COMPUTING CUP-PRODUCTS IN INTEGER COHOMOLOGY OF HILBERT SCHEMES OF POINTS ON K3 SURFACES

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Abstract. We study the images of cup products in integer cohomology of the Hilbert scheme of n points on a K3 surface.

1. Preliminaries

Definition 1.1. Let n be a natural number. A partition of n is a sequence $\lambda = (\lambda_1 \geq \ldots \geq \lambda_k > 0)$ of natural numbers such that $\sum_i \lambda_i = n$. It is convenient to write $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$ as a sequence of multiplicities. We define the weight $\|\lambda\| := \sum m_i i = n$ and the length $|\lambda| := \sum_i m_i = k$. We also define $z_\lambda := \prod_i i^{m_i} m_i!$.

Definition 1.2. Let Λ be the ring of symmetric functions. Let m_λ and p_λ denote the monomial and the power sum symmetric functions. They are both indexed by partitions and form a basis of Λ. They are linearly related by $p_\lambda = \sum_\mu \psi_{\lambda\mu} m_\mu$, the sum being over partitions with the same weight as λ, hence finite. The base change matrix $(\psi_{\lambda\mu})$ has integral entries, but its inverse $(\psi^{-1}_{\lambda\mu})$ has not. For example, $p_{(2,1,1)} = 2m_{(2,1,1)} + 2m_{(2,2)} + 2m_{(3,1)} + m_{(4)}$ but $m_{(2,1,1)} = \frac{1}{2}p_{(2,1,1)} - \frac{1}{2}m_{(2,2)} - p_{(3,1)} + p_{(4)}$. A method to determine the coefficients $(\psi_{\lambda\mu})$ is given in [2, Sect. 3.7].

Definition 1.3. Let S be a projective K3 surface. We fix integral bases 1 of $H^0(S,\mathbb{Z})$, x of $H^4(X,\mathbb{Z})$ and $\alpha_1, \ldots, \alpha_{22}$ of $H^2(S,\mathbb{Z})$. The cup product induces a symmetric bilinear form B_{H^2} on $H^2(X,\mathbb{Z})$ and thus the structure of a lattive isomorphic to $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$, *i.e.* three times the hyperbolic lattice and two times the negative E_8 lattice. We may extend B_{H^2} to a symmetric non-degenerate bilinear form on $H^*(S,\mathbb{Z})$ by setting $B(1,1) = 0,\ B(1,\alpha_i) = 0,\ B(1,x) = 1,\ B(x,x) = 0.$

Definition 1.4. B induces a form $B \otimes B$ on $\operatorname{Sym}^2 H^*(S,\mathbb{Z})$. So the cup-product

$$\mu : \operatorname{Sym}^2 H^*(S,\mathbb{Z}) \longrightarrow H^*(S,\mathbb{Z})$$

has an adjoint comultiplication Δ that is coassociative, given by:

$$\Delta : H^*(S,\mathbb{Z}) \longrightarrow \operatorname{Sym}^2 H^*(S,\mathbb{Z}), \quad \Delta = (B \otimes B)^{-1} \mu^T B$$

The image of 1 under the composite map $\mu(\Delta(1)) = B(\Delta(1), \Delta(1)) = 24x$, denoted by e is called the Euler Class.

We denote by $S^{[n]}$ the Hilbert scheme of n points on S, *i.e.* the classifying space of all zero-dimensional closed subschemes of length n, which is smooth. A classical result by Nakajima gives an explicit description of $H^*(S^{[n]},\mathbb{Q})$ in terms of creation operators $\mathfrak{q}_l(\beta)$, $\beta \in H^*(S,\mathbb{Q})$, acting on the direct sum $\bigoplus_n H^*(S^{[n]},\mathbb{Q})$. An integral basis for $H^*(S^{[n]},\mathbb{Z})$ in terms of Nakajima's operators was given by Qin–Wang:

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Theorem 1.5. [7, Thm. 5.4.] *Let $\mathfrak{m}_{\nu,\alpha} := \sum_\rho \psi_{\nu\rho}^{-1}\,\mathfrak{q}(\alpha)$, with coefficients as in Definition 1.2. The classes*

$$\frac{1}{z_\lambda}\mathfrak{q}_\lambda(1)\mathfrak{q}_\mu(x)\mathfrak{m}_{\nu^1,\alpha_1}\ldots\mathfrak{m}_{\nu^{22},\alpha_{22}}|0\rangle, \quad \|\lambda\|+\|\mu\|+\sum_{i=1}^{22}\|\nu^i\| = n$$

form an integral basis for $H^*(S^{[n]},\mathbb{Z})$. Here, λ, μ, ν^i are partitions.

