FACULTY OF SCIENCE

Radiation-Hydrodynamics with MPI-AMVRAC: Massive-Star Atmospheres and Winds

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Preface

Thank you everybody

Summary

Something something modeling stellar winds

Summary in layman's terms

Somethin something shiny point in sky

List of abbreviations and symbols

Abbreviations

HD	Hydrodynamics
RHD	Radiohydrodynamics
LTE	local thermal equilibrium
CAK	Castor, Abott and Klein
FLD	Flux limited diffusion
MPI	message passing interface
AMR	adaptive mesh refinement
VAC	versatile advection code
PDE	partial differential equations
BH	black hole
NS	neutron star
WD	white dwarf

General

c	speed of light
$M_\odot$	solar mass
$R_\odot$	solar radius
$L_\odot$	solar luminosity

Hydrodynamics

ρ	density
$\vec{v}$	velocity
e	gas energy density
p	gas pressure
γ	adiabatic index
a_{adiab}	adiabatic soundspeed
T	gas temperature
∇	gradient
$\vec{\nabla}$	divergent
$S_{\rho,\vec{v},e}$	sourceterms for HD-equations
g_{grav}	gravitational acceleration
g_{rad}	radiative acceleration

Radiation hydrodynamics

Γ	Eddington factor
ν	frequency
Ω	solid angle
κ_ν	opacity at frequency ν
κ	Rosseland mean opacity
χ_ν	absorption coefficient at frequency ν
η_ν	emission coefficient at frequency ν
$\vec{n}$	unit vector along line of sight
S_ν	source function
B_ν	plack function
I_ν, I	frequency dependent and integrated intensity
J_ν, J	mean intensity
E_ν, E	radiation energy density
$\vec{F}_\nu$,$\vec{F}$	radiation flux
H_ν, H	1^{st} moment of intensity
P_ν, P	radiation pressure
K_ν, K	2^{nd} moment of intensity
H_{eff}	scale height
g_eff	effective acceleration
τ	Optical dept

CAK and FLD

l_{sob}	sobolev length
g_{line}	line acceleration
κ_e	electron opacity
q	
t	
λ	flux limiter
D	diffusion coefficient

Numerics

Δx	difference in quantity x
N, M	dimension of grid
$\tilde{x}$	quantity x in dimensionless units
x_0	the normalised value of quantity x
$x_{i,j}^n$	quantity x at timestep n in cell i, j

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Chapter 1

Introduction

High mass stars are the drivers of dynamical and chemical evolutions of galaxies throughout the universe, including our own Milky Way. Massive stars live a short but exciting lives, finally ending in giant supernova explosions. After their death, they leave behind exotic remnants such as black holes (BH) or neutron stars (NS). Whether a massive star ends as either NS or BH depends a lot on the amounts of mass expelled during their lifetime due to stellar winds. Research in stellar winds and atmospheres of massive stars will give clues about the unexpected BH mass distribution observed by the LIGO collaboration [?], to which data is being added as we speak. This thesis focusses on making radiation-hydrodynamical models of the highly structured stellar winds and atmospheres of massive stars. Next to modelling stellar winds and atmospheres, a main result of this thesis are computer codes written to be used with mpi-AMRVAC which will be used in research on radiation dominated processes.

In various systems it is important to take into account a radiation field when considering gas dynamics ([?], ...). Radiation can exert forces to, for example, accelerate stellar winds and transport energy like in the non-convective stellar radiation zones. The equations describing gas dynamics with a radiation will evolve on two different timescales, a radiation field generally evolves much faster than gas, so solving them isn't trivial. This thesis describes radiation hydrodynamics (RHD) in two different regimes: CAK-theory in hot stellar winds and flux limited diffusion in stellar atmospheres.

The importance of good models for radiation hydrodynamics is clear in a multitude of astrophysical systems, a few are listed below. An important concept in these sort of systems is the Eddington factor Γ, this is the ratio between the radiative and gravitational acceleration.

$$\Gamma = \frac{\kappa F}{c g_{grav}} \tag{1.1}$$

Where F is the radiation energy flux, κ the absorption opacity, c the speed of light in vacuum and g_{grav} the magnitude of the local gravitational acceleration. If the Eddington factor describing a system gets bigger, the importance for taking into account radiation grows as wel.

- **stellar winds**
  Stellar winds can be subdivided in three types: pressure driven for solar type stars,

dust driven for cool massive stars and radiation driven for hot massive stars. The driving force behind these radiation driven winds is, as the name sugests, radiation force. Hot massive stars can undergo massloss of $\sim 10^{-6} M_{\odot}/yr$ due to radiation driven wind, this of course will have an impact on their evolution.
Stars with a luminosity of $10^6 L_{\odot}$ and a stellar mass of $80 M_{\odot}$ will have an Eddington factor $\Gamma \sim 50$ near the surface, radiation plays an essential role in their dynamics. CAK-theory Is an analytical model describing these winds, and the first half of this thesis was spent on modeling this.

- **stellar atmospheres**

- **accretion disks**
  Accretion disks appear in all sorts of objects: young stellar objects, pre-main sequence objects, various types of stars, around BH's, NS's, active galactic nuclei and so on. The gas in these disks is often heated due to internal collisions or accretion on the central object, which gives rise to a radiation field. This radiation field can highly impact the shape and energy distribution within the disk due to ablation and radiative heating [**?**]. In turn, depending on the type of disk, this can have an impact on the end products of star and planetary formation

- **star formation**

- **very massive stars**

- **Eruptive variables**
  Systems that exceed the Eddington limit ($\Gamma > 1$) are called super-Eddington systems. A prime example for such a system is η Carinae. η Carinae is what is known as an eruptive variable. It is a star which underwent an enormous mass loss in the middle of the 19th century, losing $10 M_{\odot}$ over the course of 10 years. One of the hypothesised drivers of this huge mass loss is radiation.

1.1 Hydrodynamics

The movement of non-isothermal gasses and fluids, in the absence of magnetic fields and radiation, are described by hydrodynamics. The hydrodynamical equations describe conservation of mass, conservation of momentum and conservation of energy in the following equations:

$$\partial_t (\rho) + \vec{\nabla} \cdot (\rho \vec{v}) = S_\rho \tag{1.2}$$

$$\partial_t (\rho \vec{v}) + \vec{\nabla} (\vec{v} \rho \vec{v} + p) = S_{\rho \vec{v}} \tag{1.3}$$

$$\partial_t (e) + \vec{\nabla} \cdot (\vec{v} e + \vec{v} p) = S_e \tag{1.4}$$

These three partial differential equations are called the continuity equation (describing mass density), the momentum equation (describing momentum density) and the energy equation (describing energy density). Above, the equations were written in their conservative form, they al have the same shape:

$$\partial_t \underbrace{u}_{\text{density}} + \vec{\nabla} \cdot \overbrace{\vec{f_u}}^{\text{density flux}} = \underbrace{S_u}_{\text{sourceterm}} \tag{1.5}$$

There are only 3 PDE's describing 4 primitive variables: ρ, $\vec{v}$, e and p. This means that there is need for an additional closure relation:

$$p = (\gamma - 1)\left(e - \frac{\rho\vec{v}^2}{2}\right) \tag{1.6}$$

The source terms in the equations describe adding or subtracting to one of the densities: S_ρ describes mass being added or subtracted from the system, $S_{\rho\vec{v}}$ describes external forces such as gravity or radiative forces and S_e describes work exerted on the system, this can be for example due to heating of the fluid. A good description of radiation hydrodynamics needs to formulate the correct source terms for the momentum equation (1.3) and gas energy equation (1.4). These source terms can depend on the other primitive variables ρ, $\vec{v}$ and e as well as free parameters describing the source of the radiation field such as the luminosity and mass of a central star.

