techniques. For all the author knows, the result for $n = 3$ is new. The freeness result for $n \geq 4$ was already proven by Markman, [34, Thm. 1.10], using a completely different method.

Proposition 22.4. *For triple products of* $H^2(S^{[n]},\mathbb{Z})$*, we have:*

$$\frac{H^6(S^{[2]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[2]},\mathbb{Z})} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}.$$

The quotient is generated by the integral class $x^{(2)}$. *Moreover,*

$$\frac{H^6(S^{[3]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[3]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 230} \oplus \left(\frac{\mathbb{Z}}{36\mathbb{Z}}\right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{72\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 254},$$

$$\frac{H^6(S^{[4]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[4]},\mathbb{Z})} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 552}.$$

For $n \geq 5$*, the quotient is free by Theorem 21.15.*

We study now cup products between classes of degree 2 and 4. The case of $S^{[3]}$ is of particular interest.

Proposition 22.5. *The cup product mapping*

$$H^2(S^{[n]},\mathbb{Z}) \otimes H^4(S^{[n]},\mathbb{Z}) \to H^6(S^{[n]},\mathbb{Z})$$

is neither injective (unless $n = 0$*) nor surjective (unless* $n \leq 2$*). We have:*

$$\frac{H^6(S^{[3]},\mathbb{Z})}{H^2(S^{[3]},\mathbb{Z}) \smile H^4(S^{[3]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}, \tag{104}$$

$$\frac{H^6(S^{[4]},\mathbb{Z})}{H^2(S^{[4]},\mathbb{Z}) \smile H^4(S^{[4]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{6\mathbb{Z}}\right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{108\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}, \tag{105}$$

$$\frac{H^6(S^{[5]},\mathbb{Z})}{H^2(S^{[5]},\mathbb{Z}) \smile H^4(S^{[5]},\mathbb{Z})} \cong \mathbb{Z}^{\oplus 22} \oplus \mathbb{Z}, \tag{106}$$

$$\frac{H^6(S^{[n]},\mathbb{Z})}{H^2(S^{[n]},\mathbb{Z}) \smile H^4(S^{[n]},\mathbb{Z})} \cong \mathbb{Z}^{\oplus 22} \oplus \mathbb{Z} \oplus \mathbb{Z}, \ n \geq 6. \tag{107}$$

In each case, the first 22 factors of the quotient are generated by the integral classes

$$\alpha_i^{(1,1,1)} - 3 \cdot \alpha_i^{(2,1)} + 3 \cdot \alpha_i^{(3)} + 3 \cdot 1^{(2)}\alpha_i^{(1,1)} - 6 \cdot 1^{(2)}\alpha_i^{(2)} + 6 \cdot 1^{(2,2)}\alpha_i^{(1)} - 3 \cdot 1^{(3)}\alpha_i^{(1)},$$

for $i = 1 \dots 22$*. Now define an integral class*

$$\begin{aligned} K := &\sum_{i \neq j} B(\alpha_i, \alpha_j) \left[\alpha_i^{(1,1)}\alpha_j^{(1)} - 2 \cdot \alpha_i^{(2)}\alpha_j^{(1)} + \frac{3}{2} \cdot 1^{(2)}\alpha_i^{(1)}\alpha_j^{(1)}\right] + \\ &+ \sum_i B(\alpha_i, \alpha_i) \left[\alpha_i^{(1,1,1)} - 2 \cdot \alpha_i^{(2,1)} + \frac{3}{2} \cdot 1^{(2)}\alpha_i^{(1,1)}\right] + x^{(2)} - 1^{(2)}x^{(1)}. \end{aligned}$$

In the case $n = 3$*, the last factor of the quotient is generated by* K*.*
In the case $n = 4$*, the class* $1^{(4)}$ *generates the 2-torsion factor and* $K - 38 \cdot 1^{(4)}$ *generates the 108-torsion factor.*
In the case $n = 5$*, the last factor of the quotient is generated by* $K - 16 \cdot 1^{(4)} + 21 \cdot 1^{(3,2)}$*.*
If $n \geq 6$*, the two last factor of the quotient are generated over the rationals by* $K + \frac{4}{3}(45 - n)1^{(2,2,2)} - (48 - n)1^{(3,2)}$ *and* $K + \frac{1}{2}(40 - n)1^{(2,2,2)} - \frac{1}{4}(48 - n)1^{(4)}$*. Over* $\mathbb{Z}$*, one has to take appropriate multiples depending on* n*, such that the coefficients become integral numbers.*

Proof. The last assertion for arbitrary n follows from Corollary 21.11. First observe that for $\boldsymbol{\alpha}^{\boldsymbol{\lambda}} \in H^2$, $\boldsymbol{\alpha}^{\boldsymbol{\mu}} \in H^4$, $\boldsymbol{\alpha}^{\boldsymbol{\nu}} \in H^6$, we have $\|\boldsymbol{\lambda}\| \leq 2$, $\|\boldsymbol{\mu}\| \leq 4$ and $\|\boldsymbol{\nu}\| \geq 2$, according to Lemma 21.7. The coefficients of the cup product matrix are thus polynomials of degree at most $2 + 4 - 2 = 4$ and it suffices to compute only a finite number of instances for n. It turns out that the maximal degree is 1 and the cokernel of the multiplication map is given as stated. □

In what follows, we compare some well-known facts about Hilbert schemes of points on K3 surfaces with our numerical calculations. This means, we have some tests that may justify the correctness of our computer program. We state now computational results for the middle cohomology group. Since $S^{[n]}$ is a projective variety of complex dimension $2n$, Poincaré duality gives $H^{2n}(S^{[n]},\mathbb{Z})$ the structure of a unimodular lattice.

Proposition 22.6. *Let* L *denote the unimodular lattice* $H^{2n}(S^{[n]},\mathbb{Z})$*. We have:*

(i) For $n = 2$*,* L *is an odd lattice of rank* 276 *and signature* 156*.*

(ii) *For $n = 3$, L is an even lattice of rank* 2554 *and signature* -1152.

(iii) *For $n = 4$, L is an odd lattice of rank* 19298 *and signature* 7082.

For n even, L is always odd.

Proof. The numerical results come from an explicit calculation. For n even, we always have the norm-1-vector given by Example 21.14, so L is odd. To obtain the signature, we could equivalently use Hirzebruch's signature theorem and compute the L-genus of $S^{[n]}$. For the signature, we need nothing but the Pontryagin numbers, which can be derived from the Chern numbers of $S^{[n]}$. These in turn are known by Ellingsrud, Göttsche and Lehn, [14, Rem. 5.5]. □

Another test is to compute the lattice structure of $H^2(S^{[2]}, \mathbb{Z})$, with bilinear form given by $(a, b) \longmapsto \int \left(a \smile b \smile 1^{(2)} \smile 1^{(2)}\right)$. The signature of this lattice is 17, as shown by Boissière, Nieper-Wißkirchen and Sarti [5, Lemma 6.9].

A Source Code for the combinatorial model of Lehn and Sorger

We give the source code for our tool implementing the integral cohomology of Hilbert schemes of points on K3 surfaces. It is available online under `https://github.com/s--kapfer/HilbK3`. We used the language Haskell, compiled with the GHC software, version 7.6.3. We make use of two external packages: PERMUTATION and MEMOTRIE. The project is divided into 4 modules.

A.1 How to use the code

The main module is in the file `HilbK3.hs`, which can be opened by GHCI for interactive use. It provides an implementation of the ring structure of $A^{[n]} = H^*(S^{[n]}, \mathbb{Q})$, for all $n \in \mathbb{N}$. It computes cup–products in reasonable time up to $n = 8$. A product of Nakajima operators is represented by a pair consisting of a partition of length k and a list of the same length, filled with indices for the basis elements of $H^*(S)$. For example, the class

$$\mathfrak{q}_3(\alpha_6)\mathfrak{q}_3(\alpha_7)\mathfrak{q}_2(x)\mathfrak{q}_1(\alpha_2)\mathfrak{q}_1(1)^2|0\rangle$$

in $H^{20}(S^{[11]})$ is written as

```
*HilbK3> (PartLambda [3,3,2,1,1,1], [6,7,23,2,0,0]) :: AnBase
```

Note that the classes $1 \in H^0(S)$ and $x \in H^4(S)$ have indices `0` and `23` in the code. The multiplication in $A^{[n]}$ is implemented by the method `multAn`.

The classes from Theorem 21.5 are represented in the same format, as shown in the following example. The multiplication in $H^*(S^{[n]}, \mathbb{Z})$ of such classes is implemented by the method `cupInt`.

Example A.1. We want to compute the results from Example 21.13. We only do one particular instance for every example, since the others are similar. By Corollary 21.11, it suffices to know the values for finitely many n to deduce the general case.