Notation 1.6. To enumerate the basis of $H^*(S^{[n]},\mathbb{Z})$, we introduce the following abbreviation:

$$1^\lambda\alpha_1^{\nu^1}\ldots\alpha_{22}^{\nu^{22}}x^\mu := \frac{1}{z_{\tilde\lambda}}\mathfrak{q}_{\tilde\lambda}(1)\mathfrak{q}_\mu(x)\mathfrak{m}_{\nu^1,\alpha_1}\ldots\mathfrak{m}_{\nu^{22},\alpha_{22}}|0\rangle$$

where the partition $\tilde\lambda$ is built from λ by appending sufficiently many Ones, such that $\|\tilde\lambda\|+\|\mu\|+\sum\|\nu^i\| = n$. If $\|\lambda\|+\|\mu\|+\sum\|\nu^i\| > n$, we put $1^\lambda\alpha_1^{\nu^1}\ldots\alpha_{22}^{\nu^{22}}x^\mu = 0$.

The ring structure of $H^*(S^{[n]},\mathbb{Q})$ has been studied in [3], where an explicit algebraic model is constructed, which we recall briefly:

Definition 1.7. [3, Sect. 2] Let π be a permutation of n letters, written as a sum of disjoint cycles. To each cycle we may associate an element of $A := H^*(S,\mathbb{Q})$. This defines an element in $A^{\otimes m}$, m being the number of cycles. So these mappings span a vector space over $\mathbb{Q}$. The space obtained by taking the direct sum over all $\pi \in S_n$ will be denoted by $A\{S_n\}$.

To define a ring structure, take two permutations π,τ, together with mappings. The result of the multiplication will be the permutation $\pi\tau$, together with a mapping of cycles. To construct the mappings to A, look first at the orbit space of the group of permutations $\langle\pi,\tau\rangle$, generated by π and τ. For each cycle of π,τ contained in one orbit B of $\langle\pi,\tau\rangle$, multiply with the associated element of A. Also multiply with a certain power of the Euler class e^g. Afterwards, apply the comultiplication Δ repeatedly on the product to get a mapping from the cycles of $\pi\tau$ contained in B to A. Here the "graph defect" g is defined as follows: Let u,v,w be the number of cycles contained in B of π, τ, $\pi\tau$, respectively. Then $g := \frac{1}{2}\left(|B|+2-u-v-w\right)$. Now follow this procedure for each orbit B.

The symmetric group S_n acts on $A\{S_n\}$ by conjugation. This action preserves the ring structure. Therefore the space of invariants $A^{[n]} := \left(A\{S_n\}\right)^{S_n}$ becomes a subring. The main theorem of Lehn and Sorger can now be stated:

Theorem 1.8. [3, Thm. 3.2.] *The following map is an isomorphism of rings:*

$$\begin{aligned} H^*(S^{[n]},\mathbb{Q}) &\longrightarrow A^{[n]} \\ \mathfrak{q}_{n_1}(\beta_1)\ldots\mathfrak{q}_{n_k}(\beta_k)|0\rangle &\longmapsto \sum_{\sigma\in S_n}\sigma a\sigma^{-1} \end{aligned}$$

with $n_1+\ldots+n_k = n$ and $a \in A\{S_n\}$ corresponds to an arbitrary permutation with k cycles of lengths $n_1,\ldots,n_k$ that are associated to the classes $\beta_1,\ldots,\beta_k \in H^*(S,\mathbb{Q})$, respectively.