Predicting fluid dynamics means solving these aforementioned equations simultaneously, this is done using numerical computer codes such as mpi-amrvac [**?**]. Solving these equations is a tricky business, and there are numerous codes and numerical schemes available. The Evolution of a simulation will depend not only on both initial and boundary conditions, but also on which physics taken into consideration (sourceterms) and even a little on which numerical methods are used, more on this in chapter 2.1.

In section **??**, an analytic expression for these source terms are calculated to simulate the conditions in line driven stellar winds. This calculation was first done by Castor, Abbott and Klein [**?**] CITE HERE and carries their initials in its name: CAK-theory.

1.2 Radiation Hydrodynamics

The last section gave a small recap of hydrodynamics. In this section, a mathematical framework is set up to combine the hydrodynamics equations with the effects of a time dependent radiation field. At the end of this we will be left with a system of partial differential equations describing the dynamics of both the gas and radiation field.
The main governing equation describing the evolution of radiation trough a medium is the radiative transfer equation:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{n} \cdot \vec{\nabla}\right) I_\nu = \eta_\nu + \kappa_\nu I_\nu \tag{1.7}$$

Where I_ν is the intensity at frequency ν along the direction of unit vector $\vec{n}$. η_ν And χ_ν are the emissivity and total opacity. This equation describes how the value of the intensity I_ν Changes when propagating trough a medium with given emissivity and opacity. This form is known as the 0^{th} order of the radiative transfer equation. The 1^{st} order radiative transfer equations are obtained by integrating over all solid angles and dividing by 4π, the 2^{nd} order one by first multiplying with $\vec{n}$ before doing the integration.

$$\frac{1}{c}\frac{\partial J_\nu}{\partial t} + \vec{\nabla}\cdot\vec{H}_\nu = \frac{1}{4\pi}\int_\Omega \eta_\nu + \kappa_\nu I_\nu d\Omega \tag{1.8}$$

$$\frac{1}{c}\frac{\partial \vec{H}_\nu}{\partial t} + \vec{\nabla}\cdot K_\nu = \frac{1}{4\pi}\int_\Omega \left(\eta_\nu + \kappa_\nu I_\nu\right)\vec{n} d\Omega \tag{1.9}$$

Where J_ν, $\vec{H}_\nu$ and K_ν are the higher order moments of the intensity. J_ν is the mean intensity, which is I_ν averaged over all solid angles and thus a scalar. $\vec{H}_\nu$ Is a vector and K_ν is a tensor of order 2. The following relations exist between the moments of intensity and some more physical quantities:

$$E_\nu = \frac{4\pi}{c} j_\nu \tag{1.10}$$

$$\vec{F}_\nu = \frac{4\pi}{c}\vec{H}_\nu \tag{1.11}$$

$$P_\nu = \frac{4\pi}{c} K_\nu \tag{1.12}$$

These are the radiative energy E_ν, the radiation flux $\vec{F}_\nu$ and the radiation pressure tensor P_ν. When assuming local thermal equilibrium (LTE), just like with the gas pressure tensor, P_ν can be written as a scalar times the unit tensor. The fraction of emissivity and total opacity is often written as the source function $S_\nu = \frac{\eta_\nu}{\chi_\nu}$.
When doing full radiative transfer, these equations are solved for an enormous amount of frequencies, corresponding to the indices $_\nu$. The situation is simplified when assuming a grey radiation field, by integrating over all frequencies. The LTE, grey radiation energy and radiation flux equations can be written in a conservative form, similar to the hydrodynamics equations:

$$\frac{\partial E}{\partial t} + \vec{\nabla}\cdot\vec{F} = c\int_\nu\int_\Omega \chi_\nu\left(S_\nu + I_\nu\right) d\nu d\Omega \tag{1.13}$$

$$\frac{\partial \vec{F}}{\partial t} + c^2\vec{\nabla}P = \frac{c}{4\pi}\int_\nu\int_\Omega \chi_\nu\left(S_\nu + I_\nu\right)\vec{n} d\nu d\Omega \tag{1.14}$$

Assuming LTE, the source function scales with the Planck function $S_\nu = \frac{4\pi}{c}B_\nu(T)$ and both $B_\nu(T)$ and χ_ν are independent of solid angle. The relations between flux, energy and the moments of intensity can be used again to replace I_ν in the source terms. Because of symmetry, $S_\nu\vec{n}$ integrated over the total solid angle will return $\vec{0}$ in the source term of the flux equation.

The absorption coefficient is the product of the gas density and the frequency dependent opacity: $\chi_\nu = \rho\kappa_\nu$. To make life easier, an Energy-meaned and Plack opacity can be defined, and in first approximation they are equal to the Rosseland mean opacity: (SOURCE?!?!?!?)

$$\frac{\int_\nu E_\nu \kappa_\nu d\nu}{\int_\nu E_\nu d\nu} = \frac{\int_\nu B_\nu \kappa_\nu d\nu}{\int_\nu B_\nu d\nu} = \kappa \tag{1.15}$$

Integration over all frequencies, replacing opacities by the Rosseland mean opacity leads to:

$$\frac{\partial E}{\partial t} + \vec{\nabla} \cdot \vec{F} = 4\pi\kappa\rho B(T) - c\kappa\rho E \tag{1.16}$$

$$\frac{\partial \vec{F}}{\partial t} + c^2 \vec{\nabla} P = c\kappa\rho\vec{F} \tag{1.17}$$

These equations are written here in the co-moving frame. Transforming to a static frame, the same frame as used in aforementioned hydro equations, an advection term is added to the density flux in both equations, $\vec{v}E$ and $\vec{v} \cdot \vec{F}$. The two source terms in the radiation energy equation are interpreted as energy exchange between the gas and the radiation field, energy leaving the radiation field heats the gas and gas is cooled by energy entering the radiation field. These so called heating and cooling source terms, $4\pi\kappa\rho B(T)$ and $c\kappa\rho E$, must be added to the gas energy equation as well.
The source term in the Flux energy equation has the same units as the gas momentum source term, this is the expression for radiation force. If momentum leaves the radiation flux there is work being done, this must also mean that there is energy leaving the radiation field. This is translated in the photon tiring term $\vec{\nabla} \cdot \vec{v}P$, which must be subtracted from the radiation energy source term.

$$\partial_t (\rho) + \vec{\nabla} \cdot (\rho\vec{v}) = 0 \tag{1.18}$$

$$\partial_t (\rho\vec{v}) + \vec{\nabla} (\vec{v}\rho\vec{v} + p) = \frac{\kappa\rho}{c}\vec{F} \tag{1.19}$$

$$\partial_t (e) + \vec{\nabla} \cdot (\vec{v}e + \vec{v}p) = -4\pi\kappa\rho B + c\kappa\rho E \tag{1.20}$$

$$\partial_t (E) + \vec{\nabla} \cdot \left(\vec{v}E + \vec{F}\right) = -\vec{\nabla} \cdot \vec{v}P + 4\pi\kappa\rho B - c\kappa\rho E \tag{1.21}$$

$$\partial_t \left(\frac{\vec{F}}{c^2}\right) + \vec{\nabla} \left(\frac{\vec{v} \cdot \vec{F}}{c^2} + P\right) = -\frac{\kappa\rho}{c}\vec{F} \tag{1.22}$$

These are the final radiation hydrodynamics equations. There are only 5 equations for 7 primitive variables: ρ, $\vec{v}$, e, p, E, $\vec{F}$ and P. Two closing relations are necessary to close the system. A first closing relation is obtained by re-using equation (1.6), a second one can be obtained by for example the *Flux limited diffusion* approximation (FLD) described in section 1.4.

