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THE THEORY OF COLLECTORS IN GASEOUS DISCHARGES

By H. M. Mott-Smith and Irving Langmuir

Abstract

When a cylindrical or spherical electrode (collector) immersed in an ionized gas is brought to a suitable potential, it becomes surrounded by a symmetrical space-charge region or "sheath" of positive or of negative ions (or electrons). Assuming that the gas pressure is so low that the proportion of ions which collide with gas molecules in the sheath is negligibly small, the current taken by the collector can be calculated in terms of the radii of the collector or sheath, the distribution of velocities among the ions arriving at the sheath boundary and the total drop of potential in the sheath. The current is independent of the actual distribution of potential in the sheath provided this distribution satisfies certain conditions.

"Orbital Motion" equations for spherical and cylindrical collectors.—General formulas for the current are derived and the calculations are then carried out for collectors in a group of ions having velocities which are (A) equal and parallel; (B) equal in magnitude but of random direction; (C) Maxwellian; (D) Maxwellian with a drift velocity superimposed. In all cases the collector current becomes practically independent of the sheath radius when this radius is large compared with that of the collector. Thus the volt-ampere characteristics of a collector of sufficiently small radius can be used to distinguish between the different types of velocity distribution. General equations are also given by means of which the velocity distribution can be calculated directly from the volt-ampere characteristics of a sphere or cylinder.

Special properties of the Maxwellian distribution.—For a collector of any shape having a convex surface, the logarithm of the current taken from a Maxwellian distribution is a linear function of the voltage difference between the collector and the gas when the collector potential is such as to retard arriving ions, but not when this potential is accelerating. This is a consequence of the following general theorem: Supposing for simplicity of statement that the surface of an electrode of any shape immersed in a Maxwellian distribution is perfectly reflecting, then the ions in the surrounding sheath will have a distribution (called D_M) of velocities and densities given by Maxwell's and Boltzmann's equations, even in the absence of collisions between the ions, provided that there are in the sheath no possible orbits in which an ion can circulate without reaching the boundary; but if such orbits exist, the distribution will be D_M except for the absence of such ions as would describe the circulating orbits. As another corollary of this theorem there is deduced an equation relating the solution of problems having inverse geometry. Finally it is indicated how the theorem can be applied to calculate the volt-ampere characteristic of A. F. Dittmer's "pierced collector" when placed in a Maxwellian distribution.

The effect of reflection of ions at the collector surface in modifying currents calculated by the preceding equations is discussed.

IN A series[1] of articles the authors have given an account of a new method of studying electrical discharges through gases at rather low pressures. The method consists in the determination of the complete volt-ampere characteristic of a small auxiliary electrode or *collector* of standard shape placed in the path of the discharge, and in the interpretation of this characteristic according to a new theory. In this way it is found possible to obtain an accurate value for the potential of the space near the collector, and much information concerning the nature, velocity and space density of the ions. This method has already been applied extensively by us in an investigation of the mercury-vapor arc,[1] and it has also been used by Compton, Turner and McCurdy in a study of the striated discharges.[2]

The idea of using a sounding electrode or "probe" is, of course, not new, but has been applied by Stark and others[3] in an attempt to measure space potentials and cathode drop in an arc. The measurements made were in most cases confined to a determination of the potential assumed by the probe when it was "floating," i. e., taking no current. As has been pointed out by one of us,[4] the conclusions drawn from these, and from similar measurements, are mostly in error because of neglect to take into account the effect of the proper motions of the ions striking the probe, and of space charge effects in the neighborhood of the collector. An exposition of the new theory, together with a condensed derivation of some of the formulas used, was given in the first of the series of articles already mentioned. This article will be referred to as "Part I" throughout the rest of the present paper, whose purpose is to complete the derivation of the equations and to extend the application of the theory to some new cases.

In many types of discharges, we observe regions where a very uniform state of ionization seems to exist. It is apparent that the densities of the different kinds of ions remain sensibly the same from point to point, and that their velocities can be described in terms of distribution functions which are independent of the space coordinates. In a region of this kind we imagine to be situated a small electrode whose potential is varied. Our problem is to calculate the current to the electrode contributed by each kind of ion as a function of the applied potential, in terms of the assumed velocity distribution functions.

[1] Langmuir and Mott-Smith, G. E. Rev. **27,** 449, 538, 616, 762, 810 (1924). See also Langmuir, G. E. Rev. **26,** 731 (1923); Science **58**, 290 (1923); J. Frank. Inst. **196,** 751 (1923).

[2] Compton, Turner and McCurdy, Phys. Rev. **24,** 597 (1924).

[3] Stark, Ann. d. Physik **18,** 212 (1905).

[4] Langmuir, J. Frank. Inst. **196,** 751 (1923).

For example, let us consider the collector to be a wire of small diameter, whose potential is made negative with respect to the region about it. The wire then repels negative ions and electrons, but attracts positive ions, and so becomes surrounded by a cylindrical positive "sheath" or region of positive space charge. This sheath is of such dimensions that the total positive charge in it equals the negative charge on the wire, so that the field of the wire does not extend beyond the edge of the sheath. The current taken by the wire therefore cannot exceed the rate at which ions arrive at the sheath edge in consequence of their proper motions.

If we suppose the negative potential of the collector to be large compared with the voltage equivalent of the ion velocities, then the sheath may be divided roughly into two parts. In the center will be a region in which is concentrated most of the drop of potential between the gas and the collector, so that in this region there will be present only positive ions and possibly a few electrons or negative ions which had exceptionally high velocity. Outside of this is a region in which both negative and positive ions are present in comparable quantities, but in which the normal conditions existing in the discharge are modified through the withdrawal of positive ions by the collector. The two regions merge into each other more or less gradually in a way depending upon the distribution of velocities among the ions. In the outer region, as a rough calculation shows, the potential approaches the space value asymptotically but never reaches it in finite distance. Actually, therefore, the sheath does not have a sharply defined edge. Since, however, the whole drop of potential in the outer region is small compared with the total, it will be convenient to take as sheath boundary the surface at which the sharp drop of potential begins, and to regard the distribution of density and of velocity of the ions as known at this surface. This convention simplifies the discussion, and as we shall see, does not restrict the validity of the equations derived in the present article. Accordingly, we shall speak of the sheath as though it had a sharp edge; the potential at this boundary we shall speak of as the space potential.

We shall further assume that the gas pressure is so low that there are only a negligible number of collisions in the sheath between ions or electrons and gas molecules. The ions in the sheath then describe free orbits, some of which end on the surface of the collector. Now if the sheath has axial symmetry so that the equipotentials are coaxial circular cylinders, it is found from simple mechanical principles that the condition for a positive or a negative ion to reach the collector depends not upon the nature of the field of force along the whole orbit, but only upon the initial and final potentials and the initial velocity of the ion on entering the sheath. The total current to the collector can thus be found by summing the contributions of the ions of different signs and initial velocities, and this current will be a function of the drop of potential in the sheath and of the sheath radius only. Another relation between these three variables can only be found by actually calculating the distribution of potential in the sheath through the use of Poisson's equation. It is thus seen that the problem of calculating the volt-ampere characteristic of the collector in general divides itself into two parts, the first of which is the purely "mechanical" problem outlined above, while the second is a "space charge" problem. From the solution of these two problems there are obtained two independent equations relating to the three variables just mentioned, and by elimination of the sheath radius between these two equations a single equation can be found expressing the current in terms of the potential.

On account of its great difficulty, the second or "space charge" part of the problem has been solved for only a few cases, so that in general we are not yet in a position to calculate the volt-ampere characteristic of a collector, unless we use experimentally determined values of the sheath size. On the other hand, the solution of the "mechanical" or "orbital motion" problem is relatively easy, and it is with this side of the question that we shall deal in the rest of the present paper. Under certain conditions, as we shall see, the current is independent of the sheath size so that the "orbital motion" equation actually gives the volt-ampere characteristic of the collector.

So far we have taken as example illustrative of the general theory, the case of a small wire charged to a negative potential. If we imagine the wire now to be charged positively there will be formed about it a sheath of negative ions and electrons moving toward the wire. The discussion of the two cases is exactly similar except for reversal of the sign of the charges and of the potential involved, and this reversal does not change the form of the "orbital motion" equations. Throughout the remainder of the paper, we therefore shall not specify the sign of the ions involved, and the term "ion" is to be taken as applying indiscriminately to positive ions, negative ions and electrons.*

The difficulties involved in the calculation of the current makes it necessary for us to restrict ourselves to cases where the geometry is

* It is assumed in the foregoing discussion that every ion which reaches the collector gives up its charge to it, i.e., that there is no reflection of either positive ions or electrons. Later in the paper we shall consider how the equations which will be derived may need to be modified to take into account the effect of reflection.

simple. The three cases which can be treated by comparatively elementary means are those of a plane collector in which edge effects are neglected, of a cylindrical collector in which the end effects are neglected, and of a spherical collector in which the disturbing effect of the connecting lead is neglected; the approximations made in neglecting these corrections amount to assuming that the current per unit area is uniform over the surface of the collector. It is seen, therefore, that we can not realize any of the theoretical cases in practice (except by the use of guard rings), although we can approximate quite closely to them by choosing proper conditions. On the whole, the cylindrical collector is the most satisfactory in that the end corrections can be made comparatively small, and the total current taken by the collector can be kept low, so as to disturb as little as possible the normal condition of the discharge. We shall treat in detail the cases of the cylindrical and the spherical collecter, and for comparison, we shall also give the equations for a plane collector receiving current on one side only for the case of a retarding potential.

