Q20. (7 marks) A sequence of complex numbers is defined by

$$z_0 = 3\sqrt{3} + 3i,$$

$$z_1 = \frac{\sqrt{3}}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,$$

and

$$z_n = 2(cos(\frac{\pi}{4}) + sin(\frac{\pi}{4})i)\frac{z_{n-1}}{|z_{n-2}|} \quad \text{for n} \in \mathbb{N} \text{ with n} \geq 2$$

(a) (2 marks) Write down the complex numbers of z_0 and z_1 in polar form. No justification is required.

For z_0 we have that $|z_0| = \sqrt{27+9} = \sqrt{36} = +/-6$, so

$$z_0 = 6(\frac{\sqrt{3}}{2} + \frac{1}{2}i) = 6(cos(\frac{\pi}{6}) + isin\frac{\pi}{6})$$

For z_1 we have that $|z_1| = \sqrt{1.5+.5} = \sqrt{2}$, so

$$z_1 = \sqrt{2}(\frac{\sqrt{3}}{2} - \frac{1}{2}i) = \sqrt{2}(cos(\frac{-\pi}{6}) + isin\frac{-\pi}{6})$$

(b) (2 marks) Determine with justification the smallest positive integer p such that, for every positive integer n, $|z_{n+p}| = |z_n|$.

if we apply PM to $|z_n|$ would obtain:

$$|z_n| = |2(cos(\frac{\pi}{4}) + sin(\frac{\pi}{4})i)|\frac{|z_{n-1}|}{|z_{n-2}|}$$

Now if we apply the modulus definition to $|2(cos(\frac{\pi}{4}) + sin(\frac{\pi}{4})i)|$ we will get

$$\sqrt{(\frac{2}{\sqrt{2}})^2 + (\frac{2}{\sqrt{2}})^2} = \sqrt{\frac{4}{2} + \frac{4}{2}} = 2$$

So now $|z_n|$ can be written as

$$|z_n| = 2\frac{|z_{n-1}|}{|z_{n-2}|}$$

Now, writing the first couple term of $|z_n|$ for $n \geq 2$ using he above relation we will obtain:

$$|z_2| = 2\frac{|z_1|}{|z_0|} = 2\frac{\sqrt{2}}{6} = \frac{\sqrt{2}}{3}$$

$$|z_3| = 2\frac{|z_2|}{|z_1|} = 2\frac{\frac{\sqrt{2}}{3}}{\sqrt{2}} = \frac{2}{3}$$

$$|z_4| = 2\frac{|z_3|}{|z_2|} = 2\frac{\frac{2}{3}}{\frac{\sqrt{2}}{3}} = \frac{4}{\sqrt{2}}$$

$$|z_5| = 2\frac{|z_4|}{|z_3|} = 2\frac{\frac{4}{\sqrt{2}}}{\frac{2}{3}} = \frac{12}{\sqrt{2}}$$

$$|z_6| = 2\frac{|z_5|}{|z_4|} = 2\frac{\frac{12}{\sqrt{2}}}{\frac{4}{\sqrt{2}}} = 6$$

$$|z_7| = 2\frac{|z_6|}{|z_5|} = 2\frac{6}{\frac{12}{\sqrt{2}}} = \sqrt{2}$$

$$|z_8| = 2\frac{|z_7|}{|z_6|} = 2\frac{\sqrt{2}}{6} = \frac{\sqrt{2}}{3}$$

$$|z_9| = 2\frac{|z_8|}{|z_7|} = 2\frac{\frac{\sqrt{2}}{3}}{\sqrt{2}} = \frac{2}{3}$$

$$|z_{10}| = 2\frac{|z_9|}{|z_8|} = 2\frac{\frac{2}{3}}{\frac{\sqrt{2}}{3}} = \frac{4}{\sqrt{2}}$$

$$|z_{11}| = 2\frac{|z_{10}|}{|z_9|} = 2\frac{\frac{4}{\sqrt{2}}}{\frac{2}{3}} = \frac{12}{\sqrt{2}}$$

