

# COMPUTING CUP-PRODUCTS IN INTEGER COHOMOLOGY OF HILBERT SCHEMES OF POINTS ON K3 SURFACES

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ABSTRACT. We study the images of cup products in integer cohomology of the Hilbert scheme of  $n$  points on a K3 surface.

## 1. PRELIMINARIES

**Definition 1.1.** Let  $n$  be a natural number. A partition of  $n$  is a sequence  $\lambda = (\lambda_1 \geq \dots \geq \lambda_k > 0)$  of natural numbers such that  $\sum_i \lambda_i = n$ . It is convenient to write  $\lambda = (1^{m_1}, 2^{m_2}, \dots)$  as a sequence of multiplicities. We define the weight  $\|\lambda\| := \sum m_i i = n$  and the length  $|\lambda| := \sum_i m_i = k$ . We also define  $z_\lambda := \prod_i i^{m_i} m_i!$ .

**Definition 1.2.** Let  $\Lambda$  be the ring of symmetric functions. Let  $m_\lambda$  and  $p_\lambda$  denote the monomial and the power sum symmetric functions. They are both indexed by partitions and form a basis of  $\Lambda$ . They are linearly related by  $p_\lambda = \sum_\mu \psi_{\lambda\mu} m_\mu$ , the sum being over partitions with the same weight as  $\lambda$ . It turns out that the base change matrix  $(\psi_{\lambda\mu})$  has integral entries, but its inverse  $(\psi_{\lambda\mu}^{-1})$  has not. For example,  $p_{(2,1,1)} = 2m_{(2,1,1)} + 2m_{(2,2)} + 2m_{(3,1)} + m_{(4)}$  but  $m_{(2,1,1)} = \frac{1}{2}p_{(2,1,1)} - \frac{1}{2}m_{(2,2)} - p_{(3,1)} + p_{(4)}$ . A method to determine the coefficients  $(\psi_{\lambda\mu})$  is given in [2, Sect. 3.7].

**Definition 1.3.** For our purposes, a lattice  $L$  is a free  $\mathbb{Z}$ -module of finite rank, equipped with a non-degenerate symmetric integral bilinear form  $B$ . The lattice  $L$  is called odd, if there exists a  $v \in L$ , such that  $B(v, v)$  is odd, otherwise it is called even. Choosing a base  $\{e_i\}_i$  of our lattice, we can write  $B$  as a symmetric matrix.  $L$  is called unimodular, if the matrix  $B$  has determinant  $\pm 1$ . The difference between the number of positive eigenvalues and the number of negative eigenvalues of  $B$  (regarded over  $\mathbb{R}$ ) is called the signature. If  $B$  has both positive and negative eigenvalues, the lattice is called indefinite. For a classification theorem for indefinite lattices, see [6].

**Definition 1.4.** Let  $S$  be a projective K3 surface. We fix integral bases 1 of  $H^0(S, \mathbb{Z})$ ,  $x$  of  $H^4(S, \mathbb{Z})$  and  $\alpha_1, \dots, \alpha_{22}$  of  $H^2(S, \mathbb{Z})$ . The cup product induces a symmetric bilinear form  $B_{H^2}$  on  $H^2(S, \mathbb{Z})$  and thus the structure of a unimodular lattice. We may extend  $B_{H^2}$  to a symmetric non-degenerate bilinear form  $B$  on  $H^*(S, \mathbb{Z})$  by setting  $B(1, 1) = 0$ ,  $B(1, \alpha_i) = 0$ ,  $B(1, x) = 1$ ,  $B(x, x) = 0$ .

It turns out that  $H^2(S, \mathbb{Z})$  has signature -16 and, by the classification theorem for indefinite unimodular lattices, is isomorphic to  $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ , i.e. three times the hyperbolic lattice and two times the negative  $E_8$  lattice.

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*Date:* October 8, 2014.

