Physica Scripta. Vol. 45, 497–503, 1992.

Probe Theory – The Orbital Motion Approach

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Received June 21, 1991; accepted December 16, 1991

Abstract

A review is given of the orbital motion limited (O.M.L.) theory of cylindrical and spherical Langmuir probes. In many cases the O.M.L. theory is invalid and the second orbital motion theory presented here, that of Bohm, Burhop and Massey, shows that the concept of an absorption radius must be introduced. The extensive numerical extensions to this theory are briefly discussed.

Recent experimental work is referred to in which measurements of ion currents to cylindrical probes showed better agreement with the ABR (radial-motion) theory than with the calculations of Laframboise. It is suggested that the mean free path must be greater than the probe radius by a large factor if the calculations of the Laframboise are to apply; the numerical value depends on the ratio of probe potential to ion temperature.

1. Introduction

The original orbital motion of probes, due to Mott-Smith and Langmuir [1] considered particle orbits within the space charge sheath surrounding a spherical or cylindrical probe. The plasma outside the sheath was assumed to be perfectly neutral, an assumption which was without foundation. In these lectures the subject is introduced without the concept of a boundary separating plasma and sheath regions. The current-voltage characteristics are calculated for both spherical and cylindrical probes, assuming that the current is limited by orbital motion. Electron currents are calculated for attracting (positive) probes and repelling (negative) probes. Similar expressions apply for the positive ion currents. In addition it is shown that the distribution of electron energies, when it differs from the Maxwellian distribution, can be determined from the second derivative of the current w.r.t. the voltage.

In many cases the above orbit motion limited (O.M.L.) theory is inapplicable because it contains the *implicit* assumption that some particles (of every energy range) graze the probe surface. It is often the case that an absorption radius exists, outside the probe, which in a sense replaces the probe radius. Particles which cross this radius are destined to hit the probe and be collected. The theory of Bohm, Burhop and Massey [2] is described below because it was a seminal paper, is instructive to read, but is apparently not well-known. These authors considered monoenergetic ions with an isotropic distribution of velocities. They also restricted their attention to the "Plasma Solution", i.e. the case where a plasma fills essentially all the space surrounding the probe, although it is quasi-neutral and not an equipotential region.

The theory was extended by Bernstein and Rabinowitz [3] who considered the full Poisson equation. It was later further extended by Laframboise [4] who included a Maxwellian distribution of attracted particles. The extensive numerical work carried out by Laframboise is only briefly referred to here because the basic physics is largely contained in the Bohm, Burhop and Massey theory.

The final part of this paper, which hitherto has been tutorial in nature, refers to some recent experimental work with cylindrical probes in which the measurements of ion currents showed much better agreement with the simple ABR radial motion theory [4] than with the more sophisticated calculations of Laframboise. It is well-known that the two theories give different results for cylindrical probes in the case where $T_i \to 0$. This has been referred to as the cold-ion paradox. The theories agree for a spherical probe as $T_i \to 0$. A proposed explanation of the results is given in terms of collisions. It is suggested that the angular momentum inherent in the Laframboise theory (even for $T_i \to 0$) is lost in collisions unless the mean free path of the ions is greater than $r_p(e|V_p|kT_i)^{1/2}$ where r_p, V_p are the probe radius and potential respectively and T_i is the ion temperature.

2. Attracting potentials: The orbital motion limited theory

Let us consider the collection of electrons by an attracting probe of cylindrical geometry i.e. one at a positive potential with respect to the surrounding plasma. If the length of the probe is large compared with its radius then the electrons move in a central field of force. Conservation of energy then gives

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mv_p^2 - eV_p \quad (1)$$

where v_p is the velocity at the probe surface. Conservation of angular momentum gives, for an electron at grazing incidence,

$$mvh_p = mr_p v_p \quad (2)$$

where h is the impact parameter (see Fig. 1). Hence

$$h_p = r_p(1 + V_p/V_0)^{1/2} \quad (3)$$

where the initial energy of the electron is eV_0. The expression given by equation (3) represents an effective radius of

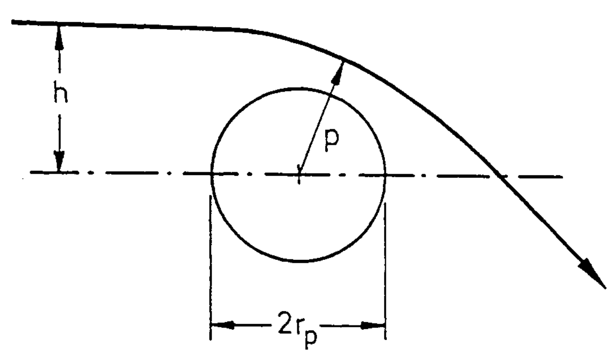

Fig. 1. Diagram illustrating the impact parameter h and the distance of closest approach p

the probe. The contribution to the current due to electrons with a narrow velocity range will be given by

$$dI = 2\pi r_p le(1 + V_p/V_0)^{1/2}(v/\pi)\, dn \tag{4}$$

where the quantity $(v/\pi)\, dn$ represents the flux crossing unit area considering electrons which move in planes perpendicular to the probe axis. We are not interested in velocity components parallel to the axis.

