

COMPUTING CUP-PRODUCTS IN INTEGER COHOMOLOGY OF HILBERT SCHEMES OF POINTS ON K3 SURFACES

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ABSTRACT. We study the images of cup products in integer cohomology of the Hilbert scheme of n points on a K3 surface.

1. PRELIMINARIES

Definition 1.1. Let n be a natural number. A partition of n is a sequence $\lambda = (\lambda_1 \geq \dots \geq \lambda_k > 0)$ of natural numbers such that $\sum_i \lambda_i = n$. It is convenient to write $\lambda = (1^{m_1}, 2^{m_2}, \dots)$ as a sequence of multiplicities. We define the weight $\|\lambda\| := \sum m_i i = n$ and the length $|\lambda| := \sum_i m_i = k$. We also define $z_\lambda := \prod_i i^{m_i} m_i!$.

Definition 1.2. Let Λ be the ring of symmetric functions. Let m_λ and p_λ denote the monomial and the power sum symmetric functions. They are both indexed by partitions and form a basis of Λ . They are linearly related by $p_\lambda = \sum_\mu \psi_{\lambda\mu} m_\mu$, the sum being over partitions with the same weight as λ . It turns out that the base change matrix $(\psi_{\lambda\mu})$ has integral entries, but its inverse $(\psi_{\lambda\mu}^{-1})$ has not. For example, $p_{(2,1,1)} = 2m_{(2,1,1)} + 2m_{(2,2)} + 2m_{(3,1)} + m_{(4)}$ but $m_{(2,1,1)} = \frac{1}{2}p_{(2,1,1)} - \frac{1}{2}m_{(2,2)} - p_{(3,1)} + p_{(4)}$. A method to determine the coefficients $(\psi_{\lambda\mu})$ is given in [2, Sect. 3.7].

Definition 1.3. For our purposes, a lattice L is a free \mathbb{Z} -module of finite rank, equipped with a non-degenerate symmetric integral bilinear form B . The lattice L is called odd, if there exists a $v \in L$, such that $B(v, v)$ is odd, otherwise it is called even. Choosing a base $\{e_i\}_i$ of our lattice, we can write B as a symmetric matrix. L is called unimodular, if the matrix B has determinant ± 1 . The difference between the number of positive eigenvalues and the number of negative eigenvalues of B (regarded over \mathbb{R}) is called the signature. If B has both positive and negative eigenvalues, the lattice is called indefinite. For a classification theorem for indefinite lattices, see [6].

Definition 1.4. Let S be a projective K3 surface. We fix integral bases 1 of $H^0(S, \mathbb{Z})$, x of $H^4(S, \mathbb{Z})$ and $\alpha_1, \dots, \alpha_{22}$ of $H^2(S, \mathbb{Z})$. The cup product induces a symmetric bilinear form B_{H^2} on $H^2(S, \mathbb{Z})$ and thus the structure of a unimodular lattice. We may extend B_{H^2} to a symmetric non-degenerate bilinear form B on $H^*(S, \mathbb{Z})$ by setting $B(1, 1) = 0$, $B(1, \alpha_i) = 0$, $B(1, x) = 1$, $B(x, x) = 0$.

It turns out that $H^2(S, \mathbb{Z})$ has signature -16 and, by the classification theorem for indefinite unimodular lattices, is isomorphic to $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$, i.e. three times the hyperbolic lattice and two times the negative E_8 lattice.

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Definition 1.5. B induces a form $B \otimes B$ on $\text{Sym}^2 H^*(S, \mathbb{Z})$. So the cup-product

$$\mu : \text{Sym}^2 H^*(S, \mathbb{Z}) \longrightarrow H^*(S, \mathbb{Z})$$

induces an adjoint comultiplication Δ that is coassociative, given by:

$$\Delta : H^*(S, \mathbb{Z}) \longrightarrow \text{Sym}^2 H^*(S, \mathbb{Z}), \quad \Delta = (B \otimes B)^{-1} \mu^T B$$

Note that this does not define a bialgebra structure. The image of 1 under the composite map $\mu(\Delta(1)) = B(\Delta(1), \Delta(1)) = 24x$, denoted by e is called the Euler Class.