I. General Equations for Cylindrical and Spherical Collectors

Taking up first the case of the *cylinder*, we may assume this to be represented by a wire whose length l is large compared with its radius r so that end corrections can be neglected. The wire is assumed to be straight, or at least such that the radius of curvature of the axis is everywhere large compared with r. Under these conditions the current per unit length i/l is sensibly the same for all parts of the wire. For the present we are also assuming that the composition of the ionized gas is uniform about the collector, and that the ion velocities are random, so that the sheath will be a circular cylinder of radius a concentric with the collector.

Considering the ions of one particular sign only, let N be the number per unit volume in a small element of volume $d\tau$ bordering the sheath. In a plane normal to the axis let u be the radial and v the tangential component of velocity of an ion, u being counted positive when directed toward the center. Then if

$$Nf(u,v)du\,dv$$

is the number of ions in $d\tau$ which have their velocity components u and v lying in specified ranges du and dv, the total number of ions which in unit time cross the sheath edge with velocities within the given limits will be

$$2\pi aNuf(u,v)du\,dv. \tag{1}$$

Let u_r, v_r be the radial and tangential velocity components of an ion arriving at the surface of the collector, and V_r the potential of the latter with respect to the sheath edge, to be taken positive when the collector attracts the ions. Then if ϵ is the charge and m the mass of an ion, we have from the laws of conservation of energy and of angular momentum the equations

$$\left.\begin{aligned}\tfrac{1}{2}m(u_r^2+v_r^2)&=\tfrac{1}{2}m(u^2+v^2)+\epsilon V\\ rv_r&=av\end{aligned}\right\} \tag{2}$$

which solved for u_r and v_r become

$$u_r^2=u^2-\left(\frac{a^2}{r^2}-1\right)v^2+2\frac{\epsilon}{m}V \tag{3}$$

$$v_r=\frac{a}{r}v. \tag{4}$$

Only those ions will be able to reach the collector for which

$$u>0,\quad u_r^2>0 \tag{5}$$*

If we plot u, v as rectangular coordinates of a point and take V to be positive, then the curve $u_r^2=0$ will be a hyperbola whose semi-axes are

$$\sqrt{2\frac{\epsilon}{m}V}\Big/\sqrt{\frac{a^2}{r^2}-1},\quad \sqrt{2\frac{\epsilon}{m}V}$$

on the v and u axes respectively.

The region for which conditions (5) are satisfied is that lying between the branches of the hyperbola and to the right of the v axis, and is shown shaded in Fig. 1. The total number of ions reaching the collector per second per unit length will be found by integrating expression (1) over this region. The figure shows that for a given value of u, v must lie between limits $-v_1$ and v_1 which are found by solving the equation $u_r^2=0$ for v, namely

$$v_1^2=\frac{r^2}{a^2-r^2}\left(u^2+2\frac{\epsilon}{m}V\right). \tag{6}$$

For V negative the curve $u_r^2=0$ becomes a hyperbola with the same axes as the previous one, but with its branches lying in the other two

* This last condition must be satisfied not only at the surface of the collector but also on any intermediate surface lying between the collector and the sheath edge; in other words the field of force must be such that the radial component of velocity does not become imaginary at any point on the orbit of an ion which satisfied condition (5). We shall tacitly assume this to be true throughout the following work, leaving the discussion of this condition to a later part of the paper.

quadrants. The region for which conditions (5) are satisfied is shown shaded in Fig. 2.

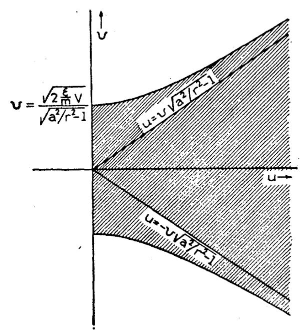

Fig. 1.

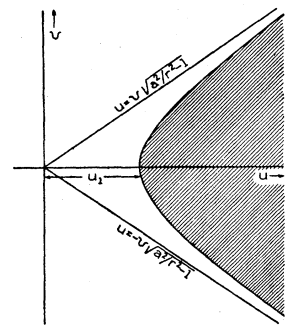

Fig. 2.

It appears that u cannot be less than a certain value u_1 given by the equation

$$u_1{}^2 = -2\frac{\epsilon}{m}V \tag{7}$$

while for any value of u greater than this, v lies between the limits $-v_1$ and v_1 defined by Eq. (6).

On multiplying expression (1) by the ionic charge ϵ and the length l of the collector and integrating over the regions indicated, we find for the total current i taken by the collector

$$i = 2\pi alN\epsilon\int_{0,\,u_1}^{\infty}\int_{-v_1}^{v_1} uf(u,v)\,dv\,du \tag{8}$$

where the lower limit of u is to be taken 0 for $V>0$ and u_1 for $V<0$. It is often convenient to replace N by an equivalent expression involving the total current crossing a unit of area at the edge of the sheath. This current I will be given by

$$I = N\epsilon\int_{0}^{\infty}\int_{-\infty}^{\infty} uf(u,v)dv\,du\,\cdot \tag{9}$$

As the radius a of the sheath becomes infinite it is found that the expression (8) approaches a limiting form, which can be obtained by the application of the rule of d'Hopital. If i_∞ is used to denote this limiting value of the current, then

$$i_\infty = \lim_{a\to\infty} 2\pi alN\epsilon\left\{-2a^2\int_{0,u_1}^{\infty} uf(u,v)\frac{dv_1}{da}du\right\}$$

On substituting the value of dv_1/da calculated from Eq. (6) into this expression and proceeding to the limit, we find

$$i_\infty = 4\pi r l N\epsilon \int_{0,\,u_1}^{\infty} u\sqrt{u^2 + 2\frac{\epsilon}{m}V}\, f(u,0)du\,. \tag{10}$$

Turning now to the case of a *spherical collector*, let r be the radius of the collector and a that of the sheath, whose boundary is assumed to be a concentric sphere. Let u, v, w be as before rectangular components of velocity of an ion at the sheath edge, u being the radial component, and let $f(u, v, w)$ be the distribution function for the three components. We replace v, w by polar coordinates q, ψ so that if g is a new function defined by

$$g(q,\, u) = \int_0^{2\pi} f(u,\, q \sin\psi,\, q\cos\psi)d\psi \tag{11}$$

then $qg(q, u)$ is the distribution function for the velocity components q and u, q being the resultant of v and w. If q_r, u_r are the values of the quantities indicated at the surface of the collector, then the mechanical relations give us equations of exactly the same form as (2), v being replaced by q and v_r by q_r. We therefore obtain for the current to the collector at potential V the expression

$$i = 4\pi N\epsilon a^2 \int_{0,\,u_1}^{\infty}\int_0^{q_1} uqg(q,\, u)dq\, du \tag{12}$$

q_1 being defined by Eq. (6) in which v_1 is to be replaced by q_1. The limiting value of the current for a/r increasing is found as before to be

$$i_\infty = 2\pi r^2 N\epsilon \int_{0,\,u_1}^{\infty} u\left(u^2 + 2\frac{\epsilon}{m}V\right)g(0,\, u)du\,\cdot \tag{13}$$

Finally, for the *plane collector* with retarding potential the condition for an ion to reach the collector is evidently that the energy component in the direction normal to the plane exceed a fixed value determined by the retarding potential. The general formulas are derived in an obvious manner, and we shall content ourselves with giving the final result for each type of velocity distribution taken up.

We proceed now to apply these general formulas to some specific examples.