(i) We do the case $n = 6,\ i = 1$.

```
*HilbK3> let i = 1 :: Int
*HilbK3> let x = (PartLambda [2,2,1,1], [0,0,0,0]) :: AnBase
*HilbK3> let y = (PartLambda [2,1,1,1,1], [i,0,0,0,0]) :: AnBase
*HilbK3> cupInt x y
[(([2-1-1-1-1],[0,23,1,0,0]),-2),(([2-2-2],[1,0,0]),1),
(([3-2-1],[1,0,0]),2),(([4-1-1],[1,0,0]),1)]
```

(ii) We do the case $n = 4,\ i = j = 1$.

```
*HilbK3> let i = 1 :: Int; let j = 1 :: Int
*HilbK3> let x = (PartLambda [2,1,1], [i,0,0]) :: AnBase
*HilbK3> let y = (PartLambda [1,1,1,1], [j,0,0,0]) :: AnBase
*HilbK3> cupInt x y
[(([2-1-1],[1,1,0]),1),(([3-1],[1,0]),1)]
```

(iii) We do the case $n = 4$.

```
*HilbK3> let d = (PartLambda [2,1,1], [0,0,0]) :: AnBase
*HilbK3> let y = (PartLambda [1,1,1,1], [23,0,0,0]) :: AnBase
*HilbK3> [ t | t <- cupInt d d, fst t == y]
[(([1-1-1-1],[23,0,0,0]),-3)]
```

(iv) We do the case $n = 5$.

```
*HilbK3> let x = (PartLambda [2,2,1], [0,0,0]) :: AnBase
*HilbK3> let y = (PartLambda [1,1,1,1,1], [23,23,0,0,0]) :: AnBase
*HilbK3> [ t | t <- cupInt x x, fst t == y]
[(([1-1-1-1-1],[23,23,0,0,0]),3)]
```

A.2 What the code does

The goal is to multiply two elements in $H^*(S^{[n]}, \mathbb{Z})$. To do this, one has to execute the following steps:

(i) Compute the base change matrices $\psi_{\rho\nu}$ and $\psi^{-1}_{\nu\rho}$ between monomial and power sum symmetric functions.

(ii) Provide a basis and the ring structure of $A = H^*(S, \mathbb{Z})$.

(iii) Create a data structure for elements in $A^{[n]}$ and $A\{S_n\}$.

(iv) Implement the multiplication in $A\{S_n\}$, *i.e.* the map $m_{\pi,\tau}$ from Definition 21.8.

(v) Implement the symmetrisation $A^{[n]} = A\{S_n\}^{S_n}$.

(vi) Use the isomorphism from Theorem 21.9 to get the ring structure of $A^{[n]}$.

(vii) Write an element in $H^*(S^{[n]}, \mathbb{Z})$ as a linear combination of products of creation operators acting on the vacuum, using Theorem 21.5.

We now describe, where to find these steps in the code.

(i) The coefficients $\psi_{\rho\nu}$ are computed by the function `monomialPower` in the module `SymmetricFunctions.hs`, using the theory from [29, Sect. 3.7]. The idea is to use the scalar product on the space of symmetric functions, so that the power sums become orthogonal: $(p_\lambda, p_\mu) = z_\lambda \delta_{\lambda\mu}$. The values for (p_λ, m_μ) are given by [29, Lemma 3.7.1], so we know how to get the matrix $\psi^{-1}_{\nu\rho}$. Since it is triangular with respect to some ordering of partitions, matrix inversion is easy.

(ii) The ring structure of $H^*(S, \mathbb{Z})$ is stored in the module `K3.hs`. The only nontrivial multiplications are the products of two elements in $H^2(S, \mathbb{Z})$, where the intersection matrix is composed by the matrices for the hyperbolic and the E_8 lattice. The cup product and the adjoint comultiplication from Definition 21.4 are implemented by the methods `cup` and `cupAd`.

(iii) The data structures for basis elements of $A^{[n]}$ and $A\{S_n\}$ are given by `AnBase` and `SnBase` in the module `HilbK3.hs`. Linear combinations of basis elements are always stored as lists of pairs, each pair consisting of a basis element and a scalar factor.

(iv) The function $m_{\pi,\tau}$ from Definition 21.8 is computed by the method `multSn`. It contains the following substeps: First, the orbits of $\langle\pi,\tau\rangle$ are computed recursively by glueing together the orbits of π if they have both non-emtpty intersection with an orbit of τ. Second, the composition $\pi\tau$ is computed using a method from the external library `Data.Permute`. Third, the functions $f^{\pi,\langle\pi,\tau\rangle}$ and $f_{\langle\pi,\tau\rangle,\pi\tau}$ using the (co–)products from `K3.hs`.

(v) The symmetrisation morphism is implemented by `toSn`. We don't konw a better way to do this than the naive approach which is summation over all elements in S_n.

(vi) The multiplication in $A^{[n]}$ is carried out by the method `multAn`.

(vii) The base change matrices between the canonical base of $A^{[n]}$ and the base of $H^*(S^{[n]},\mathbb{Z})$ are given by `creaInt` and `intCrea`. By composing `multAn` with these matrices, one gets the desired multiplication in $H^*(S^{[n]},\mathbb{Z})$, called `cupInt`.

A.3 Module for cup product structure of K3 surfaces

Here the hyperbolic and the E_8 lattice and the bilinear form on the cohomology of a K3 surface are defined. Furthermore, cup products and their adjoints are implemented.

```
-- a module for the integer cohomology structure of a K3 surface
module K3 (
  K3Domain,
  degK3,
  rangeK3,
  oneK3, xK3,
  cupLSparse,
  cupAdLSparse
  ) where

import Data.Array
import Data.List
import Data.MemoTrie

-- type for indexing the cohomology base
type K3Domain = Int

rangeK3 = [0..23] :: [K3Domain]

oneK3 = 0 :: K3Domain
xK3 = 23 :: K3Domain

rangeK3Deg :: Int -> [K3Domain]
rangeK3Deg 0 = [0]
rangeK3Deg 2 = [1..22]
rangeK3Deg 4 = [23]
rangeK3Deg _ = []

delta i j = if i==j then 1 else 0
```

```
-- degree of the element of H^*(S), indexed by i
degK3 :: (Num d) => K3Domain -> d
degK3 0 = 0
degK3 23 = 4
degK3 i = if i>0 && i < 23 then 2 else error "Not␣a␣K3␣index"

-- the negative e8 intersection matrix
e8 = array ((1,1),(8,8)) $
  zip [(i,j) | i <- [1..8],j <-[1..8]] [
  -2, 1, 0, 0, 0, 0, 0, 0,
  1, -2, 1, 0, 0, 0, 0, 0,
  0, 1, -2, 1, 0, 0, 0, 0,
  0, 0, 1, -2, 1, 0, 0, 0,
  0, 0, 0, 1, -2, 1, 1, 0,
  0, 0, 0, 0, 1, -2, 0, 1,
  0, 0, 0, 0, 1, 0, -2, 0,
  0, 0, 0, 0, 0, 1, 0, -2 :: Int]

-- the inverse matrix of e8
inve8 = array ((1,1),(8,8)) $
  zip [(i,j) | i <- [1..8],j <-[1..8]] [
  -2, -3, -4, -5, -6, -4, -3, -2,
  -3, -6, -8,-10,-12, -8, -6, -4,
  -4, -8,-12,-15,-18,-12, -9, -6,
  -5,-10,-15,-20,-24,-16,-12, -8,
  -6,-12,-18,-24,-30,-20,-15,-10,
  -4, -8,-12,-16,-20,-14,-10, -7,
  -3, -6, -9,-12,-15,-10, -8, -5,
  -2, -4, -6, -8,-10, -7, -5, -4 :: Int]

-- hyperbolic lattice
u 1 2 = 1
u 2 1 = 1
u 1 1 = 0
u 2 2 = 0
u i j = undefined

-- cup product pairing for K3 cohomology
bilK3 :: K3Domain -> K3Domain -> Int
bilK3 ii jj = let
  (i,j) = (min ii jj, max ii jj)
  in
  if (i < 0) || (j > 23) then undefined else
  if (i == 0) then delta j 23 else
  if (i >= 1) && (j <= 2) then u i j else
  if (i >= 3) && (j <= 4) then u (i-2) (j-2) else
  if (i >= 5) && (j <= 6) then u (i-4) (j-4) else
  if (i >= 7) && (j <= 14) then e8 ! ((i-6), (j-6)) else
  if (i >= 15) && (j<= 22) then e8 ! ((i-14), (j-14))  else
  0

-- inverse matrix to cup product pairing
bilK3inv :: K3Domain -> K3Domain -> Int
bilK3inv ii jj = let
  (i,j) = (min ii jj, max ii jj)
  in
  if (i < 0) || (j > 23) then undefined else
  if (i == 0) then delta j 23 else
  if (i >= 1) && (j <= 2) then u i j else
  if (i >= 3) && (j <= 4) then u (i-2) (j-2) else
  if (i >= 5) && (j <= 6) then u (i-4) (j-4) else
  if (i >= 7) && (j <= 14) then inve8 ! ((i-6), (j-6)) else
  if (i >= 15) && (j<= 22) then inve8 ! ((i-14), (j-14))  else
  0

-- cup product with two factors
-- a_i * a_j = sum [cup k (i,j) * a_k | k<- rangeK3]
cup :: K3Domain -> (K3Domain,K3Domain) -> Int
cup = memo2 r where
  r k (0,i) = delta k i
```

```
  r k (i,0) = delta k i
  r _ (i,23) = 0
  r _ (23,i) = 0
  r 23 (i,j) =  bilK3 i j
  r _ _ = 0

-- indices where the cup product does not vanish
cupNonZeros :: [ (K3Domain,(K3Domain,K3Domain)) ]
cupNonZeros = [ (k,(i,j)) | i<-rangeK3, j<-rangeK3, k<-rangeK3, cup k (i,j) /= 0]

-- cup product of a list of factors
cupLSparse :: [K3Domain] -> [(K3Domain,Int)]
cupLSparse = cu . filter (/=oneK3) where
  cu [] = [(oneK3,1)]; cu [i] = [(i,1)]
  cu [i,j] = [(k,z) | k<-rangeK3, let z = cup k (i,j), z/=0]
  cu _ = []

-- comultiplication, adjoint to the cup product
-- Del a_k = sum [cupAd (i,j) k * a_i 'tensor' a_k | i<-rangeK3, j<-rangeK3]
cupAd :: (K3Domain,K3Domain) -> K3Domain -> Int
cupAd = memo2 ad where
  ad (i,j) k = negate $ sum [bilK3inv i ii * bilK3inv j jj
    * cup kk (ii,jj) * bilK3 kk k |(kk,(ii,jj)) <- cupNonZeros ]

-- n-fold comultiplication
cupAdLSparse :: Int -> K3Domain -> [([K3Domain],Int)]
cupAdLSparse = memo2 cals where
  cals 0 k = if k == xK3 then [([],1)] else []
  cals 1 k = [([k], 1)]
  cals 2 k = [([i,j],ca) | i<-rangeK3, j<-rangeK3, let ca = cupAd (i,j) k, ca /=0]
  cals n k = clean [(i:r,v*w) |([i,j],w)<-cupAdLSparse 2 k, (r,v)<-cupAdLSparse(n-1) j]
  clean = map (\g -> (fst$head g, sum$(map snd g))). groupBy cg.sortBy cs
  cs = (.fst).compare.fst; cg = (.fst).(==).fst
```

