MEAN CURVATURE PROBLEMS IN $\mathbb{R}^3$

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Contents

These lecture notes are to be seen as an introduction to some problems in geometric analysis. As the name suggests, it is a mathematical domain at the interface of geometry and analysis, and which requires both. The first chapter will delve with the geometric matters in details, but we will only briefly recap the useful analytic notions. This choice is motivated by several reasons:

- A course on PDE techniques has been given in the first semester, and the notes are available ([**?**]). This course is *not* a prerequisite, but can serve as a basis for the concepts we will use.
- The analytic tools and PDE techniques will be *applied* when in classical situations, *explored* when considering the limit cases.
- Exploring the geometric formalism is more useful and insightful than exposing the analytic properties, since the geometry will guide the analysis of the situations.

These are *very* rough notes used as support for an advanced lecture. They have not been peer-reviewed, and are not yet up to research standards. Any reader is absolutely welcome to offer suggestions or point out typos.

Notations

- In all these notes, we will adopt the Einstein conventions where we sum over repeated upper/lower indexes: $T^{ap}S_{pb} := \sum T^{ap}S_{bp}$.
- We will use several notations for the partial derivatives. When considering a system of coordinates (x_i) we will favor $\partial_i u$. When working on a domain described by (x, y) we will favour u_x, u_y, when the fonction has no index, and $\partial_x u_1, \partial_y u_1$ when it has. We will denote multi-derivatives using multi-indexes. For instance : $\partial_{(2,1,0)}u = \partial_1\partial_1\partial_2 u$.
- $L^p(\Omega)$ denotes the Lebesgue space. For $1 \leq p \leq \infty$, $L^p(\Omega)$ is the set (of equivalence classes) of measurable real valued functions f such that $\|f\|_{L^p(\Omega)} < \infty$

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p dx^1 \dots dx^n \right)^{\frac{1}{p}}, \ \text{ if } 1 \leq p < \infty$$

$$\|f\|_{L^\infty(\Omega)} := \inf\{M \in [0, \infty] \,|\, |f(x)| \leq M \text{ a.e.}\}.$$

- $L^p_{\text{loc}}(\Omega)$ denotes the space of measurable functions which are L^p on every compact subset of Ω.

- $W^{k,p}(\Omega)$ denotes the Sobolev space:

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) | \forall s \in \mathbb{N}^m,\, |s| \leq k,\, \partial_s f \in L^p(\Omega)\}.$$

  On this space we have the norm:

$$\|f\|_{W^{k,p}(\Omega)} := \sum_{|s|\leq k} \|\partial_s f\|_{L^p(\Omega)}\,.$$

1 A reminder on surfaces

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In this chapter, we will briefly introduce the basic objects which we will use and on which we will work, as well as the formalism we will adopt. We refer the reader to [?], [?] [?], [?] or [?] for more details on the manifold formalism and deeper explorations of differential geometry.

1.1 Surfaces in $\mathbb{R}^3$

1.1.1 Intrinsic and extrinsic surfaces

Manifolds

Let us define our notion of surfaces: first an abstract and general one (a manifold of dimension 2), and then one more specific to the euclidean case (immersed surface), and thus more adapted to our purposes.

Definition 1.1.1. *A smooth manifold of dimension n is a topological Hausdorff space M such that there exists an open covering $\cup_i \Omega_i$ of M and a corresponding family of homeomorphisms $\varphi_i : \Omega_i \to O_i \subset \mathbb{R}^n$ such that, for any $\Omega_i \cap \Omega_j \neq \emptyset$ the applications*

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(\Omega_i \cap \Omega_j) \to \varphi_j(\Omega_i \cap \Omega_j)$$

are C^∞ diffeomorphisms.

The couples (Ω_i, φ_i) are called *local charts* of the manifold, while $(x_1, \dots, x_n) = \varphi(p)$ are the *local coordinates*. The $\varphi_j \circ \varphi_i^{-1}$ are called the change of coordinates (or change of charts). A covering family of (Ω_i, φ_i) is called an atlas. In the present notes, we will focus on the 2-dimensional case.

Definition 1.1.2. *A surface is a smooth manifold of dimension 2.*

Beyond the mathematical formulation, a manifold is conceptually a family of open sets of $\mathbb{R}^n$ assembled together by the chart changes (the confetti model: the open domains of the charts are as many confettis assembled together in the manner described by the change of coordinates to form the manifold).

The manifold structure is enough for differential calculus. In particular, one can extend the classical notions of smooth functions, of the rank of a map, of immersions and submersions:

Definition 1.1.3. *Let M and N be smooth manifolds of dimensions m and n respectively, and $f : M \to N$ a continuous application.*

- *f is differentiable at $x \in M$ if, for all local charts (Ω, φ) of M containing x and (U, ψ) of N containing $f(x)$, $\psi \circ f \circ \varphi^{-1} : \varphi(\Omega) \to \psi(U)$ is differentiable at $\varphi(x)$. If f is differentiable at every $x \in M$, f is differentiable on M. The rank of f at x is then the rank of $\psi \circ f \circ \varphi^{-1}$ at $\varphi(x)$, which does not depend on the chosen charts.*

- *If f is differentiable on M, f is an immersion if $rk(f)(x) = m$, for all $x \in M$, a submersion if $rk(f)(x) = n$, for all $x \in M$. If f is an immersion and a homeomorphism on its image, it is an embedding.*

- *$f \in C^k(M, N)$ if, for all local charts (Ω, φ) of M and (U, ψ) of N, $\psi \circ f \circ \varphi^{-1} : \varphi(\Omega) \to \psi(U)$ is $C^k(\varphi(\Omega), \psi(U))$.*

- *f is a C^k diffeomorphism if f is a bijection and both f and f^{-1} are C^k.*

Extrinsic surfaces

Although the smooth manifold formalism is self-contained (and we will make use of it in several occasions), we will work predominantly with surfaces in $\mathbb{R}^3$, using a particular case of the submanifold notion:

Definition 1.1.4. *A surface in $\mathbb{R}^3$ is the image of a smooth manifold Σ of dimension 2 by a C^∞ immersion $\Phi : \Sigma \to \mathbb{R}^3$. If, in addition, Φ is an embedding, the surface is embedded.*

In essence, when considering a surface in $\mathbb{R}^3$ we in fact consider a *parametrized* surface $\Phi(\Sigma)$ inside $\mathbb{R}^3$, with the parametrization Φ defined on a manifold Σ. Looking at the parametrization in a local conformal chart (Ω, φ) on Σ yields a parametrization on an open set of $\mathbb{R}^2$: $\phi := \Phi \circ \varphi^{-1} : U \to \mathbb{R}^3$ of a part of the surface $\Phi(\Sigma)$. In practice, this formalism thus allows us to locally study a surface in $\mathbb{R}^3$ as parametrized open sets of $\mathbb{R}^2$, which we assemble together based on the chart changes, allowing for simple local descriptions while laying the burden of the topological structure on the manifold and its coordinate changes.

Let us give core examples:

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Example 1.1.1. • *A disk of radius $r > 0$ in $\mathbb{R}^2$ is a one chart manifold of dimension 2. The flat disk and the hemisphere are two surfaces in $\mathbb{R}^3$ parametrized over the unit disk $\mathbb{D}$ by, respectively*

$$\Phi:\,(x,y)\in\mathbb{D}\mapsto(x,y,0),$$

and

$$\Phi:\,(x,y)\in\mathbb{D}\mapsto\left(x,y,\sqrt{1-x^2-y2}\right).$$

- *An annulus $\mathbb{D}_R\backslash\mathbb{D}_r\subset\mathbb{R}^2$ is also a one chart manifold of dimension 2. On it, one can parametrize a piece of a cylinder, or of a catenoïd by respectively:*

$$\Phi:\,(x,y)\in\mathbb{D}\backslash\mathbb{D}_r\mapsto\left(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}},\sqrt{x^2+y^2}\right),$$

and

$$\Phi:\,(x,y)\in\mathbb{D}\backslash\mathbb{D}_r\mapsto\left(\operatorname{ch}\left(\sqrt{x^2+y^2}\right)\frac{x}{\sqrt{x^2+y^2}},\operatorname{ch}\left(\sqrt{x^2+y^2}\right)\frac{y}{\sqrt{x^2+y^2}},\sqrt{x^2+y^2}\right).$$

- *The euclidean sphere $\mathbb{S}^2:=\{(x,y,z)\in\mathbb{R}^3|x^2+y^2+z^2=1\}$ is a manifold of dimension 2, with two charts given by the stereographic projection from the north and south poles:*

$$\pi_N:\begin{cases}\mathbb{S}^2\backslash\{N=(0,0,1)\}\to\mathbb{R}^2\\(x,y,z)\mapsto\left(\dfrac{x}{1-z},\dfrac{y}{1-z}\right),\end{cases}$$

$$\pi_S:\begin{cases}\mathbb{S}^2\backslash\{S=(0,0,-1)\}\to\mathbb{R}^2\\(x,y,z)\mapsto\left(\dfrac{x}{1+z},\dfrac{y}{1+z}\right).\end{cases}$$

One can parametrize the euclidean sphere as a surface in $\mathbb{R}^3$ through the inclusion: $I:\,p\in\mathbb{S}^2\mapsto p\in\mathbb{R}^3$. This inclusion can be parametrized in the two previous charts leading to two local parametrizations of the embedded sphere in $\mathbb{R}^3$:

$$\phi_N:=\pi_N^{-1}=\begin{cases}\mathbb{R}^2\to\mathbb{S}^2\backslash\{N=(0,0,1)\}\\(x,y)\mapsto\dfrac{1}{1+x^2+y^2}\left(2x,2y,x^2+y^2-1\right),\end{cases}$$

$$\phi_S:=\pi_S^{-1}=\begin{cases}\mathbb{R}^2\to\mathbb{S}^2\backslash\{S=(0,0,-1)\}\\(x,y)\mapsto\dfrac{1}{1+x^2+y^2}\left(2x,2y,1-x^2-y^2\right),\end{cases}$$

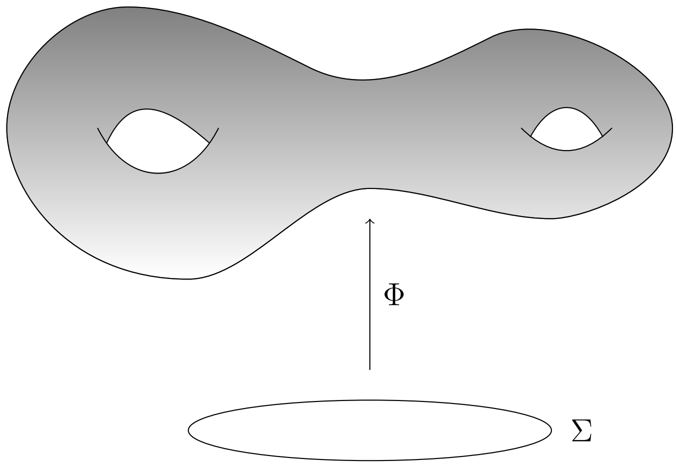

Figure 1: An extrinsic surface

- *The straight torus is a one chart manifold defined as* $\mathbb{R}^2 \backslash \mathbb{Z}^2$ *(the chart then simply associates to* $[(x, y)]$ *its representant in* $[0,1]^2$*). One can parametrize the Willmore torus on this straight torus with the following parametrization in the local chart:*

$$\phi_W := (x,y) \in [0,1]^2 \mapsto \begin{pmatrix} \left(1+\frac{\sqrt{2}}{2}\cos(2\pi x)\right)\cos(2\pi y) \\ \left(1+\frac{\sqrt{2}}{2}\cos(2\pi x)\right)\sin(2\pi y) \\ \frac{\sqrt{2}}{2}\sin(2\pi y) \end{pmatrix}.$$

  In fact, one can parametrize a broad family of immersed tori in $\mathbb{R}^3$ *on this straight torus:*

$$\phi_{a,b} := (x,y) \in [0,1]^2 \mapsto \begin{pmatrix} (a+b\cos(2\pi x))\cos(2\pi y) \\ (a+b\cos(2\pi x))\sin(2\pi y) \\ b\sin(2\pi y) \end{pmatrix}.$$

Remark 1.1.1. *These examples illustrate that several surfaces in* $\mathbb{R}^3$ *can be parametrized over the same manifold of dimension 2. In addition, looking at the parametrizations for the piece of the cylinder and the catenoïd shows how our description of our immersed surface depends on our choice of charts (and thus on the manifold structure). Indeed, another choice of coordinates for the annulus is the polar coordinates:* $(\rho, \theta) \in [r, 1) \times (-\pi, \pi]$*. For these, the parametrization of the cylinder become much simpler; respectively:* $(\cos\theta, \sin\theta, r)$ *and* $(\mathrm{ch}(r)\cos\theta, \mathrm{ch}(r)\sin\theta, r)$*. Choosing the best parametrization, that is the best description of the immersed surfaces in* $\mathbb{R}^3$*, is often a first step in the analysis of a differential geometry problem and will be done through* gauge choices *(see section 1.3.5).*

To avoid ambiguities of the term "surface", which can both mean a manifold of dimension 2 or an immersion of a manifold in $\mathbb{R}^3$, we will call the first case an intrinsic surface (as all objects are defined without referencing a larger space) and the second one an extrinsic surface (since objects are defined as part of an encompassing euclidean space).

We will keep working on the intrinsic and extrinsic formalisms in the following subsection which will aim to introduce a notion of differentiability on the manifolds.

1.1.2 Tangent space, tangent field, tangent map

Tangent vectors

Considering a parametrization Φ of an intrinsic surface Σ, and looking at it in a local chart, it is easy to differentiate the resulting local parametrization ϕ in the local coordinates: $\nabla\phi = \begin{pmatrix}\partial_x\phi \\ \partial_y\phi\end{pmatrix}$. However, this gradient is only defined in this local chart (the only setting where the local coordinates (x, y) are defined on Σ). Extending it requires introducing a linear structure on the intrinsic surface. We will, in fact, do so for all smooth manifolds.

Definition 1.1.5. *Let M be a smooth manifold of dimension n. Let $p \in M$. A tangent vector to M at p is an application $v : C^\infty(M) \to \mathbb{R}$ satisfying:*

- $\mathbb{R}$*-linearity:* $v.(af + bg) = av.f + bv.g$.
- *Leibniz:* $v.(fg) = v.fg(p) + f(p)v.g$.

We call tangent space at p and denote T_pM the vector space of tangent vectors to M at p.

In essence, a tangent vector is simply a way to differentiate a $\mathbb{R}$-valued fonction on the manifold. The vectorial structure is highlighted in local coordinates:

Definition 1.1.6. *Let (Ω, φ) be a local chart around $p \in M$ and $(x_1, \dots, x_n)$ be the associated coordinates. Then, for any $f \in C^\infty(M)$:*

$$\partial_{x_i} f(p) = \partial_i \left(f \circ \varphi^{-1}\right)(\varphi(p)).$$

The family $(\partial_{x_1}, \dots, \partial_{x_n})(p)$ forms a basis for the tangent space T_pM:

$$\forall v \in T_pM,\ v = v^i \partial_{x_i}(p).$$

We can switch basis with the coordinate changes:

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Proposition 1.1.1. *Let $p \in M$ and φ, ψ be two local charts around p and $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ the associated coordinates. Then:* $\partial_{x_i} = \partial_i \left[\psi \circ \varphi^{-1}\right]^j \partial_{y_j}$.

Proof. Let $f \in C^\infty(M)$:

$$\begin{aligned}\partial_{x_i}.f(p) &= \partial_i \left(f \circ \varphi^{-1}\right)(\varphi(p)) = \partial_i \left(\left[f \circ \psi^{-1}\right] \circ \left[\psi \circ \varphi^{-1}\right]\right)(\varphi(p)) \\ &= \partial_i \left[\psi \circ \varphi^{-1}\right]^j (\varphi(p)) \partial_j \left(f \circ \psi^{-1}\right)(\psi(p)) = \partial_i \left[\psi \circ \varphi^{-1}\right]^j \partial_{y_j} f.\end{aligned}$$

□

As a consequence, if $v \in T_pM$ and $v = v^i \partial_{x_i} = \tilde{v}^i \partial_{y_i}$:

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$$\tilde{v}^i = v^p \partial_p \left[\psi \circ \varphi^{-1}\right]^i = J^i_p v^p, \tag{1}$$

where $J = \left(\partial_p \left[\psi \circ \varphi^{-1}\right]^i\right)^{i=1,\ldots,n}_{p=1,\ldots,n}$ is the Jacobian matrix of the coordinate change diffeomorphism.

In fact, the ∂_{x_i} are functions which send a point p to a tangent vector at p i.e. vector fields.

Vector fields

Definition 1.1.7. *Let M be a smooth manifold of dimension n. A vector field X on M is a smooth map such that for every $p \in M$, $X(p) \in T_pM$. In local coordinates: $(x_1, \ldots, x_n)$ there exists a n-uplet $(X^1, \ldots, X^n)$ such that:*

$$X = X^i(x_1, \ldots, x_n)\partial_{x_i},$$

or, in other words, for any $p \in M$ and $f \in C^\infty(M)$:

$$X(p).f = X^i(p_1, \ldots, p_n)\partial_{x_i} f(p).$$

We will denote $\chi(M)$ the set of vector fields on M.

If we consider two sets of local coordinates $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ around a point p and denote respectively X^i and $\tilde{X}^i$ the expression in the x and y coordinates:

$$\begin{aligned}X = X^i \partial_{x_i} &= X^i \partial_i \left[\psi \circ \varphi^{-1}\right]^j \partial_{y_j} = \left(X^p \partial_p \left[\psi \circ \varphi^{-1}\right]^i\right) \partial_{y_i} \\ &= \tilde{X}^i \partial_{y_i}.\end{aligned}$$

So, for X to be properly defined on the manifold, its expressions in local coordinates must satisfy the following change of coordinates formula:

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$$\tilde{X}^i = X^p \partial_p \left[\psi \circ \varphi^{-1}\right]^i = J^i_p X^p. \tag{2}$$

We say that X is contravariant.

Remark 1.1.2. *Going back to the notion of a manifold as "a string of pieces of $\mathbb{R}^n$ assembled with the chart changes", the coordinate change described in* (2) *is exactly the one needed to make sure that a family of vectorial functions of $\mathbb{R}^n$ $(X^i)_{i=1\dots n}(x_1,\dots,x_n)$ assemble coherently into a vector field on the manifold.*

This contravariant change of coordinates law is constraining. For instance, one cannot simply differentiate a vector as one would do with a function: if $X, V \in \chi(M)$, the application defined in local coordinates as $V^p\partial_p X^i$ is not a vector field since it is not contravariant. Indeed, with the same formalism as above:

$$\tilde{V}^p\partial_{y_p}\tilde{X}^i = J^p_q V^q \partial_{y_p}\left(J^i_l X^l\right) = J^i_l V^q J^p_q \partial_{y_p} X^l + V^q J^p_q X^l \partial_{y_p} J^i_l = J^i_l V^q \partial_{x_q} X^l + V^q X^l \partial_q J^i_l.$$

However, since: $\partial_q J^i_l = \partial_q\partial_l\left[\psi\circ\varphi^{-1}\right]^i = \partial_l\partial_q\left[\psi\circ\varphi^{-1}\right]^i = \partial_l J^i_q$, the parasit term is symmetric, and can thus be eliminated by antisymmetrizing the expression, yielding the Lie bracket:

Definition 1.1.8. *For any $V, W \in \chi(M)$, the Lie bracket of V and W, denoted $[V,W]$ is the vector field defined as:*

$$[V,W]_p(f) = V_p(W.f) - W_p(V.f).$$

In local coordinates:

$$[V,W]^i = V^p\partial_p W^i - W^p\partial_p V^i.$$

One can check that the Lie Bracket satisfies the following properties:

Proposition 1.1.2.

- *$\mathbb{R}$-bilinearity:* $[aV + bW, X] = a[V,X] + b[W,X]$,
- *Skew-symmetry:* $[W,V] = -[V,W]$,
- *Jacobi identity:* $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$.

Proof. The first two equalities can be read directly on the formula with the properties of the derivation. For the Jacobi identity, let us consider three vector fields

X, Y, Z and write in local coordinates:

$$\begin{aligned}
T &:= ([X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]])^i \\
&= X^q\partial_q\left(Y^p\partial_p Z^i - Z^p\partial_p Y^i\right) - \left(Y^p\partial_p Z^q - Z^p\partial_p Y^q\right)\partial_q X^i \\
&+ Y^q\partial_q\left(Z^p\partial_p X^i - X^p\partial_p Z^i\right) - \left(Z^p\partial_p X^q - X^p\partial_p Z^q\right)\partial_q Y^i \\
&+ Z^q\partial_q\left(X^p\partial_p Y^i - Y^p\partial_p X^i\right) - \left(X^p\partial_p Y^q - Y^p\partial_p X^q\right)\partial_q Z^i \\
&= X^q\partial_q Y^p\partial_p Z^i - X^q\partial_q Z^p\partial_p Y^i - Y^p\partial_p Z^q\partial_q X^i \\
&+ Z^p\partial_p Y^q\partial_q X^i + \partial_{pq}Z^i Y^p X^q - \partial_{pq}Y^i Z^p X^q \\
&+ Y^q\partial_q Z^p\partial_p X^i - Y^q\partial_q X^p\partial_p Z^i - Z^p\partial_p X^q\partial_q Y^i \\
&+ X^p\partial_p Z^q\partial_q Y^i + \partial_{pq}X^i Z^p Y^q - \partial_{pq}Z^i X^p Z^q \\
&+ Z^q\partial_q X^p\partial_p Y^i - Z^q\partial_q Y^p\partial_p X^i - X^p\partial_p Y^q\partial_q Z^i \\
&+ Y^p\partial_p X^q\partial_q Z^i + \partial_{pq}Y^i X^p Z^q - \partial_{pq}X^i Y^p Z^q \\
&= 0,
\end{aligned}$$

since in the last line, all the terms cancel two by two. □

The Lie bracket is not C^∞ linear:

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$$[fV, W] = fV.(W.) - W.(fV.) = fV.(W.) - W.fV. - fW.(V.) = f[V,W] - W.fV. \qquad (3)$$

In addition, if $(\partial_{x_1}, \dots, \partial_{x_n})$ is a local frame derived from local coordinates, the corresponding Lie brackets are:

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$$\forall i, j \quad \left[\partial_{x_i}, \partial_{x_j}\right] = 0, \qquad (4)$$

owing to the commutativity of differentiations in $\mathbb{R}^n$.

Tangent map

Now that we have our linear structure on a differentiable manifold, we introduce the tangent application:

Definition 1.1.9. *Let $\Phi : M \to N$ be a smooth function between two manifolds. For each $p \in M$, the linear map*

$$d\Phi_p : \begin{cases} T_pM \to T_{\Phi(p)}N \\ v \mapsto v_\Phi : (g \mapsto v.(g \circ \Phi)) \end{cases}$$

is called the differential *(or* tangent*) map of Φ at p. The function which to p associates $d\Phi_p$ is the differential (or tangent) map of Φ.*

Proof. We first need to show that the differential map is well defined, that is that v_Φ is a tangent vector at $\Phi(p)$ of N when v is tangent to M at p.

- $\mathbb{R}$-linearity: $v_\Phi.(af+bg) = v.\left([af+bg]\circ\Phi\right) = v.\left(af\circ\Phi + bg\circ\Phi\right) = av.\left(f\circ\Phi\right) + bv.\left(g\circ\Phi\right) = av_\Phi.f + bv_\Phi.g$.
- Leibniz: $v_\Phi.(fg) = v.\left([fg]\circ\Phi\right) = v.\left(f\circ\Phi g\circ\Phi\right) = v.(f\circ\Phi)g(\Phi(p)) + f(\Phi(p))v.(g\circ\Phi) = v_\phi.fg(\Phi(p)) + f(\Phi(p))v_\Phi.g$.

One can then quickly show that it is linear: $(\lambda v + w)_\Phi.f = (\lambda v + w).f\circ\Phi = \lambda v.f\circ\Phi + w.f\circ\Phi = \lambda v_\Phi.f + w_\Phi.f$. □

Let us compare this tangent map to the differential in local coordinates. To that end, let us consider $\Phi : M \to N$ a smooth function, and study it in local coordinates $\varphi = (x_1,\dots,x_n)$ around $p\in M$ and $\psi = (y_1,\dots,y_n)$ around $\Phi(p)\in N$. Then, $\psi\circ\Phi\circ\varphi^{-1}$ is an application between a neighborhood of $\varphi(p)$ in $\mathbb{R}^m$ and of $\psi(\Phi(p))$ in $\mathbb{R}^n$. This application can then be differentiated in the classical sense, and its Jacobian matrix is $J_{\psi\circ\Phi\circ\varphi^{-1}} = \left(\partial_j\left(\psi\circ\Phi\circ\varphi^{-1}\right)^i\right)_{j=1\dots m}^{i=1\dots n}$. On the other hand, let us consider a tangent vector $v\in T_pM$. In the local coordinates, $v = v^i\partial_{x_i}$. Then

$$\begin{aligned}
\left(d\Phi_p(v)\right).f &= v.\left(f\circ\Phi\right) = v^i\partial_{x_i}\left(f\circ\Phi\right)\\
&= v^j\partial_j\left(f\circ\Phi\circ\varphi^{-1}\right)(\varphi(p)) = v^j\partial_j\left(\left[f\circ\psi^{-1}\right]\circ\left[\psi\circ\Phi\circ\varphi^{-1}\right]\right)(\varphi(p))\\
&= v^j\partial_i\left[f\circ\psi^{-1}\right]\left(\psi\circ\Phi\circ\varphi^{-1}(\varphi(p))\right)\partial_j\left[\psi\circ\Phi\circ\varphi^{-1}\right]^i(\varphi(p))\\
&= \left(J_{\psi\circ\Phi\circ\varphi^{-1}}(\varphi(p))\right)^i_j v^j\partial_{y_i}f,
\end{aligned}$$

meaning that one can express $d\Phi_p(v)$ in local coordinates around $\Phi(p)$:

$$d\Phi_p(v)^i = \left(J_{\psi\circ\Phi\circ\varphi^{-1}}(\varphi(p))\right)^i_j v^j. \qquad (5) \quad \{260120211743\}$$

In conclusion, the expression of the tangent map in local coordinates coïncide with the differential of the expression of the map in local coordinates. For all local studies of an application, it will thus be enough to consider the derivatives in a local chart.

Example 1.1.2. *Let M be a smooth manifold and I an interval of $\mathbb{R}$. A curve in M is a smooth map $\gamma : I \to M$. Since I is a manifold of dimension 1 with an atlas made out of one single chart (the inclusion in $t\in I\mapsto t\in\mathbb{R}$), its tangent map is entirely described by its action on ∂_t:*

$$\gamma'(t).f := d\gamma_t(\partial_t).f = \partial_t.\left(f(\gamma(t))\right).$$

We will call $\gamma'(t)$ the speed of γ at t.

Remark 1.1.3. *In fact, defining curves in a local chart around a point, one can show that each tangent vector is the speed of a curve on the manifold. This gives a more "physical" interpretation for the tangent vectors: an intrinsic notion of vectorial speed for objects moving on the manifold* {2601202117

Tangent plane of an immersed surface

We can apply these considerations to an extrinsic surface $\Phi(\Sigma)$:

{290120211124} **Definition 1.1.10.** *Let Σ be an intrinsic surface and $\Phi(\Sigma)$ an extrinsic surface. The tangent plane at $\Phi(p)$ of $\Phi(\Sigma)$ is $T_{\Phi(p)}\Phi(\Sigma) = d\Phi_p(T_p\Sigma) \subset \mathbb{R}^3$. If ϕ is a local parametrization of the surface around $\Phi(p)$, then $T_{\Phi(p)}\Phi(\Sigma) = \mathrm{span}(\partial_x\phi, \partial_y\phi)$.*

Proof. These two definitions coïncide thanks to (5): with $\psi = Id$ the "chart" in $\mathbb{R}^3$ and (x, y) local coordinates on Σ, one has, if $v = v^1\partial_x + v^2\partial_y$:

$$d\Phi_p(v) = v^1\partial_x\phi + v^2\partial_y\phi.$$

□

Remark 1.1.4. *Since Φ is assumed to be an immersion, the tangent plane at $\Phi(p)$ is indeed a plane (that is, of dimension 2).*

Remark 1.1.5. *Remark 1.1.3 stands in the extrinsic case: the tangent plane of $\Phi(\Sigma)$ at $\Phi(p)$ is the set of the speed of curves in $\gamma \in \Phi(\Sigma) \subset \mathbb{R}^3$. In fact, γ is locally the image of a curve $\tilde{\gamma}$ on Σ, and $\gamma'(t) = d\Phi_{\tilde{\gamma}(t)}(\tilde{\gamma}'(t))$.*

1.1.3 Riemannian manifolds, Tensorial calculus

Computing the length of a curve $\gamma : I \to \Phi(\Sigma)$ on an extrinsic surface can be done in a classical manner: we consider it as a curve in $\mathbb{R}^3$, approximate it by a broken line on intervals of vanishing length, and compute the limit of the length of this broken line:

$$L(\gamma) = \int_I |\gamma'(t)|\, dt.$$

Doing the same on an intrinsic surface requires us to assign a meaning to $|\gamma'(t)|$ for $\gamma(t) \in \Sigma$, that is, to define a notion of norm for tangent vectors.

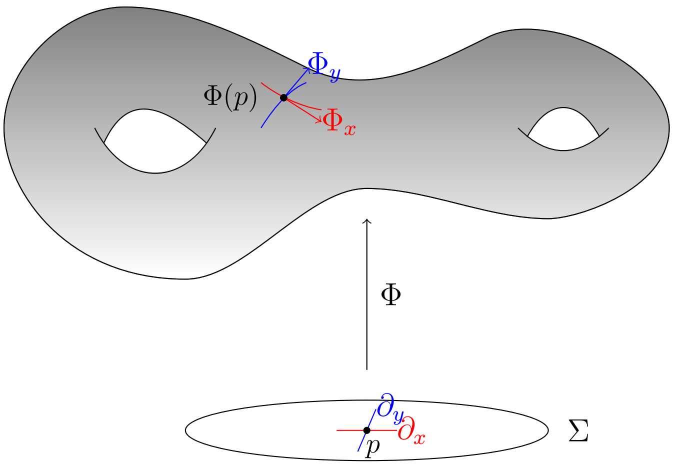

Figure 2: Intrinsic and extrinsic tangent vectors

Riemannian metric

Definition 1.1.11. *Let M be a smooth manifold of dimension n. A Riemannian metric g (also called a metric tensor) is an application which to $p \in M$ associates a scalar product on T_pM. In local coordinates $(x^1, \dots, x^n)$, for any $v = v^i \partial_{x_i}$ and $w = w^j \partial_{x_j}$:*

$$g(p)\langle v, w\rangle = g_{ij}(p) v^i w^j,$$

which we will denote $g = g_{ij} dx^i dx^j$.

Definition 1.1.12. *A manifold M endowed with a metric tensor g is a Riemannian manifold (M, g).*

Remark 1.1.6. *Given the definition and the chain rule from proposition 1.1.1, the local expression for the metric g must obey a precise change of coordinates formula. Indeed if we consider φ and ψ two intersecting charts, and $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ the associated coordinates, v and w two tangent vectors at $p \in M$, then*

$$v = v^i \partial_{x_i} = \left(v^j \partial_j \left[\psi \circ \varphi^{-1}\right]^i \right) \partial_{y_i} =: \tilde{v}^i \partial_{y_i}$$
$$w = w^i \partial_{x_i} = \left(w^j \partial_j \left[\psi \circ \varphi^{-1}\right]^i \right) \partial_{y_i} =: \tilde{w}^i \partial_{y_i}.$$

Then, denoting g_{ij} in the x coordinates and $\tilde{g}_{ij}$ in the y coordinates:

$$\begin{aligned} g(p)\langle v, w\rangle &= \tilde{g}_{ij} \tilde{v}^i \tilde{w}^j = \tilde{g}_{ij} v^p \partial_p \left[\psi \circ \varphi^{-1}\right]^i w^q \partial_q \left(\psi \circ \varphi^{-1}\right)^j \\ &= \left(\tilde{g}_{ij} \partial_p \left[\psi \circ \varphi^{-1}\right]^i \partial_q \left[\psi \circ \varphi^{-1}\right]^j \right) v^p w^q = g_{pq} v^p w^q. \end{aligned}$$

Thus, the expression of g in local coordinates must satisfy the following change of coordinates law:

$$g_{ij} = \tilde{g}_{pq}\partial_i \left(\psi \circ \varphi^{-1}\right)^p \partial_j \left(\psi \circ \varphi^{-1}\right)^q = \tilde{g}_{pq} J^p_i J^q_j. \quad (6)$$ {2701202116

One can compare (6) *with* (1)*: they are "inverted":*

{280120211117} $$\tilde{g}_{ij} = g_{pq} \left(J^{-1}\right)^p_i \left(J^{-1}\right)^q_j. \quad (7)$$

The metric is said to be 2-covariant.

Remark 1.1.7. *Going back to the notion of a manifold as "a string of pieces of $\mathbb{R}^n$ assembled with the chart changes", the coordinate change described in* (6) *is exactly the one needed to make sure that a family of positive definite matrixes $g_{ij}(x_1, \ldots, x_n)$ assemble coherently into a metric tensor on the manifold.*

1-forms

We will extend this idea to the notion of tensors. Let us first introduce the 1-forms on a manifold:

Definition 1.1.13. *A* 1*-form θ on a smooth manifold M is a smooth map such that, for every $p \in M$, $\theta(p) \in (T_pM)^*$. We denote $\chi(M)^*$ the set of* 1*-forms on M.*

In a local coordinate system $(x_1, \ldots, x_n)$, the family $(\partial_{x_1}, \ldots, \partial_{x_n})$ is a basis for vector fields expressed in this chart. We will denote $(dx^1, \ldots, dx^n)$ its dual basis. Any 1-form θ can thus be written as:

$$\theta = \theta_i dx^i.$$

If $X = X^i \partial_{x_i} \in \chi(M)$, $\theta_p(X(p)) = \theta_i X^i$. If $(y_1, \ldots, y_n)$ is another local set of coordinates, the θ_i must obey a covariant formula:

$$\tilde{\theta}_i \tilde{v}^i = \tilde{\theta}_i \left(v^p \partial_p \left[\psi \circ \varphi^{-1}\right]^i\right) = \left(\tilde{\theta}_q \partial_i \left[\psi \circ \varphi^{-1}\right]^q\right) v^i = \theta_i v^i,$$

that is:

{270120211856} $$\theta_i = \tilde{\theta}_q \partial_i \left[\psi \circ \varphi^{-1}\right]^q = \tilde{\theta}_q J^q_i, \quad (8)$$

or, in other words

{28012021131} $$\tilde{\theta}_i = \theta_p \left(J^{-1}\right)^p_i. \quad (9)$$

The notation dx^i is not artificial: it is the differential of the coordinate functions:

Proposition 1.1.3. *If $f \in C^\infty(M)$, then the tangent map df is a* 1*-form on M which satisfies $df_p(X) = X(p).f$, for any $X \in \chi(M)$.*

Proof. With $f : M \to \mathbb{R}$, $df_p : T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}$, and is thus a 1-form. By definition, it satisfies the desired formula. $\square$

So, on the domain of a local chart, one can define the function x^i which to a point $p \in M$ associates the i-th coordinate of this point in the associated coordinate system. Its differential is then exactly the dx^i introduced above: $dx^i\left(\partial_{x_j}\right) = \partial_j(x^i(x_1, \dots, x_n)) = \delta^i_j$.

Remark 1.1.8. *Technically, our coordinate functions are defined only on an open of M and are thus not in $C^\infty(M)$. We can go around this difficulty by multiplying x^i by a "bump function" η with support in the domain of the chart, and worth 1 on an open subset of this domain. The previous consideration then stands on this open subset.*

Tensors on a manifold

One can seek to extend the notion of 1-forms to 2-forms to encompass our definition of metrics, we will go one step further and introduce tensors on a manifold:

Definition 1.1.14. *A r-contravariant, s-covariant tensor (denoted from now on (r, s) tensor) on a smooth manifold M is a $C^\infty(M)$ multilinear tensor*

$$T : \left(\chi(M)^*\right)^r \times \left(\chi(M)\right)^s \to C^\infty(M).$$

In local coordinates $(x_1, \dots, x_n)$ T is written as:

$$T = T^{i_1,\dots,i_r}{}_{j_1,\dots,j_s} \partial_{x_{i_1}} \dots \partial_{x_{i_r}} dx^{j_1} \dots dx^{j_s},$$

meaning that if $\theta^1, \dots, \theta^r$ are r 1-forms and $X_1, \dots X_s$ s vector fields written in the coordinates $(x_1, \dots, x_n)$:

$$\theta^\tau = \theta^\tau_p dx^p$$
$$X_\tau = X^q_\tau \partial_{x_q},$$

then

$$T\left(\theta^1, \dots, \theta^r, X_1, \dots, X_s\right) = T^{i_1,\dots,i_r}{}_{j_1,\dots,j_s} \theta^1_{i_1} \dots \theta^r_{i_r} X^{j_1}_1 \dots X^{j_s}_s.$$

In the words of B. O'Neill ([**?**]): "A (r, s) tensor is a multilinear machine which when fed r 1-forms and s vector fields produces a real-valued function".

As the name suggests, a (r, s) tensor will follow a r-contravariant s-covariant change of coordinates law. Indeed, let us consider two sets of coordinate $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$ and 1 forms $\theta^1, \dots, \theta^r$ and vector fields $X_1, \dots X_s$ whose expressions in the local coordinates $(x_1, \ldots, x_n)$ are denoted θ_p^i (respectively X_i^p) and $\tilde{\theta}_p^i$ (respectively $\tilde{X}_i^p$) in the local coordinates $(y_1, \ldots, y_n)$. Then:

$$\begin{aligned}
\mathcal{T} :&= T\left(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s\right) \\
&= T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s} \theta^1_{i_1} \ldots \theta^r_{i_r} X_1^{j_1} \ldots X_s^{j_s} \\
&= \tilde{T}^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s} \tilde{\theta}^1_{i_1} \ldots \tilde{\theta}^r_{i_r} \tilde{X}_1^{j_1} \ldots \tilde{X}_s^{j_s} \\
&= \tilde{T}^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s} \theta^1_{p_1} \left(J^{-1}\right)^{p_1}_{i_1} \ldots \theta^r_{p_r} \left(J^{-1}\right)^{p_r}_{i_r} J^{j_1}_{q_1} X_1^{q_1} \ldots J^{j_s}_{q_s} X_s^{q_s} \\
&= \left[\left(J^{-1}\right)^{p_1}_{i_1} \ldots \left(J^{-1}\right)^{p_r}_{i_r} \tilde{T}^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s} J^{j_1}_{q_1} \ldots J^{j_s}_{q_s}\right] \theta^1_{p_1} \ldots \theta^r_{p_r} X_1^{q_1} \ldots X_s^{q_s},
\end{aligned}$$

which yields:

{280120211217}
$$\tilde{T}^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s} = J^{i_1}_{p_1} \ldots J^{i_r}_{p_r} T^{p_1,\ldots,p_r}{}_{q_1,\ldots,q_s} \left(J^{-1}\right)^{q_1}_{j_1} \ldots \left(J^{-1}\right)^{q_s}_{j_s}. \tag{10}$$

In plain words, the r upper indexes change coordinates in a contravariant manner while the s lower ones do so in a covariant fashion. This is characteristic of (r, s) tensors: if $(\Omega_\tau, \varphi_\tau)$ is an atlas of M and $T_\tau{}^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}$ a family of functions defined on each Ω_τ satisfying (10), then the function

$$T : \begin{cases} \left(\chi(M)^*\right)^r \times \left(\chi(M)\right)^s \to C^\infty(M) \\ \left(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s\right) \mapsto \left[p \to T_\tau{}^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(p_1, \ldots, p_n) \theta^1_{i_1} \ldots \theta^2_{i_r} X_1^{j_1} \ldots X_s^{j_s}\right] \end{cases}$$

does not depend on the chosen chart $(\Omega_\tau, \varphi_\tau)$ such that $p \in \Omega_\tau$, and is thus well defined on M.

Remark 1.1.9. *Tensors once more mobilize the confetti representation: one defines them locally in local coordinates systems and then glue them together to assemble a coherent object on the manifold. If the process is well done (i.e if it satisfies* (10)*) the object does not depend on the "confettis" used to assemble the manifold.*

Remark 1.1.10. *It is insightful to consider the use of tensors in General Relativity: in these modelizations the universe is a manifold and the quantities measured by an observer (distance from him, time given by his clock) correspond to a local system of coordinates. The Einstein principle then states that the laws of this universe must be expressed by tensorial equations, that is that the observed phenomena must not depend on the observer, on the chart they used.*

Example 1.1.3.
- *A vector field on M is a $(1, 0)$ tensor.*
- *A 1-form on M is a $(0, 1)$ tensor.*
- *A Riemannian metric g on M is a $(0, 2)$ tensor.*

- *A m differential form on a manifold is a skew-symmetric $(0,m)$ tensor.*
- *The volume form on a Riemannian manifold defined as* $\mathrm{dvol}_g(V_1,\dots,V_n) = \det(V_i^j)\det(g_{ij})^{\frac{1}{2}}$ *is a n differential form. In local coordinates* $\mathrm{dvol}_g = |g|^{\frac{1}{2}} dx^1 \wedge \dots \wedge dx^n$*, where* $|g| := \det(g_{ij})$.
- *The inverse of a Riemannian metric g on M is a $(2,0)$ tensor denoted in local coordinates $g^{pq}\partial_{x_p}\partial_{x_q}$.*
- *The first derivatives of a function $f \in C^\infty(M)$ form a $(0,1)$ tensor: the differential map df.*
- *The second derivatives do* not *form a $(0,2)$ tensor, they do not follow a proper change of coordinates law.*

Proof. The first three examples have been studied above and they do satisfy the right coordinate changes. A m-form is by definition a map which sends $p \in M$ to a skew-symmetric linear map on $(T_pM)^m$, hence a skew-symmetric tensor. The volume form $d\mathrm{vol}_g(V_1,\dots,V_n) = \det(V_i^j)|g|^{\frac{1}{2}}$ does not depend on the choice of coordinates. Indeed, if $V_i = V_i^j\partial_{x_i} = \tilde{V}_i^j\partial_{y_j}$, $\tilde{V}_i^j = (JV)_{ij}$ while $\tilde{g}_{ij} = (J^{-1}gJ^{-1})_{ij}$. Thus:

$$\begin{aligned} d\mathrm{vol}_g(V_1,\dots,V_n) &= \det(\tilde{V}_i^j)\det(g_{ij})^{\frac{1}{2}} = \det((JV)_i^j)\det(J^{-1}gJ^{-1})^{\frac{1}{2}} \\ &= \det(V_i^j)\det(g_{ij})^{\frac{1}{2}}. \end{aligned}$$

One can then check the formula in local coordinates, which is thus n-covariant.

Let us consider the fourth. In every local coordinates, g_{ij} yields a symmetric definite positive matrix which is then invertible into g^{pq}. We then only need to show that it changes coordinates in a $(2,0)$ covariant manner to define it as a tensor. Let us thus consider the expression $\tilde{g}_{ij}$ in another set of coordinates and introduce $\tilde{g}^{ij} = J_p^i J_q^j g^{pq}$. Then, given (7):

$$\begin{aligned} \tilde{g}_{ik}\tilde{g}^{kj} &= g_{pq}\left(J^{-1}\right)_i^p\left(J^{-1}\right)_k^q J_u^k J_v^j g^{uv} \\ &= g_{pq}\left(J^{-1}\right)_i^p J_v^j\delta_u^q g^{uv} = g_{pq}g^{qv}\left(J^{-1}\right)_i^p J_v^j = \delta_p^v\left(J^{-1}\right)_i^p J_v^j = \left(J^{-1}\right)_i^v J_v^j \\ &= \delta_i^j, \end{aligned}$$

with δ_b^a the Kronecker symbol. Similarly, one checks $\tilde{g}^{ik}\tilde{g}_{kj} = \delta_j^i$ which yields that $\tilde{g}^{\cdot\cdot} = (\tilde{g}_{\cdot\cdot})^{-1}$, and thus that g^{-1} satisfies a $(2,0)$ contravariant change of coordinates law, and is thus a $(2,0)$ tensor.

Similarly, if $f \in C^\infty(M)$, φ and ψ are two charts associated to the local coordinates $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$, then:

$$\begin{aligned} \partial_{x_i} f &= \partial_i\left(f\circ\varphi^{-1}\right) = \partial_i\left(\left[f\circ\psi^{-1}\right]\circ\left[\psi\circ\varphi^{-1}\right]\right) \\ &= \partial_i\left[\psi\circ\varphi^{-1}\right]^j\partial_{y_j}f, \end{aligned} \tag{11}$$ {010220211137}

which is the covariant desired formula.

On the other hand, if we take another derivative of (11):

{010220211155}
$$\begin{aligned}\partial_{x_i}\partial_{x_j} f &= \partial_i\partial_j\left(f\circ\varphi^{-1}\right) = \partial_i\left(\partial_j\left[\psi\circ\varphi^{-1}\right]^p\partial_p\left(f\circ\psi^{-1}\right)\circ\left[\psi\circ\varphi^{-1}\right]\right)\\ &= \partial_i\left[\psi\circ\varphi^{-1}\right]^q\partial_j\left[\psi\circ\varphi^{-1}\right]^p\partial_{pq}\left(f\circ\psi^{-1}\right) + \partial_i\partial_j\left[\psi\circ\varphi^{-1}\right]^p\partial_p\left(f\circ\psi^{-1}\right),\end{aligned} \tag{12}$$

which is not a covariant law because of the parasit term $\partial_i\partial_j\left[\psi\circ\varphi^{-1}\right]^p\partial_p\left(f\circ\psi^{-1}\right)$. Thus second derivatives cannot be defined globally without exploiting further the Riemannian structure (see the Levi-Civita connection below). □

Partition of unity

One can build explicitely a Riemannian metric using an analytic tool on manifolds: the partitions of unity.

Definition 1.1.15. *A* smooth partition of unity *on a manifold M is a collection of smooth real-valued functions $\{f_\alpha : \alpha \in A\}$ such that:*

1. $0 \le f_\alpha \le 1$ *for all* $\alpha \in A$,
2. $\{\mathrm{supp} f_\alpha : \alpha \in A\}$ *is locally finite,*
3. $\sum_\alpha f_\alpha = 1$.

A smooth partition of unity is said to be subordinate *to an open covering $\cup_i\Omega_i$ of M if each* $\mathrm{supp} f_\alpha$ *is contained in some Ω_i.*

We will admit the existence of partitions of unity and refer the reader to [**?**] and [**?**] for the proof:

{theopartunity}

Theorem 1.1.4. *Let M be a smooth manifold. For any open covering of M, there exists a subordinate partition of unity on M.*

Such partitions yield:

Proposition 1.1.5. *Let M be a smooth manifold of dimension n. There exists a metric tensor g defined on M.*

Proof. Let $(\Omega_\tau, \varphi_\tau)_{\tau\in\mathcal{T}}$ be an atlas of M, and f_τ a partition of unity on M subordinate to the covering $\cup_\tau\Omega_\tau$. For each chart φ_τ let us denote $(x_1^\tau, \ldots, x_n^\tau)$ the associated coordinates. Then given any family $\left(g_{ij}^\tau\right)_{i,j=1\ldots n}$ of symmetric definite positive matrixes one can introduce the following metric on M:

$$g = \sum_\tau f_\tau g_{ij}^\tau dx_\tau^i dx_\tau^j.$$

Since f_τ is subordinate to the atlas, its support is entirely contained in Ω_τ, that is the domain on which the local coordinates are defined, and since the f_τ form a partition of unity the sum is in fact finite around each point. Thus, the metric is indeed well defined on M. □

Once more, we encounter the idea of building an object on a manifold piecewise, and then assembling all the separate parts, here using a partition of unity to negate the interactions. In practice the metrics we will consider will be induced by the immersion in the euclidean space. This requires introducing a bit more formalism on tensors and metrics on Riemannian manifolds.

Tensorial manipulations

In a Riemannian manifold, the metric yields a scalar product on each tangent space, and thus a natural isomorphism between T_pM and its dual $(T_pM)^*$: to $v \in T_pM$ we associate $v_* = g(p)\langle v, .\rangle \in (T_pM)^*$ or in local coordinates $v_p dx^p = g_{pq} v^q dx^p$. The inverse diffeomorphism sends the 1-form $\theta = \theta_q dx^q$ to the vector $\theta^* = g^{pq}\theta_q \partial_{x_p}$.

Such an isomorphism can be applied to 1-forms and vector fields, and generalized to tensors on M:

Definition 1.1.16. *Let (M, g) be a Riemannian manifold of dimension n. There exists a canonical isomorphism ι_g sending a (r, s) tensor into a $(r-1, s+1)$ tensor defined by:*

$$\iota_g T\left(\tilde{X}_1, \theta^2, \dots, \theta^r, X_1, \dots, X_s\right) = T\left(g\langle X_1, .\rangle, \theta^2, \dots, \theta^r, X_1, \dots, X_s\right).$$

In local coordinates:

$$T_{i_1}{}^{i_2\dots i_r}{}_{j_1\dots j_s} = g_{i_1 s} T^{s i_2 \dots i_r}{}_{j_1\dots j_s}.$$

Practically, this means that one can lower (respectively raise) indexes for tensors by multiplying them by g (respectively g^{-1}).

Example 1.1.4. $g_q^p = g^{pl} g_{lq} = \delta_q^p$. *From now on, in tensorial contexts, we will favor g_q^p instead of the Kronecker symbols.*

Definition 1.1.17.
- *Let M be a smooth manifold of dimension n. For any $1 \leq a$, $1 \leq b$ the contraction C_b^a is an application that sends a (r, s) (with $a \leq r$ and $b \leq s$) tensor to a $(r-1, s-1)$ tensor in the following manner:*

$$C_b^a(T)^{i_1\dots i_{r-1}}{}_{j_1\dots j_{s-1}} = T^{i_1\dots i_{a-1} p i_{a+1}\dots i_{r-1}}{}_{j_1\dots j_{b-1} p j_{b+1}\dots j_{s-1}},$$

  where we recall that we sum over the repeated index p.

- *If (M, g) is a Riemannian manifold and T a (r, s) tensor, we denote:*

$$|T|_g^2 = T^{i_1\dots i_r}{}_{j_1\dots j_s} T_{i_1\dots i_r}{}^{j_1\dots j_s} = g_{i_1 p_1}\dots g_{i_r p_r} g^{j_1 q_1}\dots g^{j_s q_s} T^{i_1\dots i_r}{}_{j_1\dots j_s} T^{p_1\dots p_r}{}_{q_1\dots q_s}.$$

Integration on a manifold

We will quickly establish how to integrate on a manifold: let f be a function defined on (M, g) a Riemannian manifold, $d\mathrm{vol}_g$ the volume form and (η_i) a partition of unity subordinate to an atlas of M. Then, the integral of f is defined as:

$$\int_M f d\mathrm{vol}_g = \sum_i \int_{\Omega_i} \eta_i f(x_1, \ldots, x_n)|g|^{\frac{1}{2}} dx^1 \ldots dx^n. \tag{13}$$

Since η is a partition of unity (and thus $\sum_i \eta_i = 1$) and since $f d\mathrm{vol}_g$ does not depend on the local chart it is expressed in, this definition of integral does not depend on the chosen atlas, nor on the chosen partition of unity.

The volume of a manifold is thus defined as

$$\mathrm{Vol}(M, g) = \int_M d\mathrm{vol}_g. \tag{14}$$

In the case of a surface, we prefer to talk about the area:

$$\mathcal{A}(\Sigma, g) = \int_\Sigma d\mathrm{vol}_g. \tag{15}$$

Another important integral gives the L^2 norm a vector: given T a (r, s) tensor, one can introduce

$$\|T\|^2_{L^2(M)} = \int_M |T|^2_g \, d\mathrm{vol}_g, \tag{16}$$

and extend the range of the Lebesgue spaces to define $L^2(M, g)$: the L^2 space on a manifold. More generally, one can define the $L^p(M, g)$ spaces for all $p \leq \infty$ in the same manner. On a compact manifold, these spaces actually do not depend on the continuous metric g, or even $g \in L^\infty$ with a uniform bound from below:

{190420211023}

$$\exists C > 0 \text{ s.t. } \frac{|v|^2}{C} \leq |v|^2_g \leq C|v|^2. \tag{17}$$

One can then take a reference smooth metric and introduce the Lebesgue spaces in this metric $L^2(M)$, before considering some less regular ones.

One can also introduce the Sobolev spaces $W^{1,2}(M)$:

{31050211131}

$$\|f\|^p_{W^{1,p}(M)} = \int_M |f|^p_g \, d\mathrm{vol}_g + \int_M |\nabla f|^p_g \, d\mathrm{vol}_g. \tag{18}$$

The previous remarks stand in this case. The classical result on Sobolev spaces are still valid in this framework.

Induced metric

Definition 1.1.18. *Let M be a m-dimensional manifold and (N, g) a n-dimensional Riemannian manifold. Let $\Phi : M \to N$ be an immersion. Then*

$$\Phi^* g(p)\langle v, w\rangle = g(\Phi(p))\langle d\Phi_p(v), d\Phi_p(w)\rangle$$

is a Riemannian metric on M called the induced (or pullback) metric.

This induced metric is particularly interesting when considering extrinsic surfaces: any immersion $\Phi : \Sigma \to \mathbb{R}^3$ will induce a metric, called the *first fundamental form* on Σ by pulling back the Euclidean metric of $\mathbb{R}^3$:

$$\forall p \in M, \forall v, w \in T_pM \quad g(p)\langle v, w\rangle = \langle d\Phi(v), d\Phi(w)\rangle.$$

In local coordinates (x, y),

$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} |\phi_x|^2 & \langle \phi_x, \phi_y\rangle \\ \langle \phi_x, \phi_y\rangle & |\phi_y|^2 \end{pmatrix}. \tag{19}$$ {050220211202}

Definition 1.1.19. *Let Σ be a surface and Φ_1, Φ_2 two immersions in $\mathbb{R}^3$. $\Phi_1(\Sigma)$ and $\Phi_2(\Sigma)$ are said to be (locally) isometric if the induced metrics are (locally) the same.*

{110220211628}

Example 1.1.5. *The plane (x, y) is a Riemannian manifold that can be embedded in $\mathbb{R}^3$ by the inclusion: $i : (x, y) \mapsto (x, y, 0)$. One can compute the induced metric in two steps:*

$$\nabla i = \begin{pmatrix} i_x \\ i_y \end{pmatrix} = \begin{pmatrix} (1, 0, 0) \\ (0, 1, 0) \end{pmatrix},$$

$$g_i = \begin{pmatrix} \langle i_x, i_x\rangle & \langle i_x, i_y\rangle \\ \langle i_x, i_y\rangle & \langle i_y, i_y\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Of course, any isometry applied to i (for instance a translation, or a rotation) will yield a globally isometric embedding. Since those are not the most spectacular cases of isometry, we will instead consider the immersed cylinder $\Phi : (x, y) \mapsto (\cos x, \sin x, y)$. The induced metric is then computed in the following manner:

$$\nabla \Phi = \begin{pmatrix} \Phi_x \\ \Phi_y \end{pmatrix} = \begin{pmatrix} (-\sin x, \cos x, 0) \\ (0, 0, 1) \end{pmatrix},$$

$$g = \begin{pmatrix} \langle \Phi_x, \Phi_x\rangle & \langle \Phi_x, \Phi_y\rangle \\ \langle \Phi_x, \Phi_y\rangle & \langle \Phi_y, \Phi_y\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The cylinder immersed by Φ and the euclidean plane are thus locally isometric in $\mathbb{R}^3$.

