SYMMETRIC POWERS OF SYMMETRIC BILINEAR FORMS, HOMOGENEOUS ORTHOGONAL POLYNOMIALS ON THE SPHERE AND AN APPLICATION IN COMPACT HYPERKÄHLER MANIFOLDS

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Abstract. We study a construction for a symmetric bilinear form on the space $\operatorname{Sym}^k V$, derived from a form on V. We point out that it is related to integrating homogeneous polynomials over a sphere and give an orthogonal basis of such polynomials. Since the construction also applies to the Beauville–Fujiki relation from Hyperkähler theory, we remark some consequences for integer cohomology of compact Hyperkähler manifolds.

1. Symmetric Bilinear Forms on Symmetric Powers

Let V be a vector space (or a free module) over a field (resp. a commutative ring) K of rank $d+1$ with basis $\{x_0, \ldots, x_d\}$, equipped with a symmetric bilinear form $\langle\ ,\ \rangle : V \times V \to K$. We will freely identify the symmetric power $\operatorname{Sym}^k V$ with the space of homogeneous polynomials $K[x_0, \ldots, x_d]_k$ of degree k.

There are at least two possibilities to define an induced bilinear form on $\operatorname{Sym}^k V$. The first one is to define on monomials:

$$(1) \qquad (\!(x_{n_1} \ldots x_{n_k}\, ,\, x_{m_1} \ldots x_{m_k})\!) := \sum_{\sigma} \prod_{i=1}^{k} \left\langle x_{n_i}, x_{m_{\sigma(i)}} \right\rangle ,$$

the sum being over all permutations of $\{1, \ldots, k\}$, as studied by McGarraghy in [7]. However, we will *not* consider this construction. Instead, we make the following

Definition 1.1. On the basis $\{x_{n_1} \ldots x_{n_k} \mid 0 \leq n_1 \leq \ldots \leq n_k \leq d\}$ of $\operatorname{Sym}^k V$, we define a symmetric bilinear form $\langle\!\langle\ ,\ \rangle\!\rangle$ by:

$$(2) \qquad \left\langle\!\!\left\langle x_{n_1} \ldots x_{n_k}\, ,\, x_{n_{k+1}} \ldots x_{n_{2k}} \right\rangle\!\!\right\rangle := \sum_{\mathcal{P}} \prod_{\{i,j\} \in \mathcal{P}} \left\langle x_{n_i}, x_{n_j} \right\rangle ,$$

where the sum is over all partitions $\mathcal{P}$ of $\{1, \ldots, 2k\}$ into pairs.

If $U \in O(V)$ is an orthogonal transformation, then the induced diagonal action of $U^{\otimes k}$ on $\operatorname{Sym}^k V$ is orthogonal in both cases. This shows that the two definitions are independent of the choice of the base of V.

Example 1.2. To contrast the two definitions, observe that in the case $k = 2$

$$(3) \qquad (\!(ab, cd)\!) = \langle a, c\rangle \langle b, d\rangle + \langle a, d\rangle \langle b, c\rangle ,$$

$$(4) \qquad \langle\!\langle ab, cd\rangle\!\rangle = \langle a, c\rangle \langle b, d\rangle + \langle a, d\rangle \langle b, c\rangle + \langle a, b\rangle \langle c, d\rangle .$$

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Remark 1.3. Note that (1) does not require symmetry of the bilinear form $\langle\ ,\ \rangle$ on V. Indeed, the definition would also be valid for a completely arbitrary bilinear form : $V \times W \to K$, yielding a bilinear form : $\mathrm{Sym}^k V \times \mathrm{Sym}^k W \to K$. On the other hand, if the form on V is not symmetric, then (2) is not well-defined.

Remark 1.4. The defining equation (2) works equally well, if the two arguments have different degree. So we can easily extend our definition to a symmetric bilinear form $\langle\!\langle\ ,\ \rangle\!\rangle : \mathrm{Sym}^* V \times \mathrm{Sym}^* V \to K$. Note that then $\mathrm{Sym}^k V$ is in general not orthogonal to $\mathrm{Sym}^l V$ unless $k-l$ is an odd number.

We wish to investigate some properties of this construction. Let G be the Gram matrix of $\langle\ ,\ \rangle$, *i.e.* $G_{ij} = \langle x_i, x_j\rangle$ and let $\mathbb{G}$ be the Gram matrix of $\langle\!\langle\ ,\ \rangle\!\rangle$. We use multi-index notation, cf. Definition 2.1.

Proposition 1.5. *Assume $K = \mathbb{R}$ and G is positive definite, so G^{-1} exists. Then $\langle\!\langle\ ,\ \rangle\!\rangle$ takes a nice integral form:*

$$\langle\!\langle x^\alpha, x^\beta \rangle\!\rangle = \frac{1}{c}\int_{\mathbb{R}^{d+1}} x^\alpha x^\beta \, d\mu(x). \tag{5}$$

Here the integration measure is $d\mu(x) = \exp\left(-\frac{1}{2}\sum_{i,j} G^{-1}_{ij} x_i x_j\right) dx$ and the normalization constant is $c = \int_{\mathbb{R}^{n+1}} d\mu(x) = \sqrt{(2\pi)^{d+1}\det G}$.

