KU LEUVEN

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Time Dependent Radiation Hydrodynamics in Stellar Winds and Atmospheres

mo' code, mo' problems

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Contents

Chapter 1

Introduction

In various systems it is important to take into account a radiation field when considering gas dynamics ([?], ...). Radiation can exert forces to, for example, accelerate stellar winds and transport energy like in the non-convective stellar radiation zones. The equations describing gas dynamics with a radiation will evolve on two different timescales, a radiation field generally evolves much faster than gas, so solving them isn't trivial. This thesis describes radiation hydrodynamics (RHD) in two different regimes: CAK-theory in hot stellar winds and flux limited diffusion in stellar atmospheres.

The importance of good models for radiation hydrodynamics is clear in a multitude of astrophysical systems, a few are listed below. An important concept in these sort of systems is the Eddington factor Γ, this is the ratio between the radiative and gravitational acceleration.

$$\Gamma = \frac{\kappa F}{c g_{grav}} \tag{1.1}$$

Where F is the radiation energy flux, κ the absorption opacity, c the speed of light in vacuum and g_{grav} the magnitude of the local gravitational acceleration. If the Eddington factor describing a system gets bigger, the importance for taking into account radiation grows as wel.

- **stellar winds**
  Stellar winds can be subdivided in three types: pressure driven for solar type stars, dust driven for cool massive stars and radiation driven for hot massive stars. The driving force behind these radiation driven winds is, as the name sugests, radiation force. Hot massive stars can undergo massloss of $\sim 10^{-6} M_{\odot}/yr$ due to radiation driven wind, this of course will have an impact on their evolution.
  Stars with a luminosity of $10^6 L_{\odot}$ and a stellar mass of $80 M_{\odot}$ will have an Eddington factor $\Gamma \sim 50$ near the surface, radiation plays an essential role in their dynamics. CAK-theory Is an analytical model describing these winds, and the first half of this thesis was spent on modeling this.

- **stellar atmospheres**

- **accretion disks**

- **star formation**

- **very massive stars**

- **Eruptive variables**
  Systems that ecceed the Eddington limit ($\Gamma > 1$) are called super-Eddington systems. A prime example for such a system is η Carinae. η Carinae is what is known as an eruptive variable. It is a star which underwent an enormous massloss in the middle of the 19th century, losing $10M_{\odot}$ over the course of 10 years. One of the hypothesised drivers of this huge massloss is radiation.

1.1 Hydrodynamics

The movement of non-isothermal gasses and fluids, in the absence of magnetic fields and radiation, are described by hydrodynamics. The hydrodynamical equations describe conservation of mass, conservation of momentum and conservation of energy in the following equations:

$$\partial_t (\rho) + \vec{\nabla} \cdot (\rho \vec{v}) = S_\rho \tag{1.2}$$

$$\partial_t (\rho \vec{v}) + \vec{\nabla} (\vec{v} \rho \vec{v} + p) = S_{\rho \vec{v}} \tag{1.3}$$

$$\partial_t (e) + \vec{\nabla} \cdot (\vec{v} e + \vec{v} p) = S_e \tag{1.4}$$

These three partial differential equations are called the continuity equation (describing mass density), the momentum equation (describing momentum density) and the energy equation (describing energy density). Above, the equations were written in their conservative form, they al have the same shape:

$$\partial_t \underbrace{u}_{\text{density}} + \vec{\nabla} \cdot \overbrace{\vec{F_u}}^{\text{density flux}} = \underbrace{S_u}_{\text{sourceterm}} \tag{1.5}$$

There are only 3 PDE's describing 4 primitive variables: ρ, $\vec{v}$, e and p. This means that there is need for an additional closure relation:

$$p = (\gamma - 1) \left(e - \frac{\rho \vec{v}^2}{2} \right) \tag{1.6}$$

The sourceterms in the equations describe adding or substracting to one of the densities: S_ρ describes mass being added or substracted from the system, $S_{\rho \vec{v}}$ describes external forces such as gravity or radiative forces and S_e describes work excerted on the system, this can be for example due to heating of the fluid.

Predicting fluid dynamics means solving these forementioned equations simultaniously, this is done using computercodes such as mpi-amrvac [?]. Solving these equations is a tricky buisness, and there are numerous codes and numerical schemes available. The Evolution of a simulation will depend not only on both initial and boundary conditions, but also on which physics taken into consideration (sourceterms) and even a little on which numerical methods are used, more on this in chapter 2.1.