1.3 Sobolev and CAK-theory

CAK-theory is a formalism developed by Castor Abbot and Klein [**?**] describing the radiative acceleration of gas g_{rad}. It is applied in the radiation driven winds of for example hot massive stars and active galactic nuclei. Let's begin by writing down the acceleration caused by free electron scattering. Consider a star with mass M_* and luminosity L_*, if only electron scattering is assumed, light gets absorbed and re-emitted equally. If κ_e is defined as the electron scattering opacity, the radiative acceleration at any radial distance r is given by:

$$g_e = \frac{\kappa_e L_*}{4\pi r^2 c} \tag{1.23}$$

The gravitational attraction of the gas is given by Newtons gravitation law and is equal to $g_{grav} = G\frac{M_*}{r^2}$. Both accelerations vary as $\frac{1}{r^2}$, so their ratio is constant as a function of radius. An Eddington parameter for a purely electron scattering radiation force is defined as Γ_e.

$$\Gamma_e = \frac{\kappa_e L_*}{4\pi G M_* c} \tag{1.24}$$

Let's now have a look at radiative acceleration of a single absorption line in a line driven wind. Consider a a set of ions absorbing at λ_0, these ions have very high velocities and the doppler shifting of the line becomes important for absorbing photons. The first ions closest to the star will absorb photons at λ_0 so the photons further from the star are in the lines' shadow. However, the photons absorbing these photons pick up a velocity δv so the can now pick up photons with a slightly higher wavelength. This mechanism leads to a steady state monotonously increasing velocity as function of distance from the stellar surface $v(r)$. The line profile isn't infinitesimally thin, photons can be absorbed for a frequency range surrounding the central wavelength λ_0, this is described by the profile function. In velocity-space this width is equal to the thermal velocity of the ions. In radius space, an exact wavelength can be absorbed in a part of space with geometrical width $l_s obdv/dr$, the Sobolove length. With this Sobolev length comes a Sobolev optical depth $\tau_{sob} = \rho\kappa l_{sob}$. With $q\kappa_e c$ describing the lines' effectiveness scaled to electron scattering opacity, and tdv/dr describing the optical depth for a purely electron scattering opacity, τ_{sob} can be written as:

$$\tau_{sob} = \frac{\rho\kappa v_{th}}{dv/dr} = qt \tag{1.25}$$

In the optically thin limit, where the amount of absorption is considered to be insignificant, the line acceleration g_{line} can be related to g_e via q. Note however that the luminosity of the star has to be weighted with the frequency at which absorption occurs.

$$g_{thin} = w_{\nu,0} q g_e = \frac{\kappa v_{th} \nu_0 L_*}{4\pi r^2 c^2} \tag{1.26}$$

Where $w_{\nu,0} = \nu_0 L_\nu / L_*$ is the weight for the frequency at which the line is absorbed. Using solutions for the radiative transfer equation, the total line acceleration can be formulated analytically.

$$g_{line} = g_{thin} \frac{1 - e^{-qt}}{qt} \tag{1.27}$$

A radiation driven wind is accelerated by a whole load of lines, so one has to sum over all line-accelerations to come to the total radiative acceleration. The density distribution of lines N as function of their strength q was approximated by Castor Abbott and Klein to be dependent on a few free parameters and the Γ fucntion.

$$q\frac{dN}{dq} = \frac{1}{\Gamma}\left(\frac{q}{\bar{Q}}\right)^{\alpha-1} \tag{1.28}$$

With this continuous function, the summation over all lines is transformed to an integration over all line strengths, and the final CAK acceleration can be written down:

$$g_{CAK} = g_e \int_0^\infty q\frac{dN}{dq}\frac{1-e^{-qt}}{qt}dq \tag{1.29}$$

$$= \frac{1}{1-\alpha}\frac{\kappa_e L_* \bar{Q}}{4\pi r^2 c}\left(\frac{dv/dr}{\rho c \bar{Q}\kappa_e}\right)^{\alpha} \tag{1.30}$$

CAK-theory is a nice, easy to implement theory. In this thesis, the main application will be simulating the wind of Massive stars. Research performed here is not exactly new, bit it's a great way to introduce radiation forces in amrvac. For me personally it was the first step in learning fortran and getting to know the workings of an advanced (M)HD-code.

1.4 Flux Limited Diffusion

As mentioned before, the full RHD equations leave us with 5 partial differential equations and one HD closure relation for 7 variables. Several methods and approximations exist for closing the system and one of them is flux limited diffusion. (GIVE EXAMPLES, CITE)

Consider the steady state solution of the radiation flux equation (1.22). In first order $\frac{v}{c^2} << 1$, so the relation between P and F is given by.

$$\vec{\nabla}P = -\frac{\kappa\rho}{c}\vec{F} \tag{1.31}$$

In the optically thick limit the eddington approximation gives us $P = \frac{1}{3}E$, so (1.31) can be written as $\vec{F} = -\frac{1}{3}\frac{c}{\kappa\rho}\vec{\nabla}E$. However, in the optically thin, free streaming limit, the density goes to zero and thus the radiation flux should go to infinity, this is unphysical. A solution is introducing the flux limiter λ, which is a factor varrying between $\frac{1}{3}$ in the optically thick regime, and 0 in the optically thin. The extra nessecary closure relation can now be written as:

$$\vec{F} = -\frac{c\lambda}{\kappa\rho}\vec{\nabla}E \tag{1.32}$$

Different formalisms for expressing λ exist, the one used in this work has been worked out in [?].

$$R = \frac{|\nabla E|}{\rho\kappa E} \tag{1.33}$$

$$\lambda = \frac{2+R}{6+3R+R^2} \tag{1.34}$$

R portraits how thick the medium is locally, it is ∞ in the free streaming limit and approaches 0 when the medium gets optically thick. λ And R also relate the radiation pressure P to the radiation energy density E via the Eddington tensor f, which is approximated as a scalar in this situation.

$$P = fE \tag{1.35}$$

$$f = \lambda + \lambda^2 R^2 \tag{1.36}$$

This means we have 7 unknowns, 5 PDE's and 3 closure relations. The momentum flux equation can be dropped and the system is self consistent within equations (1.18), (1.19), (1.20), (1.21), (1.6), (1.32) and (1.35).
The radiation flux is eliminated from the radiation gas equation 1.21:

$$\partial_t (E) + \vec{\nabla} \cdot \left(\vec{v}E - \frac{c\lambda}{\kappa\rho} \nabla E \right) = -\vec{\nabla} \cdot \vec{v}P + 4\pi\kappa\rho B - c\kappa\rho E \tag{1.37}$$

Flux limited diffusion is a useful tool which can be used to probe the 2D or even 3D structure of for example stellar winds and atmospheres. Not only the energy and momentum source terms are calculated, but also a direct observable: the radiation flux. In this thesis, the main application for FLD will lie within simulating instabilities in an isothermal atmosphere of a massive star.

Chapter 2

Methodology

The CAK and FLD equations described in the introduction are not to be solved analytically. Instead, they are solved using advanced computer codes. This chapter aims to describe how RHD the equations are being solved in a computer code. The code at hand is `mpi-AMRVAC` which, except for optically thin radiative cooling, is incompatible with any radiation effects. Additions made during this thesis are a novelty for the code and can be used in the simulations of a multitude of astrophysical processes.