A.4 Module for handling partitions

This module defines the data structures and elementary methods to handle partitions. We define both partitions written as descending sequences of integers (λ-notation) and as sequences of multiplicities (α-notation).

```
{-# LANGUAGE TypeOperators, TypeFamilies #-}

-- implements data structure and basic functions for partitions
module Partitions where

import Data.Permute
import Data.Maybe
import qualified Data.List
import Data.MemoTrie

class (Eq a, HasTrie a) => Partition a where
  -- length of a partition
  partLength :: Integral i => a -> i

  -- weight of a partition
  partWeight :: Integral i => a -> i

  -- degree of a partition = weight - length
  partDegree :: Integral i => a -> i
  partDegree p = partWeight p - partLength p

  -- the z, occuring in all papers
  partZ :: Integral i => a -> i
  partZ = partZ.partAsAlpha
```

```haskell
  -- conjugated partition
  partConj :: a -> a
  partConj = res . partAsAlpha where
    make l (m:r) = l : make (l-m) r
    make _ [] = []
    res (PartAlpha r) = partFromLambda $ PartLambda $ make (sum r) r

  -- empty partition
  partEmpty :: a

  -- transformation to alpha-notation
  partAsAlpha :: a -> PartitionAlpha
  -- transformation from alpha-notation
  partFromAlpha :: PartitionAlpha -> a
  -- transformation to lambda-notation
  partAsLambda :: a -> PartitionLambda Int
  -- transformation from lambda-notation
  partFromLambda :: (Integral i, HasTrie i) => PartitionLambda i -> a

  -- all permutationens of a certain cycle type
  partAllPerms :: a -> [Permute]
```

```haskell
-- data type for partitiones in alpha-notation
-- (list of multiplicities)
newtype PartitionAlpha = PartAlpha { alphList::[Int] }

-- reimplementation of the zipWith function
zipAlpha op (PartAlpha a) (PartAlpha b) = PartAlpha $ z a b where
  z (x:a) (y:b) = op x y : z a b
  z [] (y:b) = op 0 y : z [] b
  z (x:a) [] = op x 0 : z a []
  z [] [] = []

-- reimplementation of the (:) operator
alphaPrepend 0 (PartAlpha []) = partEmpty
alphaPrepend i (PartAlpha  r) = PartAlpha (i:r)

-- all partitions of a given weight
partOfWeight :: Int -> [PartitionAlpha]
partOfWeight = let
  build n 1 acc = [alphaPrepend n acc]
  build n c acc = concat [ build (n-i*c) (c-1) (alphaPrepend i acc) | i<-[0..div n c]]
  a 0 = [PartAlpha []]
  a w =  if w<0 then [] else  build w w partEmpty
  in memo a

-- all partitions of given weight and length
partOfWeightLength = let
  build 0 0 _ = [partEmpty]
  build w 0 _ = []
  build w l c = if l > w || c>w then [] else
    concat [ map (alphaPrepend i) $ build (w-i*c) (l-i) (c+1)
      | i <- [0..min l $ div w c]]
  a w l = if w<0 || l<0 then [] else build w l 1
  in memo2 a

-- determines the cycle type of a permutation
cycleType :: Permute -> PartitionAlpha
cycleType p = let
  lengths = Data.List.sort $ map Data.List.length $ cycles p
  count i 0 [] = partEmpty
  count i m [] = PartAlpha [m]
  count i m (x:r) = if x==i then count i (m+1) r
    else alphaPrepend m (count (i+1) 0 (x:r))
  in count 1 0 lengths

-- constructs a permutation from a partition
partPermute :: Partition a => a -> Permute
```

```
partPermute = let
  make l n acc (PartAlpha x) = f x where
    f [] = cyclesPermute n acc
    f (0:r) = make (l+1) n acc $ PartAlpha r
    f (i:r) = make l (n+l) ([n..n+l-1]:acc) $ PartAlpha ((i-1):r)
  in make 1 0 [] . partAsAlpha

instance Partition PartitionAlpha where
  partWeight (PartAlpha r) = fromIntegral $ sum $ zipWith (*) r [1..]
  partLength (PartAlpha r) = fromIntegral $ sum r
  partEmpty = PartAlpha []
  partZ (PartAlpha l) = foldr (*) 1 $
    zipWith (\a i-> factorial a*i^a) (map fromIntegral l) [1..] where
      factorial n = if n==0 then 1 else n*factorial(n-1)
  partAsAlpha = id
  partFromAlpha = id
  partAsLambda (PartAlpha l) = PartLambda $ reverse $ f 1 l where
    f i [] = []
    f i (0:r) = f (i+1) r
    f i (m:r) = i : f i ((m-1):r)
  partFromLambda = lambdaToAlpha
  partAllPerms = partAllPerms . partAsLambda

instance Eq PartitionAlpha where
  PartAlpha p == PartAlpha q = findEq p q where
    findEq [] [] = True
    findEq (a:p) (b:q) = (a==b) && findEq p q
    findEq [] q = isZero q
    findEq p [] = isZero p
    isZero = all (==0)

instance Ord PartitionAlpha where
  compare a1 a2 = compare (partAsLambda a1) (partAsLambda a2)

instance Show PartitionAlpha where
  show p = let
    leftBracket = "(|"
    rightBracket = "|)"
    rest [] = rightBracket
    rest [i] = show i ++ rightBracket
    rest (i:q) = show i ++ "," ++ rest q
    in leftBracket ++ rest (alphList p)

instance HasTrie PartitionAlpha where
  newtype PartitionAlpha :->: a = TrieType { unTrieType :: [Int] :->: a }
  trie f = TrieType $ trie $ f . PartAlpha
  untrie f = untrie (unTrieType f) . alphList
  enumerate f = map (\(a,b) -> (PartAlpha a,b)) $ enumerate (unTrieType f)
```

```
-- data type for partitions in lambda-notation
-- (descending list of positive numbers)
newtype PartitionLambda i = PartLambda { lamList :: [i] }

lambdaToAlpha :: Integral i => PartitionLambda i -> PartitionAlpha
lambdaToAlpha (PartLambda []) = PartAlpha[]
lambdaToAlpha (PartLambda (s:p)) = lta 1 s p [] where
  lta _ 0 _ a = PartAlpha a
  lta m c [] a = lta 0 (c-1) [] (m:a)
  lta m c (s:p) a = if c==s then lta (m+1) c p a else
    lta 0 (c-1) (s:p) (m:a)

instance (Integral i, HasTrie i) => Partition (PartitionLambda i) where
  partWeight (PartLambda r) = fromIntegral $ sum r
  partLength (PartLambda r) = fromIntegral $ length r
  partEmpty = PartLambda []
  partAsAlpha = lambdaToAlpha
  partAsLambda (PartLambda r) = PartLambda $ map fromIntegral r
  partFromAlpha (PartAlpha l) = PartLambda $ reverse $ f 1 l where
```

```
    f i [] = []
    f i (0:r) = f (i+1) r
    f i (m:r) = i : f i ((m-1):r)
  partFromLambda (PartLambda r) = PartLambda $ map fromIntegral r
  partAllPerms (PartLambda l) = it $ Just $ permute $ partWeight $ PartLambda l where
    it (Just p) = if Data.List.sort (map length $ cycles p) == r
      then p : it (next p) else it (next p)
    it Nothing = []
    r = map fromIntegral $ reverse l

instance (Eq i, Num i) => Eq (PartitionLambda i) where
  PartLambda p == PartLambda q = findEq p q where
    findEq [] [] = True
    findEq (a:p) (b:q) = (a==b) && findEq p q
    findEq [] q = isZero q
    findEq p [] = isZero p
    isZero = all (==0)

instance (Ord i, Num i) => Ord (PartitionLambda i) where
  compare p1 p2 = if weighteq == EQ then compare l1 l2 else weighteq where
    (PartLambda l1, PartLambda l2) = (p1, p2)
    weighteq = compare (sum l1) (sum l2)

instance (Show i) => Show (PartitionLambda i) where
  show (PartLambda p) = "[" ++ s ++ "]" where
    s = concat $ Data.List.intersperse "-" $ map show p

instance HasTrie i => HasTrie (PartitionLambda i) where
  newtype (PartitionLambda i) :->: a = TrieTypeL { unTrieTypeL :: [i] :->: a }
  trie f = TrieTypeL $ trie $ f . PartLambda
  untrie f = untrie (unTrieTypeL f) . lamList
  enumerate f = map (\(a,b) -> (PartLambda a,b)) $ enumerate (unTrieTypeL f)
```

