# COMPUTING CUP-PRODUCTS IN INTEGER COHOMOLOGY OF HILBERT SCHEMES OF POINTS ON K3 SURFACES

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Abstract. We study the images of cup products in integer cohomology of the Hilbert scheme of $n$ points on a K3 surface.

## 1. Preliminaries

**Definition 1.1.** Let $n$ be a natural number. A partition of $n$ is a sequence $\lambda = (\lambda_1 \geq \ldots \geq \lambda_k > 0)$ of natural numbers such that $\sum_i \lambda_i = n$. It is convenient to write $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$ as a sequence of multiplicities. We define the weight $\|\lambda\| := \sum m_i i = n$ and the length $|\lambda| := \sum_i m_i = k$. We also define $z_\lambda := \prod_i i^{m_i} m_i!$.

**Definition 1.2.** Let $S$ be a projective K3 surface. We fix integral bases 1 of $H^0(S,\mathbb{Z})$, $x$ of $H^4(X,\mathbb{Z})$ and $\alpha_1, \ldots, \alpha_{22}$ of $H^2(S,\mathbb{Z})$. The cup product induces a symmetric bilinear form $B_{H^2}$ on $H^2(X,\mathbb{Z})$. Written as a symmetric matrix with respect to this basis, $B_{H^2}$ looks like

$$B_{H^2} = \begin{pmatrix} U &&&& \\ & U &&& \\ && U && \\ &&& E & \\ &&&& E \end{pmatrix},$$

where $U$ stands for the intersection matrix of the hyperbolic lattice and $E$ stands for the negative matrix of the $E_8$ lattice, *i.e.*

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \end{pmatrix}.$$

We may extend $B_{H^2}$ to a symmetric non-degenerate bilinear form on $H^*(S,\mathbb{Z})$ by setting $B(1,1) = 0,\ B(1,\alpha_i) = 0,\ B(1,x) = 1,\ B(x,x) = 0$.

**Definition 1.3.** $B$ induces a form $B \otimes B$ on $\operatorname{Sym}^2 H^*(S,\mathbb{Z})$. So the cup-product

$$\mu : \operatorname{Sym}^2 H^*(S,\mathbb{Z}) \longrightarrow H^*(S,\mathbb{Z})$$

has an adjoint comultiplication $\Delta$ that is coassociative, given by:

$$\Delta : H^*(S,\mathbb{Z}) \longrightarrow \operatorname{Sym}^2 H^*(S,\mathbb{Z}), \quad \Delta = (B \otimes B)^{-1} \mu^T B$$

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The image of 1 under the composite map $\mu(\Delta(1)) = B(\Delta(1), \Delta(1)) = 24x$, denoted by $e$ is called the Euler Class.

We denote by $S^{[n]}$ the Hilbert scheme of $n$ points on $S$, *i.e.* the classifying space of all zero-dimensional closed subschemes of length $n$, which is smooth. A classical result by Nakajima gives an explicit description of $H^*(S^{[n]}, \mathbb{Q})$ in terms of creation operators $\mathfrak{q}_l(\beta)$, $\beta \in H^*(S, \mathbb{Q})$, acting on the direct sum $\bigoplus_n H^*(S^{[n]}, \mathbb{Q})$. An integral basis for $H^*(S^{[n]}, \mathbb{Z})$ in terms of Nakajima's operators was given by Qin–Wang:

**Theorem 1.4.** [6, Thm. 5.4.] *The classes*

$$\frac{1}{z_\lambda}\mathfrak{q}_\lambda(1)\mathfrak{q}_\mu(x)\mathfrak{m}_{\nu^1,\alpha_1}\dots\mathfrak{m}_{\nu^{22},\alpha_{22}}|0\rangle, \quad \|\lambda\| + \|\mu\| + \sum_{i=1}^{22}\|\nu^i\| = n$$

*form an integral basis for $H^*(S^{[n]}, \mathbb{Z})$. Here, $\lambda$, $\mu$, $\nu^i$ are partitions. The symbol $\mathfrak{q}$ stands for Nakajima's creation operator. The relation of $\mathfrak{m}_{\nu,\alpha}$ to $\mathfrak{q}_{\tilde{\nu}}(\alpha)$ is the same as the monomial symmetric functions $m_\nu$ to the power sum symmetric functions $p_{\tilde{\nu}}$.*

**Notation 1.5.** To enumerate the basis of $H^*(S^{[n]}, \mathbb{Z})$, we introduce the following abbreviation:

$$1^\lambda\alpha_1^{\nu_1}\dots\alpha_{22}^{\nu_{22}}x^\mu := \frac{1}{z_{\tilde{\lambda}}}\mathfrak{q}_{\tilde{\lambda}}(1)\mathfrak{q}_\mu(x)\mathfrak{m}_{\nu^1,\alpha_1}\dots\mathfrak{m}_{\nu^{22},\alpha_{22}}|0\rangle$$

where the partition $\tilde{\lambda}$ is built from $\lambda$ by appending sufficiently many Ones, such that $\|\tilde{\lambda}\| + \|\mu\| + \sum\|\nu^i\| = n$.

