Integer cohomology of compact Hyperkähler manifolds

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Géométrie Algébrique en Liberté
Trieste, 27 June 2014

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Motivation

Let X be a compact Hyperkähler manifold of complex dimension $2m$ or, equivalently, an IHS manifold. Why should we be interested in $H^*(X, \mathbb{Z})$?

- It feels more geometric.
- Comparing $H^*(X, \mathbb{Z})$ with $H^*(X, \mathbb{C})$ gives us information about X, e.g. on projectivity.
- We obtain restrictions to possible automorphisms of our manifold X.
- ...

Question

Which constructions in $H^*(X, \mathbb{R}/\mathbb{C})$ carry over to $H^*(X, \mathbb{Z})$?

Beauville–Bogomolov form

As an example, consider the quadratic Beauville–Bogomolov form $q_X : H^*(X, \mathbb{R}) \to \mathbb{R}$

Theorem (Fujiki)

$q_X(\alpha)^m = c \int_X \alpha^{2m}$ for some $c \in \mathbb{R}$.

Corollary

q_X can be renormalized to yield a primitive integral quadratic form: $H^*(X, \mathbb{Z}) \to \mathbb{Z}$.

Hodge numbers for K3 surfaces

For S a compact Hyperkähler manifold of complex dimension two, i.e. a K3 surface, we have

$$
\begin{array}{ccccc}
 & & h^{p,q}(S) & & \\
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
\qquad\qquad
\begin{array}{c}
h^k(S,\mathbb{Z}) \\
1 \\
0 \\
22 \\
0 \\
1
\end{array}
$$

and the intersection pairing on H^2 is isomorphic to $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$, where U and E_8 are the bilinear forms corresponding to the hyperbolic resp. E_8 lattice.

Hodge numbers for Hilbert schemes

Denote $S^{[n]}$ the Hilbert scheme of n points on the K3 surface S. Then, the Hodge decomposition for $S^{[2]}$ is given by:

$$h^{p,q}(S^{[2]})$$

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 1 & & 21 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 21 & & 232 & & 21 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 1 & & 21 & & 1 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

$$h^k(S^{[2]}, \mathbb{Z})$$

$$\begin{array}{c}
1 \\ 0 \\ 23 \\ 0 \\ 276 \\ 0 \\ 23 \\ 0 \\ 1
\end{array}$$

and there are formulae for all $S^{[n]}$ due to Göttsche.

Betti numbers

k	$S^{[1]}$	$S^{[2]}$	$S^{[3]}$	$S^{[4]}$	$S^{[5]}$	$S^{[6]}$
0	1	1	1	1	1	1
2	22	23	23	23	23	23
4	1	276	299	300	300	300
6		23	2554	2852	2875	2876
8		1	299	19298	22127	22426
10			23	2852	125604	147431
12			1	300	22127	727606

For k fixed, these numbers stabilize for n big enough.

Cohomology of Hilbert schemes

Theorem (Nakajima)

For each $m \geq 1$ and each $\alpha \in H^j(S, \mathbb{Q})$, there is an operator

$$\mathfrak{a}_{-m}(\alpha) : H^i(S^{[n]}, \mathbb{Q}) \longrightarrow H^{i+j+2m-2}(S^{[n+m]}, \mathbb{Q})$$

and these operators, applied to $H^*(S^{[0]}, \mathbb{Q}) \cong \mathbb{Q}$, span the entire cohomology of all Hilbert schemes $S^{[n]}$.

Construction of the operators: Use the incidence scheme

$$\mathcal{I} \;\subset\; S^{[n]} \times S \times S^{[n+m]}$$

and define, using Poincaré duality,

$$\mathfrak{a}_{-m}(\alpha)\beta \;\overset{\text{P.D.}}{=}\; p_{3*}\left((p_1^*\beta \cup p_2^*\alpha) \cap [\mathcal{I}]\right).$$

Integral basis for $H^*(S^{[n]}, \mathbb{Z})$

This also works in integer cohomology!