For comparison, let us look at the parametrization of the sphere via stereographical parametrization:

$$\phi := \pi_N^{-1} = \begin{cases} \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{N = (0,0,1)\} \\ (x,y) \mapsto \dfrac{1}{1+x^2+y^2}\left(2x, 2y, x^2+y^2-1\right). \end{cases}$$

Then

$$\phi_x = \frac{2}{(1+r^2)^2}\begin{pmatrix} 1+y^2-x^2 \\ -2xy \\ 2x \end{pmatrix},$$

$$\phi_y = \frac{2}{(1+r^2)^2}\begin{pmatrix} -2xy \\ 1+x^2-y^2 \\ 2y \end{pmatrix},$$

and thus

$$\langle \phi_x, \phi_x \rangle = \langle \phi_y, \phi_y \rangle = \frac{4}{(1+r^2)^2}$$

$$\langle \phi_x, \phi_y \rangle = 0,$$

meaning that the induced metric is:

$$g = \frac{4}{(1+r^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So while this parametrization yields a metric which at each point is homothetic to the euclidean scalar product, they definitely differ. We will come back to these conformal parametrizations *later.*

These two examples are telling, regarding the difference between intrinsic and extrinsic geometry: the cylinder and the plane are locally isometric, meaning locally undistinguishable using intrinsic geometric notions. Philosophically speaking an amoeba with geometric notions would not be able to tell whether it lives on a plane or on a cylinder, while it will definitely tell between a sphere and plane. However, they clearly do not have the same shape as surfaces in $\mathbb{R}^3$: there must be some extrinsic notion of curvature to distinguish the two.

1.2 Second fundamental form, curvature

1.2.1 Extrinsic definitions

{250520211337}

Gauss map

Let us consider Σ a surface and Φ an immersion in $\mathbb{R}^3$. Recalling definition 1.1.10, one can see that the tangent planes of the immersed surface form a family of planes in $\mathbb{R}^3$ which envelopes the surface and molds its shape (the way rolling fingers over a stone allows you to guess its shape). The variations of the tangent planes will thus provide an insight on the shape the immersed surface. Since these planes are of codimension 1 they are characterized by their normal vector, with a choice to be made in the orientation (outward or inward pointing). That this choice can be made in a coherent manner on the whole immersed surface is not guaranteed, and in fact requires the underlying manifold to be oriented.

Definition 1.2.1. *An atlas on a manifold M of dimension n is said to be orientable if all the chart changes are orientables as diffeomorphisms between opens of $\mathbb{R}^n$. If there exists an orientable atlas on M, M is said to be an orientable manifold.*

In the following, almost all the surfaces we will consider will be orientable in order to define the Gauss map globally. On non-orientable surfaces, all the subsequent considerations and computations will stand *locally.*

Definition 1.2.2. *Let Σ be a smooth orientable surface and $\Phi : \Sigma \to \mathbb{R}^3$ an immersion. The Gauss map, denoted $\vec{n}$, is the unit normal to the surface, expressed in a local chart φ of an orientable atlas:*

$$\vec{n} \circ \varphi^{-1} = \frac{\phi_x \times \phi_y}{|\phi_x \times \phi_y|}. \tag{20}$$ {100220211840}

Here $\times$ denotes the classical vectorial product in $\mathbb{R}^3$. In the following, we will lighten the notations and stop differentiating $\vec{n}$ as an application on the manifold and $\vec{n} \circ \varphi^{-1}$ the application in local coordinates. The denominator in formula (20) can be linked to the volume element, since by definition of the vectorial product:

$$|\phi_x \times \phi_y| = \sqrt{\langle \phi_x, \phi_x \rangle \langle \phi_y, \phi_y \rangle - \langle \phi_x, \phi_y \rangle \langle \phi_y, \phi_x \rangle} = |g|^{\frac{1}{2}}.$$

One can then rewrite:

$$\begin{aligned} \vec{n} &= |g|^{-\frac{1}{2}} \phi_x \times \phi_y \\ d\mathrm{vol}_g &= |\phi_x \times \phi_y| \, dx \wedge dy. \end{aligned} \tag{21}$$ {100220211845}

The second fundamental form and the curvatures

By design, $\vec{n}$ characterizes the tangent plane: $T_p\Sigma \simeq T_{\Phi(p)}\Phi(\Sigma) = (\vec{n}(p))^{\perp}$. The differential of the Gauss map is thus an application:

$$d\vec{n}_p : T_p\Sigma \to \mathbb{R}^3.$$

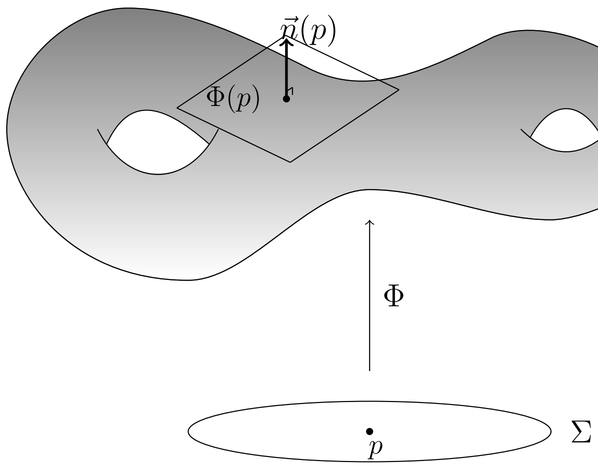

Figure 3: The Gauss map is an extrinsic map

However, since $|\vec{n}|^2 = 1$, for any $v \in T_p\Sigma$:

$$\langle d\vec{n}_p(v), \vec{n}(p)\rangle = 0. \tag{22}$$

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Thus, $d\vec{n}_p$ can be considered as a linear endomorphism on the tangent plane: the shape operator, which, applied to the metric yields the second fundamental form.

Definition 1.2.3. *The second fundamental form of Σ is defined as*

$$\forall X, Y \in T_p\Sigma \quad A(p)(X,Y) = -\langle d\vec{n}_p(X), Y\rangle.$$

The second fundamental form is thus a $(0,2)$ tensor on the manifold.

In a local coordinate chart, (22) implies that $\nabla\vec{n}(p) = \begin{pmatrix} \vec{n}_x \\ \vec{n}_y \end{pmatrix} = \begin{pmatrix} d\vec{n}_p(\partial_x) \\ d\vec{n}_p(\partial_y) \end{pmatrix}$ can be expressed in the basis (ϕ_x, ϕ_y):

$$\nabla\vec{n}(p) = \begin{pmatrix} \langle\vec{n}_x, \phi_x\rangle & \langle\vec{n}_x, \phi_y\rangle \\ \langle\vec{n}_y, \phi_x\rangle & \langle\vec{n}_y, \phi_y\rangle \end{pmatrix} g^{-1}\nabla\phi.$$

The second fundamental form is then expressed in the local coordinates:

$$\begin{aligned} A\left(a\partial_x + b\partial_y, c\partial_x + d\partial_y\right) &= -\langle a\vec{n}_x + b\vec{n}_y, c\phi_x + d\phi_y\rangle \\ &= -\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} \langle\vec{n}_x, \phi_x\rangle & \langle\vec{n}_x, \phi_y\rangle \\ \langle\vec{n}_y, \phi_x\rangle & \langle\vec{n}_y, \phi_y\rangle \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}, \end{aligned}$$

or in other words:

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$$A_{ij} = -\langle\partial_i\vec{n}, \partial_j\phi\rangle. \tag{23}$$

This then yields a simplified expression of $\nabla\vec{n}$ using tensorial language:

0220211042} $$\nabla\vec{n} = -Ag^{-1}\nabla\phi = -\left(A_i^j \partial_j \phi\right)_{i=1,2}. \tag{24}$$

A fundamental result, shown by C. Gauss (see [**?**]), revolves around the symmetry of the second fundamental form.

Proposition 1.2.1. *The second fundamental form of an immersed surface is symmetric:*

$$\forall p \in \Sigma, \quad \forall u, v \in T_p\Sigma, \quad A_p(u,v) = A_p(v,u).$$

Proof. Let us consider the expression of the second fundamental form in local coordinates: $A_{ij} = -\langle \partial_i \vec{n}, \partial_j \phi\rangle = -\partial_i\left(\langle \vec{n}, \partial_j\phi\rangle\right) + \langle \vec{n}, \partial_{ij}\phi\rangle = \langle \vec{n}, \partial_{ij}\phi\rangle$, since, by design: $\langle \vec{n}, \partial_j\phi\rangle = 0$. Since, in addition, for all i, j $\partial_{ij}\phi = \partial_{ji}\phi$, then A is symmetric. □

Remark 1.2.1. *Looking at the coordinate change law* (12) *shows that the parasite term which prevents $\partial_{ij}\phi$ from being a tensor on the surface is* tangent*, and thus, disappears when taking the scalar product against the Gauss map $\vec{n}$, yielding again that A is a* $(0,2)$ *tensor on Σ.*

Since the operator $d\vec{n}_p$ is self-adjoint for all $p \in \Sigma$, it is diagonalizable and has two eigenvalues corresponding to two orthogonal directions: the principal curvatures.

Definition 1.2.4. *The principal curvatures of Σ at p are the two eigenvalues $\kappa_1(p)$ and $\kappa_2(p)$ of $d\vec{n}_p$. If $\kappa_1(p) = \kappa_2(p)$, p is said to be umbilical. If p is not umbilical, the corresponding eigenvectors are called directions of principal curvatures. Then, the Gauss curvature is defined as:*

$$K := \kappa_1\kappa_2, \tag{25}$$

while the mean curvature is

$$H := \frac{\kappa_1 + \kappa_2}{2}, \tag{26}$$ {190220210909}

and the tracefree second fundamental form is defined in an orthonormal eigenvector basis as:

$$\mathring{A} := \begin{pmatrix} \frac{\kappa_1 - \kappa_2}{2} & 0 \\ 0 & \frac{\kappa_2 - \kappa_1}{2} \end{pmatrix}. \tag{27}$$

In other words, in local coordinates one can express the previous curvatures as:

$$K := \det_g(A) = \det\left(g^{-1}A\right) = A_1^1 A_2^2 - A_2^1 A_1^2, \tag{28}$$

$$H := \frac{1}{2}\mathrm{Tr}_g\left(A\right) = \frac{1}{2}\mathrm{Tr}\left(g^{-1}A\right) = \frac{A_1^1 + A_2^2}{2}, \tag{29}$$

$$\mathring{A}_{ij} = A_{ij} - Hg_{ij}. \tag{30}$$

In differential geometry, a usual (and more compact) notation system for the second fundamental form is:

$$A = \begin{pmatrix} e & f \\ f & g \end{pmatrix}, \tag{31}$$

with

$$\begin{aligned} e &= -\langle \vec{n}_x, \phi_x \rangle = \langle \vec{n}, \phi_{xx} \rangle \\ f &= -\langle \vec{n}_x, \phi_y \rangle = -\langle \vec{n}_y, \phi_x \rangle = \langle \vec{n}, \phi_{xy} \rangle \\ g &= -\langle \vec{n}_y, \phi_y \rangle = \langle \vec{n}, \phi_{yy} \rangle. \end{aligned} \tag{32}$$

With these notations one has the following formulas for K and H:

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$$\begin{aligned} K &= \frac{eg - f^2}{EG - F^2} \\ H &= \frac{1}{2}\frac{eG + gE - 2fF}{EG - F^2}. \end{aligned} \tag{33}$$

If the expression of K is rather palatable, we will avoid using this formula for H in the general case.

Remark 1.2.2. *The principal curvatures determine the local shape of the surface: if they have the same sign, the surface looks locally like a paraboloïd, if they have opposite signs they look like a mountain pass (we also talk about saddle points).*

Before playing with these notions in the tensorial language we absolutely must mention a *remarkable* theorem from Gauss ([**?**]):

Theorem 1.2.2. *The Gauss curvature is a metric invariant: if two surfaces are (locally) isometric then they (locally) have the same Gauss curvature.*

Proof. We will only sketch the proof here. It starts by writing:

$$\begin{aligned} \phi_{xx} &= U_1\phi_x + V_1\phi_y + e\vec{n} \\ \phi_{xy} &= U_2\phi_x + V_2\phi_y + f\vec{n} \\ \phi_{yy} &= U_3\phi_x + V_3\phi_y + g\vec{n}, \end{aligned}$$

where the U_i and V_i can be expressed as derivatives of E, F and G, for instance:

$$U_1 = \frac{G\langle \phi_{xx}, \phi_x \rangle - F\langle \phi_{xx}, \phi_y \rangle}{EG - F^2} = \frac{\frac{1}{2}GE_x + \frac{1}{2}FE_y - FF_x}{EG - F^2},$$

with, for instance

$$\langle \phi_{xx}, \phi_y \rangle = (\langle \phi_x, \phi_y \rangle)_x - \langle \phi_x, \phi_{xy} \rangle = (\langle \phi_x, \phi_y \rangle)_x - \frac{1}{2}(\langle \phi_x, \phi_x \rangle)_y = F_x - \frac{1}{2}E_y.$$

One can then compute:

$$\begin{aligned}\langle \phi_{xx}, \phi_{yy} \rangle - |\phi_{xy}|^2 &= (\langle \phi_{xx}, \phi_{yy} \rangle)_y - \langle \phi_{xxy}, \phi_y \rangle - (\langle \phi_{xy}, \phi_y \rangle)_x + \langle \phi_{xxy}, \phi_y \rangle \\ &= F_{xy} - \frac{1}{2}E_{yy} - \frac{1}{2}G_{xx}.\end{aligned}$$

Now since $ef - g^2$ can be expressed as $\langle \phi_{xx}, \phi_{yy} \rangle - |\phi_{xy}|^2$ the scalar products of their tangent parts, this yields the following:

$$\begin{aligned}K = &\frac{2F_{xy} - E_{yy} - G_{xx}}{2(EG - F^2)} \\ &+ \frac{G(E_y)^2 + E(G_x)^2 + GG_xE_x + EE_yG_y}{4(EG - F^2)} \\ &+ \frac{F(E_xG_y - E_yG_x - 2F_xG_x - 2F_yE_y + 4F_xF_y)}{4(EG - F^2)} \\ &- \frac{GE_xF_y + EG_yGF_x}{2(EG - F^2)}.\end{aligned}$$

This formula is too ugly to be of any practical use, but it shows that the Gauss curvature can be expressed as a function of the coefficients of the metric, and its derivative. If two immersions induce the same metric, that is if they are isometric, their Gauss curvature must then be the same, which proves the theorem. □

The previous theorem (called *theorema egregium*, litterally remarkable theorem) thus shows that the Gauss curvature is not an extrinsic quantity, and can in fact be introduced only with the metric structure, without referring to an immersion in a space. This points us toward considering the mean curvature and the second fundamental form as better objects to measure the extrinsic geometry of an immersed surface. Let us illustrate this on the plane, cylinder, and the sphere:

Example 1.2.1. *For the flat plane embedded in $\mathbb{R}^3$, the Gauss map is $\vec{n}_i = (0, 0, 1)$, which means that the second fundamental form is $A_{ij} = \langle \vec{n}_i, \partial_{ij} i \rangle = 0$, and thus:*

$$\begin{aligned}K = H &= 0 \\ \mathring{A} &= 0.\end{aligned}$$

Let us compare with the cylinder: $\vec{n}_\Phi = (\cos x, \sin x, 0)$, and thus:

$$\begin{aligned}A &= \begin{pmatrix} \langle \vec{n}_\Phi, \Phi_{xx} \rangle & \langle \vec{n}_\Phi, \Phi_{xy} \rangle \\ \langle \vec{n}_\Phi, \Phi_{xy} \rangle & \langle \vec{n}_\Phi, \Phi_{yy} \rangle \end{pmatrix} \\ &= \begin{pmatrix} \langle (\cos x, \sin x, 0), -(\cos x, \sin x, 0) \rangle & \langle (\cos x, \sin x, 0), 0 \rangle \\ \langle (\cos x, \sin x, 0), 0 \rangle & \langle (\cos x, \sin x, 0), 0 \rangle \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Thus $K=0$, $H=-\frac{1}{2}$, $\mathring{A}=\begin{pmatrix}-\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}$.

With the sphere:

$$\begin{aligned}
\phi_x\times\phi_y &= \frac{4}{(1+r^2)^4}\begin{pmatrix}1+y^2-x^2\\ -2xy\\ 2x\end{pmatrix}\times\begin{pmatrix}-2xy\\ 1+x^2-y^2\\ 2y\end{pmatrix}\\
&= \frac{4}{(1+r^2)^4}\begin{pmatrix}-4xy^2-2x(1+x^2-y^2)\\ -4x^2y-2y(1+y^2-x^2)\\ (1+y^2-x^2)\left(1-(y^2-x^2)\right)-4x^2y^2\end{pmatrix}\\
&= \frac{4}{(1+r^2)^4}\begin{pmatrix}-2x-2x^3-2xy^2\\ -2y-2y^3-2x^2y\\ 1-r^4\end{pmatrix}\\
&= -\frac{4}{(1+r^2)^3}\begin{pmatrix}2x\\ 2y\\ r^2-1\end{pmatrix},
\end{aligned}$$

which implies that $\vec{n}=-\phi$. *The second fundamental form is then:*

$$A=\frac{4}{(1+r^2)^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

which gives the following: $K=1$, $H=1$, $\mathring{A}=0$.

Looking at these examples, H and $\mathring{A}$ seem to be two good notions for an extrinsic study of surfaces. The problems we will study will thus revolve around these two quantities, first with second order problems, and then with a fourth order one. This will require us to consider variations of curvatures and of the second fundamental form, that is, variations of tensors. As illustrated in the case of the second order derivative, this cannot be done by local differentiations, and will require us to introduce the Levi-Civita connection.

Before concluding this chapter, we must mention a result that we will use but will not prove as it uses techniques and ideas that are not the focus of this course: the Gauss-Bonnet theorem (see section 4.5 of [**?**] for a proof and detailed explanations).

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Theorem 1.2.3. *Let* Σ *be an orientable compact surface. Then:*

$$\int_\Sigma K d\mathrm{vol}_g = 2\pi\chi(\Sigma) = 2\pi(2-2g),$$

where ξ *is the Euler characteristic and* g *the genus of* Σ.

Both the Euler characteristic and the genus are *topological invariant.* On a parametrized surface the genus g is "the number of holes" in the surface (0 for spheres and ellipsoids, 1 for tori...). We once more recommend [**?**] for more details.

1.2.2 Tensorial differentiation

Levi-Civita connection

Definition 1.2.5. *A connection D on a smooth manifold M is an application: $D: \chi(M) \times \chi(M) \to \chi(M)$ such that:*

- *$D_V W$ is $C^\infty(M)$ linear in V.*
- *$D_V W$ is $\mathbb{R}$ linear in W.*
- *$D_V(fW) = (V.f)W + f D_V W$ for any $f \in C^\infty(M)$.*

Then $D_V W$ is called the covariant derivative of W with respect to V for the connection D.

Example 1.2.2. *Let us consider (M, g) a Riemannian manifold, and introduce the Levi-Civita connection with the* Koszul formula*:*

$$\forall V, W, X \in \chi(M), \quad 2\langle \nabla_V W, X\rangle = V.(\langle W, X\rangle) + W.(\langle X, V\rangle) - X.(\langle V, W\rangle) \\ - \langle V, [W, X]\rangle + \langle W, [X, V]\rangle + \langle X, [V, W]\rangle.$$

One can check that it does satisfy the definition of a connection: if $V, W \in \chi(M)$, for any $X \in \chi(M)$ and $f \in C^\infty(M)$,

$$\begin{aligned} 2\langle \nabla_{fV} W, X\rangle &= fV.(\langle W, X\rangle) + W.(\langle X, fV\rangle) - X.(\langle fV, W\rangle) \\ &\quad - \langle V, [W, fX]\rangle + \langle W, [X, fV]\rangle + \langle X, [fV, W]\rangle \\ &= fV.(\langle W, X\rangle) + W.f\langle X, V\rangle + fW.(\langle X, V\rangle) - X.f\langle V, W\rangle - fX.(\langle V, W\rangle) \\ &\quad - f\langle V, [W, X]\rangle + f\langle W, [X, V]\rangle + X.f\langle W, [X, V]\rangle + f\langle X, [V, W]\rangle - W.f\langle X, V\rangle \\ &= 2f\langle \nabla_V W, X\rangle, \end{aligned}$$

using the property (3) *of the Lie bracket, while:*

$$\begin{aligned} 2\langle \nabla_V (fW), X\rangle &= V.(\langle fW, X\rangle) + fW.(\langle X, V\rangle) - X.(\langle V, fW\rangle) \\ &\quad - \langle V, [fW, X]\rangle + \langle fW, [X, V]\rangle + \langle X, [V, fW]\rangle \\ &= V.f\langle W, X\rangle + fV.(\langle W, X\rangle) + fW.(\langle X, V\rangle) - X.f\langle V, W\rangle - fX.(\langle V, W\rangle) \\ &\quad - f\langle V, [W, X]\rangle + X.f\langle V, W\rangle + f\langle W, [X, V]\rangle + V.f\langle X, W\rangle + f\langle X, [V, W]\rangle \\ &= 2\langle V.fW + f\nabla_V W, X\rangle, \end{aligned}$$

once more thanks to (3).

Fundamentally:

$$\begin{aligned}\langle \nabla_V W, X\rangle + \langle W, \nabla_V X\rangle = \frac{1}{2}&\left(V.(\langle W, X\rangle) + W.(\langle X, V\rangle) - X.(\langle V, W\rangle) - \langle V, [W, X]\rangle\right.\\ &+\langle W, [X, V]\rangle + \langle X, [V, W]\rangle + V.(\langle X, W\rangle) + X.(\langle W, V\rangle) - W.(\langle V, X\rangle)\\ &\left.+\langle V, [W, X]\rangle + \langle X, [W, V]\rangle + \langle W, [V, X]\rangle\right)\\ &= V.\left(\langle X, W\rangle\right),\end{aligned} \tag{34}$$

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and

$$\begin{aligned}\langle \nabla_V W - \nabla_W V, X\rangle = \frac{1}{2}&\left(V.(\langle W, X\rangle) + W.(\langle X, V\rangle) - X.(\langle V, W\rangle) - \langle V, [W, X]\rangle\right.\\ &+\langle W, [X, V]\rangle + \langle X, [V, W]\rangle - W.(\langle X, V\rangle) - V.(\langle X, W\rangle) + X.(\langle V, W\rangle)\\ &\left.+\langle W, [V, X]\rangle - \langle X, [W, V]\rangle - \langle V, [X, W]\rangle\right)\\ &= \langle [V, W], X\rangle.\end{aligned} \tag{35}$$

{030220211642}

Thus, the Levi-Civita connection is remarkable in that it is compatible both with with the Lie Bracket and the metric g.

The exceptional nature of the Levi-Civita connection is highlighted by what has been called "the miracle of Riemannian geometry":

Theorem 1.2.4. *Let (M, g) be a Riemannian manifold. There exists a unique connection ∇ such that:*

- $[V, W] = \nabla_V W - \nabla_W V$,
- $X.\left(\langle V, W\rangle\right) = \langle \nabla_X V, W\rangle + \langle V, \nabla_X W\rangle$.

Proof. The Levi-Civita connection ensures existence. For the uniqueness, let us consider D a connection satisfying the two hypotheses and look at:

$$\begin{aligned}&V.(\langle W, X\rangle) + W.(\langle X, V\rangle) - X.(\langle V, W\rangle) - \langle V, [W, X]\rangle + \langle W, [X, V]\rangle + \langle X, [V, W]\rangle =\\ &\langle D_V W, X\rangle + \langle W, D_V X\rangle + \langle D_W X, V\rangle + \langle X, D_W V\rangle - \langle D_X V, W\rangle - \langle V, D_X W\rangle\\ &-\langle V, D_W X\rangle + \langle V, D_X W\rangle + \langle W, D_X V\rangle - \langle W, D_V X\rangle + \langle X, D_V W\rangle - \langle X, D_W V\rangle\\ &= 2\langle D_V W, X\rangle.\end{aligned}$$

We recover the Koszul formula, which means that D is the Levi-Civita connection. □

The 'miracle' is that the Riemannian structure is precisely what is needed to select one connection among all the others. We will of course use this connection to differentiate *tensor fields*.

Expression in local coordinates

Proposition 1.2.5. *Let M be a manifold, and D any connection. In a local chart, we define the Christoffel symbols Γ_{ij}^k as:*

$$D_{\partial_{x_i}}\partial_{x_j} = \Gamma_{ij}^k \partial_{x_k}.$$

Then for any V and W:

$$D_V W = V^i \left(\partial_i W^k + \Gamma_{ij}^k W^j\right) \partial_{x_k}. \tag{36}$$ {040220211622}

Proof. We will use the linear properties of the connection:

$$\begin{aligned} D_V W &= D_{V^i \partial_{x_i}} \left(W^j \partial_{x_j}\right) = V^i D_{\partial_{x_i}} \left(W^j \partial_{x_j}\right) \\ &= V^i \partial_{x_i}.W^j \partial_{x_j} + V^i W^j \nabla_{\partial_{x_i}} \partial_{x_j} \\ &= V^i \partial_i W^j \partial_{x_j} + V^i W^j \Gamma_{ij}^k \partial_{x_k} \\ &= V^i \left(\partial_i W^k + \Gamma_{ij}^k W^j\right) \partial_{x_k}. \end{aligned}$$

□

The computations in local coordinates will thus be done using those Christoffel symbols. They are *not* tensors.

Example 1.2.3. *If (M, g) is a Riemannian manifolds, the Christoffel symbols associated to the Levi-Civita connection in local coordinates $(x_1, \ldots, x_n)$ are:*

$$\Gamma_{ij}^k = \frac{g^{kl}}{2} \left(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}\right). \tag{37}$$ {020220211529}

Indeed, since $[\partial_{x_i}, \partial_{x_j}] = 0$, the Koszul formula yields:

$$\begin{aligned} \langle \nabla_{\partial_{x_i}} \partial_{x_j}, \partial_{x_k} \rangle &= \Gamma_{ij}^l \langle \partial_{x_l}, \partial_{x_k} \rangle = \Gamma_{ij}^l g_{lk} \\ &= \frac{1}{2} \left(\partial_{x_i} \left(\langle \partial_{x_j}, \partial_{x_k} \rangle\right) + \partial_{x_j} \left(\langle \partial_{x_i}, \partial_{x_k} \rangle\right) - \partial_{x_k} \left(\langle \partial_{x_i}, \partial_{x_j} \rangle\right)\right) \\ &= \frac{1}{2} \left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right), \end{aligned}$$

which yields the desired result.

Remark 1.2.3. *One can use* (36) *to check that Γ_{ij}^k is* not *a tensor. Since we know the change of coordinate law for ∂W^i (see* (12) *for the case when $W^i = g^{ij} \partial_j f$) and that it falls short of a contravariant law due to a parasit term, the Christoffel symbol must induce the exact compensation to cancel out the parasit term.*

We will denote $\nabla_i W^j = \partial_i W^j + \Gamma_{ip}^j W^p$. They form the coordinates of the $(1,1)$ gradient tensor: $\nabla W(V, \theta) = \theta\left(\nabla_V W\right)$. With the above expression, we see that we can consider the connection as an operator turning a $(1,0)$ tensor into a $(1,1)$. One can quickly broaden its scope to 1-form by duality:

$$\forall V, X \in \chi(M),\, \theta \in (\chi(M))^* \quad V.\left(\theta(X)\right) = \left(\nabla_V \theta\right)(X) + \theta\left(\nabla_V X\right). \tag{38}$$ {0402202116

In local coordinates:

$$\begin{aligned} V.\left(\theta(X)\right) = V.\left(\theta_i X^i\right) &= V^p \partial_p \theta_i X^i + V^p \theta_i \partial_p X^i \\ &= \left(\nabla_V \theta\right)(X) + \theta_i\left(V^p \partial_p X^i + \Gamma_{pq}^i V^p X^q\right), \end{aligned}$$

that is:

{040220211646} $$\left(\nabla_V \theta\right)(X) = V^p\left(\partial_p \theta_i - \Gamma_{pi}^q \theta_q\right) X^i. \tag{39}$$

The connection can thus also be turned to an operator sending a $(0,1)$ tensor to a $(0,2)$ tensor: $\nabla\theta\,(V, W) = \nabla_V \theta(W)$ whose expression in local coordinates is $\nabla_i \theta_j = \partial_i \theta_j - \Gamma_{ij}^p \theta_p$. In particular, if f is a smooth function defined on the manifold:

$$\nabla_{ij} f := \nabla_i\left(\nabla_j f\right) = \partial_{ij} f - \Gamma_{ij}^k \partial_k f. \tag{40}$$

On these basic principles we extend the covariant derivative with the Levi-Civita connection on (p, q) tensors:

Definition 1.2.6. *Let (M, g) be a Riemannian manifold and T a (p, q) tensor. The covariant derivative of T, noted ∇T is defined with the following formula:*

$$\begin{aligned} &\nabla T\left(V, \theta^1, \ldots, \theta^p, X_1, \ldots, X_q\right) \\ &\quad = V.\left[T\left(\theta^1, \ldots, \theta^p, X_1, \ldots, X_q\right)\right] - T\left(\nabla_V \theta^1, \ldots, \theta^p, X_1, \ldots, X_q\right) - \ldots \\ &\quad - T\left(\theta^1, \ldots, \nabla_V \theta^p, X_1, \ldots, X_q\right) - T\left(\theta^1, \ldots, \theta^p, \nabla_V X_1, \ldots, X_q\right) - \ldots \\ &\quad - T\left(\theta^1, \ldots, \theta^p, X_1, \ldots, \nabla_V X_q\right) \end{aligned}$$

In local coordinates:

$$\begin{aligned} \nabla_i T^{a1\ldots a_p}{}_{b_1\ldots b_q} = \partial_i T^{a1\ldots a_p}{}_{b_1\ldots b_q} &+ \Gamma_{il}^{a_1} T^{l\ldots a_p}{}_{b_1\ldots b_q} + \cdots + \Gamma_{il}^{a_p} T^{a1\ldots l}{}_{b_1\ldots b_q} \\ &- \Gamma_{ib_1}^l T^{a_1\ldots a_p}{}_{l\ldots b_q} - \cdots - \Gamma_{ib_q}^l T^{a1\ldots a_p}{}_{b_1\ldots l}. \end{aligned}$$

Proof. The expression in local coordinates is derived in the same way as for (39). □

Remark 1.2.4. *It is insightful to understand the mechanism behind the changes of coordinates: since T obeys a (p, q) change of coordinates laws, ∂T will generate a parasit term ∂J (respectively $\partial\left(J^{-1}\right)$) for each upper (respectively lower) index, then compensated with the parasit term from the Christoffel symbol (with the relevant sign), in order to obtain the $(p, q+1)$ law.*

This covariant derivative follows the expected law from a derivation, and then some:

Proposition 1.2.6.
- *$\nabla (T.S) = \nabla T.S + T.\nabla S$ where the . denotes a sum on any number of upper indexes of T and lower indexes of S.*
- *$\nabla g = 0$, $\nabla g^{-1} = 0$.*
- *One can lower or raise indexes after the covariant derivative at will. For instance: $g^{ik}\nabla_a T_{bk} = \nabla_a T_b^i = g_{bc}\nabla_a T^{ci}$.*

Proof.
- The idea is that the derivatives will naturally yield the formula, while the Christoffel symbols term with the upper and lower indexes will compensate each other. We let the interested check what is nothing more than a computation in the general case, while we will illustrate it with three indexes:
$$\begin{aligned}\nabla_i T^k_{\ lp} S^{pa}{}_b &= \partial_i T^k_{\ lp} S^{pa}{}_b + \Gamma^k_{is} T^s_{\ lp} S^{pa}{}_b - \Gamma^s_{il} T^k_{\ sp} S^{pa}{}_b - \Gamma^s_{ip} T^k_{\ ls} S^{pa}{}_b \\ T^k_{\ lp} \nabla_i S^{pa}{}_b &= T^k_{\ lp}\partial_i S^{pa}{}_b + \Gamma^p_{is} T^k_{\ lp} S^{sa}{}_b + \Gamma^a_{is} T^k_{\ lp} S^{ps}{}_b - \Gamma^s_{ib} T^k_{\ lp} S^{pa}{}_s \\ &= T^k_{\ lp}\partial_i S^{pa}{}_b + \Gamma^s_{ip} T^k_{\ ls} S^{pa}{}_b + \Gamma^a_{is} T^k_{\ lp} S^{ps}{}_b - \Gamma^s_{ib} T^k_{\ lp} S^{pa}{}_s,\end{aligned}$$
where, in the last line, we switched two mute indexes s and p. Summing then yields:
$$\nabla_i T^k_{\ lp} S^{pa}{}_b + T^k_{\ lp}\nabla_i S^{pa}{}_b = \nabla_i (T.S)^{k\ a}_{\ l\ b}.$$
- In local coordinates:
$$\begin{aligned}\nabla_p g_{ij} &= \partial_p g_{ij} - \Gamma^k_{pi} g_{kj} - \Gamma^k_{pj} g_{ik} \\ &= \partial_p g_{ij} - \frac{g^{kl}g_{kj}}{2}\left(\partial_p g_{il} + \partial_i g_{pl} - \partial_l g_{ip}\right) - \frac{g^{kl}g_{ki}}{2}\left(\partial_p g_{jl} + \partial_j g_{pl} - \partial_l g_{jp}\right) \\ &= \partial_p g_{ij} - \frac{1}{2}\left(\partial_p g_{ij} + \partial_i g_{pj} - \partial_j g_{ip} + \partial_p g_{ji} + \partial_j g_{pi} - \partial_i g_{jp}\right) \\ &= 0.\end{aligned}$$
The same reasoning works on g^{-1}.
- This property is a natural consequence of the other two.

□

1.2.3 Application to the second fundamental form

{section12022021O

In this subsection, we will play around the tensorial formalisms to obtain formulas and links between all those curvature terms. Beyond the formulas themselves, which are both interesting and fairly useful for computations in a generic chart, the analytic and geometric interpretations are pivotal to understand the underlying mechanisms hidden behind the formalism of extrinsic surfaces.

The second fundamental form in tensors

Let us consider an immersion $\Phi : \Sigma \to \mathbb{R}^3$ of a surface in the Euclidean space. Let us denote g the induced metric and recall its expressions in local coordinates (see (19)):

$$g_{ij} = \langle \partial_i \Phi, \partial_j \Phi \rangle = \langle \nabla_i \Phi, \nabla_j \Phi \rangle.$$

Let us compute:

$$\begin{aligned}
\nabla_{ij}\Phi &= \partial_{ij}\Phi - \Gamma_{ij}^k \nabla_k \Phi = \partial_{ij}\Phi - \Gamma_{ij}^k \partial_k \Phi \\
&= \partial_{ij}\Phi - \frac{g^{kl}}{2}\left(\partial_i\left(\langle \partial_j\Phi, \partial_l\Phi\rangle\right) + \partial_j\left(\langle \partial_i\Phi, \partial_l\Phi\rangle\right) - \partial_l\left(\langle \partial_i\Phi, \partial_j\Phi\rangle\right)\right)\partial_k\Phi \\
&= \partial_{ij}\Phi - \frac{g^{kl}}{2}\left(\langle \partial_{ij}\Phi, \partial_l\Phi\rangle + \langle \partial_{il}\Phi, \partial_j\Phi\rangle + \langle \partial_{ij}\Phi, \partial_l\Phi\rangle + \langle \partial_{jl}\Phi, \partial_i\Phi\rangle\right. \\
&\left. - \langle \partial_{li}\Phi, \partial_j\Phi\rangle - \langle \partial_{lj}\Phi, \partial_i\Phi\rangle\right)\partial_k\Phi \\
&= \partial_{ij}\Phi - \frac{g^{kl}}{2}\left(\langle \partial_{ij}\Phi, \partial_l\Phi\rangle + \langle \partial_{il}\Phi, \partial_j\Phi\rangle + \langle \partial_{ij}\Phi, \partial_l\Phi\rangle + \langle \partial_{jl}\Phi, \partial_i\Phi\rangle\right. \\
&\left. - \langle \partial_{li}\Phi, \partial_j\Phi\rangle - \langle \partial_{lj}\Phi, \partial_i\Phi\rangle\right)\partial_k\Phi \\
&= \partial_{ij}\Phi - g^{kl}\langle \partial_{ij}\Phi, \partial_l\Phi\rangle \partial_k\Phi,
\end{aligned}$$

where we can recognize $g^{kl}\langle \partial_{ij}\Phi, \partial_l\Phi\rangle\partial_k\Phi$ as the tangent part of $\partial_{ij}\Phi$ (denoted $\pi_T(\partial_{ij}\Phi)$); indeed:

$$\begin{aligned}
\forall u \in \{1,2\} \quad \langle \partial_{ij}\Phi - g^{kl}\langle \partial_{ij}\Phi, \partial_l\Phi\rangle\partial_k\Phi, \partial_u\Phi\rangle &= \langle \partial_{ij}\Phi, \partial_u\Phi\rangle - g^{kl}\langle \partial_{ij}\Phi, \partial_l\Phi\rangle\langle \partial_k\Phi, \partial_u\Phi\rangle \\
&= \langle \partial_{ij}\Phi, \partial_u\Phi\rangle - g^{kl}\langle \partial_{ij}\Phi, \partial_l\Phi\rangle g_{ku} \\
&= \langle \partial_{ij}\Phi, \partial_u\Phi\rangle - g_u^l\langle \partial_{ij}\Phi, \partial_l\Phi\rangle \\
&= \langle \partial_{ij}\Phi, \partial_u\Phi\rangle - \langle \partial_{ij}\Phi, \partial_u\Phi\rangle = 0.
\end{aligned}$$

Thus, denoting $\vec{A}_{ij} = A_{ij}\vec{n} = \langle \partial_{ij}\Phi, \vec{n}\rangle\vec{n}$, one has

{100220211034} $$\vec{A}_{ij} = \nabla_{ij}\Phi = A_{ij}\vec{n}. \tag{41}$$

This formula is a convenient manner to compute the variation of the curvatures along a perturbation Φ_t, as is required when looking for the Euler-Lagrange equations associated to Lagrangians of the shape $\int \mathrm{Curv}^p$. It is also very insightful to understand the connection. Let $\vec{V}$ be a vector field on $\Phi(\Sigma)$ corresponding to the vector V on Σ. Then:

$$\nabla_i \vec{V} = \partial_i \vec{V} = \nabla_i\left(V^p \nabla_p \Phi\right) = \nabla_i V^p \nabla_p \Phi + V^p \nabla_{pi}\Phi = \nabla_i V^p \nabla_p \Phi + V^p \pi_{\vec{n}}(\partial_{ip}\Phi),$$

thus

$$\nabla_i V^p \nabla_p \Phi = \partial_i \vec{V} - \pi_{\vec{n}}\left(\partial_i \vec{V}\right) = \pi_T(\partial_i \vec{V}).$$

The connection, when interpreted as an operator on a surface in $\mathbb{R}^3$ thus works in the following manner:

- We differentiate the resulting fonction in $\mathbb{R}^3$ in the natural manner.
- We project on the tangent space to recover an intrinsic object on the surface.

Mean, tracefree, total curvature

From this, one can quickly compute the mean curvature: $\vec{H} := H\vec{n} = \frac{\vec{A}_i^i}{2} = \frac{1}{2}g^{ip}\nabla_{pi}\Phi = \frac{1}{2}\nabla_i^i\Phi$, where we can recognize a Laplacian (called Laplace-Beltrami operator) in the last term:

$$\Delta_g \Phi = 2H\vec{n}. \tag{42}$$ {100220211040}

This alone justifies our interest in the mean curvature: it is linked with a Laplacian of the immersion, an elliptic operator, and thus is an analytically meaningful quantity on which estimates can be used as starting points for elliptic regularity.

Now, setting our sights to the tracefree second fundamental form $\mathring{A}_{ij} = A_{ij} - Hg_{ij}$, one can immediately see $\mathring{A}_i^i = A_i^i - Hg_i^i = 2H - 2H = 0$, that is, the tracefree second fundamental form is indeed tracefree. This gives us a decomposition of $|A|_g^2$:

$$\begin{aligned}
|A|_g^2 = A_j^i A_i^j &= \left(\mathring{A}_j^i + Hg_j^i\right)\left(\mathring{A}_i^j + Hg_i^j\right) \\
&= \mathring{A}_j^i\mathring{A}_i^j + Hg_j^i\mathring{A}_i^j + Hg_i^j\mathring{A}_j^i + H^2 g_j^i g_i^j \\
&= \left|\mathring{A}\right|_g^2 + 2H\mathring{A}_i^i + H^2 g_i^i \\
&= \left|\mathring{A}\right|_g^2 + 2H^2.
\end{aligned}$$

Applying it to $\nabla_i\vec{n} = -A_i^j\nabla_j\Phi$:

$$\begin{aligned}
|\nabla\vec{n}|_g^2 = g^{ij}\langle\nabla_i\vec{n}, \nabla_j\vec{n}\rangle &= g^{ij}A_i^pA_j^q\langle\nabla_p\Phi, \nabla_q\Phi\rangle \\
&= A_i^pA^{iq}g_{pq} = |A|_g^2 = \left|\mathring{A}\right|_g^2 + 2H^2.
\end{aligned}$$

In essence:

$$|\nabla\vec{n}|_g^2 = |A|_g^2 = \left|\mathring{A}\right|_g^2 + 2H^2. \tag{43}$$ {100220211230}

Behind (43) is the principle that the tracefree-tracefull decomposition is orthogonal, and thus, as linear operators, $\mathring{A}$ and Hg will act independantly. This must not be understood to mean independance as *functions*.

Gauss-Codazzi

Indeed, let us consider the second derivatives of $\vec{n}$:

$$\begin{aligned}\nabla_{ij}\vec{n} &:= \nabla_i\left(\nabla_j\vec{n}\right) = -\nabla_i\left(A_{jk}\nabla^k\Phi\right) = -\nabla_i A_{jk}\nabla^k\Phi - A_{jk}A_i^k\vec{n}\\ \nabla_{ji}\vec{n} &:= \nabla_j\left(\nabla_i\vec{n}\right) = -\nabla_j\left(A_{ik}\nabla^k\Phi\right) = -\nabla_j A_{ik}\nabla^k\Phi - A_{ik}A_j^k\vec{n}.\end{aligned} \tag{44}$$ {1002202113

However, let us notice that $\nabla_{ij}\vec{n} = \nabla_{ji}\vec{n}$, indeed:

$$\nabla_{ij}\vec{n} = \partial_{ij}\vec{n} - \Gamma_{ij}^k\partial_k\vec{n} = \partial_{ji}\vec{n} - \Gamma_{ji}^k\partial_k\vec{n} = \nabla_{ji}\vec{n},$$

where we have used a fundamental property of the Christoffel symbols $\Gamma_{ij}^k = \Gamma_{ji}^j$ that can be read directly from the formulas in local coordinates. It must be noted however that this equality stands true for *functions*, that is 0-tensors, only. Applied to any (p,q) tensor with $p+q>0$, the operators ∇_i and ∇_j do not commute.

Identifying the tangent parts of both equalities in (44) then yields the Gauss-Codazzi equalities:

{100220211303}
$$\forall i,j,k\in\{1,2\} \quad \nabla_i A_{jk} = \nabla_j A_{ik}. \tag{45}$$

One can give (45) a more insightful form:

$$\begin{aligned}\nabla_i H &= \frac{1}{2}\nabla_i A_j^j = \frac{1}{2}g^{jk}\nabla_i A_{jk} = \frac{1}{2}g^{jk}\nabla_j A_{ik} = \frac{1}{2}\nabla_j A_i^j\\ &= \frac{1}{2}\nabla_j \mathring{A}_i^j + \frac{1}{2}\nabla_j H g_i^j = \frac{1}{2}\nabla_j\mathring{A}_i^j + \frac{1}{2}\nabla_i H,\end{aligned}$$

or in other words:

{100220211314}
$$\nabla_i H = \nabla_j \mathring{A}_i^j = \nabla^j \mathring{A}_{ij}. \tag{46}$$

Building a second fundamental form is thus not just about assembling a trace and a tracefree part, they must satisfy this compatibility condition.

Tracing in (44) also allows to recover a *structural elliptic equation* for the Gauss map:

$$\begin{aligned}\Delta_g\vec{n} = \nabla_i^i\vec{n} &= -\nabla^i A_{ik}\nabla^k\Phi - A_{ik}A^{ik}\vec{n}\\ &= -\nabla_k A_i^i\nabla^k\Phi - |A|_g^2\,\vec{n}\\ &= -2\nabla_k H\nabla^k\Phi - |A|_g^2\,\vec{n},\end{aligned}$$

which we reformulate:

{100220211431}
$$\Delta_g\vec{n} = -\left|\nabla\vec{n}\right|_g^2\vec{n} - 2\nabla H.\nabla\Phi. \tag{47}$$

We can rephrase this in a more remarkable fashion: denoting $\mathrm{div}_g(V) = \nabla_i V^i$ the divergence operator on tensors, one can notice that:

$$2\nabla H.\nabla\Phi = 2\mathrm{div}_g\left(H\nabla\Phi\right) - 2H\Delta_g\Phi = 2\mathrm{div}_g\left(H\nabla\Phi\right) - 4H^2\vec{n}.$$

Injecting this into (47) yields:

$$\Delta_g \vec{n} = -\left(|\nabla \vec{n}|^2 - 4H^2\right)\vec{n} - 2\mathrm{div}_g\left(H\nabla\Phi\right)$$

Let us work on the normal term:

$$\begin{aligned}
|\nabla \vec{n}|_g^2 - 4H^2 = |A|_g^2 - 4H^2 = \left|\mathring{A}\right|_g^2 - 2H^2, \\
&= \left(A_1^1\right)^2 + \left(A_2^2\right)^2 + 2A_2^1A_1^2 - 4\left(\frac{A_1^1 + A_2^2}{2}\right)^2 \\
&= 2\left(A_2^1A_1^2 - A_1^1A_2^2\right) = -2\mathrm{det}_g(A) = -2K,
\end{aligned}$$

where we used (43) in the first line, and the formal definition of H as the half trace and K as the determinant in the second line. Further, denoting the *curl* of $\vec{n}$:

$$\left(\nabla_g^\perp \vec{n}\right)^k = \frac{1}{\sqrt{|g|}}\epsilon^{kl}\nabla_l\vec{n}, \tag{48}$$

where

$$\left(\epsilon^{kl}\right)_{kl=1..n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

then one can compute, with (21):

$$\begin{aligned}
\nabla\vec{n} \times \nabla_g^\perp\vec{n} &= \nabla_k\vec{n} \times \left(\nabla_g^\perp\vec{n}\right)^k \\
&= -\frac{1}{\sqrt{|g|}}\left(A_1^1\nabla_1\Phi + A_1^2\nabla_2\Phi\right) \times \left(A_2^1\nabla_1\Phi + A_2^2\nabla_2\Phi\right) \\
&+ \frac{1}{\sqrt{|g|}}\left(A_2^1\nabla_1\Phi + A_2^2\nabla_2\Phi\right) \times \left(A_1^1\nabla_1\Phi + A_1^2\nabla_2\Phi\right) \\
&= -\frac{2}{\sqrt{|g|}}\left(A_1^1A_2^2 - A_1^2A_2^1\right)\nabla_1\Phi \times \nabla_2\Phi \\
&= -2K\vec{n}.
\end{aligned}$$

Injecting this into (47) then ensures:

$$\Delta_g\vec{n} = -\nabla\vec{n} \times \nabla_g^\perp\vec{n} - 2\mathrm{div}_g\left(H\nabla\Phi\right). \tag{49}$$ {100220211544}

The terms $\nabla.\nabla_g^\perp$ are said to be in Jacobian form, and in this course we will explore their analytic potential, and their mathematical origins.

We have, on our merry way shown the formula:

$$\left|\mathring{A}\right|_g^2 - 2H^2 = -2K, \tag{50}$$ {100220211555}

which will feature prominently when considering the Willmore energy.

A note on g operators

The operators div_g, Δ_g, $\nabla_g^\perp$ follow the same properties as their euclidean counterpart. For instance:

{1502202108

Lemma 1.2.1. *Let V be a simply connected domain in $\mathbb{R}^2$ and $A \in \chi(V)$ be a local vector field such that $\mathrm{div}_g(A) = 0$. Then there exists a function f such that: $A = \nabla_g^\perp f$.*

Proof. The result stands if $g = \xi$, that is if we take div and $\nabla^\perp$ instead of div_g and $\nabla_g^\perp$ by a straightforward application of the Poincaré lemma. Now, by definition of div_g and the Christoffel symbols:

$$\begin{aligned}
\mathrm{div}_g A &= \nabla_1 A^1 + \nabla_2 A^2 = \partial_1 A^1 + \Gamma^1_{1p} A^p + \partial_2 A^2 + \Gamma^2_{2p} A^p \\
&= \mathrm{div} A + \left(\Gamma^1_{11} + \Gamma^2_{21}\right) A^1 + \left(\Gamma^1_{12} + \Gamma^2_{22}\right) A^2 \\
&= \mathrm{div} A + \frac{1}{2}\left(g^{11}\partial_1 g_{11} + g^{12}(2\partial_1 g_{12} - \partial_2 g_{11}) + g^{21}\partial_2 g_{11} + g^{22}\partial_1 g_{22}\right) A^1 \\
&+ \frac{1}{2}\left(g^{11}\partial_2 g_{11} + g^{12}\partial_1 g_{22} + g^{21}(2\partial_2 g_{12} - \partial_1 g_{22}) + g^{22}\partial_2 g_{22}\right) A^2 \\
&= \mathrm{div} A + \frac{1}{2}\mathrm{Tr}(g^{-1}\partial_1 g) A^1 + \frac{1}{2}\mathrm{Tr}(g^{-1}\partial_2 g) A^2 \\
&= \mathrm{div} A + |g|^{-\frac{1}{2}}\partial_1(|g|^{\frac{1}{2}}) A^1 + |g|^{-\frac{1}{2}}\partial_2(|g|^{\frac{1}{2}}) A^2 \\
&= |g|^{-\frac{1}{2}}\mathrm{div}\left(|g|^{\frac{1}{2}} A\right).
\end{aligned}$$

Thus, applying the Poincaré lemma to the equation $\mathrm{div}_g A = |g|^{-\frac{1}{2}}\mathrm{div}\left(|g|^{\frac{1}{2}} A\right) = 0$ yields:

$$A = |g|^{-\frac{1}{2}}\nabla^\perp f = \nabla_g^\perp f.$$

□

{190420211018}

Remark 1.2.5. *Similarly, one can show that div_g satisfies the Green-Stokes theorem on a manifold. From then on, we will use it when needed, on a manifold or on a subset of $\mathbb{R}^n$.*

1.3 In local charts

While it is important to master the tensorial formalism for global matters on the manifold, we will in the following do the major part of our studies in local charts. Adapting the chart to the problem (and making sure such charts are available!) is then a pivotal preliminary part of the work. In the present subsection we will thus go over a few choices of simple charts, before detailing the most convenient for our problem: local conformal charts. One notable absent will be the exponential charts.

Indeed, these play a pivotal role in Riemannian geometry, and are unavoidable tools to study intrinsic objects in *any* dimension. They are however of limited interest in extrinsic geometry. We thus cannot avoid mentioning them, but we will do so briefly, in dimension 2 and without proofs.

Theorem 1.3.1. *Let Σ be a surface and $\Phi : \Sigma \to \mathbb{R}^3$ an immersion. Then:*

- *For any $p \in \Sigma$, there exists a local chart $\varphi : \mathbb{D} \to V$ such that the induced local parametrization $\phi : \mathbb{D} \to \mathbb{R}^3$ satisfies:*

$$\begin{aligned}&\phi(0) = \Phi(p)\\ &g(0) = Id; \partial_i g(0) = 0.\end{aligned}$$

- *For any $q \in \Sigma$, there exists a local chart $\varphi : \mathbb{D} \to V$ such that the induced local parametrization $\phi : \mathbb{D} \to \mathbb{R}^3$ satisfies:*

$$g = \begin{pmatrix} 1 & 0 \\ 0 & G \end{pmatrix}.$$

Taking the second parametrization in polar coordinates (r, θ), one can give the following formulas:

$$\begin{aligned}H &= \frac{1}{2}\left(e + \frac{g}{G}\right)\\ K &= -\frac{1}{2\sqrt{G}}\left(\frac{G_r}{G}\right)_r.\end{aligned}$$

One can compare these formulas with (33) and the one given by the *theorema egregium*: while they are much simpler, they still make computations somewhat uncomfortable.

1.3.1 Local graphs over the tangent space

Definition

Definition 1.3.1. *Let U be an open of $\mathbb{R}^2$, and $u : U \to \mathbb{R}$ a smooth application. Then $\phi : U \to \mathbb{R}^3$ is the graph parametrization of the surface if $\phi(x, y) = (x, y, u(x, y))$.*

One can then quickly compute the tangent and normal vectors:

$$\phi_x = (1, 0, u_x),\ \phi_y = (0, 1, u_y),\ \vec{n}(x, y) = \frac{1}{\sqrt{1 + |\nabla u|^2}}(-u_x, -u_y, 1). \tag{51}$$ {110220211107}

The induced metric is then:

$$g = \begin{pmatrix} 1+u_x^2 & u_x u_y \\ u_x u_y & 1+u_y^2 \end{pmatrix}. \tag{52}$$ {1102202111

To compute the second fundamental form, one has two possibilities: $\langle \partial^2\phi, \vec{n}\rangle$, $-\langle \partial\phi, \partial\vec{n}\rangle$. Since the Gauss map is a normalized vector, its derivatives are often less conveniently computable than the second derivatives of the parametrization. This is the case here (but it should not be taken as a general rule-see the stereographic projection of the sphere):

{110220211125}

$$A = \frac{1}{1+|\nabla u|^2}\begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix} = \frac{\nabla^2 u}{1+|\nabla u|^2}. \tag{53}$$

One can then compute the mean and Gauss curvatures:

$$\begin{aligned} H &= \frac{1}{2\left(1+|\nabla u|^2\right)^{\frac{3}{2}}}\left[\left(1+(u_x)^2\right)u_{yy} + \left(1+(u_y)^2\right)u_{xx} - 2u_x u_y u_{xy}\right] \\ K &= \frac{u_{xx}u_{yy} - (u_{xy})^2}{\left(1+|\nabla u|^2\right)^2}. \end{aligned} \tag{54}$$

The formula for H can be manipulated to give it a much more interesting shape:

{170220211631}

$$H = \frac{1}{2}\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{55}$$

Remark 1.3.1. *Here we must stress that all the differential operators are taken as operators on real valued functions. For instance,* $\mathrm{div}\begin{pmatrix} V^1 \\ V^2 \end{pmatrix} = V^1_x + V^2_y$ *is not to be confused with the tensorial operator* div_g*. These notations are not arbitrary:* div *is simply the tensorial divergence on* $\mathbb{R}^2$ *imbued with its euclidean metric.*

Availability

A graph parametrization thus offers us rather simple (albeit non-linear) formulas for the curvatures and the metric involving remarkable operators. While it is not the simplest one to consider, it is one of the most flexible available:

{170220211737}

Theorem 1.3.2. *Let* Σ *be a surface,* $\Phi : \Sigma \to \mathbb{R}^3$ *an immersion and* $p \in \Sigma$*. Then, one can parametrize locally* $\Phi(\Sigma)$ *around* p *above its tangent plane at* p*. More precisely, after a rotation and a translation of* $\mathbb{R}^3$*, one can parametrize locally* Σ *by*

$$\phi \begin{cases} U \to \mathbb{R}^3 \\ (x,y) \mapsto (x,y,u(x,y)), \end{cases}$$

where $u \in C^\infty(U)$ *satisfies* $u(0) = 0$ *and* $\nabla u(0) = 0$*.*

Proof. Let $\psi : V \to \Phi(\Sigma)$ be a local parametrization of $\Phi(\Sigma)$ such that $\psi(0) = \Phi(p)$. Up to a translation one can assume that $\psi(0) = \Phi(p) = 0$, and up to a rotation one can assume that the normal at 0 is $(0,0,-1)$, that is that the tangent plane at 0 is the horizontal plane of equation $z = 0$. Let us now consider $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ the projection on the first two coordinates (that is $\pi(x,y,z) = (x,y)$). Let us now consider $F = \pi \circ \psi : V \to T_{\psi(0)}\psi(V)$. One can quickly check that $dF(0)$ is of rank 2 and thus that it is invertible. With the local inversion theorem, there exists a neighborhood $U \subset V$ of 0 such that F is a diffeomorphism. If we now set $\phi := \psi \circ F^{-1}$, we obtain a local parametrization of the surface. By construction $\pi \circ \phi = Id$, which does yield:

$$\phi(x,y) = (x, y, u(x,y)).$$

Knowing that $\phi(0,0) = 0$ and $\vec{n}(0,0) = (0,0,-1)$ (look at formula (51)) one has $u(0) = 0$ and $\nabla u(0) = 0$. □

This proof does not depend on the dimension and can be adapted to function for a manifold of any dimension n immersed in any $\mathbb{R}^m$. This easy generalization makes these kind of parametrizations a useful tool to keep in mind for their flexibility.