II. Collector Characteristics for Particular Types of Velocity Distributions

(A) *Velocities equal in magnitude and parallel in direction.* This case, which corresponds to that of a "beam" of ions falling on the collector, strictly does not come under the scope of our general formulas, because the sheath formed will not be symmetrical. The results in this case can be obtained by the simpler method used in "Part I," and are given here for the sake of completeness. If V_0 is the voltage equivalent of the ion velocities, we have for a *cylindrical collector* placed with its axis normal to the path of the ions

$$i = 2rlI\sqrt{1+V/V_0}, \quad \text{for} \quad V > -V_0 \tag{14a}$$

$$= 0 \quad , \quad \text{for} \quad V < -V_0 \tag{14b}$$

I being the current per unit area conveyed by the ions outside the sheath. For the spherical collector

$$i = \pi r^2 I\left(1+\frac{V}{V_0}\right), \quad \text{for } V > -V_0 \tag{15a}$$

$$= 0 \quad , \quad \text{for } V < -V_0. \tag{15b}$$

In each case the current is independent of the size of the sheath. This ceases to be true, however, when the collector potential is so strongly accelerating that the collector receives the entire current entering the sheath. As the voltage increases the current increases according to the above formulas until it reaches this saturation value, after which it remains constant unless the sheath changes in size.

For the plane collector the current evidently remains zero until the retarding potential $-V$ becomes less than the energy component of an ion in the direction of the normal to the surface. This energy component is $V_0\cos^2\alpha$ where is angle between the normal and the path of the ions outside the sheath.

(B) *Velocities of uniform magnitude but with directions distributed at random in space.* If c is the common magnitude of the velocities, the velocities of ions in an element of volume will be represented in the u, v, w diagram by points uniformly distributed over the surface of a sphere of radius c. By considering the projections of these points on the plane it is readily seen that the distribution function for the components u, v is given by

$$f(u, v) = \begin{cases} 1/2\pi c^2\sqrt{1-(u^2+v^2)/c^2}, & \text{for } u^2+v^2 < c^2 \\ 0 & , \quad \text{for } u^2+v^2 > c^2. \end{cases} \tag{16}$$

On substituting this in the general formula (9) we find for the current density

$$I = N\epsilon c/4 \tag{17}$$

while from (8) the current to a *cylindrical collector* is

$$i=0 \qquad , \text{ for } V<-V_0 \tag{18a}$$

$$=2\pi rlI(1+V/V_0) \qquad , \text{ for } -V_0\leqq V<0 \tag{18b}$$

$$=4alI\left\{\sin^{-1}\frac{r}{\sqrt{(a^2-r^2)}}\sqrt{\frac{V}{V_0}}+\frac{r}{a}\left(1+\frac{V}{V_0}\right)\sin^{-1}\sqrt{\frac{V_0-\dfrac{r^2}{a^2-r^2}V}{V_0+V}}\right\} \tag{18c}$$

$$\text{for } 0\leqq V\leqq\left(\frac{a^2}{r^2}-1\right)V_0 \tag{18d}$$

$$=2\pi alI \qquad , \text{ for } V>\left(\frac{a^2}{r^2}-1\right)V_0 \tag{18e}$$

where V_0 is the voltage equivalent of c. We may describe these results briefly by saying that for retarding voltages the current is independent of the sheath radius and is a linear function of the voltage; for accelerating voltages it is expressed by a function which gradually changes from one nearly linear in V to one nearly linear in $\sqrt{V}$ as V increases; while for a certain value of V it reaches the saturation value (last equation). The limiting value of the current for a very large sheath is found by (10) to be in the case of an accelerating potential

$$i_\infty=4rlI\left\{\sqrt{\frac{V}{V_0}}+\left(1+\frac{V}{V_0}\right)\sin^{-1}\sqrt{\frac{V_0}{V+V_0}}\right\} \tag{19}$$

while for retarding potentials the current has the same value as that given by (18b), since this is independent of the sheath radius. If we put

$$x^2=1+\frac{V}{V_0} \tag{20}$$

then Eq. (19) can be expanded as follows

$$i_\infty=8rlIx\left(1-\frac{1}{6x^2}-\frac{1}{40x^4}-\frac{1}{116x^6}-\cdots\right) \tag{21}$$

which for x sufficiently large reduces to

$$i=8rlIx=8rlI\sqrt{1+\frac{V}{V_0}}\,. \tag{22}$$

Thus for large accelerating voltages and wires of small diameter, the square of the current is a linear function of the voltage.

With the *spherical collector* it is evident from considerations of symmetry that the formulas for the current will be the same as those (15) given for the velocity distribution of type (A), except that the projected area πr^2 of the collector must be replaced by its superficial area $4\pi r^2$.

For a *plane collector* of area A the current is

$$i = AI\left(1 + \frac{V}{V_0}\right), \text{ for } V_0 < V < 0 \cdot \tag{23}$$

Fig. 3 illustrates the characteristics for the present kind of velocity distribution of a plane electrode, a cylinder of small radius and a sphere of small radius. In the last two cases the values plotted are those of i_∞/i_o, where i_o is the current taken by the collector for $V = 0$. The abscissa is the potential of the collector with respect to the space, counted positive when the collector attracts the ions; the ordinate is the value of the current at the given voltage, divided by the value of the current taken by the collector when it is at space potential.

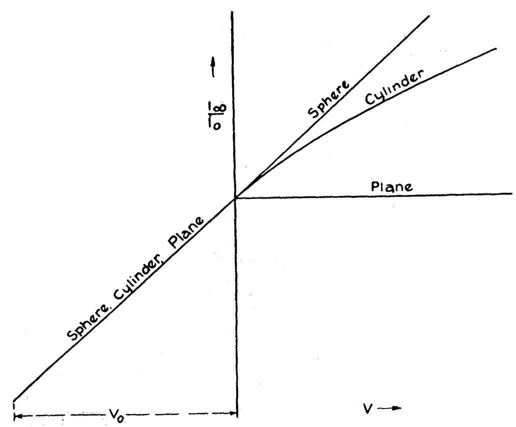

FIG. 3.

(C) *Maxwellian distribution of velocities.* If T is the temperature of the distribution and k is Boltzmann's constant, then the distribution function for the velocity components, u, v is

$$f(u,\ v) = \frac{m}{2\pi kT}\, e^{-(m/2kT)\,(u^2+v^2)}\ . \tag{24}$$

On substituting in (9) we obtain the well known formula of the kinetic theory for the drift current I of the ions

$$I = N\epsilon\sqrt{\frac{kT}{2\pi m}}\ \cdot \tag{25}$$

In evaluating the integrals for the current i taken by a *cylindrical collector*, we make the substitution

$$\frac{\epsilon V}{kT} = \eta \tag{26}$$

which defines a new variable replacing the voltage. Further we use new variables of integration defined by

$$u\sqrt{\frac{m}{2kT}} = x; \quad v\sqrt{\frac{m}{2kT}} = y\,. \tag{27}$$

Substituting the value of $f(u, v)$ in formula (8) and making these substitutions, we find

$$i = 8\sqrt{\pi}\, alI \int_{0,\sqrt{-\eta}}^{\infty} \int_{0}^{r\sqrt{x^2+\eta}/\sqrt{a^2-r^2}} xe^{-(x^2+y^2)}dy dx$$

An integration by parts reduces this to

$$i = 8\sqrt{\pi} alI \left\{ -\frac{e^{-x^2}}{2} \int_0^{r\sqrt{x^2+\eta}/\sqrt{a^2-r^2}} e^{-y^2}dy \right]_{x=0}^{x=\infty}$$
$$+\frac{1}{r}\, \frac{re^{-r^2\eta/(a^2-r^2)}}{\sqrt{a^2-r^2}} \int_0^{\infty} \frac{x}{\sqrt{x^2+\eta}}\, e^{-a^2x^2/(a^2-r^2)}\, dx \right\}.$$

In the second integral make the substitution

$$x^2+\eta = \frac{a^2-r^2}{a^2}y^2$$

then

$$i = 4\sqrt{\pi}\, alI \left\{ \int_0^{\sqrt{r^2\eta/(a^2-r^2)},\,0} e^{-y^2}dy + \frac{r}{a}e^{\eta} \int_{\sqrt{a^2\eta/(a^2-r^2)}}^{\infty} e^{-y^2}dy \right\}$$

or making use of the error function defined by

$$erf\, x = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-y^2}dy,$$

$$i = 2\pi rlI \left\{ \frac{a}{r}\left[1 - erf\sqrt{\frac{r^2\eta}{a^2-r^2}} \right] + erf\sqrt{\frac{a^2\eta}{a^2-r^2}}\right., \tag{28a}$$

$$\text{for } \eta > 0 \tag{28a}$$

$$= 2\pi rlI \cdot e^{\eta}, \qquad \text{for } \eta < 0\,. \tag{28b}$$

Thus, for retarding potentials the current to the cylinder is independent of the sheath radius and its logarithm is a linear function of the collector voltage, the slope of this linear function being ϵ/kT as seen from (26). For accelerating voltages the curves expressing i as a function of η for a given value of a/r are roughly parabolic in shape for small values of η, but have an asymptote which they approach as η becomes large, the ordinate of this asymptote being the total current which enters the sheath. The larger the value of a/r, the greater is the value of η required to bring the current up to a given fraction of the saturation value. The limiting value approached by the current as a/r becomes infinite is by formula (10)

$$i_\infty = 2\pi r l I \left\{ \frac{2}{\sqrt{\pi}} \sqrt{\eta} + e^\eta erf \sqrt{\eta} \right\}. \tag{29}$$