In section ??, an analytic expression for these sourceterms are calculated to simulate the conditions in line driven stellar winds. This calculation was first done by Castor, Abbot and Klein CITE HERE and carries their initals in its name: CAK-theory.

1.2 Radiation Hydrodynamics

The last section gave a small recap of hydrodynamics. In this section, a mathematicial framework is set up to combine the hydrodynamics equations with the effects of a time dependent radiation field. At the end of this we will be left with a system of partial differential equations describing the dynamics of both the gas and radiation field.
The main governing equation describing the evolution of radiation trough a medium is the radiative transfer equation:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{n}\cdot\vec{\nabla}\right) I_\nu = \eta_\nu + \kappa_\nu I_\nu \tag{1.7}$$

Where I_ν is the intensity at frequency ν along the direction of unit vector $\vec{n}$. η_ν And χ_ν are the emissivity and total opacity. This equation describes how the value of the intensity I_ν Changes when propagating trough a medium with given emissivity and opacity. This form is known as the 0^{th} order of the radiative transfer equation. The 1^{st} order radiative transfer equations are obtained by integrating over all solid angles and dividing by 4π, the 2^{nd} order one by first multiplying with $\vec{n}$ before doing the integration.

$$\frac{1}{c}\frac{\partial J_\nu}{\partial t} + \vec{\nabla}\cdot\vec{H}_\nu = \frac{1}{4\pi}\int_\Omega \eta_\nu + \kappa_\nu I_\nu d\Omega \tag{1.8}$$

$$\frac{1}{c}\frac{\partial \vec{H}_\nu}{\partial t} + \vec{\nabla}\cdot K_\nu = \frac{1}{4\pi}\int_\Omega \left(\eta_\nu + \kappa_\nu I_\nu\right)\vec{n} d\Omega \tag{1.9}$$

Where J_ν, $\vec{H}_\nu$ and K_ν are the higher order moments of the intensity. J_ν is the mean intensity, which is I_ν averaged over all solid angles and thus a scalar. $\vec{H}_\nu$ Is a vector and K_ν is a tensor of order 2. The following relations exist between the moments of intensity and some more physical quantities:

$$E_\nu = \frac{4\pi}{c} j_\nu \tag{1.10}$$

$$\vec{F}_\nu = \frac{4\pi}{c}\vec{H}_\nu \tag{1.11}$$

$$P_\nu = \frac{4\pi}{c} K_\nu \tag{1.12}$$

These are the radiative energy E_ν, the radiation flux $\vec{F_\nu}$ and the radiation pressure tensor P_ν. When asuming local thermal equilibrium (LTE), just like with the gass pressure tensor, P_ν can be written as a scalar times the unit tensor. The fraction of emissivity and total opacity is often written as the sourcefunction $S_\nu = \frac{\eta_\nu}{\chi_\nu}$.
When doing full radiative transfer, these equations are solved for an enormous amount of frequecies, corresponding to the indices $_\nu$. The situation is simplified when assuming a grey radiation field, by integrating over all frequencies. The LTE, grey radiation energy and radiation flux equations can be written in a conservative form, similar to the hydrodynamics equations:

$$\frac{\partial E}{\partial t} + \vec{\nabla} \cdot \vec{F} = c \int_\nu \int_\Omega \chi_\nu \left(S_\nu + I_\nu\right) d\nu d\Omega \tag{1.13}$$

$$\frac{\partial \vec{F}}{\partial t} + c^2 \vec{\nabla} P = \frac{c}{4\pi} \int_\nu \int_\Omega \chi_\nu \left(S_\nu + I_\nu\right) \vec{n} d\nu d\Omega \tag{1.14}$$

Assuming LTE, the sourcefunction scales with the Planck function $S_\nu = \frac{4\pi}{c} B_\nu(T)$ and both $B_\nu(T)$ and χ_ν are independent of solid angle. The relations between flux, energy and the moments of intensity can be used again to replace I_ν in the sourceterms. Because of symmetry, $S_\nu \vec{n}$ integrated over the total solid angle will return $\vec{0}$ in the sourceterm of the flux equation.
The absorption coefficient is the product of the gas density and the frequency dependent opacity: $\chi_\nu = \rho\kappa_\nu$. To make life easier, an Energy-meaned and Plack opacity can be defined, and in first approximation they are equal to the Rosseland mean opacity: (SOURCE?!?!?!?)