If we consider a Maxwellian distribution of velocities then the two-dimensional version of the distribution takes the form

$$dn = n_0\left(\frac{m}{2\pi kT}\right)e^{-mv^2/2kT} 2\pi v\, dv \tag{5}$$

so that

$$dI = \frac{2n_0 r_p lme}{kT} v^2 e^{-mv^2/2kT}(1 + V_p/V_0)^{1/2}\, dv \tag{6}$$

which can be written as

$$dI = 4n_0 r_p le\left(\frac{2kT}{m}\right)^{1/2} xe^{-x^2}(x^2 + a^2)^{1/2}\, dx \tag{7}$$

where $x^2 = mv^2/2kT$ and $a^2 = eV_p/kT$.

The total current is now obtained by integration, so that

$$I = 4n_0 r_p le\left(\frac{2kT}{M}\right)^{1/2} \int_0^\infty xe^{-x^2}(x^2 + a^2)^{1/2}\, dx \tag{8}$$

The integral can be readily evaluated* to give

$$\int f(x)\, dx = \frac{\sqrt{\pi}}{4}\left(\frac{2\eta}{\sqrt{\pi}} + e^\eta \operatorname{erfc}(\sqrt{\eta})\right) \tag{9}$$

The final expression for the current is now given by

$$I = 2\pi n_0 r_p le\left(\frac{kT}{2\pi m}\right)^{1/2}\left(\frac{2\sqrt{\eta}}{\sqrt{\pi}} + e^\eta \operatorname{erfc}\sqrt{\eta}\right) \tag{10}$$

where $\eta = eV_p/kT$.

Fig. 2 shows a plot of $[(2\sqrt{\eta}/\sqrt{\pi}) + e^\eta \operatorname{erfc}\sqrt{\eta}]$ together with a plot of $2(1 + \eta)^{1/2}/\sqrt{\pi}$.

It is seen that the curves are indistinguishable for values of η greater than 2. The expression for the current can therefore be written in the form

$$I = 2\pi n_0 r_p le\left(\frac{kT}{2\pi m}\right)^{1/2} \frac{2}{\sqrt{\pi}}\left(1 + \frac{eV_p}{kT}\right)^{1/2} \tag{11}$$

when $eV_p/kT \geqslant 2$. A plot of I^2 versus V_p should therefore be a straight line. The slope of the curve yields n_0 and the intercept on the current axis will then give T_e, the value of n_0 being known.

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* Determination of the integral $\int_0^\infty xe^{-x^2}(x^2 + a^2)^{1/2}\, dx$, let $x^2 + a^2 + u^2$, then $2x\, dx = 2u\, du$

$$\int_a^\infty ue^{-u^2+a^2}\, du = [-\tfrac{1}{2}ue^{-u^2+a^2}]_a^\infty + \frac{1}{2}\int_a^\infty e^{-u^2+a^2}\, du$$

$$= \frac{a}{2} + \frac{\sqrt{\pi}}{4} \operatorname{erfc}(a)e^{a^2}$$

$$= \frac{\sqrt{\pi}}{4}\left[\frac{2a}{\sqrt{\pi}} + e^{a^2} \operatorname{erfc}(a)\right]$$

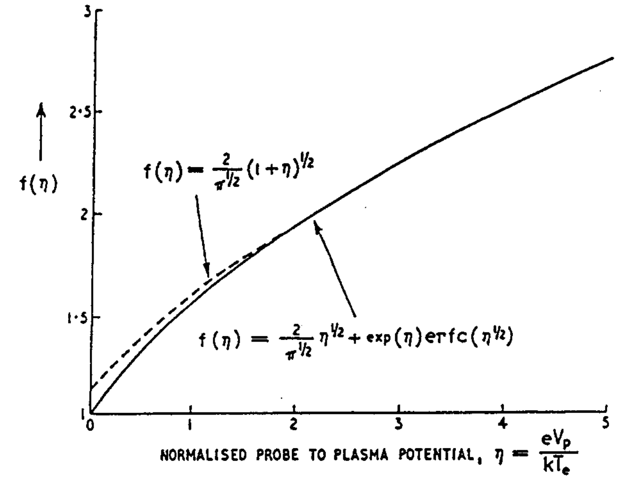

Fig. 2. A plot of the functional form of eq. (10) together with a plot of the function $2(1 + \eta)^{1/2}/\pi^{1/2}$. It is seen that they are indistinguishable for $\eta > 2$

The corresponding expression for a spherical probe is found to be

$$I = 4\pi r_p^2 n_0 e(kT/\pi m)^{1/2}(1 + V_p/V_0) \tag{12}$$

It is clear that the corresponding equations for positive ion collection are

$$I = 2\pi n_0 r_p le(kT_i/2\pi M)^{1/2} \frac{2}{\sqrt{\pi}}\left(1 - \frac{eV_p}{kT_i}\right)^{1/2} \tag{13}$$

and

$$I = 4\pi r_p^2 n_0 e(kT_i/2\pi M)^{1/2}(1 - eV_p/kT_i) \tag{14}$$

for the cylindrical and the spherical probe respectively.

These equations were first derived by Mott-Smith and Langmuir [1] for the case of a large "sheath". The approach used here is that of Lea and Allen [5].