We denote by $S^{[n]}$ the Hilbert scheme of n points on S , *i.e.* the classifying space of all zero-dimensional closed subschemes of length n . $S^{[0]}$ consists of a single point and $S^{[1]} = S$. Fogarty proved, that the Hilbert scheme is a smooth variety. A theorem by Nakajima gives an explicit description of the vector space structure of $H^*(S^{[n]}, \mathbb{Q})$ in terms of creation operators

$$\mathbf{q}_l(\beta) : H^*(S^{[n]}, \mathbb{Q}) \longrightarrow H^{*+k+l-1}(S^{[n+l]}, \mathbb{Q}),$$

where $\beta \in H^k(S, \mathbb{Q})$, acting on the direct sum $\mathbb{H} := \bigoplus_n H^*(S^{[n]}, \mathbb{Q})$. The operators $\mathbf{q}_l(\beta)$ are linear, commute with each other, and the images of the vacuum vector $|0\rangle$, defined as the generator of $H^0(S^{[0]}, \mathbb{Q}) \cong \mathbb{Q}$, under the polynomial algebra generated by the creation operators span \mathbb{H} as a vector space. It is convenient to abbreviate $\mathbf{q}_{l_1}(\beta) \dots \mathbf{q}_{l_k}(\beta) =: \mathbf{q}_\lambda(\beta)$, where the partition λ is composed by the l_i .

An integral basis for $H^*(S^{[n]}, \mathbb{Z})$ in terms of Nakajima's operators was given by Qin–Wang:

Theorem 1.6. [8, Thm. 5.4.] *Let $\mathbf{m}_{\nu, \alpha} := \sum_\rho \psi_{\nu\rho}^{-1} \mathbf{q}(\alpha)$, with coefficients $\psi_{\nu\rho}^{-1}$ as in Definition 1.2. The classes*

$$\frac{1}{z_\lambda} \mathbf{q}_\lambda(1) \mathbf{q}_\mu(x) \mathbf{m}_{\nu^1, \alpha_1} \dots \mathbf{m}_{\nu^{22}, \alpha_{22}} |0\rangle, \quad \|\lambda\| + \|\mu\| + \sum_{i=1}^{22} \|\nu^i\| = n$$

form an integral basis for $H^(S^{[n]}, \mathbb{Z})$. Here, λ, μ, ν^i are partitions.*

Notation 1.7. To enumerate the basis of $H^*(S^{[n]}, \mathbb{Z})$, we introduce the following abbreviation:

$$\boldsymbol{\alpha}^\lambda := 1^{\lambda^0} \alpha_1^{\lambda^1} \dots \alpha_{22}^{\lambda^{22}} x^{\lambda^{23}} := \frac{1}{z_{\widetilde{\lambda^0}}} \mathbf{q}_{\widetilde{\lambda^0}}(1) \mathbf{q}_{\lambda^{23}}(x) \mathbf{m}_{\lambda^1, \alpha_1} \dots \mathbf{m}_{\lambda^{22}, \alpha_{22}} |0\rangle$$

where the partition $\widetilde{\lambda^0}$ is built from λ^0 by appending sufficiently many Ones, such that $\|\widetilde{\lambda^0}\| + \sum_{i \geq 1} \|\lambda^i\| = n$. If $\sum_{n \geq 0} \|\lambda^i\| > n$, we put $\boldsymbol{\alpha}^\lambda = 0$. Thus we can interpret $\boldsymbol{\alpha}^\lambda$ as an element of $H^*(S^{[n]}, \mathbb{Z})$ for arbitrary n . We say that the symbol $\boldsymbol{\alpha}^\lambda$ is reduced, if λ^0 contains no Ones. We define also $\|\lambda\| := \sum_{n \geq 0} \|\lambda^i\|$.

Lemma 1.8. *Let $\boldsymbol{\alpha}^\lambda$ represent a class of cohomological degree $2k$. If $\boldsymbol{\alpha}^\lambda$ is reduced, then $\frac{k}{2} \leq \|\lambda\| \leq 2k$.*

Proof. This is a simple combinatorial observation. The lower bound is witnessed by $x^{(\frac{k}{2})}$ (if k is even) and the upper bound is witnessed by $1^{(2^k)}$. \square

The ring structure of $H^*(S^{[n]}, \mathbb{Q})$ has been studied by Lehn and Sorger in [3], where an explicit algebraic model is constructed, which we recall briefly:

Definition 1.9. [3, Sect. 2] Let π be a permutation of n letters, written as a sum of disjoint cycles. To each cycle we may associate an element of $A := H^*(S, \mathbb{Q})$. This defines an element in $A^{\otimes m}$, m being the number of cycles. So these mappings span a vector space over \mathbb{Q} . The space obtained by taking the direct sum over all $\pi \in S_n$ will be denoted by $A\{S_n\}$.

