COMPUTING CUP-PRODUCTS IN INTEGER COHOMOLOGY OF HILBERT SCHEMES OF POINTS ON K3 SURFACES

SIMON KAPFER

Abstract. We study the cokernels of the cup product in integer cohomology of the Hilbert scheme of n points on a K3 surface. The main difference to cohomology with rational coefficients is the presence of torsion, depending on n. We particularly discuss the case $n = 3$.

1. Preliminaries

Let S be a K3 surface. We fix integral bases 1 of $H^0(S,\mathbb{Z})$, x of $H^4(X,\mathbb{Z})$ and $\alpha_1,\dots,\alpha_{22}$ of $H^2(S,\mathbb{Z})$ such that the cup product pairing on $H^2(X,\mathbb{Z})$, written as a symmetric matrix with respect to this basis, looks like

$$\begin{pmatrix} U & & & & \\ & U & & & \\ & & U & & \\ & & & E & \\ & & & & E \end{pmatrix},$$

where U stands for the intersection matrix of the hyperbolic lattice and E stands for the negative matrix of the E_8 lattice, *i.e.*

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \end{pmatrix}.$$

We denote by $S^{[n]}$ the Hilbert scheme of n points on S. An integral basis for $H^*(S^{[n]},\mathbb{Z})$ in terms of Nakajima's operators was given by Qin–Wang:

Theorem 1.1. [39, Thm. 5.4.] *The classes*

$$\frac{1}{z_\lambda}\mathfrak{q}_\lambda(1)\mathfrak{q}_\mu(x)\mathfrak{m}_{\nu^1,\alpha_1}\dots\mathfrak{m}_{\nu^{22},\alpha_{22}}|0\rangle, \quad \|\lambda\| + \|\mu\| + \sum_{i=1}^{22}\|\nu^i\| = n$$

form an integral basis for $H^*(S^{[n]},\mathbb{Z})$. Here, λ, μ, ν^i are partitions, $\|\cdot\|$ means the weight of a partition i.e. $\|\lambda\| = \sum_i m_i i$ and $z_\lambda := \prod_i i^{m_i} m_i!$, if $\lambda = (1^{m_1}, 2^{m_2}, \dots)$. The symbol $\mathfrak{q}$ stands for Nakajima's creation operator. The relation of $\mathfrak{m}_{\nu,\alpha}$ to $\mathfrak{q}_{\tilde{\nu}}(\alpha)$

Date: August 25, 2014.

is the same as the monomial symmetric functions m_ν to the power sum symmetric functions $p_{\tilde{\nu}}$.

2. Computational results

The ring structure of $H^*(S^{[n]},\mathbb{Q})$ has been studied in [24]. Since $H^{\mathrm{odd}}(S^{[n]},\mathbb{Z}) = 0$ and $H^{\mathrm{even}}(S^{[n]},\mathbb{Z})$ is torsion-free by [27], we can also apply these results to $H^*(S^{[n]},\mathbb{Z})$. A basis for cohomology with integer coefficients was given by Qin–Wang in [39] which allows us to compute explicitly the image of $H^k(S^{[n]},\mathbb{Z}) \cup H^l(S^{[n]},\mathbb{Z})$ in $H^{k+l}(S^{[n]},\mathbb{Z})$. We obtain:

Proposition 2.1. *If X is deformation equivalent to $S^{[3]}$, then:*

$$\frac{H^4(X,\mathbb{Z})}{\operatorname{Sym}^2 H^2(X,\mathbb{Z})} \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 23}$$

The torsion part of the quotient is generated by the integral class $\frac{1}{3}\mathfrak{q}_{(3)}(1)|0\rangle$.

$$\frac{H^6(X,\mathbb{Z})}{H^2(X,\mathbb{Z}) \cup H^4(X,\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12}$$

This quotient is generated by the 12 integral classes $\mathfrak{m}_{(1^3),\alpha_i}|0\rangle$, where $i \in \{1,2,3,4,5,6,8,9,11,16,17,19\}$.

