COMPUTING CUP-PRODUCTS IN INTEGER COHOMOLOGY OF HILBERT SCHEMES OF POINTS ON K3 SURFACES

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Abstract. We study the images of cup products in integer cohomology of the Hilbert scheme of n points on a K3 surface.

1. Preliminaries

Definition 1.1. Let n be a natural number. A partition of n is a sequence $\lambda = (\lambda_1 \geq \ldots \geq \lambda_k > 0)$ of natural numbers such that $\sum_i \lambda_i = n$. It is convenient to write $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$ as a sequence of multiplicities. We define the weight $\|\lambda\| := \sum m_i i = n$ and the length $|\lambda| := \sum_i m_i = k$. We also define $z_\lambda := \prod_i i^{m_i} m_i!$.

Definition 1.2. Let Λ be the ring of symmetric functions. Let m_λ and p_λ denote the monomial and the power sum symmetric functions. They are both indexed by partitions and form a basis of Λ. They are linearly related by $p_\lambda = \sum_\mu \psi_{\lambda\mu} m_\mu$, the sum being over partitions with the same weight as λ. It turns out that the base change matrix $(\psi_{\lambda\mu})$ has integral entries, but its inverse $(\psi^{-1}_{\lambda\mu})$ has not. For example, $p_{(2,1,1)} = 2m_{(2,1,1)} + 2m_{(2,2)} + 2m_{(3,1)} + m_{(4)}$ but $m_{(2,1,1)} = \frac{1}{2}p_{(2,1,1)} - \frac{1}{2}m_{(2,2)} - p_{(3,1)} + p_{(4)}$. A method to determine the coefficients $(\psi_{\lambda\mu})$ is given in [2, Sect. 3.7].

Definition 1.3. For our purposes, a lattice L is a free $\mathbb{Z}$-module of finite rank, equipped with a non-degenerate symmetric integral bilinear form B. The lattice L is called odd, if there exists a $v \in L$, such that $B(v, v)$ is odd, otherwise it is called even. Choosing a base $\{e_i\}_i$ of our lattice, we can write B as a symmetric matrix. L is called unimodular, if the matrix B has determinant ± 1. The difference between the number of positive eigenvalues and the number of negative eigenvalues of B (regarded over $\mathbb{R}$) is called the signature. If B has both positive and negative eigenvalues, the lattice is called indefinite. For a classification theorem for indefinite lattices, see [6].

Definition 1.4. Let S be a projective K3 surface. We fix integral bases 1 of $H^0(S, \mathbb{Z})$, x of $H^4(S, \mathbb{Z})$ and $\alpha_1, \ldots, \alpha_{22}$ of $H^2(S, \mathbb{Z})$. The cup product induces a symmetric bilinear form B_{H^2} on $H^2(S, \mathbb{Z})$ and thus the structure of a unimodular lattice. We may extend B_{H^2} to a symmetric non-degenerate bilinear form B on $H^*(S, \mathbb{Z})$ by setting $B(1, 1) = 0,\ B(1, \alpha_i) = 0,\ B(1, x) = 1,\ B(x, x) = 0$.

It turns out that $H^2(S, \mathbb{Z})$ has signature -16 and, by the classification theorem for indefinite unimodular lattices, is isomorphic to $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$, *i.e.* three times the hyperbolic lattice and two times the negative E_8 lattice.

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Definition 1.5. B induces a form $B \otimes B$ on $\mathrm{Sym}^2 H^*(S,\mathbb{Z})$. So the cup-product

$$\mu : \mathrm{Sym}^2 H^*(S,\mathbb{Z}) \longrightarrow H^*(S,\mathbb{Z})$$

induces an adjoint comultiplication Δ that is coassociative, given by:

$$\Delta : H^*(S,\mathbb{Z}) \longrightarrow \mathrm{Sym}^2 H^*(S,\mathbb{Z}), \quad \Delta = (B\otimes B)^{-1}\mu^T B$$

Note that this does not define a bialgebra structure. The image of 1 under the composite map $\mu(\Delta(1)) = B(\Delta(1),\Delta(1)) = 24x$, denoted by e is called the Euler Class.