3. Retarding potentials

3.1. *Maxwellian velocity distribution*

In the case of a retarding potential eq. (8) is replaced by

$$I = 4n_0 r_p le\left(\frac{2kT}{m}\right)^{1/2} \int_a^\infty xe^{-x^2}(x^2 - a^2)^{1/2}\, dx \tag{15}$$

The effective radius of the probe is less than r_p and only those electrons with an energy greater than $(-eV_p)$ can reach the probe. In the above expression $a^2 = -eV_p/kT_e$.

Let $(x^2 - a^2) = u^2$, then $2x\, dx = 2u\, du$, and the above equation can be written in the following form.

$$I = 4n_0 r_p le(2kT_e/m)^{1/2} \int_a^\infty u^2 e^{-u^2-a^2}\, du \tag{16}$$

The integral can be readily evaluated to give

$$I = 2\pi r_p n_0 le(kT_e/2\pi m)^{1/2} e^{eV_p/kT_e}. \tag{17}$$

This is the classical result obtained in a different way by Langmuir. Note that we have not shown that the electron density is reduced by a factor of $\exp(eV_p/kT_e)$. Our calculation refers to the current collected by the probe. Langmuir's approach has been described in other lectures [6].

3.2. *Determination of the velocity distribution function*

Let us consider a spherical probe. The current collected can be written as

$$I = 4\pi r_p^2 e \int_{-eV_p}^{\infty} (1 + eV_p/E)\phi(E)\sqrt{\frac{2E}{m}}\frac{dE}{4} \tag{18}$$

where the energy distribution $\phi(E)$ is arbitrary. We shall now differentiate twice with respect to V_p, recalling Liebnitz's rule, i.e. if

$$I(\alpha) = \int_a^b f(x, \alpha)\, dx$$

then

$$\frac{dI}{d\alpha} = f(b, \alpha)\frac{db}{d\alpha} - f(a, \alpha)\frac{da}{d\alpha} + \int_a^b \frac{\partial f(x, \alpha)}{\partial x}\, dx \tag{19}$$

so that

$$\frac{d^2 I}{dV_p^2} = 0 - 0 + \int_{-eV_p}^{\infty} \frac{e^2 \pi r_p^2}{E}\phi(E)\left(\frac{2E}{m}\right)^{1/2} dE \tag{20}$$

and

$$\frac{d^2 I}{dV_p^2} = -e^2 \pi r_p^2 \phi(E)\left(\frac{2e}{-mV_p}\right)^{1/2} \tag{21}$$

Thus the distribution of electron energies can be found from the second differential of the current with respect to the voltage:

$$\phi(E) = -\frac{1}{e\pi r_p^2}\left(\frac{-mV_p}{2e}\right)^{1/2}\frac{d^2 I}{dV_p^2} \tag{22}$$

This is known as the Druyvesteyn method [7]. It can be shown to be valid for any shape of probe (excluding reentrant areas), but the velocity distribution function must be spherically symmetrical. A simple derivation due to Kagan and Perel [8] is reported in the textbook by Swift and Schwar [9].

4. The concept of the absorption radius

4.1. *Limitations of the O.M.L. theory*

The theory described in the above paragraphs contains an implicit assumption. It was assumed, when considering an attracting probe, that some of the particles hit the probe at grazing incidence. This may not be the case. An absorption radius might exist, outside the probe, such that particles which cross it are destined to hit the probe.

Let us consider the case of ion collection, and focus our attention on ions within a narrow energy range. If all imaginary cylindrical (or spherical) surfaces outside the probe are 'grazed' by ions then the corresponding impact parameters must all be greater than h_p, i.e.

$$r\left(1 - \frac{V}{V_0}\right)^{1/2} > r_p\left(1 - \frac{V_p}{V_0}\right)^{1/2} \tag{23}$$

Rearranging, the condition for no absorption radius to exist can be written

$$\frac{V_0 - V}{V_0 - V_p} > \left(\frac{r_p}{r}\right)^2 \tag{24}$$

If this condition is to hold for all of the ions, including those with small initial energies then

$$\frac{V}{V_p} > \left(\frac{r_p}{r}\right)^2 \tag{25}$$

In a dense plasma this condition does not hold. We then have an absorption radius (for each ion energy) which in effect replaces the probe radius. Whether or not the inequality holds in a particular case is difficult to say. We need to solve Poisson's equation.

4.2. *The theory of Bohm, Burhop and Massey*

We shall now discuss the theory for spherical probes developed by Bohm, Burhop and Massey [2] in which the ions have a random motion in space, in the unperturbed plasma, but they are still considered to be mono-energetic. We shall now have to consider the orbital motion of the ions as illustrated in Fig. 1. The orbit is in the plasma field and *not* in the sheath as in the historical paper of Mott-Smith and Langmuir [1]. At any point of the orbit the kinetic energy of the ion is $e(V_0 - V)$ where eV_0 is the initial energy of the ions; the potential is taken to be zero far away from the probe. At the distance of closest approach p the principle of conservation of angular momentum states that

$$hV_0^{1/2} = p(V_0 - V)^{1/2} \tag{26}$$

which is a relation between p and the impact parameter h. We must now consider the case illustrated in Fig. 3 where the impact parameter h has a minimum value h_m for some value of p, denoted by r_A. Ions with an impact parameter greater than h_m will perform an orbit around the probe and then leave the perturbed region. On the other hand ions with smaller impact parameters have no distance of closest approach and they will be accelerated towards the probe and be captured. The minimum value h_m is therefore the radius of the effective target area of the probe. Figure 4 illustrates typical orbits. The ion current collected by the probe will be the product of the "random" current density and the