Proof. We make use of the content in Section 2. First, observe that both sides of the equation are invariant under orthogonal transformations of the base space $\mathbb{R}^{d+1}$. We may therefore assume that $G = \mathrm{diag}\,(a_0, \dots, a_d)$ is a diagonal matrix. Then the integral splits nicely:

$$\begin{aligned} \frac{1}{c}\int_{\mathbb{R}^{d+1}} x^\alpha x^\beta d\mu(x) &= \frac{1}{c}\prod_{i=0}^{d}\int_{-\infty}^{\infty} x_i^{\alpha_i+\beta_i} e^{-\frac{1}{2a_i}x_i^2}\, dx_i \\ &= \frac{1}{c}\prod_{i=0}^{d} a_i^{\frac{\alpha_i+\beta_i+1}{2}} \int_{-\infty}^{\infty} x^{\alpha_i+\beta_i} e^{-\frac{1}{2}x^2}\, dx \\ &= \begin{cases} \displaystyle\prod_{i=0}^{d} a_i^{\frac{\alpha_i+\beta_i}{2}} (\alpha_i+\beta_i-1)!! & \text{if all } \alpha_i+\beta_i \text{ are even,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand, if G is diagonal, then every partition into pairs of the multiset $\{n^{\alpha_n+\beta_n} \,|\, 0 \le n \le d\}$ that contains a pair of two different numbers will not contribute to the sum in the defining equation (2). It is not hard to see that we obtain the same formula for $\langle\!\langle x^\alpha, x^\beta \rangle\!\rangle$. □

The next theorem gives a formula for the determinant of $\mathbb{G}$. This is of particular interest when $K = \mathbb{Z}$, because in this case we are doing lattice theory, and $\det \mathbb{G}$ is an important lattice-theoretic invariant, called the discriminant of the lattice. We prove the theorem for $K = \mathbb{R}$, using the above proposition, but the final formula is purely algebraic. So we conclude firstly, by analytic continuation, that we can drop the positive definiteness for G, which is an open condition. Secondly, we see that the coefficients are all integer, so the formula is universal in the sense that it holds over any commutative ring K.

Theorem 1.6. *The determinant of the Gram matrix $\mathbb{G}$ of $\langle\!\langle\ ,\ \rangle\!\rangle$ is:*

$$\det(\mathbb{G}) = \det(G)^{\binom{d+k}{d+1}}\,\theta_{d,k} \tag{6}$$

where $\theta_{d,k}$ is a combinatorial factor given by:

$$\theta_{d,k} = \begin{cases} \prod_{i=1}^{k} i^{\binom{k-i+d}{d}d} \prod_{\substack{i=1 \\ i \text{ odd}}}^{2k+d-1} i^{\binom{k-i+d}{d}} & \text{if } d \text{ is even,} \\ \prod_{i=1}^{k} i^{\binom{k-i+d}{d}d} \prod_{i=1}^{k+\frac{d-1}{2}} i^{\binom{k-i+d}{d}-\binom{k-2i+d}{d}} & \text{if } d \text{ is odd.} \end{cases} \tag{7}$$

Remark 1.7. If d or k is small, this simplifies as follows:

$$\theta_{d,0} = \theta_{d,1} = 1, \qquad \theta_{d,2} = 2^d(d+3),$$
$$\theta_{0,k} = (2k-1)!!, \qquad \theta_{1,k} = (k!)^{k+1}.$$

Proof. Note that one can think of $\sqrt{\det \mathbb{G}}$ as the volume of the parallelotope that is spanned by the monomials in some polynomial vector space. Let us first check, what happens if we apply a coordinate transformation $x \mapsto \tilde{x}$ that changes the last coordinate by $\tilde{x}_d = \gamma x_d$ and leaves the other coordinates invariant. We clearly have: $\tilde{x}^\alpha = \gamma^{\alpha_d} x^\alpha$. Let $\tilde{G}$ and $\tilde{\mathbb{G}}$ be the Gram matrices corresponding to the new coordinates. Extracting the factor $\gamma^2 =: a$, we get:

$$\frac{\det \tilde{\mathbb{G}}}{\det \mathbb{G}} = \prod_{|\alpha|=k} a^{\alpha_d} = \prod_{i=0}^{k} \prod_{|\alpha'|=k-i} a^i = \prod_{i=0}^{k} a^{i\binom{d-1+k-i}{d-1}} \overset{(10)}{=} a^{\binom{d+k}{d+1}}.$$

Now we can reduce the proof to the case when G is the identity matrix. Then the statement is established by Corollary 3.14. By an orthogonal transformation we may assume that $G = \operatorname{diag}(a_0, \dots, a_d)$ is a diagonal matrix. Then we apply successively coordinate transformations that map x_i to $\frac{x_i}{\sqrt{a_i}}$. We get a factor $(a_0 \dots a_d)^{\binom{d+k}{d+1}} = \det G^{\binom{d+k}{d+1}}$. □

2. Terminology and helper formulas

In this section we give a few standard definitions and recall some facts on elementary calculus. We also mention some technical formulas needed for our proofs.