**Definition 1.5.**  $B$  induces a form  $B \otimes B$  on  $\text{Sym}^2 H^*(S, \mathbb{Z})$ . So the cup-product

$$\mu : \text{Sym}^2 H^*(S, \mathbb{Z}) \longrightarrow H^*(S, \mathbb{Z})$$

induces an adjoint comultiplication  $\Delta$  that is coassociative, given by:

$$\Delta : H^*(S, \mathbb{Z}) \longrightarrow \text{Sym}^2 H^*(S, \mathbb{Z}), \quad \Delta = (B \otimes B)^{-1} \mu^T B$$

Note that this does not define a bialgebra structure. The image of 1 under the composite map  $\mu(\Delta(1)) = B(\Delta(1), \Delta(1)) = 24x$ , denoted by  $e$  is called the Euler Class.

We denote by  $S^{[n]}$  the Hilbert scheme of  $n$  points on  $S$ , *i.e.* the classifying space of all zero-dimensional closed subschemes of length  $n$ .  $S^{[0]}$  consists of a single point and  $S^{[1]} = S$ . Fogarty proved, that the Hilbert scheme is a smooth variety. A theorem by Nakajima gives an explicit description of the vector space structure of  $H^*(S^{[n]}, \mathbb{Q})$  in terms of creation operators

$$\mathbf{q}_l(\beta) : H^*(S^{[n]}, \mathbb{Q}) \longrightarrow H^{*+k+l-1}(S^{[n+l]}, \mathbb{Q}),$$

where  $\beta \in H^k(S, \mathbb{Q})$ , acting on the direct sum  $\mathbb{H} := \bigoplus_n H^*(S^{[n]}, \mathbb{Q})$ . The operators  $\mathbf{q}_l(\beta)$  are linear, commute with each other, and the images of the vacuum vector  $|0\rangle$ , defined as the generator of  $H^0(S^{[0]}, \mathbb{Q}) \cong \mathbb{Q}$ , under the polynomial algebra generated by the creation operators span  $\mathbb{H}$  as a vector space. It is convenient to abbreviate  $\mathbf{q}_{l_1}(\beta) \dots \mathbf{q}_{l_k}(\beta) =: \mathbf{q}_\lambda(\beta)$ , where the partition  $\lambda$  is composed by the  $l_i$ .

An integral basis for  $H^*(S^{[n]}, \mathbb{Z})$  in terms of Nakajima's operators was given by Qin–Wang:

**Theorem 1.6.** [8, Thm. 5.4.] *Let  $\mathbf{m}_{\nu, \alpha} := \sum_\rho \psi_{\nu\rho}^{-1} \mathbf{q}(\alpha)$ , with coefficients  $\psi_{\nu\rho}^{-1}$  as in Definition 1.2. The classes*

$$\frac{1}{z_\lambda} \mathbf{q}_\lambda(1) \mathbf{q}_\mu(x) \mathbf{m}_{\nu^1, \alpha_1} \dots \mathbf{m}_{\nu^{22}, \alpha_{22}} |0\rangle, \quad \|\lambda\| + \|\mu\| + \sum_{i=1}^{22} \|\nu^i\| = n$$

*form an integral basis for  $H^*(S^{[n]}, \mathbb{Z})$ . Here,  $\lambda, \mu, \nu^i$  are partitions.*

**Notation 1.7.** To enumerate the basis of  $H^*(S^{[n]}, \mathbb{Z})$ , we introduce the following abbreviation:

$$\boldsymbol{\alpha}^\lambda := 1^{\lambda^0} \alpha_1^{\lambda^1} \dots \alpha_{22}^{\lambda^{22}} x^{\lambda^{23}} := \frac{1}{z_{\widetilde{\lambda^0}}} \mathbf{q}_{\widetilde{\lambda^0}}(1) \mathbf{q}_{\lambda^{23}}(x) \mathbf{m}_{\lambda^1, \alpha_1} \dots \mathbf{m}_{\lambda^{22}, \alpha_{22}} |0\rangle$$

where the partition  $\widetilde{\lambda^0}$  is built from  $\lambda^0$  by appending sufficiently many Ones, such that  $\|\widetilde{\lambda^0}\| + \sum_{i \geq 1} \|\lambda^i\| = n$ . If  $\sum_{n \geq 0} \|\lambda^i\| > n$ , we put  $\boldsymbol{\alpha}^\lambda = 0$ . Thus we can interpret  $\boldsymbol{\alpha}^\lambda$  as an element of  $H^*(S^{[n]}, \mathbb{Z})$  for arbitrary  $n$ . We say that the symbol  $\boldsymbol{\alpha}^\lambda$  is reduced, if  $\lambda^0$  contains no Ones. We define also  $\|\lambda\| := \sum_{n \geq 0} \|\lambda^i\|$ .

