THE ABSENCE OF COMPLEX SPHERES

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ABSTRACT. This is the written version of a talk given at the KAUS conference in Göteborg in January of 2011. We provide a bird's eye view of the problem of making even dimensional spheres into complex manifolds.

1. THE LEGO OF GEOMETRY

As mathematicians discovered in the mid-19th century there are three basic types of geometries: Euclidean, hyperbolic and projective. Roughly speaking these types are characterized by the number of lines which are parallel to a given line and pass through the same point. This numer is one, infinite and none, respectively.

We can build models of each of these geometries in both real and complex differential geometry. These models serve as the fundamental examples of differential manifolds – as the Lego blocks of geometry. These models are:

	Euclidean	hyperbolic	projective
$\mathbb{R}$	$\mathbb{R}^n$ with the flat metric g_{flat}	$H^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ with the hyperbolic metric g_{hyp}	$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ with the standard round metric g_{round}
$\mathbb{C}$	$\mathbb{C}^n$ with the flat metric h_{flat}	$B^n = \{z \in \mathbb{C}^n \mid \|z\| < 1\}$ with the Bergman metric h_{Berg}	$\mathbb{P}^n$ with the Fubini-Study metric h_{FS}

Note that there is a certain asymmetry in this list: while the Euclidean and hyperbolic models are the same on the real and complex sides, then the projective models differ:

Proposition. *The sphere S^{2n} is topologically different from $\mathbb{P}^n$ if $n \geq 2$.*

Proof: We compare their cohomology groups, which are topological invariants. On one hand we have

$$H^i(S^{2n}, \mathbb{R}) = \begin{cases} \mathbb{R}, & \text{if } i = 0 \text{ or } i = 2n \\ 0, & \text{otherwise} \end{cases}$$

but on the other

$$H^i(\mathbb{P}^n, \mathbb{R}) = \begin{cases} \mathbb{R}, & \text{if } i \text{ is even} \\ 0, & \text{if } i \text{ is odd.} \end{cases}$$

So if $n \geq 2$ then $H^2(S^{2n}, \mathbb{R}) = 0 \neq \mathbb{R} = H^2(\mathbb{P}^n, \mathbb{R})$. □

Remark — Even the real projective spaces differ, as $\mathbb{RP}^{2n}$ is topologically different from $\mathbb{CP}^n$. For one their fundamental groups differ, but the real projective space is actually non-orientable for n even, while any complex manifold is orientable.

This disrepancy in the projective Lego blocks is interesting, and naturally raises:

Question. Where are the complex spheres?

2. Complex spheres

Given a necessarily even dimensional sphere S^{2n} we want to know if it can be made into a complex manifold. At least one case is familiar to us:

Proposition. *The sphere S^2 can be made into a complex manifold, which is necessarily biholomorphic to $\mathbb{P}^1$.*

Proof: Use the stethoscopic projection to define a complex structure on the upper and lower hemispheres of S^2. The associated chart transition function is

$$\begin{cases} \mathbb{C}^* & \longrightarrow & \mathbb{C}^* \\ z & \longmapsto & 1/z \end{cases}$$

which is holomorphic, so S^2 is a complex manifold.

The genus of this manifold is 0, so it admits a non-constant meromorphic map $f : S^2 \to \mathbb{P}^1$ with a unique simple pole. Such a map is necessarily an isomorphism. □

The sphere S^2 with its complex structure is of course just the Riemann sphere, or the one point compactification $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Now, how about the next case, the one of S^4?

Proposition. *The sphere S^4 cannot be made into a complex manifold.*

Proof: Suppose that it did. The its first Chern class would be $c_1 = 0$, as it lives in $H^2(S^4, \mathbb{Z}) = 0$. The second Chern class is then determined by

$$\int_{S^4} c_2 = \sum_{k=0}^{4} (-1)^k h^k(S^4, \mathbb{Z}) = 1 - 0 + 0 - 0 + 1 = 2.$$

The Chern character of the manifold is then

$$Ch(S^4) = 2 - c_2$$

and the Todd class is

$$Td(S^4) = 1 + \frac{1}{12} c_2.$$

The Riemann-Roch theorem then gives

$$\chi(O_{S^4}) = \int_{S^4} Ch(S^4)\, Td(S^4) = \int_{S^4} \left(\frac{1}{6} c_2 - c_2\right) = -\frac{5}{3},$$

which is absurd as the holomorphic Euler characteristic $\chi(O_{S^4})$ is an integer. □

Similar ideas exploiting links between the complex structure and topology of a manifold permit to show that:

Theorem (Borel-Serre, 1953)**.** *The only spheres that can possibly be made into complex manifolds are S^2, S^4 and S^6.*

In fact Borel and Serre prove that these spheres are the only ones that can possibly admit an *almost complex structure* – morally speaking such a thing defines multiplication by i on the tangent bundle of the manifold – which is a necessary condition for being a complex manifold.

3. The six-sphere

By Borel-Serre and the results of the last section, the only place a complex sphere can hide is in six real dimensions. Of course we could call our search off if we knew that S^6 admits no almost complex structure, but in fact one can be constrcutred by a clever use of the octonions.