Definition 2.1. For a multi-index $\alpha = (\alpha_0, \dots, \alpha_d)$ we define: $x^\alpha := x_0^{\alpha_0} \dots x_d^{\alpha_d}$. The degree is defined by $|\alpha| := \sum \alpha_i$, the factorial is $\alpha! := \prod \alpha_i!$. Further, we set $\alpha' := (\alpha_0, \dots, \alpha_{d-1})$. We introduce the lexicographical ordering on multi-indices: $\alpha < \beta$ iff $\alpha_d < \beta_d$ or $(\alpha_d = \beta_d) \wedge (\alpha' < \beta')$.

Definition 2.2. The binomial coefficient for nonnegative integers k and arbitrary z is defined as: $\binom{z}{k} := \frac{z(z-1)\dots(z-k+1)}{k!}$. For negative k we set $\binom{z}{k} := 0$. Thus we have $\binom{-z}{k} = (-1)^k\binom{z+k-1}{k}$. The rank of $\operatorname{Sym}^k K^{d+1}$ equals $\binom{d+k}{k} = \binom{d+k}{d}$.

We introduce the difference operator $\Delta f(n) := f(n+1) - f(n)$. It has the following properties similar to the differential operator:

$$\sum_{i=0}^{n} \Delta(f) = f\Big|_0^{n+1} = f(n+1) - f(0) \qquad \text{(telescoping sum)} \tag{8}$$

$$\Delta(fg)(n) = f(n+1)\Delta g(n) + g(n)\Delta f(n) \qquad \text{(product rule)} \tag{9}$$

$$\sum_{i=0}^{n} g(i)\Delta f(i) = (fg)\Big|_0^{n+1} - \sum_{i=0}^{n} f(i+1)\Delta g(i) \qquad \text{(summation by parts)} \tag{10}$$

This often applies to the binomial coefficient, since we have:

$$\Delta\tbinom{n}{k} = \tbinom{n+1}{n} - \tbinom{n}{k} = \tbinom{n}{k-1}. \tag{11}$$

The following identity for integers $d, k, m \geq 0$ is proven by induction over k:

$$\prod_{j=0}^{k} (k-j)!^{\binom{j+d-1}{d-1}} = \prod_{i=1}^{k} i^{\binom{k-i+d}{d}} \tag{12}$$

We also need the following identity:

$$\sum_{\substack{i=1 \\ i \text{ even}}}^{2k+d+1} \tbinom{k-i+d}{d-1} = \begin{cases} 0 & \text{if } d \text{ is even,} \\ \binom{k+d}{d} & \text{if } d \text{ is odd,} \end{cases} \tag{13}$$

which is proven by splitting the sum into:

$$\sum_{\substack{i=1 \\ i \text{ even}}}^{k+1} \tbinom{k-i+d}{d-1} + \sum_{\substack{i=k+d+1 \\ i \text{ even}}}^{2k+d+1} \tbinom{k-i+d}{d-1} = \sum_{\substack{i=1 \\ k-i \text{ even}}}^{k+1} \tbinom{i+d-2}{d-1} + (-1)^{d-1} \sum_{\substack{i=1 \\ k+d+i \text{ even}}}^{k+1} \tbinom{i+d-2}{d-1}.$$

Definition 2.3. We define the double factorial for $n \geq -1$ by

$$n!! := \prod_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (n-2i) = n(n-2)(n-4)\dots$$

Clearly, $(n-1)!!\, n!! = n!$ and $(2n)!! = 2^n n!$.

Proposition 2.4. *The number of partitions of the set $\{1, \dots, 2k\}$ into pairs equals $(2k-1)!! = \frac{(2k)!}{2^k k!}$.*

Proof. Given such a partition, look at the pair that contains the element 1. There are $2k-1$ possible partners for this element; removing the pair leaves a partition of a $(2k-2)$–elementary set into pairs. Then proceed by induction. $\square$

Denote $\Gamma(t) := \int_0^\infty r^{t-1} e^{-r} dr$ the gamma function. It satisfies:

$$n! = \Gamma(n+1), \qquad (2n-1)!!\sqrt{\pi} = 2^n \Gamma\left(n + \tfrac{1}{2}\right), \tag{14}$$

$$n!\sqrt{\pi} = 2^n \Gamma\left(\tfrac{n}{2} + 1\right) \Gamma\left(\tfrac{n+1}{2}\right), \tag{15}$$

$$\int_0^\infty r^s e^{-\frac{1}{2}r^2} dr = 2^{\frac{s-1}{2}} \Gamma\left(\tfrac{s+1}{2}\right). \tag{16}$$

Using symmetry, it follows immediately that:

$$
\begin{aligned}
(17) \qquad \int_{\mathbb{R}^{d+1}} x^\alpha x^\beta e^{-\frac{1}{2}\|x\|^2} dx &= \prod_{i=0}^{d} \int_{-\infty}^{\infty} x_i^{\alpha_i+\beta_i} e^{-\frac{1}{2}x_i^2} dx_i \\
&= \begin{cases} (2\pi)^{\frac{d+1}{2}} \prod_{i=0}^{d} (\alpha_i+\beta_i-1)!! & \text{if all } \alpha_i+\beta_i \text{ are even,} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

The reader may also consult [4].