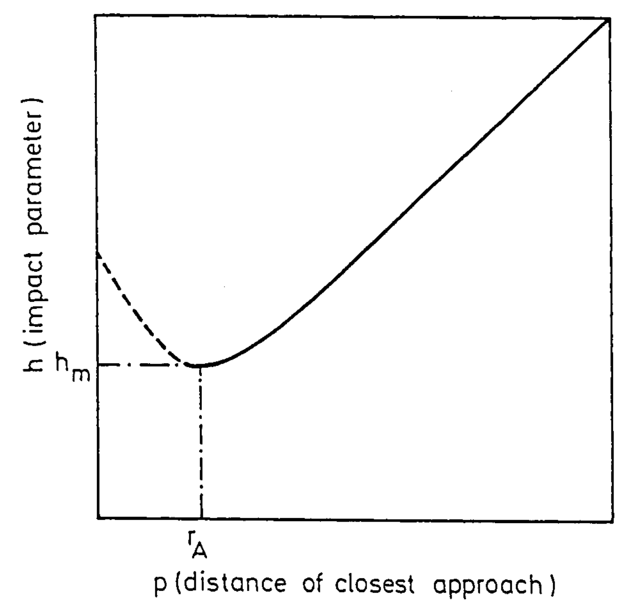

Fig. 3. The function $h = p(1 - V/V_0)^{1/2}$ showing the case where there is a minimum impact parameter for ions which describe an orbit with a distance of closest approach p

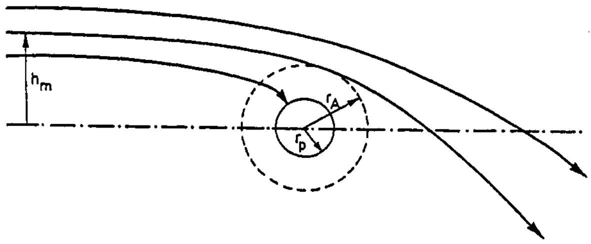

Fig. 4. Diagram showing the orbits of ions which are deflected by the electric field but not captured by the probe, together with the orbit of an ion which hits the probe. Ions with an impact parameter h_m graze the (mathematical) surface of radius r_A

area of the effective target presented by the probe, these are $n_0 e(2eV_0/M)^{1/2}/4$ and $4\pi h_m^2$ respectively so that

$$I = n_0 e(2eV_0/M)^{1/2}\pi h_m^2 \tag{27}$$

or

$$I = n_0 e(2eV_0/M)^{1/2}\pi r_A^2[1 - V(r_A)/V_0] \tag{28}$$

using equation (1) to obtain an expression for h_m.

It is now necessary to obtain an expression for the ion current in terms of the *local* ion density. Some care is required here because the expression depends on whether $r > r_A$ or $r < r_A$. When we consider $r < r_A$ we shall see ions moving towards the probe, considering their radial velocity components. On the other hand at radii greater than r_A there will be a two-way traffic because some ions are deflected by the electric field, but not captured by the probe, i.e. they do not hit the absorption surface (radius r_A) shown in Fig. 4.

The principle of conservation of angular momentum states in general that

$$h(2eV_0/M)^{1/2} = rv_t \tag{29}$$

and conservation of energy states that

$$\tfrac{1}{2}Mv_r^2 + \tfrac{1}{2}Mv_t^2 = eV_0 - eV \tag{30}$$

Where v_r and v_t are the radial the transverse components of velocity and V is negative as usual. Combining (29) and (30) gives

$$h = r\left(1 - \frac{V}{V_0} - \frac{mv_r^2}{2eV_0}\right)^{1/2} \tag{31}$$

which can be substituted into equation (27) to give

$$I = n_0 e(2eV_0/M)^{1/2}\pi r^2\left(1 - \frac{V}{V_0} - \frac{Mv_r^2}{2eV_0}\right) \tag{32}$$

Differentiation with respect to v_r at a fixed $V(r)$ gives

$$dI = n_0 e(2M/eV_0)^{1/2}\pi r^2 v_r\, dv_r$$

where we have dropped a negative sign. This increment of current can be divided by $4\pi r^2 ev_r$ to give

$$dn_i = \frac{n_0}{4}\left(\frac{2M}{eV_0}\right)^{1/2} dv_r \tag{33}$$

i.e. our velocity distribution function is constant, between certain velocity limits to be discussed.