$$\frac{\int_\nu E_\nu \kappa_\nu d\nu}{\int_\nu E_\nu d\nu} = \frac{\int_\nu B_\nu \kappa_\nu d\nu}{\int_\nu B_\nu d\nu} = \kappa \tag{1.15}$$

Integration over all frequencies, replacing opacities by the Rosseland mean opacity leads to:

$$\frac{\partial E}{\partial t} + \vec{\nabla} \cdot \vec{F} = 4\pi\kappa\rho B(T) - c\kappa\rho E \tag{1.16}$$

$$\frac{\partial \vec{F}}{\partial t} + c^2 \vec{\nabla} P = c\kappa\rho \vec{F} \tag{1.17}$$

These equations are written here in the co-moving frame. Transforming to a static frame, the same frame as used in forementioned hydro equations, an advection term is added to the density flux in both equations, $\vec{v}E$ and $\vec{v} \cdot \vec{F}$. The two sourceterms in the radiation energy equation are interpreted as energy exchange between the gas and the radiation field, energy leaving the radiation field heats the gas and gas is cooled by energy entering the radiation field. These so called heating and cooling sourceterms, $4\pi\kappa\rho B(T)$ and $c\kappa\rho E$, must be added to the gas energy equation as well.
The sourceterm in the Flux energy equation has the same units as the gas momentum sourceterm, this is the expression for radiation force. If momentum leaves the radiation flux there is work being done, this must also mean that there is energy leaving the radiation field. This is translated in the photon tiring term $\vec{\nabla} \cdot \vec{v} P$, which must be substracted from the radiation energy sourceterm.

$$\partial_t(\rho) + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \tag{1.18}$$
$$\partial_t(\rho\vec{v}) + \vec{\nabla}(\vec{v}\rho\vec{v} + p) = \frac{\kappa\rho}{c}\vec{F} \tag{1.19}$$
$$\partial_t(e) + \vec{\nabla}\cdot(\vec{v}e + \vec{v}p) = -4\pi\kappa\rho B + c\kappa\rho E \tag{1.20}$$
$$\partial_t(E) + \vec{\nabla}\cdot\left(\vec{v}E + \vec{F}\right) = -\vec{\nabla}\cdot\vec{v}P + 4\pi\kappa\rho B - c\kappa\rho E \tag{1.21}$$
$$\partial_t\left(\frac{\vec{F}}{c^2}\right) + \vec{\nabla}\left(\frac{\vec{v}\cdot\vec{F}}{c^2} + P\right) = -\frac{\kappa\rho}{c}\vec{F} \tag{1.22}$$

These are the final radiation hydrodynamics equations. There are only 5 equations for 7 primitive variables: ρ, $\vec{v}$, e, p, E, $\vec{F}$ and P. Two closing relations are nessecary to close the system. A first closing relation is obtained by re-using equation (1.6), a second one can be obtained by for example the *Flux limited diffusion* approximation (FLD) described in section 1.4.

1.3 Sobolev and CAK-theory

1.4 Flux Limited Diffusion

As mentioned before, the full RHD equations leave us with 5 partial differential equations and one HD closure relation for 7 variables. Several methods and approximations exist for closing the system and one of them is flux limited diffusion. (GIVE EXAMPLES, CITE)

Consider the steady state solution of the radiation flux equation (1.22). In first order $\frac{v}{c^2} << 1$, so the relation between P and F is given by.

$$\vec{\nabla}P = -\frac{\kappa\rho}{c}\vec{F} \tag{1.23}$$

In the optically thick limit the eddington approximation gives us $P = \frac{1}{3}E$, so (1.23) can be written as $\vec{F} = -\frac{1}{3}\frac{c}{\kappa\rho}\vec{\nabla}E$. However, in the optically thin, free streaming limit, the density goes to zero and thus the radiation flux should go to infinity, this is unphysical. A solution is introducing the flux limiter λ, which is a factor varrying between $\frac{1}{3}$ in the optically thick regime, and 0 in the optically thin. The extra nessecary closure relation can now be written as:

$$\vec{F} = -\frac{c\lambda}{\kappa\rho}\vec{\nabla}E \tag{1.24}$$

Different formalisms for expressing λ exist, the one used in this work has been worked out in [?].