For values of η greater than 2 it is found that this formula can be replaced with good approximation by the following

$$i_\infty = 2\pi r l I \cdot \frac{2}{\sqrt{\pi}} \sqrt{1+\eta} \tag{30}$$

which after substituting from (26) takes the following form

$$\left(\frac{i_\infty}{2\pi r l I}\right)^2 = \frac{4}{\pi}\left(1+\frac{\epsilon V}{kT}\right). \tag{31}$$

If therefore we plot the square of the current per unit area to a wire of small radius against the applied voltage, we obtain in the case where the distribution of velocities among the ions is Maxwellian a straight line, provided the voltage is accelerating and not too small.[5] The intercept of this line on the voltage axis is

$$V_1 = -kT/\epsilon \tag{32}$$

so that from it we can deduce the origin of potentials, **i. e.**, the space potential. The slope of the line is

$$S = \frac{4}{\pi} \frac{\epsilon}{kT} I^2$$

which in virtue of (2) can be written

$$S = \frac{2}{\pi^2} \frac{\epsilon}{m} (N\epsilon)^2. \tag{33}$$

Since the constants e, m are known, this equation enables us to calculate N, the number of ions per unit volume in the ionized gas.

[5] A figure showing the actual form of this graph over the entire voltage range together with a table [calculated from Eq. (29)] which can be used to construct the graph in the lower range of voltages is given in Part III of the authors' original article [G. E. Rev. **27,** 617 (1924)].

It is interesting to compare these results with those obtained under case (B), as given by Eq. (22). There we also found that the square of the current to a small cylindrical collector is a linear function of the applied voltage for sufficiently large accelerating potentials. If from Eq. (22) we calculate the slope of this linear function and insert the expression for I from (17), we obtain precisely the result given by (33). We would therefore be unable to distinguish between a Maxwellian distribution and distribution (B) or (A) by studying the characteristic of a small wire with accelerating potentials. On the other hand, for retarding potentials the characteristics are entirely different, the current being in one case an exponential and in the other a linear function of the voltage.

In the case of the *spherical collector*, we start with the Maxwellian distribution function for the three components u, v, w which is

$$f(u, v, w) = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-(m/2kT)(u^2+v^2+w^2)} \tag{34}$$

from which we find according to (11)

$$g(q, u) = \frac{1}{\sqrt{2\pi}}\left(\frac{m}{kT}\right)^{3/2} e^{-(m/2kT)(u^2+q^2)}\,. \tag{35}$$

Substituting in (12) and making use of the variable η defined by (26), we obtain on evaluating the integrals.

$$i = 4\pi a^2 I\left\{1-\frac{a^2-r^2}{a^2}e^{-r^2\eta/(a^2-r^2)}\right\},\ \text{for } \eta>0 \tag{36a}$$

$$= 4\pi r^2 I\cdot e^{\eta},\qquad \text{for } \eta<0\,. \tag{36b}$$

As in the case of a *cylindrical collector*, the current for retarding potential is independent of the sheath radius and its logarithm is a linear function of the voltage. For accelerating potentials the difference between the current and its saturation value $4\pi a^2 I$ decreases exponentially with the voltage. The limiting form of Eq. (36a) as η approaches infinity can be found directly or by the use of Eq. (13). It is

$$i_\infty = 4\pi r^2 I(1+\eta)\,. \tag{37}$$

For very small spheres the current is therefore a linear function of the voltage when the latter is accelerating. Thus the form of the characteristic is identical in this region with that obtained for spheres under case (B), in contradistinction to the characteristic of a cylinder for a Maxwellian distribution, which as we have seen merely approaches asymptotically the characteristic for distribution (B) as the accelerating voltage is increased.

The volt ampere characteristic of a plane electrode of area A with a Maxwellian distribution is

$$i = IAe^{\eta}, \text{ for } \eta < 0\,. \tag{38}$$

Thus the cylinder, the sphere and the plane all have the same characteristic for retarding voltage in a Maxwellian distribution of ions. As we shall see later, this same characteristic is in fact possessed by a collector of any shape whatever.

The curves of Fig. 4 illustrate the characteristics of the three forms of collectors with a Maxwellian distribution.

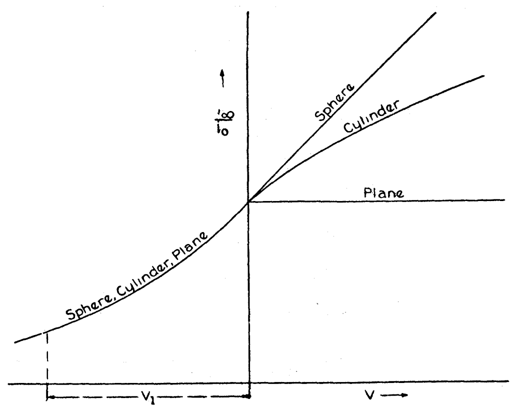

FIG. 4.

(D) *Distribution which is Maxwellian with superimposed drift.* In certain types of discharge there are groups of ions which presumably have the velocity distribution of a gas with "mass-motion." For instance, in the case of a mercury-vapor arc passing through a tube of uniform diameter, collectors are found to have characteristics which if interpreted according to the results of the last section would indicate that the free electrons has a nearly perfect Maxwellian distribution of velocities. This would imply that there was no net transport of electrons in any direction, but actually the electrons must be drifting steadily toward the anode. It becomes important therefore to find what interpretation should be put upon the collector characteristics in view of this fact.

In the case just cited the drift velocity is usually small or at any rate of the same order of magnitude as the average absolute velocity of the Maxwellian distribution. In other cases there exist in discharges beams of electrons which have a common high drift velocity on which is superimposed a Maxwellian distribution of relatively low temperature. Both of these cases will be treated in the following.

If the drift velocity is u_0, then the distribution in question is Maxwellian when referred to a set of coordinate axes moving with the velocity u_0. Let u', v' be rectangular velocity components with respect to a set of axes fixed in space, u' being taken in the direction of u_0. Then the distribution function for the components u', v' is

$$f(u',v') = \frac{m}{2\pi kT} e^{-(m/2kT)[(u'-u_0)^2+v'^2]} . \tag{39}$$

We assume that a *cylindrical collector* is placed with its axis perpendicular to the direction of u_0 and choose the direction of v' to be also perpendicular to the axis. The radial and tangential velocity components u and v of an ion at the edge of the sheath are now functions of its position on the circumference. Let an ion be situated at a point on the circumference such that the directions of u and u_0 make an angle θ with each other, then

$$\left.\begin{aligned} u' &= u\cos\theta + v\sin\theta \\ v' &= -u\sin\theta + v\cos\theta \\ du'dv' &= du\,dv \end{aligned}\right\} . \tag{40}$$

In terms of the new coordinates the distribution function becomes

$$f(u, v, \theta) = \frac{m}{2\pi kT} exp\left\{-\frac{m}{2kT}[u^2+v^2+u_0^2-2u_0(u\cos\theta+v\sin\theta)]\right\}. \tag{41}$$

Since conditions are now not symmetrical about the collector, the general formulas (8), (9) and (10) need to be somewhat modified. It is clear that the sheath itself will be no longer symmetrical and concentric with the collector, so that the exact solution would be very difficult if not impossible. Fortunately the case of most interest is that of sheaths whose dimensions are large compared with r and for such sheaths the actual shape can have but little influence on the current taken by the collector.* We can therefore in the case of a large sheath consider it to be circular, and the current to the collector will be found by averaging over the circumference the current given by Eq. (10), i. e.

$$i_\infty = 2rlN\epsilon \int_0^{2\pi} \int_{0,u_1}^{\infty} u\sqrt{u^2+2\frac{\epsilon}{m}V}\, f(u, 0, \theta)\, du d\theta$$

which on substitution of the value of f becomes

* This point will be more fully discussed in Section III.

$$i_\infty = 2rlN\epsilon \frac{m}{\pi k T}\int_0^\pi \int_{0,u_1}^\infty u\sqrt{u^2+2\frac{\epsilon}{m}V}\, e^{-(m/2kT)\,(u^2+u_0^2-2u_0u\cos\theta)}du d\theta. \quad (42)$$

It is convenient to introduce here the *random* current I_r which is defined as the current per unit area transported in one direction through a plane moving with the velocity u_0. This current will therefore be given by Eq. (25).