$$|z_{12}| = 2\frac{|z_{11}|}{|z_{10}|} = 2\frac{\frac{12}{\sqrt{2}}}{\frac{4}{\sqrt{2}}} = 6$$

$$|z_{13}| = 2\frac{|z_{12}|}{|z_{11}|} = 2\frac{6}{\frac{12}{\sqrt{2}}} = \sqrt{2}$$

$$|z_{14}| = 2\frac{|z_{13}|}{|z_{12}|} = 2\frac{\sqrt{2}}{6} = \frac{\sqrt{2}}{3}$$

And with these terms you can see that $|z_2| = |z_8| = |z_{14}|$ and thus the sequence repeats after a period of 6. Thus, the smallest integer n, such that $z_{n+p} = |z_n|$ is when $p = 6$.

(c) (3 marks) Determine with justification the smallest positive integer q such that, for every positive integer n, $z_{n+q} = z_n$.

Since we know that z_n can be represented in a polar form as $|z_n| * (cos\theta_n + isin\theta_n)$. Now, in order to get $z_{n+q} = z_n$, we must first have that both $|z_n + q| = |z_n|$ and that $\theta_{n+q} = \theta_n$.

Thus now from part b) where we found that $|z_{n+q}| = |z_n|$ were $q = 6$. Again from part b), we have found out that

$$z_n = 2(cos(\frac{\pi}{4} + \theta_{n-1}) + sin(\frac{\pi}{4} + \theta_{n-1})i)\frac{z_{n-1}}{|z_{n-2}|}$$

Now when we observe this expression, we see that θ_n is given by $\frac{\pi}{4} + \theta_{n-1}$ for $n \geq 1$. Now, when we write the first terms of θ_n, using the above relationship, we obtain:

$$\begin{aligned}
\theta_1 &= \frac{-\pi}{6} = \frac{11\pi}{6} \\
\theta_2 &= \frac{\pi}{4} + \frac{11\pi}{6} = \frac{25\pi}{12} \\
\theta_3 &= \frac{\pi}{4} + \frac{25\pi}{12} = \frac{7\pi}{3} \\
\theta_4 &= \frac{\pi}{4} + \frac{7\pi}{3} = \frac{31\pi}{12} \\
\theta_5 &= \frac{\pi}{4} + \frac{31\pi}{12} = \frac{17\pi}{6} \\
\theta_6 &= \frac{\pi}{4} + \frac{17\pi}{6} = \frac{37\pi}{12} \\
\theta_7 &= \frac{\pi}{4} + \frac{37\pi}{12} = \frac{10\pi}{3} \\
\theta_8 &= \frac{\pi}{4} + \frac{10\pi}{3} = \frac{43\pi}{12} \\
\theta_9 &= \frac{\pi}{4} + \frac{43\pi}{12} = \frac{23\pi}{6} \\
\theta_{10} &= \frac{\pi}{4} + \frac{23\pi}{6} = \frac{49\pi}{12} = \frac{49\pi}{12} - 2\pi = \frac{25\pi}{12} = \theta_2
\end{aligned}$$

Now as we can see, $\theta_2 = \theta_{10}$. This then means that it will keep repeating because θ_n is given by $\frac{\pi}{4} + \theta_{n-1}$ as-well as depending on the value of θ_{n-1}. Since we return to the θ_2, we would than see that this result will happen infinitely many times. So, then we know this this will repeat for every 8 results.

Now, remember from b) that we got $|z_{n+q}| = |z^n|$ for every 6 results.

So to put everything together, the point at which both $|z_{n+q} = z_n|$ and $\theta_{n+q} = \theta_n$ is given by taking the lowest common multiple or (LCM) of both 6 and 8m which would then be 24. Thus, the smallest positive integer q such that for every positive integer n, $z_{n+q} = z_n$ is 24.