Chapter 2

Methodology

2.1 mpi-amrvac

Modeling of the winds and atmospheres in this thesis is done in *mpi-AMRVAC* (Message Passing Interface - Adaptive Mesh Refinement Versatile Advection Code) ([?]), a multidimensional, adaptive mesh (magneto-)hydrodynamics code. In the code, several explicit solvers can be used to solve the conservative hydrodynamical equations (**??**) with sourceterms defined by the user. This code was chosen mainly because it is developed in house at the centre for mathematical plasma astrophysics (cmpa), which made a close colaboration with the developpers possible.
To run the code, several scripts are provided:

- **Sourcecode:** The sourcecode consists of al the algorithms to solve the equations, aply boundary conditions, refine the mesh, read input and write output, etc. Inside the sourcecode reside several physics modules, e.g. a module for graity, dust, radiative cooling and viscosity.

- **user module:** In the user module, the user defines the initial conditions. Also, this is the place to add additional subroutines for userdefined sourceterm, boundary conditions, timestep-calculations and extra outputvariables among others. The subroutines in this file are automaticly called by the sourcecode.

- **parameter file:** The parameter file is where the computational parameters are defined: grid size and resolution, simulation time, which numerical schemes to use, what type of boundary conditions and how many output files the user wants are only a few to be named. This file is basicly a bunch of knobs for the user.

In the following sections, the integration of CAK-theory and FLD in mpi-AMRVAC will be explained.

2.2 CAK-theory

Moddeling CAK-winds consists of two steps: assuming an isothermal gas and adding the g_{cak} and g_{grav} sourceterms to the momentum equation.

The isothermal gas assumption is made by switching off the solving for the energy equation in the parameter file. The code now neglects equations (1.4) and (**??**), and replaces them with a new closure relation:

$$p = c_{adiabatic} p^{\gamma} \tag{2.1}$$

The two free parameters $c_{adiabatic}$ and γ are set to .... and .... to match the conditions in a typical CAK wind. CITE HERE

The momentum sourceterms are added in the user module.

2.3 Flux Limited Diffusion

Solving the Radiation hydrodynamics equations is more complicated. An entirely new equation has to be solved and a bunch of new variables and parameteres have to be defined. Given the complexity of the problem it is solved in a new physics module in the sourcecode. Equation (**??**) is trivial and (1.19) only needs a sourceterm which can be computed with the help of the FLD closure relation (1.24). The sourceterms in the energy equation (1.4) evolve on a radiation timescale instead of a hydrodynamical timescale, for this reason they wil be implemented with an implicit scheme, see section 2.3.4. The radiation energy equation is entirely new to the code, and again because of the timescale difference, it will be solved partially implicitly, in an operator split manner.

In AMRVAC, the timestep at which to evolve the primitive variables is computed based on the characteristic velocities in the system. These are obtained by computing the eigenvalues of the jacobian flux matrix $A_{i,j} = \frac{\partial F_i}{\partial u_j}$, where F_i and u_j are the fluxes and conserved quantities in the conservative equations (1.5). These eigenvalues are $c_{adiab} + v$, c_{adiab} and $c_{adiab} - v$, where $c_{adiab} = \sqrt{\gamma \frac{p_{gas}}{\rho}}$ is the adiabatic soundspeed. During computations, waves must not travel accross cells, thus the timestep should be smaller then the time it takes for an acoustic wave with velocity $c_{adiab} + v$ to travel a distance Δx which is the width of a cell.
In RHD environments there is an extra type of waves, namely radio-acoustic waves which can travel faster than the soundspeed. The timestep in the RHD calculations will be calculated as a function of $\sqrt{\gamma \frac{p_{gas}+P_{rad}}{\rho}}$ instead of $\sqrt{\gamma \frac{p_{gas}}{\rho}}$.

$$dt = min \left(\frac{\Delta x}{v + \sqrt{\gamma \frac{p_{gas}+P_{rad}}{\rho}}} \right) \tag{2.2}$$

