COMPUTING CUP-PRODUCTS IN INTEGER COHOMOLOGY OF HILBERT SCHEMES OF POINTS ON K3 SURFACES

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Abstract. We study the images of cup products in integer cohomology of the Hilbert scheme of n points on a K3 surface.

1. Preliminaries

Definition 1.1. Let n be a natural number. A partition of n is a sequence $\lambda = (\lambda_1 \geq \ldots \geq \lambda_k > 0)$ of natural numbers such that $\sum_i \lambda_i = n$. It is convenient to write $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$ as a sequence of multiplicities. We define the weight $\|\lambda\| := \sum m_i i = n$ and the length $|\lambda| := \sum_i m_i = k$. We also define $z_\lambda := \prod_i i^{m_i} m_i!$.

Definition 1.2. Let S be a projective K3 surface. We fix integral bases 1 of $H^0(S,\mathbb{Z})$, x of $H^4(X,\mathbb{Z})$ and $\alpha_1, \ldots, \alpha_{22}$ of $H^2(S,\mathbb{Z})$. The cup product induces a symmetric bilinear form B_{H^2} on $H^2(X,\mathbb{Z})$. Written as a symmetric matrix with respect to this basis, B_{H^2} looks like

$$B_{H^2} = \begin{pmatrix} U & & & & \\ & U & & & \\ & & U & & \\ & & & E & \\ & & & & E \end{pmatrix},$$

where U stands for the intersection matrix of the hyperbolic lattice and E stands for the negative matrix of the E_8 lattice, *i.e.*

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \end{pmatrix}.$$

We may extend B_{H^2} to a symmetric non-degenerate bilinear form on $H^*(S,\mathbb{Z})$ by setting $B(1,1) = 0,\ B(1,\alpha_i) = 0,\ B(1,x) = 1,\ B(x,x) = 0.$

Definition 1.3. B induces a form $B \otimes B$ on $\operatorname{Sym}^2 H^*(S,\mathbb{Z})$. So the cup-product

$$\mu : \operatorname{Sym}^2 H^*(S,\mathbb{Z}) \longrightarrow H^*(S,\mathbb{Z})$$

has an adjoint comultiplication Δ that is coassociative, given by:

$$\Delta : H^*(S,\mathbb{Z}) \longrightarrow \operatorname{Sym}^2 H^*(S,\mathbb{Z}), \quad \Delta = (B \otimes B)^{-1} \mu^T B$$

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The image of 1 under the composite map $\mu(\Delta(1)) = B(\Delta(1), \Delta(1)) = 24x$, denoted by e is called the Euler Class.

We denote by $S^{[n]}$ the Hilbert scheme of n points on S, *i.e.* the classifying space of all zero-dimensional closed subschemes of length n, which is smooth. A classical result by Nakajima gives an explicit description of $H^*(S^{[n]}, \mathbb{Q})$ in terms of creation operators $\mathfrak{q}_l(\beta)$, $\beta \in H^*(S, \mathbb{Q})$, acting on the direct sum $\bigoplus_n H^*(S^{[n]}, \mathbb{Q})$. An integral basis for $H^*(S^{[n]}, \mathbb{Z})$ in terms of Nakajima's operators was given by Qin–Wang:

Theorem 1.4. [6, Thm. 5.4.] *The classes*

$$\frac{1}{z_\lambda}\mathfrak{q}_\lambda(1)\mathfrak{q}_\mu(x)\mathfrak{m}_{\nu^1,\alpha_1}\dots\mathfrak{m}_{\nu^{22},\alpha_{22}}|0\rangle, \quad \|\lambda\| + \|\mu\| + \sum_{i=1}^{22}\|\nu^i\| = n$$

form an integral basis for $H^*(S^{[n]}, \mathbb{Z})$. Here, λ, μ, ν^i are partitions. The symbol $\mathfrak{q}$ stands for Nakajima's creation operator. The relation of $\mathfrak{m}_{\nu,\alpha}$ to $\mathfrak{q}_{\tilde{\nu}}(\alpha)$ is the same as the monomial symmetric functions m_ν to the power sum symmetric functions $p_{\tilde{\nu}}$.