The ring structure of $H^*(S^{[n]}, \mathbb{Q})$ has been studied in [2], where an explicit algebraic model is constructed, which we recall briefly:

**Definition 1.6.** [2, Sect. 2] Let $\pi$ be a permutation of $n$ letters, written as a sum of disjoint cycles. To each cycle we may associate an element of $A := H^*(S, \mathbb{Q})$. This defines an element in $A^{\otimes m}$, $m$ being the number of cycles. So these mappings span a vector space over $\mathbb{Q}$. The space obtained by taking the direct sum over all $\pi \in S_n$ will be denoted by $A\{S_n\}$.

To define a ring structure, take two permutations $\pi, \tau$, together with mappings. The result of the multiplication will be the permutation $\pi\tau$, together with a mapping of cycles. Now, look first at the orbit space of the group of permutations $\langle\pi, \tau\rangle$, generated by $\pi$ and $\tau$. For each cycle of $\pi, \tau$ contained in one orbit $B$ of $\langle\pi, \tau\rangle$, multiply with the associated element of $A$. Also multiply with a certain power of the Euler class $e^g$. Afterwards, apply the comultiplication $\Delta$ repeatedly on the product to get a mapping from the cycles of $\pi\tau$ contained in $B$ to $A$. Here the "graph defect" $g$ is defined as follows: Let $u, v, w$ be the number of cycles contained in $B$ of $\pi$, $\tau$, $\pi\tau$, respectively. Then $g := \frac{1}{2}(|B| + 2 - u - v - w)$. Now follow this procedure for each orbit $B$.

The symmetric group $S_n$ acts on $A\{S_n\}$ by conjugation. This action preserves the ring structure. Therefore the space of invariants $A^{[n]} := (A\{S_n\})^{S_n}$ becomes a subring. The main theorem of Lehn and Sorger can now be stated:

**Theorem 1.7.** [2, Thm. 3.2.] *The following map is an isomorphism of rings:*

$$H^*(S^{[n]}, \mathbb{Q}) \longrightarrow A^{[n]}$$
$$\mathfrak{q}_{n_1}(\beta_1) \dots \mathfrak{q}_{n_k}(\beta_k)|0\rangle \longmapsto \sum_{\sigma \in S_n} \sigma a \sigma^{-1}$$

*with $n_1 + \ldots + n_k = n$ and $a \in A\{S_n\}$ corresponds to an arbitrary permutation with k cycles of lengths $n_1, \ldots, n_k$ that are associated to the classes $\beta_1, \ldots, \beta_k \in H^*(S, \mathbb{Q})$, respectively.*

Since $H^{\mathrm{odd}}(S^{[n]}, \mathbb{Z}) = 0$ and $H^{\mathrm{even}}(S^{[n]}, \mathbb{Z})$ is torsion-free by [3], we can apply these results to $H^*(S^{[n]}, \mathbb{Z})$ to determine the multiplicative structure of cohomology with integer coefficients.

## 2. Computational results

The algebra generated by classes of degree 2 is an interesting object to study. For cohomology with complex coefficients, Verbitsky has proven that the algebra generated by $H^2(X, \mathbb{C})$

*Remark* 2.1. Denote $h^k(S^{[n]})$ the rank of $H^k(S^{[n]}, \mathbb{Z})$. We have:

- $h^2(S^{[n]}) = 23$ for $n \geq 2$.
- $h^4(S^{[n]}) = 276,\ 299,\ 300$ for $n = 2, 3, \geq 4$ resp.
- $h^6(S^{[n]}) = 23,\ 2554,\ 2852,\ 2875,\ 2876$ for $n = 2, 3, 4, 5, \geq 6$ resp.

We now give some results in low degrees, obtained by computing multiplication matrices with respect to the integral basis and a then reducing to Smith normal form (both done by a computer).