This is the maximum timestep at which the advection can take place. The diffusion of the radiation energy field on the other hand will generally happen much quicker, depending on the gradient of the E-field and absorbtion coeficient.

$$\partial_t (E) + \underbrace{\vec{\nabla} \cdot (\vec{v}E)}_{1} = -\underbrace{\vec{\nabla} \cdot \vec{F}}_{2} - \underbrace{\vec{\nabla} \cdot \vec{v}P + 4\pi\kappa\rho B - c\kappa\rho E}_{3} \tag{2.3}$$

$$\tag{2.4}$$

- **1: The advection term**

  $$\partial_t E + \vec{\nabla} \cdot (\vec{v}E) = 0 \tag{2.5}$$

  can be handeled using already exsisting amrvac routines, this part has a similar shape to the conservative form of the hd equations (1.5). This advection is the movement of the radiation field with the gas, its a proces evolving on a similar timescale as the advection of other quantities. Implementation is as trivial as defining the riemann fluxes and calling the necesary, already existing, subroutines.

- **2: The diffusion term**

  $$\partial_t E = -\vec{\nabla} \cdot \vec{F} \tag{2.6}$$

  can, by using the fld closing relation, be written as:

  $$\partial_t E = -\vec{\nabla} \cdot \left(-\frac{\lambda c}{\kappa\rho} \nabla E \right) \tag{2.7}$$

  Trying to solve this equation togheter with the other hydro equations gives rise to two problems. First: the timescale on which the radiation energry E evolves is much shorter than the hydrodynamical timescale on which ρ, $\vec{v}$ and e evolve, and second: this is a hyperbolic equation, which makes it hard to solve with the explicit schemes which are used by amrvac to solve the other equations. A way to overcome these problems is solving for the diffusion implicitly. The technique used in developping the new physics module is an Alternative Direction Implicit scheme (ADI), and is based on the Zeus-2D FLD module [**?**].

- **3: The photon tiring and the radiation heating and cooling sourceterms**

  $$\partial_t E = -\vec{\nabla} \cdot \vec{v}P + 4\pi\kappa\rho B - c\kappa\rho E \tag{2.8}$$

  are also evolving on the faster timescale, hence they must be solved implicitly. This is done with a scheme similar to the technique found in [**?**], which will be explained below.

2.3.1 Elliptic vs Hyperbolic

2.3.2 ADI

The diffusion part of the radiative energy equation wil be solved using the Alternating Direction Implicit (ADI) scheme. This is a numerical scheme which solves the equation implicitly in the first spatial direction an explicitly in the second for half a time step, and then implicitly in the second spatial direction and explicitly in the first for another half timestep. Lets first simplify the equation by defining the diffusion coefficient $D = \frac{\lambda c}{\kappa \rho}$.

$$\partial_t E = \vec{\nabla} \cdot (D \nabla E) \tag{2.9}$$

Lets now construct a nummerical scheme. First, the left hand side of (2.9) is writen in a discrete form. Let $E_{i,j}$ be the radiative energy in cell (i, j) on a grid with Δx grid spacing in the x-direction and Δy grid spacing in the y-direction. $D_{i,j}$ is the cell centered diffusion coefficient, whilst $D1_{i,j} = \frac{1}{2}(D_{i+1,j} + D_{i,j})$ and $D2_{i,j} = \frac{1}{2}(D_{i,j+1} + D_{i,j})$ are the cell faced values. To transform from cell center to cell face it is simplest to take the mean of the two surounding cells, taking the mean of the six sourrounding cells is also an option.

$$\begin{aligned}
\left(\vec{\nabla} \cdot (D \nabla E)\right)_{i,j} &= \frac{D1_{i+1,j}(\nabla E)_x - D1_{i,j}(\nabla E)_x}{\Delta x} && (2.10)\\
&+ \frac{D2_{i,j+1}(\nabla E)_y - D2_{i,j}(\nabla E)_y}{\Delta y} && (2.11)\\
&= \frac{D1_{i+1,j}(E_{i+1,j} - E_{i,j}) - D1_{i,j}(E_{i,j} - E_{i-1,j})}{\Delta x^2} && (2.12)\\
&+ \frac{D2_{i,j+1}(E_{i,j+j} - E_{i,j}) - D2_{i,j}(E_{i,j} - E_{i,j-1})}{\Delta y^2} && (2.13)\\
& && (2.14)
\end{aligned}$$