A.5 Module for coefficients on Symmetric Functions

This module provides nothing but the base change matrices $\psi_{\lambda\mu}$ and $\psi^{-1}_{\mu\lambda}$ from Definition 21.2.

```
-- A module implementing base change matrices for symmetric functions
module SymmetricFunctions(
  monomialPower,
  powerMonomial,
  factorial
  ) where

import Data.List
import Data.MemoTrie
import Data.Ratio
import Partitions

-- binomial coefficients
choose n k = ch1 n k where
  ch1 = memo2 ch
  ch 0 0 = 1
  ch n k = if n<0 || k<0 then 0 else if k> div n 2 + 1 then ch1 n (n-k) else
    ch1(n-1) k + ch1 (n-1) (k-1)

-- multinomial coefficients
multinomial 0 [] = 1
multinomial n [] = 0
multinomial n (k:r) = choose n k * multinomial (n-k) r

-- factorial function
factorial 0 = 1
factorial n = n*factorial(n-1)
```

```
-- http://www.mat.univie.ac.at/~slc/wpapers/s68vortrag/ALCoursSf2.pdf , p. 48
-- scalar product between monomial symmetric functions and power sums
monomialScalarPower moI poI = (s * partZ poI) `div` quo where
  mI = partAsAlpha moI
  s = sum[a* moebius b | (a,b)<-finerPart mI (partAsLambda poI)]
  quo = product[factorial i| let PartAlpha l =mI, i<-l]
  nUnder 0 [] = [[]]
  nUnder n [] = []
  nUnder n (r:profile) = concat[map (i:) $ nUnder (n-i) profile | i<-[0..min n r]]
  finerPart (PartAlpha a) (PartLambda l) = nub [(a`div` sym sb,sb)
    | (a,b)<-fp 1 a l, let sb = sort b] where
    sym = s 0 []
    s n acc [] = factorial n
    s n acc (a:o) = if a==acc then s (n+1) acc o else factorial n * s 1 a o
    fp i [] l = if all (==0) l then [(1,[[]|x<-l])] else []
    fp i (0:ar) l = fp (i+1) ar l
    fp i (m:ar) l = [(v*multinomial m p,addprof p op)
      | p <- nUnder m (map (flip div i) l),
      (v,op) <- fp (i+1) ar (zipWith (\j mm -> j-mm*i) l p)] where
        addprof = zipWith (\mm l -> replicate mm i ++ l)
  moebius l = product [(-1)^c * factorial c | m<-l, let c = length m - 1]

-- base change matrix from monomials to power sums
-- no integer coefficients
-- m_j = sum [ p_i * powerMonomial i j | i<-partitions]
powerMonomial :: (Partition a, Partition b) => a->b->Ratio Int
powerMonomial poI moI = monomialScalarPower moI poI % partZ poI

-- base change matrix from power sums to monomials
-- p_j = sum [m_i * monomialPower i j | i<-partitions]
monomialPower :: (Partition a, Partition b, Num i) => a->b->i
monomialPower lambda mu = fromIntegral $ numerator $
  memoizedMonomialPower (partAsLambda lambda) (partAsLambda mu)
memoizedMonomialPower = memo2 mmp1 where
  mmp1 l m  = if partWeight l == partWeight m then mmp2 (partWeight m) l m else 0
  mmp2 w l m = invertLowerDiag (map partAsLambda $ partOfWeight w) powerMonomial l m

-- inversion of lower triangular matrix
invertLowerDiag vs a = ild where
  ild = memo2 inv
  delta i j = if i==j then 1 else 0
  inv i j | i<j = 0
    | otherwise = (delta i j - sum [a i k * ild k j | k<-vs, i>k , k>= j]) / a i i
```

A.6 Module implementing cup products for Hilbert schemes

This is our main module. We implement the algebraic model developed by Lehn and Sorger and the change of base due to Qin and Wang. The cup product on the Hilbert scheme is computed by the function `cupInt`.

```
-- implements the cup product according to Lehn-Sorger and Qin-Wang
module HilbK3 where

import Data.Array
import Data.MemoTrie
import Data.Permute hiding (sort,sortBy)
import Data.List
import qualified Data.IntMap as IntMap
import qualified Data.Set as Set
import Data.Ratio
import K3
import Partitions
import SymmetricFunctions

-- elements in A^[n] are indexed by partitions, with attached elements of the base K3
```

```haskell
-- is also used for indexing H^*(Hilb, Z)
type AnBase = (PartitionLambda Int, [K3Domain])

-- elements in A{S_n} are indexed by permutations, in cycle notation,
-- where to each cycle an element of the base K3 is attached, see L-S (2.5)
type SnBase = [([Int],K3Domain)]

-- an equivalent to partZ with painted partitions
-- counts multiplicites that occur, when the symmetrization operator is applied
anZ :: AnBase -> Int
anZ (PartLambda l, k) = comp 1 (0,undefined) 0 $ zip l k where
  comp acc old m (e@(x,_):r) | e==old = comp (acc*x) old (m+1) r
    | otherwise = comp (acc*x*factorial m) e 1 r
  comp acc _ m [] = factorial m * acc

-- injection of A^[n] in A{S_n}, see L-S 2.8
-- returns a symmetrized vector of A{S_n}
toSn :: AnBase -> ([SnBase],Int)
toSn = makeSn where
  allPerms = memo p where
    p n = map (array (0,n-1). zip [0..]) (permutations [0..n-1])
  shape l = (map (forth IntMap.!) l, IntMap.fromList $ zip [1..] sl) where
    sl = map head$ group $ sort l;
    forth = IntMap.fromList$ zip sl [1..]
  symmetrize :: AnBase -> ([[([Int],K3Domain)]],Int)
  symmetrize (part,l) = (perms, toInt $ factorial n % length perms)  where
    perms = nub [sortSn$ zipWith (\c cb ->(ordCycle $ map(p!)c, cb) ) cyc l
      | p <- allPerms n]
    cyc = sortBy ((.length).flip compare.length) $ cycles $ partPermute part
    n = partWeight part
  ordCycle cyc = take l $ drop p $ cycle cyc where
    (m,p,l) = foldl findMax (-1,-1,0) cyc
    findMax (m,p,l) ce = if m<ce then (ce,l,l+1) else (m,p,l+1)
  sortSn = sortBy compareSn  where
    compareSn (cyc1,class1) (cyc2,class2) = let
      cL = compare l2 $ length cyc1 ; l2 = length cyc2
      cC = compare class2 class1
      in if cL /= EQ then cL else
        if cC /= EQ then cC else compare cyc2 cyc1
  mSym = memo symmetrize
  makeSn (part,l) = ([ [(z,im IntMap.! k) | (z,k) <- op ]|op <- res],m)  where
    (repl,im) = shape l
    (res,m) = mSym (part,repl)

-- multiplication in A{S_n}k, see L-S, Prop 2.13
multSn :: SnBase -> SnBase -> [(SnBase,Int)]
multSn l1 l2 = tensor $ map m cmno where
  -- determines the orbits of the group generated by pi, tau
  commonOrbits :: Permute -> Permute -> [[Int]]
  commonOrbits pi tau = Data.List.sortBy ((.length).compare.length) orl where
    orl = foldr (uni [][]) (cycles pi) (cycles tau)
    uni i ni c []  = i:ni
    uni i ni c (k:o) = if Data.List.intersect c k == []
      then uni i (k:ni) c o else uni (i++k) ni c o
  pi1 = cyclesPermute n $ cy1 ; cy1 = map fst l1; n = sum $ map length cy1
  pi2 = cyclesPermute n $ map fst l2
  set1 = map (\(a,b)->(Set.fromList a,b)) l1;
  set2 = map (\(a,b)->(Set.fromList a,b)) l2
  compose s t = swapsPermute (max (size s) (size t)) (swaps s ++ swaps t)
  tau = compose pi1 pi2
  cyt = cycles tau ;
  cmno = map Set.fromList $ commonOrbits pi1 pi2;
  m or = fdown where
    sset12 = [xv | xv <-set1++set2, Set.isSubsetOf (fst xv) or]
    -- fup and fdown correspond to the images of the maps described in L-S (2.8)
    fup = cupLSparse $ map snd sset12 ++ replicate def xK3
    t = [c | c<-cyt, Set.isSubsetOf (Set.fromList c) or]
    fdown = [(zip t l,v*w*24^def)| (r,v) <- fup, (l,w)<-cupAdLSparse(length t) r]
    def = toInt ((Set.size or + 2 - length sset12 - length t)%2)
```

```haskell
-- tensor product for a list of arguments
tensor :: Num a => [[([b],a)]] -> [([b],a)]
tensor [] = [([],1)]
tensor (t:r) = [(y++x,w*v) |(x,v)<-tensor r, (y,w) <- t ]

-- multiplication in A^[n]
multAn :: AnBase -> AnBase -> [(AnBase,Int)]
multAn a = multb where
  (asl,m) = toSn a
  toAn sn =(PartLambda l, k) where
    (l,k)= unzip$ sortBy (flip compare)$ map (\(c,k)->(length c,k)) sn
  multb (pb,lb) = map ungroup$ groupBy ((.fst).(==).fst) $sort elems where
    ungroup g@((an,_):_) = (an, m*(sum $ map snd g) )
    bs = zip (sortBy ((.length).flip compare.length) $cycles $ partPermute pb) lb
    elems = [(toAn cs,v) | as <- asl, (cs,v) <- multSn as bs]

-- integer base to ordinary base, see Q-W, Thm 1.1
intCrea :: AnBase -> [(AnBase,Ratio Int)]
intCrea = map makeAn. tensor. construct where
  memopM = memo pM
  pM pa = [(pl,v)| p@(PartLambda pl)<-map partAsLambda$ partOfWeight (partWeight pa),
    let v = powerMonomial p pa, v/=0]
  construct pl = onePart pl : xPart pl :
    [ [(zip l $ repeat a,v)| (l,v)<- memopM (subpart pl a)] |a<-[1..22]]
  onePart pl = [(zip l$ repeat oneK3, 1%partZ p)] where
    p@(PartLambda l) = subpart pl oneK3
  xPart pl = [(zip l$ repeat xK3, 1)] where
    (PartLambda l) = subpart pl xK3
  makeAn (list,v) = ((PartLambda x,y),v) where
    (x,y) = unzip$ sortBy (flip compare) list

-- ordinary base to integer base, see Q-W, Thm 1.1
creaInt :: AnBase -> [(AnBase, Int)]
creaInt = map makeAn. tensor. construct where
  memomP = memo mP
  mP pa = [(pl,v)| p@(PartLambda pl)<-map partAsLambda$ partOfWeight (partWeight pa),
    let v = monomialPower p pa, v/=0]
  construct pl = onePart pl : xPart pl :
    [ [(zip l $ repeat a,v)| (l,v)<- memomP (subpart pl a)] |a<-[1..22]]
  onePart pl = [(zip l$ repeat oneK3, partZ p)] where
    p@(PartLambda l) = subpart pl oneK3
  xPart pl = [(zip l$ repeat xK3, 1)] where
    (PartLambda l) = subpart pl xK3
  makeAn (list,v) = ((PartLambda x,y),v) where
    (x,y) = unzip$ sortBy (flip compare) list

-- cup product for integral classes
cupInt :: AnBase -> AnBase -> [(AnBase,Int)]
cupInt a b = [(s,toInt z)| (s,z) <- y] where
  ia = intCrea a; ib = intCrea b
  x = sparseNub [(e,v*w*fromIntegral z) | (p,v) <- ia,
    let m = multAn p, (q,w) <- ib, (e,z)<- m q]
  y = sparseNub [(s,v*fromIntegral w) | (e,v) <- x, (s,w) <- creaInt e]

-- helper function, adds duplicates in a sparse vector
sparseNub :: (Num a) => [(AnBase, a)] -> [(AnBase,a)]
sparseNub = map (\g->(fst$head g, sum $map snd g)).groupBy ((.fst).(==).fst).
  sortBy ((.fst).compare.fst)

-- cup product for integral classes from a list of factors
cupIntList :: [AnBase] -> [(AnBase,Int)]
cupIntList = makeInt. ci . cL where
  cL [b] = intCrea b
  cL (b:r) = x where
    ib = intCrea b
    x = sparseNub [(e,v*w*fromIntegral z) |
      (p,v) <- cL r, let m = multAn p, (q,w) <- ib, (e,z)<-m q]
  makeInt l = [(e,toInt z) | (e,z) <- l]
  ci l = sparseNub [(s,v*fromIntegral w) | (e,v) <- l, (s,w) <- creaInt e]
```

```
-- degree of a base element of cohomology
degHilbK3 :: AnBase -> Int
degHilbK3 (lam,a) = 2*partDegree lam + sum [degK3 i | i<- a]

-- base elements in Hilb^n(K3) of degree d
hilbBase :: Int -> Int -> [AnBase]
hilbBase = memo2 hb where
  hb n d = sort $map ((\(a,b)->(PartLambda a,b)).unzip) $ hilbOperators n d

-- all possible combinations of creation operators of weight n and degree d
hilbOperators :: Int -> Int -> [[ (Int,K3Domain) ]]
hilbOperators = memo2 hb where
  hb 0 0 = [[]] -- empty product of operators
  hb n d = if n<0 || odd d || d<0 then [] else
    nub $ map (Data.List.sortBy (flip compare)) $ f n d
  f n d = [(nn,oneK3):x | nn <-[1..n], x<-hilbOperators(n-nn)(d-2*nn+2)] ++
    [(nn,a):x | nn<-[1..n], a <-[1..22], x<-hilbOperators(n-nn)(d-2*nn)] ++
    [(nn,xK3):x | nn <-[1..n], x<-hilbOperators(n-nn)(d-2*nn-2)]

-- helper function
subpart :: AnBase -> K3Domain -> PartitionLambda Int
subpart (PartLambda pl,l) a = PartLambda $ sb pl l where
  sb [] _ = []
  sb pl [] = sb pl [0,0..]
  sb (e:pl) (la:l) = if la == a then e: sb pl l else sb pl l

-- converts from Rational to Int
toInt :: Ratio Int -> Int
toInt q = if n ==1 then z else error "not integral" where
  (z,n) =(numerator q, denominator q)
```