1.3.2 Local conformal charts, local parametrizations

We will now introduce the type of local parametrizations more adapted to our purposes: local conformal charts.

Conformal maps

{250520211732}

Definition 1.3.2. *Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and $f : M_1 \to M_2$ a C^1 map. Then f is* conformal *if and only if there exists $\lambda : M_1 \to \mathbb{R}$ called the conformal factor such that:*

$$f^* g_2 = e^{2\lambda} g_1.$$

Remark 1.3.2. *A conformal transformation thus simply multiplies the metric by a strictly positive function. There are different ways to express this conformal factor and one should not hesitate to write κ^2 instead of $\mathrm{e}^{2\lambda}$ to obtain the simplest formulat for the situation.*

Example 1.3.1.
- *The induced metric for an immersion $\Phi : \Sigma \to \mathbb{R}^3$ is the pullback of the euclidean metric ξ of $\mathbb{R}^3$.*
- *Isometries of $\mathbb{R}^3$ (so translations and rotations) are conformal diffeomorphisms of $\mathbb{R}^3$ imbued with the euclidean metric ξ.*

- *Dilations of* $\mathbb{R}^3$ *are conformal diffeomorphisms of* $\mathbb{R}^3$ *imbued with the euclidean metric* ξ*. The conformal factor is then the logarithm of the dilation factor.*

- *The inversion* $\iota : x \to \frac{x}{|x|^2}$ *is a conformal diffeomorphism of* $\mathbb{R}^3 \cup \{\infty\}$ *imbued with the euclidean metric. Indeed let us look at*

$$\partial_1 \iota(x) = \frac{1}{|x|^4} \begin{pmatrix} x_2^2 + x_3^2 - x_1^2 \\ -2x_1x_2 \\ -2x_1x_3 \end{pmatrix}$$
$$\partial_2 \iota(x) = \frac{1}{|x|^4} \begin{pmatrix} -2x_1x_2 \\ x_1^2 + x_3^2 - x_2^2 \\ -2x_2x_3 \end{pmatrix}$$
$$\partial_3 \iota(x) = \frac{1}{|x|^4} \begin{pmatrix} -2x_1x_3 \\ -2x_2x_3 \\ x_1^2 + x_2^2 - x_3^2 \end{pmatrix}.$$

and thus

$$\iota^*\xi = \begin{pmatrix} \langle \partial_1 \iota, \partial_1 \iota \rangle & \langle \partial_1 \iota, \partial_2 \iota \rangle & \langle \partial_1 \iota, \partial_3 \iota \rangle \\ \langle \partial_2 \iota, \partial_1 \iota \rangle & \langle \partial_2 \iota, \partial_2 \iota \rangle & \langle \partial_2 \iota, \partial_3 \iota \rangle \\ \langle \partial_3 \iota, \partial_1 \iota \rangle & \langle \partial_3 \iota, \partial_2 \iota \rangle & \langle \partial_3 \iota, \partial_3 \iota \rangle \end{pmatrix} = \frac{1}{|x|^4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- *The stereographic projection (see example 1.1.5)* $\pi : (\mathbb{R}^2, \xi) \to (\mathbb{S}^2, \xi_{\mathbb{S}^2})$ *is a conformal map.*

The examples 2,3 and 4 are fundamental to understand the conformal diffeomorphisms of $\mathbb{R}^3 \cup \{\infty\}$:

{260220211225}

Theorem 1.3.3. *For* $n \geq 3$*, the conformal diffeomorphisms of* $\mathbb{R}^n \cup \{\infty\}$ *form a group (called the conformal group) which is spanned by the translations, rotations, dilations and inversions.*

The previous theorem is called the Liouville theorem (we direct the interested reader to theorem 1.1.1 of [**?**]).

Conformal charts

We will come back to conformal map between spaces, and more particularly to conformal diffeomorphisms of $\mathbb{R}^3$ later. For now, let us talk about conformal charts of a manifold.

Definition 1.3.3. *Let (M, g) be a Riemannian manifold of dimension n. A chart (U, φ) is a conformal chart of (M, g) if and only if $\varphi : (U, g) \to (\mathbb{R}^n, \xi)$ is conformal, with ξ denoting the euclidean metric of $\mathbb{R}^n$. In other words, there exists Λ such that*

$$\varphi^* \xi = e^{2\Lambda} g.$$

This immediately gives us the notion of a conformal parametrization:

Definition 1.3.4. *Let Σ be a surface and $\Phi : \Sigma \to \mathbb{R}^3$ an immersion. Let us denote g the induced metric. If (U, φ) is a conformal chart for (Σ, g), then the resulting parametrization $\phi = \Phi \circ \varphi^{-1} : (\varphi(U), \xi) \to (\mathbb{R}^3, \xi)$ is a conformal map. Such a parametrization is a local conformal parametrization. We also say that we are looking at the immersion Φ in a local conformal chart.*

Let us now explore the potential of such parametrizations. Let ϕ be a conformal parametrization defined on a disk D. Then, by definition, there exists $\lambda : D \to \mathbb{R}$ such that the induced metric g satisfies

$$g = e^{2\lambda} \mathrm{Id}.$$

In other words:

$$\langle \phi_x, \phi_y \rangle = 0, \ \langle \phi_x, \phi_x \rangle = \langle \phi_y, \phi_y \rangle = e^{2\lambda}. \tag{56}$$

This implies notably that $(\phi_x, \phi_y)(z)$ (respectively $e^{-\lambda}(\phi_x, \phi_y)(z)$) forms an orthogonal (respectively orthonormal) basis of $T_{\phi(z)}\phi(D)$ for all $z \in D$. We call such moving basis on the tangent planes *moving frames*.

Having a ready to go orthogonal moving frame is one of the massive advantages of the conformal parametrizations (with deep implications, see the Coulomb frame next subsection). It notably implies that:

$$\begin{aligned}
&|\phi_x \times \phi_y| = e^{2\lambda} \quad \vec{n} = e^{-2\lambda} \left(\phi_x \times \phi_y \right) \\
&\langle \phi_{xx}, \phi_x \rangle = \frac{1}{2} \left(|\phi_x|^2 \right)_x = \lambda_x e^{2\lambda} \\
&\langle \phi_{xy}, \phi_x \rangle = \frac{1}{2} \left(|\phi_x|^2 \right)_y = \lambda_y e^{2\lambda} \\
&\langle \phi_{xy}, \phi_y \rangle = \frac{1}{2} \left(|\phi_y|^2 \right)_x = \lambda_x e^{2\lambda} \\
&\langle \phi_{yy}, \phi_y \rangle = \frac{1}{2} \left(|\phi_y|^2 \right)_y = \lambda_y e^{2\lambda} \\
&\langle \phi_{xx}, \phi_y \rangle = \left(\langle \phi_x, \phi_y \rangle \right)_x - \langle \phi_x, \phi_{xy} \rangle = -\lambda_y e^{2\lambda} \\
&\langle \phi_{yy}, \phi_x \rangle = \left(\langle \phi_x, \phi_y \rangle \right)_y - \langle \phi_y, \phi_{xy} \rangle = -\lambda_x e^{2\lambda},
\end{aligned}$$

which yields:

$$\begin{aligned}\phi_{xx} &= \lambda_x\phi_x - \lambda_y\phi_y + e\vec{n}\\ \phi_{xy} &= \lambda_y\phi_x + \lambda_x\phi_y + f\vec{n}\\ \phi_{yy} &= -\lambda_x\phi_x + \lambda_y\phi_y + g\vec{n},\end{aligned} \tag{57}$$ {1202202107

where we recall the notation for the second fundamental form:

$$A = \begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

The Gauss and the mean curvature are thus written as follows:

{120220210802} $$\begin{aligned}K &= e^{-4\lambda}\left(eg - f^2\right)\\ H &= \frac{e+g}{2}e^{-2\lambda}\end{aligned} \tag{58}$$

while the tracefree second fundamental form is expressed as:

{100220210803} $$\mathring{A} = \begin{pmatrix} \frac{e-g}{2} & f \\ f & \frac{g-e}{2} \end{pmatrix}, \tag{59}$$

and the variations of the Gauss map are written

{120220210806} $$\begin{aligned}\vec{n}_x &= -ee^{-2\lambda}\phi_x - fe^{-2\lambda}\phi_y\\ \vec{n}_y &= -fe^{-2\lambda}\phi_x - ge^{-2\lambda}\phi_y.\end{aligned} \tag{60}$$

A good exercise throughout this section is to compare the tensorial formulas from section 1.2.3 with those obtained in the conformal formalism to better understand their inner workings. For instance (24) can be linked with (60), the upper indexes in the tensorial formula becoming a $e^{-2\lambda}$, that is, g^{-1} in the conformal chart.

Summing the first and the third line in (57) yields: $\phi_{xx}+\phi_{yy} = \Delta\phi = (e+g)\vec{n} = 2He^{2\lambda}\vec{n}$, which yields the avatar of (42) in local conformal charts:

{120220210913} $$\Delta\phi = 2He^{2\lambda}\vec{n} = 2H\phi_x \times \phi_y = H\nabla^{\perp}\phi \times \nabla\phi, \tag{61}$$

where $\nabla^{\perp}\phi = \begin{pmatrix} -\phi_y \\ \phi_x \end{pmatrix} = \nabla^{\perp}_{\xi}\phi$. Let us recall that in local charts all the operators are the classical euclidean operators, which represent the major gain from their tensorial counterparts: we only have to consider the laplacian, instead of a more complex elliptic operator.

Let us keep computing:

$$\begin{aligned}
\vec{n}_{xx} &= -(e_x - 2\lambda_x e)e^{-2\lambda}\phi_x - ee^{-2\lambda}\phi_{xx} - (f_x - 2\lambda_x f)e^{-2\lambda}\phi_y - fe^{-2\lambda}\phi_{xy} \\
&= -(e_x - \lambda_x e + \lambda_y f)e^{-2\lambda}\phi_x - (f_x - \lambda_y e - \lambda_x f)\phi_y - (e^2 + f^2)e^{-2\lambda}\vec{n} \\
\vec{n}_{xy} &= -(e_y - \lambda_y e - \lambda_x f)e^{-2\lambda}\phi_x - (f_y - \lambda_y f + \lambda_x e)e^{-2\lambda}\phi_y - (ef + fg)e^{-2\lambda}\vec{n} \\
\vec{n}_{yx} &= -(f_x - \lambda_x f + \lambda_y g)e^{-2\lambda}\phi_x - (g_x - \lambda_x g - \lambda_y f)e^{-2\lambda}\phi_y - (fe + gf)e^{-2\lambda}\vec{n} \\
\vec{n}_{yy} &= -(f_y - \lambda_y f - \lambda_x g)e^{-2\lambda}\phi_x - (g_y - \lambda_y g + \lambda_x f)e^{-2\lambda}\phi_y - (f^2 + g^2)e^{-2\lambda}\vec{n}.
\end{aligned} \tag{62}$$

0220211013}

Since $\vec{n}_{xy} = \vec{n}_{yx}$, one has

$$\begin{aligned}
e_y - \lambda_y e - \lambda_x f &= f_x - \lambda_x f + \lambda_y g \\
f_y - \lambda_y f + \lambda_x e &= g_x - \lambda_x g - \lambda_y f,
\end{aligned}$$

or in other words:

$$\begin{aligned}
e_y - \lambda_y e &= f_x + \lambda_y g \\
f_y + \lambda_x e &= g_x - \lambda_x g.
\end{aligned}$$

This is the equivalent of the Gauss-Codazzi formula (45). In this case however it seems less palatable due to the derivatives of the conformal factor, but one must remember that they are hidden in the formula for the Levi-Civita connection in the Christoffel symbols. It can actually be given a nicer form, by writing $e = \frac{e+g}{2} + \frac{e-g}{2}$. When that is done, one has:

$$\begin{aligned}
e_y = \left(\frac{e+g}{2}\right)_y + \left(\frac{e-g}{2}\right)_y &= f_x + 2\lambda_y \left(\frac{e+g}{2}\right) \\
\implies H_y = \left(\frac{e+g}{2}e^{-2\lambda}\right)_y &= \left(f_x - \left(\frac{e-g}{2}\right)_y\right)e^{-2\lambda}.
\end{aligned}$$

Working similarly on the second equality with $g = \frac{e+g}{2} - \frac{e-g}{2}$, one has:

$$\begin{aligned}
g_x = \left(\frac{e+g}{2}\right)_x - \left(\frac{e-g}{2}\right)_x &= f_y + 2\lambda_x \left(\frac{e+g}{2}\right) \\
\implies H_x = \left(\frac{e+g}{2}e^{-2\lambda}\right)_y &= \left(f_y + \left(\frac{e-g}{2}\right)_x\right)e^{-2\lambda}.
\end{aligned}$$

The Gauss-Codazzi in a local conformal chart thus states:

$$\nabla H = \begin{pmatrix} f_y + \left(\frac{e-g}{2}\right)_x \\ f_x - \left(\frac{e-g}{2}\right)_y \end{pmatrix} e^{-2\lambda}, \tag{63}$$ {120220211008}

which is a direct translation of (46).

To recover the structure equation on the Gauss map let us sum the first and last line of (62):

$$\begin{aligned}
\Delta\vec{n} &= -\left(e_x + f_y - \lambda_x(e+g)\right) e^{-2\lambda}\phi_x - \left(g_y + f_x - \lambda_y(e+g)\right) e^{-2\lambda}\phi_y - (e^2+2f^2+g^2)e^{-2\lambda}\vec{n} \\
&= -\left(\left(\frac{e+g}{2}\right)_x - 2\lambda_x\frac{e+g}{2} + f_y + \left(\frac{e-g}{2}\right)_x\right) e^{-2\lambda}\phi_x \\
&\quad - \left(\left(\frac{e+g}{2}\right)_y - 2\lambda_y\frac{e+g}{2} + f_x - \left(\frac{e-g}{2}\right)_y\right) e^{-2\lambda}\phi_y - \left|\nabla\vec{n}\right|^2\vec{n} \\
&= -2H_x\phi_x - 2H_y\phi_y - \left|\nabla\vec{n}\right|^2\vec{n},
\end{aligned}$$

using (63) to obtain the last line. The analog of (47) thus stands in the following manner:

{120220211054} $$\Delta\vec{n} = -\left|\nabla\vec{n}\right|^2\vec{n} - 2\nabla H.\nabla\phi. \tag{64}$$

This equation is transformed into the avatar of (49) in the same manner: $2\nabla H.\nabla\phi = 2\mathrm{div}\,(H\nabla\phi) - 4H^2e^{2\lambda}\vec{n}$, and $|\nabla\vec{n}|^2 - 4H^2e^{2\lambda} = \left(e^2+2f^2+g^2-4\left(\frac{e+g}{2}\right)^2\right)e^{-2\lambda} = 2(f^2-eg)e^{-2\lambda} = -2Ke^{2\lambda}$. Computing:

$$\begin{aligned}
\nabla\vec{n}\times\nabla^\perp\vec{n} &= -2\vec{n}_y\times\vec{n}_x = 2e^{-4\lambda}\left(f\phi_x + g\phi_y\right)\times\left(e\phi_x + f\phi_y\right) \\
&= -2e^{-2\lambda}(eg-f^2)\vec{n} = \left(|\nabla\vec{n}|^2 - 4H^2e^{2\lambda}\right)\vec{n}.
\end{aligned}$$

Injecting these considerations into (64) yields:

{120220211059} $$\begin{aligned}
\Delta\vec{n} &= -\left(|\nabla\vec{n}|^2 - 4H^2e^{2\lambda}\right)\vec{n} - 2\mathrm{div}\,(H\nabla\phi) \\
&= -\nabla\vec{n}\times\nabla^\perp\vec{n} - 2\mathrm{div}\,(H\nabla\phi)\,.
\end{aligned} \tag{65}$$

Once more the structure of this equation is pivotal.

Gauss-Codazzi and the structure equation on $\vec{n}$ come from comparing the second derivatives of the Gauss map. Doing the same thing on ϕ yields very little, as the underlying structure there is already accounted in the symmetry of the second fundamental form. However, for the third derivatives, one can recover the Liouville equation:

Theorem 1.3.4. *Let $\phi : \mathbb{D} \to \mathbb{R}^3$ be a conformal parametrization of conformal factor λ. Then, it satisfies the* Liouville equation*:*

$$\Delta\lambda = -Ke^{2\lambda}.$$

Proof. As announced, let us compute:

$$\begin{aligned}
\phi_{xxy} &= \lambda_{xy}\phi_x + \lambda_x\phi_{xy} - \lambda_{yy}\phi_y - \lambda_y\phi_{yy} + e_y\vec{n} + e\vec{n}_y \\
&= \left(\lambda_{xy} + \lambda_x\lambda_y + \lambda_y\lambda_x - efe^{-2\lambda}\right)\phi_x + \left(-\lambda_{yy} + \lambda_x^2 - \lambda_y^2 - ege^{-2\lambda}\right)\phi_y \\
&+ \left(e_y + \lambda_x f - \lambda_y g\right)\vec{n} \\
\phi_{xyx} &= \lambda_{yx}\phi_x + \lambda_y\phi_{xx} + \lambda_{xx}\phi_y + \lambda_x\phi_{yx} + f_x\vec{n} + f\vec{n}_x \\
&= \left(\lambda_{xy} + \lambda_y\lambda_x + \lambda_x\lambda_y - fee^{-2\lambda}\right)\phi_x + \left(\lambda_{xx} - \lambda_y^2 + \lambda_x^2 - f^2e^{-2\lambda}\right)\phi_y \\
&+ \left(f_x + \lambda_y e + \lambda_x f\right)\vec{n}.
\end{aligned}$$

Comparing the two ϕ_y terms yields:

$$\Delta\lambda = -(eg - f^2)e^{-2\lambda} = -Ke^{2\lambda}.$$

□

This equation justifies by itself the interest of the conformal charts: we recover the theorema egregium formula in a magnificiently remarkable shape on which an analyst can say a lot of things.

Of course the question now is: can we always work locally in a conformal chart? The answer is yes for smooth surfaces:

{150220211158}

Theorem 1.3.5. *Let Σ be a surface, and $\Phi : \Sigma \to \mathbb{R}^3$ an immersion such that the immersed metric does not degenerate (i.e. satisfies* (17)*). For any $\Phi(p) \in \Phi(\Sigma)$, there exists a conformal parametrization of the surface around $\Phi(p)$.*

1.3.3 Proof of theorem 1.3.5: Chern moving frame method

We will subdivise the proof in several steps. It is adapted from the proof of theorem 2.9 in [?], with core ideas by S. Chern and F. Hélein (see [?]). It is interesting to point out that the aforementioned proof is written using the differential forms formalism. While it shortens the proof, it tends to hide the intricate mechanisms on the surface behind the rules of differential forms. This proof is as explicit as possible in the hope that all those nuances are made visible.

Step 1: Coulomb frames

Let us first recall that a frame on a surface is an application $\vec{e} = (\vec{e_1}, \vec{e_2}) : \Sigma \to (\mathbb{R}^3)^2$ such that for all $p \in \Sigma$, $(e_1(p), e_2(p))$ is a basis of $T_{\Phi(p)}\Phi(\Sigma)$. Such a frame is called orthonormal if, for all p, $(\vec{e_1}, \vec{e_2})(p)$ is an orthonormal basis. In a local conformal chart, one enjoys an obvious orthonormal frame: $\vec{e_1} = e^{-\lambda}\phi_x$, $\vec{e_2} =$

$e^{-\lambda}\phi_y$. From (57) one can prove:

$$\partial_x \vec{e_1} = -\lambda_y \vec{e_2} + e e^{-\lambda}\vec{n}$$
$$\partial_y \vec{e_1} = \lambda_x \vec{e_2} + f e^{-\lambda}\vec{n},$$

which yields:

$$\langle \nabla \vec{e_1}, \vec{e_2}\rangle = -\langle \vec{e_1}, \nabla \vec{e_2}\rangle = \nabla^\perp \lambda. \tag{66}$$

One can then link the conformal character of a parametrization (and its conformal factor) to a differential property of a moving frame. Conversely, let us consider ϕ a local parametrization (not necessarily conformal) of the surface on a simply connected domain Ω, denote g its induced metric, and assume there exists a local moving frame $(\vec{e_1}, \vec{e_2})$ satisfying:

{150220210831} $$\operatorname{div}_g \left(\langle \nabla_g \vec{e_1}, \vec{e_2}\rangle\right) = 0. \tag{67}$$

Such a frame is called a *Coulomb frame*. Since this proof relies on playing with the two formalisms (tensorial, and in $\mathbb{R}^3$), we will distinguish ∇_g and ∇, even on functions, despite the burdening of the notations. One should keep in mind that while $(\nabla_g)_i f = \partial_i f = (\nabla f)_i$, $(\nabla_g)^i f = (g^{-1}\nabla f)^i$.

We will show that the existence of such a Coulomb frame allows one to reparametrize the surface conformally. The sketch of the proof is rather simple:

- the Coulomb frame $(\vec{e_1}, \vec{e_2})$ in $\mathbb{R}^3$ induces a similarly peculiar frame (U, V) on the underlying Riemann surface.
- These tensor fields are associated to two duals forms u, v on the surface.
- We wish to show that by multiplying by a properly chosen function (of course built with the suspected conformal factor λ) one can write u and v as coordinates forms $d\tilde{x}$, $d\tilde{y}$, and reparametrize with these coordinates to obtain a conformal parametrization.

Applying lemma 1.2.1 to equation (67) thus ensures that there exists $\lambda \in C^\infty(M)$ such that:

$$g^{-1}\langle \nabla \vec{e_1}, \vec{e_2}\rangle = |g|^{-\frac{1}{2}}\nabla^\perp \lambda.$$

One should pay attention to the g^{-1}, necessary since lemma 1.2.1 is written with a *vector*, that is with upper indexes.

Since $\phi_x(p)$, $\phi_y(p)$ always forms a (non orthogonal) frame for the surface, one can write $(\vec{e_1}, \vec{e_2})$ in the $(\partial_i \phi)$ basis thanks to two tensors U, V:

$$\vec{e_1} = U^1 \phi_x + U^2 \phi_y = U^i \nabla_i \phi = U^i (\nabla_g)_i \phi$$
$$\vec{e_2} = V^1 \phi_x + V^2 \phi_y = V^i \nabla_i \phi = V^i (\nabla_g)_i \phi.$$

Remark 1.3.3. *The vectors U, V are simply the pullback of the vectors $\vec{e_1}$ and $\vec{e_2}$ in $\mathbb{R}^3$ by the immersion ϕ: they form their* intrinsic *counterpart. The forms $u = U_i dx^i$, $v = V_j dx^j$ are their dual forms. As announced, we seek to integrate them up to a multiplicative factor.*

One can then compute:

$$\nabla_i \vec{e_1} = (\nabla_g)_i \, \vec{e_1} = (\nabla_g)_i \left(U^p (\nabla_g)_p \phi \right) = (\nabla_g)_i U^p (\nabla_g)_p \phi + A_{ip} U^p \vec{n}.$$

This implies that:

$$\begin{aligned} \langle \nabla_i \vec{e_1}, \vec{e_2} \rangle &= \langle (\nabla_g)_i U^p (\nabla_g)_p \phi, V^q (\nabla_g)_q \phi \rangle = (\nabla_g)_i U^p V^q g_{pq} \\ &= (\nabla_g)_i U_j V^j = \left(|g|^{-\frac{1}{2}} g \nabla^\perp \lambda \right). \end{aligned} \tag{68}$$ {150220210910}

On the other hand, since $\langle \vec{e_1}, \vec{e_1} \rangle = 1$, $\langle \nabla_i \vec{e_1}, \vec{e_1} \rangle = \frac{1}{2} \nabla_i \left(\langle \vec{e_1}, \vec{e_1} \rangle \right) = 0$, which yields in a very similar fashion to (68):

$$(\nabla_g)_i U_j U^j = 0. \tag{69}$$ {150220210911}

Let us now assemble (68) and (69) with matricial notations:

$$\begin{pmatrix} (\nabla_g)_1 U_1 & (\nabla_g)_1 U_2 \\ (\nabla_g)_2 U_1 & (\nabla_g)_2 U_2 \end{pmatrix} \begin{pmatrix} U^1 & V^1 \\ U^2 & V^2 \end{pmatrix} = |g|^{-\frac{1}{2}} \begin{pmatrix} 0 & -g_{11}\lambda_y + g_{12}\lambda_x \\ 0 & -g_{12}\lambda_y + g_{22}\lambda_x \end{pmatrix}.$$

This thus implies:

$$\begin{aligned} \begin{pmatrix} (\nabla_g)_1 U_1 & (\nabla_g)_1 U_2 \\ (\nabla_g)_2 U_1 & (\nabla_g)_2 U_2 \end{pmatrix} &= \frac{|g|^{-\frac{1}{2}}}{U^1 V^2 - U^2 V^1} \begin{pmatrix} 0 & -g_{11}\lambda_y + g_{12}\lambda_x \\ 0 & -g_{12}\lambda_y + g_{22}\lambda_x \end{pmatrix} \begin{pmatrix} V^2 & -V^1 \\ -U^2 & U^1 \end{pmatrix} \\ &= \frac{|g|^{-\frac{1}{2}}}{U^1 V^2 - U^2 V^1} \begin{pmatrix} \star & U^1(-g_{11}\lambda_y + g_{12}\lambda_x) \\ -U^2(-g_{12}\lambda_y + g_{22}\lambda_x) & \star \end{pmatrix}. \end{aligned} \tag{70}$$ {150220211017}

Since $(\vec{e_1}, \vec{e_2})$ forms a direct orthonormal frame of the surface, $\vec{e_1} \times \vec{e_2} = \vec{n}$. However:

$$\vec{e_1} \times \vec{e_2} = \left(U^i \partial_i \phi \right) \times \left(V^j \partial_j \phi \right) = U^1 V^2 \phi_x \times \phi_y - U^2 V^1 \phi_x \times \phi_y = |g|^{\frac{1}{2}} (U^1 V^2 - U^2 V^1) \vec{n} = \vec{n},$$

which yields

$$|g|^{\frac{1}{2}} (U^1 V^2 - U^2 V^1) = 1. \tag{71}$$ {150220211016}

Injecting (71) into (70) yields:

$$\begin{pmatrix} (\nabla_g)_1 U_1 & (\nabla_g)_1 U_2 \\ (\nabla_g)_2 U_1 & (\nabla_g)_2 U_2 \end{pmatrix} = \begin{pmatrix} \star & -U^1(-g_{11}\lambda_y + g_{12}\lambda_x) \\ U^2(-g_{12}\lambda_y + g_{22}\lambda_x) & \star \end{pmatrix}.$$

Thus:

$$
\begin{aligned}
(\nabla_g)_1 U_2 - (\nabla_g)_2 U_1 &= \partial_1 U_2 - \Gamma_{12}^k U_k - \partial_2 U_1 + \Gamma_{21}^k U_k = \partial_x U_2 - \partial_y U_1 \\
&= U^1(-g_{11}\lambda_y + g_{12}\lambda_x) + U^2(-g_{12}\lambda_y + g_{22}\lambda_x) \\
&= -\left(U^1 g_{11} + U^2 g_{12}\right)\lambda_y + (U^1 g_{12} + U^2 g_{22})\lambda_x \\
&= -U_1\lambda_y + U_2\lambda_x,
\end{aligned}
$$

which implies:

$$
\partial_x\left(U_2 e^{-\lambda}\right) - \partial_y\left(U_1 e^{-\lambda}\right) = \operatorname{div}\begin{pmatrix} U_2 e^{-\lambda} \\ -U_1 e^{-\lambda}\end{pmatrix} = 0.
$$

We can thus find a function φ_1 such that: $\begin{pmatrix} U_2 e^{-\lambda} \\ -U_1 e^{-\lambda}\end{pmatrix} = -\nabla^\perp\varphi_1$, that is $Ue^{-\lambda} = \nabla\varphi_1$. Exchanging the roles of U and V we also find a function φ_2 such that $Ve^{-\lambda} = \nabla\varphi_2$. If we denote $\varphi = (\varphi_1, \varphi_2)$, we have thus built a local diffeomorphism (since by construction $\nabla\varphi = (U, V)e^{-\lambda}$ is of rank 2) between subsets of $\mathbb{R}^2$. We can compute its derivative:

$$
\nabla\varphi = e^{-\lambda}\begin{pmatrix} U_1 & U_2 \\ V_1 & V_2 \end{pmatrix},
$$

and that of its inverse:

$$
\nabla\varphi^{-1} = \frac{e^{\lambda\circ\varphi^{-1}}}{U_1V_2 - U_2V_1}\begin{pmatrix} V_2 & -U_2 \\ -V_1 & U_1 \end{pmatrix}.
$$

However, since $(\vec{e_1}, \vec{e_2})$ is a orthonormal family in $\mathbb{R}^3$ (and thus (U, V) an orthonormal family on (Ω, g)):

$$
\begin{cases}
V_1V^1 + V_2V^2 = 1 \\
U_1V^1 + U_2V^2 = 0 \\
V_1U^1 + V_2U^2 = 0 \\
U_1U^1 + U_2U^2 = 1,
\end{cases}
$$

which notably implies

$$
\begin{cases}
(U_1V_2 - U_2V_1)U^1 = V_2 \\
(U_1V_2 - U_2V_1)U^2 = -V_1 \\
(U_1V_2 - U_2V_1)V^1 = -U_2 \\
(U_1V_2 - U_2V_1)V^2 = U_1,
\end{cases}
$$

which yields:

$$
\nabla\varphi^{-1} = e^{\lambda\circ\varphi^{-1}}\begin{pmatrix} U^1 & V^1 \\ U^2 & V^2 \end{pmatrix}.
$$

Thus, if we now consider $\tilde{\phi} = \phi \circ \varphi^{-1}$, one has:

$$\begin{aligned}
\tilde{\phi}_x &= e^{\lambda\circ\varphi^{-1}} U^1 \phi_x \circ \varphi^{-1} + e^{\lambda\circ\varphi^{-1}} U^2 \phi_y \circ \varphi^{-1} = e^{\lambda\circ\varphi^{-1}} \vec{e_1} \\
\tilde{\phi}_y &= e^{\lambda\circ\varphi^{-1}} V^1 \phi_x \circ \varphi^{-1} + e^{\lambda\circ\varphi^{-1}} V^2 \phi_y \circ \varphi^{-1} = e^{\lambda\circ\varphi^{-1}} \vec{e_2}.
\end{aligned}$$

Which shows that $\tilde{\phi}$ is a conformal parametrisation defined on $\varphi^{-1}(\Omega)$, and thus, locally on a disk around any point of Ω.

Building a Coulomb frame: the Coulomb gauge

We have thus shown that building a Coulomb frame is enough to ensure that one can reparametrize the surface conformally. The next step is clear: given a parametrization ϕ defined on a neighborhood $\Omega \subset \mathbb{R}^2$ and inducing a metric g, how can we build a Coulomb frame?

One can build an orthonormal frame for ϕ by applying an orthonormalization procedure on the frame (ϕ_x, ϕ_y). There thus exists an orthonormal frame $(\vec{f_1}, \vec{f_2})$. Any other orthonormal frame $e_\theta = (\vec{e_{1\theta}}, \vec{e_{2\theta}})$ is thus written:

$$\begin{pmatrix} \vec{e_{1\theta}} \\ \vec{e_{2\theta}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{f_1} \\ \vec{f_2} \end{pmatrix}.$$

The Coulomb term is thus modified in the following manner:

$$\langle \vec{e_{2\theta}}, \nabla \vec{e_{1\theta}} \rangle = \nabla\theta + \langle \vec{f_2}, \nabla \vec{f_1} \rangle.$$

Finding a Coulomb frame working on (Ω, g) thus amounts to finding θ such that:

$$\operatorname{div}_g \left(\nabla\theta + \langle \vec{f_2}, \nabla \vec{f_1} \rangle \right) = 0.$$

One of the techniques to find a solution to this equation is to consider it as the Euler-Lagrange equation of an energy, of which one merely has to find a critical point (usually a minimum). Determining the proper energy can be a tricky affair which requires intuition, flair and experience. In our case, for equations of the form $\operatorname{div}(\nabla u + f)$, an integrand $|\nabla u + f|^2$ is a good start. Let us thus introduce:

$$E(\theta) = \int_\Omega \left| \nabla\theta + \langle \vec{f_2}, \nabla \vec{f_1} \rangle \right|_g^2 d\mathrm{vol}_g.$$

Let us compute its Euler-Lagrange equation: for any $\mu \in W^{1,2}(\Omega)$ and $t \in \mathbb{R}$:

$$\begin{aligned}
E(\theta + t\mu) &= \int_\Omega \left| \nabla\theta + \langle \vec{f_2}, \nabla \vec{f_1} \rangle + t\nabla\mu \right|_g^2 d\mathrm{vol}_g \\
&= \int_\Omega \left| \nabla\theta + \langle \vec{f_2}, \nabla \vec{f_1} \rangle \right|_g^2 d\mathrm{vol}_g + 2t \int_\Omega \left(\nabla\theta + \langle \vec{f_2}, \nabla \vec{f_1} \rangle \right) . \nabla\mu d\mathrm{vol}_g + O(t^2).
\end{aligned}$$

Any critical point, θ must then satisfy $\left.\frac{dE(\theta+t\mu)}{dt}\right|_{t=0} = 0$, that is:

$$\forall\varphi \in W^{1,2}(\Omega) \quad \int_\Omega \left(\nabla\theta + \langle \vec{f}_2, \nabla \vec{f}_1\rangle\right).\nabla\mu d\text{vol}_g = 0.$$

Thanks to the Green formula for div_g (see remark 1.2.5), one can see this is merely the weak formulation of the Euler-Lagrange equation of E:

{1602202117339}

$$\begin{cases} \text{div}_g \left(\nabla\theta + \langle \vec{f}_2, \nabla \vec{f}_1\rangle\right) = 0 \\ \partial_\nu\theta + \langle \vec{f}_2, \partial_\nu \vec{f}_1\rangle = 0. \end{cases} \tag{72}$$

Classically, we will thus find a Couloumb frame by minimizing the energy E. The procedure is standard:

- show that the functional is bounded from below, and thus that the infimum exists and is finite,
- show that a bound on the functional implies a bound on the $W^{1,2}$ norm (we say that the functional is coercive)
- consider a minimizing sequence ϕ_n, since $\|\phi\|_{W^{1,2}}$ is bounded. By Riesz theorem, $\phi_n \to \phi$ weakly in $W^{1,2}$ up to extraction
- Show that ϕ minimizes the functional (usually because the weak limit can only shed energy, not gain any). Do not hesitate to use the Sobolev embeddings to obtain strong convergences in weaker spaces

In this case, since the equation is expressed with g operators, we will have to navigate between g-Sobolev spaces and classical Sobolev spaces. To that end, let us assume that g satisfies (17).

Let us apply this minimizing procedure:

- Since $E(\theta) = \int_\Omega \left|\nabla\theta + \langle \vec{f}_2, \nabla \vec{f}_1\rangle\right|_g^2 d\text{vol}_g$, the energy is positive, and thus bounded from below.
- Let us denote $m_0 = \inf_{\theta\in W^{1,2}(\Omega)} E(\theta)$ and consider $\theta_n \in W^{1,2}(\Omega)$ a minimizing sequence. Since for any $c \in \mathbb{R}$, $E(\theta + c) = E(\theta)$, one can assume the θ_n are of null average. Thanks to (17):

$$\begin{aligned}\|\nabla\theta_n\|_{L^2(\Omega)} &\leq C(g)\|\nabla\theta_n\|_{L^2_g(\Omega)} \leq C(g)\sqrt{\int_\Omega |\nabla\theta_n|^2 d\text{vol}_g} \\ &\leq C(g)\left[\sqrt{\int_\Omega \left|\langle \vec{f}_2, \nabla \vec{f}_1\rangle\right|^2 d\text{vol}_g} + \sqrt{E(\theta_n)}\right] \leq M < \infty.\end{aligned}$$

Using the Poincaré-Wirtinger inequality, since the θ_n are assumed to be of null average, we have:

$$\begin{aligned}\|\theta_n\|_{W^{1,2}(\Omega)} &\leq \|\theta_n\|_{L^2(\Omega)} + \|\nabla\theta_n\|_{L^2(\Omega)}\\ &\leq C(\Omega)\,\|\nabla\theta_n\|_{L^2(\Omega)} + \|\nabla\theta_n\|_{L^2(\Omega)}\\ &\leq C(\Omega, M, g).\end{aligned}$$

By weak compactness inside the Sobolev spaces, there exists a minimizer $\theta \in W^{1,2}(\Omega)$ of E such that $\theta_n \to \theta$ weakly in $W^{1,2}(\Omega)$.

- Let us show that θ is a weak minimizer of E:

$$\begin{aligned}E(\theta) = \int_\Omega \left|\nabla\theta + \langle \vec{f}_2, \nabla\vec{f}_1\rangle\right|_g^2 d\mathrm{vol}_g = \int_\Omega |\nabla\theta|_g^2\, d\mathrm{vol}_g + 2\int_\Omega \nabla\theta.\langle \vec{f}_2, \nabla\vec{f}_1\rangle d\mathrm{vol}_g\\ + \int_\Omega \left|\vec{f}_2, \nabla\vec{f}_1\rangle\right|_g^2 d\mathrm{vol}_g.\end{aligned}$$

Since $\nabla\theta_n \to \nabla\theta$ weakly in $L^2(\Omega)$, $\int_\Omega \nabla\theta_n.\langle \vec{f}_2, \nabla\vec{f}_1\rangle d\mathrm{vol}_g \to \int_\Omega \nabla\theta.\langle \vec{f}_2, \nabla\vec{f}_1\rangle d\mathrm{vol}_g$. In addition:

$$\begin{aligned}\int_\Omega |\nabla\theta_n|_g^2\, d\mathrm{vol}_g = \int_\Omega |\nabla(\theta_n - \nabla\theta)|_g^2\, d\mathrm{vol}_g +\\ 2\int_\Omega \nabla\theta_n.\nabla\theta d\mathrm{vol}_g - \int_\Omega |\nabla\theta|_g^2\, d\mathrm{vol}_g.\end{aligned}$$

Once more, since $\nabla\theta_n \to \nabla\theta$ weakly in $L^2(\Omega)$, $2\int_\Omega \nabla\theta_n.\nabla\theta d\mathrm{vol}_g - \int_\Omega |\nabla\theta|_g^2\, d\mathrm{vol}_g \to \int_\Omega |\nabla\theta|_g^2\, d\mathrm{vol}_g$.
Thus:

$$\liminf \int_\Omega |\nabla\theta_n|_g^2\, d\mathrm{vol}_g \geq \int_\Omega |\nabla\theta|_g^2\, d\mathrm{vol}_g.$$

This implies that:

$$E(\theta) \leq \liminf E(\theta_n) = m_0,$$

and thus that θ is a minimizer of E.It must then satisfy (72), and e_θ is a Coulomb frame which induces a conformal parametrization.

Remark 1.3.4. *This proof is a posterchild for many geometric analysis results: a geometric study of the problem, reducing it into a PDE problem, solved using a classical PDE technique (here a minimization procedure).*

Remark 1.3.5. *This proof, in the spirit of this whole chapter, did not care much about the regularity of the functions: every involved function and frame was assumed smooth. It is interesting however to look at the proof and realize that every* *major tool employed (Poincaré lemma, minimization, Calderon-Zygmund) can be applied to weak functions. Provided there exists a first frame* $(f_1, f_2) \in L^2$*, one can check that the proof works for an immersion* $\phi \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$*. For a more regular immersion, since* (72) *is an elliptic equation in divergence form, applying classical Calderon-Zygmund theory (see theorem 9.11 in [?], or theorems 13.9-13.11 of [?]) ensures the regularity of* θ*, and thus of* e_θ *and of the induced conformal parametrization.*

Remark 1.3.6. *This proof also provides a perspective on what a conformal parametrization fundamentally is. To describe a surface, one has the choice among all the available parametrizations. To simplify, we choose a* gauge condition *which reduces the* redundant degrees of freedom *(that is: all the parametrizations that describe the same object in the end). Here this gauge condition is the Coulomb condition* (67)*, which is the Euler-Lagrange equation of a Lagrangian that one can see as the tangent energy of the frame. Indeed, if* $e = (\vec{e_1}, \vec{e_2})$ *is a frame, one has:*

$$\begin{aligned} \nabla \vec{e_1} &= \langle \nabla \vec{e_1}, \vec{e_2} \rangle \vec{e_2} + \textit{normal part} \\ \nabla \vec{e_2} &= \langle \nabla \vec{e_2}, \vec{e_1} \rangle \vec{e_1} + \textit{normal part} = -\langle \nabla \vec{e_1}, \vec{e_1} \rangle \vec{e_1} + \textit{normal part.} \end{aligned}$$

Thus among all the frames, we chose the one with the less redundant energy (since we are interested in the extrinsic quantities), yielding a Coulomb frame that we can integrate into a conformal parametrization. Taking a simpler parametrization would impose conditions on the conformal factor, and thus, through the Liouville equation, impact the geometry, and not just the description of the geometry. Notice that the only degree of liberty left is adding a constant to θ*, which amounts to applying a constant rotation in* $\mathbb{R}^3$ *to the frame. This degree is not problematic as it is easily apprehended, and blocked.*

Remark 1.3.7. *This result* cannot *be extended beyond the dimension* 2*. It is a good exercise to go through it and keep track of all the times we used the dimensional hypothesis.*

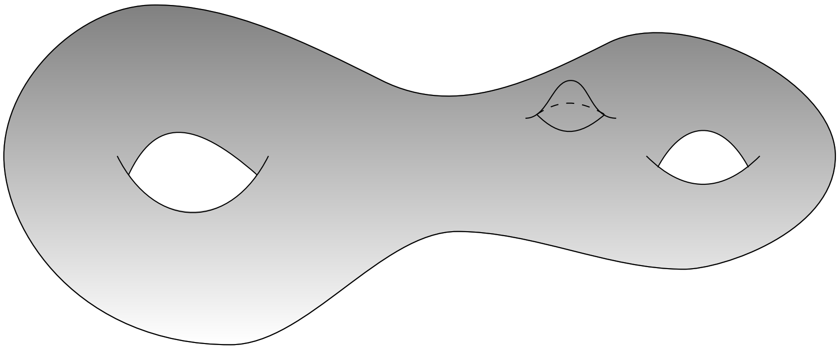

Figure 4: The perturbation of a surface {figureperturbati

2 The minimal problem

2.1 Minimal surfaces

2.1.1 Minimizing surface {080320211548}

The theory of minimal surfaces is linked with the study of elastic sheets: that is a sheet of a material with elastic properties, trying to span the least space possible. A prominent example is given by soap films, which behave as 2-dimensional elastica adopting the shape with the least area.

To modelize this situation, let us consider a surface Σ (possibly with a boundary) and an immersion $\Phi : \Sigma \to \mathbb{R}^3$. If g denotes the induced metric, the area spanned by Φ is:

$$\mathcal{A}(\Phi) = \int_\Sigma d\mathrm{vol}_g.$$

If we consider a smooth compactly supported perturbation of Φ (see figure 4): $\Phi_\varepsilon = \Phi + \varepsilon\Psi$ with $\Psi \in C_c^\infty(\Sigma)$, then, in any local chart, one can compute:

$$\begin{aligned}
&\nabla_i \Phi_\varepsilon = \nabla_i \Phi + \varepsilon \nabla_i \Psi \\
&(g_\varepsilon)_{ij} = \langle \nabla_i \Phi_\varepsilon, \nabla_j \Phi_\varepsilon \rangle = \langle \nabla_i \Phi, \nabla_j \Phi \rangle + \varepsilon \left(\langle \nabla_i \Phi, \nabla_j \Psi \rangle + \langle \nabla_j \Phi, \nabla_i \Psi \rangle \right) + o(\varepsilon).
\end{aligned}$$

If we denote $\delta = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}$, one then has:

$$\delta\left(|g_\varepsilon|^{\frac{1}{2}}\right) = \frac{1}{2|g|^{\frac{1}{2}}}\delta\left(|g_\varepsilon|\right) = \frac{|g|^{\frac{1}{2}}}{2}\mathrm{Tr}_g\left(g^{-1}\delta g_\varepsilon\right) = |g|^{\frac{1}{2}}\langle\nabla^i\Phi, \nabla_i\Psi\rangle,$$

which yields:

$$\begin{aligned}\delta\left(d\mathrm{vol}_g\right) &= \langle\nabla\Phi.\nabla\Psi\rangle d\mathrm{vol}_g = \left[\mathrm{div}_g\left(\langle\Psi, \nabla\Phi\rangle\right) - \langle\Psi, \Delta_g\Phi\rangle\right] d\mathrm{vol}_g \\ &= \left[\mathrm{div}_g\left(\langle\Psi, \nabla\Phi\rangle\right) - 2H\langle\Psi, \vec{n}\rangle\right] d\mathrm{vol}_g\end{aligned}$$

In conclusion,

$$\delta\left(\mathcal{A}(\Phi_\varepsilon)\right) = \int_\Sigma \delta\left(d\mathrm{vol}_g\right) = \int_\Sigma \left[\mathrm{div}_g\left(\langle\Psi, \nabla\Phi\rangle\right) - 2H\langle\Psi, \vec{n}\rangle\right] d\mathrm{vol}_g.$$

With a compactly supported perturbation Ψ (or on a manifold without a boundary) one can integrate by parts the first term and conclude

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$$\delta\left(\mathcal{A}(\Phi_\varepsilon)\right) = -2\int_\Sigma H\langle\Psi, \vec{n}\rangle d\mathrm{vol}_g = -\int_\Sigma \langle\Psi, \Delta_g\Phi\rangle d\mathrm{vol}_g. \tag{73}$$

For any critical point Φ of $\mathcal{A}$ (and *a fortiori* for minimizers), taking $\Psi = H\eta\vec{n}$ with η a bump function around a point yields $H\vec{n} = \Delta_g\Phi = 0$. We can then introduce the definition of a minimal surface:

Definition 2.1.1. *An immersed surface in $\mathbb{R}^3$ is a* minimal surface *if one of the following equivalent conditions is satisfied:*

1. *$H = 0$,*
2. *The immersed surface is a critical point of the area for any compactly supported variation.*

Despite the intuitions that led us to this definition, a minimal surface does not necessarily realize a minimum of the area globally. For instance, if we consider two superposed circles, one can find two minimal surfaces leaning on these two disks: the catenoïd, and the two disjoint flat disks. Which one has minimal area depends on the distance between the disks. For two close circles, the catenoïd will use less surface, while if they are very distant from one another, the disks will become preferable.

This can actually be observed: as mentioned above, soap films behave as elastica: they will form minimizing surfaces. Immerging two circular iron wires in soap water allows a soap film to lie on these two circles. If the circles are close enough, the soap will form a soap catenoïd. Taking them apart will stretch this catenoïd to the breaking point, where it will then become two disks.

Remark 2.1.1. *A very insightful comparison can be made with geodesics of a surface: geodesic curves are locally minimizing the length (and thus solve the corresponding Euler-Lagrange equation), but not globally. On a euclidean sphere, the geodesics are the equatorial circles, and any pair of points can be linked by two geodesics (the two "sides" of the equatorial circle), one shorter than the other, one minimizing the length, the other only a critical point.*

Knowing when a minimal surface is a minimizer is actually a rather complex problem, and requires studying the functional a bit more in depth. In this particular case, the graph formulation is particularly useful:

{170220211725}

Theorem 2.1.1. *Let Ω be a smooth convex domain of $\mathbb{R}^2$ and $\phi(x,y) = (x, y, u(x,y))$ a $C^\infty(\bar{\Omega})$ minimal graph over Ω. Then, for any immersed surface $\Psi(\tilde{\Sigma})$ such that $\partial\Psi(\tilde{\Sigma}) = \partial\phi(\Omega)$, then:*

$$\mathcal{A}(\phi(\Omega)) \leq \mathcal{A}(\Psi(\tilde{\Sigma})).$$

Proof. To prove the result for any immersed surface, we will foliate the cylinder $\Omega \times \mathbb{R}$ with translations of the minimal graph:

$$\phi_t := (x, y, u(x,y) + t).$$

For any $t \in \mathbb{R}$, the normal to $\phi_t(\Omega)$ coincides with the normal to $\phi(\Omega)$: $\vec{n}_t(x,y) = \vec{n}(x,y) = \frac{(-u_x, -u_y, 1)}{\sqrt{1+|\nabla u|^2}}$. Thus, since any point $(x,y,z) \in \Omega \times \mathbb{R}$ can be written as a point on one of the minimal graphs of the foliation $(x, y, u(x,y)+t)$, we can define a normal field on the cylinder:

$$X(x,y,z) = \frac{(-u_x, -u_y, 1)}{\sqrt{1+|\nabla u|^2}}.$$

Since ϕ is minimal, i.e. of null mean curvature, using (55) one has:

$$\mathrm{div}X = X_x + X_y + X_z = -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -H = 0.$$

Step 1: If $\Psi(\tilde{\Sigma}) \subset \Omega \times \mathbb{R}$
Let us consider such a surface, and the volume V between $\phi(\Omega)$ and $\Psi(\tilde{\Sigma})$ inside the cylinder. Then, using Stokes theorem:

$$\int_{\phi(\Omega)} \langle X, \vec{n}\rangle - \int_{\Psi(\tilde{\Sigma})} \langle X, \vec{n}_\Psi\rangle = \int_V \mathrm{div}(X) = 0.$$

However, since $X = \vec{n}$ on $\phi(\Omega)$, one has: $\langle X, \vec{n}\rangle = 1$, and $\langle X, \vec{n}_\Psi\rangle \leq 1$. Thus:

$$\mathcal{A}(\phi(\Omega)) = \int_\Omega \mathrm{dvol}_g = \int_{\phi(\Omega)} \langle X, \vec{n}\rangle = \int_{\Psi(\tilde{\Sigma})} \langle X, \vec{n}_\Psi\rangle \leq \int_{\tilde{\Sigma}} \mathrm{dvol}_{g_\Psi} \leq \mathcal{A}(\Psi(\tilde{\Sigma})).$$

Which concludes the proof.
Step 2: If $\Psi(\tilde{\Sigma})$ is not contained inside the cylinder
If we denote $\pi_{\mathcal{C}}$ the projection on the cylinder, it is easy to see that $\pi_{\mathcal{C}}$ is a contraction-that is Lipschitz with constant strictly lower than 1. Thus, $\mathcal{A}(\pi_{\mathcal{C}}(\Psi(\tilde{\Sigma}))) \leq \mathcal{A}(\Psi(\tilde{\Sigma}))$, and applying the previous step to $\pi_{\mathcal{C}}(\Psi(\tilde{\Sigma}))$ concludes the proof. □

This proof is based on ideas by H. Schwarz [**?**] (vol 1, p 223-269) which can also be found in [**?**] (p 82-84), and compiled in (french) lecture notes by O. Druet ([**?**]). Another proof can be found in [**?**] (p 2-3).

Theorem 2.1.1 must reinforce the idea that minimal does not mean minimizing. Determining whether or not a minimal surface is minimizing requires advanced techniques and, as evidenced by the proof, is deeply tied to the idea of foliating the ambiant space by minimal surfaces. The minimal terminology is however justified:

Theorem 2.1.2. *If $\Phi : \Sigma \to \mathbb{R}^3$ is an immersed surface, then for any $p \in \Sigma$, there exists a neighborhood of p such that $\phi(\Sigma)$ is area-minimizing among all surfaces with the same boundary:*

$$\exists p \in \Omega \quad \mathcal{A}(\Phi(\Omega)) \leq \mathcal{A}(\Psi(\tilde{\Sigma})) \quad \forall\, \partial\Psi(\tilde{\Sigma}) = \partial\phi(\Omega).$$

Proof. Taking any $p \in \Sigma$, theorem 1.3.2 ensures there exists a local graph parametrization on a disk while 2.1.1 shows it is minimizing on this disk. □

In other words: minimal surfaces are local minimizers.

Remark 2.1.2. *Theorem 2.1.1 is a nice display of the main interest of the graph representation: they are easier to grasp geometrically, and thus easier to build geometric reasonings on. The foliation idea is another pivotal concept in geometric analysis which can be found in many contexts and many domains.*

2.1.2 Analysis of minimal surfaces
{220220211137}

Study in a local conformal chart

From (26), one can quickly see that any point of a minimal surface is a saddle point. In addition, thanks to (61) and theorem 1.3.5, we can conclude that any minimal surface can be locally parametrized by an immersion $\phi : \mathbb{D} \to \mathbb{R}^3$ satisfying

$$\begin{aligned}&|\phi_x|^2 - |\phi_y|^2 = \langle \phi_x, \phi_y \rangle = 0\\ &\Delta\phi = 0.\end{aligned}$$

This implies that the conformal parametrizations of minimal surfaces are harmonic function into $\mathbb{R}^3$. Harmonic functions are truly remarkable, and we will recall some of their properties, and apply them to minimal surfaces in a conformal parametrization. First of all is the mean value property:

Lemma 2.1.1. *Let Ω be an open subset of $\mathbb{R}^n$. We say that a function u satisfies the mean value property if, for all $x \in \Omega$ and open balls $B_R(x) \subset \Omega$, one has:*

$$u(x) = \frac{1}{|B_R(x)|} \int_{B_R(x)} u(y)dy = \frac{1}{|\partial B_R(x)|} \int_{\partial B_R(x)} u(y)d\sigma(y).$$

Then:

$$u \in C^2(\Omega) \text{ and } \Delta u = 0 \Leftrightarrow u \in C^0(\Omega) \text{ and satisfies the mean value property.}$$

Proof. Let us first prove that the two mean values coïncide when u satisfies the mean value property.

Indeed, since $|\partial B_R(x)| = \omega_{n-1}R^{n-1}$ with ω_{n-1} being the volume of the unit sphere in $\mathbb{R}^n$, if

$$u(x) = \frac{1}{|\partial B_R(x)|} \int_{\partial B_R(x)} u(y)d\sigma(y),$$

then integrating on r yields:

$$\frac{R^n}{n}u(x) = \int_0^R r^{n-1}u(x)dr = \frac{1}{\omega_{n-1}} \int_0^R \left(\int_{\partial B_R(x)} u(y)d\sigma(y) \right) = \frac{1}{\omega_{n-1}} \int_{B_R(x)} u(y)dy,$$

where the last equality was obtained by writing the integral in polar coordinates. Recognizing $|B_R(x)| = \frac{\omega_{n-1}R^n}{n}$ yields the first implication.