For the implicit scheme the diffusion coefficients are chosen at the previous timestep, as they are to be computed from a known radiation energy. Evolving (2.9) over a timestep Δt from time n to time $n + 1$ means solving the following equations:

$$\frac{E^{n+1} - E^n}{\Delta t} = \vec{\nabla} \cdot \left(D^n \nabla E^{n+1}\right) \tag{2.15}$$

As mentioned before, the ADI scheme first solves half a timestep implicitly in for example the x-direction and explicitly in the y-direction:

$$\begin{aligned}
\frac{E_{i,j}^{n+\frac{1}{2}} - E_{i,j}^n}{\Delta t} &= \frac{D1_{i+1,j}^n}{\Delta x^2}(E_{i+1,j}^{n+\frac{1}{2}} - E_{i,j}^{n+\frac{1}{2}}) && (2.16)\\
&- \frac{D1_{i,j}^n}{\Delta x^2}(E_{i,j}^{n+\frac{1}{2}} - E_{i-1,j}^{n+\frac{1}{2}}) && (2.17)\\
&+ \frac{D2_{i,j+1}^n}{\Delta y^2}(E_{i,j+j}^n - E_{i,j}^n) && (2.18)\\
&- \frac{D2_{i,j}^n}{\Delta y^2}(E_{i,j}^n - E_{i,j-1}^n) && (2.19)
\end{aligned}$$

In a more simple form, this can be written as:

$$\left(1 + \frac{\Delta t D1_{i+1,j}^n}{\Delta x^2} + \frac{\Delta t D1_{i,j}^n}{\Delta x^2}\right) E_{i,j}^{n+\frac{1}{2}} - \frac{\Delta t D1_{i+1,j}^n}{\Delta x^2} E_{i+1,j}^{n+\frac{1}{2}} - \frac{\Delta t D1_{i,j}^n}{\Delta x^2} E_{i-1,j}^{n+\frac{1}{2}} = b_{i,j} \tag{2.20}$$

where $b_{i,j} = \left(1 + \frac{\Delta t D2^n_{i,j+1}}{\Delta y^2} + \frac{\Delta t D2^n_{i,j}}{\Delta y^2}\right) E^n_{i,j} + \frac{\Delta t D2^n_{i,j+1}}{\Delta y^2} E^n_{i,j+1} + \frac{\Delta t D2^n_{i,j}}{\Delta y^2} E^n_{i,j-1}$ Or in matrix form:

$$\begin{bmatrix} \ddots & \ddots & & & \\ \ddots & \left(1 + \frac{\Delta t D1^n_{i,j}}{\Delta x^2} + \frac{\Delta t D1^n_{i-1,j}}{\Delta x^2}\right) & -\frac{\Delta t D1^n_{i,j}}{\Delta x^2} & & \\ & -\frac{\Delta t D1^n_{i,j}}{\Delta x^2} & \left(1 + \frac{\Delta t D1^n_{i+1,j}}{\Delta x^2} + \frac{\Delta t D1^n_{i,j}}{\Delta x^2}\right) & -\frac{\Delta t D1^n_{i,j}}{\Delta x^2} & \\ & & -\frac{\Delta t D1^n_{i+1,j}}{\Delta x^2} & \left(1 + \frac{\Delta t D1^n_{i+2,j}}{\Delta x^2} + \frac{\Delta t D1^n_{i+1,j}}{\Delta x^2}\right) & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \tag{2.21}$$

$$\times \begin{bmatrix} \vdots \\ E^{n+\frac{1}{2}}_{i-1,j} \\ E^{n+\frac{1}{2}}_{i,j} \\ E^{n+\frac{1}{2}}_{i+1,j} \\ \vdots \end{bmatrix} = \begin{bmatrix} \cdots & b_{i,j-1} & b_{i,j} & b_{i,j+1} & \cdots \end{bmatrix} \tag{2.22}$$

This is, depending on boundary conditions, a tridiagonal matrix. In amrvac, this matrix system is solved on the entire computational domain and the innermost layer of ghost-cells. The boundary conditions on the radiation energy field are applied after the halved timestep. This way, the matrix can be kept tridiagonal independent of those boundary conditions. The tridiagonal system is then solved using Thomas' algorithm, which is a simplified form of gaussian elimination. If the computational grid is N cells by M cells this is an $(N+2) \times (N+2)$ system, and there are $M+2$ such systems to be solved to evolve half a timestep.

