

COMPUTING CUP-PRODUCTS IN INTEGER COHOMOLOGY OF HILBERT SCHEMES OF POINTS ON K3 SURFACES

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ABSTRACT. We study the images of cup products in integer cohomology of the Hilbert scheme of n points on a K3 surface.

1. PRELIMINARIES

Definition 1.1. Let n be a natural number. A partition of n is a sequence $\lambda = (\lambda_1 \geq \dots \geq \lambda_k > 0)$ of natural numbers such that $\sum_i \lambda_i = n$. It is convenient to write $\lambda = (1^{m_1}, 2^{m_2}, \dots)$ as a sequence of multiplicities. We define the weight $\|\lambda\| := \sum m_i i = n$ and the length $|\lambda| := \sum_i m_i = k$. We also define $z_\lambda := \prod_i i^{m_i} m_i!$.

Definition 1.2. Let S be a projective K3 surface. We fix integral bases 1 of $H^0(S, \mathbb{Z})$, x of $H^4(X, \mathbb{Z})$ and $\alpha_1, \dots, \alpha_{22}$ of $H^2(S, \mathbb{Z})$. The cup product induces a symmetric bilinear form B_{H^2} on $H^2(X, \mathbb{Z})$ and thus the structure of a lattice isomorphic to $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$, *i.e.* three times the hyperbolic lattice and two times the negative E_8 lattice. We may extend B_{H^2} to a symmetric non-degenerate bilinear form on $H^*(S, \mathbb{Z})$ by setting $B(1, 1) = 0$, $B(1, \alpha_i) = 0$, $B(1, x) = 1$, $B(x, x) = 0$.

Definition 1.3. B induces a form $B \otimes B$ on $\text{Sym}^2 H^*(S, \mathbb{Z})$. So the cup-product

$$\mu : \text{Sym}^2 H^*(S, \mathbb{Z}) \longrightarrow H^*(S, \mathbb{Z})$$

has an adjoint comultiplication Δ that is coassociative, given by:

$$\Delta : H^*(S, \mathbb{Z}) \longrightarrow \text{Sym}^2 H^*(S, \mathbb{Z}), \quad \Delta = (B \otimes B)^{-1} \mu^T B$$

The image of 1 under the composite map $\mu(\Delta(1)) = B(\Delta(1), \Delta(1)) = 24x$, denoted by e is called the Euler Class.

We denote by $S^{[n]}$ the Hilbert scheme of n points on S , *i.e.* the classifying space of all zero-dimensional closed subschemes of length n , which is smooth. A classical result by Nakajima gives an explicit description of $H^*(S^{[n]}, \mathbb{Q})$ in terms of creation operators $\mathbf{q}_l(\beta)$, $\beta \in H^*(S, \mathbb{Q})$, acting on the direct sum $\bigoplus_n H^*(S^{[n]}, \mathbb{Q})$. An integral basis for $H^*(S^{[n]}, \mathbb{Z})$ in terms of Nakajima's operators was given by Qin–Wang:

Theorem 1.4. [6, Thm. 5.4.] *The classes*

$$\frac{1}{z_\lambda} \mathbf{q}_\lambda(1) \mathbf{q}_\mu(x) \mathbf{m}_{\nu^1, \alpha_1} \dots \mathbf{m}_{\nu^{22}, \alpha_{22}} |0\rangle, \quad \|\lambda\| + \|\mu\| + \sum_{i=1}^{22} \|\nu^i\| = n$$

form an integral basis for $H^(S^{[n]}, \mathbb{Z})$. Here, λ, μ, ν^i are partitions. The symbol \mathbf{q} stands for Nakajima's creation operator. The relation of $\mathbf{m}_{\nu, \alpha}$ to $\mathbf{q}_{\bar{\nu}}(\alpha)$ is the same*

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as the monomial symmetric functions m_ν to the power sum symmetric functions $p_{\tilde{\nu}}$.

Notation 1.5. To enumerate the basis of $H^*(S^{[n]}, \mathbb{Z})$, we introduce the following abbreviation:

$$1^\lambda \alpha_1^{\nu^1} \dots \alpha_{22}^{\nu^{22}} x^\mu := \frac{1}{z_{\tilde{\lambda}}} \mathfrak{q}_{\tilde{\lambda}}(1) \mathfrak{q}_\mu(x) \mathfrak{m}_{\nu^1, \alpha_1} \dots \mathfrak{m}_{\nu^{22}, \alpha_{22}} |0\rangle$$

where the partition $\tilde{\lambda}$ is built from λ by appending sufficiently many Ones, such that $\|\tilde{\lambda}\| + \|\mu\| + \sum \|\nu^i\| = n$. If $\|\lambda\| + \|\mu\| + \sum \|\nu^i\| > n$, we put $1^\lambda \alpha_1^{\nu^1} \dots \alpha_{22}^{\nu^{22}} x^\mu = 0$.