We denote by $S^{[n]}$ the Hilbert scheme of n points on S, *i.e.* the classifying space of all zero-dimensional closed subschemes of length n. $S^{[0]}$ consists of a single point and $S^{[1]} = S$. Fogarty proved, that the Hilbert scheme is a smooth variety. A theorem by Nakajima gives an explicit description of the vector space structure of $H^*(S^{[n]},\mathbb{Q})$ in terms of creation operators

$$\mathfrak{q}_l(\beta) : H^*(S^{[n]},\mathbb{Q}) \longrightarrow H^{*+k+l-1}(S^{[n+l]},\mathbb{Q}),$$

where $\beta \in H^k(S,\mathbb{Q})$, acting on the direct sum $\mathbb{H} := \bigoplus_n H^*(S^{[n]},\mathbb{Q})$. The operators $\mathfrak{q}_l(\beta)$ are linear, commute with each other, and the images of the vacuum vector $|0\rangle$, defined as the generator of $H^0(S^{[0]},\mathbb{Q}) \cong \mathbb{Q}$, under the polynomial algebra generated by the creation operators span $\mathbb{H}$ as a vector space. It is convenient to abbreviate $\mathfrak{q}_{l_1}(\beta)\ldots\mathfrak{q}_{l_k}(\beta) =: \mathfrak{q}_\lambda(\beta)$, where the partition λ is composed by the l_i.

An integral basis for $H^*(S^{[n]},\mathbb{Z})$ in terms of Nakajima's operators was given by Qin–Wang:

Theorem 1.6. [8, Thm. 5.4.] *Let $\mathfrak{m}_{\nu,\alpha} := \sum_\rho \psi^{-1}_{\nu\rho}\,\mathfrak{q}(\alpha)$, with coefficients $\psi^{-1}_{\nu\rho}$ as in Definition 1.2. The classes*

$$\frac{1}{z_\lambda}\mathfrak{q}_\lambda(1)\mathfrak{q}_\mu(x)\mathfrak{m}_{\nu^1,\alpha_1}\ldots\mathfrak{m}_{\nu^{22},\alpha_{22}}|0\rangle, \quad \|\lambda\| + \|\mu\| + \sum_{i=1}^{22}\|\nu^i\| = n$$

form an integral basis for $H^*(S^{[n]},\mathbb{Z})$. Here, λ, μ, ν^i are partitions.

Notation 1.7. To enumerate the basis of $H^*(S^{[n]},\mathbb{Z})$, we introduce the following abbreviation:

$$1^\lambda \alpha_1^{\nu^1}\ldots\alpha_{22}^{\nu^{22}} x^\mu := \frac{1}{z_{\tilde\lambda}}\mathfrak{q}_{\tilde\lambda}(1)\mathfrak{q}_\mu(x)\mathfrak{m}_{\nu^1,\alpha_1}\ldots\mathfrak{m}_{\nu^{22},\alpha_{22}}|0\rangle$$

where the partition $\tilde\lambda$ is built from λ by appending sufficiently many Ones, such that $\|\tilde\lambda\| + \|\mu\| + \sum\|\nu^i\| = n$. If $\|\lambda\| + \|\mu\| + \sum\|\nu^i\| > n$, we put $1^\lambda\alpha_1^{\nu^1}\ldots\alpha_{22}^{\nu^{22}}x^\mu = 0$.

The ring structure of $H^*(S^{[n]},\mathbb{Q})$ has been studied by Lehn and Sorger in [3], where an explicit algebraic model is constructed, which we recall briefly:

Definition 1.8. [3, Sect. 2] Let π be a permutation of n letters, written as a sum of disjoint cycles. To each cycle we may associate an element of $A := H^*(S,\mathbb{Q})$. This defines an element in $A^{\otimes m}$, m being the number of cycles. So these mappings span a vector space over $\mathbb{Q}$. The space obtained by taking the direct sum over all $\pi \in S_n$ will be denoted by $A\{S_n\}$.

To define a ring structure, take two permutations π,τ, together with mappings. The result of the multiplication will be the permutation $\pi\tau$, together with a mapping of cycles. To construct the mappings to A, look first at the orbit space of the group of permutations $\langle\pi,\tau\rangle$, generated by π and τ. For each cycle of π,τ contained in one orbit B of $\langle \pi, \tau \rangle$, multiply with the associated element of A. Also multiply with a certain power of the Euler class e^g. Afterwards, apply the comultiplication Δ repeatedly on the product to get a mapping from the cycles of $\pi\tau$ contained in B to A. Here the "graph defect" g is defined as follows: Let u, v, w be the number of cycles contained in B of $\pi,\ \tau,\ \pi\tau$, respectively. Then $g := \frac{1}{2}\left(|B| + 2 - u - v - w\right)$. Now follow this procedure for each orbit B.