Lemma 2.5. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be an integrable homogeneous function of degree k, that is $f(sx) = s^k f(x)$. Using polar coordinates $(r, \omega) = (\|x\|, \frac{x}{\|x\|})$, we get:*

$$
\begin{aligned}
\int_{\mathbb{R}^{d+1}} f(x) e^{-\frac{1}{2}\|x\|^2} dx &= \int_{\mathbb{S}^d} \int_0^\infty f(r\omega) r^d e^{-\frac{1}{2}r^2} dr d\omega \\
&= 2^{\frac{k+d-1}{2}} \Gamma\left(\tfrac{k+d+1}{2}\right) \int_{\mathbb{S}^d} f(\omega) d\omega.
\end{aligned}
$$

3. Homogeneous Orthogonal Polynomials on the sphere

In this section we will construct a basis for the space of homogeneous polynomials of degree k in $d+1$ variables, $\mathbb{R}[x_0, \ldots, x_d]_k$, that is orthogonal with respect to the bilinear form given by

$$\langle\!\langle f, g \rangle\!\rangle = \int_{\mathbb{R}^{d+1}} f(x) g(x) d\mu(x),$$

where the measure is $d\mu(x) = (2\pi)^{-\frac{d+1}{2}} e^{-\frac{1}{2}\|x\|^2} dx$. In order to do this, we wish to apply the Gram-Schmidt process to the (lexicographically ordered) monomial basis $(x^\alpha)_{|\alpha|=k}$. Our result is stated in Subsection 3.3.

Remark 3.1. In view of Lemma 2.5, we could equivalently integrate the homogeneous polynomials over the unit sphere $\mathbb{S}^d$. This is the reason why we call them orthogonal on the sphere.

Remark 3.2. If the homogeneity constraint was dropped, the answer to the problem would be much simpler: A basis of $\langle\!\langle\ ,\ \rangle\!\rangle$-orthogonal polynomials is given by products $H_{\alpha_0}(x_0) \ldots H_{\alpha_d}(x_d)$ of Hermite polynomials in one variable, see also [3, Sect. 2.3.4].

3.1. **Generalities on orthogonal polynomials in one variable.** Given a nondegenerate symmetric bilinear form on the space of polynomials $K[x]$, one may ask for a basis of polynomials $(p_n)_n$ that are mutually orthogonal with respect to that form. To find such a basis, one could start with the monomial basis $(x^n)_n$ and apply some version of the Gram–Schmidt algorithm. The result should be an infinite lower triangular matrix T such that $p_n = \sum_j T_{nj} x^j$. If our bilinear form now depends only on the product of its two arguments, the procedure simplifies as follows:

Let $\mathcal{L}$ be a linear functional such that the induced bilinear form $(f, g) = \mathcal{L}(fg)$ is nondegenerate when restricted to $K[x]_{\leq n}$, the space of polynomials of bounded degree, for all $n \geq 0$. Let $(p_n)_n$ be the associated sequence of monic orthogonal polynomials, *i.e.* the leading term of $p_n(x)$ is x^n and $(p_k, p_n) = 0$ for $k \neq n$. Then we have

Theorem 3.3. [2, Thm. 4.1] *There are constants c_n, d_n such that*

$$p_0(x) = 1, \qquad p_{n+1}(x) = (x - c_n)p_n(x) - d_n p_{n-1}(x).$$

But also the converse is true:

Theorem 3.4 (Favard's theorem)**.** [2, Thm. 4.4] *Let $(p_n)_n$ be a sequence of polynomials, such that $\deg p_n = n$ and the following three-term recurrence holds:*

$$p_0(x) = 1, \qquad p_{n+1}(x) = (x - c_n)p_n(x) - d_n p_{n-1}(x).$$

Then there exists a unique linear functional $\mathcal{L}$ such that $\mathcal{L}(1) = 1$ and $\mathcal{L}(p_k p_n) = 0$ for $k \neq n$.

Theorem 3.5. [2, Thm. 4.2] *In this situation, we have for $n \geq 1$:*

$$\mathcal{L}(p_n^2) = d_n \mathcal{L}(p_{n-1}^2).$$

Since we shall deal with finite polynomial families, we need a little modification of Favard's theorem:

Remark 3.6. If $(p_n)_{n \leq N}$ is a finite sequence that satisfies a three-term recurrence as above, then we can always extend it to an infinite sequence by choosing arbitrary constants c_n, d_n for $n \geq N$. But for every such extension, the resulting functional $\mathcal{L}$ from Favard's theorem will satisfy $\mathcal{L}(1) = 1$ and $\mathcal{L}(p_n) = \mathcal{L}(p_n p_0) = 0$ for $n \geq 1$. So $\mathcal{L}$ will always be uniquely determined on $K[x]_{\leq N}$, the space of degree-bounded polynomials.