B Source Code for the operator model

We give the source code for our tool implementing the rational cohomology of Hilbert schemes of points on surfaces using Nakajima operators. We follow the notational conventions of [30]. The description given there in Section 3 allows to deduce an algorithm for operator actions on $\mathbb{H}$. The Chern classes of tangent bundles of Hilbert schemes is computed with the help of the description from [2, Section 3].

In contrast to the code in the previous section, we are not restricted to K3 surfaces. Indeed, the surface may have cohomology in odd degree as well as a non-vanishing canonical class. On the other hand, the implemention of the cup product in general is slower than the model for K3 surfaces.

B.1 How to use the code

The main module is `LS_Operators.hs`, which can be opened with ghci. We implemented the actions of the following operators:

- The Nakajima operators $\mathfrak{p}_n(a)$ and $L_n(a)$ from [30] are given by `P n a` and `L n a`.
- The differential operator ∂ is given by `Del`.
- The multiplication operators $\mathfrak{G}_k(a)$ from [32] related to Chern characters correspond to `Ch k a`.
- The Chern character $\mathfrak{ch}T$ of the tangent bundle from [2] in degree k corresponds to `ChT k`.

Example B.1. To evaluate the action of a single operator product on the vacuum, use the command `nakaState` to show the result in terms of Nakajima operators:

```
*LS_Operators>  let a = P2 0 in nakaState [P (-4) a, P (-2) a]
1 % 1 *         p_4(P2 0) p_2(P2 0) |0>
*LS_Operators>  let a = P2 0 in nakaState [Del,L(-3) a]
6 % 1 *         p_3(P2 2) |0> +
3 % 1 *         p_2(P2 1) p_1(P2 2) |0> +
3 % 1 *         p_2(P2 2) p_1(P2 1) |0> +
(-3) % 1 *      p_1(P2 0) p_1(P2 2) p_1(P2 2) |0> +
(-3) % 1 *      p_1(P2 1) p_1(P2 1) p_1(P2 2) |0>
```

```
*LS_Operators>  let a = P2 0 in
                        nakaState [ChT 2, P(-1) a, P(-1) a, P(-1) a]
2 % 1 *         p_3(P2 0) |0> +
18 % 1 *        p_2(P2 1) p_1(P2 0) |0> +
(-9) % 1 *      p_1(P2 0) p_1(P2 1) p_1(P2 1) |0>
```

B.2 What the code does

An important observation is that we do not need to know explicitly the commutator of $\mathfrak{G}_k(a)$ with $\mathfrak{p}_n(b)$ to compute the action of $\mathfrak{G}_k(a)$ on $\mathbb{H}$. Indeed, every element of $\mathbb{H}$ can be written as the action on the vacuum of either a polynomial in Nakajima operators $\mathfrak{p}_n(a)$ or of a polynomial in the operators ∂ and $\mathfrak{p}_{-1}(a)$. We call the two representations `nakaState` and `delState`, respectively. For the action of a Nakajima operator, the first one is more appropriate, while a multiplication operator acts better on the second one (multiplication commutes with ∂ and the commutators with $\mathfrak{p}_n(a)$ are known). In addition, the necessary commutation relations to switch between the two representations are known. This is the guiding philosophy for the algorithms contained in `LS_Operators.hs`.

The other module, `LS_Frobenius.hs` contains nothing but the definition of a graded Frobenius algebra according to [30, Section 2.1] and some instances, namely the cohomologies of K3 surfaces, complex tori and projective space $\mathbb{CP}^2$.

The datatype that models $\mathbb{H}$ is called `State`. It consists of linear combinations of ordered operator products, implemented as lists of pairs, containing the product (as a list) and the scalar.