The maximum radial velocity relates to a particle colliding head-on, with no tangential velocity, viz.

$$v_{max} = \left(\frac{2eV_0}{M}\right)^{1/2}\left(1 - \frac{V}{V_0}\right)^{1/2} \tag{34}$$

The critical radial velocity, i.e. that of the ions which just reach the surface of absorption, corresponds to $h = h_m$, and eq. (31) yields

$$v_{crit} = \left(\frac{2eV_0}{M}\right)^{1/2}\left[\left(1 - \frac{V}{V_0}\right) - \frac{r_A^2}{r^2}\left(1 - \frac{V(r_A)}{V_0}\right)\right]^{1/2} \tag{35}$$

Let us first consider radii greater than r_A. Integration of eq. (33) gives

$$n_i = \frac{n_0}{4}\left(\frac{2M}{eV_0}\right)^{1/2}\left(2\int_0^{v_c} dv_r + \int_{v_c}^{v_m} dv_r\right) \tag{36}$$

where the first integral refers to the particles which are not captured by the probe. The second integral refers to those which are destined to hit the probe. Thus

$$n_i = \frac{n_0}{4}\left(\frac{2M}{eV_0}\right)^{1/2}(v_c + v_m)$$

or

$$n_i = \frac{n_0}{2}\left\{\left(1 - \frac{V}{V_0}\right)^{1/2} + \left[\left(1 - \frac{V}{V_0}\right) - \frac{r_A^2}{r^2}\left(1 - \frac{V(r_A)}{V_0}\right)\right]^{1/2}\right\} \tag{37}$$

for $r > r_A$. Within the radius of absorption eq. (36) is replaced by

$$n_i = \frac{n_0}{4}\left(\frac{2M}{eV_0}\right)^{1/2}\int_{v_c}^{v_m} dv_r \tag{38}$$

because we have only one group of particles to deal with, i.e. those moving towards the probe, thus

$$n_i = \frac{n_0}{4}\left(\frac{2M}{eV_0}\right)^{1/2}(v_m - v_c)$$

or

$$n_i = \frac{n_0}{2}\left\{\left(1 - \frac{V}{V_0}\right)^{1/2} - \left[\left(1 - \frac{V}{V_0}\right) - \frac{r_A^2}{r^2}\left(1 - \frac{V(r_A)}{V_0}\right)\right]^{1/2}\right\} \tag{39}$$

for $r < r_A$. In both cases, outside and inside r_A, we can write

$$\left(1 - \frac{V(r_A)}{V_0}\right)\frac{r_A^2}{r^2} = 4e^{eV/kT_e}\left[\left(1 - \frac{V}{V_0}\right)^{1/2} - e^{eV/kT_e}\right] \tag{40}$$

where we have used (37) and (39) together with the plasma condition that $n_i = n_e = n_0 \exp(eV/kT_e)$. At $r = r_A$ eq. (40) reduces to

$$\left(1 - \frac{V(r_A)}{V_0}\right)^{1/2} = 2e^{eV(r_A)/kT_e} \tag{41}$$

which enables us to calculate $V(r_A)$ for a given ion energy eV_0.

The values of V_A corresponding to $eV_0/kTe = 0.01$ and $eV_0/kT_e = 0.5$ are $-2.8V_0$ and $-0.79V_0$ respectively. It is interesting to note that $V_A \to -3V_0$ as $eV_0/kT_e \to 0$.

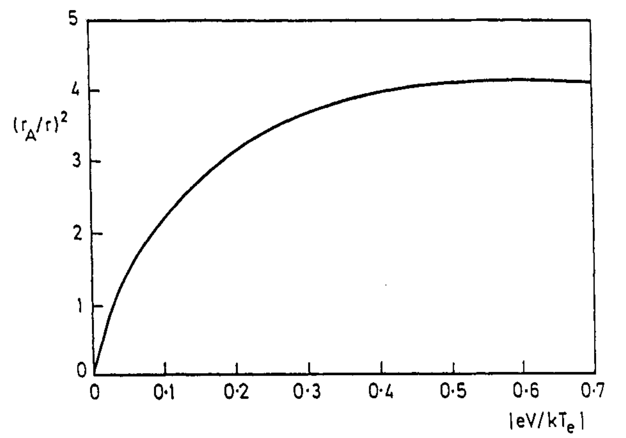

Fig. 5. The variation of $(r_A/r)^2$ with $|eV/kT_e|$ for the case where $eV_0/kT_e = 0.01$ (from eq. 40)

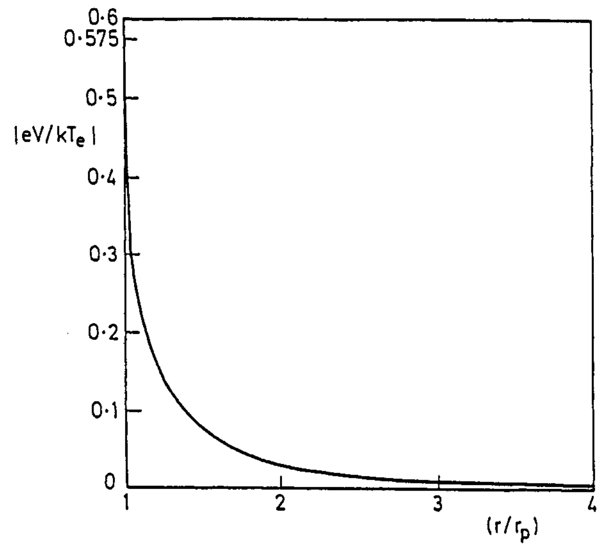

Fig. 7. The potential distribution near a spherical probe for $eV_0/kT_e = 0.01$; the modulus of eV/kT_e is shown, V being negative. The calculation is for the thin sheath case

The plasma solution is obtained by solving eq. (40) subject to the boundary condition first used by Tonks and Langmuir [10] that $dV/dr \rightarrow \infty$ at the plasma boundary, i.e. at $r = r_p$ radius of the probe. This can be done in the following manner. Curves of $(r_A/r)^2$ against eV/kT_e can be plotted as shown in Figs 5 and 6 for the two values of eV_0/kT_e. These curves have maxima when $dr/dV = 0$ or $dV/dr = \infty$ so that the corresponding values of $(r_A/r)^2$ yield the values of $(r_A/r_p)^2$. The numerical results are $(r_A/r_p)^2 = 4.2$ for $eV_0/kT_e = 0.01$ and $(r_A/r_p)^2 = 1.17$ for $eV_0/kT_e = 0.5$. Equation (40) can now be plotted and the results are shown in Figs 7 and 8.