The symmetric group S_n acts on $A\{S_n\}$ by conjugation. This action preserves the ring structure. Therefore the space of invariants $A^{[n]} := \left(A\{S_n\}\right)^{S_n}$ becomes a subring. The main theorem of [3] can now be stated:

Theorem 1.9. [3, Thm. 3.2.] *The following map is an isomorphism of rings:*

$$\begin{aligned} H^*(S^{[n]}, \mathbb{Q}) &\longrightarrow A^{[n]} \\ \mathfrak{q}_{n_1}(\beta_1)\ldots\mathfrak{q}_{n_k}(\beta_k)|0\rangle &\longmapsto \sum_{\sigma\in S_n} \sigma a \sigma^{-1} \end{aligned}$$

with $n_1+\ldots+n_k = n$ and $a \in A\{S_n\}$ corresponds to an arbitrary permutation with k cycles of lengths $n_1, \ldots, n_k$ that are associated to the classes $\beta_1, \ldots, \beta_k \in H^*(S, \mathbb{Q})$, respectively.

Since $H^{\mathrm{odd}}(S^{[n]}, \mathbb{Z}) = 0$ and $H^{\mathrm{even}}(S^{[n]}, \mathbb{Z})$ is torsion-free by [4], we can apply these results to $H^*(S^{[n]}, \mathbb{Z})$ to determine the multiplicative structure of cohomology with integer coefficients. It turns out, that it is somehow independent of n. More precisely, we have the following stability theorem due to Li, Qin and Wang:

Theorem 1.10. [8, Thm. 2.1] *Let $Q_1, \ldots, Q_s$ be products of creation operators, i.e. $Q_i = \prod_j \mathfrak{q}_{\lambda_{i,j}}(\beta_{i,j})$ for some partitions $\lambda_{i,j}$ and classes $\beta_{i,j} \in H^*(S,\mathbb{Z})$. Set $n_i := \sum_j \|\lambda_{i,j}\|$. Then the cup product $\prod_{i=1}^s \left(\frac{1}{(n-n_i)!}\mathfrak{q}_{n-n_i}(1)\, Q_i\, |0\rangle\right)$ is equal to a finite linear combination of classes of the form $\frac{1}{(n-m)!}\mathfrak{q}_{n-m}(1)\prod_j \mathfrak{q}_{\mu_j}(\gamma_j)|0\rangle$, with $\gamma \in H^*(S,\mathbb{Z})$, $m = \sum_j \|\mu_j\|$, whose coefficients are independent of n. We have the upper bound $m \le \sum_i n_i$.*

Corollary 1.11. *Let λ^0, μ^0, ν^0 be partitions containing no Ones. Then the coefficients c_i of the cup product in $H^*(S^{[n]}, \mathbb{Z})$*

$$1^{\lambda^0}\alpha_1^{\lambda^1}\ldots\alpha_{22}^{\lambda^{22}}x^{\lambda^{23}} \smile 1^{\mu^0}\alpha_1^{\mu^1}\ldots\alpha_{22}^{\mu^{22}}x^{\mu^{23}} = \sum_i c_i\cdot 1^{\nu_i^0}\alpha_1^{\nu_i^1}\ldots\alpha_{22}^{\nu_i^{22}}x^{\nu_i^{23}}$$

are polynomials in n of degree at most $\sum_{j=0}^{23}\|\lambda^j\| + \|\mu^j\| - \left\|\nu_i^j\right\|$.