B.3 Module for graded Frobenius algebras

```
{-# LANGUAGE GeneralizedNewtypeDeriving, TypeOperators, TypeFamilies #-}
module LS_Frobenius
  where

import Data.Array
import Data.List
import Data.MemoTrie

-- the d. We are dealing with surfaces, so d=2.
gfa_d = 2 :: Int

class (Ix k,Ord k)=> GradedFrobeniusAlgebra k where
  gfa_deg :: k -> Int
  gfa_base :: [k]
  gfa_baseOfDeg :: Int -> [k]
  gfa_1 :: Num a => [(k,a)]
  gfa_K :: Num a => [(k,a)]
  gfa_T :: Num a => k -> a
  gfa_mult :: Num a => k -> k -> [(k,a)]
  gfa_bilinear :: Num a => k -> k -> a
```

```
  gfa_bilinear i j = sum [ gfa_T k * x | (k,x) <- gfa_mult i j ]
  gfa_bilinearInverse :: Num a => k -> [(k, a)]

-- Tensor product
instance (GradedFrobeniusAlgebra k, GradedFrobeniusAlgebra k')
    => GradedFrobeniusAlgebra (k,k') where
  gfa_deg (i,j) = gfa_deg i + gfa_deg j
  gfa_base = [(i,j) | i<-gfa_base, j<-gfa_base]
  gfa_baseOfDeg n = [(i,j) | i<-gfa_base, j<-gfa_baseOfDeg (n-gfa_deg i) ]
  gfa_1 = [((i,j),x*y) | (i,x) <- gfa_1, (j,y) <- gfa_1]
  gfa_K = [ ((i,j),x*y) | (i,x) <- gfa_K, (j,y) <- gfa_1] ++
      [ ((i,j),x*y) | (i,x) <- gfa_1, (j,y) <- gfa_K]
  gfa_T (i,j) = gfa_T i * gfa_T j
  gfa_mult (i,j) (k,l) = [((m,n),ep*x*y) | (m,x)<-gfa_mult i k, (n,y)<-gfa_mult j l] where
    ep = if odd (gfa_deg j * gfa_deg k) then (-1) else 1
  gfa_bilinearInverse (i,j) = [ ((k,l),ep k *x*y) |
    (k,x) <- gfa_bilinearInverse i, (l,y) <-gfa_bilinearInverse j] where
      ep k = if odd (gfa_deg k * gfa_deg j) then (-1) else 1

-- base for Symmetric tensors
gfa_symBase n = cs n gfa_base where
  f [] = []
  f l@(a:b) = (a, if odd (gfa_deg a) then b else l ) : f b
  cs 0 _ = [[]]
  cs k b@(a:r) = [ x:t | (x,r) <- f b, t <- cs (k-1) r]
-- power is n, degree is k
gfa_symBaseOfDeg n k = csd n k gfa_base where
  f [] = []
  f l@(a:b) = (a, d, if odd d then b else l) : f b where d = gfa_deg a
  csd 0 0 _ = [[]]
  csd n k b@(a:r) = if n<= 0 then [] else
    [ x:t | (x,d,r) <- f b, t <- csd (n-1) (k-d) r]


gfa_multList [] = gfa_1
gfa_multList [i] = [(i,1)]
gfa_multList [i,j] = gfa_mult i j
gfa_multList (i:r) = sparseNub [ (k,y*x) | (j,x)<-gfa_multList r, (k,y)<-gfa_mult i j ]

gfa_adjoint f = adj where
  b i = [(j,x) | j<-gfa_base, let x = gfa_bilinear j i, x/=0]
  ftrans = accumArray (flip (:)) [] (head gfa_base, last gfa_base)
      [ (j,(i,x)) | i <-gfa_base, (j,x) <- f i]
  ftb i = sparseNub [ (k,y*x) | (j,x) <- b i, (k,y) <- ftrans!j]
  adj i = sparseNub [ (k,y*x) | (j,x) <- ftb i, (k,y) <- gfa_bilinearInverse j]

gfa_comult :: (GradedFrobeniusAlgebra k, Ix k,Num a,Eq a) => k -> [((k,k),a)]
gfa_comult = gfa_adjoint (uncurry gfa_mult)

gfa_comultN 0 a = [([a],1)]
gfa_comultN n a = let
  rec = gfa_comultN (n-1) a
  in sparseNub [ (c:d:r, x*y) | (b:r,x) <- rec, ((c,d),y) <- gfa_comult b]

gfa_euler :: (GradedFrobeniusAlgebra k, Ix k, Num a) => [(k,a)]
gfa_euler = [(k,fromIntegral x) | (k,x)<-e] where
  e = sparseNub [ (k,y*x*v) | (u,v) <- gfa_1, (ij,x) <- gfa_comult u, (k,y) <- uncurry gfa_mult ij]

sparseNub [] = []
sparseNub l = sn (head sl) (tail sl) where
  sl = sortBy ((.fst).compare.fst) l
  sn (i,x) ((j,y):r) = if i==j then sn (i,x+y) r else app (i,x) $ sn (j,y) r
  sn ix [] = app ix []
  app (i,x) r = if x==0 then r else (i,x) : r

scal 0 _ = []
scal a l = [ (p,a*x) | (p,x) <- l]

-- Cohomology of K3 surfaces
newtype K3Domain = K3 Int deriving (Enum,Eq,Num,Ord,Ix)
```

```
instance Show K3Domain where show (K3 i) = show i
instance GradedFrobeniusAlgebra K3Domain where
  gfa_deg (K3 0) = -2
  gfa_deg (K3 23) = 2
  gfa_deg i = if 1<=i && i<=22 then 0 else undefined

  gfa_1 = [(0,1)]
  gfa_K = []

  gfa_T (K3 23) = -1
  gfa_T _ = 0

  gfa_base = [0..23]
  gfa_baseOfDeg 0 = [1..22]
  gfa_baseOfDeg (-2) = [0]
  gfa_baseOfDeg 2 = [23]
  gfa_baseOfDeg _ = []

  gfa_mult (K3 0) i = [(i,1)]
  gfa_mult i (K3 0) = [(i,1)]
  gfa_mult (K3 23) _ = []
  gfa_mult _ (K3 23) = []
  gfa_mult (K3 i) (K3 j) =  [(23, fromIntegral $ bilK3_func i j)]

  gfa_bilinearInverse (K3 i) = [(K3 j,fromIntegral x) | j<-[0..23], let x =bilK3inv_func i j, x/=0]

delta i j = if i==j then 1 else 0

e8 = array ((1,1),(8,8)) $
  zip [(i,j) | i <- [1..8],j <-[1..8]] [
  -2, 1, 0, 0, 0, 0, 0, 0,
  1, -2, 1, 0, 0, 0, 0, 0,
  0, 1, -2, 1, 0, 0, 0, 0,
  0, 0, 1, -2, 1, 0, 0, 0,
  0, 0, 0, 1, -2, 1, 1, 0,
  0, 0, 0, 0, 1, -2, 0, 1,
  0, 0, 0, 0, 1, 0, -2, 0,
  0, 0, 0, 0, 0, 1, 0, -2 :: Int]

inve8= array ((1,1),(8,8)) $
  zip [(i,j) | i <- [1..8],j <-[1..8]]
      [-2, -3, -4, -5, -6, -4, -3, -2, -3, -6, -8, -10, -12, -8, -6, -4,
    -4, -8, -12, -15, -18, -12, -9, -6, -5, -10, -15, -20, -24, -16, -12,
    -8, -6, -12, -18, -24, -30, -20, -15, -10, -4, -8, -12, -16, -20,
    -14, -10, -7, -3, -6, -9, -12, -15, -10, -8, -5, -2, -4, -6, -8,
    -10, -7, -5, -4 :: Int]

-- Bilinear form on K3 surfaces
bilK3_func ii jj = let
  (i,j) = (min ii jj, max ii jj)
  u 1 2 = 1
  u 2 1 = 1
  u 1 1 = 0
  u 2 2 = 0
  u i j = undefined
  in
  if (i < 0) || (j > 23) then undefined else
  if (i == 0) then delta j 23 else
  if (i >= 1) && (j <= 2) then u i j else
  if (i >= 3) && (j <= 4) then u (i-2) (j-2) else
  if (i >= 5) && (j <= 6) then u (i-4) (j-4) else
  if (i >= 7) && (j <= 14) then e8 ! ((i-6), (j-6)) else
  if (i >= 15) && (j<= 22) then e8 ! ((i-14), (j-14))  else
  0 :: Int

-- Inverse bilinear form
bilK3inv_func ii jj = let
  (i,j) = (min ii jj, max ii jj)
  u 1 2 = 1
  u 2 1 = 1
```

```
  u 1 1 = 0
  u 2 2 = 0
  u i j = undefined
  in
  if (i < 0) || (j > 23) then undefined else
  if (i == 0) then delta j 23 else
  if (i >= 1) && (j <= 2) then u i j else
  if (i >= 3) && (j <= 4) then u (i-2) (j-2) else
  if (i >= 5) && (j <= 6) then u (i-4) (j-4) else
  if (i >= 7) && (j <= 14) then inve8 ! ((i-6), (j-6)) else
  if (i >= 15) && (j<= 22) then inve8 ! ((i-14), (j-14))  else
  0 :: Int

-- Cohomology of projective space
newtype P2Domain = P2 Int deriving (Show,Eq,Ord,Ix,Num)
instance GradedFrobeniusAlgebra P2Domain where
  gfa_deg (P2 0) = -2
  gfa_deg (P2 1) = 0
  gfa_deg (P2 2) = 2

  gfa_1 = [(P2 0,1)]
  gfa_K = [(P2 1,-3)]

  gfa_T (P2 2) = -1
  gfa_T _ = 0

  gfa_base = [0,1,2]
  gfa_baseOfDeg 0 = [1]
  gfa_baseOfDeg (-2) = [0]
  gfa_baseOfDeg 2 = [2]
  gfa_baseOfDeg _ = []

  gfa_mult (P2 0) i = [(i,1)]
  gfa_mult i (P2 0) = [(i,1)]
  gfa_mult (P2 2) _ = []
  gfa_mult _ (P2 2) = []
  gfa_mult (P2 1) (P2 1) =  [(2, 1)]

  gfa_bilinearInverse i =  [(2-i, 1)]


-- Cohomology of complex torus
newtype TorusDomain = Tor Int deriving (Enum,Eq,Num,Ord,Ix)
instance Show TorusDomain where show (Tor i) = show i
instance GradedFrobeniusAlgebra TorusDomain where
  gfa_deg (Tor i) =
    if i<0 then undefined else
    if i<=0 then -2 else
    if i<=4 then -1 else
    if i<=10 then 0 else
    if i<=14 then 1 else
    if i==15 then 2 else undefined

  gfa_1 = [(0,1)]
  gfa_K = []

  gfa_T (Tor 15) = -1
  gfa_T _ = 0

  gfa_base = [0..15]
  gfa_baseOfDeg (-2) = [0]
  gfa_baseOfDeg (-1) = [1..4]
  gfa_baseOfDeg 0 = [5..10]
  gfa_baseOfDeg 1 = [11..14]
  gfa_baseOfDeg 2 = [15]
  gfa_baseOfDeg _ = []

  gfa_mult i j = if k<0 then [] else [(k,fromIntegral x)] where (k,x)= torusMultArray!(i,j)

  gfa_bilinearInverse i = [(15-i,gfa_bilinear i (15-i))]
```

```
torusMultArray = listArray ((0,0),(15,15)) mults where
  toLists = listArray (0,15) list
  list = [[],[1],[2],[3],[4],[1,2],[1,3],[1,4],[2,3],[2,4],
      [3,4],[1,2,3],[1,2,4],[1,3,4],[2,3,4],[1,2,3,4::Int]]
  fromLists x = findIndex (x==) list
  mults = [ check (toLists!i ++ toLists!j)  | i<-[0..15],j<-[0..15]]
  check ab = case fromLists $ sort ab of
    Nothing -> (-1,0)
    Just i -> (Tor i, sign ab)
  sign [] = 1; sign (i:r) = sign r * signum (product [j-i | j<-r])
```