**Lemma 1.8.** *Let  $\boldsymbol{\alpha}^\lambda$  represent a class of cohomological degree  $2k$ . If  $\boldsymbol{\alpha}^\lambda$  is reduced, then  $\frac{k}{2} \leq \|\lambda\| \leq 2k$ .*

*Proof.* This is a simple combinatorial observation. The lower bound is witnessed by  $x^{(\frac{k}{2})}$  (if  $k$  is even) and the upper bound is witnessed by  $1^{(2^k)}$ .  $\square$

The ring structure of  $H^*(S^{[n]}, \mathbb{Q})$  has been studied by Lehn and Sorger in [3], where an explicit algebraic model is constructed, which we recall briefly:

**Definition 1.9.** [3, Sect. 2] Let  $\pi$  be a permutation of  $n$  letters, written as a sum of disjoint cycles. To each cycle we may associate an element of  $A := H^*(S, \mathbb{Q})$ . This defines an element in  $A^{\otimes m}$ ,  $m$  being the number of cycles. So these mappings span a vector space over  $\mathbb{Q}$ . The space obtained by taking the direct sum over all  $\pi \in S_n$  will be denoted by  $A\{S_n\}$ .

To define a ring structure, take two permutations  $\pi, \tau$ , together with mappings. The result of the multiplication will be the permutation  $\pi\tau$ , together with a mapping of cycles. To construct the mappings to  $A$ , look first at the orbit space of the group of permutations  $\langle \pi, \tau \rangle$ , generated by  $\pi$  and  $\tau$ . For each cycle of  $\pi, \tau$  contained in one orbit  $B$  of  $\langle \pi, \tau \rangle$ , multiply with the associated element of  $A$ . Also multiply with a certain power of the Euler class  $e^g$ . Afterwards, apply the comultiplication  $\Delta$  repeatedly on the product to get a mapping from the cycles of  $\pi\tau$  contained in  $B$  to  $A$ . Here the "graph defect"  $g$  is defined as follows: Let  $u, v, w$  be the number of cycles contained in  $B$  of  $\pi, \tau, \pi\tau$ , respectively. Then  $g := \frac{1}{2}(|B| + 2 - u - v - w)$ . Now follow this procedure for each orbit  $B$ .

The symmetric group  $S_n$  acts on  $A\{S_n\}$  by conjugation. This action preserves the ring structure. Therefore the space of invariants  $A^{[n]} := (A\{S_n\})^{S_n}$  becomes a subring. The main theorem of [3] can now be stated:

**Theorem 1.10.** [3, Thm. 3.2.] *The following map is an isomorphism of rings:*

$$H^*(S^{[n]}, \mathbb{Q}) \longrightarrow A^{[n]}$$

$$\mathbf{q}_{n_1}(\beta_1) \dots \mathbf{q}_{n_k}(\beta_k) | 0 \rangle \longmapsto \sum_{\sigma \in S_n} \sigma a \sigma^{-1}$$

with  $n_1 + \dots + n_k = n$  and  $a \in A\{S_n\}$  corresponds to an arbitrary permutation with  $k$  cycles of lengths  $n_1, \dots, n_k$  that are associated to the classes  $\beta_1, \dots, \beta_k \in H^*(S, \mathbb{Q})$ , respectively.

Since  $H^{\text{odd}}(S^{[n]}, \mathbb{Z}) = 0$  and  $H^{\text{even}}(S^{[n]}, \mathbb{Z})$  is torsion-free by [4], we can apply these results to  $H^*(S^{[n]}, \mathbb{Z})$  to determine the multiplicative structure of cohomology with integer coefficients. It turns out, that it is somehow independent of  $n$ . More precisely, we have the following stability theorem, derived from Li, Qin and Wang:

**Theorem 1.11.** [8, Thm. 2.1] *Let  $Q_1, \dots, Q_s$  be products of creation operators, i.e.  $Q_i = \prod_j \mathbf{q}_{\lambda_{i,j}}(\beta_{i,j})$  for some partitions  $\lambda_{i,j}$  and classes  $\beta_{i,j} \in H^*(S, \mathbb{Z})$ . Set  $n_i := \sum_j \|\lambda_{i,j}\|$ . Then the cup product  $\prod_{i=1}^s \left( \frac{1}{(n-n_i)!} \mathbf{q}_{n-n_i}(1) Q_i | 0 \rangle \right)$  is equal to a finite linear combination of classes of the form  $\frac{1}{(n-m)!} \mathbf{q}_{n-m}(1) \prod_j \mathbf{q}_{\mu_j}(\gamma_j) | 0 \rangle$ , with  $\gamma \in H^*(S, \mathbb{Z})$ ,  $m = \sum_j \|\mu_j\|$ , whose coefficients are independent of  $n$ . We have the upper bound  $m \leq \sum_i n_i$ .*

**Corollary 1.12.** *Let  $\alpha^\lambda, \alpha^\mu, \alpha^\nu$  be reduced. Assume  $n \geq \|\lambda\|, \|\mu\|$ . Then the coefficients  $c_\nu^{\lambda\mu}$  of the cup product in  $H^*(S^{[n]}, \mathbb{Z})$*

$$\alpha^\lambda \smile \alpha^\mu = \sum_\nu c_\nu^{\lambda\mu} \alpha^\nu$$

*are polynomials in  $n$  of degree at most  $\|\lambda\| + \|\mu\| - \|\nu\|$ .*

*Proof.* Set  $Q_\lambda := \mathbf{q}_{\lambda^0}(1) \mathbf{q}_{\lambda^{23}}(x) \prod_{1 \leq j \leq 22} \mathbf{q}_{\lambda^j}(\alpha_j)$  and  $n_\lambda := \|\lambda\|$ . Then we have:  $\alpha^\lambda = \frac{1}{(n-n_\lambda)! z_{\lambda^0}} \mathbf{q}_{n-n_\lambda}(1) Q_\lambda | 0 \rangle$  and  $\alpha^\mu = \frac{1}{(n-n_\mu)! z_{\mu^0}} \mathbf{q}_{n-n_\mu}(1) Q_\mu | 0 \rangle$ . Thus the coefficient  $c_\nu^{\lambda\mu}$  in the product expansion is a constant, which depends on  $\|\lambda\|, \|\mu\|, \|\nu\|$ ,

but not on  $n$ , multiplied with  $\frac{(n-n_\nu)!}{(n-m)!}$  for a certain  $m \leq n_\lambda + n_\mu$ . This is a polynomial of degree  $m - n_\nu \leq n_\lambda + n_\mu - n_\nu = \|\lambda\| + \|\mu\| - \|\nu\|$ .  $\square$

*Remark 1.13.* The above condition,  $n \geq \|\lambda\|, \|\mu\|$ , seems to be unnecessary. This implies, that if  $n < \|\lambda\|$  or  $n < \|\mu\|$  but  $n \geq \|\nu\|$ , the polynomial  $c_\nu^{\lambda\mu}$  is zero at  $n$ .

*Example 1.14.* We now give some explicit examples for illustration.

- (1)  $1^{(2,2)} \smile \alpha_i^{(2)} = 2 \cdot 1^{(2)} \alpha_i^{(1)} x^{(1)} + 1^{(2,2)} \alpha_i^{(2)} + 2 \cdot 1^{(2)} \alpha_i^{(3)} + \alpha_i^{(4)}$
- (2) Let  $i, j$  be indices, such that  $B(\alpha_i, \alpha_j) = 1$ ,  $B(\alpha_i, \alpha_i) = 0 = B(\alpha_j, \alpha_j)$  and let  $k \geq 0$ . Set  $\alpha^\lambda := \alpha_i^{(1)} \alpha_j^{(1)} x^{(1^k)}$  and  $\alpha^\nu := x^{(1^{2k})}$ . Then  $c_\nu^{\lambda\lambda} = 1$ .

## 2. COMPUTATIONAL RESULTS

We now give some results in low degrees, obtained by computing multiplication matrices with respect to the integral basis and a then reducing to Smith normal form (both done by a computer).