Recall that the octonions are the eight dimensional real algebra $\mathbb{O}$ spanned by the elements $1, e_1, \ldots, e_7$, where $e_j^2 = -1$ for all j. We will not attempt to describe the various products $e_j e_k$ here, but instead suggest the reader familiarize himself with the Fano plane mnenonic. The octonions form a non-commutative, non-associative algebra, which may be equipped with an inner product and a norm. We introduce the convention that if $x = x_0 + x_1 e_1 + \ldots + x_7 e_7$ is an element of $\mathbb{O}$, then the real part of x is $\operatorname{Re} x := x_0$.

Proposition. *The six-sphere may be identified with the set* $\{x \in \mathbb{O} \,|\, x^2 = -1\}$.

Proof: If $x^2 = -1$, then $x\bar{x} = \|x^2\| = 1$, and cancellation gives $\bar{x} = -x$. It follows that the real part $x_0 = 0$, so x is an element of the space $\{x \in \mathbb{O} \,|\, \operatorname{Re} x = 0\}$. This is a seven-dimensional real vector space.

The equation $\|x\|^2 = 1$ now simply reads

$$x_1^2 + \ldots + x_7^2 = 1,$$

which is nothing but the defining equation for S^6 in $\mathbb{R}^7$. □

Proposition. *The sphere S^6 admits an almost complex structure.*

Proof: First we determine the tangent space of the six-sphere. We use the above to calculate that

$$T_{S^6,x} = \{h \in \mathbb{O} \,|\, \langle x, h \rangle = hx + xh = 0\}.$$

Now we simply define the desired structure pointwise by

$$\begin{cases} J(x) : T_{S^6,x} & \longrightarrow & T_{S^6,x} \\ \qquad\quad h & \longmapsto & xh \end{cases}.$$

This gives a well defined smooth endomorphism of T_{S^6}, which satisfies $J^2 = -\operatorname{id}$ because $x^2 = -1$. □

We can now make the tangent bundle of S^6 into a complex vector bundle by defining $i\xi = J(\xi)$.

An almost complex structure defines a complex structure on a manifold, and thus a complex manifold, if and only if it satisfies an integrability condition. Morally, this integrability condition measures how far the transition functions of the manifold are from being holomorphic. This condition is satisfied if and only if a certain tensor, the Nijenhuis tensor of the almost complex structure, vanishes. One can calculate that this tensor does not vanish in the above case, so the almost complex structure we have defined is not integrable.

This still leaves open the question of whether some other almost complex structure on S^6 is integrable. At the time of writing, this is still an open question[1].

4. Exotic projective spaces

So what would happen if the sphere S^6 were a complex manifold? Various people have looked into this, for example we have:

Theorem ([CDP98]). *If S^6 is a complex manifold then it admits no non-constant meromorphic functions.*

One interpretation of this theorem is that S^6 would be very far from being an algebraic manifold. It would also be very resistant to trancedental methods, because as $H^2(S^6, \mathbb{R}) = 0$ then S^6 cannot be a Kähler manifold.

Somewhat more troubling is what the existence of a complex structure on S^6 would lead to. Suppose that such a thing exists, and denote by $\varepsilon : X \to S^6$ the blow-up of S^6 in a point.

Proposition (Hirzebruch, 1954). *The manifold X is an exotic projective space, i.e. diffeomorphic, but not biholomorphic, to the projective space $\mathbb{P}^n$.*

Proof: Diffeomorphically X is the connected sum of S^6 and $\mathbb{P}^3$ with its orientation reversed. Spheres act as neutral elements for the connected sum, and in odd dimensions $\mathbb{P}^n$ is diffeomorphic to itself with its orientation reversed via conjugation of its coordinates, so

$$X \simeq_{C^\infty} S^6 \,\#\, \overline{\mathbb{P}^3} \simeq_{C^\infty} \mathbb{P}^3$$

as promised.

The first Chern class of $\mathbb{P}^3$ is $4h$, where $h = c_1(\mathcal{O}_{\mathbb{P}^3}(1))$, and S^6 has zero first Chern class as $H^6(S^6, \mathbb{Z}) = 0$. It follows (see [GH94, pp. 608]) that

$$c_1(X) = \varepsilon^* c_1(S^6) - 2h = -2h,$$

so X and $\mathbb{P}^3$ have different Chern classes, hence they cannot be biholomorphic. □

The situation is even more violent that this, as the exotic projective space X cannot be Kähler; a result of Kodaira and Yau says that any Kähler manifold which is homeomorphic to $\mathbb{P}^n$ is actually $\mathbb{P}^n$.

5. Why is this problem difficult?

It is not without reason that the problem of the existence of a complex structure J on S^6 has been open for about 60 years. If such a thing exists then:

- The sphere S^6 admits no meromorphic functions and is not Kähler, thus rendering useless most modern complex and algebro-geometric tools.
- The structure J cannot be compatible with the standard round metric on S^6, which poses problems for any differential geometric approach.

[1] G. Etesi [Ete11] has a preprint where he claims to answer this question positively. His article has yet to be peer-reviewed.

- The fundamental groups of the sphere are $\pi_i(S^6) = 0$ for $0 < i < 6$, so the problem cannot be reduced to a simpler one by realizing S^6 as the total space of a fibration over a smaller manifold.
- The holomorphic automorphism group $\mathrm{Aut}_{\mathcal{O}_{S^6}} S^6$ cannot act transitively on S^6.

The moral is that this problem is very trancendental, asymmetric and inhomogeneous. Any problem with one of these characteristics is hard by modern standards; a problem possessing all three would seem to require a very delicate approach.

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