B.4 Module for the commutator algebra

```
{-# LANGUAGE GeneralizedNewtypeDeriving, ParallelListComp #-}
module LS_Operators
  where

import LS_Frobenius

data VertexOperator k = P Int k | L Int k | Del | Ch Int k | GV Int k | ChT Int
    deriving (Show,Eq,Ord)

newtype State a k = Vak { unVak:: [([VertexOperator k],a)] }

weight (P n _) = -n
weight (L n _) = -n
weight Del = 0
weight (Ch _ _) = 0
weight (GV _ _) = 0
weight (ChT _) = 0

degree (P n k) = gfa_deg k
degree (L n k) = gfa_deg k + gfa_d
degree Del = 2
degree (Ch n k) = gfa_deg k + 2*n
degree (GV n k) = gfa_deg k + 2*n
degree (ChT n) = 2*n


data ActsOn = DelState | Both | NakaState deriving (Show,Eq)


actsOn (P _ _) = Both
actsOn Del = Both
actsOn (L _ _) = NakaState
actsOn (Ch _ _) = DelState
actsOn (GV _ _) = DelState
actsOn (ChT _) = DelState


actOnNakaVac p@(P n _) = Vak $ if n<0 then [([p],1)] else []
actOnNakaVac (L n k) = Vak $ sparseNub [(o,y*x/2) |
  nu <- [n+1 .. -1], ((a,b),x) <- gfa_comult k, (o,y) <-unVak$nakaState $op nu a b ] where
  op nu a b = if n-nu > 0 then [P nu a, P (n-nu) b] else [P (n-nu) a, P nu b]
actOnNakaVac Del = Vak []

actOnDelVac p@(P n k) = Vak $ if n >= 0 then [] else scal ( 1/ fac) $ rec n where
  fac = fromIntegral $ product [n+1 .. -1]
  rec (-1) = [([P(-1) k],1)]
  rec n = sparseNub [ t | (o,x) <- rec (n+1), (oo,y) <- p', t<-[(oo++o,x*y),(o++oo,-x*y)]]
  p' = [ ([Del,P(-1) a], x) | (a,x) <-gfa_1 ] ++ [ ([P(-1) a,Del], -x) | (a,x) <-gfa_1 ]
actOnDelVac Del = Vak  []
actOnDelVac (Ch _ _) = Vak  []
actOnDelVac (GV _ _) = Vak  []
actOnDelVac (ChT _) = Vak []

-- ad(Del)^n(op)/n!
facDiff n op = let
```

```
  bins 0 = [1]
  bins n = zipWith (-) (b++[0]) (0:b) where b = bins (n-1)
  ders = scanr (:) [] $ replicate (fromIntegral n) Del
  dels = scanl (flip (:)) [] $ replicate (fromIntegral n) Del
  fac = product [1..fromIntegral n]
  in [ (d1++[op]++d2,fromIntegral b/fac) | d1<-ders | d2 <- dels | b <- bins n]

ad n u v = let
  rec = ad (n-1) u v
  --com = [ ([Del,P(-1) 0],1) , ([P(-1) 0,Del], -1)]
  new = [ z | (x,a) <- rec, (y,b) <- u, z <- [(x++y,-a*b),(y++x,a*b)]]
  in if n==0 then v else sparseNub new

commutator (P n a) (P m b) = if n+m==0 then [ ([], gfa_bilinear a b*fromIntegral n) ] else []
commutator Del (P n a) = ([L n a], fromIntegral n) :
  [ ([P n c],x*y*sc) | (b,x) <- gfa_K, (c,y) <- gfa_mult b a] where
  sc = fromIntegral $ (-n*(abs n - 1) ) `div` 2
commutator p@(P _ _) Del = scal (-1) $ commutator Del p
commutator (L n a) (P m b) = [ ([P (n+m) c], x*fromIntegral(-m)) | (c,x) <- gfa_mult a b ]
commutator (Ch _ _) Del = []
commutator (Ch n a) (P (-1) y) =
  [ (c,x*fromRational z) | (b,x) <- gfa_mult a y, (c,z) <- facDiff n (P (-1) b) ]
commutator (GV _ _) Del = []
commutator (GV n a) (P (-1) y) = if odd n then [(s,negate x) | (s,x) <-csn] else csn where
  csn = commutator (Ch n a) (P (-1) y)
commutator (ChT _) Del = []
commutator (ChT n) p@(P (-1) y) =  sparseNub $ first ++ second ++ third ++ fourth ++ fifth where
  k2= [(c,2*x*xx*z) | (a,x) <-gfa_K, (b,xx) <-gfa_K, (c,z) <- gfa_mult a b]
  todd_Inv_y = [[(y,1)], [(b,x*xx/2) | (a,x)<-gfa_K, (b,xx) <- gfa_mult a y],
    sparseNub [(b,x*xx) | (a,x)<-scal (1/6) k2 ++ scal (1/12) gfa_euler, (b,xx) <- gfa_mult a y]]
  exp_K_y = [[(y,1)], [ (b,-x*xx) | (a,x) <- gfa_K, (b,xx) <- gfa_mult a y] ,
    [ (b,x*xx/2)| (a,x) <-k2, (b,xx) <- gfa_mult a y ] ]
  expTodd_y = zipWith scal [1,-1,1] todd_Inv_y
  first = [ (c,x) | (c,x) <-facDiff n p ]
  second = [ ( o++[GV gn b2], x*xx*z) | k <- [0..2] ,
       (a,x) <- todd_Inv_y!!k, ((b1,b2),xx) <- gfa_comult a,
    nu <- [0..n-k-2], let gn = n-nu-k-2, (o,z) <- facDiff nu (P (-1) b1) ]
  third = [ (c,x*xx*(-1)^nu) | nu<-[max (n-2) 0..n],
    (a,x) <- exp_K_y !! (n-nu), (c,xx) <-facDiff nu (P (-1) a) ]
  fourth = [ ( o++[Ch gn b2], x*xx*z*(-1)^nu) | k <- [0..2] ,
       (a,x) <- expTodd_y!!k, ((b1,b2),xx) <- gfa_comult a,
    nu <- [0..n-k-2], let gn = n-nu-k-2, (o,z) <- facDiff nu (P (-1) b1) ]
  fifth = if n==2 then [ ([P(-1) b], x*xx) | (a,x) <- gfa_euler, (b,xx) <- gfa_mult a y] else []


showOperatorList [] = "|0>"
showOperatorList (Del:r) = "D␣" ++ showOperatorList r
showOperatorList (P n k:r) = sh ++ showOperatorList r where
  sh = (if n<0 then "p_"++show(-n)else "p"++show n)++"("++show k++")␣"
showOperatorList (L n k:r) = sh ++ showOperatorList r where
  sh = (if n<0 then "L_"++show(-n)else "L"++show n)++"("++show k++")␣"
showOperatorList (Ch n k:r) = "ch"++show n++"["++show k++ "]␣" ++ showOperatorList r

instance (Show a, Show k) => Show (State a k) where
  show (Vak []) = "0"
  show (Vak [(l,x)]) = show x ++ "␣*␣\t"++showOperatorList l
  show (Vak ((l,x):r)) = show x ++ "␣*␣\t"++showOperatorList l ++ "␣+\n"++show(Vak r)



-- Operator product acting on Vacuum. Result is given in terms of deltas and P(-1) operators.
delState :: (GradedFrobeniusAlgebra k, Ord k) => [VertexOperator k] -> State Rational k

delState [] = Vak [([],1)]
delState (o:r) = if actsOn o == NakaState then toDel $ nakaState (o:r) else result where
  result = Vak $ sparseNub[ (q,x*y) | (s,x) <-unVak$ delState r, (q,y) <- unVak $ commuteIn s]
  commuteIn [] = actOnDelVac o
  commuteIn (pd:r) = case (o,pd) of
    (Del,_) -> Vak [ (Del:pd:r,1) ]
    (P (-1) _, Del) -> Vak [ (o:pd:r,1) ]
```

```
    (P (-1) a, P (-1) b) -> if a <= b then Vak [(o:pd:r,1)] else Vak cI
    _ -> Vak cI
    where
    cI = case comm of [] -> ted; _ -> sparseNub $ ted ++ comm
    ted = [(pd:q,x*sign) | (q,x) <- unVak $ commuteIn r]
    comm =  [ (ds,x*y) | (q,x) <- commutator o pd, (ds,y) <- unVak $ delState (q++r) ]
    sign= if odd (degree pd) && odd (degree o) then -1 else 1


-- Operator product acting on Vacuum. Result is given in terms of creation operators.
nakaState :: (GradedFrobeniusAlgebra k, Ord k) => [VertexOperator k] -> State Rational k

nakaState [] = Vak [([],1)]
nakaState (o:r) = if actsOn o == DelState then toNaka $ delState (o:r) else result where
  result = Vak $ sparseNub[ (q,x*y) | (s,x) <- unVak $ nakaState r, (q,y) <- unVak $ commuteIn s]
  nakaSort p [] = ([p],1)
  nakaSort p (q:r) = case (odd (degree p)&& odd (degree q), compare p q) of
    (True,EQ) -> (p:q:r,0)
    (v, GT) -> (q:n, if v then -s else s) where (n,s) = nakaSort p r
    _ -> (p:q:r,1)
  commuteIn [] = actOnNakaVac o
  commuteIn (p:r) = case (o,p) of
    (P _ _, P _ _) -> if o<p then Vak [(o:p:r,1)] else Vak cI
    _ -> Vak cI
    where
    cI = case comm of [] -> ted; _ -> sparseNub $ ted ++ comm
    ted = [(n,x*sign*sign2) | (q,x) <- unVak $ commuteIn r, let (n,sign2)=nakaSort p q]
    comm =  [ (ds,x*y) | (q,x) <- commutator o p, (ds,y) <- unVak $ nakaState (q++r) ]
    sign= if odd (degree p) && odd (degree o) then -1 else 1

-- Transforms state representations
toDel (Vak l)  = Vak $ sparseNub[ (p,x*y)|(o,x) <- l, (p,y) <- unVak$delState o]
toNaka (Vak l) = Vak $ sparseNub[ (p,x*y)|(o,x) <- l, (p,y) <- unVak$nakaState o]

scale a (Vak sta) = Vak $ scal a sta
add (Vak s) (Vak t) = Vak $ sparseNub $ s ++ t

multLists l stat = toNaka $ ml l stat where
  ml [] stat = stat
  ml (l:r) stat = let
    Vak s = ml r stat
    ns = sparseNub[ (t,x*y*z) | (a,x) <- s, (o,y) <- l, (t,z)<- unVak $ delState (o++a) ]
    in Vak ns

-- Chern classes related to ChT
cT = (!!) c where
  c = [([],1::Rational)] : [if odd k then [] else cc k | k<-[1..] ]
  cc k = [ (ChT (2*i):o, x*fact(2*i)/fromIntegral(-k) ) | i<-[1..div k 2], (o,x) <- cT (k-2*i) ]
  fact n = fromIntegral $ product [1..n]
```