Proof. Assume $n \ge \sum\|\lambda^j\|, \sum\|\mu^j\|$. Set $Q_1 := \mathfrak{q}_{\lambda^0}(1)\mathfrak{q}_{\lambda^{23}}(x)\prod_{1\le j\le 22}\mathfrak{q}_{\lambda^j}(\alpha_j)$, $Q_2 := \mathfrak{q}_{\mu^0}(1)\mathfrak{q}_{\mu^{23}}(x)\prod_{1\le j\le 22}\mathfrak{q}_{\mu^j}(\alpha_j)$, $Q_3 := \mathfrak{q}_{\nu_i^0}(1)\mathfrak{q}_{\nu_i^{23}}(x)\prod_{1\le j\le 22}\mathfrak{q}_{\nu_i^j}(\alpha_j)$. Let n_1, n_2, n_3 be defined as above. Then: $1^{\lambda^0}\alpha_1^{\lambda^1}\ldots\alpha_{22}^{\lambda^{22}}x^{\lambda^{23}} \smile 1^{\mu^0}\alpha_1^{\mu^1}\ldots\alpha_{22}^{\mu^{22}}x^{\mu^{23}} = \frac{1}{(n-n_1)!\,z_{\lambda^0}}\mathfrak{q}_{n-n_1}(1)Q_1|0\rangle \smile \frac{1}{(n-n_2)!\,z_{\mu^0}}\mathfrak{q}_{n-n_2}(1)Q_2|0\rangle$. Thus the coefficient c_i in the product expansion is some multiple of $\frac{(n-n_3)!}{(n-m)!}$ for a certain $n \le n_1 + n_2$. This is a polynomial of degree $m - n_3 \le n_1 + n_2 - n_3$.

2. Computational results

We now give some results in low degrees, obtained by computing multiplication matrices with respect to the integral basis and a then reducing to Smith normal form (both done by a computer).

Remark 2.1. Denote $h^k(S^{[n]})$ the rank of $H^k(S^{[n]},\mathbb{Z})$. We have:

- $h^2(S^{[n]}) = 23$ for $n \geq 2$.
- $h^4(S^{[n]}) = 276,\ 299,\ 300$ for $n = 2, 3, \geq 4$ resp.
- $h^6(S^{[n]}) = 23,\ 2554,\ 2852,\ 2875,\ 2876$ for $n = 2, 3, 4, 5, \geq 6$ resp.

The algebra generated by classes of degree 2 is an interesting object to study. For cohomology with complex coefficients, Verbitsky has proven in [9] that the cup product mapping from $\operatorname{Sym}^k H^2(S^{[n]},\mathbb{C})$ to $H^{2k}(S^{[n]},\mathbb{C})$ is injective for $k \leq n$. One concludes that this also holds for integral coefficients.

Proposition 2.2. *We identify* $\operatorname{Sym}^2 H^2(S^{[n]},\mathbb{Z})$ *with its image in* $H^4(S^{[n]},\mathbb{Z})$ *under the cup product mapping. Then:*

$$\frac{H^4(S^{[2]},\mathbb{Z})}{\operatorname{Sym}^2 H^2(S^{[2]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23} \oplus \frac{\mathbb{Z}}{5\mathbb{Z}}, \tag{1}$$

$$\frac{H^4(S^{[3]},\mathbb{Z})}{\operatorname{Sym}^2 H^2(S^{[3]},\mathbb{Z})} \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 23}, \tag{2}$$

$$\frac{H^4(S^{[n]},\mathbb{Z})}{\operatorname{Sym}^2 H^2(S^{[n]},\mathbb{Z})} \cong \mathbb{Z}^{\oplus 24}, \quad \textit{for } n \geq 4. \tag{3}$$

The 3-torsion part is generated by the integral class $1^{(3)}$.

Remark 2.3. The torsion in the case $n = 2$ was also computed by Boissière, Nieper-Wißkirchen and Sarti, [1, Prop. 3] using similar techniques. The result for $n = 3$ seems to be new. The freeness result for $n \geq 4$ was already proven by Markman, [5, Thm. 1.10], using a completely different method.

Proposition 2.4. *Studying triple products of* $H^2(S^{[n]},\mathbb{Z})$*, we get:*

$$\frac{H^6(S^{[2]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[2]},\mathbb{Z})} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$$

The quotient is generated by the integral class $1^{(2)}$.

$$\frac{H^6(S^{[3]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[3]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 230} \oplus \left(\frac{\mathbb{Z}}{36\mathbb{Z}}\right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{72\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 507}$$

$$\frac{H^6(S^{[4]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[4]},\mathbb{Z})} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 552}$$

For $n \geq 5$*, the quotient is free.*

We study now cup products between classes of degree 2 and 4. The case of $S^{[3]}$ is of particular interest.