Conversely, if $R^n u(x) = \frac{n}{\omega_{n-1}} \int_{B_R(x)} u(y)dy$, differentiating this equality one can check that:

$$nR^{n-1}u(x) = \frac{n}{\omega_{n-1}} \int_{\partial B_R(x)} u(y)d\sigma_y,$$

which yields the second implication, and thus the equivalence between the two mean value conditions.

Let now u be a C^2 function, $x \in \Omega$ and $\bar{u}(s) = \frac{1}{\omega_{n-1}s^{n-1}} \int_{B_s(x)} u(y)d\sigma_y$ for s small enough. One can then check that $\bar{u}$ is differentiable and that its derivative is

$$\bar{u}'(s) = \frac{1}{\omega_{n-1}s^{n-1}} \int_{\partial B_R(x)} u_r(y)d\sigma_y = \frac{1}{\omega_{n-1}s^{n-1}} \int_{\partial B_R(x)} \partial_\nu u(y)d\sigma_y.$$

Now integrating by parts yields

$$\bar{u}'(s) = \frac{1}{\omega_{n-1}s^{n-1}} \int_{\partial B_R(x)} \partial_\nu u(y)d\sigma_y = \frac{1}{\omega_{n-1}s^{n-1}} \int_{B_R(x)} \Delta u(y)dy, \tag{74}$$ {180220211204}

Thus if u is harmonic, $\bar{u}' = 0$ which implies that $\bar{u}(s) = \lim_{r\to 0} \bar{u}(s) = u(x)$ for all s, which shows that u satisfies the mean value property.

Now, assume that u is a continuous function satisfying the mean value property. Let $\varphi \in C_c^\infty(B_1(0))$ be a smooth radial bump such that $\int_{B_0(1)} \varphi(x)dx = 1$. Let $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi\left(\frac{x}{\varepsilon}\right)$ and $u_\varepsilon(x) := u \star \varphi_\varepsilon(x) = \int_\Omega u(y)\varphi_\varepsilon(y-x)dy$. Then:

$$
\begin{aligned}
u_\varepsilon(x) &= \int_\Omega u(y)\varphi_\varepsilon(y-x)dy = \int_{x+\Omega} u(x+y)\varphi_\varepsilon(y)dy\\
&= \varepsilon^{-n}\int_{B_\varepsilon(y)} u(x+y)\varphi\left(\frac{y}{\varepsilon}\right)dy\\
&= \int_{B_1(0)} u(x+\varepsilon y)\varphi(y)dy\\
&= \int_0^1 \left(\int_{\partial B_r(0)} u(x+\varepsilon y)\varphi(y)d\sigma_y\right)dr\\
&= \int_0^1 \varphi(r)\left(\int_{\partial B_r(0)} u(x+\varepsilon y)d\sigma_y\right)dr\\
&= \int_0^1 \varphi(r)\left(\int_{\partial B_\varepsilon(x)} u(y)d\sigma_y\right)dr\\
&= u(x)\omega_{n-1}\int_0^1 r^{n-1}\varphi(r)dr = u(x)\int_{B_1(0)}\varphi(y)dy = u(x).
\end{aligned}
$$

Now, since u_ε is smooth (as the convolution of a smooth kernel and a L^1_{loc} function), u is also smooth. This implies that (74) stands, and since $\bar{u}$ is a constant function, $\Delta u = 0$, which concludes the proof. $\square$

Notably, this lemma implies that any harmonic function is immediately smooth. In other words, any immersed minimal surface is smooth. Thanks to the elliptic regularity theory this can be extended to the notion of weak immersions, but we will come back to it. We can push this regularity even further:

Proposition 2.1.3. *Let u be a harmonic function on $B_R(0) \subset \mathbb{R}^n$. Then, for any multi-index α of length $|\alpha| = k$:*

{180220211319}
$$
|\partial_\alpha u(0)| \le \frac{n^k e^{k-1} k!}{R^k}\left(\sup_{B_R(0)} |u|\right). \tag{75}
$$

Thus, any harmonic function is analytic.

Proof. If u is harmonic, $\partial_i \Delta u = 0 = \Delta \partial_i u$, and then $\partial_i u$ is harmonic. By induction any $\partial_\alpha u$ is also harmonic and satisfies the mean value property:

$$
\partial_i u(0) = \frac{n}{\omega_{n-1} r^n}\int_{B_r(0)} \partial_i u(y)dy = \frac{n}{\omega_{n-1} r^n}\int_{\partial B_0(r)} u(y)\nu_i d\sigma_y
$$

with an integration by parts. Thus, taking $r \to R$:

$$|\partial_i u(0)| \leq \frac{n}{\omega_{n-1}R^n}\left(\sup_{B_R(0)}|u|\right)\omega_{n-1}R^{n-1} = \frac{n}{R}\left(\sup_{B_R(0)}|u|\right).$$

Similarly, for any multi index α and $r < R$:

$$|\partial_i\partial_\alpha u(0)| \leq \frac{n}{r}\left(\sup_{B_r(0)}|\partial_\alpha u|\right). \tag{76}$$ {180220211310}

Thus, if we assume that the proposition stands for k, one has, by applying it to any $u(x+.)$ on $B_{R-r}(x)$ with $|x| = r$, we have:

$$|\partial_\alpha u(x)| \leq \frac{n^k e^{k-1} k!}{(R-r)^k}\left(\sup_{B_R(0)}|u|\right).$$

Thus, applying (76) on a ball of radius r_0:

$$\begin{aligned}|\partial_i\partial_\alpha u(0)| \leq \frac{n}{r}\left(\sup_{B_r(0)}|\partial_\alpha u|\right) &\leq \frac{n}{r_0}\frac{n^k e^{k-1} k!}{(R-r_0)^k}\left(\sup_{B_R(0)}|u|\right)\\ &\leq \frac{n^{k+1}e^{k-1}k!}{R^{k+1}}\frac{(k+1)^{k+1}}{k^k}\left(\sup_{B_R(0)}|u|\right)\\ &\leq \frac{n^{k+1}e^k(k+1)!}{R^{k+1}}\left(\sup_{B_R(0)}|u|\right),\end{aligned}$$

by taking $r_0 = \frac{1}{k+1}R$. This concludes the proof by induction. □

A consequence of this proposition is the Liouville theorem:

Corollary 2.1.1. *Any bounded harmonic function defined on $\mathbb{R}^n$ is constant.*

Proof. Taking $k = 1$ and $R \to \infty$ in (75) yields $|\nabla u|(0) = 0$. Translating the domain of definition extends it to any $x \in \mathbb{R}^n$, which shows the result. □

This translates directly to surfaces, assuming that they are parametrized conformally over a plane:

{190220210925}

Corollary 2.1.2. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ be a conformal parametrization of a minimal surface. Then $\phi(\mathbb{R}^2)$ is not bounded.*

We will also say that $\Phi(\mathbb{R})$ is not compact. That the surface is parametrized over a plane is not without topological consequences, and limits the range of such a result.

Another result on harmonic maps one can hope to extend to minimal surfaces is the maximum principle.

{1802202110

Theorem 2.1.4. *Let $\Omega \subset \mathbb{R}^2$ be a connected domain and assume that $u \in C^2(\Omega)$ is a real-valued harmonic function. If u attains its supremum/infimum in Ω, then u is constant.*

Proof. The proof relies on the mean-value equality: if $x \in \Omega$ is a maximizer for u, then one can find a small disk $B_r(x) \subset \Omega$ on which one has:

{180220211001}
$$\sup_{\Omega} u = u(x) = \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)dy \leq \frac{1}{|B_r(x)|}\int_{B_r(x)} u(x)dy \leq u(x), \tag{77}$$

since by definition $u(y) \leq \sup_\Omega u = u(x)$. We are thus in the equality case of (77), which implies that $u(x) = u(y)\, \forall y \in B_r(x)$. The set $\{y \text{ s.t. } u(y) = u(x)\}$ is thus open, and closed (since u is continuous). Since Ω is connected, $u = u(x)$ on the whole of Ω. Working with $-u$ yields the result for the infimum. $\square$

Thus for a harmonic map, the extrema are reached on the boundary, not on the inside. However, it is hard to apply such a result to parametrized surfaces. Indeed, the parametrizations take value in $\mathbb{R}^3$, not $\mathbb{R}$. One can apply it to the coordinates in $\mathbb{R}^3$ and deduce a few results on conformally parametrized minimal surfaces, but these coordinates are arbitrary (one could change them by rotating), which makes interpreting geometrically these results somewhat awkward. Still, one can obtain a uniqueness result for conformal parametrizations:

{180220211904}

Proposition 2.1.5. *Let Ω be a connected domain of $\mathbb{R}^2$ and ϕ, ψ two conformal parametrizations of minimal surfaces such that $\phi_{|\partial\Omega} = \psi_{|\partial\Omega}$. Then $\phi = \psi$.*

With a graph formulation, the geometric interpretation is much easier. If, in this case the operator is not the Laplacian, it is *elliptic.*

Definition 2.1.2. *Let Ω be a smooth domain of $\mathbb{R}^n$. The operator $L : C^2(\Omega) \to C^0(\Omega)$ defined by $Lu = a^{ij}\partial_{ij}u + b^i\partial_i u + cu$ is uniformly elliptic if there exists $\lambda > 0$ such that, for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, $a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$.*

Example 2.1.1. *Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and $L_u\phi = \operatorname{div}\left(\frac{\nabla\phi}{\sqrt{1+|\nabla u|^2}}\right)$. Then $a^{ij} = \frac{\delta_{ij}}{\sqrt{1+|\nabla u|^2}}$, and*

$$\frac{\delta_{ij}}{\sqrt{1+|\nabla u|^2}}\xi^i\xi^j \geq \frac{|\xi|^2}{\sqrt{1+\|\nabla u\|_\infty^2}}.$$

Thus L_u is uniformly elliptic.

A graph parametrization ϕ over a compact domain Ω can then be written as $L_\phi \phi = 0$. It is thus a solution of a uniformly elliptic equation.

These elliptic operators are "almost" Laplacian, or close enough to allow for maximum principles.

Theorem 2.1.6. *Weak Maximum Principle Let $u \in C^2(\Omega) \cap C^0\left(\bar{\Omega}\right)$ and L a uniformly elliptic operator with $c(x) \leq 0$ in Ω, a, b and $c \in C^0(\bar{\Omega})$ such that $Lu \geq 0$. If $\max_{\bar{\Omega}} u \geq 0$, then u reaches its maximum on $\partial\Omega$.* {230220210948}

Proof. If we first assume that the inequality $Lu > 0$ is strict, and assume by contradiction that the maximum is reached for $x_0 \in \Omega$, at this point $\nabla u(x) = 0$, while $\nabla^2 u(x)$ is semi definite negative. Since L is uniformly elliptic (a^{ij}) is semi definite positive, and thus $a^{ij}\partial_{ij} u \leq 0$. With the sign assumptions on c and u, one has $Lu(x) \geq 0$, which contradicts the assumption.

If the inequality is no longer strict, considering $u_\varepsilon(x) = u + \varepsilon e^{\alpha x_1}$, one has $Lu_\varepsilon = Lu + \varepsilon e^{\alpha x_1}\left(\alpha^2 a^{11} + \alpha b^1 + c\right)$. Since $\lambda > 0$, there exists $\alpha > 0$ big enough such that $Lu_\varepsilon > 0$. the result then stands for u_ε. One concludes:

$$\max_{\bar{\Omega}} u \leq \max_{\bar{\Omega}} u_\varepsilon \leq \max_{\partial\Omega} u + \varepsilon \max_{\partial\Omega} e^{\alpha x_1}.$$

Taking $\varepsilon \to 0$ yields the result. □

Since the mean curvature operator for graphs is a uniformly elliptic operator, one can deduce a maximum principle, with a geometric application:

{190220211018}

Proposition 2.1.7. *Let Ω be a bounded domain of $\mathbb{R}^n$. Let $u \in C^2(\bar{\Omega})$ be a solution of*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0.$$

Then

$$\min_{\partial\Omega} u \leq u \leq \max_{\partial\Omega} u.$$

In particular, if ϕ is the graph of a minimal surface, it reaches its highest and lowest point on its boundary.

One can improve the maximum into a comparison principle for the mean curvature operator for graphs:

{180220211945}

Theorem 2.1.8. *Let $\Omega \subset \mathbb{R}^n$ a bounded domain, and let us denote L the mean curvature operator for graphs on Ω. If $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy*

$$L(u) \geq L(v) \text{ in } \Omega \text{ and } u \leq v \text{ on } \partial\Omega,$$

Then $u \leq v$ in Ω.

In particular, if ϕ and ψ are two minimal graphs over Ω such that $\phi_{|\partial\Omega} \leq \psi_{|\partial\Omega}$, then $\phi(\Omega)$ is below $\psi(\Omega)$.

Proof. The second part of the theorem follows directly from the first. To prove the latter, let us set $w = u - v$ and $\kappa(f) = \sqrt{1 + |\nabla f|^2}$. Then:

$$\begin{aligned} L(u) - L(v) &= \operatorname{div}\left(\frac{\nabla u}{\kappa(u)} - \frac{\nabla v}{\kappa(v)}\right) \\ &= \operatorname{div}\left(\frac{\nabla w}{\kappa(u)} + \left(\frac{1}{\kappa(u)} - \frac{1}{\kappa(v)}\right)\nabla v\right) \\ &= \operatorname{div}\left(\frac{\nabla w}{\kappa(u)} + \frac{\kappa(v)^2 - \kappa(u)^2}{\kappa(u)\kappa(v)(\kappa(u)+\kappa(v))}\nabla v\right) \\ &= \operatorname{div}\left(\frac{\nabla w}{\kappa(u)} + \frac{\partial_j w \partial^j (u+v)}{\kappa(u)\kappa(v)(\kappa(u)+\kappa(v))}\nabla v\right). \end{aligned}$$

Thus w satisfies $\mathcal{L}(w) \geq 0$ with $\mathcal{L}(f) = A^{ij}\partial_{ij} f + B^i \partial_i f$. Computing explicitely the A^{ij} one can show it is uniformly elliptic, which concludes the proof. □

Remark 2.1.3. *There exists a whole ecosystem of maxima principles, sometimes with associated comparison principles, applied to a wide array of elliptic equations. We will then mention Hopf's maximum principle (theorem 3.5 of [?]):*

{200520211457}

Theorem 2.1.9. *Let L be a uniformly elliptic operator with $c = 0$. Then, assuming $Lu \geq 0$, if u achieves its maximum in the interior of Ω, u is a constant.*

The proof itself relies on another maximum principle which gives that the normal derivatives on the boundary are strictly positive (lemma 3.4 of [?]), applied to an interior critical point.

Remark 2.1.4. *It must be noticed that both proposition 2.1.5 and theorem 2.1.8 do not by themselves ensure the uniqueness of a minimal surfaces leaning on a given curve. Not only is the parametrization fixed in both cases, but the boundary conditions are* analytic*; they weigh on the parametrizations of the boundary, not on the geometric image of $\partial\Omega$. Technically, applying a diffeomorphism of Ω to a minimal parametrization ϕ puts it outside the domain of the theorem. This invariance at the source will come back later and is a classical issue of geometric analysis where we study* descriptions *of geometric objects.*

We now have all the tools to extend corollary 2.1.2 to any minimal surface without a boundary:

0220211023}

Theorem 2.1.10. *Let Σ be a surface without boundary and $\Phi : \Sigma \to \mathbb{R}^3$ a minimal immersion. Then $\Phi(\Sigma)$ is not compact.*

Proof. Let us reason by contradiction and assume that $\Phi(\Sigma)$ is compact. Let Π_k be the horizontal plane of $\mathbb{R}^3$ of equation $z = k$. Since $\Phi(\Sigma)$ is compact, for k big enough, $\Phi(\Sigma) \cap \Pi_k = \emptyset$. When k spans all of $\mathbb{R}$, the planes Π_k will cover the whole space. Let us then define $k_0 = \sup\{k | \Phi(\Sigma) \cap \Pi_k \neq \emptyset\}$. Since $\Phi(\Sigma)$ is compact, there exists a point $p \in \Sigma$ such that $\Phi(p) \in \Phi(\Sigma) \cap \Pi_{k_0}$. If we parametrize locally $\Phi(\Sigma)$ around p by a graph on a disk $\mathbb{D}$: $\phi(x, y) = (x, y, u(x, y))$, $\phi(0) = \Phi(p)$. The function u is then a solution of the mean curvature operator for graphs, reaching its maximum at 0 (by definition of k_0). Using theorem 2.1.9 (since the divergence operator has no constant term) implies that u is a constant. Thus Φ parametrizes a plane around p, and since minimal surfaces are analytic, $\Phi(\Sigma)$ is a plane, and thus not compact. □

Remark 2.1.5. *The proof of theorem 2.1.10 used a simple case of the moving plane technique, which is very useful in geometric analysis. It is also typical of the kind of techniques employed to bridge the gap between the analytical results and the geometric ones.*

2.1.3 Enneper-Weierstrass representation

Definition

If we consider u a harmonic function defined on a simply connected open of $\mathbb{R}^2$: $\operatorname{div}(\nabla u) = 0$, which implies that there exists (a unique, up to a constant) *dual harmonic function* $u^\star$ such that: $\nabla u = \nabla^\perp u^\star$.

Now if ϕ is a conformal parametrization of a minimal surface on a simply connected domain, one can define its dual harmonic function $\phi^\star$ coordinate by coordinate:

$$\nabla \phi = \nabla^\perp \phi^\star.$$

Then:

$$\begin{aligned} |\phi^\star_x|^2 = |\phi_y|^2 = |\phi_x|^2 = |-\phi^\star_y|^2 = |\phi^\star_y|^2 \\ \langle \phi^\star_x, \phi^\star_y \rangle = \langle \phi_y, -\phi_x \rangle = -\langle \phi_x, \phi_y \rangle = 0. \end{aligned}$$

The dual harmonic function $\phi^\star$ is thus not merely a harmonic function in $\mathbb{R}^3$, but a conformal immersion with the same conformal factor, that is a parametrization of a minimal surface that we will call the dual minimal surface:

Proposition 2.1.11. *Let $\phi \in C^\infty(\Omega, \mathbb{R}^3)$ a conformal parametrization of a minimal surface, defined on a simply connected domain of $\mathbb{R}^2$. Its dual harmonic* *function conformally parametrizes a minimal surface of* $\mathbb{R}^3$ *called the dual minimal surface. It is defined uniquely, up to translation.*

If we now introduce $\Psi := \frac{\phi - i\phi^\star}{2}$, one can check that it satisfies the Cauchy-Riemann equations coordinates by coordinates and thus that it defines a holomorphic function in $\mathbb{C}^3$. In addition

$$\begin{aligned}
\Psi' &= \frac{1}{4}\left(\partial_x \Psi - i\partial_y \Psi\right) = \frac{1}{4}\left(\phi_x - i\phi_y - i\phi_x^\star - \phi_y^\star\right) = \frac{1}{4}\left(\phi_x - i\phi_y - i\phi_y + \phi_x\right) \\
&= \frac{1}{2}\left(\phi_x - i\phi_y\right) = \phi_z \\
\langle \Psi', \Psi' \rangle &= \frac{1}{4}\left\langle \phi_x - i\phi_y, \phi_x - i\phi_y \right\rangle = \frac{|\phi_x|^2 - |\phi_y|^2 - 2i\langle \phi_x, \phi_y \rangle}{4} = 0.
\end{aligned}$$

The function Ψ is thus a holomorphic *isotropic curve*. These considerations yield the following result:

Proposition 2.1.12. *Let* Ω *be a simply connected domain of* $\mathbb{C}$*. Then for any conformal parametrization of minimal surfaces* $\phi : \Omega \to \mathbb{R}^3$ *there exists* $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ *a triplet of holomorphic functions on* Ω *satisfying* $\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 0$ *(we say that* Φ *is a* null curve*) and a constant* ϕ_0*:*

$$\forall z \in \Omega \quad \phi(z) = \phi_0 + \Re\left(\int_{z_0}^{z} \Phi(\xi) d\xi\right).$$

Its dual surface is then:

$$\phi^\star(z) = \phi_0^\star + \Im\left(\int_{z_0}^{z} \Phi(\xi) d\xi\right).$$

Of course, one then simply has to represent all null curves Φ to represent all the minimal surfaces parametrized over a simply connected domain of $\mathbb{C}$.

Lemma 2.1.2. *For every holomorphic null curve* $\Phi \neq 0$*, there exists* f *a holomorphic function and* g *a meromorphic function on* Ω *such that* fg^2 *is holomorphic and such that:*

$$\Phi_1 = \frac{f}{2}(1 - g^2),\ \Phi_2 = \frac{if}{2}(1 + g^2),\ \Phi_3 = fg.$$

Proof. If we consider such a holomorphic null curve, one can factorize:

$$0 = \Phi_1^2 + \Phi_2^2 + \Phi_3^2 = \left(\Phi_1 - i\Phi_2\right)\left(\Phi_1 + i\Phi_2\right) + \Phi_3^2,$$

and if we set $f = \Phi_1 - i\Phi_2$ and $g = \frac{\Phi_3}{\Phi_1 - i\Phi_2}$, then the above equality yields $f^2g^2 + f(\Phi_1 + i\Phi_2) = 0$, which ensures that $fg^2 = -(\Phi_1 + i\Phi_2)$ is a holomorphic function. Then:

$$\begin{aligned}
\Phi_1 &= \frac{1}{2}(\Phi_1 - i\Phi_2 + \Phi_1 + i\Phi_2) = \frac{1}{2}\left(f - fg^2\right) = \frac{f}{2}(1-g^2) \\
\Phi_2 &= \frac{i}{2}(\Phi_1 - i\Phi_2 - \Phi_1 - i\Phi_2) = \frac{i}{2}\left(f + fg^2\right) = \frac{if}{2}(1+g^2) \\
\Phi_3 &= \frac{\Phi_3}{\Phi_1 - i\Phi_2}(\Phi_1 - i\Phi_2) = fg.
\end{aligned}$$

□

All this yields the Enneper-Weierstrass representation for minimal surfaces:

Theorem 2.1.13. *Any minimal surface conformally parametrized on a simply connected domain Ω can be written:*

$$\phi(z) = \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \phi(z_0) + \int_{z_0}^{z} \begin{pmatrix}\frac{1}{2}f(1-g^2)\\ \frac{i}{2}f(1+g^2)\\ fg\end{pmatrix} d\xi,$$

with f a holomorphic function and g a meromorphic function on Ω such that fg^2 is holomorphic

Such representations are useful mainly to build examples of minimal surfaces with certain desired characteristics. We will offer an illustration of such a use in the end of this subsection. Another way to employ the Enneper-Weierstrass is to use the full description of local parametrizations to obtain local results on minimal surfaces.

Geometric interpretation

There is another way to obtain the Enneper-Weierstrass representation that makes its geometric interpretation easier (albeit it does not mention the dual minimal surface). If we again consider a conformal parametrization of a minimal surface on a simply connected domain: $\phi : \Omega \to \mathbb{R}^3$, one has $\Delta\phi = 0$. In complex formulation, the laplacian is written $\partial_{\bar{z}}\partial_z$. The harmonic function equation then becomes: $\partial_{\bar{z}}(\phi_z) = 0$ i.e. ϕ_z is holomorphic. In addition if we compute:

$$\langle \phi_z, \phi_z \rangle = \frac{1}{4}\langle \phi_x - i\phi_y, \phi_x - i\phi_y \rangle = \frac{1}{4}\left(|\phi_x|^2 - |\phi_y|^2 - 2i\langle \phi_x, \phi_y \rangle\right) = 0.$$

We recover what was alluded in the previous considerations: $\phi_z = \Psi'$ is exactly the holomorphic nullcurve integrated in the Enneper-Weierstrass representation:

$$\phi_z = \begin{pmatrix} \frac{1}{2} f(1-g^2) \\ \frac{i}{2} f(1+g^2) \\ fg \end{pmatrix}.$$

Then:

$$\begin{aligned} |\phi_z|^2 &= \frac{1}{4}\langle \phi_x - i\phi_y, \phi_x + i\phi_y\rangle = \frac{1}{4}\left(|\phi_x|^2 + |\phi_y|^2\right) = \frac{e^{2\lambda}}{2} \\ &= |f|^2\left(\frac{|1-g^2|^2}{4} + \frac{|1+g^2|}{4} + |g|^2\right) = \frac{|f|^2}{2}\left(1+|g|^4+2|g|^2\right) = \frac{\left(|f|(1+|g|^2)\right)^2}{2} \end{aligned}$$

$$\begin{aligned} \phi_z \times \phi_{\bar z} &= \frac{1}{4}(\phi_x - i\phi_y)\times(\phi_x + i\phi_y) = \frac{i}{2}\phi_x \times \phi_y = \frac{ie^{2\lambda}}{2}\vec{n} \\ &= \begin{pmatrix} \frac{1}{2} f(1-g^2) \\ \frac{i}{2} f(1+g^2) \\ fg \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} \bar f(1-\bar g^2) \\ -\frac{i}{2} \bar f(1+\bar g^2) \\ \bar f \bar g \end{pmatrix} = \begin{pmatrix} \frac{i}{2}|f|^2(\bar g + |g|^2 g + g + |g|^2\bar g) \\ \frac{1}{2}|f|^2(g - |g|^2\bar g - \bar g + |g|^2 g) \\ -\frac{i}{4}|f|^2(1-|g|^4) \end{pmatrix} \\ &= \frac{i|f|^2(1+|g|^2)^2}{2}\begin{pmatrix} g+\bar g \\ \frac{g-\bar g}{i} \\ |g|^2-1 \end{pmatrix} = \frac{ie^{2\lambda}}{2}\begin{pmatrix} 2\Re(g) \\ 2\Im(g) \\ |g|^2-1 \end{pmatrix} = \frac{ie^{2\lambda}}{2}\pi_N^{-1}(g), \end{aligned}$$

where π_N denotes the stereographic projection on the sphere. Thus g is the conformal Gauss map seen in the stereographic local chart of the sphere. At poles of g, $\vec{n}$ is at the north pole. On the other hand, f is a metric term, with $|f|$ the conformal factor between the metric on the euclidean sphere and the induced metric.

One consequence to have in mind is that poles and zeros of g are inconsequential: they can be shifted by rotating the image surface in $\mathbb{R}^3$. However, ϕ_z cancels out at a point where f has a zero but g has no pole. This can be seen as a condition to produce immersed minimal surfaces, but we would in fact rather relax our conditions to exploit the full potential of the Enneper-Weierstrass representation and allow ϕ to lose its immersed status. Let us then consider $\phi : \Omega \to \mathbb{R}^3$ a conformal harmonic map. Since ϕ_z is holomorphic, it can only be 0 at a finite number of points. Thus, away from a finite number of points, ϕ is an immersion, and necessarily a conformal parametrization of a minimal surface. At the points z_0 where $\phi_z(z_0) = 0$, since ϕ_z is holomorphic and satisfies $\langle \phi_z, \phi_z\rangle = 0$, there exists an integer m and a vector $\vec{A} \in \mathbb{C}^3$ such that:

{190220211502}

$$\begin{aligned} &\langle \vec{V}, \vec{V}\rangle = 0 \\ &\phi_z(z) = \vec{V}(z-z_0)^m + O\left(|z-z_0|^{m+1}\right). \end{aligned} \tag{78}$$

For minimal surfaces, the immersed property can only fail at a finite number of isolated points, and in a specific manner, conditioned by their analytic nature. In that case z_0 is called a *branch point* of multiplicity $m+1$. Even at this singular point, (78) ensures the tangent plan is well defined (spanned by $\Re(\vec{A}), \Im(\vec{A})$), which allows one to extend the Gauss map by continuity at those branch points.

Similarly, if we consider f meromorphic and not just holomorphic, we have to consider some non-bounded ϕ_z, ie ϕ_z meromorphic. Around a pole z_0 of f, there exists a negative integer $-m$ and a complex vector $\vec{A}$ such that:

$$\begin{aligned} &\langle \vec{A}, \vec{A} \rangle = 0 \\ &\phi_z(z) = \vec{A}(z-z_0)^{-m} + O(|z-z_0|^{1-m}). \end{aligned}$$

We call such a z_0 an *end* of the surface.

When studying minimal surfaces, we can thus broaden our scope to branched non-compact immersed surfaces:

Definition 2.1.3. *Let Σ be a compact Riemannian surface. We say that $\phi : \Sigma \to \mathbb{R}^3$ is a branched non-compact immersed surface if ϕ is a compact immersion away from a finite number of points z_i for which there exists an isotropic complex vector $\vec{A}_i$, an integer $m_i \neq 0$ and a local conformal chart centered at z_i such that, in this chart*

$$\phi_z(z) = \vec{A}_i(z-z_i)^{m_i} + O(|z-z_i|^{m_i+1}).$$

If m_i is strictly positive, z_0 is a branch point of multiplicity m_i+1. If $m_i = -1$, z_0 is an immersed catenoïd end. If $m_i < -1$, z_0 is an end of multiplicity $-m_i - 1$. If $m_i < -2$, it is a branched end.

In all cases, the Gauss map can be extended by continuity across the singular points.

For such a ϕ, we call $\phi(\Sigma)$ a branched non compact immersed surface.

The Enneper-Weierstrass representation can then deploy into the whole space of minimal, non-compact branched immersion.

Theorem 2.1.14. *Any branched non-compact minimal surface conformally parametrized on a simply connected domain Ω can be written:*

$$\phi(z) = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \phi(z_0) + \int_{z_0}^{z} \begin{pmatrix} \frac{1}{2} f(1-g^2) \\ \frac{i}{2} f(1+g^2) \\ fg \end{pmatrix} d\xi,$$

where f and g are two meromorphic functions.

Remark 2.1.6. *Not all failing immersions are branched: here it is a consequence of the analytic nature of minimal surfaces. When studying non analytic problems, one might have to consider more exotic singularities, like a conical singularity where $\vec{n}$ (and thus the tangent plane) is not well defined at the singular point.*

Examples of minimal surfaces:

On simply connected domains, building a minimal surface is merely a matter of finding a proper f and g. We here give a few classical examples.

1. Taking $f = 1$ and g a constant yields a plane of normal $\pi_N(g)$.
2. Taking $g = z$ and $f = 1$ yields the Enneper surface: equivalently an immersed plane with $\phi \simeq r^3$ at infinity or a branched non compact sphere with a branched end of multiplicity 3.
3. Taking $g = \frac{i}{z}$ and $f = 1$ yields a catenoïd: an immersed sphere with two catenoïd ends.
4. Taking $g = \frac{i}{z}$ and f a complex constant yields an helicoïd.
5. The Scherk doubly periodic surface is obtained with $f = \frac{2}{1-z^4}$, $g = z$.

2.2 Plateau problem in $\mathbb{R}^3$

The Plateau problem (from the Belgian mathematician and physicist Joseph Plateau) goes back to the physical interpretation of minimal surfaces as a soap film: given a closed exterior iron wire support, does there exist a soap film which leans on it? In mathematical terms, given an exterior closed curve Γ in $\mathbb{R}^3$, does there exist a minimal immersed surface such that $\phi : \Omega \to \mathbb{R}^3$ such that $\phi(\partial\Omega) = \Gamma$?

2.2.1 Minimal graphs

We will first consider a simpler problem as a warm-up: let Ω be a smooth bounded domain, and φ a function on $\bar{\Omega}$. We seek a solution to

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$$\begin{cases} L(u) := \operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \text{ in } \Omega \\ u = \varphi \text{ on } \partial\Omega. \end{cases} \tag{79}$$

If u is a solution of (79), then $\phi(x, y) = (x, y, u(x, y))$ is a solution of the Plateau problem for the exterior curve $\Gamma = (x, y, \varphi(x, y))_{|\partial\Omega}$. In other words, if the exterior curve is a graph, we naturally try to solve the Plateau problem by a minimal graph. We will, in fact, prove:

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Theorem 2.2.1. *Let Ω be a smooth bounded convex domain of $\mathbb{R}^2$ and $\varphi \in C^{2,\alpha}(\bar{\Omega})$. Then there exists a unique solution $u \in C^{2,\alpha}(\bar{\Omega})$ to the graph Plateau problem* (79).

We have already studied the operator L in section 2.1.2, and we know that if we fix $u \in C^1(\bar{\Omega})$, $L = L_u$ is uniformly elliptic. From theorem 2.1.8, one can deduce the uniqueness of a C^2 solution:

Corollary 2.2.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$. The solution of theorem 2.2.1 is unique, if it exists.*

Proof. We simply apply theorem 2.1.8 with $L(u) = L(v)$ and $u = v$ on the boundary. □

We will then need to show the existence of a solution to (79) using a *non-linear continuity method.*

Strategy of the proof: fixed point

As has been mentioned above, the main issue of the operator L is its non-linearity, which has been avoided by dissociating the occurences of u. Indeed, if given a function v, one can find a solution of $L_v(f) = 0$, which yields an operator $T : v \mapsto f$. A solution of (79) then satisfies $L_u(u) = 0$, that is $T(u) = u$: it is a fixed point of the operator T.

More precisely: if Ω is a smooth bounded domain of $\mathbb{R}^2$, and L an elliptic operator:

$$L(u) = a^{ij}(x, u, \nabla u)\partial_{ij}u + b(x, u, \nabla u), \tag{80}$$ {220220211732}

with $a^{ij}, b \in C^{0,\alpha}_{\mathrm{loc}}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^2)$. Let us assume in addition that:

$$a^{ij}(x, t, p)\xi_i\xi_j > 0 \text{ for all } x \in \bar{\Omega},\ t \in \mathbb{R},\ \xi \in \mathbb{R}^3. \tag{81}$$ {220220211739}

Given $\varphi \in C^{2,\alpha}(\bar{\Omega})$, we wish to solve

$$\begin{cases} L(u) = 0 \text{ in } \Omega \\ u = \varphi \text{ on } \partial\Omega. \end{cases} \tag{82}$$ {220220211441}

The minimal graph operator satisfies such conditions. Let us try to formalize the idea of solving the problem by a fixed point problem. First, let us recall without proof a classical result of the Schauder elliptic theory which ensures that the equation can be solved straightforwardly for a linear operator:

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Theorem 2.2.2. *Let Ω be a smooth domain of $\mathbb{R}^2$ and $\mathcal{L}$ a uniformly elliptic operator on Ω given by*

$$\mathcal{L}u(x) = \alpha^{ij}(x)\partial_{ij}u(x) + \beta^i(x)\partial_i u(x) + \gamma(x)u(x),$$

where the coefficients α^{ij}, β^i *and* γ *are* $C^{k,\alpha}(\bar{\Omega})$, *with* $\gamma \leq 0$ *and* $\lambda|\xi|^2 \leq \alpha^{ij}\xi_i\xi_j$. *Let us consider* $f \in C^{k,\alpha}(\bar{\Omega})$ *and* $\varphi \in C^{2,\alpha}(\bar{\Omega})$ *and the equation*

$$\begin{cases} \mathcal{L}(u) = f \ \mathit{in}\ \Omega \\ u = \varphi \ \mathit{on}\ \partial\Omega. \end{cases}$$

This equation admits a unique solution $u \in C^{k,\alpha}(\bar{\Omega})$, *such that*

$$\|u\|_{C^{k+2,\alpha}(\bar{\Omega})} \leq C(\Omega, \lambda, \|\alpha^{ij}\|_{C^{k,\alpha}(\bar{\Omega})}, \|\beta^i\|_{C^{k,\alpha}(\bar{\Omega})}, \|\gamma\|_{C^{k,\alpha}(\bar{\Omega})}) \left[\|f\|_{C^{k,\alpha}(\bar{\Omega})} + \|\varphi\|_{C^{k+2,\alpha}(\bar{\Omega})}\right].$$

This exact theorem corresponds to theorem 3.15 in [?] and has in fact two parts: the existence, proven by a Perron method, and the estimates, which are the elliptic estimates obtained through a Green representation formula. Both parts can be found in [?] (respectively section 5 and 6), [?], [?]. It must be emphasized how essential the Holder hypotheses are for the elliptic estimates.

If we fix $\beta < 1$, we can thus define the operator:

$$T : \begin{cases} C^{1,\beta}(\bar{\Omega}) \to C^{1,\beta}(\bar{\Omega}) \\ v \mapsto u, \end{cases}$$

where u is the unique solution to the linear elliptic operator $a^{ij}(x, v(x), \nabla v(x))\partial_{ij}u + b(x, v(x), \nabla v(x)) = 0$ with the boundary condition φ. This operator T is well defined thanks to theorem 2.2.2 since the coefficients $a^{ij}(x, v(x), \nabla v(x))$ and the right-hand term $b(x, v(x), \nabla v(x))$ belong in $C^{0,\alpha\beta}(\bar{\Omega})$, with the boundary term $\varphi \in C^{2,\alpha}(\bar{\Omega}) \subset C^{2,\alpha\beta}(\bar{\Omega})$. In fact, the Schauder estimates imply that $T(v)$ belongs to $C^{2,\alpha\beta}(\bar{\Omega}) \subset C^{1,\beta}(\bar{\Omega})$. Any fixed point of T is then a solution of (82) in $C^{2,\alpha\beta}(\bar{\Omega})$.

This is not the regularity announced, but it can be gained back by injecting the solution into the equations. Indeed, if one has such a solution $u \in C^{2,\alpha\beta}(\bar{\Omega})$, ∇u is a Lipschitz function. One can check that this increases the regularity for $a^{ij}(x, v(x), \nabla v(x))$ and $b(x, v(x), \nabla v(x))$ from the original $C^{0,\alpha\beta}(\bar{\Omega})$ to $C^{0,\alpha}(\bar{\Omega})$. Applying the Schauder estimates once more yields that the solution u is in $C^{2,\alpha}(\bar{\Omega})$, as announced.

Remark 2.2.1. *Such a procedure of going through the equation to increase a regularity is called a* bootstrap. *It is a cornerstone of many PDE regularity reasonings.*

There remains to show that T has a fixed point. By theorem of Arzela-Ascoli, T is a compact operator. In finite dimensions we would use Brouwer fixed point theorem to conclude. Here however, T acts on a functional space, which is of infinite dimension. One can however go back to a finite dimension fixed point with the *Schauder fixed point theorem*:

Theorem 2.2.3. *Let $\mathcal{B}$ be a Banach space, and E a closed convex subset of $\mathcal{B}$. Let $T : E \to E$ be a continuous map such that $\overline{T(E)}$ is compact, then T has a fixed point.*

Proof. Choosing N an integer, since $\overline{T(E)}$ is compact, there exists a finite number of balls B_i centered at points x_i such that

$$T(E) \subset \cup_{i=1}^N B_i.$$

If we denote E_N the convex hull of the $T(x_i)$, any point of E_N can be written $x = \sum \lambda_i x_i$ with $\sum \lambda_i = 1$. Such a convex hull is thus homeomorphic to a closed ball in a finite dimensional vector space. In addition, since E is convex, $E_N \subset E$, and $T(E_N) \subset T(E)$. Thus, we can define

$$P_N : \begin{cases} T(E) \to E_N \\ T(x) \mapsto \dfrac{\sum_{i=1}^{k_N} d(T(x), T(E)\backslash B_i) T(x_i)}{\sum_{i=1}^{k_N} d(T(x), T(E)\backslash B_i)}. \end{cases}$$

Such a P_N is continuous, and so is $P_N \circ T$. Applying Brouwer's theorem to $P_N \circ T$ on E_N then yields a fixed point $P_N \circ T(y_N) = y_N$. and satisfies:

$$P_N \circ T(x) - T(x) = \frac{\sum_{i=1}^{k_N} d(T(x), T(E)\backslash B_i) \left[T(x_i) - T(x)\right]}{\sum_{i=1}^{k_N} d(T(x), T(E)\backslash B_i)}.$$

In the previous expression, the only non zero terms in the sum are those i such that $x \in B_i$, and thus for which $|T(x) - T(x_i)| \leq \frac{1}{N}$. This yields:

$$|P_N \circ T(x) - T(x)| \leq \frac{1}{N} \text{ for all } x \in E.$$

Taking $N \to \infty$, since $\overline{T(E)}$ is compact, up to extraction y_N converges towards $y_0 \in \overline{T(E)} \subset E$, (E is closed). If we apply the previous estimate with $x = y_N$ and take it to the limit, we obtain that y_0 is a fixed point (T is continuous). □

The Schauder fixed point theorem is a good basis to work on. However, working on convex sets whose image is compact is a bit restrictive. We can replace this assumption by a set of a priori estimates:

Theorem 2.2.4. *Let $\mathcal{B}$ be a Banach space, let $T : \mathcal{B} \to \mathcal{B}$ be a compact map such that there exists $M > 0$ such that*

$$\forall x \in \mathcal{B} \text{ and } 0 \leq \sigma \leq 1 \text{ s.t. } x = \sigma T(x),\ \|x\|_{\mathcal{B}} \leq M.$$

Then T has a fixed point.

Proof. If we take $\alpha > M$ and set

$$T_\alpha(x) := \begin{cases} T(x) \text{ if } \|T(x)\| \leq \alpha \\ \dfrac{\alpha T(x)}{\|T(x)\|} \text{ if } \|T(x)\| > \alpha. \end{cases}$$

This map is continuous, compact and sends the ball B_α (which is closed and convex) into itself. Applying the Schauder fixed point theorem, there exists a fixed point for T_α. If $\|T(x)\| > \alpha$, then $x = \sigma T(x)$ with $\sigma = \frac{\alpha}{\|T(x)\|}$, thus by the a priori estimate $\left\|\frac{\alpha T(x)}{\|T(x)\|}\right\| = \|T(x)\| = \|x\| \leq M$, which contradicts $\alpha > M$. Thus $\|T(x)\| \leq \alpha$ and $T(x) = x$. □

For an operator of shape (80), $\sigma T(u) = u$ is equivalent to

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$$\begin{cases} a^{ij}(x,u,\nabla u)\partial_{ij}u + \sigma b(x,u,\nabla u) = L_\sigma(u) = 0 \text{ in } \Omega \\ u = \sigma\varphi \text{ on } \partial\Omega. \end{cases} \tag{83}$$

In the end, we have shown the following theorem, of which theorem 2.2.1 is but a corollary:

Theorem 2.2.5. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$ and L an operator of shape* (80)*, with $C^{0,\alpha}$ coefficients and satisfying* (81)*. If there exists $0 < \beta < 1$ and $M > 0$ such that all C^2 solutions of* (83) *for a given $0 \leq \sigma \leq 1$ satisfy*

$$\|u\|_{C^{1,\beta}(\bar{\Omega})} \leq M,$$

then the equation (82) *has a solution $u \in C^{2,\alpha}(\bar{\Omega})$.*

So to prove the existence of a solution to (82), one only has to show an a priori estimate in any $C^{1,\beta}$.

Pulling back from the general case and going back to the minimal graph problem, we will decompose the $C^{1,\beta}$ estimate:

- The uniform estimate :$\|u\|_{L^\infty(\Omega)} \leq M$
- The control of the derivative on the boundary: $\sup_{\partial\Omega} |\nabla u| \leq M$
- The uniform interior estimate for the derivative: $\sup\limits_{\Omega} |\nabla u| \leq M$
- The Holder estimate for the derivative: $\sup_{x,y\in\bar{\Omega}} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^\beta}$

In the following, we set ourselves in the context of theorem 2.2.1 with L the minimal graph operator (which notably implies that $b = 0$).

Uniform estimate

For this estimate, we have a maximum principle: applying theorem 2.1.6 to the minimal graph problem yields:

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u = \max_{\partial\Omega} \varphi \leq M,$$

which proves the result.

Control of the derivative on the boundary

This estimate uses a classical technique: *barrier functions*. It revolves around building functions which can be used in the comparison principle to constrain the solution around a point of the boundary.

In our case let us consider $x_0 \in \partial\Omega$. To simplify notations we will assume that up to translations that $x_0 = 0$, and up to rotations that the tangent plan is horizontal. Since $\partial\Omega$ is assumed to be strictly convex, there exists $C_\Omega > 0$ such that:

$$\forall x = (x_1, x_2) \in \partial\Omega,\ |x|^2 \leq C_\Omega x_1.$$

If we set $\psi = Ax_1 + \varphi(0) + \partial_1\varphi(0)x_1 + \partial_2\varphi(0)x_2$, $\nabla\psi$ is a constant vector, and thus $L(\psi) = 0$ (recall that $b = 0$ in this case). In addition, since $u = \psi$ on the boundary:

$$|u(x) - (\varphi(0) + \partial_1\varphi(0)x_1 + \partial_2\varphi(0)x_2)\,| \leq \left(\sup_\Omega |\nabla^2\varphi|\right)|x|^2 \quad \forall x \in \partial\Omega.$$

Choosing $A = C_\Omega\left(\sup_\Omega |\nabla^2\varphi|\right)$ yields, for all $x \in \partial\Omega$

$$\psi_1 = \varphi(0)+\partial_1\varphi(0)x_1+\partial_2\varphi(0)x_2-Ax_1 \leq u(x) \leq Ax_1+\varphi(0)+\partial_1\varphi(0)x_1+\partial_2\varphi(0)x_2 = \psi_2.$$

Applying the comparison principle (theorem 2.1.8) yields $\psi_1(x) \leq u(x) \leq \psi_2(x)$, for all $x \in \Omega$. This implies:

$$|u(x) - \varphi(0) - \partial_1\varphi(0)x_1 - \partial_2\varphi(0)x_2| \leq A|x_1| \leq C_\Omega\left(\sup_\Omega |\nabla^2\varphi|\right)|x|.$$

This thus implies:

$$|\nabla u(0)| \leq C_\Omega\left(\sup_\Omega |\nabla^2\varphi|\right) + |\nabla\varphi(0)|\,.$$

This pointwise equality can be repeated on any point of the boundary to yield the desired control:

$$\sup_{\partial\Omega} |\nabla u| \leq C_\Omega\left(\sup_\Omega |\nabla^2\varphi| + \sup_\Omega |\nabla\varphi|\right).$$

Uniform estimate on the interior for the derivative

We will show the estimate by controlling $\sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|}$. Since

$$\sup_{\Omega} |\nabla u| \leq \sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|},$$

this will show the estimates.

Let us consider $x, y \in \Omega$ and set $\tau = y - x$, $\Omega_\tau = \{z + \tau, z \in \Omega\}$. If we define $u_\tau(z) = u(z - \tau)$ on Ω_τ, one has $L(u_\tau) = 0$. The function u_τ is thus admissible for the comparison principle (theorem 2.1.8 applied on $\Omega \cap \Omega_\tau$), which yields:

$$\sup_{z\in\Omega\cap\Omega_\tau} |u(z) - u_\tau(z)| \leq \sup_{z\in\partial(\Omega\cap\Omega_\tau)} |u(z) - u_\tau(z)|.$$

Taking $z = x$, one has $|u(x) - u(y)| = |u(x) - u_\tau(x)| \leq \sup_{z\in\partial(\Omega\cap\Omega_\tau)} |u(z) - u_\tau(z)|$. Since $\partial(\Omega \cap \Omega_\tau) = \partial\Omega \cup \partial\Omega_\tau$, one can deduce:

$$|u(x) - u(y)| \leq \sup_{z\in\partial\Omega, z'\in\Omega, |z-z'|=\tau=|x-y|} |u(z) - u(z')|,$$

and necessarily:

$$\frac{|u(x) - u(y)|}{|x-y|} \leq \sup_{z\in\partial\Omega, z'\in\Omega, |z-z'|=\tau=|x-y|} \frac{|u(z) - u(z')|}{|z-z'|} \leq \sup_{z\in\partial\Omega} |\nabla u|(z)).$$

This is true for all x, y, and thus yields the desired estimate:

$$\sup_{\Omega} |\nabla u| \leq C_\Omega \left(\sup_{\Omega} |\nabla^2 \varphi| + \sup_{\Omega} |\nabla \varphi| \right).$$

The Holder estimate for the derivative

The Holder estimate is more delicate. One might want to use the Schauder estimates to jump from a C^1 control (and thus $C^{0,\alpha}$) to a $C^{1,\alpha}$. This however requires a Holder control on the coefficients a^{ij}, and thus in our case, a $C^{1,\alpha}$ control on u, which is exactly what we want to show... We say that such a situation is critical. Critical problems require sharper estimates, and often, a more refined theory.

In this case this will require a De Giorgi-Nash-Moser type result, based on a Moser-Trudinger iteration with estimates up to the boundary. Since we will not use these tools in the following, we will not detail the associated theorems, and refer the reader to the chapter 16 of [?].

2.2.2 Energy minimization and Gauge forcing

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Now that we have seen what the problem looked like when framed in a the graph manner, and had an idea of how to solve it using flexible PDE techniques, let us go back to the generic problem. Let Γ be a Jordan curve in $\mathbb{R}^3$ and $\mathbb{D}$ the unit disk of $\mathbb{R}^2$. We seek a parametrization of a minimal surface $\phi : \mathbb{D} \to \mathbb{R}^3$ such that:

- $\phi(\mathbb{D}) = \Sigma$ is minimal,
- $\phi_{|\partial\mathbb{D}}$ parametrizes Γ.

In other words, we look for a minimal disk (in the topological sense) whose boundary is Γ. Given the definition of a minimal surface, it is only natural to look for a *minimizing* disk, that is to find a parametrization ϕ with the right boundary condition minimizing the area formula under the proper boundary constraint:

$$\mathcal{A}(\phi) = \int_{\mathbb{D}} d\mathrm{vol}_g = \int_{\mathbb{D}} |g|^{\frac{1}{2}} dxdy = \int_{\mathbb{D}} |\phi_x \times \phi_y| dxdy.$$

To solve this minimizing problem, we need to formalize the boundary constraint. If we take a smooth parametrization $\gamma : \partial\mathbb{D} = \mathbb{S}^1 \simeq [0, 2\pi) \to \mathbb{R}^3$, ϕ parametrizes Γ if it is a reparametrization of γ, i.e. if there is a continuous monotone function θ such that $\phi_{|\partial\mathbb{D}} = \gamma \circ \theta$. Since minimization procedures are done with the Sobolev spaces frame (to make use of their weak compactness properties) by $\phi_{|\partial\mathbb{D}}$ means the *trace* of ϕ. Let us recall that:

$$T \begin{cases} C^0(\mathbb{D}) \to C^0(\partial\mathbb{D}) \\ \qquad\qquad v \mapsto v_{|\partial\mathbb{D}} \end{cases}$$

defines a linear continuous surjective operator which can be extended into a linear continuous surjective operator defined $W^{1,2}(\mathbb{D})$and with values in $W^{\frac{1}{2},2}(\mathbb{D})$, defined in the Fourier formalism as the set of functions $u = \sum_{n=0}^{+\infty} a_n e^{in\theta}$ such that

$$\|u\|^2_{W^{\frac{1}{2},2}(\mathbb{D})} \le \sum_{n=0}^{+\infty} (1+n)|a_n|^2 < +\infty.$$

The condition $\phi_{|\partial\mathbb{D}} = \gamma \circ \theta$ is thus well defined, and if we denote:

$$\mathcal{C}(\Gamma) = \{\phi \in W^{1,2}(\mathbb{D}, \mathbb{R}^3) \text{ s.t. } \phi_{|\partial\mathbb{D}} = \gamma \circ \theta, \text{ with } \theta \text{ continuous increasing}\}$$

We can then look for a minimizer of $\mathcal{A}(\phi)$ inside this class $\mathcal{C}(\Gamma)$:

$$\mathcal{A}(\phi_0) = \inf_{\phi \in \mathcal{C}(\Gamma)} \mathcal{A}(\phi).$$

The procedure is classical:

- show that the functional is bounded from below, and thus that the infimum exists and is finite,
- show that a bound on the functional implies a bound on the $W^{1,2}$ norm (we say that the functional is coercive)
- consider a minimizing sequence ϕ_n, since $\mathcal{A}(\phi_n)$ is bounded, $\|\phi\|_{W^{1,2}}$ is bounded. By Riesz theorem, $\phi_n \to \phi$ weakly in $W^{1,2}$ up to extraction
- Show that ϕ minimizes the functional (usually because the weak limit can only shed energy, not gain any). Do not hesitate to use the Sobolev embeddings to obtain strong convergences in weaker spaces

However, in this case, this procedure is hopeless due to the invariances displayed by the area functional. Indeed, the area is a *geometric quantity*, it does not depend on the chosen parametrization of our disk: for all $\Theta \in \text{Diff}(\mathbb{D})$: $\mathcal{A}(\phi \circ \Theta) = \mathcal{A}(\phi)$. So, in our minimizing procedure, even if we get a parametrization ϕ_0 of the minimizing surface from the very beginning, the minimizing sequence could be $\phi_k = \phi_0 \circ \Theta_k$, with Θ_k a noncompact sequence of diffeomorphisms of the disk which thus prevents the parametrizations ϕ_k from converging (while the resulting surface remains the same).

We are thus facing a classical issue in geometric analysis: our problem has redundant degrees of liberty which make the analysis aspect of the problem much more complex, here by inducing some noncompactness. This issue here will be solved by *forcing a gauge choice*, here a Coulomb gauge. If ϕ is a conformal parametrization, the functional $\mathcal{A}(\phi) = \int_{\mathbb{D}} e^{2\lambda} dxdy = \frac{1}{2}\int_{\mathbb{D}} |\nabla \phi|^2 dxdy$, where we can recognize the Dirichlet energy of ϕ in the last integral. Further, for any arbitrary parametrization:

$$
\begin{aligned}
|\nabla \phi|^2 = |\phi_x|^2 + |\phi_y|^2 &= \sqrt{\left(|\phi_x|^2 + |\phi_y|^2\right)^2} = \sqrt{\left(|\phi_x|^2 - |\phi_y|^2\right)^2 + 4|\phi_x|^2|\phi_y|^2} \\
&= \sqrt{\left(|\phi_x|^2 - |\phi_y|^2\right)^2 + 4\langle \phi_x, \phi_y\rangle^2 + 4|g|} = 2|g|^{\frac{1}{2}} \sqrt{1 + \frac{1}{4|g|}\left|\begin{pmatrix} |\phi_x|^2 - |\phi_y|^2 \\ 2\langle \phi_x, \phi_y\rangle \end{pmatrix}\right|^2}.
\end{aligned}
$$

One can recognize that the vector $\begin{pmatrix} |\phi_x|^2 - |\phi_y|^2 \\ 2\langle \phi_x, \phi_y\rangle \end{pmatrix}$ cancel out if and only if the parametrization is conformal. So if decide to minimize the Dirichlet energy instead of the area functional, the minimizing sequence will be forced to become conformal, and to minimize the area at the same time! We will force the minimizer to adopt the Coulomb gauge in a conformal forcing procedure. Another reason to prefer working with the Dirichlet energy (which is in fact a more analytic description of what has already been mentioned) is that it is no longer invariant by reparametrizations.

Dirichlet energy and conformal invariance

Introducing the Dirichlet energy

$$\mathcal{D}(\phi) = \frac{1}{2}\int_{\mathbb{D}} |\nabla \phi|^2 dxdy,$$

we will thus study the minimization procedure:

$$\mu = \inf_{\phi \in \mathcal{C}(\Gamma)} \mathcal{D}(\phi).$$

Remark 2.2.2. *The idea of working with the Dirichlet energy instead of with the Area dates back from the 1930s and the (independant) works of Douglas and Rado. It can also be traced back to the study of geodesics on a surface where a similar trick is used to force the parametrization by arc length.*

The preliminary work is not over yet though. Indeed, while $\mathcal{D}$ is no longer invariant by any diffeomorphism of the disk, it is still invariant by *conformal diffeomorphisms*, meaning one still has to deal with redundant degrees of liberty.