The ring structure of $H^*(S^{[n]}, \mathbb{Q})$ has been studied in [2], where an explicit algebraic model is constructed, which we recall briefly:

Definition 1.6. [2, Sect. 2] Let π be a permutation of n letters, written as a sum of disjoint cycles. To each cycle we may associate an element of $A := H^*(S, \mathbb{Q})$. This defines an element in $A^{\otimes m}$, m being the number of cycles. So these mappings span a vector space over \mathbb{Q} . The space obtained by taking the direct sum over all $\pi \in S_n$ will be denoted by $A\{S_n\}$.

To define a ring structure, take two permutations π, τ , together with mappings. The result of the multiplication will be the permutation $\pi\tau$, together with a mapping of cycles. Now, look first at the orbit space of the group of permutations $\langle \pi, \tau \rangle$, generated by π and τ . For each cycle of π, τ contained in one orbit B of $\langle \pi, \tau \rangle$, multiply with the associated element of A . Also multiply with a certain power of the Euler class e^g . Afterwards, apply the comultiplication Δ repeatedly on the product to get a mapping from the cycles of $\pi\tau$ contained in B to A . Here the "graph defect" g is defined as follows: Let u, v, w be the number of cycles contained in B of $\pi, \tau, \pi\tau$, respectively. Then $g := \frac{1}{2}(|B| + 2 - u - v - w)$. Now follow this procedure for each orbit B .

The symmetric group S_n acts on $A\{S_n\}$ by conjugation. This action preserves the ring structure. Therefore the space of invariants $A^{[n]} := (A\{S_n\})^{S_n}$ becomes a subring. The main theorem of Lehn and Sorger can now be stated:

Theorem 1.7. [2, Thm. 3.2.] *The following map is an isomorphism of rings:*

$$H^*(S^{[n]}, \mathbb{Q}) \longrightarrow A^{[n]} \\ \mathfrak{q}_{n_1}(\beta_1) \dots \mathfrak{q}_{n_k}(\beta_k) |0\rangle \longmapsto \sum_{\sigma \in S_n} \sigma a \sigma^{-1}$$

with $n_1 + \dots + n_k = n$ and $a \in A\{S_n\}$ corresponds to an arbitrary permutation with k cycles of lengths n_1, \dots, n_k that are associated to the classes $\beta_1, \dots, \beta_k \in H^*(S, \mathbb{Q})$, respectively.

Since $H^{\text{odd}}(S^{[n]}, \mathbb{Z}) = 0$ and $H^{\text{even}}(S^{[n]}, \mathbb{Z})$ is torsion-free by [3], we can apply these results to $H^*(S^{[n]}, \mathbb{Z})$ to determine the multiplicative structure of cohomology with integer coefficients.

2. COMPUTATIONAL RESULTS

The algebra generated by classes of degree 2 is an interesting object to study. For cohomology with complex coefficients, Verbitsky has proven that the algebra generated by $H^2(X, \mathbb{C})$

Remark 2.1. Denote $h^k(S^{[n]})$ the rank of $H^k(S^{[n]}, \mathbb{Z})$. We have:

- $h^2(S^{[n]}) = 23$ for $n \geq 2$.
- $h^4(S^{[n]}) = 276, 299, 300$ for $n = 2, 3, \geq 4$ resp.
- $h^6(S^{[n]}) = 23, 2554, 2852, 2875, 2876$ for $n = 2, 3, 4, 5, \geq 6$ resp.

We now give some results in low degrees, obtained by computing multiplication matrices with respect to the integral basis and a then reducing to Smith normal form (both done by a computer).

Proposition 2.2. *Studying the image of $\text{Sym}^2 H^2$ in H^4 , we obtain:*

$$\frac{H^4(S^{[2]}, \mathbb{Z})}{\text{Sym}^2 H^2(S^{[2]}, \mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}} \right)^{\oplus 23} \oplus \frac{\mathbb{Z}}{5\mathbb{Z}}$$

This was already known to Boissière, Nieper-Wißkirchen and Sarti, [?, Prop. 3].

$$\frac{H^4(S^{[3]}, \mathbb{Z})}{\text{Sym}^2 H^2(S^{[3]}, \mathbb{Z})} \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 23}$$

The torsion part of the quotient is generated by the integral class $\frac{1}{3}\mathbf{q}_{(3)}(1)|0\rangle$.

$$\frac{H^4(S^{[n]}, \mathbb{Z})}{\text{Sym}^2 H^2(S^{[n]}, \mathbb{Z})} \cong \mathbb{Z}^{\oplus 24}, \quad \text{for } n \geq 4.$$

This was already proven by Markman, [4, Thm. 1.10].