C Divisible classes in $H^4(K_2(A),\mathbb{Z})$

Here we list the divisible classes in from Section 17. The results are obtained by using a computer based calculation.

Proposition C.1. *The 31 following classes of Π' are divisible by 3 in $H^4(K_2(A),\mathbb{Z})$ and their thirds span a $\mathbb{F}_3$-vector space of dimension 31 in $\frac{\Pi'^{sat}}{\Pi'}$.*

$$\sum_{\tau\in\Lambda}\Big(Z_\tau - Z_{\tau+\tau'}\Big),\ \textit{with}$$

(i) $\Lambda = \left\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right\rangle$ *and* $0\neq\tau'\in P^\perp = \left\langle \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\rangle$,

(ii) $\Lambda = \left\langle \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\rangle$ *and* $0\neq\tau'\in P^\perp = \left\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right\rangle \setminus \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$,

(iii) $\Lambda = \left\langle \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\\1\end{pmatrix} \right\rangle$ *and* $\tau'\in \left\{ \begin{pmatrix}0\\1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\0\\0\\2\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\2\\2\end{pmatrix} \right\}$,

(iv) $\Lambda = \left\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix} \right\rangle$ *and* $\tau'\in \left\{ \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\\2\end{pmatrix}, \begin{pmatrix}1\\0\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\2\\1\end{pmatrix} \right\}$,

(v) $\Lambda = \left\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\end{pmatrix} \right\rangle$ *and* $\tau'\in \left\{ \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix} \right\}$,

(vi) $\Lambda = \left\langle \begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix} \right\rangle$ *and* $\tau'\in \left\{ \begin{pmatrix}0\\1\\0\\2\end{pmatrix}, \begin{pmatrix}1\\0\\2\\2\end{pmatrix} \right\}$,

(vii) $\Lambda = \left\langle \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix} \right\rangle$ *and* $\tau'\in \left\{ \begin{pmatrix}0\\1\\0\\2\end{pmatrix}, \begin{pmatrix}1\\0\\2\\0\end{pmatrix} \right\}$,

(viii) $\Lambda = \left\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\2\end{pmatrix} \right\rangle$ *and* $\tau' = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$,

(ix) $\Lambda = \left\langle \begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\\2\end{pmatrix} \right\rangle$ *and* $\tau' = \begin{pmatrix}1\\1\\0\\2\end{pmatrix}$.

Proposition C.2. *We use Notation 15.10. The 19 following classes are divisible by 3 in $H^4(K_2(A),\mathbb{Z})$ and their thirds span a sub-vector space of dimension 19 of $\frac{H^4(K_2(A),\mathbb{Z})}{\mathrm{Sym}^{sat}\oplus\Pi'^{sat}}$.*

(i) $u_2^2 + \sum_{\tau\in\Lambda} Z_\tau - Z_0$, *for* $\Lambda = \left\langle \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \right\rangle$,

(ii) $v_2^2+v_2u_2+u_2^2+\sum_{\tau\in\Lambda}Z_\tau-Z_0$, *for* $\Lambda=\left\langle\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\right\rangle$,

(iii) $w_2^2+w_2u_2+u_2^2+\sum_{\tau\in\Lambda}Z_\tau-Z_0$, *for* $\Lambda=\left\langle\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\1\end{pmatrix}\right\rangle$,

(iv) $w_2^2-w_2u_2+u_2^2+\sum_{\tau\in\Lambda}Z_\tau-Z_0$, *for* $\Lambda=\left\langle\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\2\end{pmatrix}\right\rangle$,

(v) $w_2^2-w_2v_2+w_2u_2+v_2^2+v_2u_2+u_2^2+\sum_{\tau\in\Lambda}Z_\tau-Z_0$, *for* $\Lambda=\left\langle\begin{pmatrix}0\\0\\1\\2\end{pmatrix},\begin{pmatrix}0\\1\\0\\1\end{pmatrix}\right\rangle$,

(vi) $w_1^2+w_1u_2+u_2^2+\sum_{\tau\in\Lambda}Z_\tau-Z_0$, *for* $\Lambda=\left\langle\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\2\\0\end{pmatrix}\right\rangle$,

(vii) $w_1^2-w_1u_2+u_2^2+\sum_{\tau\in\Lambda}Z_\tau-Z_0$, *for* $\Lambda=\left\langle\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\right\rangle$,

(viii) $v_1^2+v_1u_2+u_2^2+\sum_{\tau\in\Lambda}Z_\tau-Z_0$, *for* $\Lambda=\left\langle\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right\rangle$,

(ix) $v_1^2-v_1u_2+u_2^2+\sum_{\tau\in\Lambda}Z_\tau-Z_0$, *for* $\Lambda=\left\langle\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\2\end{pmatrix}\right\rangle$,

(x) $v_1^2+v_1w_1-v_1u_2+w_1^2+w_1u_2+u_2^2+\sum_{\tau\in\Lambda}Z_\tau-Z_0$, *for* $\Lambda=\left\langle\begin{pmatrix}0\\0\\1\\2\end{pmatrix},\begin{pmatrix}1\\0\\0\\2\end{pmatrix}\right\rangle$,

(xi) $v_1^2+v_1w_1-v_1w_2-v_1v_2+v_1u_2+w_1^2+w_1w_2+w_1v_2-w_1u_2+w_2^2-w_2v_2+w_2u_2+v_2^2+v_2u_2+u_2^2+\sum_{\tau\in\Lambda}Z_\tau-Z_0$, *for* $\Lambda=\left\langle\begin{pmatrix}0\\0\\1\\2\end{pmatrix},\begin{pmatrix}1\\1\\0\\1\end{pmatrix}\right\rangle$,

(xii) $v_1^2-v_1w_1+v_1w_2-v_1v_2+v_1u_2+w_1^2+w_1w_2-w_1v_2+w_1u_2+w_2^2+w_2v_2-w_2u_2+v_2^2+v_2u_2+u_2^2+\sum_{\tau\in\Lambda}Z_\tau-Z_0$, *for* $\Lambda=\left\langle\begin{pmatrix}0\\0\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\0\\1\end{pmatrix}\right\rangle$,

(xiii) $u_1^2+\sum_{\tau\in\Lambda}Z_\tau-Z_0$, *for* $\Lambda=\left\langle\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\rangle$,

(xiv) $u_1^2-u_1v_2+v_2^2+\sum_{\tau\in\Lambda}Z_\tau-Z_0$, *for* $\Lambda=\left\langle\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right\rangle$,

(xv) $u_1^2+u_1v_2+v_2^2+\sum_{\tau\in\Lambda}Z_\tau-Z_0$, *for* $\Lambda=\left\langle\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\2\end{pmatrix}\right\rangle$,

(xvi) $u_1^2+u_1w_1+w_1^2+\sum_{\tau\in\Lambda}Z_\tau-Z_0$, *for* $\Lambda=\left\langle\begin{pmatrix}0\\1\\0\\2\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\rangle$, *(xvii)* $u_1^2 + u_1w_1 - u_1v_2 + w_1^2 + w_1v_2 + v_2^2 + \sum_{\tau\in\Lambda} Z_\tau - Z_0$, *for* $\Lambda = \left\langle \begin{pmatrix} 0\\1\\0\\2 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\1 \end{pmatrix} \right\rangle$,

(xviii) $u_1^2 - u_1w_1 + u_1w_2 - u_1u_2 + w_1^2 + w_1w_2 - w_1u_2 + w_2^2 + w_2u_2 + u_2^2 + \sum_{\tau\in\Lambda} Z_\tau - Z_0$, *for* $\Lambda = \left\langle \begin{pmatrix} 0\\1\\0\\1 \end{pmatrix}, \begin{pmatrix} 1\\0\\1\\0 \end{pmatrix} \right\rangle$,

(xix) $u_1^2 + u_1v_1 - u_1w_1 + v_1^2 + v_1w_1 + w_1^2 + \sum_{\tau\in\Lambda} Z_\tau - Z_0$, *for* $\Lambda = \left\langle \begin{pmatrix} 0\\1\\2\\1 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\0 \end{pmatrix} \right\rangle$.

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