2.1 mpi-amrvac

Modelling of the winds and atmospheres in this thesis is done in `mpi-AMRVAC` (Message Passing Interface - Adaptive Mesh Refinement Versatile Advection Code) ([?]), a multi-dimensional, adaptive mesh (magneto-)hydrodynamics code. In the code, several explicit solvers can be used to solve the conservative hydrodynamical equations (**??**) with source terms defined by the user. This code was chosen mainly because it is developed in house at the centre for mathematical plasma astrophysics (cmpa), which made a close collaboration with the developers possible.
To run the code, several scripts are provided:

- **Source code:** The source code consists of al the algorithms to solve the equations, apply boundary conditions, refine the mesh, read input and write output, etc... Inside the source code reside several physics modules, e.g. a module for gravity, dust, radiative cooling and viscosity.

- **user module:** In the user module, the user defines the initial conditions. Also, this is the place to add additional subroutines for user defined source terms, boundary conditions, time-step calculations and extra output variables among others. The subroutines in this file are automatically called by the source code.

- **parameter file:** The parameter file is where the computational parameters are defined: grid size and resolution, simulation time, which numerical schemes to use, what type of boundary conditions and how many output files the user wants are only a few to be named. This file is basically a bunch of knobs for the user.

In the following sections the integration of CAK-theory and FLD in mpi-AMRVAC will be explained, but first an introduction to numerical HD.

2.2 HD in mpi-AMRVAC

`MPI-AMRVAC` Solves systems of hyperbolic PDE's, for example the HD-equations (1.2), (1.3) and (1.4). To solve them, the code makes use of a finite volume approach. In this section, the basic ideas of this approach are explained. Remember the shape of the HD equations in their conservative form, in one dimension they look like:

$$\partial_t u + \partial_x f_u = 0 \tag{2.1}$$

where u is the conserved quantity and F_u is its relevant flux. In earlier more primitive methods such as the finite difference methods, this equation would be discretisised by writing down the derivatives in a discrete way at each cell center. A finite volume method on the other hand treats the value of u as an average over a small volume instead as a value at a single point. The flux F_u is a quantity existing at the cell faces. Discretization for a finite volume method thus begins by taking the average of u in a finite volume cell by means of integration:

$$\int_{t^n}^{t^{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (\partial_t u + \partial_x f_u)\, dt dx = 0 \tag{2.2}$$

Define U as the space averaged value for u and F as the time averaged value for f_u, the above equation can be written as:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}\right) \tag{2.3}$$

A good solver will give an accurate yet computationally cheap expression for the time integrated fluxes$F^n_{i+\frac{1}{2}}$ and $F^n_{i-\frac{1}{2}}$, which are not known analytically. Estimating the values of F is done by a *Riemann solver*. `MPI-AMRVAC` has a few of these Riemann solvers, including hll, hllc, roe, tvdlf, etc.. The approximations for $F^n_{i+\frac{1}{2}}$ will depend on u and f_u.

2.3 CAK-theory

Modelling CAK-winds consists of two steps: assuming an isothermal gas and adding the g_{cak} and g_{grav} source terms to the momentum equation.

The isothermal gas assumption is made by switching off the solving for the energy equation in the parameter file. The code now neglects equations (1.4) and (**??**), and replaces them with a new closure relation:

$$p = c_{adiabatic} p^{\gamma} \tag{2.4}$$

The two free parameters c_{adiab} and γ are set to .... and .... to match the conditions in a typical CAK wind. CITE HERE

The momentum source terms are added in the user module. All free parameters are defined, and dv/dr is calculated using a centred finite difference.

$$S_{\rho\vec{v},i,j} = \rho_{i,j} \frac{1}{1-\alpha} \frac{\kappa_e L_* \bar{Q}}{4\pi r_{i,j}^2 c} \left(\frac{1}{\rho_{i,j} c \bar{Q} \kappa_e} \right)^{\alpha} \frac{v_{i+1} - v_{i-1}}{r_{i+1} - r_{i-1}} \tag{2.5}$$

2.4 Flux Limited Diffusion

Solving the Radiation hydrodynamics equations is more complicated. An entirely new equation has to be solved and a bunch of new variables and parameters have to be defined. Given the complexity of the problem it is solved in a new physics module in the source code. Equation (**??**) is trivial and (1.19) only needs a source term which can be computed with the help of the FLD closure relation (1.32). The source terms in the energy equation (1.4) evolve on a radiation timescale instead of a hydrodynamical timescale, for this reason they will be implemented with an implicit scheme, see section 2.4.4. The radiation energy equation is entirely new to the code, and again because of the timescale difference, it will be solved partially implicitly, in an operator split manner.

$$\partial_t (E) + \underbrace{\vec{\nabla} \cdot (\vec{v}E)}_{1} = -\underbrace{\vec{\nabla} \cdot \vec{F}}_{2} - \underbrace{\vec{\nabla} \cdot \vec{v}P + 4\pi\kappa\rho B - c\kappa\rho E}_{3} \tag{2.6}$$

$$\tag{2.7}$$

- **1: The advection term**

$$\partial_t E + \vec{\nabla} \cdot (\vec{v}E) = 0 \tag{2.8}$$

can be handled using already existing amrvac routines, this part has a similar shape to the conservative form of the hd equations (2.1). This advection is the movement of the radiation field with the gas, it's a process evolving on a similar timescale as the advection of other quantities. Implementation is as trivial as defining the fluxes f and calling the necessary, already existing, subroutines. The code will use the same Riemann solver as used for the other HD variables.

- **2: The diffusion term**

$$\partial_t E = -\vec{\nabla} \cdot \vec{F} \tag{2.9}$$

can, by using the fld closing relation, be written as:

$$\partial_t E = -\vec{\nabla} \cdot \left(-\frac{\lambda c}{\kappa\rho} \nabla E \right) \tag{2.10}$$

Trying to solve this equation together with the other hydro equations gives rise to two problems. First: the timescale on which the radiation energy E evolves is much shorter than the hydrodynamical timescale on which ρ, $\vec{v}$ and e evolve, and second: this is a hyperbolic equation, which makes it hard to solve with the explicit schemes which are used by amrvac to solve the other equations. A way to overcome these problems is solving for the diffusion implicitly. The technique used in developing the new physics module is an Alternative Direction Implicit scheme (ADI), and is based on the Zeus-2D FLD module [**?**].

- **3: The photon tiring and the radiation heating and cooling sourceterms**

$$\partial_t E = -\vec{\nabla} \cdot \vec{v} P + 4\pi\kappa\rho B - c\kappa\rho E \tag{2.11}$$

are also evolving on the faster timescale, hence they must be solved implicitly. This is done with an implicit bisection scheme similar to the technique found in [**?**], which will be explained below.

In AMRVAC, the time step at which to evolve the primitive variables is computed based on the characteristic velocities in the system. These are obtained by computing the eigenvalues of the Jacobian flux matrix $A_{i,j} = \frac{\partial F_i}{\partial u_j}$, where F_i and u_j are the fluxes and conserved quantities in the conservative equations (1.5). These eigenvalues are $c_{adiab} + v$, c_{adiab} and $c_{adiab} - v$, where $c_{adiab} = \sqrt{\gamma \frac{p_{gas}}{\rho}}$ is the adiabatic soundspeed. During computations, waves must not travel across cells, thus the time step should be smaller then the time it takes for an acoustic wave with velocity $c_{adiab} + v$ to travel a distance Δx which is the width of a cell.