To define a ring structure, take two permutations π, τ , together with mappings. The result of the multiplication will be the permutation $\pi\tau$, together with a mapping of cycles. To construct the mappings to A , look first at the orbit space of the group of permutations $\langle \pi, \tau \rangle$, generated by π and τ . For each cycle of π, τ contained in one orbit B of $\langle \pi, \tau \rangle$, multiply with the associated element of A . Also multiply with a certain power of the Euler class e^g . Afterwards, apply the comultiplication Δ repeatedly on the product to get a mapping from the cycles of $\pi\tau$ contained in B to A . Here the "graph defect" g is defined as follows: Let u, v, w be the number of cycles contained in B of $\pi, \tau, \pi\tau$, respectively. Then $g := \frac{1}{2}(|B| + 2 - u - v - w)$. Now follow this procedure for each orbit B .

The symmetric group S_n acts on $A\{S_n\}$ by conjugation. This action preserves the ring structure. Therefore the space of invariants $A^{[n]} := (A\{S_n\})^{S_n}$ becomes a subring. The main theorem of [3] can now be stated:

Theorem 1.10. [3, Thm. 3.2.] *The following map is an isomorphism of rings:*

$$H^*(S^{[n]}, \mathbb{Q}) \longrightarrow A^{[n]}$$

$$\mathbf{q}_{n_1}(\beta_1) \dots \mathbf{q}_{n_k}(\beta_k) | 0 \rangle \longmapsto \sum_{\sigma \in S_n} \sigma a \sigma^{-1}$$

with $n_1 + \dots + n_k = n$ and $a \in A\{S_n\}$ corresponds to an arbitrary permutation with k cycles of lengths n_1, \dots, n_k that are associated to the classes $\beta_1, \dots, \beta_k \in H^*(S, \mathbb{Q})$, respectively.

Since $H^{\text{odd}}(S^{[n]}, \mathbb{Z}) = 0$ and $H^{\text{even}}(S^{[n]}, \mathbb{Z})$ is torsion-free by [4], we can apply these results to $H^*(S^{[n]}, \mathbb{Z})$ to determine the multiplicative structure of cohomology with integer coefficients. It turns out, that it is somehow independent of n . More precisely, we have the following stability theorem due to Li, Qin and Wang:

Theorem 1.11. [8, Thm. 2.1] *Let Q_1, \dots, Q_s be products of creation operators, i.e. $Q_i = \prod_j \mathbf{q}_{\lambda_{i,j}}(\beta_{i,j})$ for some partitions $\lambda_{i,j}$ and classes $\beta_{i,j} \in H^*(S, \mathbb{Z})$. Set $n_i := \sum_j \|\lambda_{i,j}\|$. Then the cup product $\prod_{i=1}^s \left(\frac{1}{(n-n_i)!} \mathbf{q}_{n-n_i}(1) Q_i | 0 \rangle \right)$ is equal to a finite linear combination of classes of the form $\frac{1}{(n-m)!} \mathbf{q}_{n-m}(1) \prod_j \mathbf{q}_{\mu_j}(\gamma_j) | 0 \rangle$, with $\gamma \in H^*(S, \mathbb{Z})$, $m = \sum_j \|\mu_j\|$, whose coefficients are independent of n . We have the upper bound $m \leq \sum_i n_i$.*

Corollary 1.12. *Let $\alpha^\lambda, \alpha^\mu, \alpha^\nu$ be reduced. Assume $n \geq \|\lambda\|, \|\mu\|$. Then the coefficients $c_\nu^{\lambda\mu}$ of the cup product in $H^*(S^{[n]}, \mathbb{Z})$*

$$\alpha^\lambda \smile \alpha^\mu = \sum_\nu c_\nu^{\lambda\mu} \alpha^\nu$$

are polynomials in n of degree at most $\|\lambda\| + \|\mu\| - \|\nu\|$.

Proof. Set $Q_\lambda := \mathbf{q}_{\lambda^0}(1) \mathbf{q}_{\lambda^{23}}(x) \prod_{1 \leq j \leq 22} \mathbf{q}_{\lambda^j}(\alpha_j)$ and $n_\lambda := \|\lambda\|$. Then we have: $\alpha^\lambda \smile \alpha^\mu = \frac{1}{(n-n_1)! z_{\lambda^0}} \mathbf{q}_{n-n_1}(1) Q_1 | 0 \rangle \smile \frac{1}{(n-n_2)! z_{\mu^0}} \mathbf{q}_{n-n_2}(1) Q_2 | 0 \rangle$. Thus the coefficient $c_\nu^{\lambda\mu}$ in the product expansion is a constant, dependent on $\|\lambda\|, \|\mu\|, \|\nu\|$,

but independent of n , multiplied with $\frac{(n-n_3)!}{(n-m)!}$ for a certain $m \leq n_1 + n_2$. This is a polynomial of degree $m - n_3 \leq n_1 + n_2 - n_3 = \|\lambda\| + \|\mu\| - \|\nu\|$. \square