Proposition 2.3. *Comparing $H^2(S^{[n]}, \mathbb{Z}) \cup H^4(S^{[n]}, \mathbb{Z})$ with $H^6(S^{[n]}, \mathbb{Z})$, we obtain:*

- (1) $\frac{H^6(S^{[2]}, \mathbb{Z})}{H^2(S^{[2]}, \mathbb{Z}) \cup H^4(S^{[2]}, \mathbb{Z})} = 0$
- (2) $\frac{H^6(S^{[3]}, \mathbb{Z})}{H^2(S^{[3]}, \mathbb{Z}) \cup H^4(S^{[3]}, \mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{3\mathbb{Z}} \right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$
- (3) $\frac{H^6(S^{[4]}, \mathbb{Z})}{H^2(S^{[4]}, \mathbb{Z}) \cup H^4(S^{[4]}, \mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{6\mathbb{Z}} \right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{108\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$
- (4) $\frac{H^6(S^{[5]}, \mathbb{Z})}{H^2(S^{[5]}, \mathbb{Z}) \cup H^4(S^{[5]}, \mathbb{Z})} \cong \mathbb{Z}^{\oplus 22} \oplus \mathbb{Z}$
- (5) $\frac{H^6(S^{[n]}, \mathbb{Z})}{H^2(S^{[n]}, \mathbb{Z}) \cup H^4(S^{[n]}, \mathbb{Z})} \cong \mathbb{Z}^{\oplus 22} \oplus \mathbb{Z} \oplus \mathbb{Z}, \quad n \geq 6.$

In each case, the first 22 parts of the quotient are generated by the integral classes $\alpha_i^{(1,1,1)} - 3 \cdot \alpha_i^{(2,1)} + 3 \cdot \alpha_i^{(3)} + 3 \cdot 1^{(2)} \alpha_i^{(1,1)} - 6 \cdot 1^{(2)} \alpha_i^{(2)} + 6 \cdot 1^{(2,2)} \alpha_i^{(1)} - 3 \cdot 1^{(3)} \alpha_i^{(1)}$, for $i = 1 \dots 22$. Now define an integral class

$$\begin{aligned} K := & \sum_{i \neq j} B(\alpha_i, \alpha_j) \left[\alpha_i^{(1,1,1)} \alpha_j^{(1)} - 2 \cdot \alpha_i^{(2)} \alpha_j^{(1)} + \frac{3}{2} \cdot 1^{(2)} \alpha_i^{(1)} \alpha_j^{(1)} \right] + \\ & + \sum_i B(\alpha_i, \alpha_i) \left[\alpha_i^{(1,1,1)} - 2 \cdot \alpha_i^{(2,1)} + \frac{3}{2} \cdot 1^{(2)} \alpha_i^{(1,1)} \right] + x^{(2)} - 1^{(2)} x^{(1)}. \end{aligned}$$

In the case $n = 3$, the last part of the quotient is generated by K .

In the case $n = 4$, the class $1^{(4)}$ generates the 2-torsion part and $K + 38 \cdot 1^{(4)}$ generates the 108-torsion part.

In the case $n = 5$, the last part of the quotient is generated by $K + 16 \cdot 1^{(4)} - 21 \cdot 1^{(3,2)}$.

If $n \geq 6$, the two last parts of the quotient are generated by some multiples of $K - \frac{4}{3}(45 - n)1^{(2,2,2)} + (48 - n)1^{(3,2)}$ and $K - \frac{1}{2}(40 - n)1^{(2,2,2)} + \frac{1}{4}(48 - n)1^{(4)}$.

Proposition 2.4. *Studying the classes obtained by products of $H^2(S^{[n]}, \mathbb{Z})$, we get:*

$$\frac{H^6(S^{[2]}, \mathbb{Z})}{\text{Sym}^3 H^2(S^{[2]}, \mathbb{Z})} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$$

The quotient is generated by the integral class $1^{(2)}$.

$$\frac{H^6(S^{[3]}, \mathbb{Z})}{\text{Sym}^3 H^2(S^{[3]}, \mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}} \right)^{\oplus 230} \oplus \left(\frac{\mathbb{Z}}{36\mathbb{Z}} \right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{72\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 507}$$

$$\frac{H^6(S^{[4]}, \mathbb{Z})}{\text{Sym}^3 H^2(S^{[4]}, \mathbb{Z})} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 552}$$

For $n \geq 5$, the quotient is free.

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