In RHD environments there is an extra type of waves, namely radio-acoustic waves which can travel faster than the sound speed. The time step in the RHD calculations will be calculated as a function of $\sqrt{\gamma \frac{p_{gas}+P_{rad}}{\rho}}$ instead of $\sqrt{\gamma \frac{p_{gas}}{\rho}}$.

$$dt = \min\left(\frac{\Delta x}{v + \sqrt{\gamma \frac{p_{gas}+P_{rad}}{\rho}}}\right) \tag{2.12}$$

This is the maximum time step at which the advection can take place. The diffusion of the radiation energy field on the other hand will generally happen much quicker, depending on the gradient of the E-field and absorption coefficient.

2.4.1 Elliptic vs Hyperbolic

2.4.2 ADI

The diffusion part of the radiative energy equation will be solved using the Alternating Direction Implicit (ADI) scheme. This is a numerical scheme which solves the equation implicitly in the first spatial direction an explicitly in the second for half a time step, and then implicitly in the second spatial direction and explicitly in the first for another half time step. Lets first simplify the equation by defining the diffusion coefficient $D = \frac{\lambda c}{\kappa\rho}$.

$$\partial_t E = \vec{\nabla} \cdot (D \nabla E) \tag{2.13}$$

Lets now construct a numerical scheme. First, the left hand side of (2.13) is written in a discrete form. Let $E_{i,j}$ be the radiative energy in cell (i, j) on a grid with Δx grid spacing in the x-direction and Δy grid spacing in the y-direction. $D_{i,j}$ is the cell centred diffusion coefficient, whilst $D1_{i,j} = \frac{1}{2}(D_{i+1,j} + D_{i,j})$ and $D2_{i,j} = \frac{1}{2}(D_{i,j+1} + D_{i,j})$ are the cell faced values. To transform from cell center to cell face it is simplest to take the mean of the two surrounding cells, taking the mean of the six surrounding cells is also an option.

$$\left(\vec{\nabla} \cdot (D\nabla E)\right)_{i,j} = \frac{D1_{i+1,j}(\nabla E)_x - D1_{i,j}(\nabla E)_x}{\Delta x} \tag{2.14}$$

$$+ \frac{D2_{i,j+1}(\nabla E)_y - D2_{i,j}(\nabla E)_y}{\Delta y} \tag{2.15}$$

$$= \frac{D1_{i+1,j}(E_{i+1,j} - E_{i,j}) - D1_{i,j}(E_{i,j} - E_{i-1,j})}{\Delta x^2} \tag{2.16}$$

$$+ \frac{D2_{i,j+1}(E_{i,j+j} - E_{i,j}) - D2_{i,j}(E_{i,j} - E_{i,j-1})}{\Delta y^2} \tag{2.17}$$

$$\tag{2.18}$$

For the implicit scheme the diffusion coefficients are chosen at the previous time step, as they are to be computed from a known radiation energy. Evolving (2.13) over a time step Δt from time n to time $n+1$ means solving the following equations:

$$\frac{E^{n+1} - E^n}{\Delta t} = \vec{\nabla} \cdot \left(D^n \nabla E^{n+1}\right) \tag{2.19}$$

As mentioned before, the ADI scheme first solves half a time step implicitly in for example the x-direction and explicitly in the y-direction:

$$\frac{E_{i,j}^{n+\frac{1}{2}} - E_{i,j}^n}{\Delta t} = \frac{D1_{i+1,j}^n}{\Delta x^2}(E_{i+1,j}^{n+\frac{1}{2}} - E_{i,j}^{n+\frac{1}{2}}) \tag{2.20}$$

$$- \frac{D1_{i,j}^n}{\Delta x^2}(E_{i,j}^{n+\frac{1}{2}} - E_{i-1,j}^{n+\frac{1}{2}}) \tag{2.21}$$

$$+ \frac{D2_{i,j+1}^n}{\Delta y^2}(E_{i,j+j}^n - E_{i,j}^n) \tag{2.22}$$

$$- \frac{D2_{i,j}^n}{\Delta y^2}(E_{i,j}^n - E_{i,j-1}^n) \tag{2.23}$$

In a more simple form, this can be written as:

$$\left(1 + \frac{\Delta t D1_{i+1,j}^n}{\Delta x^2} + \frac{\Delta t D1_{i,j}^n}{\Delta x^2}\right) E_{i,j}^{n+\frac{1}{2}} - \frac{\Delta t D1_{i+1,j}^n}{\Delta x^2} E_{i+1,j}^{n+\frac{1}{2}} - \frac{\Delta t D1_{i,j}^n}{\Delta x^2} E_{i-1,j}^{n+\frac{1}{2}} = b_{i,j} \tag{2.24}$$

where $b_{i,j} = \left(1 + \frac{\Delta t D2_{i,j+1}^n}{\Delta y^2} + \frac{\Delta t D2_{i,j}^n}{\Delta y^2}\right) E_{i,j}^n + \frac{\Delta t D2_{i,j+1}^n}{\Delta y^2} E_{i,j+1}^n + \frac{\Delta t D2_{i,j}^n}{\Delta y^2} E_{i,j-1}^n$ Or in matrix form:

$$\begin{bmatrix} \ddots & \ddots & & & \\ \ddots & \left(1+\frac{\Delta t D1^n_{i,j}}{\Delta x^2}+\frac{\Delta t D1^n_{i-1,j}}{\Delta x^2}\right) & -\frac{\Delta t D1^n_{i,j}}{\Delta x^2} & & \\ & -\frac{\Delta t D1^n_{i,j}}{\Delta x^2} & \left(1+\frac{\Delta t D1^n_{i+1,j}}{\Delta x^2}+\frac{\Delta t D1^n_{i,j}}{\Delta x^2}\right) & -\frac{\Delta t D1^n_{i,j}}{\Delta x^2} & \\ & & -\frac{\Delta t D1^n_{i+1,j}}{\Delta x^2} & \left(1+\frac{\Delta t D1^n_{i+2,j}}{\Delta x^2}+\frac{\Delta t D1^n_{i+1,j}}{\Delta x^2}\right) & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \tag{2.25}$$

$$\times \begin{bmatrix} \vdots \\ E^{n+\frac{1}{2}}_{i-1,j} \\ E^{n+\frac{1}{2}}_{i,j} \\ E^{n+\frac{1}{2}}_{i+1,j} \\ \vdots \end{bmatrix} = \begin{bmatrix} \cdots & b_{i,j-1} & b_{i,j} & b_{i,j+1} & \cdots \end{bmatrix} \tag{2.26}$$

This is, depending on boundary conditions, a tridiagonal matrix. In amrvac, this matrix system is solved on the entire computational domain and the innermost layer of ghost cells. The boundary conditions on the radiation energy field are applied after the halved time step. This way, the matrix can be kept tridiagonal independent of those boundary conditions. The tridiagonal system is then solved using Thomas' algorithm, which is a simplified form of Gaussian elimination. If the computational grid is N cells by M cells this is an $(N+2)\times(N+2)$ system, and there are $M+2$ such systems to be solved to evolve half a time step.

The second half of the timestep will be solved implicit in the y-direction and explicit in x-direction. Notice again the diffusion coefficient can only be updated at the end of the cycle:

$$\frac{E^{n+1}_{i,j}-E^{n+\frac{1}{2}}_{i,j}}{\Delta t} = \frac{D1^n_{i+1,j}}{\Delta x^2}(E^{n+\frac{1}{2}}_{i+1,j}-E^{n+\frac{1}{2}}_{i,j}) \tag{2.27}$$

$$-\frac{D1^n_{i,j}}{\Delta x^2}(E^{n+\frac{1}{2}}_{i,j}-E^{n+\frac{1}{2}}_{i-1,j}) \tag{2.28}$$

$$+\frac{D2^n_{i,j+1}}{\Delta y^2}(E^{n+1}_{i,j+j}-E^{n+1}_{i,j}) \tag{2.29}$$

$$-\frac{D2^n_{i,j}}{\Delta y^2}(E^{n+1}_{i,j}-E^{n+1}_{i,j-1}) \tag{2.30}$$

And this system, of course, has a matrix notation completely similar to the one in the first half of the time step, which can be solved in a completely similar way. This time there are $N+2$ matrices of size $(M+2)\times(M+2)$ to be solved.