Remark 1.13. The above condition $n \geq \|\lambda\|, \|\mu\|$ seems to be unnecessary. This means, that if $n < \|\lambda\|$ or $n < \|\mu\|$ but $n \geq \|\nu\|$, the polynomial $c_\nu^{\lambda\mu}$ is zero at n .

2. COMPUTATIONAL RESULTS

We now give some results in low degrees, obtained by computing multiplication matrices with respect to the integral basis and a then reducing to Smith normal form (both done by a computer).

Remark 2.1. Denote $h^k(S^{[n]})$ the rank of $H^k(S^{[n]}, \mathbb{Z})$. We have:

- $h^2(S^{[n]}) = 23$ for $n \geq 2$.
- $h^4(S^{[n]}) = 276, 299, 300$ for $n = 2, 3, \geq 4$ resp.
- $h^6(S^{[n]}) = 23, 2554, 2852, 2875, 2876$ for $n = 2, 3, 4, 5, \geq 6$ resp.

The algebra generated by classes of degree 2 is an interesting object to study. For cohomology with complex coefficients, Verbitsky has proven in [9] that the cup product mapping from $\text{Sym}^k H^2(S^{[n]}, \mathbb{C})$ to $H^{2k}(S^{[n]}, \mathbb{C})$ is injective for $k \leq n$. One concludes that this also holds for integral coefficients.

Proposition 2.2. *We identify $\text{Sym}^2 H^2(S^{[n]}, \mathbb{Z})$ with its image in $H^4(S^{[n]}, \mathbb{Z})$ under the cup product mapping. Then:*

$$\begin{aligned} (1) \quad & \frac{H^4(S^{[2]}, \mathbb{Z})}{\text{Sym}^2 H^2(S^{[2]}, \mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}} \right)^{\oplus 23} \oplus \frac{\mathbb{Z}}{5\mathbb{Z}}, \\ (2) \quad & \frac{H^4(S^{[3]}, \mathbb{Z})}{\text{Sym}^2 H^2(S^{[3]}, \mathbb{Z})} \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 23}, \\ (3) \quad & \frac{H^4(S^{[n]}, \mathbb{Z})}{\text{Sym}^2 H^2(S^{[n]}, \mathbb{Z})} \cong \mathbb{Z}^{\oplus 24}, \quad \text{for } n \geq 4. \end{aligned}$$

The 3-torsion part is generated by the integral class $1^{(3)}$.

Remark 2.3. The torsion in the case $n = 2$ was also computed by Boissière, Nieper-Wißkirchen and Sarti, [1, Prop. 3] using similar techniques. The result for $n = 3$ seems to be new. The freeness result for $n \geq 4$ was already proven by Markman, [5, Thm. 1.10], using a completely different method.

Proposition 2.4. *Studying triple products of $H^2(S^{[n]}, \mathbb{Z})$, we get:*

$$\frac{H^6(S^{[2]}, \mathbb{Z})}{\text{Sym}^3 H^2(S^{[2]}, \mathbb{Z})} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$$

The quotient is generated by the integral class $1^{(2)}$.

$$\begin{aligned} \frac{H^6(S^{[3]}, \mathbb{Z})}{\text{Sym}^3 H^2(S^{[3]}, \mathbb{Z})} &\cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}} \right)^{\oplus 230} \oplus \left(\frac{\mathbb{Z}}{36\mathbb{Z}} \right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{72\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 507} \\ \frac{H^6(S^{[4]}, \mathbb{Z})}{\text{Sym}^3 H^2(S^{[4]}, \mathbb{Z})} &\cong \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 552} \end{aligned}$$

For $n \geq 5$, the quotient is free.

We study now cup products between classes of degree 2 and 4. The case of $S^{[3]}$ is of particular interest.