3.2. **A polynomial family.**

Definition 3.7. Let n, m be integers with $0 \leq 2n \leq m + 1$, a condition that we always will assume silently. We define polynomials p_n^m of degree n:

$$p_n^m(x) := \sum_{\substack{l=0 \\ n-l \text{ even}}}^{n} \frac{n!\,(m-2n)!!\,(-1)^{\frac{n-l}{2}}}{l!\,(m-n-l)!!\,(n-l)!!}\, x^l. \tag{18}$$

It is straightforward to check that the following two identities hold.

Lemma 3.8. *For $n \geq 1$, we have a trigonometric differential relation:*

$$\frac{d}{d\omega}\Big[p_{n-1}^{m-2}\big(\tan(\omega)\big)\cos(\omega)^{m-1}\Big] = (n-m)\, p_n^m\big(\tan(\omega)\big)\cos(\omega)^{m-1} \tag{19}$$

□

Proposition 3.9. *For $0 \leq 2n \leq m - 1$, we have a three-term recurrence:*

$$p_0^m(x) = 1, \qquad p_1^m(x) = x, \qquad p_{n+1}^m(x) = x p_n^m(x) - d_n^m p_{n-1}^m(x), \tag{20}$$

where $d_n^m := \frac{n(m-n+1)}{(m-2n)(m-2n+2)}$. □

Theorem 3.10. *We define, for $m \geq 1$, a linear functional $\mathcal{L}$ on the vector space of polynomials of degree less than m, by setting*

$$\mathcal{L} : f \longmapsto \int_0^\infty \int_{-\infty}^\infty z^{m-1} f\left(\frac{y}{z}\right) e^{-\frac{y^2+z^2}{2}}\, dy dz. \tag{21}$$

Then the p_n^m form a set of orthogonal polynomials with respect to the induced bilinear form, i.e. for $k \neq n$, $k+n \leq m-1$ we have $\mathcal{L}(p_k^m p_n^m) = 0$ and for $2n \leq m-1$:

$$\mathcal{L}(p_n^m p_n^m) = 2^{\frac{3}{2}m - \frac{1}{2} - 2n} \frac{n!}{(m-n)!} \Gamma\left(\frac{m}{2} - n\right)\Gamma\left(\frac{m}{2} - n + 1\right)\Gamma\left(\frac{m+1}{2}\right). \tag{22}$$

Proof. Since p_n^m satisfy the three-term relation in Prop. 3.9, by Favard's theorem and Remark 3.6, there exists a unique functional $\mathcal{L}'$ with $\mathcal{L}'(1) = 1$, such that the p_n^m form an orthogonal basis with respect to the bilinear form induced by $\mathcal{L}'$. We claim that $\mathcal{L}$ is a scalar multiple of $\mathcal{L}'$. Since $(p_n^m)_n$ is a basis of the space of polynomials, we must show that, for $n \geq 1$, $\mathcal{L}(p_n^m) = \mathcal{L}(p_n^m p_0^m) = 0$. Using polar coordinates $(y, z) = (r\cos\omega, r\sin\omega)$ and Lemma 3.8, we get:

$$\begin{aligned}\mathcal{L}(p_n^m) &= \int_0^\pi \int_0^\infty p_n^m\big(\tfrac{\cos\omega}{\sin\omega}\big)\sin(\omega)^{m-1} r^m e^{-\frac{r^2}{2}}\,dr d\omega \\ &= \int_0^\infty r^m e^{-\frac{r^2}{2}}\,dr \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (-1)^n p_n^m\big(\tan(\omega)\big)\cos(\omega)^{m-1} d\omega \\ &= 2^{\frac{m-1}{2}}\Gamma\left(\tfrac{m+1}{2}\right)\left[\tfrac{(-1)^n}{n-m} p_{n-1}^{m-2}\big(\tan(\omega)\big)\cos(\omega)^{m-1}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = 0,\end{aligned}$$

while $\mathcal{L}(1) = 2^{\frac{m-1}{2}}\sqrt{\pi}\,\Gamma\big(\frac{m}{2}\big) = 2^{\frac{3m-1}{2}}\frac{1}{m!}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{m+1}{2}\right)$ by (17) and (15). To verify that equation (22) holds for $n \geq 1$, too, check that the right hand side satisfies $\mathcal{L}(p_n^m p_n^m) = d_n^m \mathcal{L}(p_{n-1}^m p_{n-1}^m)$, the recurrence from Theorem 3.5. □

Corollary 3.11. *$\mathcal{L}(x^k p_n^m) = 0$ for $k < n$ and $\mathcal{L}(x^n p_n^m) = \mathcal{L}(p_n^m p_n^m)$.*

Proof. The first assertion is established by the theorem in the case $k = 0$. Now the three-term recurrence from Proposition 3.9 allows us to inductively conclude that $\mathcal{L}(x^k p_n^m) = \mathcal{L}(x^{k-1} p_{n+1}^m) + d_n^m \mathcal{L}(x^{k-1} p_{n-1}^m) = 0$. The second assertion is trivial in the case $n \leq 1$. For $n \geq 1$, the same argument as above yields now $\mathcal{L}(x^n p_n^m) = \mathcal{L}(x^{n-1} p_{n+1}^m) + d_n^m \mathcal{L}(x^{n-1} p_{n-1}^m) = d_n^m \mathcal{L}(x^{n-1} p_{n-1}^m)$, so $\mathcal{L}(x^n p_n^m)$ and $\mathcal{L}(p_n^m p_n^m)$ must be equal because they satisfy the same recurrence relation. □

