COMPUTING CUP-PRODUCTS IN INTEGER COHOMOLOGY OF HILBERT SCHEMES OF POINTS ON K3 SURFACES

SIMON KAPFER

Abstract. We study the images of cup products in integer cohomology of the Hilbert scheme of n points on a K3 surface.

1. Preliminaries

Definition 1.1. Let S be a projective K3 surface. We fix integral bases 1 of $H^0(S,\mathbb{Z})$, x of $H^4(X,\mathbb{Z})$ and $\alpha_1,\dots,\alpha_{22}$ of $H^2(S,\mathbb{Z})$. The cup product induces a symmetric bilinear form B_{H^2} on $H^2(X,\mathbb{Z})$, written as a symmetric matrix with respect to this basis, looks like

$$B_{H^2} = \begin{pmatrix} U & & & & \\ & U & & & \\ & & U & & \\ & & & E & \\ & & & & E \end{pmatrix},$$

where U stands for the intersection matrix of the hyperbolic lattice and E stands for the negative matrix of the E_8 lattice, *i.e.*

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \end{pmatrix}.$$

We may extend B_{H^2} to a symmetric non-degenerate bilinear form on $H^*(S,\mathbb{Z})$ by setting $B(1,1)=0,\ B(1,\alpha_i)=0,\ B(1,x)=1,\ B(x,x)=0$.

Definition 1.2. B induces a form $B\otimes B$ on $\operatorname{Sym}^2 H^*(S,\mathbb{Z})$. So the cup-product

$$\mu : \operatorname{Sym}^2 H^*(S,\mathbb{Z}) \longrightarrow H^*(S,\mathbb{Z})$$

has an adjoint comultiplication Δ, given by:

$$\Delta : H^*(S,\mathbb{Z}) \longrightarrow \operatorname{Sym}^2 H^*(S,\mathbb{Z}), \quad \Delta = (B\otimes B)^{-1}\mu^T B$$

The image of 1 under the composite map $\mu(\Delta(1)) = B(\Delta(1),\Delta(1)) = 24x$, denoted by e is called the Euler Class.

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We denote by $S^{[n]}$ the Hilbert scheme of n points on S, *i.e.* the classifying space of all zero-dimensional closed subschemes of length n, which is smooth. A classical result by Nakajima gives an explicit description of $H^*(S^{[n]},\mathbb{Q})$ in terms of creation operators $\mathfrak{q}_l(\beta)$, $\beta\in H^*(S,\mathbb{Q})$, acting on the direct sum $\bigoplus_n H^*(S^{[n]},\mathbb{Q})$. An integral basis for $H^*(S^{[n]},\mathbb{Z})$ in terms of Nakajima's operators was given by Qin–Wang:

Theorem 1.3. [5, Thm. 5.4.] *The classes*

$$\frac{1}{z_\lambda}\mathfrak{q}_\lambda(1)\mathfrak{q}_\mu(x)\mathfrak{m}_{\nu^1,\alpha_1}\dots\mathfrak{m}_{\nu^{22},\alpha_{22}}|0\rangle,\quad \|\lambda\|+\|\mu\|+\sum_{i=1}^{22}\|\nu^i\|=n$$

form an integral basis for $H^*(S^{[n]},\mathbb{Z})$. Here, λ, μ, ν^i are partitions, $\|\cdot\|$ means the weight of a partition i.e. $\|\lambda\|=\sum_i m_i i$ and $z_\lambda:=\prod_i i^{m_i}m_i!$, if $\lambda=(1^{m_1},2^{m_2},\dots)$. The symbol $\mathfrak{q}$ stands for Nakajima's creation operator. The relation of $\mathfrak{m}_{\nu,\alpha}$ to $\mathfrak{q}_{\tilde\nu}(\alpha)$ is the same as the monomial symmetric functions m_ν to the power sum symmetric functions $p_{\tilde\nu}$.

The ring structure of $H^*(S^{[n]},\mathbb{Q})$ has been studied in [2], where an explicit algebraic model is constructed. Since $H^{\mathrm{odd}}(S^{[n]},\mathbb{Z})=0$ and $H^{\mathrm{even}}(S^{[n]},\mathbb{Z})$ is torsion-free by [3], we can also apply these results to $H^*(S^{[n]},\mathbb{Z})$ to determine the multiplicative structure of cohomology with integer coefficients.

2. Computational results

With the help of a computer, we are able to compute arbitrary products in $H^*(S^{[n]},\mathbb{Z})$. We give some results in low degrees. The algebra generated by classes of degree 2 is an interesting object to study. For cohomology with complex coefficients, Verbitsky has proven that the algebra generated by $H^2(X,\mathbb{C})$

Notation 2.1. To enumerate the basis of $H^*(S^{[n]},\mathbb{Z})$, we introduce the following abbreviation:

$$1^\lambda\alpha_1^{\nu_1}\dots\alpha_{22}^{\nu_{22}}x^\mu := \frac{1}{z_{\tilde\lambda}}\mathfrak{q}_{\tilde\lambda}(1)\mathfrak{q}_\mu(x)\mathfrak{m}_{\nu^1,\alpha_1}\dots\mathfrak{m}_{\nu^{22},\alpha_{22}}|0\rangle$$

where the partition $\tilde\lambda$ is built from λ by appending sufficiently many Ones, such that $\|\tilde\lambda\|+\|\mu\|+\sum\|\nu^i\|=n$.

By computing multiplication matrices with respect to the integral basis and a reduction to Smith normal form (with the help of a computer), images of cup products can be explored.