Notation 1.5. To enumerate the basis of $H^*(S^{[n]}, \mathbb{Z})$, we introduce the following abbreviation:

$$1^\lambda \alpha_1^{\nu^1}\dots\alpha_{22}^{\nu^{22}} x^\mu := \frac{1}{z_{\tilde{\lambda}}}\mathfrak{q}_{\tilde{\lambda}}(1)\mathfrak{q}_\mu(x)\mathfrak{m}_{\nu^1,\alpha_1}\dots\mathfrak{m}_{\nu^{22},\alpha_{22}}|0\rangle$$

where the partition $\tilde{\lambda}$ is built from λ by appending sufficiently many Ones, such that $\|\tilde{\lambda}\| + \|\mu\| + \sum\|\nu^i\| = n$. If $\|\lambda\| + \|\mu\| + \sum\|\nu^i\| > n$, we put $1^\lambda \alpha_1^{\nu^1}\dots\alpha_{22}^{\nu^{22}} x^\mu = 0$.

The ring structure of $H^*(S^{[n]}, \mathbb{Q})$ has been studied in [2], where an explicit algebraic model is constructed, which we recall briefly:

Definition 1.6. [2, Sect. 2] Let π be a permutation of n letters, written as a sum of disjoint cycles. To each cycle we may associate an element of $A := H^*(S, \mathbb{Q})$. This defines an element in $A^{\otimes m}$, m being the number of cycles. So these mappings span a vector space over $\mathbb{Q}$. The space obtained by taking the direct sum over all $\pi \in S_n$ will be denoted by $A\{S_n\}$.

To define a ring structure, take two permutations π, τ, together with mappings. The result of the multiplication will be the permutation $\pi\tau$, together with a mapping of cycles. Now, look first at the orbit space of the group of permutations $\langle\pi, \tau\rangle$, generated by π and τ. For each cycle of π, τ contained in one orbit B of $\langle\pi, \tau\rangle$, multiply with the associated element of A. Also multiply with a certain power of the Euler class e^g. Afterwards, apply the comultiplication Δ repeatedly on the product to get a mapping from the cycles of $\pi\tau$ contained in B to A. Here the "graph defect" g is defined as follows: Let u, v, w be the number of cycles contained in B of π, τ, $\pi\tau$, respectively. Then $g := \frac{1}{2}(|B| + 2 - u - v - w)$. Now follow this procedure for each orbit B.

The symmetric group S_n acts on $A\{S_n\}$ by conjugation. This action preserves the ring structure. Therefore the space of invariants $A^{[n]} := (A\{S_n\})^{S_n}$ becomes a subring. The main theorem of Lehn and Sorger can now be stated:

Theorem 1.7. [2, Thm. 3.2.] *The following map is an isomorphism of rings:*

$$\begin{aligned} H^*(S^{[n]},\mathbb{Q}) &\longrightarrow A^{[n]} \\ \mathfrak{q}_{n_1}(\beta_1)\ldots\mathfrak{q}_{n_k}(\beta_k)|0\rangle &\longmapsto \sum_{\sigma\in S_n}\sigma a\sigma^{-1} \end{aligned}$$

with $n_1+\ldots+n_k=n$ and $a\in A\{S_n\}$ corresponds to an arbitrary permutation with k cycles of lengths $n_1,\ldots,n_k$ that are associated to the classes $\beta_1,\ldots,\beta_k\in H^*(S,\mathbb{Q})$, respectively.

Since $H^{\mathrm{odd}}(S^{[n]},\mathbb{Z})=0$ and $H^{\mathrm{even}}(S^{[n]},\mathbb{Z})$ is torsion-free by [3], we can apply these results to $H^*(S^{[n]},\mathbb{Z})$ to determine the multiplicative structure of cohomology with integer coefficients.