The ion currents are readily computed from eq. (28) and are

$$I = 0.57 n_0 e(kT_e/M)^{1/2} A \tag{42}$$

and

$$I = 0.54 n_0 e(kT_e/M)^{1/2} A \tag{43}$$

for $eV_0/kT_e = 0.01$ and 0.5 respectively. Thus the ion current is very insensitive to variations in the energy of the ions. It does in fact *decrease* slightly with increasing "ion temperature".

It is of interest to calculate the quantity $\langle 1/v_r^2 \rangle$ at the plasma boundary.

Using eq. (33) this is seen to be given by

$$\left\langle \frac{1}{v_r^2} \right\rangle = \int_{v_c}^{v_m} \frac{dv}{v_r^2} \Big/ \int_{v_c}^{v_m} dv_r = \frac{1}{v_c v_m} \tag{44}$$

where we have to substitute for the relevant values of v_c and v_m i.e. those at $r = r_p$. Thus

$$\left\langle \frac{1}{v_r^2} \right\rangle^{-1} = [v_c][v_m] \tag{45}$$

where the square brackets denote the values of v_c and v_m, given by eqs (35) and (34) respectively, evaluated at $r = r_p$. Differentiation of eq. (39) with respect to V gives dn_i/dV in the absorption region and putting $dV/dr \rightarrow \infty$ to obtain

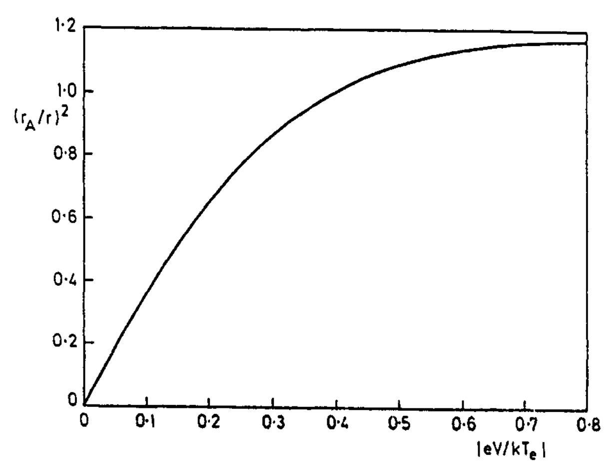

Fig. 6. The variation of $(r_A/r)^2$ with $|eV/kT_e|$ for the case where $eV_0/kT_e = 0.5$

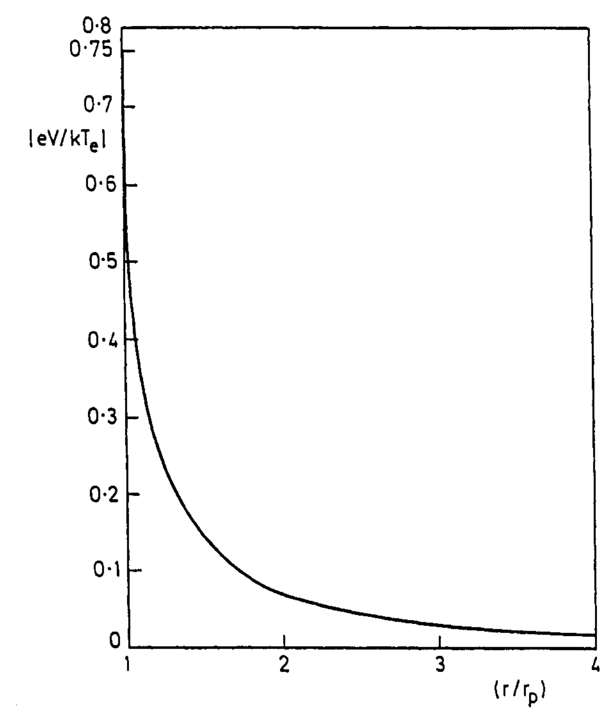

Fig. 8. The potential distribution near a spherical probe for $eV_0/kT_e = 0.5$; the modulus of eV/kT_e is shown, V being negative. The calculation is for the thin sheath case

dn_i/dV at the plasma boundary one obtains

$$\frac{dn_i}{dV} = \frac{n_0}{4V_0}\left\{\frac{[v_m]-[v_c]}{[v_m][v_c]}\right\}\left(\frac{2eV_0}{M}\right)^{1/2} \tag{46}$$

or

$$\frac{dn_i}{dV} = \frac{n_i e}{M[v_m][v_c]} \tag{47}$$

but in the plasma $n_i = n_e = n_0 \exp(eV/kT_e)$ so that the L.H.S. is equal to $n_i e/kT_e$ and therefore

$$M[v_m][v_c] = kT_e$$

use of eq. (45) then gives the result

$$\tfrac{1}{2}M\left\langle\frac{1}{v_r^2}\right\rangle^{-1} = \tfrac{1}{2}kT_e \tag{48}$$

which is a generalized form of the Bohm criterion [11]. Thus the different results obtained for different values of (eV_0/kT_e) do coalesce when expressed in this particular form.