The second half of the timestep will be solved implicit in the y-direction and explicit in x-direction. Notice again the diffusion coefficient can only be updated at the end of the cycle:

$$\frac{E^{n+1}_{i,j} - E^{n+\frac{1}{2}}_{i,j}}{\Delta t} = \frac{D1^n_{i+1,j}}{\Delta x^2}(E^{n+\frac{1}{2}}_{i+1,j} - E^{n+\frac{1}{2}}_{i,j}) \tag{2.23}$$

$$- \frac{D1^n_{i,j}}{\Delta x^2}(E^{n+\frac{1}{2}}_{i,j} - E^{n+\frac{1}{2}}_{i-1,j}) \tag{2.24}$$

$$+ \frac{D2^n_{i,j+1}}{\Delta y^2}(E^{n+1}_{i,j+j} - E^{n+1}_{i,j}) \tag{2.25}$$

$$- \frac{D2^n_{i,j}}{\Delta y^2}(E^{n+1}_{i,j} - E^{n+1}_{i,j-1}) \tag{2.26}$$

And this system, of course, has a matrix notation completely similar to the one in the first half of the timestep, which can be solved in a completely similar way. This time there are $N+2$ matrices of size $(M+2) \times (M+2)$ to be solved.

2.3.3 pseudo-timestepping

2.3.4 Bisection Implicit scheme

Solving for the sourceterms in the gas and radiation energy equations happens with another implicit scheme.

$$e^{n+1} - e^n = \Delta t \left(-4\kappa\sigma \left(\frac{(\gamma-1)e^{n+1}}{\rho} \right)^4 + c\kappa E^{n+1} \right) \tag{2.27}$$

$$E^{n+1} - E^n = \Delta t \left(+4\kappa\sigma \left(\frac{(\gamma-1)e^{n+1}}{\rho} \right)^4 - c\kappa E^{n+1} - \nabla \vec{v} P^{n+1} \right) \tag{2.28}$$

This can be rewritten as:

$$e^{n+1} - e^n = -a_1 \left(e^{n+1}\right)^4 + a_2 E^{n+1} \tag{2.29}$$

$$E^{n+1} - E^n = a_1 \left(e^{n+1}\right)^4 - a_2 E^{n+1} - a_3 E^{n+1} \tag{2.30}$$

where $a_1 = 4\kappa\sigma \left(\frac{(\gamma-1)}{\rho^{n+1}}\right)^4 \Delta t$, $a_2 = c\kappa\Delta t$ and $a_3 = \frac{\nabla \vec{v} P^{n+1}}{E^{n+1}} \Delta t$. Manipulation of the equations returns:

$$\left(e^{n+1}\right)^4 + \frac{1+a_2+a_3}{a_1+a_3} e^{n+1} - \frac{(1+a_2+a_3)e^n + a_2 E^n}{a_1+a_3} = 0 \tag{2.31}$$

$$E^{n+1} = \frac{a_1 \left(e^{n+1}\right)^4 + E^n}{1+a_2+a_3} \tag{2.32}$$

Equation (2.32) is a 4^{th} degree polynomial in e^{n+1}, with a single root between 0 and $\frac{1+a_2+a_3}{a_1+a_3}$. This is solved for e^{n+1} using the bisection method to calculate the contribution of the radiative heating and cooling to the gas energy. e^{n+1} Is then plugged in equation (**??**) to find the contribution of the radiative heating and cooling and the photon tiring to the radiation energy.

2.4 visualisation

Chapter 3

Results

3.1 CAK-Theory

qoiscaoziuoqioru,cozue,c,ozc,uuraco,

3.2 Flux Limited Diffusion

3.2.1 Testcase 1

3.2.2 Testcase 2

3.2.3 Isothermal Atmosphere

3.2.4 Strange mode instabilities

Chapter 4

Conclusions

qsdfqhofjqmoj

Chapter 5

Future Work

5.1 Alternative Implicit Schemes

5.2 MPI

5.3 AMR

5.4 Non-Isothermal atmospheres

5.5 Super Eddinton Limit

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