MULTIPLICATIVE STRUCTURE OF INTEGER COHOMOLOGY OF HILBERT SCHEMES OF POINTS ON K3 SURFACES

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Abstract. We study the cokernels of the cup product in integer cohomology of the Hilbert scheme of n points on a K3 surface. The main difference to cohomology with rational coefficients is the presence of torsion, depending on n. We particularly discuss the case $n = 3$.

1. Preliminaries

Let S be a K3 surface. We fix integral bases 1 of $H^0(S,\mathbb{Z})$, x of $H^4(X,\mathbb{Z})$ and $\alpha_1,\dots,\alpha_{22}$ of $H^2(S,\mathbb{Z})$ such that the cup product pairing on $H^2(X,\mathbb{Z})$, written as a symmetric matrix with respect to this basis, looks like

$$\begin{pmatrix} U & & & & \\ & U & & & \\ & & U & & \\ & & & E & \\ & & & & E \end{pmatrix},$$

where U stands for the intersection matrix of the hyperbolic lattice and E stands for the negative matrix of the E_8 lattice, *i.e.*

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \end{pmatrix}.$$

We denote by $S^{[n]}$ the Hilbert scheme of n points on S. An integral basis for $H^*(S^{[n]},\mathbb{Z})$ was given by Qin–Wang:

Theorem 1.1. [39, Thm. 5.4.] *The classes*

$$\frac{1}{z_\lambda}\mathfrak{q}_\lambda(1)\mathfrak{q}_\mu(x)\mathfrak{m}_{\nu^1,\alpha_1}\dots\mathfrak{m}_{\nu^{22},\alpha_{22}}|0\rangle, \quad |\lambda| + |\mu| + \sum_{i=1}^{22}|\nu^i| = n$$

form an integral basis for $H^*(S^{[n]},\mathbb{Z})$. Here, λ, μ, ν^i are partitions, $|\cdot|$ means the weight of a partition and

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2. Introduction

The ring structure of $H^*(S^{[n]}, \mathbb{Q})$ has been studied in [24]. Since $H^{\text{odd}}(S^{[n]}, \mathbb{Z}) = 0$ and $H^{\text{even}}(S^{[n]}, \mathbb{Z})$ is torsion-free by [27], we can also apply these results to $H^*(S^{[n]}, \mathbb{Z})$. A basis for cohomology with integer coefficients was given by Qin–Wang in [39] which allows us to compute explicitly the image of $H^k(S^{[n]}, \mathbb{Z}) \cup H^l(S^{[n]}, \mathbb{Z})$ in $H^{k+l}(S^{[n]}, \mathbb{Z})$. We obtain:

Theorem 2.1. *If X is deformation equivalent to $S^{[3]}$, then:*

$$\begin{aligned}
\frac{H^4(X,\mathbb{Z})}{\operatorname{Sym}^2 H^2(X,\mathbb{Z})} &\cong \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 23} \\
\frac{H^6(X,\mathbb{Z})}{H^2(X,\mathbb{Z}) \cup H^4(X,\mathbb{Z})} &\cong \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12} \\
\frac{H^6(X,\mathbb{Z})}{\operatorname{Sym}^3 H^2(X,\mathbb{Z})} &\cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 2} \oplus \left(\frac{\mathbb{Z}}{4\mathbb{Z}}\right)^{\oplus 21} \oplus \mathbb{Z}^{\oplus 2278}
\end{aligned}$$

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