3.3. **Homogeneous orthogonal polynomials.** We are now ready to give the desired basis of homogeneous polynomials that are orthogonal on the sphere. Inspired by the procedure for spherical harmonics, see [3, p. 35], we make the following

Definition 3.12. For multi-indices $\alpha = (\alpha_0, \ldots, \alpha_d)$ we recursively define homogeneous polynomials h_α of degree $|\alpha|$ by $h_{(\alpha_0)}(x) := x_0^{\alpha_0}$ and, for $d \geq 1$,

$$h_\alpha(x) := p_{\alpha_d}^{2|\alpha|+d}\left(\frac{x_d}{r}\right) r^{\alpha_d} h_{\alpha'}(x'),$$

where we have set $r = \sqrt{x_0^2 + \ldots + x_{d-1}^2} = \|x'\|$.

Theorem 3.13. *Fix a degree k. Then the $h_\alpha(x)$, $|\alpha| = k$ form an orthogonal basis of $\mathbb{R}[x_0, \ldots, x_d]_k$ that comes from a Gram–Schmidt process applied to the monomials. More precisely, we have:*

$$\langle\!\langle h_\alpha, h_\alpha \rangle\!\rangle = \alpha_d!\,\frac{(2|\alpha'|+d)!!\;(2|\alpha|+d-1)!!}{(|\alpha'|+|\alpha|+d)!}\,\langle\!\langle h_{\alpha'}, h_{\alpha'} \rangle\!\rangle\,, \tag{23}$$

$$\langle\!\langle h_\alpha, h_\beta \rangle\!\rangle = 0 \quad \textit{for } \alpha \neq \beta, \tag{24}$$

$$\langle\!\langle x^\alpha, h_\alpha \rangle\!\rangle = \langle\!\langle h_\alpha, h_\alpha \rangle\!\rangle\,, \tag{25}$$

$$\langle\!\langle x^\alpha, h_\beta \rangle\!\rangle = 0 \quad \textit{for } \alpha < \beta. \tag{26}$$

Proof. For equation (23), we use polar coordinates on $\mathbb{R}^d$ to compute:

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}} \left[p_{\alpha_d}^{2|\alpha|+d}\left(\frac{x_d}{r}\right) r^{\alpha_d} h_{\alpha'}(x)\right]^2 e^{-\frac{1}{2}\|x\|^2} dx \\
&= \int_0^\infty \int_{\mathbb{R}} \left[p_{\alpha_d}^{2|\alpha|+d}\left(\frac{x_d}{r}\right)\right]^2 r^{2|\alpha'|+2\alpha_d+d-1} e^{-\frac{r^2+x_d^2}{2}} dx_d dr \int_{\mathbb{S}^{d-1}} \left[h_{\alpha'}(\omega)\right]^2 d\omega \\
&\overset{(22)}{=} \frac{\alpha_d!\, 2^{2|\alpha'|+|\alpha|+\frac{3}{2}d-\frac{1}{2}}}{(|\alpha|+|\alpha'|+d)!} \Gamma\big(|\alpha'|+\tfrac{d}{2}\big)\, \Gamma\big(|\alpha'|+\tfrac{d}{2}+1\big)\, \Gamma\big(|\alpha|+\tfrac{d+1}{2}\big) \int_{\mathbb{S}^{d-1}} \left[h_{\alpha'}(\omega)\right]^2 d\omega \\
&\overset{\text{Lemma 2.5}}{=} \alpha_d!\, 2^{|\alpha'|+|\alpha|+d+\frac{1}{2}} \frac{\Gamma\big(|\alpha'|+\frac{d}{2}+1\big)\, \Gamma\big(|\alpha|+\frac{d+1}{2}\big)}{(|\alpha'|+|\alpha|+d)!} \int_{\mathbb{R}^d} \left[h_{\alpha'}(x')\right]^2 dx' \\
&\overset{(14)}{=} \alpha_d! \frac{(2|\alpha'|+d)!!\; (2|\alpha|+d-1)!!}{(|\alpha'|+|\alpha|+d)!} \sqrt{2\pi} \int_{\mathbb{R}^d} \left[h_{\alpha'}(x')\right]^2 dx'.
\end{aligned}$$

For the proof of (24), we may assume that $\alpha_d \neq \beta_d$. Then we use the calculation above to see that Thm. 3.10 now implies the vanishing of the integral. Equations (25) and (26) follow from Corollary 3.11 in the same way. □