Proposition 2.5. *The cup product mapping : $H^2(S^{[n]}, \mathbb{Z}) \otimes H^4(S^{[n]}, \mathbb{Z}) \rightarrow H^6(S^{[n]}, \mathbb{Z})$ is neither injective (unless $n = 0$) nor surjective (unless $n \leq 2$). We have:*

$$(4) \quad \frac{H^6(S^{[3]}, \mathbb{Z})}{H^2(S^{[3]}, \mathbb{Z}) \smile H^4(S^{[3]}, \mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{3\mathbb{Z}} \right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$$

$$(5) \quad \frac{H^6(S^{[4]}, \mathbb{Z})}{H^2(S^{[4]}, \mathbb{Z}) \smile H^4(S^{[4]}, \mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{6\mathbb{Z}} \right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{108\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$$

$$(6) \quad \frac{H^6(S^{[5]}, \mathbb{Z})}{H^2(S^{[5]}, \mathbb{Z}) \smile H^4(S^{[5]}, \mathbb{Z})} \cong \mathbb{Z}^{\oplus 22} \oplus \mathbb{Z}$$

$$(7) \quad \frac{H^6(S^{[n]}, \mathbb{Z})}{H^2(S^{[n]}, \mathbb{Z}) \smile H^4(S^{[n]}, \mathbb{Z})} \cong \mathbb{Z}^{\oplus 22} \oplus \mathbb{Z} \oplus \mathbb{Z}, \quad n \geq 6.$$

In each case, the first 22 parts of the quotient are generated by the integral classes

$$\alpha_i^{(1,1,1)} - 3 \cdot \alpha_i^{(2,1)} + 3 \cdot \alpha_i^{(3)} + 3 \cdot 1^{(2)} \alpha_i^{(1,1)} - 6 \cdot 1^{(2)} \alpha_i^{(2)} + 6 \cdot 1^{(2,2)} \alpha_i^{(1)} - 3 \cdot 1^{(3)} \alpha_i^{(1)},$$

for $i = 1 \dots 22$. Now define an integral class

$$\begin{aligned} K := & \sum_{i \neq j} B(\alpha_i, \alpha_j) \left[\alpha_i^{(1,1,1)} \alpha_j^{(1)} - 2 \cdot \alpha_i^{(2)} \alpha_j^{(1)} + \frac{3}{2} \cdot 1^{(2)} \alpha_i^{(1)} \alpha_j^{(1)} \right] + \\ & + \sum_i B(\alpha_i, \alpha_i) \left[\alpha_i^{(1,1,1)} - 2 \cdot \alpha_i^{(2,1)} + \frac{3}{2} \cdot 1^{(2)} \alpha_i^{(1,1)} \right] + x^{(2)} - 1^{(2)} x^{(1)}. \end{aligned}$$

In the case $n = 3$, the last part of the quotient is generated by K .

In the case $n = 4$, the class $1^{(4)}$ generates the 2-torsion part and $K + 38 \cdot 1^{(4)}$ generates the 108-torsion part.

In the case $n = 5$, the last part of the quotient is generated by $K + 16 \cdot 1^{(4)} - 21 \cdot 1^{(3,2)}$.

If $n \geq 6$, the two last parts of the quotient are generated by some multiples of $K - \frac{4}{3}(45 - n)1^{(2,2,2)} + (48 - n)1^{(3,2)}$ and $K - \frac{1}{2}(40 - n)1^{(2,2,2)} + \frac{1}{4}(48 - n)1^{(4)}$.

Proof. The last assertion for arbitrary n follows from Corollary 1.12. First observe that for $\alpha^\lambda \in H^2$, $\alpha^\mu \in H^4$, $\alpha^\nu \in H^6$, we have $\|\lambda\| \leq 2$, $\|\mu\| \leq 4$ and $\|\nu\| \geq 2$, according to Lemma 1.8. The coefficient of the cup product matrix are thus polynomials of degree at most $2 + 4 - 2 = 4$ and it suffices to compute only a finite number of instances for n . It turns out, that the maximal degree is 1 and its cokernel is given as stated. \square

Remark 2.6. Observe that the generators of the quotients are independent of the choice of the base of $H^2(S, \mathbb{Z})$.

We give now some computational results for the middle cohomology group. Since $S^{[n]}$ is a projective variety of complex dimension $2n$, Poincaré duality gives $H^{2n}(S^{[n]}, \mathbb{Z})$ the structure of an unimodular lattice.

Proposition 2.7. *Let L denote the lattice $H^{2n}(S^{[n]}, \mathbb{Z})$. We have:*

- (1) For $n = 2$, L is an odd lattice of rank 276 and signature 124.
- (2) For $n = 3$, L is an even lattice of rank 2554 and signature -640.
- (3) For $n = 4$, L is an odd lattice of rank 19298 and signature ...

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