*Remark 2.1.* Denote  $h^k(S^{[n]})$  the rank of  $H^k(S^{[n]}, \mathbb{Z})$ . We have:

- $h^2(S^{[n]}) = 23$  for  $n \geq 2$ .
- $h^4(S^{[n]}) = 276, 299, 300$  for  $n = 2, 3, \geq 4$  resp.
- $h^6(S^{[n]}) = 23, 2554, 2852, 2875, 2876$  for  $n = 2, 3, 4, 5, \geq 6$  resp.

The algebra generated by classes of degree 2 is an interesting object to study. For cohomology with complex coefficients, Verbitsky has proven in [9] that the cup product mapping from  $\text{Sym}^k H^2(S^{[n]}, \mathbb{C})$  to  $H^{2k}(S^{[n]}, \mathbb{C})$  is injective for  $k \leq n$ . One concludes that this also holds for integral coefficients.

**Proposition 2.2.** *We identify  $\text{Sym}^2 H^2(S^{[n]}, \mathbb{Z})$  with its image in  $H^4(S^{[n]}, \mathbb{Z})$  under the cup product mapping. Then:*

$$\begin{aligned}
 (1) \quad & \frac{H^4(S^{[2]}, \mathbb{Z})}{\text{Sym}^2 H^2(S^{[2]}, \mathbb{Z})} \cong \left( \frac{\mathbb{Z}}{2\mathbb{Z}} \right)^{\oplus 23} \oplus \frac{\mathbb{Z}}{5\mathbb{Z}}, \\
 (2) \quad & \frac{H^4(S^{[3]}, \mathbb{Z})}{\text{Sym}^2 H^2(S^{[3]}, \mathbb{Z})} \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 23}, \\
 (3) \quad & \frac{H^4(S^{[n]}, \mathbb{Z})}{\text{Sym}^2 H^2(S^{[n]}, \mathbb{Z})} \cong \mathbb{Z}^{\oplus 24}, \quad \text{for } n \geq 4.
 \end{aligned}$$

*The 3-torsion part is generated by the integral class  $1^{(3)}$ .*

*Remark 2.3.* The torsion in the case  $n = 2$  was also computed by Boissière, Nieper-Wißkirchen and Sarti, [1, Prop. 3] using similar techniques. The result for  $n = 3$  seems to be new. The freeness result for  $n \geq 4$  was already proven by Markman, [5, Thm. 1.10], using a completely different method.

**Proposition 2.4.** *For triple products of  $H^2(S^{[n]}, \mathbb{Z})$ , we have:*

$$\frac{H^6(S^{[2]}, \mathbb{Z})}{\text{Sym}^3 H^2(S^{[2]}, \mathbb{Z})} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$$

*The quotient is generated by the integral class  $1^{(2)}$ .*

$$\frac{H^6(S^{[3]}, \mathbb{Z})}{\text{Sym}^3 H^2(S^{[3]}, \mathbb{Z})} \cong \left( \frac{\mathbb{Z}}{2\mathbb{Z}} \right)^{\oplus 230} \oplus \left( \frac{\mathbb{Z}}{36\mathbb{Z}} \right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{72\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 507}$$

$$\frac{H^6(S^{[4]}, \mathbb{Z})}{\text{Sym}^3 H^2(S^{[4]}, \mathbb{Z})} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 552}$$

For  $n \geq 5$ , the quotient is free.

We study now cup products between classes of degree 2 and 4. The case of  $S^{[3]}$  is of particular interest.

**Proposition 2.5.** *The cup product mapping :  $H^2(S^{[n]}, \mathbb{Z}) \otimes H^4(S^{[n]}, \mathbb{Z}) \rightarrow H^6(S^{[n]}, \mathbb{Z})$  is neither injective (unless  $n = 0$ ) nor surjective (unless  $n \leq 2$ ). We have:*

$$(4) \quad \frac{H^6(S^{[3]}, \mathbb{Z})}{H^2(S^{[3]}, \mathbb{Z}) \smile H^4(S^{[3]}, \mathbb{Z})} \cong \left( \frac{\mathbb{Z}}{3\mathbb{Z}} \right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$$

$$(5) \quad \frac{H^6(S^{[4]}, \mathbb{Z})}{H^2(S^{[4]}, \mathbb{Z}) \smile H^4(S^{[4]}, \mathbb{Z})} \cong \left( \frac{\mathbb{Z}}{6\mathbb{Z}} \right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{108\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$$