2. Computational results

The algebra generated by classes of degree 2 is an interesting object to study. For cohomology with complex coefficients, Verbitsky has proven that the algebra generated by $H^2(X,\mathbb{C})$

Remark 2.1. Denote $h^k(S^{[n]})$ the rank of $H^k(S^{[n]},\mathbb{Z})$. We have:

- $h^2(S^{[n]})=23$ for $n\geq 2$.
- $h^4(S^{[n]})=276,\ 299,\ 300$ for $n=2,3,\geq 4$ resp.
- $h^6(S^{[n]})=23,\ 2554,\ 2852,\ 2875,\ 2876$ for $n=2,3,4,5,\geq 6$ resp.

We now give some results in low degrees, obtained by computing multiplication matrices with respect to the integral basis and a then reducing to Smith normal form (both done by a computer).

Proposition 2.2. *Studying the image of* $\mathrm{Sym}^2 H^2$ *in* H^4*, we obtain:*

$$\frac{H^4(S^{[2]},\mathbb{Z})}{\mathrm{Sym}^2 H^2(S^{[2]},\mathbb{Z})}\cong\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23}\oplus\frac{\mathbb{Z}}{5\mathbb{Z}}$$

This was already known to Boissière, Nieper-Wißkirchen and Sarti, [**?**, Prop. 3].

$$\frac{H^4(S^{[3]},\mathbb{Z})}{\mathrm{Sym}^2 H^2(S^{[3]},\mathbb{Z})}\cong\frac{\mathbb{Z}}{3\mathbb{Z}}\oplus\mathbb{Z}^{\oplus 23}$$

The torsion part of the quotient is generated by the integral class $\frac{1}{3}\mathfrak{q}_{(3)}(1)|0\rangle$.

$$\frac{H^4(S^{[n]},\mathbb{Z})}{\mathrm{Sym}^2 H^2(S^{[n]},\mathbb{Z})}\cong\mathbb{Z}^{\oplus 24},\quad \textit{for } n\geq 4.$$

This was already proven by Markman, [4, Thm. 1.10].

Proposition 2.3. *Comparing $H^2(S^{[n]},\mathbb{Z})\cup H^4(S^{[n]},\mathbb{Z})$ with $H^6(S^{[n]},\mathbb{Z})$, we obtain:*

$$\frac{H^6(S^{[2]},\mathbb{Z})}{H^2(S^{[2]},\mathbb{Z})\cup H^4(S^{[2]},\mathbb{Z})}=0 \tag{1}$$

$$\frac{H^6(S^{[3]},\mathbb{Z})}{H^2(S^{[3]},\mathbb{Z})\cup H^4(S^{[3]},\mathbb{Z})}\cong\left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 22}\oplus\frac{\mathbb{Z}}{3\mathbb{Z}} \tag{2}$$

$$\frac{H^6(S^{[4]},\mathbb{Z})}{H^2(S^{[4]},\mathbb{Z})\cup H^4(S^{[4]},\mathbb{Z})}\cong\left(\frac{\mathbb{Z}}{6\mathbb{Z}}\right)^{\oplus 22}\oplus\frac{\mathbb{Z}}{108\mathbb{Z}}\oplus\frac{\mathbb{Z}}{2\mathbb{Z}} \tag{3}$$

$$\frac{H^6(S^{[5]},\mathbb{Z})}{H^2(S^{[5]},\mathbb{Z})\cup H^4(S^{[5]},\mathbb{Z})}\cong\mathbb{Z}^{\oplus 22}\oplus\mathbb{Z} \tag{4}$$

$$\frac{H^6(S^{[n]},\mathbb{Z})}{H^2(S^{[n]},\mathbb{Z})\cup H^4(S^{[n]},\mathbb{Z})}\cong\mathbb{Z}^{\oplus 22}\oplus\mathbb{Z}\oplus\mathbb{Z},\ n\geq 6. \tag{5}$$