The accuracy of this scheme can be checked by computing and comparing both the left hand side and right had side of equation (2.19) after calculating E^{n+1}. The error is defined by taking the maximum value of the difference of the LHS and RHS weighted with the ratio of the radiative energy and the hydrodynamical time step:

$$Error = \max_{i,j} \left(\frac{\frac{E^{n+1}-E^n}{\Delta t} - \vec{\nabla} \cdot (D \nabla E)}{E^n/dt} \right) \tag{2.31}$$

Test calculations show that this error is often rather large, depending on the distribution of the radiation and density field in the simulation space. A solution is given in [?] by means of pseudo-time stepping. This method will be explained in the next section.

2.4.3 pseudo-timestepping

Instead of solving equation (2.13) a new variable is introduced, the *pseudotimestep* w. The ADI-method is used to solve

$$\partial_w E = \partial_t E - \vec{\nabla} \cdot (D \nabla E) \tag{2.32}$$

which reduces to (2.13) when $\partial_w E = 0$. The pseudo-time step Δw can be adjusted independently from the hydrodynamical time step and is increasing in size to converge toward a w-stationary state where $\partial_w E = 0$ [?].

The implicit scheme to evolve half a pseudo-timestep from m to $m + \frac{1}{2}$, implicit in the x-direction, looks like:

$$\frac{E_{i,j}^{m+\frac{1}{2}} - E_{i,j}^m}{\Delta w} = \frac{E_{i,j}^{m+\frac{1}{2}} - E_{i,j}^n}{\Delta t} \tag{2.33}$$

$$- \frac{D1_{i+1,j}^n}{\Delta x^2} (E_{i+1,j}^{m+\frac{1}{2}} - E_{i,j}^{m+\frac{1}{2}}) \tag{2.34}$$

$$+ \frac{D1_{i,j}^n}{\Delta x^2} (E_{i,j}^{m+\frac{1}{2}} - E_{i-1,j}^{m+\frac{1}{2}}) \tag{2.35}$$

$$- \frac{D2_{i,j+1}^n}{\Delta y^2} (E_{i,j+j}^m - E_{i,j}^m) \tag{2.36}$$

$$+ \frac{D2_{i,j}^n}{\Delta y^2} (E_{i,j}^m - E_{i,j-1}^m) \tag{2.37}$$

Where, again, the Diffusion coefficient is considered constant throughout the timestep. This corresponds to a similar tridiagonal matrix equation which has to be solved for every pseudo-timestep. The size of the pseudo-timestep Δw is exponentially increasing in size per iteration $m = 1...W$ and is given by:

$$\Delta w = \Delta w_0 \left(\frac{\Delta w_1}{\Delta w_0} \right)^{\frac{m-1}{W-1}} \tag{2.38}$$

Δw_0 And Δw_1 are one quarter of the size of a grid cell and one quarter of the size of the numerical domain, respectively. The error of this combined ADI-pseudo-time step method is measured after W pseudo-time steps using (2.31), if it is too large W and Δw_1 are increased whilst Δw_0 is decreased. If this still doesn't suffice, the above scheme can be applied twice to half a hydrodynamical time step, four times to a quarter hydrodynamical time step, ...

2.4.4 Bisection Implicit scheme

Solving for the source terms in the gas and radiation energy equations happens with another implicit scheme.

$$e^{n+1} - e^n = \Delta t \left(-4\kappa\sigma \left(\frac{(\gamma-1)e^{n+1}}{\rho} \right)^4 + c\kappa E^{n+1} \right) \tag{2.39}$$

$$E^{n+1} - E^n = \Delta t \left(+4\kappa\sigma \left(\frac{(\gamma-1)e^{n+1}}{\rho} \right)^4 - c\kappa E^{n+1} - \nabla\vec{v}P^{n+1} \right) \tag{2.40}$$

This can be rewritten as:

$$e^{n+1} - e^n = -a_1 \left(e^{n+1}\right)^4 + a_2 E^{n+1} \tag{2.41}$$

$$E^{n+1} - E^n = a_1 \left(e^{n+1}\right)^4 - a_2 E^{n+1} - a_3 E^{n+1} \tag{2.42}$$

where $a_1 = 4\kappa\sigma \left(\frac{(\gamma-1)}{\rho^{n+1}}\right)^4 \Delta t$, $a_2 = c\kappa\Delta t$ and $a_3 = \frac{\nabla\vec{v}P^{n+1}}{E^{n+1}}\Delta t$. Manipulation of the equations returns:

$$E^{n+1} = \frac{a_1 \left(e^{n+1}\right)^4 + E^n}{1 + a_2 + a_3} \tag{2.43}$$

And

$$\left(e^{n+1}\right)^4 + \frac{1 + a_2 + a_3}{a_1 + a_3} e^{n+1} - \frac{(1 + a_2 + a_3)e^n + a_2 E^n}{a_1 + a_3} = 0 \tag{2.44}$$

Equation (2.44) is a 4^{th} degree polynomial in e^{n+1}, with a single root between 0 and $\frac{1+a_2+a_3}{a_1+a_3}$. This is solved for e^{n+1} using the bisection method to calculate the contribution of the radiative heating and cooling to the gas energy. The newly calculated e^{n+1} is then plugged in equation (2.43) to find the contribution of the radiative heating and cooling, and the photon tiring to the radiation energy.

2.5 Dimensionless problem

Computationally, floating point values are the most precise around unity. For this reason, it is preferable to rescale physical quantities such as ρ, $\vec{v}$, e, ... with values typical to the simulation domain. AMRVAC already does port of this for us, in the user module, one can define either the units for either number density, temperature and length (N_0, T_0, l_0) or number density, velocity and length (N_0, v_0, l_0). The other units will be then computed from these quantities using the following relations:

$$\rho_0 = \mu m_p N_0 \tag{2.45}$$

$$p_0 = \gamma N_0 k_b T_0 \tag{2.46}$$

$$v_0 = \sqrt{\frac{p_0}{\rho_0}} \tag{2.47}$$

$$t_0 = \frac{l_0}{v_0} \tag{2.48}$$

Calculations in amrvac are always done with unit less quantities, in HD these are: $\tilde{\rho} = \frac{\rho}{\rho_0}$, $\tilde{\vec{v}} = \frac{\vec{v}}{\vec{v}_0}$, $\tilde{p} = \frac{p}{p_0}$ and $\tilde{e} = \frac{e}{p_0}$. Using this, we can transform the continuity equation (1.2) to dimensionless units:

$$\frac{t_0}{\rho_0}\frac{\partial}{\partial t}\rho + \frac{t_0}{\rho_0}\vec{\nabla}\cdot(\rho\vec{v}) = \frac{t_0}{\rho_0}S_\rho \tag{2.49}$$

$$\frac{\partial}{\partial \tilde{t}}\tilde{\rho} + \tilde{\nabla}\left(\tilde{\vec{v}}\tilde{\rho}\right) = \tilde{S}_\rho \tag{2.50}$$