Proposition 2.2.6. *Let us denote* $\mathrm{Conf}^+(\mathbb{D})$ *the set of direct conformal diffeomorphisms of the disk, that is the set of direct diffeomorphisms* Θ *satisfying* $|\Theta_x^2| - |\Theta_y|^2 = \langle \Theta_x, \Theta_y \rangle = 0$.

Then, for any $u \in W^{1,2}(\mathbb{D})$*, then* $\mathcal{D}(u) = \mathcal{D}(u \circ \Theta)$ *for any* $\Theta \in \mathrm{Conf}^+(\mathbb{D})$.

Proof. Given $\Theta \in \mathrm{Conf}^+(\mathbb{D})$, one has:

$$\begin{aligned}&(\Theta_x^1)^2 + (\Theta_x^2)^2 = (\Theta_y^1)^2 + (\Theta_y^2)^2 = \chi^2\\ &\Theta_x^1\Theta_y^1 + \Theta_x^2\Theta_y^2 = 0.\end{aligned}$$

So $\left(\frac{\Theta_x}{\chi^2}, \frac{\Theta_y}{\chi^2}\right)$ forms a *direct* orthonormal family, and thus

$$\Theta_x^1 = \Theta_y^2,\ \Theta_x^2 = -\Theta_y^1. \tag{84}$$ {260220211148}

One can then simply compute:

$$\begin{aligned}(u \circ \Theta)_x &= u_x \circ \Theta\Theta_x^1 + u_y \circ \Theta\Theta_x^2\\ (u \circ \Theta)_y &= u_x \circ \Theta\Theta_y^1 + u_y \circ \Theta\Theta_y^2,\end{aligned}$$

which yields:

$$\begin{aligned}|\nabla(u \circ \Theta)|^2 &= |(u \circ \Theta)_x|^2 + |(u \circ \Theta)_y|^2\\ &= |u_x \circ \Theta|^2\left((\Theta_x^1)^2 + (\Theta_y^1)^2\right) + |u_y \circ \Theta|^2\left((\Theta_x^2)^2 + (\Theta_y^2)^2\right)\\ &+ 2\langle u_x \circ \Theta, u_y \circ \Theta\rangle\left(\Theta_x^1\Theta_x^2 + \Theta_y^1\Theta_y^2\right)\\ &= |u_x \circ \Theta|^2\left((\Theta_x^1)^2 + (-\Theta_x^2)^2\right) + |u_y \circ \Theta|^2\left((\Theta_y^1)^2 + (\Theta_y^2)^2\right)\\ &+ 2\langle u_x \circ \Theta, u_y \circ \Theta\rangle\left(-\Theta_x^1\Theta_y^1 + \Theta_y^1\Theta_x^1\right)\\ &= \chi^2 |\nabla u|^2 \circ \Theta = \left(\Theta_x^1\Theta_y^2 - \Theta_y^1\Theta_x^2\right)|\nabla u|^2 \circ \Theta = \mathrm{Jac}(\Theta)\,|\nabla u|^2 \circ \Theta.\end{aligned}$$

Thus

$$\mathcal{D}(u \circ \Theta) = \int_{\mathbb{D}} |\nabla (u \circ \Theta)|^2 dxdy = \int_{\mathbb{D}} \mathrm{Jac}(\Theta) \, |\nabla u|^2 \circ \Theta = \int_{\mathbb{D}} |\nabla u|^2 dxdy$$
$$= \mathcal{D}(u).$$

□

To understand this new invariance group, the complex formulation is very useful: if we write $\Theta(x+yi) := \Theta^1(x,y) + i\Theta^2(x,y)$, then

$$\begin{aligned}\Theta_{\bar{z}}(z) &= \frac{1}{2}\left(\Theta^1_x + i\Theta^1_y\right) + \frac{i}{2}\left(\Theta^2_x + i\Theta^2_y\right)\\ &= \frac{1}{2}\left(\Theta^1_x - \Theta^2_y\right) + \frac{i}{2}\left(\Theta^1_y + \Theta^2_x\right)\\ &= 0,\end{aligned}$$

thanks to (84). So conformal diffeomorphisms between opens of $\mathbb{R}^2$ are holomorphic. Conversely, one can check that holomorphic functions yield conformal maps.

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Proposition 2.2.7. *Conformal diffeomorphisms of opens of $\mathbb{R}^2$ are holomorphic diffeomorphisms between opens of $\mathbb{C}$.*

Remark 2.2.3. *Compare proposition 2.2.7 and the Liouville theorem (theorem 1.3.3), and the relative size of the conformal groups. This must be related to the specific position of the dimension 2 when considering the conformal parametrization.*

For our particular problem, we are only interested in the conformal diffeomorphisms of the unit disk, which are perfectly described thanks to Möbius transformations:

Theorem 2.2.8.

$$\mathrm{Conf}(\mathbb{D}) = \left\{ \Theta(z) = e^{i\theta}\frac{a+z}{1+\bar{a}z},\ |a|<1,\ \theta \in \mathbb{R} \right\}.$$

Proof. First, we can check that such map $F_{\theta,a}(z) = e^{i\theta}\frac{a+z}{1+\bar{a}z}$ are conformal diffeomorphisms of the disk: they are holomorphic and thus conformal, while:

$$\forall z, z' \in \mathbb{D},\ F_{\theta,a}(z) = z' \Leftrightarrow z = e^{-i\theta}\frac{z' - ae^{i\theta}}{1 - \overline{ae^{i\theta}}z'} = F_{-\theta, ae^{i\theta}}(z').$$

This yields one of the inclusions. For the other, let us consider $\Theta \in \mathrm{Conf}(\mathbb{D})$. Let us set $-a = \Theta^{-1}(0)$ and $\tilde{\Theta}(z) = \Theta \circ F_{0,-a}(z)$, so that $\tilde{\Theta}$ is a holomorphic function on the disk satisfying $\tilde{\Theta}(0) = 0$. Applying Schwarz lemma to $\tilde{\Theta}$ yields $|\tilde{\Theta}(z)| \leq |z|$ while applying it to $\tilde{\Theta}^{-1}$ yields $|z| \leq |\tilde{\Theta}(z)|$. Thus, the equality case in the Schwarz lemma yields $\tilde{\Theta}(z) = e^{i\theta} z$ and thus: $\Theta(z) = e^{i\theta} F_{0,a}(z) = F_{\theta,a}(z)$, which concludes the proof. $\square$

While this invariance group is well described, it is non compact ($a \to a_0 \in \partial\mathbb{D}$). We thus need to kill this invariance group. Since it is decribed by three free parameters ($\Re(a)$, $\Im(a)$, θ), it will be enough to fix three points of the parametrization. We will thus minimize $\mathcal{D}$ on

$$\tilde{\mathcal{C}}(\Gamma) = \left\{ \phi \in \mathcal{C}(\Gamma) \text{ with } \theta(1) = 1,\, \theta\left(e^{\frac{2i\pi}{3}}\right) = e^{\frac{2i\pi}{3}},\, \theta\left(e^{\frac{4i\pi}{3}}\right) = e^{\frac{4i\pi}{3}} \right\}.$$

By restricting ourselves to $\tilde{\mathcal{C}}(\Gamma)$ we do not exclude any exterior curve and parametrized disks leaning on this curve, we just limit ourselves to some parametrizations. Indeed, given $\phi \in \mathcal{C}(\Gamma)$, there exists θ such that $\phi_{|\partial\mathbb{D}} = \gamma \circ \theta$. For any $F_{\psi,a}$, $(\phi \circ F_{\psi,a})_{|\partial\mathbb{D}} = \gamma \circ (\theta \circ F_{\psi,a})$, and if we can find $\psi \in [0, 2\pi)$ and $a \in \mathbb{C}$ such that:

$$(\theta \circ F_{\psi,a})(1) = 1,\, (\theta \circ F_{\psi,a})\left(e^{\frac{2i\pi}{3}}\right) = e^{\frac{2i\pi}{3}},\, (\theta \circ F_{\psi,a})\left(e^{\frac{4i\pi}{3}}\right) = e^{\frac{4i\pi}{3}}, \tag{85}$$ {260220211801}

then $\phi \circ F_{\psi,a} \in \tilde{\mathcal{C}}(\Gamma)$. The condition (85) amounts to solving a system of three equations with three degrees of freedom, which can be done straightforwardly.

Thus, any minimizing procedure on $\tilde{\mathcal{C}}(\Gamma)$ will produce the same geometric result as a minimization on $\mathcal{C}(\Gamma)$, without the redundancy due to the conformal invariance. The analytic gain will be revealed thanks to the following Courant-Lebesgue lemma:

Lemma 2.2.1. *Let $u \in W^{1,2}(\mathbb{D}, \mathbb{R}^3)$, $z_0 \in \overline{\mathbb{D}}$ and $0 < \delta < 1$. There exists $\delta \leq \rho \leq \sqrt{\delta}$ such that, if s denotes the arc length on $\partial B_\rho(z_0) \cap \mathbb{D}$, one has:*

$$\int_{\partial B_\rho(z_0)\cap\mathbb{D}} |\partial_s u|^2 \, ds \leq \frac{2}{\rho |\ln \rho|} \int_{\mathbb{D}} |\nabla u|^2 dxdy.$$

Proof. Working from a simple estimate:

$$\begin{aligned}
\int_{\mathbb{D}} |\nabla u|^2 dxdy &\geq \int\int_{\left(B_{\sqrt{\delta}}(z_0)\setminus B_\delta(z_0)\right)\cap\mathbb{D}} |\nabla u|^2 dxdy \\
&\geq \int_\delta^{\sqrt{\delta}} \left(\rho \int_{\partial B_\rho(z_0)\cap\mathbb{D}} \left[|\partial_s u|^2 + |\partial_\nu u|^2 \right] ds \right) d\rho \geq \int_\delta^{\sqrt{\delta}} \left(\int_{\partial B_\rho(z_0)\cap\mathbb{D}} |\partial_s u|^2 ds \right) d\rho \\
&\geq \inf_{\delta<\rho<\sqrt{\delta}} \left(\rho \int_{\partial B_\rho(z_0)\cap\mathbb{D}} |\partial_s u|^2 ds \right) \int_\delta^{\sqrt{\delta}} \frac{d\rho}{\rho} \\
&\geq \inf_{\delta<\rho<\sqrt{\delta}} \left(\rho \int_{\partial B_\rho(z_0)\cap\mathbb{D}} |\partial_s u|^2 ds \right) \frac{|\ln \delta|}{2}.
\end{aligned}$$

Taking the minimum, one finds $\delta \leq \rho \leq \sqrt{\delta}$ such that:

$$\rho \int_{\partial B_\rho(z_0)\cap\mathbb{D}} |\partial_s u|^2 ds \leq \frac{2}{|\ln\delta|}\int_{\mathbb{D}} |\nabla u|^2 dxdy \leq \frac{2}{|\ln\rho|}\int_{\mathbb{D}} |\nabla u|^2 dxdy,$$

since $\rho \geq \delta$. □

This analytic Courant-Lebesgue lemma will translate in our context into the following *"three points lemma"* which will ensure a uniform control on the modulus of continuitiy on the boundary:

{020320210920} **Lemma 2.2.2.** *Let $\phi \in \tilde{\mathcal{C}}(\Gamma)$ such that $\int_{\mathbb{D}} |\nabla u|^2 dxdy \leq E_0$ and $\varepsilon > 0$. There exists $\delta(\varepsilon, E_0) > 0$ such that:*

$$z_1, z_2 \in \partial\mathbb{D},\ |z_1 - z_2| \leq \delta \implies |\phi(z_1) - \phi(z_2)| \leq \varepsilon.$$

Proof. Since $\phi \in \tilde{\mathcal{C}}(\Gamma)$, $\phi(1) = \gamma(1)$, $\phi(e^{\frac{2i\pi}{3}}) = \gamma(e^{\frac{2i\pi}{3}})$, $\phi(e^{\frac{4i\pi}{3}}) = \gamma(e^{\frac{4i\pi}{3}})$ are fixed, with γ a reference parametrization of Γ. Let us thus consider $\varepsilon_0(\Gamma)$ such that: for any $p \in \mathbb{R}^3$, $B_{\varepsilon_0(\Gamma)}(p)$ contains at most of the the $\gamma(1)$, $\gamma(e^{\frac{2i\pi}{3}})$, $\gamma(e^{\frac{4i\pi}{3}})$.

Let us consider $0 < \varepsilon < \varepsilon_0$. Since Γ is a Jordan curve (and thus non self intersecting), there exists $0 < \varepsilon_1 < \varepsilon$ such that for any $p, q \in \Gamma$ satisfying $|p - q| \leq \varepsilon$, there exists a curve $\tilde{\Gamma} \subset \Gamma$ of extremities p and q contained inside a ball of radius ε. Indeed, by contradiction, if for all n, there exists $p_n = \gamma(a_n), q_n = \gamma(b_n) \in \Gamma$ with $|p_n - q_n| \leq \frac{1}{n}$ such that for any curve $\tilde{\Gamma}$ linking the two points, $\tilde{\Gamma}$ is not contained in a ball of radius ε. Since Γ is bounded, all these points lie inside a compact, and thus converge up to extraction: $p_n \to p$, $a_n \to a$, $q_n \to q$, $b_n \to b$, with $p = q$. Since Γ does not self-intersect, necessarily, $a = b$. And since γ is continuous, $\gamma([a_n, b_n])$ lies inside a ball of radius ε, for n big enough, which shows the intermediary result.

Since $\varepsilon < \varepsilon_0$, any such arc contains at most one of the three fixed points $\gamma(1)$, $\gamma(e^{\frac{2i\pi}{3}})$, $\gamma(e^{\frac{4i\pi}{3}})$.

Let now $0 < \delta < \frac{1}{2}$ such that:

$$|\ln\delta| \geq \frac{4\pi E_0}{\varepsilon_1^2}.$$

For any $z_1 \in \mathbb{D}$, using the Courant-Lebesgue lemma, there exists $\delta \leq \rho \leq \sqrt{\delta}$ such that:

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$$\int_{\partial B_\rho(z_1)\cap\mathbb{D}} |\partial_s \phi|^2 ds \leq \frac{2E_0}{\rho|\ln\rho|}. \tag{86}$$

Taking w_1, w_2 the intersection points of $\partial B_\rho(z_1)$ with the unit circle, one has:

$$|\phi(w_1) - \phi(w_2)|^2 \leq \left(\int_{\partial B_\rho(z_1)\cap\mathbb{D}} |\partial_s \phi| ds \right)^2$$
$$\leq \pi\rho \int_{\partial \mathbb{D}_\rho(z_1)\cap\mathbb{D}} |\partial_s \phi|^2 ds,$$

thanks to the Hölder inequality. Applying (86), this yields:

$$|\phi(w_1) - \phi(w_2)|^2 \leq \frac{2\pi E_0}{|\ln \rho|} \leq \frac{2\pi E_0}{|\ln \delta|} \leq \varepsilon_1^2.$$

In addition, since $\delta < \frac{1}{2}$, the smallest portion of the circle linking w_1 and w_2 (the small circle $\mathcal{C}_1$, which is then contained in a disk of radius δ $B_\delta(z_1)$) contains at most one of of the three 1, $e^{\frac{2i\pi}{3}}$ and $e^{\frac{4i\pi}{3}}$, while the biggest one ($\mathcal{C}_2$) contains at least two of them. The image of $\mathcal{C}_1$ contains thus at most one of the three fixed points $\gamma(1)$, $\gamma(e^{\frac{2i\pi}{3}})$, $\gamma(e^{\frac{4i\pi}{3}})$, while the image of $\mathcal{C}_2$ contains at least two. Applying the claim, to $p = \phi(w_1)$ and $q = \phi(w_2)$ ensures that one of the two images is contained inside a ball of radius ε. Since such a ball contains at most one of the three fixed points $\gamma(1)$, $\gamma(e^{\frac{2i\pi}{3}})$, $\gamma(e^{\frac{4i\pi}{3}})$, one has:

$$\phi(\mathcal{C}_1) \subset B_\varepsilon,$$

which yields the desired inequality for all v_1, $v_2 \in \mathcal{C}_1$. Since the previous reasoning stands independantly of z_1, this yields the result on the whole circle. $\square$

Remark 2.2.4. *The three fixed points hypothesis is pivotal: it avoids the problematic configuration of the boundary being parametrized on only a small arc circle, of a concentration of the parametrization around a point, which would dilate the modulus of continuity.*

Existence of a conformal solution

Let us now do the minimization procedure. Let us denote

$$\mu = \inf_{\phi \in \mathcal{C}(\Gamma)} \mathcal{D}(\phi).$$

Let us consider $\phi_k \in \mathcal{C}(\Gamma)$ a minimizing sequence, that is satisfying $\mathcal{D}(\phi_k) \to \mu$. As has been shown above (see (85)D) up to the composition by a conformal diffeomorphism, one can assume that $\phi_k \in \tilde{\mathcal{C}}(\Gamma)$, and thus, thanks to the three points lemma (lemma 2.2.2) that its modulus of continuity on the boundary is uniformly bounded in k. Thus, using the Arzela-Ascoli theorem, up to extraction, there exists $\varphi_0 \in C^0(\partial\mathbb{D})$ such that $\phi_{k|\partial\mathbb{D}} \to \varphi_0$ in $C^0(\partial\mathbb{D})$. The work on the conformal invariance and the idea of fixing three points to exhaust all the degrees of liberty in the invariance group have thus allowed a good convergence on the boundary: should we manage to show a convergence on the interior, the limit would then be in the right space $\tilde{\mathcal{C}}(\Gamma)$.

To ensure the convergence in the open disk $\mathbb{D}$, one classically wants to bound the $W^{1,2}(\mathbb{D})$ norm uniformly, and apply the Riesz compactness theorem to deduce a weak convergence in $W^{1,2}(\mathbb{D})$. Here the difficulty (also very typical) comes from the fact that we only enjoy a bound on the Dirichlet energy, and that we need to adjoint it a control on the L^2 norm of ϕ. This is usually done with the *Poincaré inequalities*. The two most classical ones are the classical *Poincaré inequality*:

Lemma 2.2.3. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$. There exists $C(\Omega)$ such that any function $u \in W_0^{1,2}(\Omega)$ satisfy*

$$\|u\|_{L^2(\Omega)} \leq C(\Omega)\|\nabla u\|_{L^2(\Omega)}.$$

and the Poincaré-Wirtinger inequality (see theorem 2, section 5.8.1 in [?])

Lemma 2.2.4. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$. There exists $C(\Omega)$ such that any function $u \in W^{1,2}(\Omega)$ satisfy*

$$\|u - \bar{u}\|_{L^2(\Omega)} \leq C(\Omega)\|\nabla u\|_{L^2(\Omega)},$$

with $\bar{u}$ the average of u on Ω.

There are, however, many variants which all rely on the same principle: a control on ∇u yields a control on u *up to killing the constant function*. Any condition that kills the constant functions is likely to yield a Poincaré inequality. In our case we will use:

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Lemma 2.2.5. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$. There exists $C(\Omega)$ such that any function $u \in W^{1,2}(\Omega)$ satisfy*

$$\|u - \bar{u}_{\partial\Omega}\|_{L^2(\Omega)} \leq C(\Omega)\|\nabla u\|_{L^2(\Omega)},$$

with $\bar{u}_{\partial\Omega} = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} u d\sigma$.

Proof. We reason by contradiction and assume there exists $u_n \in W^{1,2}(\Omega)$ such that:

$$\|u_n - \overline{u_n}_{\partial\Omega}\|_{L^2(\Omega)} \geq n\|\nabla u_n\|_{L^2(\Omega)}.$$

Up to translation (and considering $\tilde{u}_n = u_n - \overline{u_n}_{\partial\Omega}$) one can assume $\overline{u_n}_{\partial\Omega} = 0$, and up to a dilation one can assume $\|u_n\|_{L^2(\Omega)} = 1$. One then has:

$$\begin{aligned} \|u_n\|_{L^2(\Omega)} &= 1 \\ \overline{u_n}_{\partial\Omega} &= 0 \\ \|\nabla u_n\|_{L^2(\Omega)} &\leq \frac{1}{n}. \end{aligned}$$

By weak compactness, $u_n \to u$ weakly in $W^{1,2}(\Omega)$ and $W^{\frac{1}{2},2}(\Omega)$, and thus strongly in $L^2(\Omega)$ and $L^2(\partial\Omega)$ using the Rellich-Kondrakov theorem. Thus:

$$\begin{aligned} \|u\|_{L^2(\Omega)} &= 1 \\ \overline{u}_{\partial\Omega} &= 0 \\ \|\nabla u\|_{L^2(\Omega)} &\leq \liminf \|\nabla u_n\|_{L^2(\Omega)} = 0. \end{aligned}$$

Thus u is a constant function (since $\|\nabla u\|_{L^2(\Omega)} = 0$), which is non-null ($\|u\|_{L^2(\Omega)} = 1$) and satisfy $\overline{u}_{\partial\Omega} = 0$, which yields a contradiction and concludes the proof. $\square$

In our case let us apply theorem 2.2.5 to ϕ_k:

$$\begin{aligned} \|\phi_k\|_{L^2(\Omega)} &\leq \|\phi_k - \overline{\phi_{k}}_{|\partial\mathbb{D}}\|_{L^2(\Omega)} + \|\overline{\phi_{k}}_{|\partial\mathbb{D}}\|_{L^2(\Omega)} \\ &\leq C(\Omega)\mathcal{D}(\phi_k) + C(\Gamma) \qquad\qquad \leq C(\Omega, \Gamma), \end{aligned}$$

since ϕ_k parametrizes the curve Γ on the boundary, and minimizes its Dirichlet energy. Using the weak compactness in $W^{1,2}(\mathbb{D})$, there exists $\phi \in W^{1,2}(\mathbb{D})$ such that, up to extraction $\phi_k \to \phi$ weakly in $W^{1,2}(\mathbb{D})$ and $W^{\frac{1}{2},2}(\partial\mathbb{D})$. Since $\phi_{k|\partial\mathbb{D}} \to \varphi_0$ on $\partial\mathbb{D}$, $\phi_{|\partial\mathbb{D}} = \varphi_0$. In short: $\phi \in \tilde{\mathcal{C}}(\Gamma)$. Thus:

$$\mu \leq \mathcal{D}(\phi) \leq \liminf \mathcal{D}(\phi_k) = \mu,$$

that is: ϕ is a minimizer of $\mathcal{D}$, and thus a critical point of $\mathcal{D}$, satisfying:

$$\begin{cases} \Delta\phi = 0 \text{ in } \mathbb{D} \\ \phi \text{ parametrizes } \Gamma \text{ on } \partial\mathbb{D}. \end{cases}$$

This concludes our minimization procedure, but not our problem. Indeed, one still needs to show that the solution thus obtained is conformal, to conclude that it is minimal. In fact, we have so far only obtained a harmonic map into $\mathbb{R}^3$, which could have been done much more quickly through a representation formula.

ϕ is conformal

To prove the conformal nature of ϕ, let us study the behavior of $\mathcal{D}$ under the action of a family of diffeomorphisms of the disk: let us define, for α a smooth function

$$\Theta_\varepsilon(x,y) = (r\cos(\theta+\varepsilon\alpha(x,y)), r\sin(\theta+\varepsilon\alpha(x,y))).$$

Since Θ_ε is a family of diffeomorphisms of the disk with $\Theta_0 = Id$, and ϕ minimizes $\mathcal{D}$, then

$$\frac{d}{d\varepsilon}\left[\phi\circ\Theta_\varepsilon\right] = 0.$$

Let us then compute:

$$\begin{aligned}\left[\phi\circ\Theta_\varepsilon\right] &= \phi\left((x,y)+\varepsilon\alpha(-y,x)+o(\varepsilon^2)\right)\\ &= \phi(x,y)+\varepsilon\alpha\left[-y\phi_x+x\phi_y\right]+o(\varepsilon^2),\end{aligned}$$

which yields:

$$\nabla\left[\phi\circ\Theta_\varepsilon\right] = \nabla\phi+\varepsilon\nabla\alpha\left[-y\phi_x+x\phi_y\right]+\varepsilon\alpha\begin{pmatrix}-y\phi_{xx}+x\phi_{xy}+\phi_y\\ -y\phi_{xy}+x\phi_{yy}-\phi_x\end{pmatrix}+o(\varepsilon),$$

which gives us:

$$\begin{aligned}\left|\nabla\left[\phi\circ\Theta_\varepsilon\right]\right|^2 &= |\nabla\phi|^2+2\varepsilon\nabla\alpha.\begin{pmatrix}x\langle\phi_x,\phi_y\rangle-y|\phi_x|^2\\ -y\langle\phi_x,\phi_y\rangle+x|\phi_y|^2\end{pmatrix}+2\varepsilon\alpha\mathrm{div}\left(\frac{|\phi_x|^2+|\phi_y|^2}{2}\begin{pmatrix}-y\\ x\end{pmatrix}\right)+o(\varepsilon)\\ &= |\nabla\phi|^2+2\varepsilon\nabla\alpha.\begin{pmatrix}x\langle\phi_x,\phi_y\rangle+y\left(\frac{|\phi_y|^2}{2}-\frac{|\phi_x|^2}{2}\right)\\ -y\langle\phi_x,\phi_y\rangle+x\left(\frac{|\phi_y|^2}{2}-\frac{|\phi_x|^2}{2}\right)\end{pmatrix}\\ &\quad+2\varepsilon\mathrm{div}\left(\alpha\frac{|\phi_x|^2+|\phi_y|^2}{2}\begin{pmatrix}-y\\ x\end{pmatrix}\right)+o(\varepsilon)\end{aligned}$$

and thus:

$$\mathcal{D}(\phi\circ\Theta_\varepsilon) = \mathcal{D}(\phi)+2\varepsilon\int_{\mathbb{D}}\nabla\alpha.\begin{pmatrix}x\langle\phi_x,\phi_y\rangle+y\left(\frac{|\phi_y|^2}{2}-\frac{|\phi_x|^2}{2}\right)\\ -y\langle\phi_x,\phi_y\rangle+x\left(\frac{|\phi_y|^2}{2}-\frac{|\phi_x|^2}{2}\right)\end{pmatrix}dxdy+o(\varepsilon).$$

Taking first $\alpha = xy$, then $\alpha = x^2-y^2$ yields:

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$$\begin{aligned}&\int_{\mathbb{D}}\langle\phi_x,\phi_y\rangle dxdy = 0\\ &\int_{\mathbb{D}}\left[|\phi_x|^2-|\phi_y|^2\right]dxdy = 0.\end{aligned} \tag{87}$$

In addition, one can show through direct computations that both $\langle \phi_x, \phi_y \rangle$ and $|\phi_x|^2 - |\phi_y|^2$ are harmonic. By the mean value formula, (87) yields:

$$\langle \phi_x, \phi_y \rangle(0) = \int_{\mathbb{D}} \langle \phi_x, \phi_y \rangle dxdy = 0$$
$$\left[|\phi_x|^2 - |\phi_y|^2\right](0) = \int_{\mathbb{D}} \left[|\phi_x|^2 - |\phi_y|^2\right] dxdy = 0.$$

The minimizer ϕ is thus conformal at the origin. But being conformal at a point is invariant by conformal transformations of a disk. Taking any conformal transformation of $\mathbb{D}$ allow one to send any x to the origin, and since $\mathcal{D}$ is conformally invariant, the previous reasonings still stand. We can thus show that ϕ is conformal on the disk, and thus a minimal surface, which solves the Plateau problem.

Remark 2.2.5. *One very important point of this proof is the dialectic produced by the invariance group: the redundant invariances produced need to be controlled to ensure the success of the minimization process, first by changing the energy and then fixing the three remaining degrees of liberty. However, once a solution is produced, these redundant degrees of liberty produce relations, formulas, and loops that can be injected into the analysis to obtain more regularity (here the conformal nature of the solution). We will encounter this idea later when considering Willmore surfaces, but such an idea is also at the basis of many reasonings in general relativity (to introduce the ADM mass notably).*

3 Constant Mean Curvature surfaces

3.1 Definition and introduction

3.1.1 The soap bubble problem

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While the theory of minimal surfaces is linked with the physical problem of *soap films*, constant mean curvature surfaces proceed from *soap bubbles*: when such a soap bubble forms, the soap behaves as an elastic closed surface striving to minimize its area while containing a set volume of fluid (we here assume that the inner gas is incompressible). Mathematically, the resulting surface solves an *isoperimetric problem*: it minimizes its area under a volume constraint.

Let us modelize this situation by considering an immersed surface in $\mathbb{R}^3$: $\Phi : \Sigma \to \mathbb{R}^3$, and let us consider a disk on this surface, and a local parametrization of this disk $\phi : \mathbb{D} \to \mathbb{R}^3$. Let us consider the cone $\mathcal{C}_\phi$ of apex 0 and leaning on $\phi(\partial\mathbb{D})$. The volume of this cone is then given by

$$\mathrm{Vol}(\mathcal{C}_\phi) = \int_0^1 \int_{\mathbb{D}} \langle \phi, \vec{n} \rangle \lambda^2 d\mathrm{vol}_g d\lambda = \frac{1}{3} \int_{\mathbb{D}} \langle \phi, \vec{n} \rangle d\mathrm{vol}_g.$$

If we consider a local perturbation of the surface $\phi_\varepsilon = \phi + \varepsilon\psi$, with $\psi \in C_c^\infty(\mathbb{D}, \mathbb{R}^3)$, the variation of the volume of $\mathcal{C}_\phi$ is then given by:

$$\begin{aligned}
\mathrm{Vol}(\mathcal{C}_{\phi_\varepsilon}) &= \frac{1}{3} \int_{\mathbb{D}} \langle \phi_\varepsilon, \vec{n}_\varepsilon \rangle d\mathrm{vol}_{g_\varepsilon} = \frac{1}{3} \int_{\mathbb{D}} \langle \phi_\varepsilon, \partial_x \phi_\varepsilon \times \partial_y \phi_\varepsilon \rangle dxdy \\
&= \mathrm{Vol}(\mathcal{C}_\phi) + \frac{\varepsilon}{3} \left[\int_{\mathbb{D}} \langle \psi, \partial_x \phi \times \partial_y \phi \rangle + \langle \phi, \partial_x \psi \times \partial_y \phi \rangle + \langle \phi, \partial_x \phi \times \partial_y \psi \rangle dxdy \right] + o(t) \\
&= \mathrm{Vol}(\mathcal{C}_\phi) + \frac{\varepsilon}{3} \left[\int_{\mathbb{D}} \langle \psi, \partial_x \phi \times \partial_y \phi \rangle + \partial_x \left(\langle \phi, \psi \times \partial_y \phi \rangle \right) + \partial_y \left(\langle \phi, \partial_x \phi \times \psi \rangle \right) \right. \\
&\quad \left. - \langle \partial_x \phi, \psi \times \partial_y \phi \rangle - \langle \phi, \psi \times \partial_{xy} \phi \rangle - \langle \partial_y \phi, \partial_x \phi \times \psi \rangle - \langle \phi, \partial_{yx} \phi \times \psi \rangle dxdy \right] + o(\varepsilon) \\
&= \mathrm{Vol}(\mathcal{C}_\phi) + \frac{\varepsilon}{3} \left[\int_{\mathbb{D}} 3 \langle \psi, \partial_x \phi \times \partial_y \phi \rangle + \mathrm{div} \begin{pmatrix} \langle \phi, \psi \times \partial_y \phi \rangle \\ \langle \phi, \partial_x \phi \times \psi \rangle \end{pmatrix} dxdy \right] + o(\varepsilon) \\
&= \mathrm{Vol}(\mathcal{C}_\phi) + \varepsilon \int_{\mathbb{D}} \langle \psi, \vec{n} \rangle d\mathrm{vol}_g + o(\varepsilon).
\end{aligned}$$

Thus, if ϕ is a minimizer of the area for a fixed $\mathrm{Vol}(\mathcal{C}_\phi)$, for all perturbations ψ such that $\mathrm{Vol}(\mathcal{C}_{\phi_\varepsilon}) = \mathrm{Vol}(\mathcal{C}_\phi)$ at the first order, $\delta(\mathrm{A}(\phi_\varepsilon)) = 0$. Applying (73) then yields:

$$\forall \psi \in C_c^\infty(\mathbb{D}) \text{ s.t. } \int_{\mathbb{D}} \langle \psi, \vec{n} \rangle d\mathrm{vol}_g = 0, \int_{\mathbb{D}} \langle \psi, \vec{n} \rangle H d\mathrm{vol}_g = 0,$$

or equivalently

$$\forall \psi \in C_c^\infty(\mathbb{D}) \text{ s.t. } \int_{\mathbb{D}} \langle \psi, \vec{n} \rangle d\mathrm{vol}_g = 0, \forall C \in \mathbb{R}, \int_{\mathbb{D}} \langle \psi, \vec{n} \rangle (H - C) d\mathrm{vol}_g = 0.$$

Taking $C = \bar{H} = \frac{\int_{\mathbb{D}} H d\mathrm{vol}_g}{\int_{\mathbb{D}} d\mathrm{vol}_g}$, and $\psi = \eta \vec{n}$ with η a smooth compactly supported function of null average, one has:

$$\forall \eta \in C_c^\infty(\mathbb{D}) \text{ s.t. } \int_{\mathbb{D}} \eta d\mathrm{vol}_g = 0, \int_{\mathbb{D}} \eta (H - \bar{H}) d\mathrm{vol}_g = 0.$$

Lemma 3.1.1. *Let $H \in L^1_{\mathrm{loc}}(\mathbb{D})$ satisfy:*

$$\forall \eta \in C_c^\infty(\mathbb{D}) \textit{ s.t. } \int_{\mathbb{D}} \eta d\mathrm{vol}_g = 0, \int_{\mathbb{D}} \eta (H - \bar{H}) d\mathrm{vol}_g = 0.$$

Then H is constant.

Proof. Let us first assume that H is continuous, and assume there exists a point $x_0 \in \mathbb{D}$ such that $(H - \bar{H})(x_0) > 0$. Then since $\int_{\mathbb{D}} (H - \bar{H}) d\mathrm{vol}_g = 0$ by construction, there exists a small open ball $B_{r_1}(x_1)$ on which $H - \bar{H} < 0$. Since H is continuous $(H - \bar{H})(x) > \frac{\varepsilon}{2}$ on a small open ball $B_{r_0}(x_0)$ centered at x, Let us consider a small bump η_1 centered on x_0 and compactly supported on $B_{r_0}(x_0)$ and worth 1 on $B_{\frac{r_1}{2}}(x_0)$, and another bump η_2 centered at x_1 and compactly supported in $B_{r_1}(x_1)$. Then $\eta = \eta_1 - \Lambda \eta_2$ is smooth and compactly supported, and with the right choice of Λ (that is $\Lambda = \frac{\int_{B_{r_0}(x_0)} \eta_1 d\mathrm{vol}_g}{\int_{B_{r_1}(x_1)} \eta_2 d\mathrm{vol}_g}$) of null g average. However one has:

$$\int_{\mathbb{D}} \eta (H - \bar{H}) d\mathrm{vol}_g \geq \frac{\varepsilon}{2} \int_{B_{r_0}(x_0)} \eta_1 d\mathrm{vol}_g > 0.$$

Thus, one has necessarily that $H = \bar{H}$. If H is not continuous and merely L^1_{loc}, one can mollify it using a kernel ξ_ε and thus obtain $F_\varepsilon = \int_{\mathbb{R}^2} \xi_\varepsilon(z - x)(H - \bar{H}) 1_{\mathbb{D}}(z) dz$ a sequence of smooth functions converging toward $(H - \bar{H}) 1_{\mathbb{D}}$ in L^1_{loc}. For η a smooth compactly supported function of null g-average, one has:

$$\begin{aligned}
\int_{\mathbb{D}} \eta(x) F_\varepsilon(x) d\mathrm{vol}_g(x) &= \int_{\mathbb{D}} \int_{\mathbb{R}^2} \eta(x) \xi_\varepsilon(z) (H - \bar{H}) 1_{\mathbb{D}}(z + x) dz d\mathrm{vol}_g(x) \\
&= \int_{\mathbb{R}^2} \xi_\varepsilon(z) \left(\int_{\mathbb{D}} \eta(z - x)(H - \bar{H}) 1_{\mathbb{D}}(x) d\mathrm{vol}_g(x) \right) dz = 0.
\end{aligned}$$

Since, for all z small enough, $\eta(z-x)$ is a smooth compactly supported function of null g-average. Thus, the result stands for all F_ε, and goes to the limit with the L^1_{loc} convergence. $\square$

Thus, we deduce that critical points of the area under a first order volume constraint are of constant mean curvature. If we go back to the physical source of our problem, one can modelize a soap bubble as a surface of constant mean curvature.

Remark 3.1.1. *There is some very annoying peculiarity in our definition of the volume: it is not invariant by translation. Indeed, given $\vec{a} \in \mathbb{R}^3$:*

$$\mathcal{C}_{\phi+\vec{a}} = \mathcal{C}_\phi + \langle \vec{a}, \int_{\mathbb{D}} \vec{n} d\text{vol}_g \rangle.$$

However, its first order variation is a translational invariant! So while the value assigned to the volume is rather arbitrary, the notion of it remaining the same up to the second order is invariant.

Remark 3.1.2. *One recovers the same dialectic as with minimal/minimizing surfaces: not all constant mean curvature surfaces minimize their area among surfaces containing a fixed volume. Such surfaces are called* isoperimetric*, and isoperimetric surfaces are indeed of constant mean curvature.*

3.1.2 Analysis of CMC surfaces

The Gauss map of a mean curvature surface

Before going into the heart of the behaviour, let us go back for a while to the soap film problem. As detailed in section 2.1.1, the soap film minimizes its area. Thus, if we consider a perturbation $\Phi + \varepsilon\psi$, $\mathcal{A}(\Phi + \varepsilon\psi) \geq \mathcal{A}(\Phi)$, which implies that the first order variation cancels out:

$$\delta\mathcal{A}(\Phi) = -\int_\Sigma \langle \psi, \Delta_g \Phi \rangle d\text{vol}_g.$$

If we stretch slightly our definitions, we can consider that Φ takes value in a manifold M of dimension m, immersed in $\mathbb{R}^n$, with $n > m$. While we have not introduced a proper definition of the mean curvature in codimension greater than one (which can be done without too much effort), all the metric quantities are *intrinsic*, and can thus be generalized to any codimension, as with the definition of the area and its variational study. But let us now assume that Φ describes a soap film in $\mathbb{R}^n$ which is constrained to remain on the immersed manifold $M \subset \mathbb{R}^n$. Then Φ must minimize the area among all surfaces in M. So, if we consider a perturbation $\Phi_\varepsilon = \Phi + \varepsilon\psi + o(\varepsilon)$, such that $\Phi_\varepsilon \in M$, one must still have $\delta\mathcal{A}(\Phi) = 0$. The computations leading to $\delta\mathcal{A}$ remains mechanically the same and lead to the same result:

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$$\delta\mathcal{A}(\Phi) = -\int_\Sigma \langle \psi, \Delta_g \Phi \rangle d\mathrm{vol}_g. \tag{88}$$

However, this time ψ cannot cover all functions in $\mathbb{R}^n$ but is constrained to be the first variation of a transformation satisfying $\Phi_\varepsilon \in M$, which means $\psi = \delta\Phi_\varepsilon \in TM$. Condition (88) then ensures that the parametrization of a soap film satisfies:

$$\Delta_g \Phi \in (TM)^\perp.$$

Definition 3.1.1. *If $M \subset \mathbb{R}^n$ is a submanifold of dimension $m < n$ and $\Phi : \Sigma \to \mathbb{R}^n$ is an immersed surface in $\mathbb{R}^n$, we say that Φ is a harmonic map in M if it satisfies, for all $p \in M$:*

$$\begin{aligned} &\Phi(p) \in M \\ &\Delta_g \Phi(p) \in \left(T_{\Phi(p)} M\right)^\perp . \end{aligned}$$

If, in addition Σ is a subset of $\mathbb{R}^2$ and if Φ is conformal, we say that Φ is minimal.

Thus, what generalizes the notion of a minimal surface to applications taking values in a submanifold are harmonic maps. If we now assume that M is a hypersurface, that is that $m = n - 1$, then at each point $P \in M$, there exists a unique vector $N(P)$ normal to M at P. The condition $\Delta_g \Phi(p) \in \left(T_{\Phi(p)} M\right)^\perp$ is then naturally equivalent to $\Delta_g \Phi(p) = C(p) N(\Phi(p))$. If we denote $X = N \circ \Phi$ as a function on Σ, and by definition of N:

$$\begin{aligned} C = \langle \Delta_g \Phi, X \rangle &= \operatorname{div}\left(\langle \nabla \Phi, X \rangle\right) - \langle \nabla \Phi . \nabla X \rangle \\ &= -\langle \nabla \Phi . \nabla X \rangle. \end{aligned}$$

If in addition, M is a euclidean hypersphere $\mathbb{S}^{n-1}$, then the normal N is merely the identity ($N(P) = P$, which implies $X = \Phi$), from which we deduce that

$$C = -|\nabla \Phi|_g^2.$$

Thus, a harmonic map into the sphere $\mathbb{S}^{n-1}$ satisfy

$$\Delta_g \Phi + |\nabla \Phi|^2 \Phi = 0.$$

If we now come back to the original problem and consider $\Phi : \Sigma \to \mathbb{R}^3$ a CMC immersion, its Gauss map is an application $\vec{n} : \Sigma \to \mathbb{R}^3$, which satisfy equation (47):

$$\Delta_g \vec{n} = -|\nabla \vec{n}|_g^2 \vec{n} - 2\nabla H . \nabla \Phi = -|\nabla \vec{n}|_g^2 \vec{n}.$$

Thus $\vec{n}$ is harmonic if and only if Φ is a CMC immersion. If we now look at the immersion in a local conformal chart $\phi : \mathbb{D} \to \mathbb{R}^3$, then

$$\begin{aligned} \vec{n}_x &= -(e\phi_x + f\phi_y)e^{-2\lambda} \\ \vec{n}_y &= -(f\phi_x + g\phi_y)e^{-2\lambda}, \end{aligned}$$

which means:

$$\begin{aligned} |\vec{n}_x|^2 - |\vec{n}_y|^2 &= (e^2 - g^2)e^{-2\lambda} = 2H(e-g) \\ \langle \vec{n}_x, \vec{n}_y \rangle &= (ef + fg)e^{-2\lambda} = 2Hf. \end{aligned}$$

From this, we deduce:

Proposition 3.1.1. *The Gauss map $\vec{n}$ of an immersion Φ is harmonic if and only if Φ is CMC. In addition, in a local conformal chart, $\vec{n}$ is minimal if and only if ϕ is minimal.*

The fact that the Gauss map of a CMC surface is harmonic has far-reaching consequences. Among them is the possibility of writing a Weierstrass representation for CMC surfaces. The harmonic map theory also forms a very interesting problem of its own. We particularly recommend [**?**] on this subject (and others). We will not detail them to save time for the Willmore problem, and will instead focus on the local analysis of the CMC equation in graph form and in conformal coordinates.

Constant Mean Curvature graphs

If we now study a constant mean curvature surface locally as a graph over its tangent plane, the equation becomes:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 2H.$$

So, in a graph form, the CMC operator is almost the same as the minimal operator, with only a different right-hand side. This allows one to write a comparison principle for CMC graphs, with the same geometric consequence as for minimal ones.

Theorem 3.1.2. *If ϕ and ψ are two CMC graphs over Ω such that*

$$\begin{aligned} & H_\phi \geq H_\psi \\ & \phi_{|\partial\Omega} \leq \psi_{|\partial\Omega}. \end{aligned}$$

Then $\phi(\Omega)$ is below $\psi(\Omega)$.

Proof. One simply has to apply the first part of theorem 2.1.8 to conclude. □

While the comparison principle remains true for CMC surfaces (and in fact a pointwise inequality between the mean curvatures without any assumption of constance yields the same result), the maximum principle fails here, and any result like proposition 2.1.7 has no hope to stand. Indeed, if we consider a spherical cap written as a graph over a disk, the highest point is reached in the center of the disk.

Study in a local conformal chart

Given an immersed constant mean curvature surface $\Phi : \Sigma \to \mathbb{R}^3$, thanks to (61) and theorem 1.3.5, one can parametrize $\Phi(\Sigma)$ locally by a conformal immersion $\phi : \mathbb{D} \to \mathbb{R}^3$ satisfying:

$$\begin{cases} \Delta\phi = 2H\phi_x \times \phi_y = H\nabla^{\perp}\phi \times \nabla\phi \\ |\phi_x|^2 - |\phi_y|^2 = \langle \phi_x, \phi_y \rangle = 0, \end{cases}$$

with H a real constant. The analysis in a local conformal chart is thus a touch more complex than the minimal case and requires us to use the *Calderón-Zygmund* theory, which we will quickly review here. The Calderón-Zygmund theorem (see theorem 9.9 of [**?**] for the proof) states:

Theorem 3.1.3. *Let $\Omega \subset\subset \mathbb{R}^2$ a smooth bounded domain and $f \in L^p(\Omega)$, $1 < p < \infty$, and let w be the Newtonian potential of f:*

$$w(x) = \frac{1}{2\pi}(\ln|.| * f) = \frac{1}{2\pi}\int_{\Omega} \ln|z - x| f(z) dz.$$

Then, w satisfies:

$$\Delta w = f \; \|w\|_{W^{2,p}(\Omega)} \leq C_p \|f\|_{L^p(\Omega)}.$$

Thus, if we consider u a weak solution in $W^{1,p}(\Omega)$ to the Dirichlet problem:

$$\begin{cases} \Delta u = f \text{ in } \Omega \\ \quad u = \varphi \text{ on } \partial\Omega, \end{cases} \tag{89}$$ {060320211212}

with $\varphi \in L^1(\partial\Omega)$, then $v = u - w$ solves:

$$\begin{cases} \Delta v = 0 \text{ in } \Omega \\ \quad v = \varphi - \mathrm{Tr}_{|\partial\Omega}(w) \text{ on } \partial\Omega. \end{cases}$$

Thus, v is harmonic on Ω and, thanks to the representation formula, is thus smooth on any $\Omega' \subset\subset \Omega$ and satisfies:

$$\begin{aligned}\|v\|_{W^{2,p}(\Omega')} &\leq C(p,\Omega',\Omega)\|\varphi - \mathrm{Tr}_{|\partial\Omega}(w)\|_{L^1(\partial\Omega)} \\ &\leq C(p,\Omega',\Omega)\left(\|\varphi\|_{L^1(\partial\Omega)} + \|w\|_{W^{2,p}(\Omega)}\right).\end{aligned}$$

One thus obtains the following interior estimates:

$$\|u\|_{W^{2,p}(\Omega')} \leq \|v\|_{W^{2,p}(\Omega')} + \|w\|_{W^{2,p}(\Omega')} \leq C(p,\Omega',\Omega)\left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^p(\Omega)}\right),$$

which we compile in the following theorem:

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Theorem 3.1.4. *Let u be a solution of the Poisson problem with the Dirichlet boundary conditions* (89)*, with $f \in L^p(\Omega)$ and $\varphi \in L^1(\partial\Omega)$. Then u satisfies, for any $\Omega' \subset\subset \Omega$:*

$$\|u\|_{W^{2,p}(\Omega')} \leq C(p,\Omega',\Omega)\left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^p(\Omega)}\right).$$

Remark 3.1.3. *This is but a brief look at the Calderón-Zygmund theory which spans a much bigger domain. For instance, one needs not limit oneself to the straight laplacian, but can apply it to a uniformly elliptic operator whose coefficients are regular enough (in the right Sobolev spaces). For instance, under good integration properties of the metric, Δ_g will fall under the the dominion of Calderón-Zygmund. We refer the reader to the whole chapter 9 of [?] for more details, or theorem 9.13 and lemma 9.17 for two precise results.*

Remark 3.1.4. *These estimates are interior estimates, on smaller domains. One can find estimates up to the boundary by imposing more regularity on the boundary condition (enough to ensure that the harmonic function is $W^{2,p}$ on the whole of Ω). Classically, this requires $\varphi \in W^{2-\frac{1}{p},p}$, or φ is the trace of a $W^{2,p}$ function. In our case, since we are looking at surfaces in local charts, interior estimates are enough, and more natural. It will be slightly different for the Plateau problem, but then the boundary estimates are natural.*

Remark 3.1.5. *However, the hypothesis $p > 1$ is absolutely pivotal. One can find a function whose Laplacian is L^1, but which is not in $W^{2,1}$. This strict inequality is precisely what makes some equations critical, and studying what happens at the boundary of this condition is both an active domain in PDE study, and one of the subject of what is to follow.*

These estimates are carried over for $f \in W^{k,p}$:

Corollary 3.1.1. *Let u be a solution of the Poisson problem with the Dirichlet boundary conditions* (89), *with $f \in W^{k,p}(\Omega)$ and $\varphi \in L^1(\partial\Omega)$. Then u satisfies, for any $\Omega' \subset\subset \Omega$:*

$$\|u\|_{W^{k+2,p}(\Omega')} \leq C(p, \Omega', \Omega) \left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{W^{k,p}(\Omega)} \right).$$

If we go back to the CMC surface in a local conformal parametrization, we can prove the following regularity result:

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Theorem 3.1.5. *Let $\phi \in W^{1,p}(\mathbb{D})$ be the local conformal parametrization of a CMC surface of mean curvature H. If $p > 2$, then ϕ is smooth on $\mathbb{D}$, and for every radius $r < 1$ and integer k:*

$$\|\phi\|_{W^{k,p}(\mathbb{D}_r)} \leq C(r, p, k, H) \|\phi\|_{W^{1,p}(\mathbb{D})}.$$

Proof. With $\phi \in W^{1,p}(\mathbb{D})$, the trace of ϕ on the boundary value is in $W^{1-\frac{1}{p},p}(\partial\mathbb{D})$, and thus necessarily in $L^1(\partial\mathbb{D})$, which takes care of the boundary condition of theorem 3.1.4. In addition, $\phi_x \times \phi_y \in L^{\frac{p}{2}}$ with $\frac{p}{2} > 1$. The parametrization ϕ thus solves an equation of the type $\Delta\phi = f(\phi) \in L^q(\mathbb{D})$, with $q > 1$. One can then apply theorem 3.1.4 for any $r < r' < 1$ and deduce:

$$\|\phi\|_{W^{2,\frac{p}{2}}(\mathbb{D}_{r'})} \leq C(r', p, H) \|\phi\|_{W^{1,p}(\mathbb{D})}.$$

Sobolev embeddings then ensure that $\phi \in W^{1,\frac{2p}{4-p}}$ if $p < 4$, and in a $C^{1,\alpha}$ space if $p > 4$. In the first case, since $\frac{2p}{4-p} > p$, we can reiterate the process. As long as $p_n < 4$, we can then increase the exponent in the iterative manner: $p_{n+1} = \frac{2p_n}{4-p_n}$. Since this sequence is strictly increasing and cannot converge if $p > 2$, we end up in a finite time with an exponent $p_m > 4$ and thus:

$$\|\phi\|_{W^{2,\frac{p_m}{2}}(\mathbb{D}_{r_m})} \leq C(r, p, H) \|\phi\|_{W^{1,p}(\mathbb{D})}.$$

Since one then has $\nabla\phi \in C^{0,\alpha}(\mathbb{D}_{r_m})$, $\phi_x \times \phi_y$ is then in $W^{1,\frac{p_m}{2}}(\mathbb{D}_{r_m})$. Thus ϕ solves an equation of the type $\Delta\phi = f(\phi) \in W^{1,q'}(\mathbb{D}_{r_m})$. We have thus increased the differentiability of the right-hand side, at the cost of restricting slightly the domaine. We can thus reiterate the previous procedure to obtain the result for all k. □

Remark 3.1.6. *The hypothesis $p > 2$ is necessary with this procedure, to be able to launch the Calderón-Zygmund bootstrap. It is however problematic in that it is more regularity than what we are used to ask for (for instance during the minimization procedure) and will require us to go through some more subtle analysis.*

3.2 Compactness through compensation

In this section, we will follow mainly the third chapter of F. Hélein's [?], itself derived from H. Brézis and J.M. Coron's [?]. We recommend greatly F. Hélein's book for a cristal clear look at these considerations.

3.2.1 Wente's lemma

We thus need to study the equation $\Delta\phi = 2H\phi_x \times \phi_y = H\nabla^\perp\phi \times \nabla\phi$ with H a constant. Here, the value of the constant is not important. Geometrically because H can be changed up to a dilation, analytically by considering $H\phi$ instead of ϕ. In addition, the vectorial nature of the mean curvature equation is not pivotal. Accordingly, we will thus consider the equation on $u \in W^{1,2}(\mathbb{D})$ for $a, b \in W^{1,2}(\mathbb{D})$:

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$$\begin{cases} \Delta u = \nabla^\perp a.\nabla b \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{90}$$

The first remark one can make on this equation is that it is a conformal invariant:

Lemma 3.2.1. *Let u be a solution of* (90) *and Θ a conformal diffeomorphism of Ω. Then, $u \circ \Theta$ solves:*

$$\begin{cases} \Delta(u\circ\Theta) = \nabla^\perp(a\circ\Theta).\nabla(b\circ\Theta) \ \textit{in } \Omega \\ u\circ\Theta = 0 \ \textit{on } \partial\Omega. \end{cases}$$

Proof. This is merely a computation. Recall (84):

$$\Theta^1_x = \Theta^2_y,\ \Theta^2_x = -\Theta^1_y.$$

Then:

$$\begin{aligned}
\partial_x(a\circ\Theta) &= a_x\circ\Theta\Theta^1_x + a_y\circ\Theta\Theta^2_x \\
\partial_y(a\circ\Theta) &= a_x\circ\Theta\Theta^1_y + a_y\circ\Theta\Theta^2_y = -a_x\circ\Theta\Theta^2_x + a_y\circ\Theta\Theta^1_x \\
\partial_x(b\circ\Theta) &= b_x\circ\Theta\Theta^1_x + b_y\circ\Theta\Theta^2_x \\
\partial_y(b\circ\Theta) &= b_x\circ\Theta\Theta^1_y + b_y\circ\Theta\Theta^2_y = -b_x\circ\Theta\Theta^2_x + b_y\circ\Theta\Theta^1_x \\
\Delta(u\circ\Theta) &= u_{xx}\circ\Theta\left[\left(\Theta^1_x\right)^2 + \left(\Theta^2_x\right)^2\right] + u_{yy}\circ\Theta\left(\Theta^2_y\right)^2 = \Delta u\circ\Theta\left[\left(\Theta^1_x\right)^2 + \left(\Theta^2_x\right)^2\right] \\
\nabla^\perp(a\circ\Theta).\nabla(b\circ\Theta) &= (a\circ\Theta)_x(b\circ\Theta)_y - (a\circ\Theta)_y(b\circ\Theta)_x \\
&= (a_xb_y - a_yb_x)\left[\left(\Theta^1_x\right)^2 + \left(\Theta^2_x\right)^2\right],
\end{aligned}$$

which concludes the proof. □

The fundamental result for all the *compactness through compensation* theory is *Wente's lemma* which gives both a $W^{1,2}$ and L^∞ estimate for solutions of (90):

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Lemma 3.2.2. *Let u solve* (90) *on $\mathbb{D}$. Then, there exists C such that u satisfies:*

$$\|u\|_{L^\infty(\mathbb{D})} + \|\nabla u\|_{L^2(\mathbb{D})} \leq C\|\nabla a\|_{L^2(\mathbb{D})}\|\nabla b\|_{L^2(\mathbb{D})}.$$

Proof. We will first prove the control for $u(0)$. The result for $\|u\|_{L^\infty(\mathbb{D})}$ will follow thanks to the conformal invariance, and will imply the control on ∇u.