Theorem (Qin, Wang)

Let $1 \in H^0(S, \mathbb{Z})$ be the canonical generator. The operators

$$\frac{1}{z_\lambda}\mathfrak{a}_{-\lambda}(1) : H^i(S^{[n]}, \mathbb{Z}) \longrightarrow H^{i+2m-2k}(S^{[n+m]}, \mathbb{Z})$$

$$\mathfrak{a}_{-m}(\alpha) : H^i(S^{[n]}, \mathbb{Z}) \longrightarrow H^{i+j+2m-2k}(S^{[n+m]}, \mathbb{Z})$$

span the integer cohomologies $H^*(S^{[n]}, \mathbb{Z})$. Here, the composition of several operators $\mathfrak{a}_{-m_i}(1)$ is denoted via a partition λ and z_λ denotes some constant depending on λ.

Algebra generated by $H^2(S^{[n]}, \mathbb{C})$

Theorem (Verbitsky)

The subalgebra generated by $H^2(S^{[n]}, \mathbb{C})$ in $H^*(S^{[n]}, \mathbb{C})$ is equal to

$$\frac{\operatorname{Sym}^* H^2(S^{[n]}, \mathbb{C})}{\langle \alpha^{n+1} \mid q(\alpha) = 0 \rangle},$$

where q is the Beauville-Bogomolov Form. In fact, this holds for any compact Hyperkähler manifold of complex dimension $2n$.

What about cohomology with integral coefficients?

Lehn, Sorger and Vasserot developed formulae for the cup product on $H^*(S^{[n]}, \mathbb{Q})$. We can use them also for computations in $H^*(S^{[n]}, \mathbb{Z})$.

Ring structure of $H^*(S^{[n]}, \mathbb{Q})$

Rough idea of the algebraic model:

- To any composition of operators $\mathfrak{a}_{-m_1}(\alpha_1)\dots\mathfrak{a}_{-m_k}(\alpha_k)$ one associates a conjugacy class of the symmetric group $\S_n$, given by the partition $\lambda = (m_1, \dots, m_k)$
- The cup product is built with the product in $\S_n$:
- E.g. $(1\,2\,3)(4\,5)\cdot(1\,4) = (1\,2\,3\,4\,5)$, but $(1\,2\,3)(4\,5)\cdot(1\,2) = (1)(2\,3)(4\,5)$.
- If two cycles ν_i, ν_j are joined together by a transposition, multiply the corresponding classes, e.g. $(1\,2\,3)_{\alpha_i}(4\,5)_{\alpha_j}\cdot(1\,4)_{\alpha_l} = (1\,2\,3\,4\,5)_{\alpha_i\cdot\alpha_j\cdot\alpha_l}$
- If a cycle is split in two by a transposition, use a map adjoint to the multiplication in $H^*(S)$.
- Sum up all such possibilities and put the constants in the right way.

Algebra generated by $H^2(S^{[n]}, \mathbb{Z})$

For integer cohomology, we get torsion for small n, e.g.

- $H^4(S^{[2]}, \mathbb{Z}) \;=\; \operatorname{Sym}^2 H^2(S^{[2]}, \mathbb{Z}) \oplus \mathbb{Z}_{2^{46}.5^2}$
- $H^4(S^{[3]}, \mathbb{Z}) \;=\; \operatorname{Sym}^2 H^2(S^{[3]}, \mathbb{Z}) \oplus \mathbb{Z}^{23} \oplus \mathbb{Z}_3$
- $H^4(S^{[n]}, \mathbb{Z}) \;=\; \operatorname{Sym}^2 H^2(S^{[n]}, \mathbb{Z}) \oplus \mathbb{Z}^{24}, \qquad n \geq 4.$

Question

Is there a geometric interpretation, e.g. for $n = 3$?

Thank you for your attention!

References

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