$$(6) \quad \frac{H^6(S^{[5]}, \mathbb{Z})}{H^2(S^{[5]}, \mathbb{Z}) \smile H^4(S^{[5]}, \mathbb{Z})} \cong \mathbb{Z}^{\oplus 22} \oplus \mathbb{Z}$$

$$(7) \quad \frac{H^6(S^{[n]}, \mathbb{Z})}{H^2(S^{[n]}, \mathbb{Z}) \smile H^4(S^{[n]}, \mathbb{Z})} \cong \mathbb{Z}^{\oplus 22} \oplus \mathbb{Z} \oplus \mathbb{Z}, \quad n \geq 6.$$

In each case, the first 22 parts of the quotient are generated by the integral classes  $\alpha_i^{(1,1,1)} - 3 \cdot \alpha_i^{(2,1)} + 3 \cdot \alpha_i^{(3)} + 3 \cdot 1^{(2)} \alpha_i^{(1,1)} - 6 \cdot 1^{(2)} \alpha_i^{(2)} + 6 \cdot 1^{(2,2)} \alpha_i^{(1)} - 3 \cdot 1^{(3)} \alpha_i^{(1)}$ , for  $i = 1 \dots 22$ . Now define an integral class

$$K := \sum_{i \neq j} B(\alpha_i, \alpha_j) \left[ \alpha_i^{(1,1)} \alpha_j^{(1)} - 2 \cdot \alpha_i^{(2)} \alpha_j^{(1)} + \frac{3}{2} \cdot 1^{(2)} \alpha_i^{(1)} \alpha_j^{(1)} \right] + \\ + \sum_i B(\alpha_i, \alpha_i) \left[ \alpha_i^{(1,1,1)} - 2 \cdot \alpha_i^{(2,1)} + \frac{3}{2} \cdot 1^{(2)} \alpha_i^{(1,1)} \right] + x^{(2)} - 1^{(2)} x^{(1)}.$$

In the case  $n = 3$ , the last part of the quotient is generated by  $K$ .

In the case  $n = 4$ , the class  $1^{(4)}$  generates the 2-torsion part and  $K + 38 \cdot 1^{(4)}$  generates the 108-torsion part.

In the case  $n = 5$ , the last part of the quotient is generated by  $K + 16 \cdot 1^{(4)} - 21 \cdot 1^{(3,2)}$ . If  $n \geq 6$ , the two last parts of the quotient are generated by some multiples of  $K - \frac{4}{3}(45 - n)1^{(2,2,2)} + (48 - n)1^{(3,2)}$  and  $K - \frac{1}{2}(40 - n)1^{(2,2,2)} + \frac{1}{4}(48 - n)1^{(4)}$ .

*Proof.* The last assertion for arbitrary  $n$  follows from Corollary 1.12. First observe that for  $\alpha^\lambda \in H^2$ ,  $\alpha^\mu \in H^4$ ,  $\alpha^\nu \in H^6$ , we have  $\|\lambda\| \leq 2$ ,  $\|\mu\| \leq 4$  and  $\|\nu\| \geq 2$ , according to Lemma 1.8. The coefficient of the cup product matrix are thus polynomials of degree at most  $2 + 4 - 2 = 4$  and it suffices to compute only a finite number of instances for  $n$ . It turns out, that the maximal degree is 1 and its cokernel is given as stated.  $\square$

*Remark 2.6.* Observe that the generators of the quotients are independent of the choice of the base of  $H^2(S, \mathbb{Z})$ .

We give now some computational results for the middle cohomology group. Since  $S^{[n]}$  is a projective variety of complex dimension  $2n$ , Poincaré duality gives  $H^{2n}(S^{[n]}, \mathbb{Z})$  the structure of an unimodular lattice.

**Proposition 2.7.** *Let  $L$  denote the lattice  $H^{2n}(S^{[n]}, \mathbb{Z})$ . We have:*

- (1) For  $n = 2$ ,  $L$  is an odd lattice of rank 276 and signature 124.

- (2) For  $n = 3$ ,  $L$  is an even lattice of rank 2554 and signature -640.
- (3) For  $n = 4$ ,  $L$  is an odd lattice of rank 19298 and signature ...

The oddity of the lattice for  $n$  even is witnessed by Example 1.14 (2).

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