Step 1: control on $u(0)$

We can start by assuming that all the involved terms are smooth. By density, the result will be extended to all $a, b \in W^{1,2}(\mathbb{D})$. Since u solves a Poisson equation with Dirichlet boundary condition, it satisfies a representation formula:

$$u(0) = \frac{1}{2\pi}\int_{\mathbb{D}} \nabla^\perp a(x).\nabla b(x) \ln\frac{1}{r} dxdy.$$

The formula becomes much more exploitable in polar coordinates:

$$\begin{aligned}
u(0) &= \frac{1}{2\pi}\int_{\mathbb{D}} \nabla^\perp a(x).\nabla b(x) \ln\frac{1}{r} dxdy \\
&= \frac{1}{2\pi}\int_{\mathbb{D}} \frac{1}{r}(a_r b_\theta - a_\theta b_r)\ln\frac{1}{r} r dr d\theta = \frac{1}{2\pi}\int_0^1 dr \int_0^{2\pi} \ln\frac{1}{r}\left[\partial_r(ab_\theta) - \partial_\theta(ab_r)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^1 dr \int_0^{2\pi} \ln\frac{1}{r}\left[\partial_r(ab_\theta) - \partial_\theta(ab_r)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^1 dr \int_0^{2\pi} \ln\frac{1}{r}\partial_r([a - \bar{a}_r]\, b_\theta) d\theta,
\end{aligned}$$

since $\int_0^{2\pi} \partial_\theta(ab_r)d\theta = 0$, and where $\bar{a}_r = \frac{1}{2\pi}\int_0^{2\pi} a d\theta$. For any $\varepsilon > 0$, one has:

$$\int_\varepsilon^1 dr \int_0^{2\pi} \ln\frac{1}{r}\partial_r([a - \bar{a}_r]\, b_\theta)d\theta = \ln\varepsilon\left(\int_0^{2\pi}[a - \bar{a}_r]\, b_\theta d\theta\right)(\varepsilon) - \int_\varepsilon^1 \frac{dr}{r}\int_0^{2\pi}[a - \bar{a}_r]\, b_\theta d\theta. \tag{91}$$

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Besides, using Hölder's inequality:

$$\begin{aligned}
\left|\int_0^{2\pi}[a - \bar{a}_r]\, b_\theta d\theta\right| &\leq \left(\int_0^{2\pi}|a - \bar{a}_r|^2 d\theta\right)^{\frac{1}{2}}\left(\int_0^{2\pi}|b_\theta|^2 d\theta\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^{2\pi}|a_\theta|^2 d\theta\right)^{\frac{1}{2}}\left(\int_0^{2\pi}|b_\theta|^2 d\theta\right)^{\frac{1}{2}}
\end{aligned}$$

with Poincaré-Wirtinger's inequality. Thus:

$$\left|\int_0^{2\pi}[a - \bar{a}_r]\, b_\theta d\theta\right| \leq r^2\left(\int_0^{2\pi}|\nabla a|^2 d\theta\right)^{\frac{1}{2}}\left(\int_0^{2\pi}|\nabla b|^2 d\theta\right)^{\frac{1}{2}}.$$

Injecting this into (91) yields:

$$\left|\int_\varepsilon^1 dr \int_0^{2\pi} \ln\frac{1}{r}\partial_r([a-\bar{a}_r]\,b_\theta)d\theta\right| \leq \varepsilon^2\ln\varepsilon\left(\int_0^{2\pi}|\nabla a|^2 d\theta\right)^{\frac{1}{2}}(\varepsilon)\left(\int_0^{2\pi}|\nabla b|^2 d\theta\right)^{\frac{1}{2}}(\varepsilon) + \int_\varepsilon^1 r\left(\int_0^{2\pi}|\nabla a|^2 d\theta\right)^{\frac{1}{2}}\left(\int_0^{2\pi}|\nabla b|^2 d\theta\right)^{\frac{1}{2}} dr.$$

To deal with the first term, let us denote $f(r) = \left(\int_0^{2\pi}|\nabla a|^2 d\theta\right)^{\frac{1}{2}}$ and $g(r) = \left(\int_0^{2\pi}|\nabla b|^2 d\theta\right)^{\frac{1}{2}}$. Since $\nabla a, \nabla b \in L^2(\mathbb{D})$, $rf^2 + rg^2 \in L^1((0,1))$. Thus, if there exists $C>0$ such that for all $\varepsilon>0$, one has: $\varepsilon^2\ln\varepsilon f(\varepsilon)g(\varepsilon)\geq C$, which implies that for all $\varepsilon>0$:

$$\varepsilon(f^2(\varepsilon)+g^2(\varepsilon)) \geq 2\varepsilon f(\varepsilon)g(\varepsilon) \geq \frac{C}{\varepsilon\ln\varepsilon}.$$

Since the right-hand side is not integrable, while the left-hand side is, this leads to a contradiction. We can (and will) then chose a sequence $\varepsilon_n \to 0$ such that: $\varepsilon_n f(\varepsilon_n)g(\varepsilon_n)\to 0$.

For these ε_n,

$$\begin{aligned}\int_{\varepsilon_n}^1 r\left(\int_0^{2\pi}|\nabla a|^2 d\theta\right)^{\frac{1}{2}}\left(\int_0^{2\pi}|\nabla b|^2 d\theta\right)^{\frac{1}{2}} dr &\leq \int_0^1\left(\int_0^{2\pi} r|\nabla a|^2 d\theta\right)^{\frac{1}{2}}\left(\int_0^{2\pi} r|\nabla b|^2 d\theta\right)^{\frac{1}{2}} dr \\ &\leq \|\nabla a\|_{L^2(\mathbb{D})}\|\nabla b\|_{L^2(\mathbb{D})}.\end{aligned}$$

Taking $\varepsilon_n\to 0$, we then obtain:

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$$|u(0)| \leq \frac{1}{2\pi}\|\nabla a\|_{L^2(\mathbb{D})}\|\nabla b\|_{L^2(\mathbb{D})}. \tag{92}$$

Step 2: control on $\|u\|_{L^\infty(\mathbb{D})}$
For any $p\in\mathbb{D}$, one can exchange p and 0 using a conformal diffeomorphism of the disk and consider the composition of u and this diffeomorphism to transfer (92) to p, and thus yields the L^∞ estimate.
Step 3: control on $\|\nabla u\|_{L^2(\mathbb{D})}$
We simply write:

$$\begin{aligned}\|\nabla u\|^2_{L^2(\mathbb{D})} \leq \left|\int_{\mathbb{D}} u\Delta u dxdy\right| &\leq \|u\|_{L^\infty(\mathbb{D})}\|\nabla^\perp a\nabla b\|_{L^1(\mathbb{D})} \\ &\leq \frac{1}{2\pi}\|\nabla a\|_{L^2(\mathbb{D})}\|\nabla b\|_{L^2(\mathbb{D})},\end{aligned}$$

which yields the final result with $C = \frac{1}{2\pi}+\frac{1}{\sqrt{2\pi}}$. □

Remark 3.2.1. *The shape of the right-hand side is entirely pivotal to this result. It can be seen in the proof, where it allows one to isolate the ∂_θ, but is perhaps much more obvious when looking at counter-examples. The equation $\Delta u = |\nabla u|^2$ has the same growth behavior as the CMC equation, but* does not *satisfy Wente-like estimates. The algebraic shape, the Jacobian shape is what makes such an equation remarkable: it allows one to compensate for the failure of the rough integral estimates and yields more compactness.*

Remark 3.2.2. *One should, at this point, go back to chapter 1 and see how often such equations are found when looking at the structure equations for surfaces.*

Remark 3.2.3. *The estimates given by 3.2.2 are invariant by conformal reparametrization. Crucially, this means that they are* scale-invariant*: the result is the same, with the same constant on any $\mathbb{D}_r$.*

Remark 3.2.4. *As explained above, the result still stands with $H \neq 1$, H* constant*. It* does not *stand with H a function, which would disturb the integration by parts.*

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Remark 3.2.5. *The question of the optimal constant in Wente's lemma is interesting on its own. We refer the reader to chapter 3 of [**?**] and will only mention that $C = \sqrt{\frac{3}{16\pi}}$.*

3.2.2 Application: ε-regularity for CMC surfaces

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We here wish to apply Wente's lemma and show a regularity result for CMC surfaces starting with mere $W^{1,2}$ controls on the immersion. Given theorem 3.1.5, it is enough to show L^p estimates to break the criticality and conclude. All the focus should then be on breaking criticality. To that end, we will recall a PDE technique called "estimates on Riesz potentials". The original proof can be found in [**?**].

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Lemma 3.2.3. *Let u be a $W^{1,2}$ solution of $\Delta u = f \in L^1$ such that: there exists $\alpha > 0$ and $C > 0$ such that for all $s > 0$ and for all $p \in \mathbb{D}_{\frac{1}{2}}$ such that*

$$\int_{B_s(p)} |f| dxdy \leq C_M s^\alpha. \tag{93}$$ {090320211800}

Then there exists $p > 2$ such that $\nabla u \in L^p\left(\mathbb{D}_{\frac{1}{4}}\right)$ and:

$$\|\nabla u\|_{L^p\left(\mathbb{D}_{\frac{1}{4}}\right)} \leq C\left(\|\nabla u\|_{L^2(\mathbb{D})} + C_M\right).$$

Proof. Since $\Delta u = f$ one can use the Green representation formula:,$\nabla u(x) = A(x) + B(x)$, where:

$$A(x) = \nabla \int_{\mathbb{D}_{\frac{1}{2}}} G(x-y) f dy$$
$$B(x) = \int_{\partial \mathbb{D}_{\frac{1}{2}}} \nabla \partial_{\vec{\nu}} G(x-y) u(y) d\sigma_y,$$

with G the Green function of the Laplacian on $\mathbb{D}_{\frac{1}{2}}$.

For $x \in \mathbb{D}_{\frac{1}{4}}$, $y \in \mathbb{D}_{\frac{1}{2}}$ $|\partial_{\vec{\nu}} G(x-y)| \leq \frac{C}{|x-y|} \leq C$. Thus

$$B(x) \leq C ||u||_{L^1\left(\partial \mathbb{D}_{\frac{1}{2}}\right)} \leq C ||\nabla u||_{L^2\left(\mathbb{D}_{\frac{1}{2}}\right)},$$

and thus, for any $p > 2$:

$$||B||_{L^p\left(\mathbb{D}_{\frac{1}{4}}\right)} \leq C ||\nabla u||_{L^2\left(\mathbb{D}_{\frac{1}{2}}\right)}.$$

The B term is thus not going to be problematic. Let us look at the A term:

$$\begin{aligned}
|A(x)| &\leq \left| \int_{\mathbb{D}_{\frac{1}{2}}} \nabla G(x-y) f(y) dy \right| \\
&\leq \int_{B_{\frac{1}{4}}(x)} |\nabla G(x-y)| f(y) dy + \int_{\mathbb{D} \backslash B_{\frac{1}{4}}(x)} |\nabla G(x-y)| |f(y)| dy \\
&\leq \sum_{n \in \mathbb{N}^*} \int_{B_{\frac{1}{4^n}}(x) \backslash B_{\frac{1}{4^{n+1}}}(x)} \frac{1}{|x-y|^{1-\frac{\alpha}{4}}} f(y) |x-y|^{-\frac{\alpha}{4}} + C \int_{\mathbb{D} \backslash B_{\frac{1}{4}}(x)} \frac{1}{|x-y|} f(y) dy \\
&\leq \sum_{n \in \mathbb{N}^*} \int_{B_{\frac{1}{4^n}}(x) \backslash B_{\frac{1}{4^{n+1}}}(x)} \frac{1}{|x-y|^{1-\frac{\alpha}{4}}} f(y) \left(\frac{1}{4^{n+1}} \right)^{-\frac{\alpha}{4}} + C \int_{\mathbb{D} \backslash B_{\frac{1}{4}}(x)} f(y) dy.
\end{aligned}$$

In the estimates above we used that for $y \in B_{\frac{1}{4^n}}(x) \backslash B_{\frac{1}{4^{n+1}}}(x)$, $|x-y| \geq \frac{1}{4^{n+1}}$. This yields:

$$\begin{aligned}
|A(x)| &\leq C \left(\sum_{n \in \mathbb{N}^*} \int_{B_{\frac{1}{4^n}}(x) \backslash B_{\frac{1}{4^{n+1}}}(x)} \frac{1}{|x-y|^{1-\frac{\alpha}{4}}} |f|(y) dy \left(\frac{1}{4^{n+1}} \right)^{\frac{\alpha}{4}} \left(\frac{1}{4^n} \right)^{-\frac{\alpha}{2}} \right. \\
&\quad \left. + ||f||_{L^2(\mathbb{D})} \right) \\
&\leq C \left(||f||_{L^2(\mathbb{D})} + \sum_{n \in \mathbb{N}^*} \left(\frac{1}{4^{n+1}} \right)^{\frac{\alpha}{4}} \left(\frac{1}{4^n} \right)^{-\frac{\alpha}{2}} \frac{1}{|.|^{1-\frac{\alpha}{4}}} *_{\mathbb{D}_{\frac{1}{4^n}}} |f|(x) \right).
\end{aligned}$$

Thus:

$$\begin{aligned}
||\nabla u||_{L^p(\mathbb{D}_{\frac{1}{4}})} &\leq C\left(||\nabla u||_{L^2\left(\mathbb{D}_{\frac{1}{2}}\right)} + \sum_{n\in\mathbb{N}^*}\left(\frac{1}{4^{n+1}}\right)^{\frac{\alpha}{4}}\left(\frac{1}{4^n}\right)^{-\frac{\alpha}{2}}\left\|\frac{1}{|.|^{1-\frac{\alpha}{4}}} *_{\mathbb{D}_{\frac{1}{4^n}}} |f|\right\|_{L^p\left(\mathbb{D}_{\frac{1}{2}}\right)}\right)\\
&\leq C\left(||\nabla u||_{L^2\left(\mathbb{D}_{\frac{1}{2}}\right)} + \sum_{n\in\mathbb{N}^*}\left(\frac{1}{4^{n+1}}\right)^{\frac{\alpha}{4}}\left(\frac{1}{4^n}\right)^{-\frac{\alpha}{2}}\left\|\frac{1}{|.|^{1-\frac{\alpha}{4}}}\right\|_{L^p(\mathbb{D}_{\frac{1}{2}})}\left(\sup_{x\in\mathbb{D}_{\frac{1}{2}}}||f||_{L^1\left(B_{\frac{1}{4^n}}(x)\right)}\right)\right)\\
&\leq C\left(||\nabla u||_{L^2\left(\mathbb{D}_{\frac{1}{2}}\right)} + \sum_{n\in\mathbb{N}^*}\left(\frac{1}{4^{n+1}}\right)^{\frac{\alpha}{4}}\left\|\frac{1}{|.|^{1-\frac{\alpha}{4}}}\right\|_{L^p(\mathbb{D}_{\frac{1}{2}})}\left(\sup_{x\in\mathbb{D}_{\frac{1}{2}}}\left(\frac{1}{4^n}\right)^{-\frac{\alpha}{2}}||f||_{L^1\left(B_{\frac{1}{4^n}}(x)\right)}\right)\right)\\
&\leq C\left(||\nabla u||_{L^2(\mathbb{D})} + C_M\left\|\frac{1}{|.|^{1-\frac{\alpha}{4}}}\right\|_{L^p\left((\mathbb{D}_{\frac{1}{2}}\right)}\right) < \infty \text{ if } 2 < p < \frac{2}{1-\frac{\alpha}{4}}.
\end{aligned}$$

Going from the first line to the second required Young's inequality. This concludes the proof. Here the Young's inequality for the convolution is not the classical one, but a modification on a subset of $\mathbb{R}^2$, with the same proof. We leave it to the reader. □

Remark 3.2.6. *In lemma 3.2.3, the added inequality is called a* Morrey estimate*, and f is in a* Morrey space*. They are a domain of study of their own and we have barely broached the subject, with a very rough proof of how to use them to break criticality.*

Remark 3.2.7. *In lemma 3.2.3, we work on $\mathbb{D}_{\frac{1}{4}}$ for convenience, but any $r < 1$ would have been enough to ensure the interior estimate on B.*

{100320211540}

Theorem 3.2.1. *Let $\phi \in W^{1,2}(\mathbb{D})$ be the local conformal parametrization of a CMC surface of mean curvature H. There exists $\varepsilon_0 > 0$ such that, if $\|\nabla\phi\|_{L^2(\mathbb{D})} \leq \varepsilon_0$, then for all $k \geq 1$ and $r < 1$*

$$\|\nabla^k\phi\|_{L^\infty(\mathbb{D}_r)} \leq C(r,k,H)\|\nabla\phi\|_{L^2(\mathbb{D})}.$$

Proof. As alluded, we wish to show a Morrey-like estimate. To simplify matters, we will assume, without loss of generality that $H = 1$. Then, let $p \in \mathbb{D}$ and $r > 0$ such that $B_r(p) \subset \mathbb{D}$. Let us write ϕ on $B_r(p)$ as: $\phi = \phi_e + \phi_h$ where ϕ_e and ϕ_h solve respectively:

$$\begin{cases}\Delta\phi_e = \nabla^\perp\phi \times \nabla\phi \text{ in } B_r(p)\\ \phi_e = 0 \text{ on } \partial B_r(p),\end{cases}$$

$$\begin{cases}\Delta\phi_h = 0 \text{ in } B_r(p)\\ \phi_e = \phi \text{ on } \partial B_r(p).\end{cases}$$

Then ϕ_e satisfies the Wente's inequality:

$$\|\nabla\phi_e\|_{L^2(B_r(p))} \leq C\|\nabla\phi\|^2_{L^2(B_r(p))} \leq C\|\nabla\phi\|_{L^2(\mathbb{D})}\|\nabla\phi\|_{L^2(B_r(p))}, \qquad (94)$$ {1003202109

and ϕ_h, as the harmonic completion of ϕ satisfies:

{100320210954}
$$\|\nabla\phi_h\|_{L^2(B_r(p))} \leq \|\nabla\phi\|_{L^2(B_r(p))}. \qquad (95)$$

This only controls the energy on the whole ball, and not the way the energy focuses. To that end we will recall a classical lemma on harmonic functions (see for instance §232 de [**?**]):

{070620211550}
Lemma 3.2.4. *If v is harmonic on $\mathbb{D}$, then for all $p \in \mathbb{D}$, $r \to \frac{1}{r^2}\int_{B_r(p)} |\nabla v|^2 dxdy$ is increasing.*

Here, this implies that:

{100320211000}
$$\|\nabla\phi_h\|^2_{L^2\left(B_{\frac{r}{2}}(p)\right)} \leq \frac{1}{4}\|\nabla\phi_h\|^2_{L^2(B_r(p))} \leq \frac{1}{4}\|\nabla\phi\|^2_{L^2(B_r(p))}. \qquad (96)$$

Thus, thanks to (94) and (96), one has:

$$\begin{aligned}
\|\nabla\phi\|_{L^2\left(B_{\frac{r}{2}}(p)\right)} &\leq \|\nabla\phi_e\|_{L^2\left(B_{\frac{r}{2}}(p)\right)} + \|\nabla\phi_h\|_{L^2\left(B_{\frac{r}{2}}(p)\right)} \\
&\leq \left(C\|\nabla\phi\|_{L^2(\mathbb{D})} + \frac{1}{2}\right)\|\nabla\phi\|_{L^2(B_r(p))} \\
&\leq \frac{3}{4}\|\nabla\phi\|_{L^2(B_r(p))},
\end{aligned}$$

if $\|\nabla\phi\|_{L^2(\mathbb{D})} \leq \varepsilon_0 = \frac{1}{4C}$. Thus, the following estimate stands:

$$\int_{B_{\frac{r}{2}}(p)} |\nabla\phi|^2 dxdy \leq \left(\frac{3}{4}\right)^2 \int_{B_r(p)} |\nabla\phi|^2 dxdy.$$

Iterating this result then yields:

$$\begin{aligned}
\int_{B_{\frac{r}{2^i}}(p)} |\nabla\phi|^2 dxdy &\leq \left(\frac{3}{4}\right)^{2i} \int_{B_r(p)} |\nabla\phi|^2 dxdy \\
&\leq C_r\left(\frac{r}{2^i}\right)^{\alpha},
\end{aligned}$$

with $\alpha := 2\log_2\left(\frac{4}{3}\right) > 0$ and $C_r = r^{-\alpha}\|\nabla\phi\|^2_{L^2(\mathbb{D})}$. This stands true for all i, all $p \in \mathbb{D}_{\frac{1}{2}}$ and $\frac{1}{4} < r < \frac{1}{2}$. For any $s < \frac{1}{2}$, choosing i such that $\frac{r}{2^{i+1}} < s < \frac{r}{2^i}$, one thus has:

$$\begin{aligned}
\int_{B_s(p)} |\nabla\phi|^2 dxdy \leq \int_{B_{\frac{r}{2^i}}(p)} |\nabla\phi|^2 dxdy &\leq C_r\left(\frac{r}{2^i}\right)^{\alpha} \\
&\leq 2^{\alpha} C_r s^{\alpha} \leq C_r\|\nabla\phi\|^2_{L^2(\mathbb{D})} s^{\alpha}.
\end{aligned}$$

Since $\left|\nabla^\perp\phi \times \nabla\phi\right| \le |\nabla\phi|^2$, the right-hand side of the equation thus satisfies a Morrey-type estimate. Applying estimates on Riesz potentials we deduce that, for a $p > 2$ and $r < 1$,

$$\|\nabla\phi\|_{L^p(\mathbb{D}_r)} \le C\|\nabla\phi\|_{L^2(\mathbb{D})}.$$

The criticality of the equation is thus broken, and applying theorem 3.1.5 allows one to bootstrap and obtain the desired result. □

The resulting inequality of 3.2.1 is *not* scale-invariant: if ϕ satisfies the same hypotheses on $\mathbb{D}_R$, the theorem can be applied to $\phi(R.)$, leading to

$$\|\nabla\left(\phi(R.)\right)\|_{L^\infty(\mathbb{D})} = R\|\nabla\phi\|_{L^2(\mathbb{D}_R)} \le C\|\nabla(\phi(R.))\|_{L^2(\mathbb{D})} = C\|\nabla\phi\|_{L^2(\mathbb{D}_R)},$$

which implies

$$\|\nabla\phi\|_{L^2(\mathbb{D}_R)} \le \frac{C}{R}\|\nabla\phi\|_{L^2(\mathbb{D}_R)}.$$

These estimates thus degenerate as $R \to 0$.

Theorem 3.2.1 is not a regularity result, but an ε*-regularity* theorem. The smallness hypothesis does not make any difference when considering a CMC surface: if $\phi \in W^{1,2}$, around any point there exists a small disk on which the small energy condition is satisfied (notice that it depends only depends on the Wente constant, and is thus scale-invariant), and on which ϕ is smooth. The CMC immersion is thus smooth in the interior of the domain. If we however consider *a sequence* of CMC immersions ϕ_n: there may be a point on which no disk satisfies the smallness energy condition *uniformly*. And since the resulting inequality is not scale invariant the ε-regularity does not yield uniform estimates at such a *concentration point*.

3.2.3 Compensation in exotic spaces

Varying the domain

In this subsection, we will further explore the compensation phenomenons to give better insight and deeper understanding of them. We will not give proofs (as they can get quite technical and are not at the heart of this course) but will try to explain where the ideas and the broad strokes come from. This subsection is based on the third chapter of [**?**]. We stress once more how insightful reading this book, or even this chapter, can be when considering such geometric problems.

First of all, we know that Wente's result is scale-invariant, the constant C does not depend on the radius of the disk. The natural question is then: is it shape invariant? The answer, remarkably, is yes, thanks to a result by F. Bethuel and J.-M. Ghidaglia (see [**?**]):

Theorem 3.2.2. *There exists a positive constant C such that for every open bounded subset Ω of $\mathbb{R}^2$, and for any $a, b \in W^{1,2}(\Omega)$, the solution u of* (90) *satisfies:*

$$\|u\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \leq C\|\nabla a\|_{L^2(\Omega)}\|\nabla b\|_{L^2(\Omega)}.$$

Actually, this result also stands on Riemann surfaces, with or without boundary! To properly phrase this let us introduce the Sobolev spaces on a manifold: let (Σ, g) be a compact Riemannian surface. The Sobolev space $W^{k,p}(\Sigma)$ is the set of measurable maps from Σ into $\mathbb{R}$ defined as

$$W^{k,p}(\Sigma) := \left\{ f \text{ measurable} : \Sigma \to \mathbb{R} \text{ s.t } \sum_{l=0}^{k} \int_\Sigma \left|\nabla_g^l f\right|_g^p d\mathrm{vol}_g < \infty \right\}.$$

Actually, on compact surfaces, this definition does not depend on the chosen metric g: any other smooth metric will yield the same space (the value of the norms however are g dependant). This give more flexibility when considering a surface Σ, one can introduce a reference metric g_0 and define the Sobolev space regarding this reference metric. One can then consider immersions in Sobolev spaces and study the induced metrics with Sobolev regularity. This will come back later when considering *weak immersions*.

If we now consider the second order problem:

{100320211719}
$$\begin{cases} \Delta_g u = \nabla_g^\perp a.\nabla b \text{ in } \Sigma \\ u = 0 \text{ on } \partial\Sigma, \text{ if } \partial\Sigma \neq \emptyset \\ u(p_0) = 0 \text{ at a certain } p_0 \in \Sigma \text{ if } \partial\Sigma = \emptyset. \end{cases} \tag{97}$$

we have the following theorem (due to a collective effort by Y. Ge [?], S. Baraket [?] and P. Topping [?]):

Theorem 3.2.3. *A solution of* (97) *with $a, b \in W^{1,2}(\Sigma)$ satisfies*

$$\|u\|_{L^\infty(\Sigma)} + \|\nabla u\|_{L^2(\Sigma)} \leq C\|\nabla a\|_{L^2(\Sigma)}\|\nabla b\|_{L^2(\Sigma)}.$$

Changing the spaces

While the L^p spaces offer a wide range of controls, they sometimes are not sharp enough. we will there introduce a refinement of these spaces, the Lorentz spaces $L^{p,q}$.

Definition 3.2.1. *Let $f : \Omega \to \mathbb{R}$ be a measurable function on an open subset of $\mathbb{R}^2$. The non-increasing rearrangement of $|f|$ on $[0, |\Omega|)$ is the only non-increasing function $f^* : [0, |\Omega|) \to \mathbb{R}$ such that*

$$\lambda\left(\{x \in \Omega \text{ s.t. } |f(x)| \geq s\}\right) = \lambda\left(\{t \in (0, |\Omega|) \text{ s.t. } f^*(t) \geq s\}\right).$$

The Lorentz space $L^{p,q}(\Omega)$, $p \in (1,\infty)$, $q \in [1,\infty]$ is the set of measurable f such that

$$|f|_{p,q} = \left[\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right]^{\frac{1}{q}} < \infty, \text{ if } q < \infty,$$

$$|f|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty \text{ if } q = \infty.$$

On these Lorentz spaces, we define the norms

$$\|f\|_{p,q} = \left[\int_0^\infty \left(t^{\frac{1}{p}} f^{**}(t)\right)^q \frac{dt}{t}\right]^{\frac{1}{q}}, \text{ if } q < \infty,$$

$$\|f\|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^{**}(t),$$

where $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$.

These definitions are quite complex, but the core idea is that the $L^{p,q}$ space forms an interpolation family for the Lebesgue spaces:

$$L^{p,1} \subset L^{p,q} \subset L^{p,q'} \subset L^{p,\infty}\ 1 < q < q'$$
$$L^{p,p} = L^p$$
$$L^{p',q'} \subset L^{p,q} \text{ if } p < p'$$

In other words, the $L^{p,q}$ with $q < p$ refine the Lebesgue spaces while for $p > q$ they form rougher spaces to interpolate. Most often, one can avoid going back to the definition of the Lorentz and can use the Marcinkiewitz interpolation theorem:

Theorem 3.2.4. *Let Ω be an open bounded subset of $\mathbb{R}$ and U an open subset of $\mathbb{R}^n$. Let r_0, r_1, p_0, p_1 be real numbers such that $1 \leq r_0 < r_1 \leq \infty$, and $1 \leq p_0 \neq p_1 \leq \infty$. Let T be a linear operator whose domain D contains*

$$\cup_{r_0 \leq r \leq r_1} L^r(\Omega),$$

and which maps continuously $L^{r_0}(\Omega)$ to $L^{p_0}(U)$, and $L^{r_1}(\Omega)$ to $L^{p_1}(U)$ with the norms

$$\forall f \in L^{r_0}(\Omega),\ \|Tf\|_{L^{p_0}(U)} \leq k_0 \|f\|_{L^{r_0}(\Omega)},$$
$$\forall f \in L^{r_1}(\Omega),\ \|Tf\|_{L^{p_1}(U)} \leq k_1 \|f\|_{L^{r_1}(\Omega)}.$$

Then, for each $1 \leq q \leq \infty$, and for every pair (p, r) such that $\exists \theta \in (0,1)$,

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1},$$

f maps continuously $L^{r,q}(\Omega)$ to $L^{p,q}(U)$, and moreover,

$$\forall f \in L^{r,q}(\Omega),\ \|Tf\|_{L^{p,q}(U)} \leq B_\theta \|f\|_{L^{r,q}(\Omega)}.$$

In our considerations we will only consider $L^{2,1}(\Omega) \subset L^2 \subset L^{2,\infty}(\Omega)$. The set $L^{2,\infty}(\Omega)$, "weak L^2 space" contains L^2 and functions slightly less regular than L^2. The set $L^{2,1}$ only contains functions more regular than L^2. For instance:

Example 3.2.1. $(x,y) \to \frac{1}{r} \in L^{2,\infty}(\mathbb{D})$ *but not in* L^2.

while one has:

{110320210916}

Proposition 3.2.5. *If* $f \in W^{1,2}(\Omega)$ *satisfies* $\nabla f \in L^{2,1}(\Omega)$. *Then* f *is continuous and uniformly bounded in* Ω.

In contrast, the Sobolev embeddings only send $W^{1,2}(\Omega)$ in L^p for all $p < \infty$, so not even in L^∞.

One should still keep in mind that the $L^{p,q}$ spaces remain fundamentally complex. For instance, counterintuitively:

Proposition 3.2.6.

$$\left\| \frac{1}{r} \right\|_{L^{2,\infty}(\mathbb{D})} = \left\| \frac{1}{r} \right\|_{L^{2,\infty}\left(\mathbb{D}_{\frac{1}{2}}\right)}$$

Let us now use this finer grid to further our understanding of the compensation phenomena:

{110320210902}

Theorem 3.2.7. *Let* Ω *be an open subset of* $\mathbb{R}^2$ *with* C^1 *boundary. If* $f \in L^1(\Omega)$, *any solution of*

$$\begin{cases} \Delta\phi = f \ \ in \ \Omega \\ \quad \phi = 0 \ \ on \ \Omega \end{cases}$$

satisfies:

$$\|\nabla\phi\|_{L^{2,\infty}(\Omega)} \leq C(\Omega)\|f\|_{L^1(\Omega)}.$$

One can then contextualise Wente's lemma in light of theorem 3.2.7: Calderón-Zygmund fails for f in L^1 and only sends ϕ in $W^{1,(2,\infty)}$ (actually the proof relies on Green's representation formula and the fact that $\frac{1}{r} \in L^{2,\infty}$). The added structure of a Jacobian right-hand side then only has to bridge the "small" gap between L^2 and $L^{2,\infty}$. In fact, it even allows one to jump from $L^{2,\infty}$, to $L^{2,1}$:

{110320210916}

Theorem 3.2.8. *Let* u *solve* (90) *on* $\mathbb{D}$. *Then, there exists* C *such that* u *satisfies:*

$$\|\nabla u\|_{L^{2,1}(\mathbb{D})} \leq C\|\nabla a\|_{L^2(\mathbb{D})}\|\nabla b\|_{L^2(\mathbb{D})}.$$

Of course, Wente's inequality follows from theorem 3.2.8 and proposition 3.2.8. Still, one can go further:

0320210918}

Theorem 3.2.9. *Let u solve* (90) *on $\mathbb{D}$ with $\nabla a \in L^{2,\infty}(\mathbb{D})$ and $\nabla b \in L^2(\mathbb{D})$. Then, there exists C such that u satisfies:*

$$\|\nabla u\|_{L^2(\mathbb{D})} \leq C\|\nabla a\|_{L^{2,\infty}(\mathbb{D})}\|\nabla b\|_{L^2(\mathbb{D})}.$$

So, the $L^{2,p}$ grid allows us to refine our understanding of the compensation phenomena: from the structureless $\Delta \in L^1 \implies \nabla \in L^{2,\infty}$, the structure allows one to jump into $\Delta = \nabla^\perp \nabla \implies \nabla \in L^{2,1}$ or to weaken the hypotheses needed to go into L^2.

The question of the reason why the structure allows this jump requires us to introduce (very quickly) another function space: the *Hardy space*. We will not go into the exact definition but the Hardy space allows one to complete Calderon-Zygmund: $\Delta \in$ Hardy $\implies W^{2,1}$, by smoothing out the singularity of the Green function. The structure $\nabla^\perp a \nabla b = \det(\nabla a, \nabla b)$ is then automatically in a Hardy space as soon as $a, b \in W^{1,2}$. This exact structure is pivotal, since the Hardy space is not stable under the multiplication by a smooth function.

3.3 The CMC Plateau problem

3.3.1 Setting, Non-existence condition

As in the minimal case, solving the Plateau problem requires finding a CMC disk leaning on a fixed exterior curve. More formally, if Γ is a smooth Jordan curve in $\mathbb{R}^3$, we seek $\phi : \mathbb{D} \to \mathbb{R}^3$ such that

- $\phi(\mathbb{D})$ is CMC
- $\phi_{|\partial\mathbb{D}}$ parametrizes Γ.

Physically, we seek the shape of a soap film leaning on a shape between two domains of different air pressures. We can solve this problem for easy configurations: if we consider Γ a circle in $\mathbb{R}^3$, any spherical cap leaning on this circle solves the Plateau problem.

To obtain a solution in the general case, one can minimize the area (or rather the Dirichlet energy, to limit the invariance group) under a volume constraint, as was done in section 3.1.1. This however requires to make sure the convergence maintains the constraint up to the limit, which can be problematic. We will thus take another approach and modify the energy to force the mean curvature to take the prescribed value: in fact we will minimize the energy

$$\mathcal{D}_V(\phi) = \frac{1}{2}\int_{\mathbb{D}} |\nabla\phi|^2 dxdy + \frac{C}{3}\int_{\mathbb{D}} \langle \phi, \phi_x \times \phi_y \rangle dxdy. \tag{98}$$ {090320210952}

As in section 2.2.2, we took the Dirichlet energy instead of the Area to force the Coulomb gauge at the limit. The added term will fix $\Delta\Phi = C$. Indeed, if we consider a perturbation $\phi_\varepsilon = \phi + \varepsilon\psi + o(\psi)$, then

$$\begin{aligned}\delta\left(\mathcal{D}_V(\phi)\right) &= \int_{\mathbb{D}} \nabla\phi.\nabla\psi dxdy + \frac{C}{3}\int_{\mathbb{D}} \langle \psi, \phi_x \times \phi_y\rangle + \langle \phi, \psi_x \times \phi_y\rangle + \langle \phi, \phi_x \times \psi_y\rangle dxdy \\ &= \int_{\mathbb{D}} \operatorname{div}\left(\nabla\phi + \begin{pmatrix}\langle \phi, \psi \times \phi_y\rangle \\ \langle \phi, \phi_x \times \psi\rangle\end{pmatrix}\right) + C\langle \psi, \phi_x \times \phi_y\rangle - \langle \Delta\phi, \psi\rangle dxdy \\ &= \int_{\mathbb{D}} \langle \psi, \Delta\phi - C\phi_x \times \phi_y\rangle dxdy.\end{aligned}$$

{110320211128} (99)

So, if we manage to find a minimizer of this energy *and show that it is conformal*, then we will have found a CMC surface of mean curvature $H = \frac{C}{2}$ parametrized conformally:

{090320210921}
$$\begin{cases} \Delta\phi = 2H\phi_x \times \phi_y \\ |\phi_x|^2 - |\phi_y|^2 = \langle \phi_x, \phi_y\rangle = 0 \\ \phi_{|\partial\mathbb{D}} = \gamma \text{ parametrizes } \Gamma. \end{cases} \tag{100}$$

More than a constant mean curvature, we are trying to solve a *prescribed* constant mean curvature surface, with an additional constraint. This may prove too much to ensure the existence. Indeed, if we go back to the elementary configuration of $\Gamma =$ circle of radius R, since the mean curvature of a sphere of radius r is $H = \frac{1}{r}$, with $H > \frac{1}{R}$ one cannot find a sphere of prescribed constant mean curvature H leaning on the circle.

In fact, one has the following non-existence result from Heinz:

{090320211039}

Theorem 3.3.1. *The Plateau problem* (100) *has no solution if*

$$|H| > \frac{L(\Gamma)}{k(\Gamma)},$$

where $k(\Gamma) = \left|\int_{\partial\mathbb{D}} \gamma \wedge \gamma_\theta d\sigma\right|$, *where* γ *is a parametrization of* Γ.

Proof. One can check that $\left|\int_{\partial\mathbb{D}} \gamma \wedge \gamma_\theta d\sigma\right|$ is invariant by reparametrizations, and thus that $k(\Gamma)$ is well defined.

If we assume that (100) has a solution ϕ, then one has:

$$2H\int_{\mathbb{D}} \phi_x \times \phi_y dxdy = \int_{\mathbb{D}} \Delta\phi dxdy = \int_{\partial\mathbb{D}} \partial_\nu\phi d\sigma.$$

Besides, as has already been remarked in (61), $2\phi_x \times \phi_y = \nabla^\perp\phi \times \nabla\phi = \operatorname{div}\left(\nabla^\perp\phi \times \phi\right)$. Injecting this in the above equality yields:

$$\int_{\partial\mathbb{D}} \partial_\nu\phi d\sigma = 2H\int_{\mathbb{D}} \phi_x \times \phi_y dxdy = H\int_{\partial_{\mathbb{D}}} \partial_\theta\phi \times \phi d\sigma,$$

which in turn implies $|H| \leq \frac{\left|\int_{\partial\mathbb{D}} \partial_\nu \phi d\sigma\right|}{k(\Gamma)} \leq \frac{\int_{\partial\mathbb{D}} |\partial_\nu \phi| d\sigma}{k(\Gamma)} \leq \frac{\int_{\partial\mathbb{D}} |\partial_\theta \phi| d\sigma}{k(\Gamma)}$ since ϕ is conformal. This concludes the proof. □

3.3.2 Preparation work

We can now try to start our minimization process: as for the minimal Plateau problem, the boundary condition will be force us to minimize within $\mathcal{C}(\Gamma)$. There remains however one major issue: the volume functional $\mathcal{V}(\phi) = \int_D \langle \phi, \phi_x \times \phi_y \rangle dxdy$ is not well defined for a mere $\phi \in W^{1,2}(\mathbb{D}, \mathbb{R}^3)$: it requires us to consider $\phi \in \mathcal{C}(\Gamma) \cap L^\infty(\mathbb{D}, \mathbb{R}^3)$. In fact, given theorem 3.3.1, one needs to take a smallness hypothesis on H to ensure the existence. We will thus assume that $\Gamma \subset B_R(0)$ and minimize $\mathcal{D}_V$ among functions in

$$\mathcal{C}_H(\Gamma) = \left\{ \phi \in \mathcal{C}(\Gamma) \cap L^\infty(\mathbb{D}) \text{ s.t. } \|\phi\|_{L^\infty(\mathbb{D})} \leq \frac{1}{H} \right\}, \tag{101}$$

where H satisfies

$$|H|R < 1. \tag{102}$$ {090320211211}

This is not an innocuous constraint: indeed should we reach a minimizer that "touches" $\frac{1}{H}$, then it would only be a critical point among all perturbations diminishing its norm, and thus, it would not be a solution of (100). A key step in the proof will be proving that the minimizing sequence *does not saturate the constraint*.

With $\phi \in \mathcal{C}_H(\Gamma)$, $\mathcal{V}(\phi) \leq \frac{1}{H}\mathcal{D}(\phi)$ and thus

$$\mathcal{D}_V(\phi) \geq \mathcal{D}(\phi) - \frac{2H}{3}\frac{1}{H}\mathcal{D}(\phi) = \frac{1}{3}\mathcal{D}(\phi). \tag{103}$$ {090320211050}

This will ensure the uniform $W^{1,2}(\mathbb{D})$ estimate using lemma 2.2.5. We now only have to control the behavior at the boundary. To that end, let us notice that:

Lemma 3.3.1. *Let $\phi \in W^{1,2}(\mathbb{D}) \cap L^\infty(\mathbb{D})$ and $\Theta \in \mathrm{Diff}^+(\mathbb{D})$. Then $\mathcal{V}(\phi \circ \Theta) = \mathcal{V}(\phi)$.*

Proof. The proof is a straightforward computation:

$$\begin{aligned}
&(\phi \circ \Theta)_x = \phi_x \circ \Theta \Theta^1_x + \phi_y \Theta^2_x \\
&(\phi \circ \Theta)_y = \phi_x \circ \Theta \Theta^1_y + \phi_y \Theta^2_y \\
&(\phi \circ \Theta)_x \times (\phi \circ \Theta)_y = (\phi_x \times \phi_y) \circ \Theta \left(\Theta^1_x \Theta^2_y - \Theta^1_y \Theta^2_x \right) = (\phi_x \times \phi_y) \circ \Theta \mathrm{Jac}(\Theta).
\end{aligned}$$

This concludes the proof. □

The volume functional is thus a geometric quantity, invariant by reparametrizations. The total energy $\mathcal{D}_V$ is thus only a conformal invariant, and the boundary control using the three point lemma will work as devised in this case and ensure the uniform continuity and thus the convergence on the boundary. A final question then remains: will the weak limit of a minimizing sequence be a minimizer i.e. what is the behavior of $\mathcal{D}_V$ under weak $W^{1,2}$ limit?

{110320211052}

Proposition 3.3.2. *If $\|\phi_k\|_{L^\infty} \leq \frac{1}{|H|}$ and $\phi_k \to \phi$ a.e., weakly in $W^{1,2}$ and strongly in L^2, with $\|\phi\|_{L^\infty} \leq \frac{1}{|H|}$, then $\mathcal{D}(u) \leq \liminf \mathcal{D}(\phi_k)$.*

Proof. Using Egorov's theorem, for any $\delta > 0$ there exists a set E_δ such that $|E_\delta| \leq \delta$ such that the convergence $\phi_k \to \phi$ is uniform on $\mathbb{D}\backslash E_\delta$. Then, if we decompose:

$$\begin{aligned}\mathcal{D}_V(\phi_k) = \frac{1}{2}\int_{E_\delta} |\nabla \phi_k|^2 dxdy + \frac{2H}{3}\int_{E_\delta} \langle \phi_k, \partial_x\phi_k \times \partial_y \phi_k\rangle dxdy \\ \frac{1}{2}\int_{\mathbb{D}\backslash E_\delta} |\nabla \phi_k|^2 dxdy + \frac{2H}{3}\int_{\mathbb{D}\backslash E_\delta} \langle \phi_k, \partial_x\phi_k \times \partial_y \phi_k\rangle dxdy\end{aligned}$$

On E_δ, since $|\phi_k| \leq \frac{1}{|H|}$, one has

$$\begin{aligned}\frac{1}{2}\int_{E_\delta} |\nabla \phi_k|^2 dxdy + \frac{2H}{3}\int_{E_\delta} \langle \phi_k, \partial_x\phi_k \times \partial_y \phi_k\rangle dxdy &\geq \frac{1}{2}\int_{E_\delta} |\nabla \phi_k|^2 dxdy \\ &- \frac{2}{3}\int_{E_\delta} |\partial_x \phi_k \times \partial_y\phi_k|\, dxdy \\ &\geq \frac{1}{6}\int_{E_\delta} |\nabla \phi_k|^2 dxdy.\end{aligned}$$

On $\mathbb{D}\backslash E_\delta$, we will use the uniform convergence to ensure the control of the volume terms:

$$\begin{aligned}\int_{\mathbb{D}\backslash E_\delta} \langle \phi_k, \partial_x\phi_k \times \partial_y \phi_k\rangle dxdy &= \int_{\mathbb{D}\backslash E_\delta} \langle \phi, \partial_x\phi_k \times \partial_y \phi_k\rangle dxdy + o(1) \\ &= \int_{\mathbb{D}\backslash E_\delta} \langle \phi, \partial_x(\phi_k - \phi) \times \partial_y (\phi_k - \phi)\rangle dxdy + \int_{\mathbb{D}\backslash E_\delta} \langle \phi, \phi_x \times \phi_y\rangle dxdy \\ &+ \int_{\mathbb{D}\backslash E_\delta} \langle \phi, \partial_x(\phi_k - \phi) \times \phi_y\rangle dxdy + \int_{\mathbb{D}\backslash E_\delta} \langle \phi, \phi_x \times \partial_y(\phi_k - \phi)\rangle dxdy \\ &+ o(1).\end{aligned}$$

Now, since $\phi_k \to \phi$ weakly in $W^{1,2}$ and $\phi\nabla\phi \in L^2$ (given that $\|\phi\|_{L^\infty} \leq \frac{1}{|H|}$) the second to last line converges toward 0. Thus:

$$\int_{\mathbb{D}\setminus E_\delta} \langle \phi_k, \partial_x\phi_k \times \partial_y\phi_k\rangle dxdy = \int_{\mathbb{D}\setminus E_\delta} \langle \phi, \partial_x(\phi_k-\phi) \times \partial_y(\phi_k-\phi)\rangle dxdy + \int_{\mathbb{D}\setminus E_\delta} \langle \phi,\phi_x \times \phi_y\rangle dxdy \\ + o(1).$$

Since, in addition, $|\phi| \leq \frac{1}{|H|}$, one has:

$$\frac{2H}{3}\int_{\mathbb{D}\setminus E_\delta} \langle \phi_k, \partial_x\phi_k \times \partial_y\phi_k\rangle dxdy \geq \frac{2H}{3}\int_{\mathbb{D}\setminus E_\delta} \langle \phi,\phi_x \times \phi_y\rangle dxdy - \frac{1}{3}\int_{\mathbb{D}\setminus E_\delta} |\nabla(\phi_k-\phi)|^2 dxdy.$$

Assembling these estimates ensure:

$$\begin{aligned}\mathcal{D}_V(\phi_k) \geq{}& \frac{1}{6}\int_{E_\delta} |\nabla\phi_k|^2 dxdy + \frac{2H}{3}\int_{\mathbb{D}\setminus E_\delta} \langle \phi, \phi_x \times \phi_y\rangle dxdy \\ &+ \frac{1}{2}\int_{\mathbb{D}\setminus E_\delta} |\nabla\phi_k|^2 dxdy - \frac{1}{3}\int_{\mathbb{D}\setminus E_\delta} |\nabla(\phi_k-\phi)|^2 dxdy + o(1),\end{aligned}$$

and thus

$$\begin{aligned}\mathcal{D}_V(\phi_k) - \mathcal{D}_V(\phi) \geq{}& \frac{1}{6}\int_{E_\delta} |\nabla\phi_k|^2 dxdy - \frac{1}{2}\int_{\mathbb{D}\setminus E_\delta} |\nabla\phi_k|^2 dxdy \\ &- \frac{1}{3}\int_{\mathbb{D}\setminus E_\delta} |\nabla(\phi_k-\phi)|^2 dxdy - \frac{2H}{3}\int_{E_\delta} \langle \phi, \phi_x \times \phi_y\rangle dxdy \qquad (104) \quad \{110320211040\} \\ &- \frac{1}{2}\int_{\mathbb{D}} |\nabla\phi|^2 dxdy + o(1).\end{aligned}$$

Writing

$$\begin{aligned}\int_{\mathbb{D}\setminus E_\delta} |\nabla\phi_k|^2 dxdy &= \int_{\mathbb{D}\setminus E_\delta} |\nabla(\phi_k-\phi)|^2 dxdy + \int_{\mathbb{D}\setminus E_\delta} |\nabla\phi|^2 dxdy + 2\int_{\mathbb{D}\setminus E_\delta} \langle \nabla\phi_k, \nabla\phi\rangle dxdy \\ &= \int_{\mathbb{D}\setminus E_\delta} |\nabla(\phi_k-\phi)|^2 dxdy + \int_{\mathbb{D}\setminus E_\delta} |\nabla\phi|^2 dxdy + o(1),\end{aligned}$$

given the weak convergence. Injecting this into (104) yields:

$$\begin{aligned}\mathcal{D}_V(\phi_k) - \mathcal{D}_V(\phi) \geq{}& \frac{1}{6}\int_{E_\delta} |\nabla\phi_k|^2 dxdy + \frac{1}{6}\int_{\mathbb{D}\setminus E_\delta} |\nabla(\phi_k-\phi)|^2 dxdy \\ &- \frac{1}{2}\int_{E_\delta} |\nabla\phi|^2 dxdy - \frac{2H}{3}\int_{E_\delta} \langle \phi, \phi_x \times \phi_y\rangle dxdy \\ &+ o(1) \\ \geq{}& \frac{1}{6}\int_{E_\delta} |\nabla\phi_k|^2 dxdy + \frac{1}{6}\int_{\mathbb{D}\setminus E_\delta} |\nabla(\phi_k-\phi)|^2 dxdy \\ &- \frac{5}{6}\int_{E_\delta} |\nabla\phi|^2 dxdy + o(1),\end{aligned}$$

still using the bound on ϕ. Thus, one has:

$$\begin{aligned}\mathcal{D}_V(\phi_k) - \mathcal{D}_V(\phi) &\geq -\frac{5}{6}\int_{E_\delta} |\nabla\phi|^2 dxdy + o(1)\\ &\geq o_\delta(1) + o(1).\end{aligned}$$

This concludes the proof. □

3.3.3 Minimizing procedure

Let us thus consider $(\phi_n)_{n\in\mathbb{N}} \in \mathcal{C}_H(\Gamma)$ minimizing $\mathcal{D}_V$ (which is indeed bounded from below thanks to (103)). Thanks to (103) and lemma 2.2.5, ϕ_n is uniformly bounded in $W^{1,2}(\mathbb{D})$, and thus converges weakly up to extraction towards $\phi \in W^{1,2}(\mathbb{D})$. Using Sobolev embeddings, ϕ_n converges strongly strongly toward ϕ in any $L^p(\mathbb{D})$ with $p < \infty$. Up to another extraction $\phi_n \to \phi$ a.e. in $\mathbb{D}$, and since $|\phi_n| \leq \frac{1}{H}$, this is carried over the limit. Using the three points lemma (and up to composition with a conformal diffeomorphism), ϕ_n is uniformly bounded on the boundary and thus converges uniformly towards $\phi_{|\partial\mathbb{D}} \in C^0(\partial\mathbb{D})$. In summary: $\phi \in \mathcal{C}_H(\Gamma)$. Proposition 3.3.2 then ensures that ϕ is a minizer of $\mathcal{D}_V$ in $\mathcal{C}_H(\Gamma)$.

To show that it does solve the H-CMC equation, we need to prove that it does not saturate the $|\phi| \leq \frac{1}{|H|}$ condition. To that end, we will show that $|\phi| \leq R$, and since $HR < 1$, this will be enough.

Let us then consider $\varphi \in C_c^\infty(\mathbb{D})$ and $\phi_\varepsilon = \phi(1-\varepsilon\varphi)$. With ε small enough, $\phi_\varepsilon \in \mathcal{C}_H(\Gamma)$. Thus, $\mathcal{D}_V(\phi_\varepsilon) \geq \mathcal{D}_V(\phi)$. Computing:

{110320211125} $$\mathcal{D}_V(\phi_\varepsilon) = \mathcal{D}_V(\phi) - \varepsilon\left(\int_{\mathbb{D}} \langle\nabla\phi.\nabla(\varphi\phi)\rangle dxdy + 2H\mathcal{R}(\phi,\varphi\phi)\right) + o(\varepsilon), \tag{105}$$

with

$$\begin{aligned}\mathcal{R}(v,u) &= \frac{1}{3}\int_{\mathbb{D}} (\langle v, v_x\wedge u_y + u_x \wedge v_y\rangle + \langle u, v_x\wedge v_y\rangle) dxdy\\ &= \int_{\mathbb{D}} \langle u, v_x \times v_y\rangle dxdy,\end{aligned}$$

working as in (99). Thus, injecting $\mathcal{D}_V(\phi_\varepsilon) \geq \mathcal{D}_V(\phi)$ into (105) yields:

$$\int_{\mathbb{D}} \langle\nabla\phi.\nabla(\varphi\phi)\rangle dxdy + 2H\mathcal{R}(\phi,\varphi\phi) \leq 0.$$

Now since $\int_{\mathbb{D}}\langle\nabla\phi.\nabla(\varphi\phi)\rangle dxdy = \frac{1}{2}\int_{\mathbb{D}}\nabla\phi.\nabla|u|^2 dxdy + \int_{\mathbb{D}}\varphi|\nabla u|^2 dxdy$, this becomes:

$$\int_{\mathbb{D}} \nabla\phi.\nabla|u|^2 + 2\varphi\left(|\nabla u|^2 + 2H\langle\phi,\phi_x\times\phi_y\rangle\right) dxdy \leq 0.$$

Once more, the control: $|\phi| \leq \frac{1}{|H|}$ yields $|\nabla u|^2 + 2H\langle \phi, \phi_x \times \phi_y \rangle \geq 0$. Thus the minimizing nature of φ yields

$$\int_{\mathbb{D}} \nabla \phi . \nabla |u|^2 dxdy \leq 0.$$

Extending this by density to $\varphi \in W_0^{1,2}$, we can apply this with $\varphi = (|\phi|^2 - R^2)_+$, which yields

$$\int_{\mathbb{D}} \left| \nabla \left(|\phi|^2 - R^2 \right)_+ \right|^2 dxdy \leq 0.$$

Thus, $(|\phi|^2 - R^2)_+$ is a constant and null on the boundary, which implies that it is zero everywhere, and thus that $|\phi|^2 \leq R^2 < \frac{1}{H^2}$. The minimizer ϕ thus does not saturate the H condition, it is a critical point of the functional, and thus solves the CMC equation. It is then smooth inside the disk (by application of theorem 3.2.1), and one can show that it is conformal using the same method as for the minimal Plateau problem.

4 Willmore surfaces

4.1 The Willmore energy

4.1.1 Elastic potential energy

Elastic energy of a curve

Let us consider the following physical situation: a perfect spring of stiffness k links a fixed point (at $x = 0$) and a mass m (at $x(t)$) and exerts a force $-kx$ on the mass. Moving the mass on a distance x requires a total work $W = \frac{1}{2}kx^2$. Since the change in Potential energy between two positions is by definition the work exerted by the forces, the potential energy can then be considered as:

$$PEE = \frac{1}{2}kx^2.$$

Let us now consider a curve $\gamma : I \to \mathbb{R}^2$. Let us consider $s \in I$ and $\delta l \ll 1$. We will modelize the piece of the curve of length $2\delta l$ centered on $\gamma(s)$ as two perfect springs of same stiffness k. Let δs_+ be such that: $\delta l = \int_s^{s+\delta s_+} |\gamma'(v)|dv$. Then:

$$\begin{aligned}
\delta l &= \int_0^{\delta s_+} |\gamma'(s+v)|dv = \int_0^{\delta s_+} \sqrt{|\gamma'(s)|^2 + 2\langle \gamma''(s), \gamma'(s)\rangle v + o(v)}dv \\
&= \int_0^{\delta s_+} |\gamma'(s)| + \left\langle \gamma''(s), \frac{\gamma'(s)}{|\gamma'(s)|} \right\rangle vdv + o(\delta s_+^2) \\
&= \delta s_+ |\gamma'(s)| + \frac{\delta s_+^2}{2} \left\langle \gamma''(s), \frac{\gamma'(s)}{|\gamma'(s)|} \right\rangle + o(\delta s_+^2),
\end{aligned}$$

which implies that $\delta s_+ = \frac{\delta l}{|\gamma'(s)|} - \frac{\delta l^2}{2|\gamma'(s)|^4}\langle \gamma''(s), \gamma'(s)\rangle + o(\delta l^2)$. Defining in a similar manner $\delta l = \int_{s-\delta s_-}^{s} |\gamma'(v)|dv$, one has: $\delta s_- = \frac{\delta l}{|\gamma'(s)|} + \frac{\delta l^2}{2|\gamma'(s)|^4}\langle \gamma''(s), \gamma'(s)\rangle + o(\delta l^2)$.