$$I_r = N\epsilon\sqrt{\frac{kT}{2\pi m}}. \quad (43)$$

The drift current I_d is the total current per cm² carried by the ions through a fixed plane perpendicular to the direction of u_0 and is given by

$$I_d = N\epsilon u_0. \quad (44)$$

We will also define a parameter α as follows

$$\alpha = \sqrt{\frac{m}{2kT}}u_0 = \frac{1}{2\sqrt{\pi}}\frac{I_d}{I_r} = \sqrt{\frac{3}{2}}\frac{u_0}{C} \quad (45)$$

where in the last expression C is the root-mean-square velocity of the Maxwellian distribution. In terms of these quantities and of the variables defined by Eqs. (26) and (27), Eq. (42) may be written

$$i_\infty = \frac{8}{\sqrt{\pi}} rlI_r e^{-\alpha^2}\int_0^\pi \int_{0,\sqrt{-\eta}}^\infty x\sqrt{x^2+\eta}\, e^{-(x^2-2\alpha x\cos\theta)}dx d\theta \quad (46)$$

The integral on the right cannot be expressed in finite terms, but for negative values of η (when the right hand lower limit for x is taken) it can be evaluated as an infinite series convergent for all α and η as follows

$$i_\infty = 2\pi rlI_r e^{\eta-\alpha^2}\sum_{p=0}^\infty \frac{(2p+1)!}{2^{2p}(p!)^2}\left(\frac{\alpha}{\sqrt{-\eta}}\right)^p i^{-p}J_p(2\alpha\sqrt{-\eta}\; i) \quad (47)$$

Here $i^{-p}J_p(xi)$ is the Bessel function of the first kind and pth order with a pure imaginary argument, and is itself a real and increasing function of x resembling the exponential.[6] The series converges very rapidly for small values of α; for instance, when $\alpha = 0.3$, corresponding to $I_d/I_r = 1$ approximately, the first three terms of the series give the result correct to within four parts in ten thousand.

In the previous case of a Maxwellian distribution the current I crossing a unit area of the sheath was identical with I_r, the random current in the gas. In the present case however, I is not the same for differ-

[6] Cf. Jahnke u. Emde "Funktionentafeln"

ferent parts of the sheath circumference, and must be replaced by an average $\bar{I}$ which is different from I_r. This average value is found by putting $\eta=0$ in (47) or (46) and dividing by the area of the collector. We thus find

$$\bar{I}=I_r\left(1+\frac{\alpha^2}{2}-\frac{\alpha^4}{16}+\frac{\alpha^6}{96}-\cdots\right). \tag{48}$$

which for small values of α can be reduced to

$$\bar{I}=I_r(1+\alpha^2/2)=I_r[1+0.040(I_d/I_r)^2]. \tag{49}$$

Thus when I_d/I_r does not exceed 1, as is usually true in the case mentioned above of the mercury-vapor arcs, the difference between $\bar{I}$ and I_r is not over four percent.

If we plot the logarithm of i_∞ calculated from (47) against η we obtain for different values of the parameter α a series of curves such as is shown in Fig. 5. For $\alpha=0$ we obtain the straight line corresponding to a

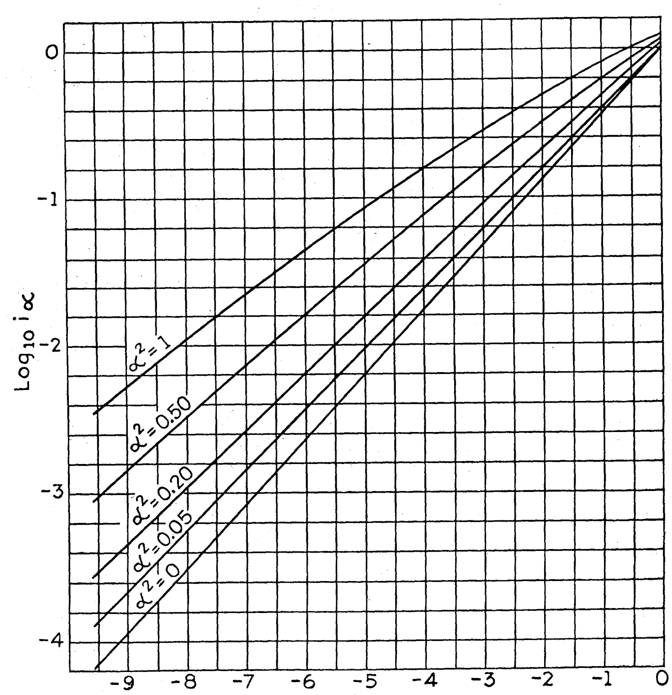

Fig. 5.

pure Maxwellian distribution. For values of α^2 between 0 and .2 the curves are still sensibly straight lines over a range of about ten thousand fold change in the current, but the slopes of these lines decrease uniformly as α increases. This range corresponds to values of the ratio of drift to random current from zero to about 1.6, so that even with these relatively large drift currents the characteristic of the collector will indicate a Maxwellian velocity distribution for the ions. The temperature determined from the characteristic will however differ from that of the existing Maxwellian distribution, and the difference can be roughly calculated from the slopes of the theoretical characteristics at the origin $\eta=0$. From the preceding equations we can deduce the following formula

$$S_0=\frac{d}{d\eta}\log i_\infty \Big|_{\eta=0}=1-\frac{I_r}{\bar{I}}\alpha^2\left(1+\frac{3\alpha^2}{4}+\frac{5\alpha^4}{32}+\cdots\right). \tag{50}$$

Since according to (48) the ratio $I_r/\bar{I}$ does not differ much from unity for $\alpha<1$, we can reduce (50) to the following for small values of α:

$$S_0=1-\alpha^2=1-\frac{1}{4\pi}(I_d/I_r)^2. \tag{51}$$

The temperature of the distribution is proportional to the reciprocal of the slope, hence we can summarize the above results in the following statement: when the temperature of the Maxwellian velocity distribution of the ions is determined from the characteristics of a small cylindrical collector for retarding voltages, the result, in case a drift-current exists, will be too large by a fractional amount which is approximately $1/4\pi(I_d/I_r)^2$.

For larger values of α it can be seen that the curve of log i vs η is not quite straight, but becomes concave toward the potential axis. As α becomes increasingly large, the expression for the current approaches a limiting form which corresponds to the case of ions having a large drift velocity on which is superimposed a small "temperature" motion. There are two cases. On the one hand, if $-\eta$ is less than α^2, the retarding voltage is small compared to the voltage equivalent of the drift velocity, and in the limit as the temperature decreases the expression for the current approaches that given by Eq. (14) for the case of a unidirectional stream of ions of equal velocities. On the other hand, when $-\eta$ is nearly equal to or greater than α a different limiting form will be reached, because in this case the temperature motion although small is nevertheless of importance when the retarding voltage is comparable with the voltage equivalent of the drift velocity. It will be found convenient to replace η by a new variable λ defined by

$$\lambda=\alpha-\sqrt{-\eta} \tag{52}$$

which in view of Eqs. (45) and (26) defining α and η can also be written

$$\lambda=\sqrt{\frac{\epsilon}{kT}}(\sqrt{V_0}-\sqrt{V}) \tag{53}$$

where V_0 is the voltage equivalent of u_0.

In terms of this variable it is found that the equation for the current can be written as a series of powers of the reciprocal of α whose first two terms only are retained in the following equation

$$i_\infty=4\sqrt{\alpha}\, rlI_r\left\{\int_0^\infty e^{-(x-\lambda)^2}\sqrt{x}\,dx+\frac{3}{2\alpha}\int_0^\infty e^{-(x-\lambda)^2}x^{3/2}dx\right\}. \tag{54}$$

For large values of α and small values of λ the expression for the current therefore approaches the limiting form

$$i_\infty=\frac{2}{\sqrt{\pi}}rlI_d\sqrt[4]{\frac{kT}{\epsilon V_0}}\int_0^\infty e^{-(x-\lambda)^2}\sqrt{x}\;dx\,. \tag{55}$$

When $\lambda=0$ we get

$$i_\infty\big|_{\lambda=0}=\frac{2}{\sqrt{\pi}}rlI_d\sqrt[4]{\frac{kT}{\epsilon V_0}}\cdot\frac{1}{2}\Gamma\left(\frac{3}{4}\right).$$

If then we put

$$F(\lambda)=\frac{2}{\Gamma\left(\frac{3}{4}\right)}\int_0^\infty e^{-(x-\lambda)^2}\sqrt{x}\;dx \tag{56}$$

the equation for the current i_∞ can be written

$$\begin{aligned} i_\infty &= \frac{\Gamma\left(\frac{3}{4}\right)}{2\sqrt{\pi}}2rlI_d\sqrt[4]{\frac{kT}{\epsilon V_0}}F(\lambda) \\ &= 0.34\,A_pI_d\sqrt[4]{\frac{kT}{\epsilon V_0}}F(\lambda) \end{aligned} \tag{57}$$

where A_p is the projected area of the collector.