Since $H^{\mathrm{odd}}(S^{[n]},\mathbb{Z}) = 0$ and $H^{\mathrm{even}}(S^{[n]},\mathbb{Z})$ is torsion-free by [4], we can apply these results to $H^*(S^{[n]},\mathbb{Z})$ to determine the multiplicative structure of cohomology with integer coefficients.

2. Computational results

We now give some results in low degrees, obtained by computing multiplication matrices with respect to the integral basis and a then reducing to Smith normal form (both done by a computer).

Remark 2.1. Denote $h^k(S^{[n]})$ the rank of $H^k(S^{[n]},\mathbb{Z})$. We have:

- $h^2(S^{[n]}) = 23$ for $n \geq 2$.
- $h^4(S^{[n]}) = 276,\ 299,\ 300$ for $n = 2, 3, \geq 4$ resp.
- $h^6(S^{[n]}) = 23,\ 2554,\ 2852,\ 2875,\ 2876$ for $n = 2, 3, 4, 5, \geq 6$ resp.

The algebra generated by classes of degree 2 is an interesting object to study. For cohomology with complex coefficients, Verbitsky has proven in [8] that the cup product mapping from $\mathrm{Sym}^k H^2(S^{[n]},\mathbb{C})$ to $H^{2k}(S^{[n]},\mathbb{C})$ is injective for $k \leq n$. One concludes that this also holds for integral coefficients.

Proposition 2.2. *Studying the image of* $\mathrm{Sym}^2 H^2$ *in* H^4*, we obtain:*

$$\frac{H^4(S^{[2]},\mathbb{Z})}{\mathrm{Sym}^2 H^2(S^{[2]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23} \oplus \frac{\mathbb{Z}}{5\mathbb{Z}}$$

This was already known to Boissière, Nieper-Wißkirchen and Sarti, [1, Prop. 3].

$$\frac{H^4(S^{[3]},\mathbb{Z})}{\mathrm{Sym}^2 H^2(S^{[3]},\mathbb{Z})} \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 23}$$

The torsion part of the quotient is generated by the integral class $1^{(3)}$.

$$\frac{H^4(S^{[n]},\mathbb{Z})}{\mathrm{Sym}^2 H^2(S^{[n]},\mathbb{Z})} \cong \mathbb{Z}^{\oplus 24}, \quad \textit{for } n \geq 4.$$

This was already proven by Markman, [5, Thm. 1.10].

Proposition 2.3. *Studying triple products of* $H^2(S^{[n]},\mathbb{Z})$*, we get:*

$$\frac{H^6(S^{[2]},\mathbb{Z})}{\mathrm{Sym}^3 H^2(S^{[2]},\mathbb{Z})} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$$

The quotient is generated by the integral class $1^{(2)}$.

$$\frac{H^6(S^{[3]},\mathbb{Z})}{\mathrm{Sym}^3 H^2(S^{[3]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 230} \oplus \left(\frac{\mathbb{Z}}{36\mathbb{Z}}\right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{72\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 507}$$

$$\frac{H^6(S^{[4]},\mathbb{Z})}{\mathrm{Sym}^3 H^2(S^{[4]},\mathbb{Z})} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 552}$$

For $n \geq 5$*, the quotient is free.*

We study now cup products between classes of degree 2 and 4. The case of $S^{[n]}$ is of particular interest.

Proposition 2.4. *Comparing* $H^2(S^{[n]},\mathbb{Z}) \cup H^4(S^{[n]},\mathbb{Z})$ *with* $H^6(S^{[n]},\mathbb{Z})$*, we obtain:*

$$\frac{H^6(S^{[2]},\mathbb{Z})}{H^2(S^{[2]},\mathbb{Z}) \cup H^4(S^{[2]},\mathbb{Z})} = 0 \tag{1}$$

$$\frac{H^6(S^{[3]},\mathbb{Z})}{H^2(S^{[3]},\mathbb{Z}) \cup H^4(S^{[3]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}} \tag{2}$$

$$\frac{H^6(S^{[4]},\mathbb{Z})}{H^2(S^{[4]},\mathbb{Z}) \cup H^4(S^{[4]},\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{6\mathbb{Z}}\right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{108\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \tag{3}$$