Then, the global forces applied to $\gamma(s)$ are:

$$\begin{aligned}
F_{\delta l} &= k\left[\gamma(s+\delta s_+) - \gamma(s)\right] + k\left[\gamma(s-\delta s_-) - \gamma(s)\right] \\
&= k\left[\gamma(s) + \gamma'(s)\delta s_+ + \frac{\delta s_+^2}{2}\gamma''(s) - \gamma(s) + \gamma(s) - \gamma'(s)\delta s_- + \frac{\delta s_-^2}{2}\gamma''(s) - \gamma(s) + o(\delta l^2)\right] \\
&= k\left[\gamma'(s)\left(\delta s_+ - \delta s_-\right) + \frac{\delta s_+^2 + \delta s_-^2}{2}\gamma''(s) + o(\delta l^2)\right] \\
&= k\delta l^2\left[\frac{\gamma''(s)}{|\gamma'(s)|^2} - \left\langle \frac{\gamma''(s)}{|\gamma'(s)|^2}, \frac{\gamma'(s)}{|\gamma'(s)|}\right\rangle \frac{\gamma'(s)}{|\gamma'(s)|}\right] + o(\delta l^2) \\
&= k\delta l^2 \kappa(s)\vec{n}(s) + o(\delta l^2),
\end{aligned}$$

where $\kappa(s)$ is the curvature of γ at the point s. In our approximation, $F_{\delta l}$ represents the force applied to the whole segment of length $2\delta l$ (modelized by a point and two springs). Assuming it is uniform on this short segment, one finds the force at s:

$$F := \frac{F}{2\delta l} = \frac{1}{2}k\kappa(s)\vec{n}(s)\delta l + o(\delta l) = \frac{1}{2}k\kappa(s)\vec{n}(s)\delta l + o(\delta l).$$

One can modelize the displacement on $[s-\delta s_-, s+\delta s_+]$ by:

$$\begin{aligned}
T_{\delta l} &:= \gamma'(s+\delta s_+) - \gamma'(s-\delta s_-) = \gamma''(s)(\delta s_+ + \delta s_-) + o(\delta l) \\
&= 2\left(\kappa(s)\vec{n}(s) + \left\langle \frac{\gamma''(s)}{\kappa(s)}, \frac{\gamma'(s)}{|\gamma'(s)|}\right\rangle \frac{\gamma'(s)}{|\gamma'(s)|}\right)\delta l + o(\delta l),
\end{aligned}$$

and from this deduce the displacement of $\gamma(s)$:

$$T = \kappa(s)\vec{n}(s) + \left\langle \frac{\gamma''(s)}{\kappa(s)}, \frac{\gamma'(s)}{|\gamma'(s)|}\right\rangle \frac{\gamma'(s)}{|\gamma'(s)|} + o(1).$$

The work on this displacement at $\gamma(s)$ is then:

$$W = \langle F, T\rangle = \frac{1}{2}k\kappa^2(s)\delta l.$$

Taking $\delta l \to 0$ and integrating overt the curve yields its elastic energy:

$$PEE := \frac{k}{2}\int \kappa^2(s)dl = \frac{k}{2}\int \kappa^2(s)|\gamma'(s)|ds = \frac{k}{2}\int_\gamma \kappa^2.$$

Remark 4.1.1. *In this modelization, we went back, in a disguised manner to a parametrization by arc length.*

Elastic energy of a surface

Let us now generalize this to *surfaces* in $\mathbb{R}^3$: let us consider $\phi : \mathbb{D} \to \mathbb{R}^3$ the parametrization of a surface in $\mathbb{R}^3$. At a point x_0, we modelize the surface as a set of small elastic curves going through x_0, which will each contribute to its elastic energy. Let then v be a vector of norm 1 for the induced metric at x_0, $\delta l \ll 1$ and let us define:

$$\gamma(s) = \phi(x_0 + sv).$$

As before, one can define the associated $\delta s^v_\pm = \frac{\delta l}{|\nabla_v \phi(x_0)|} + o(\delta l) = \delta l + o(\delta l)$ since by definition $\langle \nabla_v \phi(x_0), \nabla_v \phi(x_0) \rangle = 1$. Thus: $\gamma'(s) = \nabla_v \phi(x_0 + sv)$, which implies that $|\gamma'(s)| = |\nabla_v \phi|(x_0 + s\delta sv) = 1 + o(\delta l)$. Similarly:

$$\gamma''(s) = \nabla^2 \phi(x_0 + sv)(v, v),$$

which implies that

$$\kappa(s) = A(x_0 + sv)(v, v).$$

Recalling that, as a self-adjoint operator, A is diagonalizable with eigenvalues κ_1, κ_2 (the principal curvatures, see section 1.2.1) reached for two orthogonal unit eigenvectors v_1, v_2, one can write: $v = \cos\theta v_1 + \sin\theta v_2$. Then the force applied on $\phi(x_0)$ by the elastic curve γ is then given by: $F_\theta = C\left(\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta\right) \vec{n}(x_0)\delta l$ where C denotes a constant. Summing over all the curves one obtains the total force applied on $\phi(x_0)$:

$$F = \int_0^\pi F_\theta \delta l d\theta = C\frac{\kappa_1 + \kappa_2}{2}\vec{n}(x_0)\delta l^2 = C\delta l^2 \frac{\mathrm{Tr}_g(A)}{2}\vec{n}(x_0) = C\delta l^2 H\vec{n}(x_0).$$

Modelizing the displacement at $\phi(x_0)$ by:

$$T = \frac{1}{\delta l^2}\int_0^\pi \left[\gamma'(\delta s_+) - \gamma'(\delta s_-)\right] = H(x_0)\vec{n}(x_0) + \text{tangent terms} + o(1).$$

The work exerted at $\phi(x_0)$ is then:

$$W = CH^2\delta l^2.$$

The Potential Elastic Energy is then proportional to $\int H^2 d\mathrm{vol}_g$.

4.1.2 Definition, conformal invariance

Definition 4.1.1. *Let $\Phi : \Sigma \to \mathbb{R}^3$ be an immersion of a surface inside $\mathbb{R}^3$. Let g denote its induced metric, and H its mean curvature. The Willmore energy of Φ is then defined as:*

$$W(\Phi) = \int_\Sigma H^2 d\mathrm{vol}_g.$$

The Willmore energy was first introduced in the XIXth century by S. Germain [?] as a way to describe the elastic behavior of surfaces in order to understand the Chladni figures. As the elastic energy of a surface, it is a quantity both important and ubiquitous in engineering matters, and in problems of shape optimization. Indeed, in biology, in order to describe the typical biconcave shape of a red blood cell P. Canham [?] and W. Helfrich [?] introduced the Canham-Helfrich energy, with a familiar leading term:

$$CHE = \int (H - \alpha)^2 + \beta \mathrm{Vol} + \gamma \mathcal{A}.$$

The Willmore energy is also involved in General Relativity (the Hawking mass) and in other complex mathematical problems like the sphere eversion. One of its property that kept drawing attention, despite the difficulty of the analysis is its conformal invariance, which we will explore here (see section 1.3.2 for the definition of conformal transformations). Let us first define another energy: *the tracefree total curvature*

$$\mathring{E} = \int_\Sigma |\mathring{A}|_g^2 d\mathrm{vol}_g. \tag{106}$$ {2505202117345}

Interestingly, $\mathring{E}$ and W only differ by a *topological term*:

{250520211735}

Proposition 4.1.1.

$$\mathring{E} = 2W - 4\pi\chi(\Sigma) = 2W - 8\pi(1-g).$$

Proof. One simply has to combine the Gauss-Bonnet theorem (theorem 1.2.3) and (50). □

This equality is fundamental for the study of the Willmore energy. Not only does it mean that the behavior of other extrinsic curvature energies will be deduced from the Willmore energy, but the invariance properties of one of the energies can be transferred to the other *as long as the topology does not change.*

Proposition 4.1.2. *Given Φ an immersion, $|\mathring{A}|^2 d\mathrm{vol}_g$ is a pointwise conformal invariant. Thus $\mathring{E}$ is a conformal invariant, and W is invariant under conformal transformations that leave the topology unchanged.*

Proof. Given Liouville theorem (see section 1.3.2), we only need to check the invariance under translations, rotations, dilations and the inversion. As isometries, translations and rotations obviously leave the geometric quantity $|\mathring{A}|^2 d\mathrm{vol}_g$ invariant, while the invariance by dilations is simply a matter of scale invariance. Indeed, if $\Phi_\lambda = \lambda\Phi$, $g_\lambda = \lambda^2 g$, $|g_\lambda|^{\frac{1}{2}} = \lambda^2 |g|^{\frac{1}{2}}$, $\mathring{A}_\lambda = \lambda\mathring{A}$, and thus:

$$\begin{aligned}|\mathring{A}_\lambda|^2 d\mathrm{vol}_{g_\lambda} &= |g_\lambda^{-1}\mathring{A}|^2 |g|^{\frac{1}{2}} dxdy \\ &= \frac{1}{\lambda^2}|\mathring{A}|_g^2 \lambda^2 |g|^{\frac{1}{2}} dxdy = |\mathring{A}|^2 d\mathrm{vol}_g.\end{aligned}$$

There only remains to check that the invariance stands for inversions. Let $\tilde{\Phi} = \iota \circ \Phi = \frac{\Phi}{|\Phi|^2}$. We compute in succession:

$$\begin{aligned}
\nabla_i \tilde{\Phi} &= \frac{\nabla_i \Phi}{|\Phi|^2} - 2\frac{\langle \nabla_i \Phi, \Phi\rangle}{|\Phi|^4}\Phi \\
\tilde{g}_{ij} &= \frac{g_{ij}}{|\Phi|^4} \\
\tilde{\vec{n}} &= \vec{n} - 2\frac{\langle \vec{n}, \Phi\rangle}{|\Phi|^2}\Phi \\
\nabla_i \tilde{\vec{n}} &= \nabla_i \vec{n} - 2\frac{\langle \nabla_i \vec{n}, \Phi\rangle}{|\Phi|^2}\Phi - 2\frac{\langle \vec{n}, \Phi\rangle}{|\Phi|^2}\left[\nabla_i \Phi - 2\frac{\langle \nabla_i \Phi, \Phi\rangle}{|\Phi|^4}\Phi\right] \\
&= -|\Phi|^2 A_i^j \nabla_j \tilde{\Phi} - 2\langle \vec{n}, \Phi\rangle \nabla_i \tilde{\Phi} \\
\tilde{A}_{ij} &= -\langle \nabla_i \tilde{\vec{n}}, \nabla_j \tilde{\Phi}\rangle = \frac{A_{ij}}{|\Phi|^2} - 2\frac{\langle \vec{n}, \Phi\rangle}{|\Phi|^4} g_{ij} \\
\tilde{\mathring{A}}_{ij} &= \frac{\mathring{A}_{ij}}{|\Phi|^2} \\
\tilde{\mathring{A}}_i^j &= |\Phi|^2 \mathring{A}_i^j \\
|\tilde{\mathring{A}}|^2 d\mathrm{vol}_{\tilde{g}} &= |\mathring{A}|^2 d\mathrm{vol}_g.
\end{aligned}$$

The rest of the proof follows from proposition 4.1.1. □

The topological condition is pivotal here. It is to be understood as *conformal transformations that do not create or destroy ends*. If it is not satisfied, W is *not* an invariant. Let us for instance consider a euclidean sphere. Its Willmore energy is $W = 4\pi$. If we however consider an inversion centered at a point on the sphere $\iota_p(x) = \frac{x-p}{|x-p|^2}$, this transformation sends it to a flat plane, of Willmore energy $W = 0 \neq 4\pi$. In fact, using an a Gauss-Bonnet formula that takes branch points and branched ends into consideration, one can find the exact variation of the Willmore energy:

{260520211000}

Theorem 4.1.3. *Let $\Phi : \Sigma \to \mathbb{R}^3$ be a branched possibly non compact immersion. Let (p_i) be its branch points of multiplicity n_i and (q_j) be its branched ends of multiplicity m_j. Then:*

$$\int_\Sigma K d\mathrm{vol}_g = 2\pi\left(\chi(\Sigma) + \sum (n_i - 1) - \sum (m_j + 1)\right).$$

We refer the reader to theorem 2.6 of [?] for a proof. The basic idea is to first prove a Gauss Bonnet formula on surfaces with boundary, and to use it to isolate singular points.

Example 4.1.1. *A plane can be seen as a surface with one end parametrized over the sphere. One can check the consistency of this point of view:*

$$\int_{\mathbb{R}^2} K d\mathrm{vol}_g = 2\pi * (2-(1+1)) = 0.$$

{260520211024}

Proposition 4.1.4. *Let $\Phi : \Sigma \to \mathbb{R}^3$ be a branched possibly non compact immersion, Θ a conformal transformation and $\tilde{\Phi} = \Theta \circ \Phi$. Then:*

$$W(\tilde{\Phi}) = W(\Phi) + 4\pi \left(\#\{\textit{destroyed ends}\} - \#\{\textit{ created ends}\}\right),$$

where the ends are counted with multiplicity.

Proof. The fundamental equality is $\mathring{E}(\tilde{\Phi}) = \mathring{E}(\Phi)$. Applying the first equality of proposition 4.1.1 then yields:

$$W(\Phi) - \int_\Sigma K_\Phi d\mathrm{vol}_g = W(\Psi) - \int_\Sigma K_{\tilde{\Phi}} d\mathrm{vol}_{\tilde{g}},$$

which ensures thanks to theorem 4.1.3:

$$W(\Psi) = W(\Phi) + 2\pi \Big(\sum (\tilde{n}_i - 1) - \sum (\tilde{m}_j + 1) - \sum (n_i - 1) + \sum (m_j + 1) \Big).$$

In the above, ends that remain ends and branch points that remain branch points are repeated twice and thus cancel out. An end of multiplicity m that becomes a branch point (of multiplicity $\tilde{n} = m$) will then add an energy of $2\pi * (\tilde{n} - 1 + m + 1) = 4\pi m$. Similarly a branch point of multiplicity n that becomes an end (of multiplicity $\tilde{m} = n$) will then contribute with $2\pi * (-\tilde{m} + 1 + n - 1) = -4\pi n$. Considering all the destroyed/created ends thus leads to the desired formula. □

Example 4.1.2.

- *Recovering the previous example: transforming a sphere into a plane creates one end and destroys none, with the following formula:*

$$W(\textit{plane}) = W(\textit{sphere}) + 4\pi(0-1) = W(\textit{sphere}) - 4\pi.$$

- *Considering the Enneper surface, and inverting it at a point* away from the surface*, one destroys one end of multiplicity* 3 *and creates none. Thus:*

$$W(\textit{inverted Enneper}) = W(\textit{Enneper}) + 4\pi * 3 = 12\pi.$$

- *More broadly: given a minimal immersion Φ inverted away from the surface, one destroys the multiplicity of Φ at ∞ ends. Thus:*

$$W(\tilde{\Phi}) = 4\pi \# \Phi^{-1}(\{\infty\}).$$

The same result stands for $\int_\Sigma |\nabla \vec{n}|^2$ instead of the Willmore energy.

Proposition 4.1.4 has deep consequences on the possible values that the Willmore energy can take. For instance, as mentioned in the exemple, the Willmore energy of *conformally minimal* surfaces (that is surfaces such that there exists an immersion turning them minimal) is a multiple of 4π. Besides, taking a compact surface and inverting it a any point p creates an end of multiplicity the multiplicity of the point $\#\Phi^{-1}(\{p\})$ and thus diminishes the Willmore energy by $4\pi\#\Phi^{-1}(\{p\})$. Since the Willmore energy of the resulting surface must be positive, necessarily:

$$W(\Phi) \geq \#\Phi^{-1}(\{p\}).$$

This must stand for any $p \in \mathbb{R}^3$, and thus yields a lower estimate for the Willmore energy:

{260520211041}

Proposition 4.1.5. *The Willmore energy of a compact branched immersion satisfies:*

$$W(\Phi) \geq 4\pi \max_{p \in \mathbb{R}^3} \#\Phi^{-1}(\{p\}).$$

A quick consequence is that the euclidean sphere realizes the minimum of the Willmore energy for compact surfaces. It is in fact the only surface to do so.

Proposition 4.1.6. *The Willmore energy of a compact surface satisfies:*

$$W \geq 4\pi.$$

The only surface that reaches this minimum is the euclidean sphere.

Proof. Applying prop 4.1.5 with any point on the surface yields the result. If Φ realizes this minimum, Φ cannot be branched (if it is, inverting it at a branch point yields a negative Willmore energy, and thus a contradiction) and is in fact embedded (if not, inverting at a point of self intersection yields the same contradiction). Taking an inversion at a point of multiplicity 1 then yields a one-ended minimal immersion: a plane. Thus, before the inversion, it was a Euclidean sphere. $\square$

Finding the minimal value of the Willmore energy by topological class is actually a very complex problem. While the minimum is known for spheres since Willmore [?], the minimum for tori was a long standing conjecture whose proof is very recent [?] and makes use of geometric measure theory tools:

Theorem 4.1.7. *The Willmore energy of compact tori is greater than $2\pi^2$. This minimum is realized by the Clifford torus.*

The minimum for genus 2 and above is still open, with very few serious candidates so far. Geometric measure theory has so far proven the most useful approach method. We will not deal with these issues any further and in fact turn our attention to the Willmore surfaces problem.

Definition 4.1.2. *An immersion is said to be Willmore if it is a critical point of the Willmore energy (or equivalently of the tracefree total curvature).*

Example 4.1.3.

- *Minimal immersions, as global minimizers of the Willmore energy are Willmore surfaces. They are however* non compact *examples of Willmore surfaces.*
- *Since the tracefree total curvature is a conformal invariant, the notion of Willmore surface is invariant by conformal transformations. Conformal transformations of Willmore surfaces thus remain Willmore surfaces. In particular, inversions of minimal surfaces are Willmore surfaces, offering us a wide variety of compact examples, as well as a motivation to look at this notion.*
- *The euclidean sphere is a Willmore surface.*
- *The Clifford torus is a Willmore surface.*

This problem is complex and interesting, and we will focus on the analysis from now on, starting by converting it into a PDE problem by finding the Euler-Lagrange equation.

4.1.3 The Willmore equations

First order analysis

Let $\Phi : \Sigma \to \mathbb{R}^3$ be an immersion, and let us consider a perturbation decomposed into a tangent and normal part:

$$\Phi_\varepsilon = \Phi + \varepsilon \Psi = \Phi + \varepsilon \left(T^p \nabla_p \Phi + N \vec{n} \right).$$

Then:

$$\begin{aligned}
\nabla_i \Phi_\varepsilon &= \nabla_i \Phi + \varepsilon \left(\nabla_i T^p \nabla_p \Phi + T^p \nabla_{ip} \Phi + \nabla_i N \vec{n} + N \nabla_i \vec{n} \right) \\
&= \nabla_i \Phi + \varepsilon \left(\left[\nabla_i T^p - A_i^p N \right] \nabla_p \Phi + \left[\nabla_i N + T^p A_{ip} \right] \vec{n} \right)
\end{aligned}$$

Recomputing the perturbation of the metric then ensures

$$\begin{aligned}
g_{\varepsilon ij} &= \langle \nabla_i \Phi_\varepsilon, \nabla_j \Phi_\varepsilon \rangle = g_{ij} + \varepsilon \left(\nabla_i T_j + \nabla_j T_i - 2A_{ij} N \right) + o(\varepsilon), \\
g_\varepsilon^{ij} &= g^{ij} - \left(\nabla^i T^j + \nabla^j T^i - 2A^{ij} N \right) \varepsilon + o(\varepsilon), \\
\delta\left(|g|\right) &= |g| \mathrm{Tr}(g^{-1}\delta g) = 2|g| \left(\mathrm{div}_g T - 2HN \right) \\
\delta\left(|g|^{\frac{1}{2}}\right) &= |g|^{\frac{1}{2}} \left(\mathrm{div}_g T - 2HN \right),
\end{aligned}$$

where we once more denoted $\delta = \frac{d}{d\varepsilon}\big|_{\varepsilon=0}$. In order to compute the perturbation of the second fundamental form, we need to compute $\vec{A}_{\varepsilon ij} = \nabla_{\varepsilon ij} \Phi_\varepsilon$, which a priori requires finding the corresponding Christoffel symbols $\Gamma_{\varepsilon ij}^{\ k}$. However, since:

$$\begin{aligned}
\vec{A}_{\varepsilon ij} &= \nabla_{\varepsilon ij} \Phi_\varepsilon = \partial_{ij} \Phi_\varepsilon - \Gamma_{\varepsilon ij}^{\ k} \nabla_k \Phi_\varepsilon \\
&= \nabla_{ij} \Phi_\varepsilon + \text{tangent terms.}
\end{aligned}$$

Since we are interested in quadratic terms in the second fundamental form, this means that the previous equality is doomed to be taken against a *normal* term, making the tangent ones irrelevant *in this first order analysis* (a second order one, for instance to study the stability, would need to account for these). Thus:

$$\begin{aligned}
\vec{A}_{\varepsilon ij} &= \nabla_{ij} \Phi_\varepsilon + \text{tangent terms} \\
&= \vec{A}_{ij} + \varepsilon \left[\nabla_{ij} N + \nabla_i T^p A_{jp} + T^p \nabla_p A_{ij} + T^p \nabla_p A_j^i + \nabla_j T^p A_{ip} - A_i^p A_{jp} N \right] \vec{n} + \text{tangent terms}
\end{aligned}$$

and

$$\begin{aligned}
\delta \vec{A}_{\varepsilon j}^{\ i} &= \left[\nabla_j^i N + \nabla^i T^p A_{jp} + \nabla_j T^p A_p^i + T^p \nabla_p A_j^i - A^{ip} A_{jp} N - \left(\nabla^i T^p + \nabla^p T^i - 2A^{ip} N \right) A_{jp} \right] \vec{n} \\
&\quad + \text{tangent terms} \\
&= \left[\nabla_j^i N + \nabla_j T^p A_p^i - \nabla^p T^i A_{jp} + T^p \nabla_p A_j^i + A^{ip} A_{jp} N \right] \vec{n} + \text{tangent terms.}
\end{aligned}$$

Then:

$$\begin{aligned}
\delta \vec{H}_\varepsilon &= \frac{1}{2} \left(\Delta_g N + |A|^2 N + 2T^p \nabla_p H \right) \vec{n} + \text{tangent terms,} \\
\delta \vec{\mathring{A}}_{\varepsilon j}^{\ i} &= \delta \left(\vec{A}_j^i - H g_j^i \right) \\
&= \left[\nabla_j^i N - \frac{1}{2} \Delta_g N g_j^i + \nabla_j T^p A_p^i - \nabla^p T^i A_{jp} + T^p \nabla_p \mathring{A}_j^i + \left(A^{ip} A_{jp} - \frac{1}{2} |A|^2 g_j^i \right) N \right] \vec{n} \\
&\quad + \text{tangent terms.}
\end{aligned}$$

Thus:

$$
\begin{aligned}
\delta\left(H_\varepsilon^2|g_\varepsilon|^{\frac{1}{2}}\right) &= \left[\left(\Delta_g N + |A|^2 N + 2T^p\nabla_p H\right) H + \left(\mathrm{div}_g T - 2HN\right) H^2\right] |g|^{\frac{1}{2}} \\
&= \left[\nabla_p\left(\nabla^p N H + T^p H^2\right) - \nabla_p N \nabla^p H + HN\left(|A|^2 - 2H^2\right)\right] |g|^{\frac{1}{2}} \\
&= \left[\nabla_p\left(\nabla^p N H - N\nabla^p H + T^p H^2\right) + N\left(\Delta_g H + \left|\mathring{A}\right|^2 H\right)\right] |g|^{\frac{1}{2}} \\
&= \left[\nabla_p\left(\nabla^p(\langle \Psi, \vec{n}\rangle) H - \langle \Psi, \vec{n}\rangle \nabla^p H + \langle \Psi, \nabla^p \Phi\rangle H^2\right) + \left\langle \Psi, \left(\Delta_g H + \left|\mathring{A}\right|^2 H\right)\vec{n}\right\rangle\right] |g|^{\frac{1}{2}} \\
&= \left[\mathrm{div}_g\left(\langle \Psi, \nabla\vec{n} H - \nabla H \vec{n} + H^2\nabla\Phi\rangle + \langle \nabla\Psi, H\vec{n}\rangle\right) + \left\langle \Psi, \left(\Delta_g H + \left|\mathring{A}\right|^2 H\right)\vec{n}\right\rangle\right] |g|^{\frac{1}{2}}.
\end{aligned}
\tag{107}
$$

0520211513}

We can then find the Euler-Lagrange equation for the Willmore energy:

{260520211516}

Theorem 4.1.8. *Willmore immersions satisfy the* Willmore equation*:*

$$\Delta_g H + H\left|\mathring{A}\right|_g^2 = 0.$$

Proof. From (107), one can deduce that:

$$
\begin{aligned}
\delta W(\Phi_\varepsilon) &= \int_\Sigma \left[\mathrm{div}\left(\dots\right) + \left\langle \Psi, \left(\Delta_g H + \left|\mathring{A}\right|^2 H\right)\vec{n}\right\rangle\right] d\mathrm{vol}_g \\
&= \int_\Sigma \left\langle \Psi, \left(\Delta_g H + \left|\mathring{A}\right|^2 H\right)\vec{n}\right\rangle d\mathrm{vol}_g.
\end{aligned}
\tag{108}
$$

{260520211623}

This stands for all Ψ, and thus yields the Willmore equation. $\square$

Similarly, we can compute the Euler-Lagrange equation of the tracefree total curvature:

$$
\begin{aligned}
\delta\left(\left|\mathring{A}\right|_g^2|g|^{\frac{1}{2}}\right) = \Bigg[2\Bigg(&\nabla_j^i N + \nabla_j T^p \mathring{A}_p^i - \nabla^p T^i \mathring{A}_{jp} + \nabla_j T^i H - \nabla_j T^i H + T^p\nabla_p \mathring{A}_j^i + \\
&\left(\mathring{A}_p^i\mathring{A}_j^p + 2H\mathring{A}_j^i + H^2 g_j^i - \frac{1}{2}\left|\mathring{A}\right|_2^2 g_j^i - H^2 g_j^i\right) N\Bigg)\mathring{A}_i^j + \left(\mathrm{div}_g T - 2HN\right)\left|\mathring{A}\right|_g^2\Bigg] |g|^{\frac{1}{2}}.
\end{aligned}
$$

Let us notice that $\left(\mathring{A}_p^i\mathring{A}_j^p\right) = \begin{pmatrix} \mathring{A}_1^1\mathring{A}_1^1 + \mathring{A}_2^1\mathring{A}_1^2 & \mathring{A}_1^1\mathring{A}_2^1 + \mathring{A}_2^1\mathring{A}_2^2 \\ \mathring{A}_1^2\mathring{A}_1^1 + \mathring{A}_2^2\mathring{A}_1^2 & \mathring{A}_1^2\mathring{A}_2^1 + \mathring{A}_2^2\mathring{A}_2^2 \end{pmatrix} = \frac{\left|\mathring{A}\right|_g^2}{2}\left(g_j^i\right)$ since $\mathring{A}_1^1 + \mathring{A}_2^2 = 0$. Injecting this into the above then yields:

$$
\begin{aligned}
\delta\left(\left|\mathring{A}\right|_g^2|g|^{\frac{1}{2}}\right) &= \left[2\nabla_j^i N\mathring{A}_i^j + 2 * \frac{\left|\mathring{A}\right|^2}{2}\left(\nabla_j T^p g_p^j - \nabla^p T^i g_{pi}\right) + 2T^p\nabla_p\mathring{A}_j^i\mathring{A}_i^j + 2H\left|\mathring{A}\right|_g^2 N + \mathrm{div}_g T\left|\mathring{A}\right|_g^2\right] |g|^{\frac{1}{2}} \\
&= \left[\nabla_p\left(2\nabla^q N\mathring{A}_q^p + T^p\left|\mathring{A}\right|_g^2\right) - 2\nabla_p N\nabla^q\mathring{A}_q^p + 2H\left|\mathring{A}\right|_g^2 N\right] |g|^{\frac{1}{2}}.
\end{aligned}
$$

Thanks to the Gauss-Codazzi equation one has:

$$\nabla^q \mathring{A}_q^p = \nabla^q A_q^p - \nabla^p H = \nabla^p A_q^q - \nabla^p H = \nabla^p H.$$

Injecting this into the above equality then yields:

$$\begin{aligned}
\delta\left(|\mathring{A}|_g^2 |g|^{\frac{1}{2}}\right) &= \left[\nabla_p\left(2\nabla^q N \mathring{A}_q^p + T^p |\mathring{A}|_g^2\right) - 2\nabla_p N \nabla^p H + 2H|\mathring{A}|_g^2 N\right] |g|^{\frac{1}{2}} \\
&= \left[\nabla_p\left(2\nabla^q N \mathring{A}_q^p - 2N\nabla^p H + T^p |\mathring{A}|_g^2\right) + 2N\left(\Delta_g H + H|\mathring{A}|_g^2\right)\right] |g|^{\frac{1}{2}} \\
&= \left[\nabla_p\left(2\nabla^q\left(\langle \Psi, \vec{n}\rangle\right)\mathring{A}_q^p - 2\langle \Psi, \vec{n}\rangle \nabla^p H + \langle \Psi, \nabla^p \Phi\rangle |\mathring{A}|_g^2\right) + 2\langle \Psi, \left(\Delta_g H + H|\mathring{A}|_g^2\right)\vec{n}\rangle\right. \\
&= \left[2\mathrm{div}_g\left(\langle \Psi, \mathring{A}\nabla\vec{n} - \nabla H\vec{n} + \frac{|\mathring{A}|^2}{2}\nabla\Phi\rangle + \langle \mathring{A}\nabla\Psi, \vec{n}\rangle\right)\right. \\
&\quad \left. +2\langle \Psi, \left(\Delta_g H + H|\mathring{A}|_g^2\right)\vec{n}\rangle\right] |g|^{\frac{1}{2}}.
\end{aligned} \tag{109}$$

{2605202117

Proceeding as in theorem 4.1.8, we recover the Willmore equation as the Euler-Lagrange equation of the tracefree total curvature. This is coherent with proposition 4.1.1: small perturbations do not change the topology, and thus any critical point of W is a critical point of $\mathring{E}$.

Analytic properties of the Willmore equation

The gap between the introduction of the Willmore energy and the analysis of the Willmore equation (first in [**?**], then in [**?**]) can be explained by the lackluster properties of the Willmore equation. Indeed, it is an equation of the type $\Delta H = H^3$ (recall that (50) allows one to express $|\mathring{A}|^2$ as a function of H and K), which typically do not behave well (the exponent 3 is supercritical in dimension 2). For instance, in the weak formalism (which we will adopt and detail below) to define the Willmore energy, one needs $H \in L^2$, which does not even assure that the Laplacian is in L^1. The Willmore equation is then outside the domain of the Calderón-Zygmund estimates, and does not enjoy a Wente-like shape, putting it a priori outside compactness through compensation phenomena.

To go around this difficulty we will use the Noether's theorem to introduce conservation laws:

Noether's theorem. *Each invariance family induces a conservation law:* $\exists Q \quad \mathrm{div}(Q) = 0$.

We will not apply this theorem exactly but we will use the general principle to introduce conservation laws and conserved quantities. Here we are studying a conformally invariant problem. We thus have 4 families of invariance to study: translations, dilations, rotations and inversions.

Translations

Let $\vec{a} \in \mathbb{R}^3$ and let us consider a Willmore immersion Φ and the perturbation by translation: $\Phi_\varepsilon = \Phi + \varepsilon\vec{a}$, that is, we took $\Psi = \vec{a}$. Since, for any $\Omega \subset \Sigma$, $\int_\Omega H^2 d\mathrm{vol}_g$ is invariant under the action of translations, $\delta\left(\int_\Omega H^2 d\mathrm{vol}_g\right) = 0$. Using (108) with $\Delta_g H + H\left|\mathring{A}\right|_g^2 = 0$ (since Φ is Willmore), this implies that:

$$\int_\Omega \mathrm{div}_g\left(\langle \vec{a}, \nabla\vec{n}H - \nabla H\vec{n} + H^2\nabla\Phi\rangle\right) d\mathrm{vol}_g = 0.$$

This stands, for all $\vec{a}$ and on all $\Omega \subset \Sigma$, and thus:

$$\mathrm{div}_g\left(\nabla\vec{n}H - \nabla H\vec{n} + H^2\nabla\Phi\right) = 0.$$

We can then introduce a first conserved quantity:

$$\mathcal{L} = -2\left(\nabla H\vec{n} - \nabla\vec{n}H - H^2\nabla\Phi\right) = -2\left(\nabla H\vec{n} + H\mathring{A}\nabla\Phi\right). \tag{110}$$ {260520211628}

On a simply connected domain (in practice, in a local conformal chart on a disk), we will write it as

$$\nabla^\perp\vec{L} = -2\left(\nabla H\vec{n} + H\mathring{A}\nabla\Phi\right) \tag{111}$$ {20520211630}

One can check that using the formula for $\mathring{A}$ yields the same conserved quantity under translations.

Remark 4.1.2. *The -2 coefficient is* historic. *In [**?**], the conservation law was introduced by hand, before being interpreted as a conservation law in [**?**]. We put this coefficient to recover the original formulation.*

Dilations

This time we take a perturbation of the type $\Phi_\varepsilon = \Phi + \varepsilon\Phi$. Then (108) yields the following conserved quantity:

$$\mathrm{div}_g\left(\frac{1}{2}\langle\Phi, \mathcal{L}\rangle + \langle\nabla\Phi, H\vec{n}\rangle\right) = 0,$$

which allows us to introduce:

$$\mathcal{S} = \langle\Phi, \mathcal{L}\rangle. \tag{112}$$ {260520211637}

On a simply connected domain, we write: $\mathcal{S} = \langle\Phi, \nabla^\perp\vec{L}\rangle = \nabla^\perp\left(\langle\Phi, \vec{L}\rangle\right) - \langle\nabla^\perp\Phi, \vec{L}\rangle$, and thus:

$$\nabla^\perp S = \langle\nabla^\perp\Phi, \vec{L}\rangle. \tag{113}$$ {260520211641}

Choosing $\left|\mathring{A}\right|^2$ yields the same conservation law.

Rotations

We consider a rotation $\Theta_{\vec{a},\varepsilon}$ of axis $\vec{a}$ and angle ε. A first order development show that:

$$\Phi_\varepsilon := \Theta_{\vec{a},\varepsilon}\Phi = \Phi + \varepsilon\vec{a}\times\Phi + o(\varepsilon).$$

The invariance of $H^2 d\mathrm{vol}_g$ then induces the following conservation law:

$$\mathrm{div}_g\left(\frac{1}{2}\langle a\times\Phi,\mathcal{L}\rangle + \langle\vec{a}\times(H\nabla\Phi),\vec{n}\rangle\right) = 0.$$

Since, using the property of the vectorial product in $\mathbb{R}^3$, one has:

$$\begin{aligned}\frac{1}{2}\langle a\times\Phi,\mathcal{L}\rangle + \langle\vec{a}\times(H\nabla\Phi),\vec{n}\rangle &= \frac{1}{2}\langle\vec{a},\Phi\times\mathcal{L}+2H\nabla\Phi\times\vec{n}\rangle\\ &= \frac{1}{2}\langle\vec{a},\Phi\times\mathcal{L}+2H\nabla^\perp\Phi\rangle.\end{aligned}$$

This stands for all $\vec{a}$, which implies:

{260520211652}
$$\mathcal{R} = \Phi\times\mathcal{L} + H\nabla^\perp\Phi. \tag{114}$$

On a simply connected domain we write:

{260520211658}
$$\nabla^\perp R = \vec{L}\times\nabla^\perp\Phi + 2H\nabla^\perp\Phi. \tag{115}$$

Remark 4.1.3. *Here the conservation law is different when we consider* $\left|\mathring{A}\right|^2$ *instead of* H^2*, but the two differ only by a* $\nabla^\perp\vec{n}$ *term and are thus equivalent. We chose to work with this one for its better analytic properties, that we will highlight below.*

Inversion

Here, since W is *not* invariant by inversions in the general case, we need to work with $\mathring{A}$. Let

$$\varphi_\varepsilon(x) = \frac{\frac{x}{|x|^2}-\varepsilon\vec{a}}{\left|\frac{x}{|x|^2}-\varepsilon\vec{a}\right|^2},$$

with $\vec{a}\in\mathbb{R}^3$. Hence

$$\Phi_\varepsilon := \varphi_\varepsilon\circ\Phi = \Phi - \varepsilon\left(|\Phi|^2\vec{a} - 2\langle\Phi,\vec{a}\rangle\Phi\right) + o(\varepsilon).$$

One can then inject this Ψ in (109) to conclude. To that end we compute

$$\begin{aligned}\left\langle\mathring{A}\nabla\Psi,\vec{n}\right\rangle &= 2\left\langle\mathring{A}\nabla\Phi,\Phi\right\rangle\langle\vec{n},\vec{a}\rangle - 2\left\langle\mathring{A}\nabla\Phi,\vec{a}\right\rangle\langle\Phi,\vec{n}\rangle\\ &= 2\left\langle\vec{a},\left\langle\mathring{A}\nabla\Phi,\Phi\right\rangle\vec{n} - \langle\Phi,\vec{n}\rangle\mathring{A}\nabla\Phi\right\rangle\\ &= 2\left\langle\vec{a},\Phi\times\left(\vec{n}\times\mathring{A}\nabla\Phi\right)\right\rangle.\end{aligned}$$

Moreover
$$\langle X, \mathcal{L}\rangle = \left\langle \vec{a}, |\Phi|^2\mathcal{L} - 2\langle \Phi, \mathcal{L}\rangle \Phi \right\rangle .$$
We deduce :
$$\forall \vec{a} \in \mathbb{R}^3 \quad \mathrm{div}_g \left(\left\langle \vec{a}, |\Phi|^2\mathcal{L} - 2\langle \Phi, \mathcal{L}\rangle \Phi + 4\Phi \times \left(\vec{n} \times \mathring{A}\nabla\Phi \right) \right\rangle \right) = 0.$$
From this, we find the fourth conserved quantity :
$$\mathcal{I} = -|\Phi|^2\mathcal{L} + 2\langle \Phi, \mathcal{L}\rangle \Phi - 4\Phi \times \left(\vec{n} \times \mathring{A}\nabla\Phi \right). \tag{116}$$ {180620191808}

On a simply connected domain:
$$\begin{aligned}
\mathcal{I} &= -|\Phi|^2\mathcal{L} + 2\langle \Phi, \mathcal{L}\rangle \Phi + 4\Phi \times \left(\mathring{A}\nabla^\perp\Phi \right) \\
&= -|\Phi|^2 \nabla^\perp \vec{L} + 2\left\langle \Phi, \nabla^\perp \vec{L} \right\rangle \Phi + 4\Phi \times \left(\mathring{A}\nabla^\perp\Phi \right) \\
&= \nabla^\perp \left(-|\Phi|^2 \vec{L} + 2\left\langle \Phi, \vec{L}\right\rangle \Phi \right) + 2\left\langle \nabla^\perp\Phi, \Phi\right\rangle \vec{L} - 2\left\langle \Phi, \vec{L}\right\rangle \nabla^\perp\Phi \\
&\quad - 2\left\langle \nabla^\perp\Phi, \vec{L}\right\rangle \Phi + 4\Phi \times \left(\mathring{A}\nabla^\perp\Phi \right) \\
&= \nabla^\perp(\dots) + 2\Phi \times \left(\vec{L} \times \nabla^\perp\Phi + 2\mathring{A}\nabla^\perp\Phi \right) - 2\nabla^\perp S\Phi \\
&= \nabla^\perp(\dots) + 2\Phi \times \left(\vec{L} \times \nabla^\perp\Phi + 2H\nabla^\perp\Phi + 2\mathring{A}\nabla^\perp\Phi - 2H\nabla^\perp\Phi \right) - 2\nabla^\perp S\Phi \\
&= \nabla^\perp(\dots) + 2\Phi \times \left(\nabla^\perp \vec{R} + 2\nabla^\perp \vec{n} \right) - 2\nabla^\perp S\Phi \\
&= \nabla^\perp(\dots) + 2\Phi \times \nabla^\perp \vec{R} - 2\nabla^\perp S\Phi + 4\Phi \times \nabla^\perp \vec{n} \\
&= \nabla^\perp(\dots) + 2\Phi \times \nabla^\perp \vec{R} - 2\nabla^\perp S\Phi + 4\nabla^\perp(\Phi \times \vec{n}) - 4\nabla^\perp\Phi \times \vec{n} \\
&= \nabla^\perp(\dots) + 2\Phi \times \nabla^\perp \vec{R} - 2\nabla^\perp S\Phi + 4\nabla\Phi.
\end{aligned} \tag{117}$$ {190620191124}

The Jacobian Willmore system

Taking $\mathrm{div}(\mathcal{I}) = 2\nabla\Phi \times \nabla^\perp\vec{R} - 2\nabla^\perp S\nabla\Phi + 4\Delta\Phi = 0$ one obtains:
$$\Delta\Phi = 2\nabla^\perp S\nabla\Phi + 2\nabla^\perp\vec{R} \times \nabla\Phi. \tag{118}$$ {260520211733}

This equation is pivotal: it shows that if we can obtain some regularity on S and $\vec{R}$, we can send it back to Φ and obtain some regularity. In addition, its Jacobian, Wente-like structure highlight that this is possible even in a critical setting thanks to compactness by compensation.

Let us then compute:

$$\begin{aligned}\left\langle \vec{n}, \nabla^{\perp}\vec{R}\right\rangle &= \left\langle \vec{n}, \vec{L}\times\nabla^{\perp}\Phi + 2H\nabla^{\perp}\Phi\right\rangle = \left\langle \vec{L}, \nabla^{\perp}\Phi\times\vec{n}\right\rangle \\ &= -\left\langle \vec{L}, \nabla\Phi\right\rangle = -\nabla S.\end{aligned}$$

Similarly

$$\begin{aligned}\vec{n}\times\nabla^{\perp}\vec{R} + \nabla^{\perp}S\vec{n} &= \vec{n}\times\left(\vec{L}\times\nabla^{\perp}\Phi + 2H\nabla^{\perp}\Phi\right) + \left\langle \vec{L}, \nabla^{\perp}\Phi\right\rangle\vec{n} \\ &= \left\langle \vec{L}, \nabla^{\perp}\Phi\right\rangle\vec{n} - \left\langle \vec{L}, \vec{n}\right\rangle\nabla^{\perp}\Phi + 2H\nabla\Phi \\ &= \vec{L}\times\left(\vec{n}\times\nabla^{\perp}\Phi\right) + 2H\nabla\Phi = \nabla\vec{R}.\end{aligned}$$

Taking the divergence of the last two equalities yield:

{190620191055}

Theorem 4.1.9. *Let $\Phi \in C^{\infty}\left(\mathbb{D}, \mathbb{R}^3\right)$ be a conformal Willmore immersion. Then*

{190620191057}

$$\begin{cases}\Delta S = -\left\langle \nabla\vec{n}, \nabla^{\perp}\vec{R}\right\rangle \\ \Delta\vec{R} = \nabla\vec{n}\times\nabla^{\perp}\vec{R} + \nabla^{\perp}S\nabla\vec{n} \\ \Delta\Phi = \frac{1}{2}\left(\nabla^{\perp}S.\nabla\Phi + \nabla^{\perp}\vec{R}\times\nabla\Phi\right).\end{cases} \tag{119}$$

Remark 4.1.4. *This system allows one to go around the Willmore equation. Its Jacobian shape will entice to use compactness through compensation to obtain estimates on S, $\vec{R}$, and thus on Φ. This shape is precisely due to its origin as a system of conservation laws. Here lies the power of Wente's lemma: while it requires a precise Jacobian shape, it can be made to follow from conservation laws, and thus invariance of the system. It is a mathematically meaningful and deep shape.*

We can now precise the analytic framework for our work: the weak immersions, which will be just enough regularity to study Willmore immersions, while being just weak enough for flexibility.

4.2 Conformal weak immersions

4.2.1 Definition

In order to define a Willmore surface, one needs to have an immersion ϕ regular enough to define:

- g the non-degenerate induced metric,

- the second fondamental form,
- the Willmore energy.

The first condition is a metric one, which requires a regularity assumption on the immersion.

{serni}

Definition 4.2.1. *Let $\Phi : \Sigma \to \mathbb{R}^3$ be an immersion of a compact manifold and let g be its induced metric. Given g_0 a smooth reference metric on Σ, we say that Φ is a* Lipschitz immersion *if it satisfies:*

1.

$$\Phi \in W^{1,\infty}(\Sigma, \mathbb{R}^3),$$

{3105202111421}

2.

$$\exists c > 0 \text{ s.t. } \forall p \in \Sigma,\ X \in T_p\Sigma \quad c^{-1} g_0(X,X) \leq g(X,X) \leq c g_0(X,X).$$

{3105202111422}

This definition does not depend on the reference metric g_0. Indeed, as seen above in (18), any two non degenerate metrics on a compact manifold are equivalent, and define the same Sobolev space. Both conditions are thus g_0 independant (with only a change of constant $c > 0$ and of the resulting norm $\|\Phi\|_{W^{1,\infty}}$) and thus define a g_0 independant notion. One might notice that this definition is flexible enough to allow branch points and branched ends at isolated points. For now, we will only consider immersions of surfaces.

With this definition, ensures that Φ is an immersion (in fact *uniformly*), and thus that the tangent plane is well defined. The Gauss map $\vec{n}$ is thus defined in the usual manner with the classical formula in local coordinates: {3105202111421}

$$\vec{n} := \frac{\phi_x \times \phi_y}{|\phi_x \times \phi_y|}.$$

One can check that both the numerator and the denominator are well defined in L^∞, and thus that $\vec{n} \in \mathbb{S}^1$. To define the second fundamental form, one needs to make sure that $\nabla \vec{n}$ is well defined. To that end, we will make the following assumption:

{310520211238}

3. $\vec{n} \in W^{1,2}$, or equivalently

$$\int_\Sigma |\nabla \vec{n}|^2_{g_0} \, d\mathrm{vol}_{g_0} < \infty.$$

Once more, since g and g_0 are equivalent metrics, one can require similarly

$$\int_\Sigma |\nabla \vec{n}|_g^2 \, d\mathrm{vol}_g < \infty.$$

This condition is enough to ensure that the Willmore energy is well defined. Indeed, recall (43):

$$|\nabla \vec{n}|_g^2 = |A|_g^2 = \left|\mathring{A}\right|_g^2 + 2H^2.$$

Thus: $W(\Phi) \leq \int_\Sigma |\nabla \vec{n}|_g^2 \, d\mathrm{vol}_g$. We thus have enough regularity to study the Willmore energy on a weak immersion:

Definition 4.2.2. *The space $\mathcal{E}_\Sigma$ of Lipschitz immersions with L^2-bounded second fundamental form is defined as:*

$$\mathcal{E}_\Sigma := \left\{ \Phi : \Sigma \to \mathbb{R}^3 \textit{ measurable satisfying } 1), 2) \textit{ and } 3) \right\}.$$

As has been said, given $\Phi \in \mathcal{E}_\Sigma$, $W(\Phi)$ (or similarly $\mathring{E}(\Phi)$) is well defined and is finite. One can thus a priori consider the critical points of W and $\mathring{E}$. However, we cannot study the Willmore equation directly. Indeed, while we control H in L^2, the constant term in the Willmore equation is in H^3, and thus a priori not even in L^1, and, as has been noticed, outside the frame of the Calderón-Zygmund theorem. We will thus need to work a bit more to define *weak Willmore immersions.*

Conformal parametrizations in the weak setting

The objective is of course to be able to use our study of the conservation laws and the resulting Willmore equations (119). To that end, we wish to show that this weak framing of the problem is adapted to a conformal formulation. To that end, we will prove:

{310520211332}

Theorem 4.2.1. *Let $\phi \in \mathcal{E}_\mathbb{D}$ such that:*

$$\int_\mathbb{D} |\nabla \vec{n}|^2 \, dx_1 dx_2 < \frac{8\pi}{3}.$$

Then there exists a homeomorphism ψ of $\mathbb{D}$, locally bi-Lipschitz, such that the map $\phi \circ \psi : \mathbb{D} \to \mathbb{R}^3$ is conformal.

We have, in fact, already proven this while showing theorem 1.3.5. Indeed the proof works short of one point:

- We first show that we can integrate a Coulomb frame into local conformal coordinates.

- We find a Coulomb frame by an energy minimization procedure to obtain a Coulomb frame with minimal energy, and thus lower than the energy of a reference Coulomb frame.

The proof of theorem 4.2.1 thus reduces to the question: given a weak immersion satisfying $\int_{\mathbb{D}} |\nabla \vec{n}|^2 \, dx_1 dx_2 < \frac{8\pi}{3}$, can we find a reference frame of bounded energy?

This was answered by F. Hélein's *controlled lifting theorem* (lemma 5.1.4 in [?]):

Theorem 4.2.2. *For every map $u \in W^{1,2}(\mathbb{D}, \mathbb{S}^2)$ such that:*

$$\|\nabla u\|_{L^2(\mathbb{D})} \leq \frac{8\pi}{3},$$

there exists a frame (e_1, e_2) in $W^{1,2}(\mathbb{D}, \mathbb{R}^3)$ such that for almost every $z \in \mathbb{D}$, (e_1, e_2) is a positively oriented basis of $u(z)^\perp$. Furthermore:

$$\|\nabla (e_1, e_2)\|_{L^2(\mathbb{D})} \leq C \|\nabla u\|_{L^2(\mathbb{D})}.$$

Proof. **Step 1:** $u \in C^\infty$

It is sufficient to show the result for smooth u. Indeed, given $u \in W^{1,2}$ by density of C^∞ in the Sobolev spaces, one can find smooth $u_n \to u$ converging in $W^{1,2}$. By weak compactness, up to extractions the associated frames (e_1^n, e_2^n) converge weakly toward a frame (e_1, e_2) of $u^\perp$. Since the weak limit only loses energy, this frame satisfies the energy estimate.

We can thus assume $u \in C^\infty$ and find a starting frame $\tilde{b} = (\tilde{e}_1, \tilde{e}_2)$ in $C^\infty \subset W^{1,2}$.

Step 2: The Coulomb minimization procedure

Starting from the frame $(\tilde{e}_1, \tilde{e}_2)$, one can find, on $\mathbb{D}_r$ a Coulomb frame $(e_{r,1}, e_{r,2})$ with the same minimization procedure as in theorem 4.2.1. The Coulomb frame thus satisfies:

$$\begin{cases} \operatorname{div}\left(\langle \nabla e_{r,2}, e_{r,1} \rangle\right) = 0 \text{ in } \mathbb{D}_r \\ \langle \partial_\nu e_{r,2}, e_{r,1} \rangle = 0 \text{ on } \partial\mathbb{D}_r \end{cases}$$

We can thus write (since $\mathbb{D}$ is simply connected):

$$\begin{aligned} \nabla^\perp f_r &= \langle \nabla e_{r,2}, e_{r,1} \rangle \text{ in } \mathbb{D}_r \\ f_r &= 0 \text{ on } \mathbb{D}_r \end{aligned}$$

, which implies that f_r solves:

$$\begin{cases} \Delta f_r = \langle \nabla^\perp e_{r,2}, \nabla e_{r,1} \rangle \text{ in } \mathbb{D}_r \\ \quad f_r = 0 \text{ on } \partial\mathbb{D}_r. \end{cases}$$

Applying Wente's lemma with the constant of remark then yields: {3105202115

$$\|\nabla f_r\|_{L^2(\mathbb{D}_r)} \leq \sqrt{\frac{3}{16\pi}}\, \|\nabla e_{r,1}\|_{L^2(\mathbb{D}_r)} \|\nabla e_{r,2}\|_{L^2(\mathbb{D}_r)} ,$$

which implies

$$2\,\|\nabla f_r\|_{L^2(\mathbb{D}_r)} \leq \sqrt{\frac{3}{16\pi}} \left(\|\nabla e_{r,1}\|^2_{L^2(\mathbb{D}_r)} + \|\nabla e_{r,2}\|^2_{L^2(\mathbb{D}_r)} \right). \tag{120}$$ {3100520211

Step 3: A priori estimate on f_r

Decomposing $\nabla(e_{1,r}, e_{2,r})$ in the $(e_{1,r}, e_{2,r}, u)$ frame yields that:

{310520211634}
$$\|\nabla e_{r,1}\|^2_{L^2(\mathbb{D}_r)} + \|\nabla e_{r,2}\|^2_{L^2(\mathbb{D}_r)} = 2\,\|\langle \nabla e_{r,2}, e_{r,1}\rangle\|^2_{L^2(\mathbb{D}_r)} + \|\nabla u\|^2_{L^2(\mathbb{D}_r)} . \tag{121}$$

Injecting this into (120) yields:

$$2\,\|\nabla f_r\|_{L^2(\mathbb{D}_r)} \leq \sqrt{\frac{3}{16\pi}} \left(2\,\|\nabla f_r\|^2_{L^2(\mathbb{D}_r)} + \|\nabla u\|^2_{L^2(\mathbb{D}_r)} \right),$$

which can be rephrased as $P(t) \geq 0$, where $t = \|\nabla f_r\|_{L^2(\mathbb{D}_r)}$ and

$$P(t) = 2t^2 - 8\sqrt{\frac{\pi}{3}}t + \|\nabla u\|^2_{L^2(\mathbb{D}_r)} .$$

Studying the sign of this polynomial of discriminant $\Delta' = \frac{16\pi}{3} - 2\,\|\nabla u\|^2_{L^2(\mathbb{D}_r)}$, since by assumption $\|\nabla u\|^2_{L^2(\mathbb{D}_r)} < \frac{8\pi}{3}$, one has:

$$\|\nabla f_r\|_{L^2(\mathbb{D}_r)} \in [0, \alpha] \cup [\beta, +\infty),$$

where α and β are the two roots of P. The objective is of course to show that $\|\nabla f_r\|_{L^2(\mathbb{D}_r)}$ lies in the first interval. The strategy will then be to show that $r \mapsto \|\nabla f_r\|_{L^2(\mathbb{D}_r)}$ is continuous, and thus lies entirely in one of the interval, and is worth 0 at 0.

Step 4: $r \mapsto \|\nabla f_r\|_{L^2(\mathbb{D}_r)}$ is continuous

Let us define $g : [0,1] \to \mathbb{R}$ by

$$\begin{cases} g(0) = 0 \\ g(r) = \|\nabla f_r\|_{L^2(\mathbb{D}_r)} \text{ if } r > 0. \end{cases}$$

Let us show that g is continuous.