Where ∇ has the units of reciprocal length and $\tilde{S}_\rho \rho_0 S_\rho$. Equivalently, for the momentum equation (1.3) and the gas energy equation (1.4):

$$\frac{\partial}{\partial \tilde{t}}\left(\tilde{\rho}\tilde{\vec{v}}\right) + \tilde{\vec{\nabla}}\left(\tilde{\vec{v}}\tilde{\rho}\tilde{\vec{v}} + \tilde{p}\right) = \tilde{S}_{\rho\vec{v}} \tag{2.51}$$

$$\frac{\partial}{\partial \tilde{t}}\tilde{e} + \tilde{\vec{\nabla}}\left(\tilde{\vec{v}}\tilde{e} + \tilde{\vec{v}}\tilde{p}\right) = \tilde{S}_e \tag{2.52}$$

$$\tag{2.53}$$

$\tilde{S}_{\rho\vec{v}} = S_{\rho\vec{v}}\frac{t_0}{\rho_0 v_0}$ and $\tilde{S}_e = S_e\frac{t_0}{e_0}$ are the momentum and energy equation source terms. For the RHD equations this will rescale the physical constants:

$$\tilde{S}_{\rho\vec{v}} = \frac{\kappa\rho}{c}\vec{F}\frac{t_0}{\rho_0 v_0} \tag{2.54}$$

$$= \frac{\tilde{\kappa}\tilde{\rho}}{\tilde{c}}\tilde{\vec{F}} \tag{2.55}$$

$$\tilde{S}_e = -4\pi\kappa\rho B\frac{t_0}{e_0} + c\kappa\rho E\frac{t_0}{e_0} \tag{2.56}$$

$$= -4\tilde{\sigma}\tilde{\kappa}\tilde{T}^4 + \tilde{c}\tilde{\rho}\tilde{E} \tag{2.57}$$

The rescaled Stefan-Boltzmann constant $\tilde{\sigma}$, the rescaled speed of light $\tilde{c}$ and the rescaled opacity $\tilde{\kappa}$ are given by $\tilde{\sigma} = \frac{\sigma T_0^4}{e_0 v_0}$, $\tilde{c} = \frac{c}{v_0}$ and $\tilde{\kappa} = \kappa v_0 t_0$. The unit of the radiation energy density and radiation flux can also be defined: $E_0 = e_0$ and $F_0 = \rho_0 v_0^3$. These are used to transform the RHD euqation for radiation energy (1.21) and the fld closure relation (1.32).

2.6 visualisation

AMRVAC output is given in binary .vtu files. Special software such as visit [] and paraview are used to open and analyse the simulation results. The standard output consists of the primitive variables at every simulation cell for a specified number of time steps. Extra output variables can be defined in the user module.

Chapter 3

Results

The first result is of course a working CAK subroutine and FLD module in mpi-AMRVAC. This is something that didn't exist before and it will open the way toward simulations of new physical regimes where radiation plays a role in the dynamics of the system. Other than writing the software, there are also some scientific results. These will be described in this chapter.

3.1 CAK-Theory

3.1.1 Massive star stellar wind

3.2 Flux Limited Diffusion

The diffusion model is first tested against some analytical results, these tests will give us some knowledge on how accurately we can interpret the simulations of physical phenomena. The diffusion term and advection term are both tested separately by comparison to an exact solution and the photon tiring, radiative cooling and radiative heating source terms are tested together versus a Runge-Kutta solver of a simplified problem.

3.2.1 Testcase 1: Advection and Diffusion

The advection problem and the diffusion problem, solved by the already existing Riemann solver and the new ADI solver respectively can be compared to an exact analytical solution. The Riemann solver is nothing new, it's just a matter of checking the correct implementation in mpi-amrvac. The problem is tested on a numerical domain with constant density, a constant velocity, a constant gas energy density and an initial radiation energy density $E_0(x, y, t)$ given by:

$$E_0(x, y, t) = 2 + \sin(2\pi x)\sin(2\pi y) \tag{3.1}$$

If diffusion and other source terms are ignored, the radiation field will evolve as

$$E^{adv}(x, y, t) = 2 + \sin(2\pi(x - v_x t))\sin(2\pi(y - v_y t)) \tag{3.2}$$

If diffusion is switched on but the advection is ignored by fixing the velocity field to $\vec{0}$ every iteration, and the diffusion coefficient is chosen constant at $D = 1$, the field evolves as

$$E^{diff}(x,y,t) = 2 + \exp(-8\pi^2 t)\sin(2\pi x)\sin(2\pi y) \tag{3.3}$$

Function $E_0(x,y,t)$ describes a series of dots of more and less radiative energy, see figure **??**. The numerical domain is chosen in such a way that there is one region of lower and one region of higher energy in each direction, $-0.5 \leq x \leq 0.5$, $0 \leq y \leq 1$. In time, the diffusion test runs until the amplitude of the dots diminish by two orders of magnitude $\exp(-8\pi^2 t) = 0.01$ and the advection test tuns until a point has passed the computational domain twice $\min(v_x, v_y)t = 2$. Boundary conditions are set periodical. Computational result $\tilde{E}$ can be compared with the analytical results to define the residuals:

$$RES^{adv} = \left| \frac{\tilde{E}^{adv} - E^{adv}}{E^{adv}} \right| \tag{3.4}$$

$$RES^{diff} = \left| \frac{\tilde{E}^{diff} - E^{diff}}{E^{diff}} \right| \tag{3.5}$$

Which are plotted for different timesteps in figure **??** and **??** together with the numerical solutions $\tilde{E}^{adv}$ and $\tilde{E}^{diff}$

3.2.2 Testcase 2: Photon Tiring, Heating and Cooling

To test the implicit bisection scheme used for adding the photon tiring, radiative heating and radiative cooling source terms, we make comparisons with an explicit Runge-Kutta solver. Of course this Runge-Kutta solver is not to be used in the actual code, because the time step chosen in the Runge-Kutta solver will be orders of magnitude smaller. The equation at hand is:

$$\frac{de}{dt} = c\rho\kappa E - 4\rho\kappa\sigma T^4 \tag{3.6}$$

Except for the gas energy density, all primitive variables ($\rho = \rho_0$, $\vec{v} = 0$ and $E = E_0$) are kept constant. The computational domain is taken as small as possible and radiative diffusion is switched off.

The system would be in radiative equilibrium when $c\rho\kappa E = 4\rho\kappa\sigma T^4$. Using $T = \frac{p}{\rho} = \frac{(\gamma-1)e}{\rho}$, on can compute the equilibrium gas energy.

$$e_{rad.eq.} = \frac{\rho}{\gamma - 1}\left(\frac{cE}{4\sigma}\right)^{\frac{1}{4}} \tag{3.7}$$

Different initial conditions are chosen for e_0 ranging from $10^2 e_{rad.eq.}$ to $10^{-10} e_{rad.eq.}$. Comparisons between the bisection method and a simple Runge-Kutta solver are plotted in figure 3.2.2. Remember that the implicit bisection method uses a time step which is several orders of magnitude larger than the explicit Runge-Kutta method.