Corollary 3.14. *Let* $D(d,k) := \det\big(\langle\!\langle x^\alpha, x^\beta \rangle\!\rangle\big)_{|\alpha|,|\beta|=k}$ *be the determinant of the Gram matrix of* $\langle\!\langle\ ,\ \rangle\!\rangle$*. Then:*

$$D(d,k) = \theta_{d,k}$$

where $\theta_{d,k}$ *is defined as in Equation (7).*

Proof. We do a double induction over k and d. First check that $D(d,0) = \theta_{d,0} = 1$ and $D(0,k) = \theta_{0,k} = (2k-1)!!$. From the above theorem it is clear that $D(d,k) = \prod_{|\alpha|=k} \langle\!\langle h_\alpha, h_\alpha \rangle\!\rangle$. Since $\{|\alpha| = k\} = \bigcup_{j=0}^k \{|\alpha'| = j\} \times \{k-j\}$, we have from Equation (23):

$$D(d,k) = \prod_{j=0}^k D(d-1,j) \prod_{|\alpha'|=j} (k-j)!\, \frac{(2j+d)!!\,(2k+d-1)!!}{(j+k+d)!},$$

hence

$$\begin{aligned}
\frac{D(d,k)}{\prod_{j=0}^k D(d-1,j)} &= \prod_{j=0}^k \left[\frac{(2j+d)!!\,(2k+d-1)!!}{(j+k+d)!}(k-j)!\right]^{\binom{j+d-1}{d-1}} \\
&\overset{(12)}{=} \prod_{j=0}^k \left[\frac{(2j+d)!!}{(j+k+d)!!}\right]^{\binom{j+d-1}{d-1}} (2k+d-1)^{\binom{k+d}{d}} \prod_{i=1}^k i^{\binom{k-i+d}{d}} =: R(d,k).
\end{aligned}$$

We will now show the principal inductive step: $\frac{D(d,k+1)}{D(d,k)D(d-1,k+1)} = \frac{\theta_{d,k+1}}{\theta_{d,k}\theta_{d-1,k+1}}$. The left hand side clearly equals

$$\begin{aligned}
\frac{R(d,k+1)}{R(d,k)} &= \frac{(2k+d+2)!!^{\binom{k+d}{d-1}}\,(2k+d+1)^{\binom{k+d+1}{d}}\,(2k+d+1)!!^{\binom{k+d}{d-1}}}{(2k+d+2)!^{\binom{k+d}{d-1}} \prod_{j=0}^k (j+k+d+1)^{\binom{j+d-1}{d-1}}} \prod_{i=1}^{k+1} i^{\binom{k-i+d}{d-1}} \\
&= \frac{(2k+d+1)^{\binom{k+d}{d}}}{\prod_{i=k+d+1}^{2k+d+1} i^{\binom{k-i+d}{d-1}}} \prod_{i=1}^{k+1} i^{\binom{k-i+d}{d-1}}.
\end{aligned}$$

If we split $\theta_{d,k} = A(d,k)B(d,k)$ with $A(d,k) := \prod_{i=1}^{k} i^{\binom{k-i+d}{d}d}$, we see that

$$\frac{A(d,k+1)}{A(d,k)A(d-1,k+1)} = \prod_{i=1}^{k+1} i^{\binom{k-i+d+1}{d}d-\binom{k-i+d}{d}d-\binom{k-i+d}{d-1}(d-1)} \overset{(11)}{=} \prod_{i=1}^{k+1} i^{\binom{k-i+d}{d-1}},$$

while the other factor $B(d,k)$ gives, for even d,

$$\frac{B(d,k+1)}{B(d,k)B(d-1,k+1)} = \frac{(2k+d+1)^{\binom{-k-1}{d}} \prod\limits_{\substack{i=1\\ i \text{ odd}}}^{2k+d+1} i^{\binom{k-i+d+1}{d}-\binom{k-i+d}{d}}}{\prod\limits_{i=1}^{k+\frac{d}{2}} i^{\binom{k-i+d}{d-1}} \prod\limits_{\substack{i=1\\ i \text{ even}}}^{2k+d} \left(\frac{i}{2}\right)^{-\binom{k-i+d}{d-1}}}$$

$$\overset{11}{=} \frac{(2k+d+1)^{\binom{k+d}{d}} \prod\limits_{i=1}^{2k+d+1} i^{\binom{k-i+d}{d-1}}}{\prod\limits_{i=1}^{k+\frac{d}{2}} i^{\binom{k-i+d}{d-1}} \prod\limits_{\substack{i=1\\ i \text{ even}}}^{2k+d} 2^{\binom{k-i+d}{d-1}}} \overset{(13)}{=} \frac{(2k+d+1)^{\binom{k+d}{d}}}{\prod\limits_{i=k+d+1}^{2k+d+1} i^{\binom{k-i+d}{d-1}}},$$

but also for odd d,

$$\frac{B(d,k+1)}{B(d,k)B(d-1,k+1)} = \frac{(k+\frac{d+1}{2})^{-\binom{-k-1}{d}} \prod\limits_{i=1}^{k+\frac{d+1}{2}} i^{\binom{k-i+d}{d-1}-\binom{k-2i+d}{d-1}}}{\prod\limits_{\substack{i=1\\ i \text{ odd}}}^{2k+d+1} i^{\binom{k-i+d}{d-1}}}$$

$$= \frac{(k+\frac{d+1}{2})^{\binom{k+d}{d}} \prod\limits_{i=1}^{k+\frac{d+1}{2}} i^{\binom{k-i+d}{d-1}} \prod\limits_{\substack{i=1\\ i \text{ even}}}^{2k+d} 2^{\binom{k-i+d}{d-1}}}{\prod\limits_{i=1}^{2k+d+1} i^{\binom{k-i+d}{d-1}}} \overset{(13)}{=} \frac{(2k+d+1)^{\binom{k+d}{d}}}{\prod\limits_{i=k+d+1}^{2k+d+1} i^{\binom{k-i+d}{d-1}}}.$$