Proposition 2.2. *Studying the image of* $\mathrm{Sym}^2 H^2$ *in* H^4*, we obtain:*

$$\frac{H^4(S^{[2]},\mathbb{Z})}{\mathrm{Sym}^2 H^2(S^{[2]},\mathbb{Z})}\cong\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23}\oplus\frac{\mathbb{Z}}{5\mathbb{Z}}$$

This was already known to Boissière, Nieper-Wißkirchen and Sarti, [1, Prop. 3].

$$\frac{H^4(S^{[3]},\mathbb{Z})}{\mathrm{Sym}^2 H^2(S^{[3]},\mathbb{Z})}\cong\frac{\mathbb{Z}}{3\mathbb{Z}}\oplus\mathbb{Z}^{\oplus 23}$$

The torsion part of the quotient is generated by the integral class $\frac{1}{3}\mathfrak{q}_{(3)}(1)|0\rangle$.

$$\frac{H^4(S^{[n]},\mathbb{Z})}{\mathrm{Sym}^2 H^2(S^{[n]},\mathbb{Z})}\cong\mathbb{Z}^{\oplus 24},\quad \textit{for } n\geq 4.$$

This was already proven by Markman, [4, Thm. 1.10].

Proposition 2.3. *Comparing $H^2(S^{[n]}, \mathbb{Z}) \cup H^4(S^{[n]}, \mathbb{Z})$ with $H^6(S^{[n]}, \mathbb{Z})$, we obtain:*

$$\frac{H^6(S^{[2]}, \mathbb{Z})}{H^2(S^{[2]}, \mathbb{Z}) \cup H^4(S^{[2]}, \mathbb{Z})} = 0 \tag{1}$$

$$\frac{H^6(S^{[3]}, \mathbb{Z})}{H^2(S^{[3]}, \mathbb{Z}) \cup H^4(S^{[3]}, \mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12} \tag{2}$$

$$\frac{H^6(S^{[4]}, \mathbb{Z})}{H^2(S^{[4]}, \mathbb{Z}) \cup H^4(S^{[4]}, \mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23} \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12} \tag{3}$$

$$\frac{H^6(S^{[5]}, \mathbb{Z})}{H^2(S^{[5]}, \mathbb{Z}) \cup H^4(S^{[5]}, \mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23} \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12} \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}}\right)^{\oplus 3} \tag{4}$$

$$\frac{H^6(S^{[n]}, \mathbb{Z})}{H^2(S^{[n]}, \mathbb{Z}) \cup H^4(S^{[n]}, \mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 22} \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^{\oplus 12} \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}}\right)^{\oplus 2} \oplus \mathbb{Z},\ n \geq 6. \tag{5}$$

- *The 3-torsion part is generated by the 12 integral classes $\alpha_i^{(1,1,1)} \in H^6$, where $i = 1, 2, 3, 4, 5, 6, 8, 9, 11, 16, 17, 19$.*
- *The 2-torsion part is generated by the 22 integral classes $\alpha_i^{(1,1,1)} + \alpha_i^{(2,1)} + \alpha_i^{(3)} + 1^{(2)}\alpha_i^{(1,1)} + 1^{(3)}\alpha_i^{(1)}$, $i = 1, \ldots, 22$ and, in the cases $n = 4, 5$, by the integral class $1^{(4)} \in H^6$.*
- *The 5-torsion part is generated by the 2 integral classes $\alpha_i^{(1,1,1)} + 2\alpha_i^{(2,1)} + 3\alpha_i^{(3)} + 4 \cdot 1^{(2)}\alpha_i^{(1,1)} + 2 \cdot 1^{(2)}\alpha_i^{(2)} + 2 \cdot 1^{(3)}\alpha_i^{(1)} + 3 \cdot 1^{(2,2)}\alpha_i^{(1)}$, $i = 13, 21$ and, in the case $n = 5$, by the integral class $1^{(4)} + 1^{(3,2)}$.*
- *The free summand is generated by the class $3 \cdot 1^{(4)} - 12 \cdot 1^{(3,2)} + 10 \cdot 1^{(2,2,2)}$.*

Proposition 2.4.

$$\frac{H^6(S^{[2]}, \mathbb{Z})}{\mathrm{Sym}^3 H^2(S^{[2]}, \mathbb{Z})} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$$

The quotient is generated by the integral class $\frac{1}{2}\mathfrak{q}_{(2)}(1)|0\rangle$.

$$\frac{H^6(S^{[3]}, \mathbb{Z})}{\mathrm{Sym}^3 H^2(S^{[3]}, \mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 230} \oplus \left(\frac{\mathbb{Z}}{36\mathbb{Z}}\right)^{\oplus 22} \oplus \frac{\mathbb{Z}}{72\mathbb{Z}} \oplus \mathbb{Z}^{\oplus 507}$$

$$\frac{H^6(S^{[4]}, \mathbb{Z})}{\mathrm{Sym}^3 H^2(S^{[4]}, \mathbb{Z})} \cong$$

$$\frac{H^6(S^{[5]}, \mathbb{Z})}{\mathrm{Sym}^3 H^2(S^{[5]}, \mathbb{Z})} \cong$$

$$\frac{H^6(S^{[n]}, \mathbb{Z})}{\mathrm{Sym}^3 H^2(S^{[n]}, \mathbb{Z})} \cong n \geq 6.$$

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Simon Kapfer, Lehrstuhl für Algebra und Zahlentheorie, Universitätsstrasse 14, D-86159 Augsburg

E-mail address: simon.kapfer@math.uni-augsburg.de