The function $F(\lambda)$ can be evaluated[7] by expansion in various series, of which the most convenient appear to be the following.

$$\frac{\Gamma\left(\frac{3}{4}\right)}{2}F(\lambda)=e^{-\lambda^2}\sum_{n=0}^{\infty}\frac{2^n}{n!}\frac{2}{2n+3}\Gamma\left(\frac{n}{2}+\frac{7}{4}\right)\lambda^n,\ \text{for}\ |\lambda|<1 \tag{58}$$

[7] The integral defining $F(\lambda)$ can be reduced to a contour integral which is expressible in terms of a certain type of the confluent hypergeometric function. Cf. Whittaker & Watson, "Modern Analysis," 3rd ed., p. 349 (1920).

$$=\sqrt{\lambda}\left[\left\{1-\frac{1}{16\lambda^2}-\frac{15}{512\lambda^4}-\cdots\right]\frac{\sqrt{\pi}}{2}erf(-\lambda)\right.$$

$$\left.+\left[\frac{93}{256\lambda}+\frac{31}{512\lambda^3}+\cdot\ \right]e^{-\lambda^2}\right\}\ ,\ \text{for}\ \lambda>1 \tag{59}$$

$$=\frac{1}{4}\sqrt{\frac{\pi}{2}}\frac{e^{-\lambda^2}}{(-\lambda)^{3/2}}\sum_{n=0}^{\infty}(-1)^n\frac{(4n+1)!}{2^{6n}(2n)!n!}\lambda^{-2n},\ \text{for}\ \lambda<-1\ . \tag{60}$$

The first series is convergent for all values of λ but is only useful in the range indicated. The last two are asymptotic expansions. From these series the approximate values given in Table I were calculated.

TABLE I

λ	$F(\lambda)$	λ	$F(\lambda)$	λ	$F(\lambda)$
−3	1.072×10^{-5}	−0.2	.730	1.0	2.685
−2	.00264	0	1.000	1.4	3.31
−1.5	.0200	0.2	1.312	1.8	3.86
−1.0	.1094	0.4	1.655	2.2	4.33
−0.6	.338	0.6	2.018	2.6	4.71
−0.4	.501	0.8	2.353	3.0	4.98

The graph of the function is shown in Fig. 6. It can be seen from Eq. (59) that when λ is large and positive $F(\lambda)$ becomes nearly proportional to $\sqrt{\lambda}$. Now λ is approximately proportional to the difference between

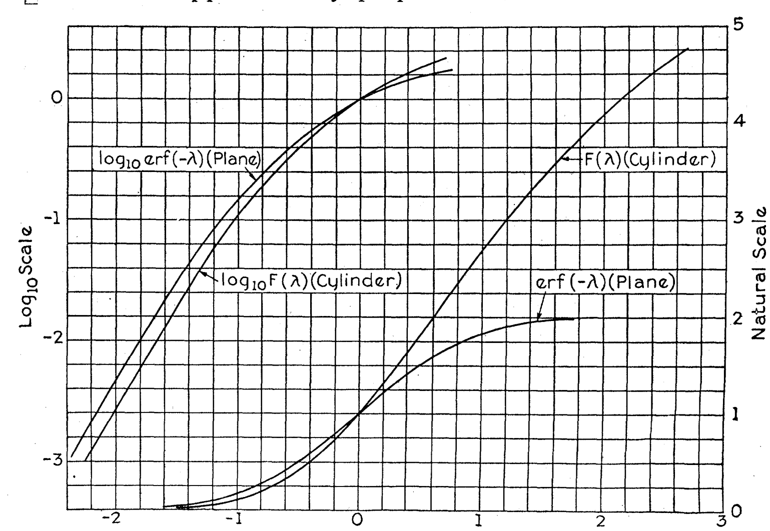

FIG. 6.

the potential of the collector and the voltage equivalent of the drift velocity of the ions. In this region, therefore, the current is approximately a parabolic function of this voltage difference. But we found [(cf. Eq. (30)] that in the case of a pure Maxwellian distribution the same relation held between the current and the potential of the collector with respect to the space, when this potential was an accelerating one. We see therefore that in the present case when the retarding voltage on the collector is less than V_0 the current is nearly the same, except for a constant factor, as though the drift velocity were annihilated and the collector voltage raised by the corresponding amount. On the other hand when λ is negative Eq. (60) shows that the current decreases very rapidly as $-\lambda$ increases, but the logarithm of the current is roughly a linear function of the square of the voltage difference λ, while with a Maxwellian distribution log i is a linear function of the first power of the corresponding voltage.

For comparison there is plotted in Fig. 6 the graph of the function $erf(-\lambda)$. This is approximately the volt-ampere characteristic of a *plane collector* placed perpendicular to the direction of the drift velocity when this velocity is large compared with the Maxwellian motion. It will be seen that for negative λ the currents to the two kinds of collectors stand in a nearly constant ratio, but for positive λ the current to the plane reaches a saturation value while that to the cylinder continued to increase parabolically. The size of the error made in using formula (57) to calculate the current when the ratio of the drift to the random current is not very large can be seen from Eq. (54). The second integral in formula (54) is approximately equal to the first one for $|\lambda| \leqq 1$ so that the percentage error caused by omitting the second integral is of the order of $100/\alpha$.

For the *spherical collector* with the present velocity distribution function there can be obtained similar equations which, however, for the sake of brevity we shall omit. We will give only the results corresponding to Eqs. (51) and (55). In the case of the spherical collector the error made in determining the temperature of a Maxwellian distribution which is modified by a small drift velocity is only two thirds as much as for the cylindrical collector. When the potential of the spherical collector is retarding and greater in absolute magnitude than the voltage equivalent V_0 of the drift velocity, the current to the collector is approximately the same as though the drift velocity were removed and the collector voltage raised by V_0 (the two currents specified being here equal instead of merely proportional as with the cylindrical collector).

III. Discussion of Equations; General Theorems; Effect of Reflection at the Collector Surface

In deriving the foregoing equations we have ignored the distribution of potential within the sheath, and have assumed that the current taken by a collector from a sheath of given dimensions depends only on the total difference of potential between the collector and the sheath boundary. But this certainly cannot be true without some restrictions on the nature of the field in the sheath. For instance, there is the obvious qualification that the potential of any point within the sheath must lie between the extreme values of the potential of the collector and of the sheath boundary. This condition is sufficient in the case of the plane collector, and of the cylindrical and spherical collectors where the ions considered are moving in a retarding field. If however ions are moving in an accelerating field toward a cylindrical or spherical collector some further condition must be added, as can be seen for example from the fact that whereas our equations for this case give a value for the current greater than i_0 the current taken at the space potential, the current for any accelerating potential would actually be reduced to i_0 if the whole voltage drop in the sheath were concentrated in a thin layer covering the collector.

Let us consider in more detail the case of the cylindrical collector at an accelerating potential. It has already been pointed out [(cf. footnote to (5)] that the conditions expressed by (5) which are satisfied by the initial velocity components of an ion reaching the collector must be supplemented by the assumption that the radial velocity component does not become imaginary at any point on the orbit of such an ion. The assumption implies some property of the field of force which we proceed to investigate. The initial velocity-components u, v satisfying conditions (5) are represented by points lying in the shaded region of Fig. 1. Now if an ion with certain values of u and v is able to reach the collector, it is clear that any ion with the same value of u but a smaller value of v will also be able to do so. It is sufficient therefore to discuss the conditions relative to the points on the hyperbolic boundary of the region in question.

If u_s, v_s are the radial and tangential velocity components of an ion at a distance s from the center, then the relation between u_s, v_s and u, v must be given by Eq. (3) when s is substituted for r, and V_s the potential at the distance s for V_r. That is

$$u_s^2 = u^2 - \left(\frac{a^2}{s^2} - 1\right) v^2 + 2\frac{\epsilon}{m} V_s \tag{61}$$

$$v_s = \frac{a}{s} v \tag{62}$$

The relation between the initial velocity components of an ion corresponding to a point on the hyperbola in Fig. 1 is given by Eq. (6). Such an ion in order to reach the collector, must satisfy for every value of s between a and r the condition

$$u_s^2 > 0 \tag{63}$$

On substituting from Eq. (6) into (61) and transposing, etc., we find that this inequality can be written

$$V_s > \frac{a^2 - s^2}{a^2 - r^2} \frac{r^2}{s^2} V_r - u^2 \left[1 - \frac{a^2 - s^2}{a^2 - r^2} \frac{r^2}{s^2} \right] \frac{m}{2\epsilon} \tag{64}$$

The quantity in brackets is always positive when s lies between a and r, so that if (64) is to be satisfied for every value of u between zero and infinity we must have

$$V_s > \frac{a^2 - s^2}{a^2 - r^2} \frac{r^2}{s^2} V_r \tag{65}$$

which can be written

$$V_s > M \left(\frac{a^2}{s^2} - 1 \right) \tag{66}$$

where M is a constant independent of s. This inequality therefore expresses the property which must be possessed by the field of force in the sheath in order that the equations which we have developed for the volt-ampere characteristic of a cylinder may apply.