$$\frac{H^6(S^{[5]},\mathbb{Z})}{H^2(S^{[5]},\mathbb{Z}) \cup H^4(S^{[5]},\mathbb{Z})} \cong \mathbb{Z}^{\oplus 22} \oplus \mathbb{Z} \tag{4}$$

$$\frac{H^6(S^{[n]},\mathbb{Z})}{H^2(S^{[n]},\mathbb{Z}) \cup H^4(S^{[n]},\mathbb{Z})} \cong \mathbb{Z}^{\oplus 22} \oplus \mathbb{Z} \oplus \mathbb{Z}, \ n \geq 6. \tag{5}$$

In each case, the first 22 parts of the quotient are generated by the integral classes $\alpha_i^{(1,1,1)} - 3 \cdot \alpha_i^{(2,1)} + 3 \cdot \alpha_i^{(3)} + 3 \cdot 1^{(2)}\alpha_i^{(1,1)} - 6 \cdot 1^{(2)}\alpha_i^{(2)} + 6 \cdot 1^{(2,2)}\alpha_i^{(1)} - 3 \cdot 1^{(3)}\alpha_i^{(1)}$, *for* $i = 1 \dots 22$*. Now define an integral class*

$$K := \sum_{i \neq j} B(\alpha_i, \alpha_j) \left[\alpha_i^{(1,1)}\alpha_j^{(1)} - 2 \cdot \alpha_i^{(2)}\alpha_j^{(1)} + \frac{3}{2} \cdot 1^{(2)}\alpha_i^{(1)}\alpha_j^{(1)}\right] + \\ + \sum_i B(\alpha_i, \alpha_i) \left[\alpha_i^{(1,1,1)} - 2 \cdot \alpha_i^{(2,1)} + \frac{3}{2} \cdot 1^{(2)}\alpha_i^{(1,1)}\right] + x^{(2)} - 1^{(2)}x^{(1)}.$$

In the case $n = 3$*, the last part of the quotient is generated by* K*.*
In the case $n = 4$*, the class* $1^{(4)}$ *generates the 2-torsion part and* $K + 38 \cdot 1^{(4)}$ *generates the 108-torsion part.*
In the case $n = 5$*, the last part of the quotient is generated by* $K + 16 \cdot 1^{(4)} - 21 \cdot 1^{(3,2)}$*.*
If $n \geq 6$*, the two last parts of the quotient are generated by some multiples of* $K - \frac{4}{3}(45-n)1^{(2,2,2)} + (48-n)1^{(3,2)}$ *and* $K - \frac{1}{2}(40-n)1^{(2,2,2)} + \frac{1}{4}(48-n)1^{(4)}$*.*

References

1. S. Boissière, M. Nieper-Wißkirchen, and A. Sarti, *Smith theory and Irreducible Holomorphic Symplectic Manifolds*, eprint arXiv:1204.4118 (2012).
2. A. Lascoux, *Symmetric functions*, Notes of the course given at Nankai University, 2001, http://www.mat.univie.ac.at/ slc/wpapers/s68vortrag/ALCoursSf2.pdf .
3. M. Lehn and C. Sorger, *The cup product of Hilbert schemes for K3 surfaces*, Invent. Math. **152** (2003), no. 2, 305–329.
4. E. Markman, *Integral generators for the cohomology ring of moduli spaces of sheaves over Poisson surfaces*, Adv. Math. **208** (2007), no. 2, 622–646.
5. E. Markman, *Integral constraints on the monodromy group of the hyperKähler resolution of a symmetric product of a K3 surface*, Internat. J. Math. **21** (2010), no. 2, 169–223.
6. H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Ann. of Math. (2) **145** (1997), no. 2, 379–388.
7. Z. Qin and W. Wang, *Integral operators and integral cohomology classes of Hilbert schemes*, Math. Ann. **331** (2005), no. 3, 669–692.
8. M. Verbitsky, *Cohomology of compact hyper-Kähler manifolds and its applications*, Geom. Funct. Anal. **6** (1996), no. 4, 601–611.

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