In each case, the first 22 parts of the quotient are generated by the integral classes

$$\alpha_i^{(1,1,1)}-3\cdot\alpha_i^{(2,1)}+3\cdot\alpha_i^{(3)}+3\cdot 1^{(2)}\alpha_i^{(1,1)}-6\cdot 1^{(2)}\alpha_i^{(2)}+6\cdot 1^{(2,2)}\alpha_i^{(1)}-3\cdot 1^{(3)}\alpha_i^{(1)},$$

for $i=1\ldots 22$. Now define an integral class

$$\begin{aligned}K:=&\sum_{i\neq j}B(\alpha_i,\alpha_j)\left[\alpha_i^{(1,1)}\alpha_j^{(1)}-2\cdot\alpha_i^{(2)}\alpha_j^{(1)}+\frac{3}{2}\cdot 1^{(2)}\alpha_i^{(1)}\alpha_j^{(1)}\right]+\\&+\sum_{i}B(\alpha_i,\alpha_i)\left[\alpha_i^{(1,1,1)}-2\cdot\alpha_i^{(2,1)}+\frac{3}{2}\cdot 1^{(2)}\alpha_i^{(1,1)}\right]+x^{(2)}-1^{(2)}x^{(1)}.\end{aligned}$$

In the case $n=3$, the last part of the quotient is generated by K.
In the case $n=4$, $1^{(4)}$ generates the 2-torsion part and $K+38\cdot 1^{(4)}$ generates the 108-torsion part.
In the case $n=5$, the last part of the quotient is generated by $K+16\cdot 1^{(4)}-21\cdot 1^{(3,2)}$.
If $n\geq 6$, the two last parts of the quotient are generated by some multiples of $K-\frac{4}{3}(45-n)1^{(2,2,2)}+(48-n)1^{(3,2)}$, resp. $K-\frac{1}{2}(40-n)1^{(2,2,2)}+\frac{1}{4}(48-n)1^{(4)}$.

Proposition 2.4.

$$\frac{H^6(S^{[2]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[2]},\mathbb{Z})}\cong\frac{\mathbb{Z}}{2\mathbb{Z}}$$

The quotient is generated by the integral class $1^{(2)}$.

$$\frac{H^6(S^{[3]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[3]},\mathbb{Z})}\cong\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 230}\oplus\left(\frac{\mathbb{Z}}{36\mathbb{Z}}\right)^{\oplus 22}\oplus\frac{\mathbb{Z}}{72\mathbb{Z}}\oplus\mathbb{Z}^{\oplus 507}$$

$$\frac{H^6(S^{[4]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[4]},\mathbb{Z})}\cong\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 276}\oplus\left(\frac{\mathbb{Z}}{6\mathbb{Z}}\right)^{\oplus 1748}\oplus\left(\frac{\mathbb{Z}}{12\mathbb{Z}}\right)^{\oplus 253}\oplus\left(\frac{\mathbb{Z}}{24\mathbb{Z}}\right)^{\oplus 23}$$

$$\frac{H^6(S^{[5]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[5]},\mathbb{Z})}\cong\left(\frac{\mathbb{Z}}{4\mathbb{Z}}\right)^{\oplus 22}\oplus\left(\frac{\mathbb{Z}}{12\mathbb{Z}}\right)^{\oplus 254}\oplus\left(\frac{\mathbb{Z}}{24\mathbb{Z}}\right)^{\oplus 2002}\oplus\left(\frac{\mathbb{Z}}{120\mathbb{Z}}\right)^{\oplus 22}$$

$$\frac{H^6(S^{[n]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[n]},\mathbb{Z})}\cong\left(\frac{\mathbb{Z}}{24\mathbb{Z}}\right)^{\oplus 276}\oplus\left(\frac{\mathbb{Z}}{120\mathbb{Z}}\right)^{\oplus 1770}\oplus\left(\frac{\mathbb{Z}}{240\mathbb{Z}}\right)^{\oplus 232}\oplus\left(\frac{\mathbb{Z}}{720\mathbb{Z}}\right)^{\oplus 22}\oplus\mathbb{Z}^{\cdot\cdot}$$

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