In the case of the spherical collector the discussion is the same except that v must be replaced by q, the resultant tangential component (regardless of direction). The conclusion reached is therefore that the equations which we have developed for the sphere will only hold if condition (65) or (66) is satisfied.

From another point of view these conditions define more exactly what we have called a "sheath-edge"; for it can easily be seen that if we assume any distribution of potential we like between a and r we can always find a cylinder (or sphere) of radius a' intermediate between a and r such that for this cylinder (sphere) *or any other of smaller radius*, condition (65) is satisfied when a' is substituted for a. In other words such a surface can be taken to be the edge of the sheath if the distribution function for the velocities of the ions crossing it is known. As far as the equations of orbital motion determine it, the *sheath edge is therefore simply a surface on which we know the velocities of the ions and within which the condition* (65) *is satisfied.*

Additional light is thrown upon this point by an alternative method of calculating the current to a cylindrical or spherical collector. Instead of considering the distribution of velocity components among the ions in an element of volume, we can resolve the whole of the ions outside the sheath into a number of swarms consisting of ions moving in parallel lines with equal velocities (i. e. the direction and magnitude of the velocities in each swarm fall within small ranges centering about given mean values), that is, into distributions each of which is of the type considered under II (A). For each swarm the collector will possess an effective target area such that every ion will be collected whose rectilinear path outside the sheath when prolonged falls on the target. The total current is then found by multiplying this target area by the current density of the swarm and adding the products for all the swarms.

The target area of the collector in general depends only upon the collector potential and the velocity of the ions in the corresponding swarm, and not upon the nature of the orbits described by the ions in the field of force of the collector. In fact, Eq. (14a) shows that the width of the target is $2i\sqrt{1+V/V_0}$. But when this quantity exceeds the width of the force field, it is evident that the latter width must be substituted for the former in computing the current contribution of the corresponding swarm of ions. Even before the ultimate limit is reached the width of the target may be restricted to a smaller limit if condition (65) is not satisfied everywhere in the field of force. In this way the radius of the force-field or some related quantity enters into the calculation, so that the total current to the collector depends upon the "sheath-radius."

It is to be noted however that for retarding potentials on the collector the calculated target width is less than the diameter of the collector. In this case, therefore, the current to the collector is independent not only of the distribution of potential in the sheath but also of the actual dimensions of the sheath. This has already been found true for all the cases treated in this article, as will be confirmed by examining Eqs. (14a), (15a), (18b), (28b), (36b).

Again in the case of accelerating potentials, for which the calculated target width is greater than the collector diameter, simple conditions must be reached when the radius of the sheath is made sufficiently large compared with that of the collector. In such case the sheath diameter will eventually become greater than the target widths calculated for all the different swarms, so that the total current to the collector once more becomes independent of the sheath radius. This explains the result already found that the current to a cylindrical or spherical collector at accelerating potential reaches a limit as the sheath radius is indefinitely increased. By the same argument the current for very large sheaths must be independent of the actual shape of the sheath, which justifies the assumption of this kind which we made in treating the cases where unsymmetrical sheaths are formed about the collector.

The equations derived in Section II hold for ions of either sign, subject to the conventions already made that the potential of the collector with respect to the space is to be counted positive when the collector attracts the ions. When ions of both signs are being collected, the contribution from the ions of each sign are simply added, in doing this we may need to use a different size of sheath for the positive current and for the negative current. The calculation of the current assumes that the sheath sizes and velocity distributions are known, but of course, the real use of the equations lies in the determination of the velocity and density distribution of the ions from the observed characteristics. This is possible when conditions are such that practically all the current comes from one group of ions, as for instance the electrons. Unless the sheath sizes are measured directly, it will further be necessary to use collectors of each shape and size that the current does not depend upon the sheath size.

With these conditions it will be possible by examination of the volt-ampere characteristic to determine whether the velocity distribution of the ions considered is one of the types treated above. But it is also interesting to consider the inverse problem of calculating directly the velocity distribution function from the volt-ampere characteristic.

For the plane electrode with retarding voltages this calculation is simple, since the slope of the curve of current vs. voltage at any point is proportional to the number of ions having a velocity component normal to the plane which lies within a chosen fixed small range centering about the value corresponding to the collector voltage. In practice however the large size of the electrode necessary to realize the conditions of the plane electrodes sometimes disturb the normal conditions of the discharge so much as to vitiate the results. It is better then to use a spherical or cylindrical collector, chosen so small that the current to it will be independent of the size and shape of the sheath.

The current taken by a *spherical collector* with retarding potentials when the sheath is very large is given by Eqs. (13) and (7).

$$i = 2\pi r^2 N\epsilon \int_{\sqrt{-2\epsilon V/m}}^{\infty} u\left(u^2 + 2\frac{\epsilon}{m}V\right) g(0,\, u)du.$$

This equation can only be applied if the ions have no drift motion so that the distribution function involves the velocities components only in the form of their resultant $\sqrt{u^2+v^2+w^2}$. By two successive differentiations the above equation is reduced to

$$\frac{d^2i}{dV^2} = 4\left(\frac{\epsilon}{m}\right)^2 \pi r^2 N\epsilon\, g\left(0, \sqrt{-2\frac{\epsilon}{m}V}\right) \tag{67}$$

Thus an analysis of the volt-ampere characteristic of the collector enables us to determine the distribution function $g(0, c)$ where c is the velocity equivalent to the potential V of the collector. This function actually involves only one variable, so that we may call it $G(c)$. Then $c^2G(c)dc$ is the fraction of the number of ions in a given volume for which the resultant velocity c falls within the range from c to $c+dc$.

For the *cylindrical collector* under the same conditions of large sheaths and a retarding potential we have from Eq. (10)

$$i(V) = 4\pi r l N\epsilon \int_{\sqrt{-2\epsilon V/m}}^{\infty} u\sqrt{u^2+2\frac{\epsilon}{m}V}\; f(u, o)du$$

On introducing a new variable of integration ψ given by

$$\psi = \frac{1}{2}\frac{m}{\epsilon}u^2$$

and putting

$$A = 4\sqrt{2}\,\pi\left(\frac{\epsilon}{m}\right)^{3/2} N\epsilon, \qquad f\left(\sqrt{2\frac{\epsilon}{m}\psi}, 0\right) = F(\psi)$$

this becomes

$$\frac{1}{A}i(V) = \int_{-V}^{\infty} \sqrt{\psi+V}\; F(\psi)d\psi$$

which on differentiation gives

$$\frac{2}{A}\frac{d}{dV}i(v) = \int_{-V}^{\infty} \frac{F(\psi)}{\sqrt{\psi+V}}\,d\psi \tag{68}$$

If we regard $i(V)$ as a known function, this constitutes an integral equation for $F(\psi)$, of a type which has been treated by Liouville. The solution can be written[8]

$$F(\psi) = \frac{4}{\pi A}\int_0^{\infty} i''(-\psi - t^2)dt \tag{69}$$

where according to the usual notation the prime denotes differentiation of i with respect to its argument. It is to be noted that this formula

[8] See for example Volterra "Les Equations Integrales, etc."

involves only the values of i corresponding to negative potentials on the collector. Other formulas equivalent to (69) can easily be derived, which are more convenient in special cases.

This equation then enables us to calculate the distribution function $F(\psi)$ from the observed characteristic of a cylindrical collector. The derivation of the original Eq. (10) assumes only that the distribution function is symmetrical in u and v. Thus $2\pi qF(\psi)dq$ is the fraction of the ions for which the resultant q of the velocity components u and v falls in a specified range dq, ψ being the voltage equivalent of q.

Special properties of the Maxwellian distribution. The equations which have been given show that when a plane, cylindrical or spherical collector is placed in a Maxwellian field of ions, the current for retarding potentials is given by the same simple formula, namely,

$$i = i_0 e^{\epsilon V/kT} \tag{70}$$

where i_0 is the current taken by the collector when it is at the same potential as the surrounding space.

Although we have assumed that the ions describe free orbits in the sheath, Eq. (70) is the same as would be derived on the assumption that the ions made collisions with each other and reached a state of statistical equilibrium, for in such a distribution the law of distribution of velocities at every point is Maxwellian, while the distribution of density is given by Boltzmann's law.[9] That is, Eq. (70) indicates that in the sheaths of the three collectors mentioned, the ions retain the distribution of velocities and of densities proper to a state of statistical equilibrium, even though they make no collisions with each other.

