# MAA042 Numerical Methods with Matlab
# Lab Assignment

Deadline: October 23, 2023

- This lab consists of five exercises.
- The lab work can be done individually or in groups of maximum three students. You were supposed to form your group and inform the instructor by September 6, 2023.
- Each group must submit one final report. However, everyone in the group must verbally be able to account for the entire report.
- In the final lab report you must show all your work and provide supporting arguments. No credit will be given for answers without explanations. A good report shall include required theories, your approaches, results, discussion and analysis of the results.
- You can write your report in Latex or MS Word. Templates are available on the Canvas page of the course.
- You have to include all your MATLAB functions and scripts in the final report. Regardless of whether you choose to write in Latex or MS Word, all MATLAB codes should be written by yourself and not included as images from a print screen or similar.
- You have to write the report in such a way that all your work is readable and understandable from the text without looking into your MATLAB codes.
- All parts of the assignment must be completed for passing grade.
- The deadline for submission of lab report is October 23, 2023. This will be followed by an oral exam for each group at the Zoom meetings which will take place on October 26-27, 2023 (exact schedule will be announced later through Canvas). Please note that participation in the oral exam is mandatory.
- There is no late submission: if no report is submitted before the deadline on October 23, 2023, then you will receive a fail (U) grade.

# 1 Numerical Differentiation

The second order central difference formula approximating $f'''(x_0)$ is given by

$$(D_0^2 f)(x_0; h) = \frac{f(x_0 + 2h) - 2f(x_0 + h) + 2f(x_0 - h) - f(x_0 - 2h)}{2h^3}. \tag{1}$$

(a) Show that the truncation error in the approximation (1) is $O(h^2)$.

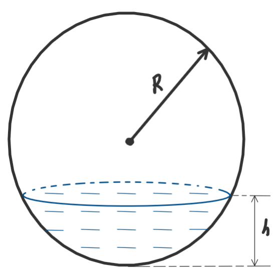
(b) Consider the function $f(x) = x^x$, which has the following third derivative

$$f'''(x) = x^x \left( (1 + \ln x)^3 + \frac{3}{x}(1 + \ln x) - \frac{1}{x^2} \right).$$

Approximate $f'''(1.5)$ by using the central difference formula (1) with $h = h_k = 2^{-k},\ k = 1, 2, \ldots, 15$. Calculate the absolute error in the approximation for each stepsize $h_k = 2^{-k},\ k = 1, 2, \ldots, 15$ and then plot the errors against the stepsizes $\{h_k\}_{k=1}^{15}$ in log-log scale. What do you observe from the plot? Does this confirm the theory for order of convergence[1]. If not, then explain why[2].

# 2 Solving Nonlinear Algebraic Equation

The spherical ball of radius $R$ is constructed from a white oak with a density $\rho$. The ball is submerged to a depth $h$ in water.




To find $h$, one needs to solve the following algebraic equation[3]

$$h^3 - 3Rh^2 + 4R^3\rho = 0.$$

Consider the problem of finding the depth $h$ if $R = 15$ cm and $\rho = 0.71$. It results in a nonlinear algebraic equation with the root in the interval $[0, 30]$.

(a) Verify graphically in MATLAB that the resulting equation has a unique root in the interval $[0, 30]$.

[1] Hint: plotting in the same frame the graph of $y = h^2$ may give you an idea about the order of convergence.

[2] Hint: make the analysis of round-off & truncation errors, see slide 10 of Lecture 4.

[3] For derivation see pp. 40-41 in the textbook.

(b) Write the code implementing the Bisection method and use it to approximate the depth $h$ within an accuracy of $10^{-15}$. How many Bisection iterations are required to achieve the approximation within an accuracy of $10^{-15}$? Store this numerical result. It can be considered as an exact (reference) solution.

(c) Write the code implementing the Newton-Raphson method and use it to approximate the depth $h$ within a relative precision of $10^{-15}$. As an initial guess take the middle point of the interval $[0, 30]$. How many Newton-Raphson steps are then needed to obtain the result? Find the absolute error of your numerical result[4].

(d) Write the code implementing the Secant method and use it to approximate the depth $h$ within a relative precision of $10^{-15}$. As the initial guesses take the endpoints of the interval $[0, 30]$. How many steps does then the Secant method need to obtain the result? Find the absolute error of your numerical result[4].

(e) Compare the computational costs of three methods used for the given root-finding problem.

# 3 Curve Fitting

A periodic signal is given by:

$$y = A\cos(2\pi x) + B\sin(3\pi x),$$

where $A$ and $B$ are constants. Data over one period is given in the following table:

| $x_k$ | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 1.1 | 1.3 | 1.5 | 1.7 | 1.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $y_k$ | 2.04 | 0.07 | $-2.52$ | 0.07 | 2.04 | $-0.18$ | $-0.78$ | 0.22 | $-0.78$ | $-0.18$ |

(a) Derive the normal equations of the least squares curve for given data.

(b) Write MATLAB script that solves the resulting normal equations to find the values of constants $A$ and $B$.

(c) Plot the resulting least squares curve over one period $[0, 2]$. In the same plot show the data points with markers.

[4] Hint: use the numerical result obtained in part (b) as a reference solution.

## 4 Numerical Integration

Consider the following integral $\int_0^1 \frac{e^x \sin x}{1+x^2} dx.$

(a) Use build-in MATLAB function **integral** to approximate the given integral. Store the obtained result, so that it can be used as an exact (reference) solution.

(b) Approximate the given integral by using Midpoint and Simpson's methods with the stepsizes $h = h_k = 2^{-k}$, $k = 1, 2, \ldots, 15$. Using the result obtained in part (a) as a reference solution, calculate the absolute errors in approximations for each stepsize $h_k$. Plot in one frame the errors against the stepsizes $\{h_k\}_{k=1}^{15}$ in log-log scale. What do you observe from the plots? Does this confirm the theory for order of convergence? If not, then explain why.

## 5 Numerical Solution of Differential Equation

Consider the following initial value problem (IVP)

$$\begin{cases} y' + 10y = 30 - 18e^{-t}, & t > 0 \\ y(0) = 0.3 \end{cases}$$

The exact solution to this IVP is $y(t) = 3 - 2e^{-t} - 0.7e^{-10t}$, $t \geq 0$.

(a) Use Matlab ODE solver *ode45* to calculate the numerical solution of given IVP on interval $[0, 10]$. Plot the numerical solution of $y(t)$ against time. Use the exact solution, to find an error in the numerical solution.

(b) Write the code implementing the fourth order Runge-Kutta method (*RK4*) given by the following Butcher table:

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

Use *RK4* method with stepsizes $\tau = \tau_k = 10^{-k}$, $k = 1, 2, 3, 4$ to find the numerical solution of given IVP on interval $[0, 10]$. Use the exact solution, to find errors in numerical solutions for each stepsize $\tau_k = 10^{-k}$, $k = 1, 2, 3, 4$ and then plot the errors against stepsizes in log-log scale. What do you observe from the plots? Does this confirm the theory for order of convergence?