Lemma 4.2.1. *There exists a constant $C_0 > 0$, depending only on u, such that for any $r_0 \in [0,1]$ and any basis $b^{r_0} = (e_1^{r_0}, e_2^{r_0})$, there exists a basis $b_r^{r_0} = (e_{r,1}^{r_0}, e_{r,2}^{r_0})$ associated with $u_r(z) = u(rz)$ such that*

$$b_{r_0}^{r_0} = b^{r_0}$$

and

$$g_{b^{r_0}}(r) := \int_{\mathbb{D}} \left|\langle \nabla e_{r,1}^{r_0}, e_{r,2}^{r_0}\rangle\right|^2 dxdy$$

is a Lipschitz function such that

$$|g_{b^{r_0}}(r) - g_{b^{r_0}}(r')| \leq C_0 \left(1 + \|\nabla b^{r_0}\|_{L^2}\right) |r - r'|.$$

Proof. Let S_r be the orthogonal symmetry around $u_r(z)$. Let $R_r^{r_0} : \mathbb{D} \to SO(3)$ be the unique solution of

$$\begin{aligned} &R_{r_0}^{r_0}(z) = \mathrm{Id}, \ \forall z \in \mathbb{D} \\ &\partial_r R_r^{r_0}(z) = -\frac{1}{2} S_r^{-1}(z) \partial_r S_r(z) R_r^{r_0}(z) \ \forall z \in \mathbb{D}. \end{aligned}$$

Thus defined, $R_r^{r_0}$ is orthogonal and $\left(R_r^{r_0}\right)^{-1} S_r R_r^{r_0}$ does not depend on r. With the initial value hypothesis we deduce that:

$$S_r R_r^{r_0} = R_r^{r_0} S_{r_0}.$$

We can thus define the frame for $u_r(z)^\perp$:

$$b_r^{r_0}(z) := \left(R_r^{r_0}(z) e_1^{r_0}, R_r^{r_0}(z) e_2^{r_0}\right).$$

Computing explicitely, we can show that for all r, r':

$$|g_{b^{r_0}}(r) - g_{b^{r_0}}(r')| \leq \left(2\|\nabla b^{r_0}\|_{L^2} + \|\nabla R_r^{r_0}\|_{L^2} + \|\nabla R_{r_0}^{r_0}\|_{L^2}\right) \left\| \left(R_r^{r_0}\right)^T \nabla R_r^{r_0} - \left(R_{r'}^{r_0}\right)^T \nabla R_{r'}^{r_0} \right\|_{L^2},$$

which yields the result since $R_r^{r_0}$ is smooth. □

By definition, the Coulomb frame minimizes the energy of the frames. Since, in adition $b^r = b_r^r$, one has:

$$g(r) = \inf_{r_0, b^{r_0}} g_{b^{r_0}}(r).$$

Since $\tilde{b}$ was assumed to be of finite energy, we can replace this infimum by

$$g(r) = \inf_{r_0, b^{r_0}, \|\nabla b^{r_0}\|_{L^2} \leq M} g_{b^{r_0}}(r).$$

As the infimum of a family of uniformly Lipschitz functions, g is thus continuous.
Step 5: Conclusion We can thus conclude that $\|\nabla f_r\|_{L^2(\mathbb{D}_r)}$ lies entirely in $[0,\alpha)$. Computing explicitely this α yields:

$$\|\nabla f_r\|_{L^2(\mathbb{D}_r)} \leq \sqrt{\frac{4\pi}{3}} - \sqrt{\frac{4\pi}{3} - \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{D}_r)}}.$$

Using $\|\nabla u\|_{L^2(\mathbb{D}_r)} \leq \frac{8\pi}{3}$, and applying the result with $r=1$ yields

$$\|\nabla f_1\|_{L^2(\mathbb{D})} \leq C\|\nabla u\|_{L^2(\mathbb{D})}.$$

Injecting this into (121) yields the desired result. □

As a result we can deduce that weak immersion with L^2 second fundamental form setting is adapted to local conformal charts: it ensures that there exists a local Coulomb frame with energy:

{310520211658}

$$\|\nabla(e_1,e_2)\|_{L^2} \leq C\|\nabla u\|_{L^2}, \tag{122}$$

and a resulting conformal parametrization.

Remark 4.2.1. *One might notice that the starting frame for the Coulomb minimization given by Hélein's controlled lifting theorem is already taken as a Coulomb frame. This is only natural since the frame most likely to satisfy an energy estimate is the frame with less energy. Besides, one might notice that we once more used the* conservation law *which is the Euler-Lagrange equation to the Coulomb energy to deduce a* Jacobian *system of equations.*

Remark 4.2.2. *One follow-up of the previous remark is that since the b frame given by Hélein's theorem is already Coulomb, one might expect that $f=\lambda$ (see section 1.3.5 for an explaination of how the conformal factor derives from the Coulomb frame). This however not exactly the case: the two differ by a harmonic function whose precise role is to ensure the boundary condition. We explore this idea in what follows.*

4.2.2 Control on the conformal factor

Not only can one parametrize weak immersions with L^2 second fundamental form in local conformal charts, but (122) suggests that the conformal factor, that is the metric, can be locally controlled.

{010620211017}

Theorem 4.2.3. *Let $\phi \in \mathcal{E}_{\mathbb{D}}$ a conformal weak immersion. Let λ denote its conformal factor and assume that*

{010620210901}

$$\|\nabla\lambda\|_{L^{2,\infty}(\mathbb{D})} \leq M < +\infty, \tag{123}$$

and

$$\|\nabla\vec{n}\|_{L^2(\mathbb{D})} \leq \sqrt{\frac{8\pi}{3}}.$$

Then there exists a constant $\Lambda \in \mathbb{R}$ such that:

$$\|\lambda - \Lambda\|_{L^\infty\left(\mathbb{D}_{\frac{1}{2}}\right)} + \|\nabla\lambda\|_{L^2\left(\mathbb{D}_{\frac{1}{2}}\right)} \leq C(M).$$

Proof. Applying a dilation $x \mapsto \rho x$ changes λ into $\lambda + \log\rho$. By applying the right dilation we can thus assume that λ is of null average over $\mathbb{D}$. Then, since by Poincaré inequality: $\forall 1 < p \leq \infty$, $\forall f \in L^p(\mathbb{D})$, $\|f - \bar{f}\|_{L^p} \leq C\|\nabla f\|_{L^p}$, applying Marcinkiewitz interpolation theorem ensures that:

$$\|\lambda\|_{L^{2,\infty}(\mathbb{D})} \leq C(M). \tag{124}$$ {070620211023}

Let us consider the Coulomb frame $b = (e_1, e_2)$ given by Hélein's controlled lifting theorem. Then, one can check that:

$$\begin{aligned}
\langle\nabla^\perp e_1, \nabla e_2\rangle &= \langle\nabla^\perp e_1, \vec{n}\rangle\langle\nabla e_2, \vec{n}\rangle \\
&= \langle e_1, \nabla^\perp\vec{n}\rangle\langle e_2, \nabla\vec{n}\rangle = \langle\vec{n}_x, e_1\rangle\langle\vec{n}_y, e_2\rangle - \langle\vec{n}_y, e_1\rangle\langle\vec{n}_x, e_2\rangle \\
&= \langle\vec{n}_x, \cos\theta e^{-\lambda}\phi_x + \sin\theta e^{-\lambda}\phi_y\rangle\langle\vec{n}_y, -\sin\theta e^{-\lambda}\phi_x + \cos\theta e^{-\lambda}\phi_y\rangle \\
&- \langle\vec{n}_y, \cos\theta e^{-\lambda}\phi_x + \sin\theta e^{-\lambda}\phi_y\rangle\langle\vec{n}_x, -\sin\theta e^{-\lambda}\phi_x + \cos\theta e^{-\lambda}\phi_y\rangle \\
&= \langle\vec{n}_x, \phi_x e^{-\lambda}\rangle\langle\vec{n}_y, \phi_y e^{-\lambda}\rangle - \langle\vec{n}_y, \phi_x e^{-\lambda}\rangle\langle\vec{n}_x, \phi_y e^{-\lambda}\rangle = Ke^{2\lambda}.
\end{aligned}$$

From the Liouville equation one obtains that the conformal factor satisfies a Wente equation: $\Delta\lambda = -\langle\nabla^\perp e_1, \nabla e_2\rangle$. To obtain the right boundary condition, let l be the harmonic function equal to λ on $\partial\mathbb{D}$, and then consider $\mu = \lambda - l$ which solves

$$\begin{cases} \Delta\mu = -\langle\nabla^\perp e_1, \nabla e_2\rangle \text{ in } \mathbb{D} \\ \quad \mu = 0 \text{ on } \partial\mathbb{D}. \end{cases}$$

Wente's lemma then ensures that:

$$\|\nabla\mu\|_{L^2(\mathbb{D})} + \|\mu\|_{L^\infty(\mathbb{D})} \leq C\|\nabla e_1\|_{L^2(\mathbb{D})}\|\nabla e_2\|_{L^2(\mathbb{D})} \leq C\|\nabla\vec{n}\|^2_{L^2(\mathbb{D})} \leq C. \tag{125}$$ {310520211748}

From the property of harmonic functions (detailed in section 3.2.2) we can say that, for all $q \leq \infty$,

$$\|\nabla l\|_{L^q\left(\mathbb{D}_{\frac{1}{2}}\right)} \leq C\|\lambda\|_{L^1(\partial\mathbb{D})}. \tag{126}$$ {310520211757}

Using a trace theorem, one has $\|\lambda\|_{L^1(\partial\mathbb{D})} \leq C\|\lambda\|_{W^{1,p}(\mathbb{D})}$ for all $p > 1$. From Marcinkiewitz interpolation theorem one can deduce, with $q = 2$:

$$\|\nabla l\|_{L^2\left(\mathbb{D}_{\frac{1}{2}}\right)} \leq C\|\lambda\|_{W^{1,(2,\infty)}(\mathbb{D})} \leq C(M)$$

thanks to (124). Injecting this into (125) yields that:

$$\|\nabla\lambda\|_{L^2\left(\mathbb{D}_{\frac{1}{2}}\right)} \leq C(M),$$

which is the first inequality we desire.

To obtain the second inequality, let Λ be the average of l over $\mathbb{D}_{\frac{1}{2}}$. Using (126) with $q=\infty$ yields:

$$\|\nabla\lambda\|_{L^\infty\left(\mathbb{D}_{\frac{1}{2}}\right)} \leq C(M).$$

By continuity of l, there exists $p \in \mathbb{D}_{\frac{1}{2}}$ such that $l(p) = \Lambda$. Writing $l(x) = l(p) + \int \nabla l(z).(x-p)$, we can deduce that:

$$\|l-\Lambda\|_{L^\infty\left(\mathbb{D}_{\frac{1}{2}}\right)} \leq C\|\nabla l\|_{L^\infty\left(\mathbb{D}_{\frac{1}{2}}\right)} \leq C(M).$$

Injecting this into (125) yields

$$\|\lambda-\Lambda\|_{L^\infty\left(\mathbb{D}_{\frac{1}{2}}\right)} \leq \|\lambda - l\|_{L^\infty\left(\mathbb{D}_{\frac{1}{2}}\right)} + \|l-\Lambda\|_{L^\infty\left(\mathbb{D}_{\frac{1}{2}}\right)} \leq C(M).$$

This is the second inequality, which concludes the proof. □

This result is pivotal for the weak analysis: it allows one to have an a priori control on the metric terms with only a curvature assumption. Thanks to this, in many a weak analysis, the metric terms can be considered as constants. Indeed, under the hypotheses of the theorem one enjoy the following Harnack inequality:

{010620210852}
$$\exists \Lambda, C \in \mathbb{R} \text{ s.t. } \forall x \in \mathbb{D}_{\frac{1}{2}} \quad \frac{e^\Lambda}{C} \leq e^{\lambda(x)} \leq Ce^\Lambda. \tag{127}$$

The constant Λ is essential to account for the invariance of the Dirichlet energy of the Gauss map by conformal transformations, and more precisely dilations. Applying a conformal transformation changes the Λ, but not the Harnack estimate.

The hypothesis (123) might seem strange. From the proof, one can see that it is necessary to properly control the harmonic part of the conformal factor on the inside of the disk. Any small control on λ would have been enough (for instance a L^1 control on the boundary). The advantage of this hypothesis is that it is *free*:

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Theorem 4.2.4. *Let (Σ, g) be a closed Riemann surface of fixed genus. Let h denote the metric with constant curvature (and volume equal to one in the torus case) in the conformal class of g and $\Phi \in \mathcal{E}(\Sigma)$ conformal, that is :*

$$\Phi^*\xi = e^{2u}h.$$

Then there exists a finite conformal atlas (U_i, Ψ_i) *and a positive constant* C *depending only on the genus of* Σ*, such that*

$$\|\nabla \lambda_i\|_{L^{2,\infty}(V_i)} \leq C \|\nabla_{\Phi^*\xi} \vec{n}\|^2_{L^2(\Sigma)},$$

with $\lambda_i = \frac{1}{2} \log \frac{|\nabla \Phi|^2}{2}$ *the conformal factor of* $\Phi \circ \Psi_i^{-1}$ *in* $V_i = \Psi_i(U_i)$*.*

We will not prove this result but merely give the broad strokes of the proof. The core concept is that the Liouville equation ensures that $\Delta\lambda$ lies in L^1 and thus that $\nabla\lambda$ is in $L^{2,\infty}$. This however must be adapted to function on surfaces of any topology. To that end one must seek special conformal charts of *disk* and *annulus* types (we refer the reader to P. Laurain and T. Rivière's theorem 3.1 of [**?**] for more details). Then one can show that the Green function has only $\frac{1}{|x|}$ estimates, which by convolution properties yield the proof. Theorem 3.1 deals with surfaces of genus higher than 1, but since spheres already have a natural two charts conformal atlas, the result stands for every genus.

Thus, up to choosing the right conformal atlas, one can locally satisfy (123) and obtain local Harnack inequalities for the conformal factor.

4.2.3 Weak Willmore immersions

Let us start by noticing that, in a local conformal chart and for a smooth immersion, the first conservation law (resulting from the invariance by translations) is equivalent to the Willmore equation. Indeed, we can compute:

$$\begin{aligned}\operatorname{div}\left(\nabla H \vec{n} + H \mathring{A} \nabla \phi\right) &= \Delta H \vec{n} + H \left(\frac{e-g}{2} e^{-2\lambda} \phi_{xx} + 2 f e^{-2\lambda} \phi_{xy} - \frac{e-g}{2} e^{-2\lambda} \phi_{yy} \right) + \text{ tangent terms} \\ &= \left(\Delta H + H \left| \mathring{A} \right|^2 \right) \vec{n} + \text{ tangent terms.}\end{aligned}$$

This means that, a priori, an immersion only need to satisfy $\operatorname{div}\left(\nabla H \vec{n} + H \mathring{A} \nabla \Phi\right) = 0$ to be a Willmore immersion. The gain is that, even for a weak immersion, the conservation law is defined in a distributional sense. Indeed, $\nabla \vec{n} \in L^2$ ensures that $H \nabla \phi \in L^2$. In a disk of small total curvature, one can apply theorem 4.2.3 and ensure that $H \in L^2$. Similarly $\mathring{A} \nabla \phi \in L^2$ Thus both $\nabla H \vec{n}$ and $H \mathring{A} \nabla \phi$ are defined in the distributional sense. From this we define:

Definition 4.2.3. *Let* $\Phi \in \mathcal{E}(\Sigma)$*,* Φ *is a weak Willmore immersion if*

$$\operatorname{div}\left(\nabla H \vec{n} + H \mathring{A} \nabla \phi\right) = 0 \tag{128}$$ {equationwillmore

holds in a distributional sense in every conformal parametrization $\Psi : \mathbb{D} \to D$ *on every neighborhood* D *of* x *, for all* $x \in \Sigma$*. Here, the operators* div, ∇ *and* $\nabla^\perp$ *are to be understood with respect to the flat metric on* $\mathbb{D}$*.*

All we need to do now is prove a regularity result to ensure that *weak Willmore immersions* are indeed Willmore immersions in the classical sense.

4.3 ε-regularity for Willmore immersions

4.3.1 Controls on $\vec{L}$

Low-regularity estimates

In order to obtain some regularity, for a Willmore immersion we wish to make use of the system (119) and some Wente inequalities to find an ε-regularity. However, this requires that $\nabla\vec{n}$, ∇S and $\nabla\vec{R}$ are in L^2. While the first one stands by assumption, the last two are far from obvious. Indeed, we need to start with the weak Willmore immersion formalism, and thus the weak Willmore equation that stands in $\mathcal{D}(\mathbb{D})$ the space of distributions. This will force us to start with low regularity estimates, in order to gain enough to apply compactness through compensation estimates.

We will then use (without proving) theorem 3 of [?].

{231120201048}

Theorem 4.3.1. *Let $f \in L^p(\mathbb{D})$ such that $\int f dxdy = 0$, $1 < p < \infty$. Then there exists some $Y \in L^\infty(\mathbb{D}) \cap W^{1,p}_0(\mathbb{D})$ such that*

$$\mathrm{div} Y = f,$$

and:

$$\|Y\|_{L^\infty(\mathbb{D})} + \|\nabla Y\|_{W^{1,p}(\mathbb{D})} \leq C\|f\|_{L^p(\mathbb{D})}.$$

This theorem is another example of compactness through compensation due to the work of J. Bourgain and H. Brézis: the estimates is not true in general, but, with the assumption on the average, one gains the additionnal regularity.

We will also need a Hodge decomposition theorem which we will not prove (as it would take us into a whole different rabbit hole) to transfer the regularity to:

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Theorem 4.3.2 (theorem 10.5.1 in [?])**.** *Let $r > 0$ and $1 < p < \infty$. For any $X \in L^p(\mathbb{D}_r, \mathbb{R}\times\mathbb{R})$ there exists $\alpha \in W^{1,p}(\mathbb{D}_r)$ such that*

$$\mathrm{div}(X) = \Delta\alpha$$

and

$$\|\alpha\|_{W^{1,p}(\mathbb{D}_r)} \leq C_p r^{-\left(1-\frac{2}{p}\right)}\|X\|_{L^p(\mathbb{D}_r)}.$$

We will thuse use Bourgain-Brézis's result to prove:

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Theorem 4.3.3. *Let $V \in \mathcal{D}'(\mathbb{R}^3)$ such that $\nabla V = \nabla^\perp A + B$ with $A \in L^2(\mathbb{D})$ and $B \in L^1(\mathbb{D}, \mathbb{R}^2)$. Then for any $r < 1$ there exists $c \in \mathbb{R}$ a constant and $C(r) > 0$ such that*

$$\|V - c\|_{L^2(\mathbb{D}_r)} \leq C(r) \left(\|A\|_{L^2(\mathbb{D})} + \|B\|_{L^1(\mathbb{D})} \right).$$

Proof. Let us first assume that V, A and B are smooth, and obtain an a priori estimate. Then, for any $U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \in W_0^{1,2}(\mathbb{D}, \mathbb{R}^2)$, one has:

$$\begin{aligned} \left| \int_{\mathbb{D}} \nabla V . U dxdy \right| &\leq \left| \int_{\mathbb{D}} \nabla^\perp A . U dxdy \right| + \left| \int_{\mathbb{D}} B . U dxdy \right| \\ &\leq \left| \int_{\mathbb{D}} A . \mathrm{div} \left(\begin{pmatrix} U_2 \\ -U_1 \end{pmatrix} \right) dxdy \right| + \left| \int_{\mathbb{D}} B . U dxdy \right| \\ &\leq \left(\|A\|_{L^2(\mathbb{D})} + \|B\|_{L^1(\mathbb{D})} \right) \left(\|U\|_{L^\infty(\mathbb{D})} + \|\nabla U\|_{L^2(\mathbb{D})} \right). \end{aligned} \tag{129}$$

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Applying theorem 4.3.1 with $f = V - \bar{V}$, with $\bar{V} = \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} V dxdy$, we can find $U \in L^\infty(\mathbb{D}) \cap W_0^{1,2}(\mathbb{D}, \mathbb{R}^2)$ such that:

$$\mathrm{div}\,(\mathrm{U}) = V - \bar{V}, \tag{130}$$

{eqdivU}

and

$$\|U\|_{L^\infty(\mathbb{D})} + \|\nabla U\|_{W^{1,p}(\mathbb{D})} \leq C \left\| V - \bar{V} \right\|_{L^2(\mathbb{D})}. \tag{131}$$

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Then, integrating by parts yields:

$$\begin{aligned} \left| \int_{\mathbb{D}} \left(V - \bar{V} \right)^2 dxdy \right| &\leq \left| \int_{\mathbb{D}} \left(V - \bar{V} \right) \mathrm{div} U dxdy \right| \\ &\leq \left| \int_{\mathbb{D}} \nabla \left(V - \bar{V} \right) U dxdy \right| \leq \left| \int_{\mathbb{D}} \nabla V U dxdy \right| \\ &\leq \left(\|A\|_{L^2(\mathbb{D})} + \|B\|_{L^1(\mathbb{D})} \right) \left(\|U\|_{L^\infty(\mathbb{D})} + \|\nabla U\|_{L^2(\mathbb{D})} \right) \\ &\leq C \left(\|A\|_{L^2(\mathbb{D})} + \|B\|_{L^1(\mathbb{D})} \right) \left\| V - \bar{V} \right\|_{L^2(\mathbb{D})}, \end{aligned} \tag{132}$$

{231120201104}

injecting first (130), then (129) and (131). From (132), one then deduces:

$$\left\| V - \bar{V} \right\|_{L^2(\mathbb{D})} \leq C \left(\|A\|_{L^2(\mathbb{D})} + \|B\|_{L^1(\mathbb{D})} \right). \tag{133}$$

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A rescaling yields (133) on any $\mathbb{D}_r$.

In the general case, we mollify A and B to approximate them by smooth functions on a smaller disk $\mathbb{D}_r$ A_n, B_n. By properties of the mollification (and of the convolution) A_n and B_n satisfy the same divergence equation as A and B, and we can thus write $\nabla V_n = \nabla^\perp A_n + B_n$, with V_n assumed to be of null average over $\mathbb{D}_r$. Then, thanks to (133) V_n satisfy:

$$\|V_n\|_{L^2(\mathbb{D}_r)} \leq C\left(\|A_n\|_{L^2(\mathbb{D}_r)} + \|B_n\|_{L^1(\mathbb{D}_r)}\right) \leq C\left(\|A\|_{L^2(\mathbb{D})} + \|B\|_{L^1(\mathbb{D})}\right). \tag{134}$$

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In addition, one can check that for all n, m $\nabla(V_n - V_m) = \nabla^\perp(A_n - A_m) + (B_n - B_m)$, and thus with (133) one has:

$$\|V_n - V_m\|_{L^2(\mathbb{D}_r)} \leq C\left(\|A_n - A_m\|_{L^2(\mathbb{D}_r)} + \|B_n - B_m\|_{L^1(\mathbb{D}_r)}\right).$$

Since A_n and B_n are Cauchy, V_n is Cauchy in L^2, and thus converges toward a $\tilde{V}$ satisfying $\nabla V = \nabla \tilde{V}$. The two quantities thus only differ by a constant. Going to the limit in (134) proves the result. It also ensures that $V - \text{cst} \in L^2$, and thus that $\bar{V}$ is well defined. Since $\bar{\tilde{V}} = \lim \bar{V}_n = 0$, one has: $V - \bar{V} = \tilde{V}$. We can then precise the nature of the constant in the final result. □

Study of the weak Willmore equation

This section is devoted to the following result, which is only a slight improvement over theorem 7.4 of [?], with a control by $H \nabla \Phi$ replacing one by $\nabla \vec{n}$ (the refined control will not be used here, but one can see in [?] how it can be applied). However, we will follow *mutatis mutandis* the previous proof.

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Theorem 4.3.4. *Let $\Phi \in \mathcal{E}(\mathbb{D}_\rho)$ be a conformal weak Willmore immersion. Let $\vec{n}$ denote its Gauss map, H its mean curvature and λ its conformal factor. We assume*

$$\|\nabla \lambda\|_{L^{2,\infty}(\mathbb{D}_\rho)} \leq M < +\infty,$$

and

$$\|\nabla \vec{n}\|_{L^2(\mathbb{D}_\rho)} \leq \sqrt{\frac{8\pi}{3}}.$$

Then for any $r < 1$ there exists a constant $\vec{\mathcal{L}} \in \mathbb{R}^3$ and a constant $C \in \mathbb{R}$ depending on r and M such that

$$\left\| e^\lambda \left(\vec{L} - \vec{\mathcal{L}}\right)\right\|_{L^2(\mathbb{D}_{r\rho})} \leq C \|H \nabla \Phi\|_{L^2(\mathbb{D}_\rho)}.$$

where $\vec{L}$ is given by (111).

Proof. We will prove the theorem on $\mathbb{D}$. The proof on $\mathbb{D}_\rho$ follows by taking $\Phi_\rho = \Phi(\rho.)$. Let $\Phi \in \mathcal{E}(\mathbb{D})$ be a conformal weak Willmore immersion, $\vec{n}$ its Gauss map,

H its mean curvature and λ its conformal factor. satisfying the hypotheses. Let $r < 1$ and $\vec{L} \in \mathcal{D}'(\mathbb{D})$ defined by (111).

Step 1: Control of the conformal factor

Applying theorem 4.2.3 we find $\Lambda \in \mathbb{R}$ and C depending on r and M such that

$$\|\lambda - \Lambda\|_{L^\infty\left(\mathbb{D}_{\frac{r+1}{2}}\right)} \leq C.$$

Consequently, λ satisfies (127) on $\mathbb{D}_{\frac{r+1}{2}}$:

$$\forall x \in \mathbb{D}_{\frac{r+1}{2}} \quad \frac{e^\Lambda}{C} \leq e^{\lambda(x)} \leq Ce^\Lambda. \tag{135}$$ {070620211317}

Step 2: Control on $\nabla \vec{L}$

From (135), one obtains:

$$\|H\vec{n}\|_{L^2\left(\mathbb{D}_{\frac{r+1}{2}}\right)} \leq C\|He^\lambda\|_{L^2\left(\mathbb{D}_{\frac{r+1}{2}}\right)} \leq Ce^{-\Lambda}\|H\nabla\Phi\|_{L^2\left(\mathbb{D}_{\frac{r+1}{2}}\right)}.$$

We can exploit this to control the right-hand side of (111). We can compute from it:

$$\begin{aligned}
\nabla^\perp \vec{L} &= -2\left(\nabla H\vec{n} + H\mathring{A}\nabla\Phi\right)\\
&= -2\left(\nabla(H\vec{n}) + H\mathring{A}\nabla\Phi - H\nabla\vec{n}\right)\\
&= -2\nabla(H\vec{n}) - 4H\mathring{A}\nabla\Phi - 2H^2\nabla\Phi.
\end{aligned}$$

We control each term of the right-hand side as follows:

$$\begin{aligned}
\|H^2\nabla\Phi\|_{L^1\left(\mathbb{D}_{\frac{r+1}{2}}\right)} &\leq \|H\nabla\Phi\|_{L^2\left(\mathbb{D}_{\frac{r+1}{2}}\right)} \|H\|_{L^2\left(\mathbb{D}_{\frac{r+1}{2}}\right)}\\
&\leq C\,\|H\nabla\Phi\|_{L^2(\mathbb{D})}\,\|H\nabla\Phi e^{-\lambda}\|_{L^2\left(\mathbb{D}_{\frac{r+1}{2}}\right)}\\
&\leq e^{-\Lambda}C\,\|H\nabla\Phi\|^2_{L^2(\mathbb{D})},
\end{aligned}$$

where we used (127) to take the conformal factor out of the L^2 norm. Applying (43) yields:

$$\|H^2\nabla\Phi\|_{L^1\left(\mathbb{D}_{\frac{r+1}{2}}\right)} \leq e^{-\Lambda}C\,\|H\nabla\Phi\|_{L^2(\mathbb{D})}.$$

Similarly:

$$\|H\mathring{A}\nabla\Phi\|_{L^1\left(\mathbb{D}_{\frac{r+1}{2}}\right)} \leq e^{-\Lambda}C\,\|H\nabla\Phi\|_{L^2(\mathbb{D})}.$$

Finally one obtains:

$$\|H\vec{n}\|_{H^{-1}\left(\mathbb{D}_{\frac{r+1}{2}}\right)} \leq \|H\vec{n}\|_{L^2\left(\mathbb{D}_{\frac{r+1}{2}}\right)} \leq Ce^{-\Lambda}\|H\nabla\Phi\|_{L^2\left(\mathbb{D}_{\frac{r+1}{2}}\right)}.$$

The last three estimates combined gives us the hypothesis of theorem 4.3.3:

$$\nabla \vec{L} \in \nabla^{\perp} L^2\left(\mathbb{D}_{\frac{r+1}{2}}\right) \bigoplus L^1\left(\mathbb{D}_{\frac{r+1}{2}}\right).$$

Step 3: Conclusion
Thanks to Step 2 and theorem 4.3.3

$$\exists \vec{\mathcal{L}} \in \mathbb{R}^3 \quad \left\| \vec{L} - \vec{\mathcal{L}} \right\|_{L^2(\mathbb{D}_r)} \leq C e^{-\Lambda} \| H \nabla \Phi \|_{L^2(\mathbb{D})}$$

with C a real constant that depends on r and M. Hence

$$\begin{aligned} \left\| \left(\vec{L} - \vec{\mathcal{L}} \right) e^{\lambda} \right\|_{L^2(\mathbb{D}_r)} &\leq e^{\Lambda} \left\| \vec{L} - \vec{\mathcal{L}} \right\|_{L^2(\mathbb{D}_r)} \\ &\leq C \| H \nabla \Phi \|_{L^2(\mathbb{D})}, \end{aligned}$$

with C as desired. This concludes the proof on $\mathbb{D}$. □

This control on $\vec{L}$ is exactly what is required: the constant is absolutely not problematic. Indeed, $\vec{L}$ is defined as the integration of a conserved quantity, and thus up to a constant. We can (and will) then chose $\vec{L}$ to ensure the L^2 estimate. This can be applied to control ∇S and $\nabla \vec{R}$.

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Corollary 4.3.1. *Under the hypotheses of theorem 4.3.4, for all $r < 1$, there exists $C(r)$ such that*

$$\| S \|_{W^{1,2}(\mathbb{D}_r)} + \| \vec{R} \|_{W^{1,2}(\mathbb{D}_r)} \leq C \| H \nabla \Phi \|_{L^2(\mathbb{D})}.$$

Proof. Let $r < 1$ and $\vec{L}$ such that

$$\left\| \vec{L} e^{\lambda} \right\|_{L^2(\mathbb{D}_r)} \leq C_1 \left\| H \nabla \Phi \right\|_{L^2(\mathbb{D})}.$$

Then S and $\vec{R}$ defined as

$$\begin{aligned} \nabla^{\perp} S &= \langle \vec{L}, \nabla \Phi \rangle \\ \nabla^{\perp} \vec{R} &= \vec{L} \times \nabla^{\perp} \Phi + 2 H \nabla^{\perp} \Phi, \end{aligned}$$

satisfy:

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$$\begin{aligned} \| \nabla S \|_{L^2(\mathbb{D}_r)} + \| \nabla \vec{R} \|_{L^2(\mathbb{D}_r)} &\leq \left\| \vec{L} e^{\lambda} \right\|_{L^{2,\infty}(\mathbb{D}_r)} + \| H \nabla \Phi \|_{L^2(\mathbb{D}_r)} \\ &\leq (C_1 + 1) \left\| H \nabla \Phi \right\|_{L^2(\mathbb{D})}. \end{aligned} \tag{136}$$

Noticing that S and $\vec{R}$ are defined up to an additive constant, we can choose S and $\vec{R}$ to be of null average value on $\mathbb{D}_{r'}$.

The Poincaré–Wirtinger's inequality yields for any $1 < p < \infty$ and any u such that $\nabla u \in L^p(\mathbb{D}_r)$:
$$\|u - \bar{u}\|_{L^p(\mathbb{D}_r)} \leq C_{p,r} \|\nabla u\|_{L^p(\mathbb{D}_r)}$$
with $C_{p,r} \in \mathbb{R}_+$ and $\bar{u}$ the mean value of u on $\mathbb{D}_{r'}$. These inequalities can be extended using Marcinkiewitz interpolation theorem (see for example theorem 3.3.3 of [?]) to $L^{2,\infty}$: there exists C_r such that for any u with $\nabla u \in L^{2,\infty}(\mathbb{D})$

$$\|u - \bar{u}\|_{L^{2,\infty}(\mathbb{D}_r)} \leq C_{r'} \|\nabla u\|_{L^{2,\infty}(\mathbb{D}_r)}.$$

Applied to S and $\vec{R}$ (which are of null mean value), this yields:

$$\|S\|_{W^{1,2}(\mathbb{D}_r)} + \|\vec{R}\|_{W^{1,2}(\mathbb{D}_r)} \leq C \|H \nabla \Phi\|_{L^2(\mathbb{D})},$$

where C depends on r. $\square$

Thus, the weak Willmore equation offers just enough regularity to recover the regularity needed for Wente's lemma. It must be pointed out that the result of J. Bourgain and H. Brézis is not the original estimate. T. Rivière first found a $L^{2,\infty}$ estimate on $\vec{L}$ (which is highly similar to the one we presented, which can be paralleled with theorem 3.2.7: without the compensation estimate one obtains a $L^{2,\infty}$ estimate, with compensation we can jump to a L^2). We can now apply this to deduce the ε-regularity theorem.

4.3.2 Control on $\nabla \vec{n}$ and Φ

We will prove the following theorem:

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Theorem 4.3.5. *Let $\Phi \in \mathcal{E}(\mathbb{D})$ be a conformal weak Willmore immersion. Let $\vec{n}$ denote its Gauss map, H its mean curvature and $\lambda = \frac{1}{2} \log \left(\frac{|\nabla \Phi|^2}{2} \right)$ its conformal factor. We assume*
$$\|\nabla \lambda\|_{L^{2,\infty}(\mathbb{D})} \leq M.$$
Then there exists $\varepsilon_0 > 0$ such that if
$$\int_{\mathbb{D}} |\nabla \vec{n}|^2 < \varepsilon_0, \tag{137}$$
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then for any $r < 1$ and for any $k \in \mathbb{N}$
$$\|\nabla^k \vec{n}\|^2_{L^\infty(\mathbb{D}_r)} \leq C \int_{\mathbb{D}} |\nabla \vec{n}|^2,$$
$$\|e^{-\lambda} \nabla^k \Phi\|^2_{L^\infty(\mathbb{D}_r)} \leq C \left(\int_{\mathbb{D}} |\nabla \vec{n}|^2 + 1 \right),$$
with C a real constant depending on r, M and k.

Proof. **Step 1 : Set-up**
Applying theorem 4.3.4 and corollary 4.3.1 on $\mathbb{D}_{r'}$ with $r < r'$ we deduce that:

$$\|S\|_{W^{1,2}(\mathbb{D}_{r'})} + \|\vec{R}\|_{W^{1,2}(\mathbb{D}_{r'})} \leq C\|H\nabla\Phi\|_{L^2(\mathbb{D})}. \tag{138}$$

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We will also use (135) which stands on $\mathbb{D}_{r'}$.

Step 2 : Breaking the criticality

We will now control properly the potentials S and $\vec{R}$ in L^q pour $q > 2$ by showing a Morrey-type estimate. Let us recall:

$$\begin{cases} \Delta S := -\langle \nabla \vec{n}, \nabla^{\perp}\vec{R}\rangle \\ \Delta \vec{R} := \nabla \vec{n} \times \nabla^{\perp}\vec{R} + \nabla^{\perp} S \nabla \vec{n} \\ \Delta \vec{\Phi} = \dfrac{1}{2}(\nabla^{\perp} S \nabla \vec{\Phi} + \nabla^{\perp}\vec{R} \times \nabla \vec{\Phi}). \end{cases}$$

Let $p \in \mathbb{D}_r$ and of radius $s < 1 - r'$. On $B_s(p)$ we write: $S = \Psi_S + v_S$, $\vec{R} = \vec{\Psi}_{\vec{R}} + \vec{v}_{\vec{R}}$ where v_S and $v_{\vec{R}}$ are harmonic and Ψ_S, $\vec{\Psi}_{\vec{R}}$ solutions of

$$\begin{cases} \Delta \Psi_S = -\langle \nabla \vec{n}, \nabla^{\perp}\vec{R}\rangle \text{ in } B_s(p) \\ \quad \Psi_S = 0 \text{ on } \partial B_s(p). \end{cases}$$

$$\begin{cases} \Delta \vec{\Psi}_{\vec{R}} = -\nabla \vec{n} \times \nabla^{\perp}\vec{R} + \nabla^{\perp} S \nabla \vec{n} \text{ in } B_s(p) \\ \quad \vec{\Psi}_{\vec{R}} = 0 \text{ on } \partial B_s(p). \end{cases}$$

With Wente's lemma we deduce that:

$$\int_{B_s(p)} |\nabla \Psi_S|^2 + |\nabla \vec{\Psi}_{\vec{R}}|^2 \leq C \left(\int_{\mathbb{D}} |\nabla \vec{n}|^2 \right) \left(\int_{\mathbb{D}} |\nabla S|^2 + |\nabla \vec{R}|^2 \right).$$

Applying lemma 3.2.4 we deduce:

$$\begin{aligned} \frac{4}{s^2} \int_{B_{\frac{s}{2}}(p)} |\nabla v_S|^2 + |\nabla \vec{v}_{\vec{R}}|^2 &\leq \frac{1}{s^2} \int_{B_s(p)} |\nabla v_S|^2 + |\nabla \vec{v}_{\vec{R}}|^2 \\ &\leq \frac{1}{s^2} \int_{B_s(p)} |\nabla S|^2 + |\nabla \vec{R}|^2. \end{aligned}$$

$$\text{Thus} \int_{B_{\frac{s}{2}}(p)} |\nabla v_S|^2 + |\nabla \vec{v}_{\vec{R}}|^2 \leq \frac{1}{4} \int_{B_s(p)} |\nabla S|^2 + |\nabla \vec{R}|^2.$$

This yields:

$$
\begin{aligned}
\int_{B_{\frac{s}{2}}(p)} |\nabla S|^2 + |\nabla \vec{R}|^2 &\leq \int_{B_{\frac{s}{2}}} |\nabla \Psi_S|^2 + |\nabla \vec{\Psi}_{\vec{R}}|^2 + \int_{B_{\frac{s}{2}}(p)} |\nabla v_S|^2 + |\nabla \vec{v}_{\vec{R}}|^2 \\
&\leq \left(C \int_{\mathbb{D}} |\nabla \vec{n}|^2 + \frac{1}{2} \right) \left(\int_{B_s(p)} |\nabla S|^2 + |\nabla \vec{R}|^2 \right) \\
&\leq \frac{3}{4} \left(\int_{B_s(p)} |\nabla S|^2 + |\nabla \vec{R}|^2 \right).
\end{aligned}
$$

Where the last inequality stands if $\int_D |\nabla \vec{n}|^2 \leq \frac{1}{4C}$ which we will assume from now on. Iterating yields:

$$
\begin{aligned}
\int_{B_{\frac{s}{2^i}}(p)} |\nabla S|^2 + |\nabla \vec{R}|^2 &\leq \left(\frac{3}{4} \right)^i \int_{B_s(p)} |\nabla S|^2 + |\nabla \vec{R}|^2 \\
&\leq C_s \left(\frac{s}{2^i} \right)^\alpha,
\end{aligned}
$$

where we set $\alpha := \log_2 \left(\frac{4}{3} \right)$ and $C_s := s^{-\alpha} \left(\int_{\mathbb{D}} |\nabla S|^2 + |\nabla \vec{R}|^2 \right)$. The above stands for all i and all $s \in [\frac{1}{4}, 1 - r')$. This yields the desired Morrey-type estimate:

$$
s^{-\alpha} \int_{B_s(p)} |\nabla S|^2 + |\nabla \vec{R}|^2 \leq C \left(\int_{\mathbb{D}} |\nabla S|^2 + |\nabla \vec{R}|^2 \right).
$$

where we used $s \geq \frac{1}{4}$ to obtain $C_s \leq C \left(\int_{\mathbb{D}} |\nabla S|^2 + |\nabla \vec{R}|^2 \right)$. Applying lemma 3.2.3 ensures that there exists $q > 2$ such that:

$$
\|\nabla S\|_{L^q\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} + \|\nabla \vec{R}\|_{L^q\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} \leq C_{r,r'} \left(\|\nabla S\|_{L^p(\mathbb{D}_{r'})} + \|\nabla \vec{R}\|_{L^q(\mathbb{D}_{r'})} \right) \leq C_{r,r'} \|\nabla \vec{n}\|_{L^2(\mathbb{D})}.
$$

Step 3: Increased regularity on H and Φ

Using the third equation of the Willmore system (119) $\Delta \Phi = \frac{1}{2} \left(\nabla^\perp S . \nabla \Phi + \nabla^\perp \vec{R} \times \nabla \Phi \right)$, we deduce that $\|\Delta \Phi\|_{L^q\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} \leq C_{r,r'} e^\Lambda \|\nabla \vec{n}\|_{L^2(\mathbb{D})}$. Applying Calderon-Zygmund yields:

$$
\|\nabla \Phi\|_{W^{2,q}\left(\mathbb{D}_{\frac{2r+r'}{3}}\right)} \leq C \left(\|\Delta \Phi\|_{L^q\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} + \|\nabla \Phi\|_{L^q\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} \right) \leq C_{r,r'} \left(\|\nabla \vec{n}\|_{L^2(\mathbb{D})} + 1 \right) \|\nabla \Phi\|_{L^2(\mathbb{D})}, \tag{139}
$$

where for the last inequality we used (135).

In addition, we can recall that $\Delta\Phi = 2e^{2\lambda}H\vec{n}$, and thus, once more thanks to (135):

$$\|H\nabla\Phi\|_{L^q\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} \leq \left\|\frac{1}{2}e^{-2\lambda}\Delta\Phi\times\nabla^{\perp}\Phi\right\|_{L^q\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} \leq C_{r,r'}e^{-\Lambda}e^{\Lambda}\|\nabla\vec{n}\|_{L^2(\mathbb{D})} \leq C_{r,r'}\|\nabla\vec{n}\|_{L^2(\mathbb{D})}.$$

Step 4: on a small ball

Let $x_0 \in \mathbb{D}_{\frac{2r+r'}{3}}$ and s small enough such that $B_s(x_0) \subset \mathbb{D}_{\frac{2r+r'}{3}}$. Setting $\Phi_s = \Phi(x_0+s)$, one can apply the previous procedure and deduce that:

$$\|H_s\nabla\Phi_s\|_{L^q\left(\mathbb{D}_{\frac{1}{2}}\right)} \leq C\|\nabla\vec{n}_s\|_{L^2(\mathbb{D})}.$$

Since $H_s(x) = H(x_0+sx)$, and $\nabla\Phi_s = \nabla(\Phi(x_0+sx)) = s\nabla\Phi(x_0+sx)$, one has:

$$\begin{aligned}\|H_s\nabla\Phi_s\|_{L^q\left(\mathbb{D}_{\frac{1}{2}}\right)} &= \left(\int_{\mathbb{D}_{\frac{1}{2}}}(s(H\nabla\Phi)(x_0+sx))^q\right)^{\frac{1}{q}}\\ &= \left(\int_{B_s(x_0)}s^{q-2}(H\nabla\Phi)(x)^q\right)^{\frac{1}{q}} = s^{1-\frac{2}{q}}\|H\nabla\Phi\|_{L^q(B_{\frac{s}{2}}(x_0))}.\end{aligned}$$

Similarly, since $\vec{n}_s = \vec{n}(x_0+sx)$, $\|\nabla\vec{n}_s\|_{L^2(\mathbb{D})} = \|\nabla\vec{n}\|_{L^2(B_s(x_0))}$. The estimate then becomes:

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$$\|H\nabla\Phi\|_{L^q(B_{\frac{s}{2}}(x_0))} \leq Cs^{-\left(1-\frac{2}{q}\right)}\|\nabla\vec{n}\|_{L^2(B_s(x_0))}. \tag{140}$$

Using corollary 4.3.2 there exists $\alpha_s \in W^{1,q}\left(\mathbb{D}_{\frac{1}{2}}\right)$ such that

{lealpha}
$$\Delta\alpha_s = \operatorname{div}\left(H_s\nabla\Phi_s\right) \tag{141}$$

and

{lecontrolealpha}
$$\|\alpha_s\|_{W^{1,q}\left(\mathbb{D}_{\frac{1}{2}}\right)} \leq C_q\|H_s\nabla\Phi_s\|_{L^q\left(\mathbb{D}_{\frac{1}{2}}\right)}. \tag{142}$$

Letting $\alpha = \alpha_s\left(\frac{x-x_0}{s}\right)$, one has:

$$\Delta\alpha(x) = \frac{1}{s^2}\operatorname{div}\left(H_s\nabla\Phi_s\right)\left(\frac{x-x_0}{s}\right) = \operatorname{div}\left(H\nabla\Phi\right)(x)$$

and:

$$
\begin{aligned}
\|\nabla\alpha\|_{L^q\left(B_{\frac{s}{2}}(x_0)\right)} &\leq \left(\int_{\mathbb{D}_{B_{\frac{s}{2}}(x_0}} \frac{1}{s^q}|\nabla\alpha_s|^q\left(\frac{x-x_0}{s}\right)\right)^{\frac{1}{q}} \\
&\leq s^{-\left(1-\frac{2}{q}\right)}\|\nabla\alpha_s\|_{\mathbb{D}_{\frac{1}{2}}} \leq s^{-\left(1-\frac{2}{q}\right)}C_q\|H_s\nabla\Phi_s\|_{L^q\left(\mathbb{D}_{\frac{1}{2}}\right)} \\
&\leq s^{-\left(1-\frac{2}{q}\right)}s^{1-\frac{2}{q}}C_q\|H\nabla\Phi\|_{L^q(B_{\frac{s}{2}}(x_0))} \\
&\leq C_q\|H\nabla\Phi\|_{L^q(B_{\frac{s}{2}}(x_0))}.
\end{aligned}
$$

Injecting (140) into the above yields

$$
\|\nabla\alpha\|_{L^q\left(B_{\frac{s}{2}}(x_0)\right)} \leq C_q\ s^{-\left(1-\frac{2}{q}\right)}\|\nabla\vec{n}\|_{L^2(B_s(x_0))}. \tag{143}
$$
{070620211955}

We can show that α satisfies a Morrey-type inequality given $m \in \mathbb{N}$ we have, thanks to Hölder:

$$
\begin{aligned}
\|\nabla\alpha\|_{L^2\left(B_{\frac{s}{2m}}(x_0)\right)} &\leq \left(\int_{B_{\frac{s}{2m}}(x_0)} |\nabla\alpha|^2\right)^{\frac{1}{2}} \\
&\leq \left(\||\nabla\alpha|^2\|_{L^{\frac{q}{2}}\left(B_{\frac{s}{2m}}(x_0)\right)}\|1\|_{L^{\frac{q}{q-2}}\left(B_{\frac{s}{2m}}(x_0)\right)}\right)^{\frac{1}{2}} \\
&\leq \left(\|\nabla\alpha\|^2_{L^q\left(B_{\frac{s}{2m}}(x_0)\right)}\left(\frac{s}{2m}\right)^{2\frac{q-2}{q}}\right)^{\frac{1}{2}} \\
&\leq \frac{1}{(2m)^{1-\frac{2}{q}}}s^{1-\frac{2}{q}}\|\nabla\alpha\|_{L^q\left(B_{\frac{s}{2}}(x_0)\right)}.
\end{aligned}
$$

Injecting (143) into the above yields

$$
\|\nabla\alpha\|_{L^2\left(B_{\frac{s}{2m}}(x_0)\right)} \leq \frac{C_q}{(2m)^{1-\frac{2}{q}}}\|\nabla\vec{n}\|_{L^2(B_s(x_0))}. \tag{144}
$$
{070620212004}

The constant C_q depends on q, but *not on* s: it is a constant introduced in the Hodge decomposition on $\mathbb{D}_{\frac{1}{2}}$.

Step 5: estimates on $\vec{n}$

To obtain estimates for $\nabla\vec{n}$ we will use equation (65):

$$
\Delta\vec{n} + \nabla\vec{n} \times \nabla^{\perp}\vec{n} + 2\mathrm{div}\left(H\nabla\Phi\right) = 0.
$$

Setting $\nu = \vec{n} - 2\alpha$ and using (144) with $m = 1$ yields

{lecontrole

$$\begin{aligned} \|\nabla \nu\|_{L^2\left(B_{\frac{s}{2}}(x_0)\right)} &\leq \|\nabla(\vec{n} - 2\alpha)\|_{L^2\left(B_{\frac{s}{2}}(x_0)\right)} \\ &\leq \|\nabla \vec{n}\|_{L^2\left(B_{\frac{s}{2}}(x_0)\right)} + 2\|\nabla \alpha\|_{L^2\left(B_{\frac{s}{2}}(x_0)\right)} \\ &\leq \|\nabla \vec{n}\|_{L^2(B_s(x_0))} + C_q \|\nabla \vec{n}\|_{L^2(B_s(x_0))} \\ &\leq C_q \|\nabla \vec{n}\|_{L^2(B_s(x_0))} . \end{aligned} \tag{145}$$

Besides, ν satisfies

$$\Delta \nu + \nabla \vec{n} \times \nabla^{\perp} \vec{n} = 0.$$

We split $\nu = \nu_1 + \nu_2$ with ν_2 harmonic and ν_1 solution of

$$\begin{cases} \Delta \nu_1 + \nabla \vec{n} \times \nabla^{\perp} \vec{n} = 0 \text{ in } B_{\frac{s}{2}}(x_0) \\ \nu_1 = 0 \text{ on } \partial B_{\frac{s}{2}}(x_0). \end{cases}$$

Using Wente's lemma we bound

{onestimenu1}

$$\|\nabla \nu_1\|_{L^2\left(B_{\frac{s}{2}}(x_0)\right)} \leq C \|\nabla \vec{n}\|^2_{L^2\left(B_{\frac{s}{2}}(x_0)\right)} \leq C\varepsilon_0 \|\nabla \vec{n}\|_{L^2\left(B_{\frac{s}{2}}(x_0)\right)}. \tag{146}$$

Applying lemma 3.2.4 we deduce that

{070620212016}

$$\|\nabla \nu_2\|^2_{L^2\left(B_{\frac{s}{2m}}(x_0)\right)} \leq \frac{1}{m^2} \|\nabla \nu_2\|^2_{L^2\left(B_{\frac{s}{2}}(x_0)\right)} \leq \frac{1}{m^2} \|\nabla \nu\|^2_{L^2\left(B_{\frac{s}{2}}(x_0)\right)} \leq \frac{C_q}{m^2} \|\nabla \vec{n}\|^2_{L^2(B_s(x_0))} . \tag{147}$$

Combining (146) and (146) yields:

{070620212018}

$$\begin{aligned} \|\nabla \nu\|_{L^2\left(B_{\frac{s}{2m}}(x_0)\right)} &\leq \|\nabla \nu_1\|_{L^2\left(B_{\frac{s}{2m}}(x_0)\right)} + \|\nabla \nu_2\|_{L^2\left(B_{\frac{s}{2m}}(x_0)\right)} \\ &\leq \left(\frac{C_q}{m} + C\varepsilon_0\right) \|\nabla \vec{n}\|_{L^2(B_s(x_0))} . \end{aligned} \tag{148}$$

Since $\|\nabla \vec{n}\|_{L^2\left(B_{\frac{s}{2m}}(x_0)\right)} \leq \|\nabla \nu\|_{L^2\left(B_{\frac{s}{2m}}(x_0)\right)} + 2\|\nabla \alpha\|_{L^2\left(B_{\frac{s}{2m}}(x_0)\right)}$, one can combine (144) and (148) to obtain:

{070620212029}

$$\|\nabla \vec{n}\|_{L^2\left(B_{\frac{s}{2m}}(x_0)\right)} \leq \left(\frac{C_q}{m} + \frac{C_q}{(2m)^{1-\frac{2}{q}}} + C\varepsilon_0\right) \|\nabla \vec{n}\|_{L^2(B_s(x_0))} . \tag{149}$$

Taking m big enough to have $\frac{C_q}{m} + \frac{C_q}{(2m)^{1-\frac{2}{q}}} \leq \frac{1}{4}$ and ε_0 such that $C\varepsilon_0 \leq \frac{1}{2}$, one obtains:

$$\|\nabla \vec{n}\|_{L^2\left(B_{\frac{s}{2m}}(x_0)\right)} \leq \frac{3}{4} \|\nabla \vec{n}\|_{L^2(B_s(x_0))} .$$

This stands for all x_0 and s, thus since the chosen m and ε_0 do not depend on s (thanks to the invariance scales of the respective quantities) we can iterate and deduce that

$$\int_{B_{\frac{s}{(2m)^i}}(p)} |\nabla \vec{n}|^2 \leq \left(\frac{3}{4}\right)^{2i} \int_{B_s(p)} |\nabla \vec{n}|^2$$
$$\leq C_s \left(\frac{s}{(2m)^i}\right)^{\alpha},$$

where we set $\alpha = \log_{2m}\left(\frac{4}{3}\right) > 0$. This yields the desired Morrey-type estimate:

$$s^{-\alpha} \int_{B_s(x_0)} |\nabla \vec{n}|^2 \leq C \int_{\mathbb{D}_{\frac{2r+r'}{3}}} |\nabla \vec{n}|^2.$$

Applying lemma 3.2.3 ensures that there exists $q' > 2$ such that:

$$\|\nabla \vec{n}\|_{L^{q'}\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} \leq C \|\nabla \vec{n}\|_{L^2(\mathbb{D})}.$$

Taking $p = \min(q, q') > 2$, one has that $\nabla \vec{n}$, ∇S, $\nabla \vec{R}$ and $\nabla \Phi$ are in L^p, with the desired estimates.

Step 6: bootstrap 1 Reinjecting the accrued regularity ensures that ΔS, $\Delta \vec{R}$, $\Delta \Phi$, $\Delta \vec{n} \in L^{\frac{p}{2}}$. If $p \geq 4$, then Calderón-Zygmund and Sobolev estimates ensure that $\nabla \vec{n}$, ∇S, $\nabla \vec{R}$, $\nabla \Phi \in L^\infty$ with the right estimates. If $p < 4$ Calderón-Zygmund and Sobolev estimates yield $\nabla \vec{n}$, ∇S, $\nabla \vec{R}$, $\nabla \Phi \in L^{\frac{2p}{4-p}}$. In a finite number of iterations in the sequence $p_{n+1} = \frac{2p_n}{4-p_n}$, we thus reach $p_{n_0} > 4$, which yields: $\nabla \vec{n}$, ∇S, $\nabla \vec{R}$, $\nabla \Phi \in L^\infty$ with the right estimates.

Step 7: bootstrap 2 If $\nabla \vec{n}$, ∇S, $\nabla \vec{R}$ and $\nabla \Phi$ are in $W^{k,p}$ for all $p < \infty$, with the desired estimates, then the equation in S and $\vec{R}$ ensures that $\Delta S, \Delta \vec{R} \in W^{k,p}$ for all $p < \infty$, and thus, by Calderón-Zygmund that $\nabla S, \nabla \vec{R} \in W^{k+1,p}$. More precisely we proceed in the following manner:

- The equation in S and $\vec{R}$ yields the result on ∇S and $\nabla \vec{R}$,
- The equation in Φ yields both the control on $\nabla \Phi$ and on $H \nabla \Phi$
- The equation in $\vec{n}$ ensures that $\Delta \vec{n} \in W^{k,p} \forall p < \infty$, which yields the result on $\vec{n}$.

The reader can trace the estimates to make sure they have the desired shape. $\square$