**Proposition 2.2.** *Studying the image of* $\mathrm{Sym}^2 H^2$ *in* $H^4$*, we obtain:*

$$\frac{H^4(S^{[2]}, \mathbb{Z})}{\mathrm{Sym}^2 H^2(S^{[2]}, \mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23} \oplus \frac{\mathbb{Z}}{5\mathbb{Z}}$$

*This was already known to Boissière, Nieper-Wißkirchen and Sarti,* [**?**, Prop. 3].

$$\frac{H^4(S^{[3]}, \mathbb{Z})}{\mathrm{Sym}^2 H^2(S^{[3]}, \mathbb{Z})} \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 23}$$

*The torsion part of the quotient is generated by the integral class* $\frac{1}{3}\mathfrak{q}_{(3)}(1)|0\rangle$.

$$\frac{H^4(S^{[n]}, \mathbb{Z})}{\mathrm{Sym}^2 H^2(S^{[n]}, \mathbb{Z})} \cong \mathbb{Z}^{\oplus 24}, \quad \textit{for } n \geq 4.$$

*This was already proven by Markman,* [4, Thm. 1.10].

**Proposition 2.3.** *Comparing $H^2(S^{[n]},\mathbb{Z})\cup H^4(S^{[n]},\mathbb{Z})$ with $H^6(S^{[n]},\mathbb{Z})$, we obtain:*

$$\frac{H^6(S^{[2]},\mathbb{Z})}{H^2(S^{[2]},\mathbb{Z})\cup H^4(S^{[2]},\mathbb{Z})} = 0 \tag{1}$$

$$\frac{H^6(S^{[3]},\mathbb{Z})}{H^2(S^{[3]},\mathbb{Z})\cup H^4(S^{[3]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12} \tag{2}$$

$$\frac{H^6(S^{[4]},\mathbb{Z})}{H^2(S^{[4]},\mathbb{Z})\cup H^4(S^{[4]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23} \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12} \tag{3}$$

$$\frac{H^6(S^{[5]},\mathbb{Z})}{H^2(S^{[5]},\mathbb{Z})\cup H^4(S^{[5]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23} \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12} \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}}\right)^{\oplus 3} \tag{4}$$

$$\frac{H^6(S^{[n]},\mathbb{Z})}{H^2(S^{[n]},\mathbb{Z})\cup H^4(S^{[n]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 22} \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12} \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}}\right)^{\oplus 2} \oplus \mathbb{Z},\ n \geq 6. \tag{5}$$

- *The 3-torsion part is generated by the 12 integral classes $\alpha_i^{(1,1,1)} \in H^6$, where $i = 1,2,3,4,5,6,8,9,11,16,17,19$.*
- *The 2-torsion part is generated by the 22 integral classes $\alpha_i^{(1,1,1)} + \alpha_i^{(2,1)} + \alpha_i^{(3)} + 1^{(2)}\alpha_i^{(1,1)} + 1^{(3)}\alpha_i^{(1)}$, $i = 1,\dots,22$ and, in the cases $n = 4,5$, by the integral class $1^{(4)} \in H^6$.*
- *The 5-torsion part is generated by the 2 integral classes $\alpha_i^{(1,1,1)} + 2\alpha_i^{(2,1)} + 3\alpha_i^{(3)} + 4\cdot 1^{(2)}\alpha_i^{(1,1)} + 2\cdot 1^{(2)}\alpha_i^{(2)} + 2\cdot 1^{(3)}\alpha_i^{(1)} + 3\cdot 1^{(2,2)}\alpha_i^{(1)}$, $i = 13, 21$ and, in the case $n = 5$, by the integral class $3\cdot 1^{(4)} + 3\cdot 1^{(3,2)}$.*
- *The free summand is generated by the class $3\cdot 1^{(4)} - 12\cdot 1^{(3,2)} + 10\cdot 1^{(2,2,2)}$.*

**Proposition 2.4.**

$$\frac{H^6(S^{[2]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[2]},\mathbb{Z})} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$$

*The quotient is generated by the integral class $1^{(2)}$.*

$$\frac{H^6(S^{[3]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[3]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 230} \oplus \left(\frac{\mathbb{Z}}{36\mathbb{Z}}\right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{72\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 507}$$

$$\frac{H^6(S^{[4]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[4]},\mathbb{Z})} \cong$$

$$\frac{H^6(S^{[5]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[5]},\mathbb{Z})} \cong$$

$$\frac{H^6(S^{[n]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[n]},\mathbb{Z})} \cong n \geq 6.$$

## References


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