MULTIPLICATIVE STRUCTURE OF INTEGER COHOMOLOGY OF HILBERT SCHEMES OF POINTS ON K3 SURFACES

SIMON KAPFER

Abstract. We study the cokernels of the cup product in integer cohomology of the Hilbert scheme of n points on a K3 surface. The main difference to cohomology with rational coefficients is the presence of torsion, depending on n. We particularly discuss the case $n = 3$.

1. Preliminaries

Let S be a K3 surface. We fix integral bases 1 of $H^0(S,\mathbb{Z})$, x of $H^4(X,\mathbb{Z})$ and $\alpha_1,\dots,\alpha_{22}$ of $H^2(S,\mathbb{Z})$ such that the cup product pairing on $H^2(X,\mathbb{Z})$, written as a symmetric matrix with respect to this basis, looks like

$$\begin{pmatrix} U & & & & \\ & U & & & \\ & & U & & \\ & & & E & \\ & & & & E \end{pmatrix},$$

where U stands for the intersection matrix of the hyperbolic lattice and E stands for the negative matrix of the E_8 lattice, *i.e.*

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \end{pmatrix}.$$

We denote by $S^{[n]}$ the Hilbert scheme of n points on S. An integral basis for $H^*(S^{[n]},\mathbb{Z})$ in terms of Nakajima's operators was given by Qin–Wang:

Theorem 1.1. [**?**, Thm. 5.4.] *The classes*

$$\frac{1}{z_\lambda}\mathfrak{q}_\lambda(1)\mathfrak{q}_\mu(x)\mathfrak{m}_{\nu^1,\alpha_1}\dots\mathfrak{m}_{\nu^{22},\alpha_{22}}|0\rangle, \quad \|\lambda\| + \|\mu\| + \sum_{i=1}^{22}\|\nu^i\| = n$$

form an integral basis for $H^*(S^{[n]},\mathbb{Z})$. Here, λ, μ, ν^i are partitions, $\|\cdot\|$ means the weight of a partition i.e. $\|\lambda\| = \sum_i m_i i$ and $z_\lambda := \prod_i i^{m_i} m_i!$, if $\lambda = (1^{m_1}, 2^{m_2}, \dots)$. The symbol $\mathfrak{q}$ stands for Nakajima's creation operator. The relation of $\mathfrak{m}_{\nu,\alpha}$ to $\mathfrak{q}_{\tilde{\nu}}(\alpha)$

Date: August 25, 2014.

is the same as the monomial symmetric functions m_ν to the power sum symmetric functions $p_{\tilde{\nu}}$.

2. Computational results

The ring structure of $H^*(S^{[n]}, \mathbb{Q})$ has been studied in [?]. Since $H^{\mathrm{odd}}(S^{[n]}, \mathbb{Z}) = 0$ and $H^{\mathrm{even}}(S^{[n]}, \mathbb{Z})$ is torsion-free by [?], we can also apply these results to $H^*(S^{[n]}, \mathbb{Z})$. A basis for cohomology with integer coefficients was given by Qin–Wang in [?] which allows us to compute explicitly the image of $H^k(S^{[n]}, \mathbb{Z}) \cup H^l(S^{[n]}, \mathbb{Z})$ in $H^{k+l}(S^{[n]}, \mathbb{Z})$. We obtain:

Proposition 2.1. *If X is deformation equivalent to $S^{[3]}$, then:*

$$\frac{H^4(X,\mathbb{Z})}{\mathrm{Sym}^2 H^2(X,\mathbb{Z})} \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 23}$$

The torsion part of the quotient is generated by the integral class $\frac{1}{3}\mathfrak{q}_{(3)}(1)|0\rangle$.

$$\frac{H^6(X,\mathbb{Z})}{H^2(X,\mathbb{Z}) \cup H^4(X,\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12}$$

This quotient is generated by the 12 integral classes $\mathfrak{m}_{(1^3),\alpha_i}|0\rangle$, where $i \in \{1,2,3,4,5,6,8,9,11,16,17,19\}$.

Proposition 2.2.

$$\frac{H^6(S^{[4]},\mathbb{Z})}{H^2(S^{[4]},\mathbb{Z}) \cup H^4(S^{[4]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23} \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12}$$

$$\frac{H^6(S^{[5]},\mathbb{Z})}{H^2(S^{[5]},\mathbb{Z}) \cup H^4(S^{[5]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23} \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12} \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}}\right)^{\oplus 3}$$

$$\frac{H^6(S^{[6]},\mathbb{Z})}{H^2(S^{[6]},\mathbb{Z}) \cup H^4(S^{[6]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23} \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12} \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}}\right)^{\oplus 2} \oplus \mathbb{Z}$$

The free summand is generated by $\left[\frac{10}{48}\mathfrak{q}_{(2^3)}(1) - \frac{12}{6}\mathfrak{q}_{(3,2,1)}(1) + \frac{3}{8}\mathfrak{q}_{(4,1^2)}(1)\right]|0\rangle$.

Proposition 2.3.

$$\frac{H^6(S^{[3]},\mathbb{Z})}{\mathrm{Sym}^3 H^2(S^{[3]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 243} \oplus \left(\frac{\mathbb{Z}}{4\mathbb{Z}}\right)^{\oplus 10} \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 3} \oplus \mathbb{Z}^{\oplus 507}$$

$$\frac{H^6(S^{[4]},\mathbb{Z})}{\mathrm{Sym}^3 H^2(S^{[4]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23} \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 2} \oplus \mathbb{Z}^{\oplus 575}$$

$$\frac{H^6(S^{[5]},\mathbb{Z})}{\mathrm{Sym}^3 H^2(S^{[5]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 22} \oplus \mathbb{Z}^{\oplus 597}$$

$$\frac{H^6(S^{[n]},\mathbb{Z})}{\mathrm{Sym}^3 H^2(S^{[n]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 22} \oplus \mathbb{Z}^{\oplus 598} \quad \textit{for } n \geq 6.$$

Simon Kapfer, Lehrstuhl für Algebra und Zahlentheorie, Universitätsstrasse 14, D-86159 Augsburg

E-mail address: simon.kapfer@math.uni-augsburg.de