We shall now verify an assumption made in the ABR theory (Allen, Boyd and Reynolds, [12]) namely that the velocities of initially *cold* ions are directed towards the centre of the probe. It has already been pointed out that the potential at the absorption radius tends to $-3V_0$ as $eV_0/kT_e \to 0$. It is also evident from eq. (28) that r_A tends to infinity at the same time. Now eq. (40) simplifies at potentials which are numerically much greater than V_0 to give

$$\left(1 - \frac{V(r_A)}{V_0}\right)r_A^2 = 4r^2 e^{eV/kT} e(-V/V_0)^{1/2} \tag{49}$$

which on multiplication by n_0 becomes

$$n_0\left(1 - \frac{V(r_A)}{V_0}\right)r_A^2 = 4nr^2(-V/V_0)^{1/2} \tag{50}$$

remembering that $n = n_0 \exp(eV/kT_e)$. Comparison of eqs (50) and (28) gives the result

$$I = 4\pi r^2 ne(-2eV/M)^{1/2} \tag{51}$$

which shows that the ions are moving radially towards the probe, thus justifying the procedure adopted in the ABR theory. Analogous calculations for the cylindrical probe give a different result, the ions are found to have appreciable tangential velocities in the cylindrical case.

The ABR (radial motion) theory for cold ions gives the following result, for the thin sheath case,

$$I = 0.61 n_0 e(kT_e/M)^{1/2} A \tag{52}$$

Comparison of eqn (42), (43) and (52) shows that the ion current is *very insensitive* to the random ion energy so that it is hardly worth while introducing the velocity distribution of the ions. In some cases measurements of ion current may be used, instead of measurement of electron current, to determine the electron density in a plasma. The ion current is much less than the saturation electron current, so that the plasma is perturbed to a lesser extent and less heat is dissipated at the probe. The electron temperature is required and can be determined from the lower part of the semilogarithmic plot of the electron current if not known from other measurements. It is interesting to note that the "plane" probe which is commonly used will actually have a field pattern, in the plasma, which is roughly hemispherical. Another point which is worth making is that Langmuir and Mott-Smith [13] found that comparable ion currents were received, whichever way the probe was facing. The conclusion reached was that the ions had a random motion whereas we now see that field penetration was responsible. Figures 7 and 8 show that the field penetrates the plasma for about 3 or 4 times the probe radius.

5. Numerical solutions of Poisson's equation

The last section described solutions obtained using the quasi-neutral plasma equation. When the space charge sheath is thin this method is very useful. If the sheath is not thin it is usually best to abandon the model which considers separate plasma and sheath regions. In principle one simply solves the space-charge equation*

$$\nabla^2 V = -\frac{(n_1 - n_e)e}{\varepsilon_0} \tag{53}$$

The expressions previously found for the ion density, both within and outside the absorption radius, can be used again, together with the Boltzmann relation for the electron density. Computations based on this model, i.e. for mono-energetic ions, were carried out by Bernstein and Rabinowitz [3] for both the spherical and the cylindrical probe. These calculations were extended by Chen [14], who also discovered a number of numerical misprints in the original paper.

Laframboise [4] further extended the theory by including a Maxwellian distribution of the *attracted* particles. The results of these calculations are widely used and the report issued by the University of Toronto is to be found in many laboratories where Langmuir probes are employed. Analytical formulae which fit Laframboise's numerical results, to within about 3 per cent have been given by Kiel and by Peterson and Talbot [17].

6. Practical limitations of the Orbital Motion Theory

Mention will be made here of some experimental work with cylindrical probes reported by Allen, Annaratone and Allen [18]. A more detailed paper by Annaratone *et al.* is in the press [19]. In experiments with R.F. discharges the probe results (for ion collection) were found to be in good agreement with the ABR theory, rather than the orbital motion theory discussed above. The ABR theory assumes radial motion only. It was developed by Allen, Boyd and Reynolds [12] and extended by Chen [14].

Comparison between both theories and the experimental results was made using a Sonin plot [20, 21, 22]. In the plot shown in Fig. 9 the ion current is normalized in the following way

$$I_i^1 = i[r_p n_e le(2kT_e/M)^{1/2}]^{-1} \tag{54}$$

where I_i^1 is the normalized ion current, i is the measured current and the other symbols have their usual meaning. The notation of Annaratone, Allen and Allen is followed

* In practice much numerical work is required, I shall not attempt to summarize the results here.