Proposition 2.2.

$$\frac{H^6(S^{[4]},\mathbb{Z})}{H^2(S^{[4]},\mathbb{Z}) \cup H^4(S^{[4]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23} \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12}$$

$$\frac{H^6(S^{[5]},\mathbb{Z})}{H^2(S^{[5]},\mathbb{Z}) \cup H^4(S^{[5]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23} \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12} \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}}\right)^{\oplus 3}$$

$$\frac{H^6(S^{[6]},\mathbb{Z})}{H^2(S^{[6]},\mathbb{Z}) \cup H^4(S^{[6]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23} \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12} \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}}\right)^{\oplus 2} \oplus \mathbb{Z}$$

The free summand is generated by $\left[\frac{10}{48}\mathfrak{q}_{(2^3)}(1) - \frac{12}{6}\mathfrak{q}_{(3,2,1)}(1) + \frac{3}{8}\mathfrak{q}_{(4,1^2)}(1)\right]|0\rangle$.

Proposition 2.3.

$$\frac{H^6(S^{[3]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[3]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 243} \oplus \left(\frac{\mathbb{Z}}{4\mathbb{Z}}\right)^{\oplus 10} \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 3} \oplus \mathbb{Z}^{\oplus 507}$$

$$\frac{H^6(S^{[4]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[4]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23} \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 2} \oplus \mathbb{Z}^{\oplus 575}$$

$$\frac{H^6(S^{[5]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[5]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 22} \oplus \mathbb{Z}^{\oplus 597}$$

$$\frac{H^6(S^{[n]},\mathbb{Z})}{\operatorname{Sym}^3 H^2(S^{[n]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 22} \oplus \mathbb{Z}^{\oplus 598} \quad \textit{for } n \geq 6.$$

References

1. C. Allday and V. Puppe, *Cohomological methods in transformation groups*, Cambridge Studies in Advanced Mathematics, vol. 32, Cambridge University Press, Cambridge, 1993. MR 1236839 (94g:55009)
2. M. Artebani, A. Sarti, and S. Taki, *Non-symplectic automorphisms of prime order on $K3$ surfaces*, Math. Z. (to appear) **268** (2011), 507–533.
3. A. Beauville, *Some remarks on Kähler manifolds with $c_1 = 0$*, Classification of algebraic and analytic manifolds (Katata, 1982), Progr. Math., vol. 39, Birkhäuser Boston, Boston, MA, 1983, pp. 1–26.
4. ———, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. **18** (1983), no. 4, 755–782 (1984).
5. A. Beauville and R. Donagi, *La variété des droites d'une hypersurface cubique de dimension* 4, C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), no. 14, 703–706.
6. S. Boissière, *Automorphismes naturels de l'espace de Douady de points sur une surface*, Canad. J. Math. **64** (2012), 3–23.
7. S. Boissière, M. Nieper-Wißkirchen, and A. Sarti, *Higher dimensional Enriques varieties and automorphisms of generalized kummer varieties*, J. Math. Pures Appl. **95** (2011), 553–563.
8. A. Borel, *Seminar on transformation groups*, With contributions by G. Bredon, E. E. Floyd, D. Montgomery, R. Palais. Annals of Mathematics Studies, No. 46, Princeton University Press, Princeton, N.J., 1960.
9. H. Brandt and O. Intrau, *Tabellen reduzierter positiver ternärer quadratischer Formen*, Abh. Sächs. Akad. Wiss. Math.-Nat. Kl. **45** (1958), no. 4, 261.
10. G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972, Pure and Applied Mathematics, Vol. 46.
11. K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original. MR 1324339 (96a:20072)
12. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1988, Reprint of the 1962 original, A Wiley-Interscience Publication. MR 1013113 (90g:16001)
13. P. Deligne, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 35, 259–278.
14. W. G. Dwyer and C. W. Wilkerson, *Smith theory revisited*, Ann. of Math. (2) **127** (1988), no. 1, 191–198.
15. A. D. Elagin, *On an equivariant derived category of bundles of projective spaces*, Tr. Mat. Inst. Steklova **264** (2009), no. Mnogomernaya Algebraicheskaya Geometriya, 63–68.
16. G. Ellingsrud, L. Göttsche, and M. Lehn, *On the cobordism class of the Hilbert scheme of a surface*, J. Algebraic Geom. **10** (2001), no. 1, 81–100.
17. A. Fujiki, *On the de Rham cohomology group of a compact Kähler symplectic manifold*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 105–165.
18. A. Garbagnati and A. Sarti, *Symplectic automorphisms of prime order on $K3$ surfaces*, J. Algebra **318** (2007), no. 1, 323–350.
19. B. Hassett, *Special cubic fourfolds*, Compositio Math. **120** (2000), no. 1, 1–23.
20. D. Huybrechts, *Compact hyper-Kähler manifolds: basic results*, Invent. Math. **135** (1999), no. 1, 63–113.
21. V. M. Kharlamov, *The topological type of nonsingular surfaces in $\mathbf{R}P^3$ of degree four*, Funct. Anal. Appl. **10** (1976), no. 4, 295–304.
22. V. A. Krasnov, *Harnack-Thom inequalities for mappings of real algebraic varieties*, Izv. Akad. Nauk SSSR Ser. Mat. **47** (1983), no. 2, 268–297.
23. ———, *Real algebraic GM-manifolds*, Izv. Ross. Akad. Nauk Ser. Mat. **62** (1998), no. 3, 39–66.
24. M. Lehn and C. Sorger, *The cup product of Hilbert schemes for $K3$ surfaces*, Invent. Math. **152** (2003), no. 2, 305–329.
25. E. Markman, *The Beauville–Bogomolov class as a characteristic class*, `arXiv:1105:3223v1`.
26. ———, *A survey of Torelli and monodromy results for holomorphic-symplectic varieties*, `arXiv:1101.4606v2`.

