Aspects of the Beauville–Fujiki relation

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Summary

For X a compact Hyperkähler manifold, $\dim X = 2n$, we construct a form $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\mathrm{Sym}^n H^2(X)$ from the Beauville–Bogomolov form on $H^2(X)$, such that the evident embedding: $\mathrm{Sym}^n H^2(X) \to H^{2n}(X)$ becomes metric.

Introduction

Let X be a compact Hyperkähler manifold of dimension $2n$. The Beauville–Fujiki relation expresses an integral symmetric bilinear form on $H^2(X,\mathbb{Z})$, called the Beauville–Bogomolov form, in terms of the Poincaré pairing on $H^{2n}(X,\mathbb{Z})$:

$$\langle \alpha, \alpha \rangle \overset{*}{=} \left(\int_X \alpha^{2n} \right)^{\frac{1}{n}}$$

Question: Is there a way to invert this procedure?

Answer: Yes, on the image of $\mathrm{Sym}^n H^2(X)$ in $H^{2n}(X)$.

B–F relation, polarized version:

$$\int_X \alpha_1 \wedge \ldots \wedge \alpha_{2n} \overset{*}{=} \sum_{\mathcal{P}} \prod_{\{i,j\} \in \mathcal{P}} \langle \alpha_i, \alpha_j \rangle .$$

The sum is over all partitions $\mathcal{P}$ of $\{1, \ldots, 2n\}$ into pairs.

We can take this as a general recipe to generate symmetric bilinear forms on symmetric powers!

*all equations are meant to hold only up to a constant factor

Generalized setting

Let V be a free module with basis $(x_i)_{0 \le i \le d}$, equipped with a symmetric bilinear form $\langle\ ,\ \rangle$. On the induced basis of $\mathrm{Sym}^n V$, we define a symmetric bilinear form $\langle\!\langle\ ,\ \rangle\!\rangle$ by:

$$\langle\!\langle x_{k_1} \ldots x_{k_n}\, ,\, x_{k_{n+1}} \ldots x_{k_{2n}} \rangle\!\rangle := \sum_{\mathcal{P}} \prod_{\{i,j\} \in \mathcal{P}} \left\langle x_{k_i}, x_{k_j} \right\rangle ,$$

where the sum is over all partitions $\mathcal{P}$ of $\{1, \ldots, 2n\}$ into pairs.

Link to real analysis

There is an alternative description, for $V = \mathbb{R}^{d+1}$ with the standard scalar product. For two homogeneous polynomials $h_1(x)$, $h_2(x)$ in $d+1$ variables, we have

$$\langle\!\langle h_1(x), h_2(x) \rangle\!\rangle \overset{*}{=} \int_{\mathbb{S}^d} h_1(\omega) h_2(\omega) d\omega,$$

with an analytic integral over the unit sphere $\mathbb{S}^d$.

Finding a basis of homogeneous polynomials orthogonal on the sphere amounts to understanding a portion of the structure of the Beauville–Fujiki relation! A such orthogonal basis

- can be constructed recursively,
- admits a computation of the discriminant of $\langle\!\langle\ ,\ \rangle\!\rangle$ in closed form.

Theorem

Let a be the discriminant of $\langle\ ,\ \rangle$ on V, where $\mathrm{rk}\, V = d+1$. Then the discriminant of $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\mathrm{Sym}^n V$ equals

$$a^{\binom{d+n}{n}} \theta$$

where the factor θ is integral and contains only prime numbers smaller than $2n+d$.

Consequence for compact HK manifolds

Seen as a lattice, $\mathrm{Sym}^n H^2(X,\mathbb{Z})$ is embedded in the unimodular Poincaré lattice $H^{2n}(X,\mathbb{Z})$. Its discriminant is composed of factors coming from:

- The discriminant of the Beauville–Bogomolov form,
- the Fujiki constant,
- the combinatorial factor θ.

Corollary

For all known examples X of compact HKM, the quotient

$$\frac{H^{2n}(X,\mathbb{Z})}{\mathrm{Sym}^n H^2(X,\mathbb{Z})}$$

contains no prime torsion factors greater than $2n + b_2 - 2$.

References

- S. Kapfer, *Symmetric Powers of Symmetric Bilinear Forms, Homogeneous Orthogonal Polynomials on the sphere and an application in Compact Hyperkähler Manifolds*, preprint 2015.
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