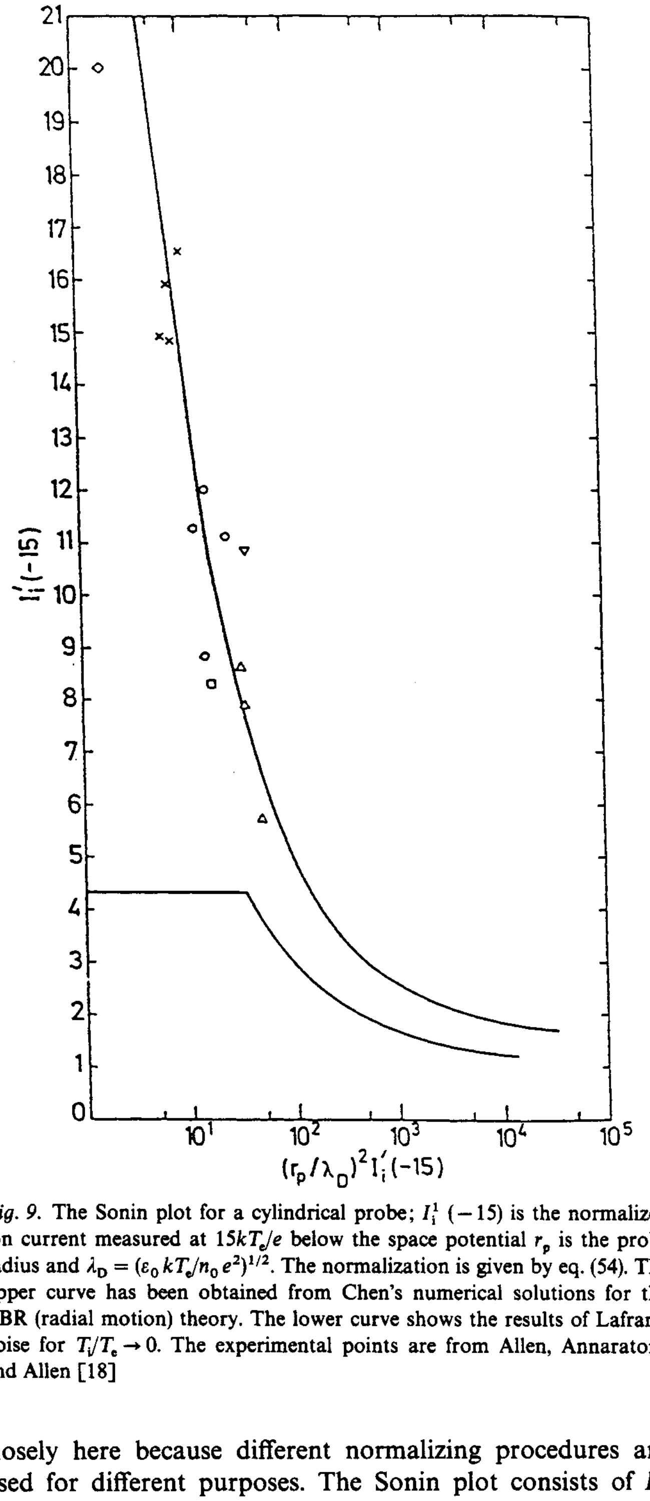

Fig. 9. The Sonin plot for a cylindrical probe; I_i^1 (-15) is the normalized ion current measured at $15kT_e/e$ below the space potential r_p is the probe radius and $\lambda_D = (\varepsilon_0 kT_e/n_0 e^2)^{1/2}$. The normalization is given by eq. (54). The upper curve has been obtained from Chen's numerical solutions for the ABR (radial motion) theory. The lower curve shows the results of Laframboise for $T_i/T_e \to 0$. The experimental points are from Allen, Annaratone and Allen [18]

closely here because different normalizing procedures are used for different purposes. The Sonin plot consists of I_i^1 measured at $15kT_e/e$ below the plasma potential, plotted against $(r_p/\lambda_D)^2 I_i^1$. The lower plot corresponds to Laframboise's calculations, the horizontal part corresponding to the orbital motion limited (O.M.L.) theory. The upper curve is that calculated assuming radial motion (ABR theory). It is seen that, for these particular experiments, the results agree with the simple ABR theory, rather than with the more sophisticated calculations of Laframboise.

Let us consider the orbital motion limited regime. The angular momentum of an ion collected by the probe is given by

$$J = M(2E_0/M)^{1/2}(1 - eV_p/E_0)^{1/2} r_p \tag{55}$$

If the initial energy of the ion is small then

$$J \simeq M(-2eV_p/M)^{1/2} r_p \tag{56}$$

Thus even if $E_0 \to 0$, the ions still have an angular momentum! This is because the effective radius of the probe for capture has become infinitely large. Clearly collisions may render this result invalid in some practical situations.

The maximum value of the angular momentum assuming that the m.f.p. λ is less than the linear dimensions of the plasma, is given by

$$J_m \simeq M(2E_0/M)^{1/2}\lambda \tag{57}$$

The orbital limited motion theory will fail if $J_m < J$ where J has been estimated, as above, on the basis of a collision-free plasma of infinite extent. Thus the O.M.L. theory will fail if

$$\lambda < r_p(-eV_p/kT_i)^{1/2} \tag{58}$$

In one particular experiment, using Argon at $0.5Pa$, $\lambda = 3.85$ mm, $r_p = 0.25$ mm but the R.H.S. of the inequality is 10.9 mm. Thus the O.M.L. theory fails. It is clear that care must be taken in using the orbital motion kind of theory. No plasma is entirely collision-free nor infinite in extent.

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