27. ———, *Integral generators for the cohomology ring of moduli spaces of sheaves over Poisson surfaces*, Adv. Math. **208** (2007), no. 2, 622–646.
28. ———, *Integral constraints on the monodromy group of the hyperKähler resolution of a symmetric product of a $K3$ surface*, Internat. J. Math. **21** (2010), no. 2, 169–223.
29. D. Markushevich, *Rational Lagrangian fibrations on punctual Hilbert schemes of $K3$ surfaces*, Manuscripta Math. **120** (2006), no. 2, 131–150.
30. J. M. Masley and H. L. Montgomery, *Cyclotomic fields with unique factorization*, J. Reine Angew. Math. **286/287** (1976), 248–256.
31. G. Mongardi, *On symplectic automorphisms of hyperkähler fourfolds*, `arXiv:1112.5073v3`.
32. ———, *Symplectic involutions on deformations of $K3^{[2]}$*, `arXiv:1107.2854`.
33. H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Ann. of Math. (2) **145** (1997), no. 2, 379–388.
34. M. Nieper-Wißkirchen, *Chern numbers and Rozansky-Witten invariants of compact hyper-Kähler manifolds*, World Scientific Publishing Co. Inc., River Edge, NJ, 2004.
35. V. V. Nikulin, *Finite groups of automorphisms of Kählerian $K3$ surfaces*, Trudy Moskov. Mat. Obshch. **38** (1979), 75–137.
36. ———, *Integral symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 1, 111–177.
37. Kieran G. O'Grady, *Irreducible symplectic 4-folds numerically equivalent to $(K3)^{[2]}$*, Commun. Contemp. Math. **10** (2008), no. 4, 553–608.
38. K. Oguiso and S. Schröer, *Enriques manifolds*, J. Reine Angew. Math. (to appear) (2011).
39. Z. Qin and W. Wang, *Integral operators and integral cohomology classes of Hilbert schemes*, Math. Ann. **331** (2005), no. 3, 669–692.
40. M. Verbitsky, *Cohomology of compact hyper-Kähler manifolds and its applications*, Geom. Funct. Anal. **6** (1996), no. 4, 601–611.

Simon Kapfer, Lehrstuhl für Algebra und Zahlentheorie, Universitätsstrasse 14, D-86159 Augsburg

E-mail address: `simon.kapfer@math.uni-augsburg.de`