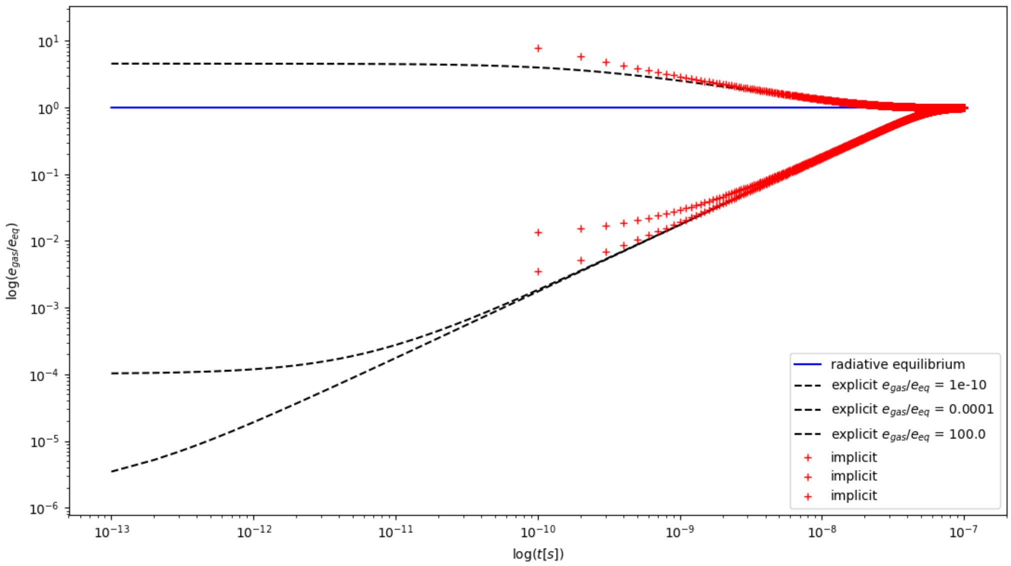

Figure 3.1: blahblahblah

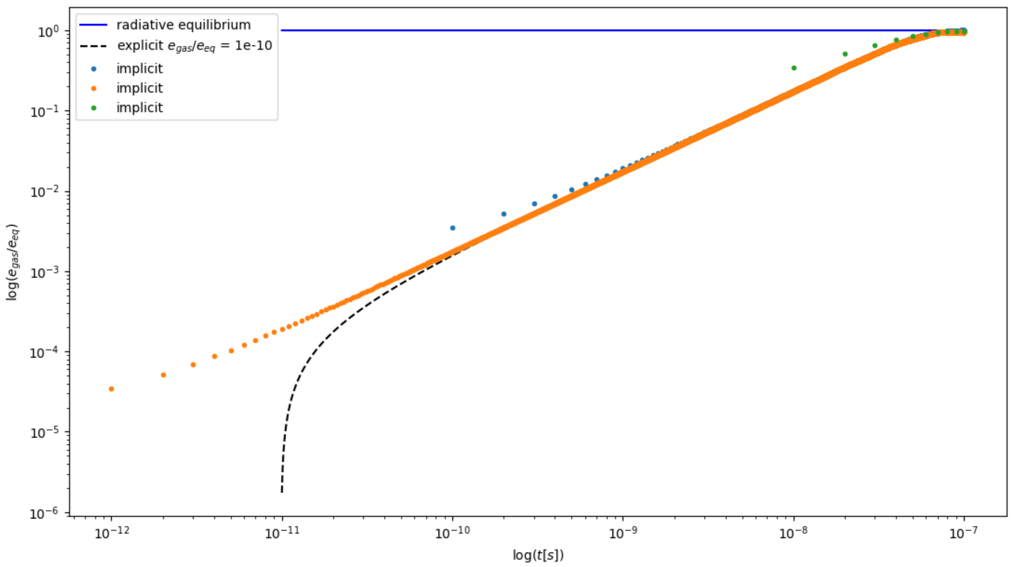

Figure 3.2: blahblahblah

3.2.3 Isothermal Atmosphere

A first practical use for the FLD module is modelling a stellar atmosphere surrounding a massive star. For convenience, the atmosphere will be considered isothermal and plane parallel, so the flux in the initial condition and the gravitational acceleration are constant. Let's also assume the only absorption an emission is done by means of electron scattering, with a constant opacity κ.

The initial conditions are crucial in stabilizing simulations. In an isothermal atmosphere, the density and gas pressure decay exponentially on the length of a scale height $H_{eff} = \frac{c^2_{sound}}{g_{eff}}$. Where $g_{eff} = g_{grav}(\Gamma - 1)$ is the sum of radiation and gravitational accelerations.

$$\rho(y) = \rho_0 \exp\left(-\frac{y}{H_{eff}}\right) \tag{3.8}$$

$$p(y) = p_0 \exp\left(-\frac{y}{H_{eff}}\right) \tag{3.9}$$

WORK OUT MORE

The velocity field is $\vec{0}$ everywhere. Boundary conditions ρ_0 and p_0 can be computed by defining the optical depth τ.

$$\tau(y) = \int_y^\infty \kappa\rho dy \tag{3.10}$$

In a static medium, the gravitational and radiative acceleration of the gas is countered by the gas pressure gradient. Concerning the radiation field, one can make a similar statement . The radiative acceleration is countered by the radiation pressure gradient:

$$\frac{dp}{dy} = -g_{grav}(1 - \Gamma) \tag{3.11}$$

$$\frac{dP}{dy} = -g_{grav}\Gamma \tag{3.12}$$

HIER MIST NOG IETS Equations (3.10) and (3.11) can be solved toward a value for p at a given optical depth. (3.12) And (3.11) can by divided by one another to get a relation between dp and dP.

$$p_0 = g_{eff}\frac{\tau_0}{\kappa} \tag{3.13}$$

$$dP = \frac{\Gamma}{\Gamma - 1}dp \tag{3.14}$$

The gas energy density can be set from the gas pressure profile. Because of the plane parallel approximation, Γ is constant as well and in the Eddington limit, $E = 3P$. Now we have a boundary condition for the radiation energy field as well. The initial conditions are given by:

$$\rho(y) = g_{eff} \frac{\tau_0}{\kappa c_{sound}^2} \exp\left(-\frac{y}{H_{eff}}\right) \tag{3.15}$$

$$\rho\vec{v} = \vec{0} \tag{3.16}$$

$$e = \frac{\tau_0}{\kappa(\gamma - 1)} \exp\left(-\frac{y}{H_{eff}}\right) \tag{3.17}$$

$$E = \frac{3\Gamma}{1-\Gamma} \frac{\tau_0}{\kappa} \exp\left(-\frac{y}{H_{eff}}\right) \tag{3.18}$$

$$\tag{3.19}$$

The parameters defining the physics of the system are the Eddington parameter Γ, the optical depth at the lower boundary τ_0 and the sound speed c_{sound}. The Eddington parameter contains information about the mass, luminosity and radius of the star, it determines whether the atmosphere will be blown away, collapsing in on itself or relax in a steady state. τ_0 Sets the physical lower boundary of the numerical domain. A high value means the model starts at in a high density environment near the core, a low value means the model simulates the outermost boundaries of the star. The sound speed determines the temperature of the atmosphere and the velocity at which waves can travel. Together with the Mass which sets the gravitational field, the sound speed also determines the scale height of the gas.

Calculations are done on a Cartesian grid, with the y-direction parallel to the radius of the star.The mass of the star is chosen at $M_* = 50M_\odot$. Using Leavits' law, we get a luminosity for a typical $50M_\odot$ star of $L_* = ...L_\odot$ and an Eddington parameter $\Gamma = ...$ The calculation domain begins at $\tau_0 = ...$ and resolves about ... scale heights in the y-direction and ... in the x-direction. The resolution of the grid is chosen such that there are ... cells per scale height. The simulation stops after .. times the hydrodynamical timescale, this is the time needed for a sound wave to travel across the computational domain.

constant flux discrepancy

comparison to mesa structure

3.2.4 Strange mode instabilities

3.2.5 Non-isothermal evolution?

Chapter 4

Conclusions

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Chapter 5

Future Work

5.1 Alternative Implicit Schemes

5.2 MPI

5.3 AMR

5.4 Non-Isothermal atmospheres

5.5 Super Eddinton Limit

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