□

4. Application in Hyperkähler geometry

Let X be a compact Hyperkähler manifold of complex dimension $2k$. These objects are also called Irreducible Holomorphic Symplectic manifolds. The second cohomology group $H^2(X,\mathbb{Z})$ comes with an integral quadratic form, called the Beauville–Bogomolov form q_X, which can be computed by an integration over some cup–product power, see [8, Subsection 2.3]:

$$\int_X \alpha^{2k} = (2k-1)!!\, c_X q_X(\alpha)^k, \qquad \alpha \in H^2(X,\mathbb{Z}). \tag{27}$$

This equation is referred to as the Beauville–Fujiki relation. The constant $c_X \in \mathbb{Q}$ is chosen such that the quadratic form q_X is indivisible and $q_X(\sigma+\bar\sigma) > 0$ for a holomorphic two-form σ with $\int_X \sigma\bar\sigma = 1$. There is an alternative description, as shown in [5, Chap. 23]. Up to a scalar factor $\tilde c$, q_X is equal to:

$$\tilde c\, q_X(\alpha) = \frac{k}{2}\int_X \alpha^2(\sigma\bar\sigma)^{k-1} + (1-k)\left(\int_X \alpha\,\sigma^{n-1}\bar\sigma^n\right)\left(\int_X \alpha\,\sigma^n\bar\sigma^{n-1}\right). \tag{28}$$

Now q_X, by polarisation, gives rise to a symmetric bilinear form $\langle\ ,\ \rangle$ on $H^2(X,\mathbb{Z})$, namely $2\langle\alpha,\beta\rangle := q_X(\alpha+\beta) - q_X(\alpha) - q_X(\beta)$. On the other hand, from (27) one deduces (again by polarisation, cf. [8, Eq. 3.2.4]) that:

$$\int_X \alpha_1 \wedge \ldots \wedge \alpha_{2k} = c_X \langle\!\langle \alpha_1 \ldots \alpha_k\,,\, \alpha_{k+1} \ldots \alpha_{2k} \rangle\!\rangle\,, \tag{29}$$

with the induced form $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\operatorname{Sym}^k H^2(X,\mathbb{Z})$, according to Definition 1.1. Since the Poincaré pairing $(\beta_1,\beta_2)_X := \int_X \beta_1 \wedge \beta_2$ gives $H^{2k}(X,\mathbb{Z})$ the structure of an unimodular lattice, we have got a primitive imbedding of lattices:

$$\Big(\operatorname{Sym}^k H^2(X,\mathbb{Z}),\ c_X\langle\!\langle\ ,\ \rangle\!\rangle\Big) \longrightarrow \Big(H^{2k}(X,\mathbb{Z}),\ (\ ,\ )_X\Big). \tag{30}$$

From this observation and Theorem 1.6, we deduce some interesting corollaries:

Corollary 4.1. *Let X be a compact Hyperkähler manifold of complex dimension $2k$. Denote h_2 resp. d_2 the rank and the discriminant of $H^2(X,\mathbb{Z})$. Then the torsion part of the quotient*

$$\frac{H^{2k}(X,\mathbb{Z})}{\operatorname{Sym}^k H^2(X,\mathbb{Z})}$$

contains no prime factors that are bigger than $2k + h_2 - 2$ and don't divide neither c_X nor d_2. □

For the known examples of compact Hyperkähler manifolds, we can refine this a bit, using [8, Table 1]:

Corollary 4.2. *The torsion part of the quotient*

$$\frac{H^{2k}(X,\mathbb{Z})}{\operatorname{Sym}^k H^2(X,\mathbb{Z})}$$

contains no prime factors bigger than

- $2k+21$*, if X is $S^{[k]}$, the Hilbert scheme of k points on a K3 surface S,*
- $2k+5$*, if X is $K^{[[k]]}$, the generalized Kummer variety of a torus,*
- 16*, if X is the 10–dimensional O'Grady manifold,*
- 6*, if X is the 6–dimensional O'Grady manifold.*

□

Remark 4.3. The cases $X = S^{[2]}$ and $X = S^{[3]}$ were already studied in [1, Prop. 6.6] and [6, Prop. 2.4], using explicit calculations. The case $X = S^{[2]}$ is particularly nice, because $\operatorname{Sym}^2 H^2(S^{[2]},\mathbb{Z})$ and $H^4(S^{[2]},\mathbb{Z})$ have the same rank. Since the rank and the discriminant of $H^2(S^{[2]},\mathbb{Z})$ are 23 and -2, Theorem 1.6 implies that the cardinality of the quotient is precisely $\sqrt{2^{24}\,2^{22}(22+3)}$.

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