Proposition 2.5. *The cup product mapping* $: H^2(S^{[n]},\mathbb{Z})\otimes H^4(S^{[n]},\mathbb{Z}) \to H^6(S^{[n]},\mathbb{Z})$ *is neither injective (unless* $n=0$*) nor surjective (unless* $n \leq 2$*). We have:*

$$\frac{H^6(S^{[3]},\mathbb{Z})}{H^2(S^{[3]},\mathbb{Z}) \smile H^4(S^{[3]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}} \tag{4}$$

$$\frac{H^6(S^{[4]},\mathbb{Z})}{H^2(S^{[4]},\mathbb{Z}) \smile H^4(S^{[4]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{6\mathbb{Z}}\right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{108\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \tag{5}$$

$$\frac{H^6(S^{[5]},\mathbb{Z})}{H^2(S^{[5]},\mathbb{Z}) \smile H^4(S^{[5]},\mathbb{Z})} \cong \mathbb{Z}^{\oplus 22} \oplus \mathbb{Z} \tag{6}$$

$$\frac{H^6(S^{[n]},\mathbb{Z})}{H^2(S^{[n]},\mathbb{Z}) \smile H^4(S^{[n]},\mathbb{Z})} \cong \mathbb{Z}^{\oplus 22} \oplus \mathbb{Z} \oplus \mathbb{Z},\ n \geq 6. \tag{7}$$

In each case, the first 22 parts of the quotient are generated by the integral classes

$$\alpha_i^{(1,1,1)} - 3\cdot\alpha_i^{(2,1)} + 3\cdot\alpha_i^{(3)} + 3\cdot 1^{(2)}\alpha_i^{(1,1)} - 6\cdot 1^{(2)}\alpha_i^{(2)} + 6\cdot 1^{(2,2)}\alpha_i^{(1)} - 3\cdot 1^{(3)}\alpha_i^{(1)},$$

for $i = 1\dots 22$*. Now define an integral class*

$$\begin{aligned} K := &\sum_{i\neq j} B(\alpha_i,\alpha_j)\left[\alpha_i^{(1,1)}\alpha_j^{(1)} - 2\cdot\alpha_i^{(2)}\alpha_j^{(1)} + \frac{3}{2}\cdot 1^{(2)}\alpha_i^{(1)}\alpha_j^{(1)}\right] + \\ &+ \sum_i B(\alpha_i,\alpha_i)\left[\alpha_i^{(1,1,1)} - 2\cdot\alpha_i^{(2,1)} + \frac{3}{2}\cdot 1^{(2)}\alpha_i^{(1,1)}\right] + x^{(2)} - 1^{(2)}x^{(1)}. \end{aligned}$$

In the case $n=3$*, the last part of the quotient is generated by* K*.*
In the case $n=4$*, the class* $1^{(4)}$ *generates the 2-torsion part and* $K + 38\cdot 1^{(4)}$ *generates the 108-torsion part.*
In the case $n=5$*, the last part of the quotient is generated by* $K + 16\cdot 1^{(4)} - 21\cdot 1^{(3,2)}$*.*
If $n \geq 6$*, the two last parts of the quotient are generated by some multiples of* $K - \frac{4}{3}(45-n)1^{(2,2,2)} + (48-n)1^{(3,2)}$ *and* $K - \frac{1}{2}(40-n)1^{(2,2,2)} + \frac{1}{4}(48-n)1^{(4)}$*.*

Proof. The last assertion for arbitrary n follows from 1.11. First, observe that if $1^{\lambda^0}\alpha_1^{\lambda^1}\dots\alpha_{22}^{\lambda^{22}}x^{\lambda^{23}}$ has cohomological degree $2k$, then $\frac{k}{2} \leq \sum \|\lambda^j\| \leq 2k$. The coefficients of the cup product martix are polynomials of degree at most $2+4-2=4$ and it suffices to compute only a finite number of instances for n. It turns out, that the maximal degree is 1 and its cokernel is given as stated. □

Remark 2.6. Observe that the generators of the quotients are independent of the choice of the base of $H^2(S,\mathbb{Z})$.

We give now some computational results for the middle cohomology group. Since $S^{[n]}$ is a projective variety of complex dimension $2n$, Poincaré duality gives $H^{2n}(S^{[n]},\mathbb{Z})$ the structure of an unimodular lattice.

Proposition 2.7. *Let* L *denote the lattice* $H^{2n}(S^{[n]},\mathbb{Z})$*. We have:*

(1) *For* $n=2$*,* L *is an odd lattice of rank 276 and signature 124.*
(2) *For* $n=3$*,* L *is an even lattice of rank 2554 and signature -640.*
(3) *For* $n=4$*,* L *is an odd lattice of rank 19298 and signature ...*

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