That this is so is well known for the plane case, and it can also be shown directly to be true for the cylindrical and spherical cases. Taking the ***cylindrical case*** first and considering only the inward-moving ions, let $I_a(u, v)dudv$ stand for the current carried across a unit area of the sheath boundary by ions having velocity components in the specified range $du\,dv$, and $I_s(u_s, v_s)\,du_s dv_s$ for the corresponding quantity at some intermediate distance s where the velocity components are u_s, v_s. For a Maxwellian distribution at the boundary we have according to Eq. (24)

$$I_a(u,v)dudv = Ae^{-(m/2kT)(u^2+v^2)}udu dv \tag{71}$$

where A is a constant. Since the total currents carried by the ions in question across the sheath boundary and the intermediate cylinder are the same, we have

$$sI_s(u_s,v_s)du_s dv_s = aI_a(u,v)dudv \tag{72}$$

[9] Compare "Part I," p. 450.

The relations between the quantities u, v, u_s, v_s are given by Eqs. (61), (62) from which we obtain the further result

$$dudv = \begin{vmatrix} \dfrac{\partial u}{\partial u_s} & \dfrac{\partial u}{\partial v_s} \\ \dfrac{\partial v}{\partial u_s} & \dfrac{\partial v}{\partial v_s} \end{vmatrix} du_s dv_s = \frac{s}{a}\frac{u_s}{u} du_s dv_s \tag{73}$$

On substituting from (71), (61), (73), in (72) we find

$$I_s(u_s, v_s) du_s dv_s = A e^{\frac{\epsilon V_s}{kT}} e^{-(m/2kT)(u_s^2+v_s^2)} u_s du_s dv_s \tag{74}$$

An examination of Eq. (61) of transformation shows that the values of v_s range from $-\infty$ to ∞ and of u_s from 0 to ∞ since we are assuming that V_s is negative. Bearing this fact in mind and comparing (74) with (71) we see that the ions at s have the same velocity distribution which they had at q, while the number of ions in a given velocity range stand at the ratio $e^{\epsilon V_s/kT}$ at the two places. The first part of this result shows that the velocity distribution at s is still Maxwellian, while the second part shows that the variation of density from point to point is governed by Boltzmann's equation; excepting of course for the fact that only one-half the ions of a complete Maxwellian distribution are included, i. e. those having a positive radial velocity component.

It is interesting to see how these results must be modified for the case when V_s is positive, so that the ions are accelerated as they move into the sheath. Eq. (74) still holds, but a consideration of the transformation Eq. (61) shows that certain values of u_s and v_s are excluded. In fact the only points in the u_s, v_s plane which correspond to real values of u and to positive values of u_s are those lying to the right of the v_s axis and outside of the ellipse

$$u_s^2 + \left(1 - \frac{s^2}{a^2}\right) v_s^2 - 2\frac{\epsilon}{m} V_s = 0 \tag{75}$$

constituting the region shown shaded in Fig. 7. We see then that even with an accelerating potential on the collector the Maxwellian nature of the distribution is not completely obliterated as the ions move into the sheath. At any point certain whole groups of ions proper to a Maxwellian distribution are absent, but the remaining ions have precisely the distribution according to velocity-coordinates which is characteristic of a complete Maxwellian distribution having the same temperature as that of the ions at the sheath boundary. Furthermore, the space density of ions in each velocity-class outside the excluded ones satisfies Boltzmann's equation.

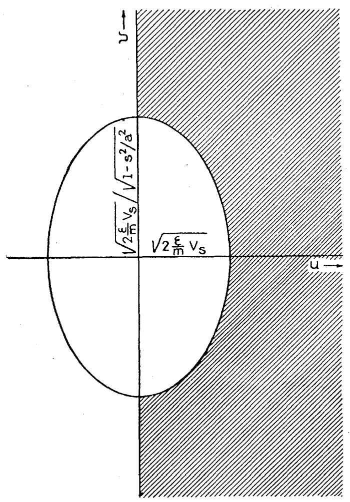

FIG. 7.

The argument for the *spherical case* is the same, except that the tangential component q_s replaces v_s. With a retarding potential on the collector the ions in the sheath will have a distribution of velocities and densities given by Maxwell's and Boltzmann's equations. With an accelerating potential, the distribution will be the same except that at any radial distance s no ions will be present for which the velocity points, having coordinates u_s, v_s, w_s, fall within an oblate ellipsoid which can be generated by revolving the ellipse of Fig. 7 about the u axis.

In view of the similarity of these results for the sheaths about the plane, the cylindrical and the spherical collector, it seems reasonable to infer that they hold for force-fields of any shape under the conditions assumed, and in fact we can state a general theorem concerning such fields of force which may be formulated in the following way. Let us consider a conservative system consisting of a large number of particles moving in an enclosed space and continually exchanging energy and momentum with each other so that a state of statistical equilibrium is reached in which there is a law of distribution of positional and velocity components which we shall, for brevity, denote by D_M (D_M being then the distribution defined by Maxwell's and Boltzmann's equations). In this space we imagine to exist a region A in which there is no interaction by collision or otherwise between the particles, and in which there is a field of force acting on the particles. Any interior boundary surfaces of A (such as the surfaces of collectors) we shall assume to be perfectly reflecting. The particles penetrating into A will then describe "orbits" under the influence of the force field and eventually return to the exterior boundary of A. If we consider not merely the orbits actually described by particles but the totality of orbits which may be followed by a particle in A with all possible modes of starting from interior points, there may be some of these paths which never carry the particles outside of A. These we will call "interior orbits." Our theorem is then as follows: if the field A has no interior orbits, the distribution of particles throughout it will be D_M; if interior orbits exist, the distribution will be D_M except for an excluded class of particles, these being the ones which would describe the interior orbits if they were present.

To prove the theorem, let us imagine that after a steady state has been reached we introduce temporarily some mechanism allowing the particles in A to interchange momenta. For instance we can introduce a sufficient number of particles which are not acted upon by the field in A, and which are initially given the distribution D_M in A and outside of A. Through collisions with these "neutral" particles the original particles, without loss of energy of momentum as a whole, will acquire a distribution which is D_M in A if it was not such before. This distribution in A will persist if the neutral particles are removed, and will remain in equilibrium with the distribution outside of A.[10]

Now it is a characteristic of D_M that it includes particles moving along every possible orbit[11] in A. If there are no interior orbits, all the orbits in A eventually return to the exterior boundary. Since in the new state of equilibrium the particles are once more moving freely in A, the new state cannot differ from the old one, so that even before the introduction of the "catalyzing" agent the distribution in A must have been D_M. If there are interior orbits, then in the new state of equilibrium there must be present circulating particles which describe these orbits. But there orbits never intersect those of the other particles in A, so that if we imagine the circulating particles to be removed, the resulting distribution must be the same as that which existed before the introduction of the catalyst. The original distribution is then seen to differ from D_M only in the way stated in the theorem.

We have so far assumed all "collectors" in to be perfectly reflecting. In this state, the current of particles moving toward a given collector P is equal to that moving away from it. Considering only the ingoing current, this may be composed of a current I_P consisting of particles which move directly from the exterior boundary of A to P, and of a current I_{PQ} of particles which reach P after having previously touched another collector or collectors Q. If we now assume that the collectors

[10] The proof of this statement may be based upon thermodynamic grounds: for according to the Second Law the system cannot depart from the state of statistical equilibrium once this has been reached through the temporary introduction of the "catalyzing" mechanism. In the usual treatment of such problems from the standpoint of statistical mechanics, the assumption that the system remains in a state of equilibrium is equivalent to the assumption that the system is "quasi-ergodig."

[11] Since the number of particles in not infinite, it is understood of course that this statement is not to be taken literally, but in the sense that a particle can be found which describes a path which differs but little from a previously chosen path.

absorb all the particles which reach them, it is seen that the current I_P will remain unchanged, while the current I_{PQ} will vanish. This indicates the way in which the results given by our theorem must be modified when they are applied to the calculation of the current to a collector in an actual case.

An interesting and important application of this theorem is the following. We consider two closed surfaces S and R, (S enclosing R), which are equipotentials in a field of force, and we imagine that particles pass in both directions through S with a Maxwellian distribution of velocities and describe free orbits in the interior. By the above theorem we know that the distribution of velocity and density of particles throughout the interior of S will be given by Maxwell's and Boltzmann's laws except for the absence of certain numbers of particles which would describe interior orbits. If we now imagine these circulating particles to be supplied, the distribution will be D_M throughout. In this distribution let i_S^R be the total current leaving S and reaching R, i_R^S the total current leaving R and reaching S, and I_S, I_R the currents *per unit area* crossing S and R respectively. Then since we are assuming that conditions are steady and that the particles meet with no obstacle within S,

$$i_S^R = i_R^S \tag{76}$$

Further, the space densities of the particles is uniform over S and uniform over R and the ratio of these densities is given by Boltzmann's equation. Since the average velocities of the particles are also equal at the two surfaces, the currents I_S and I_R must be related by the Boltzmann equation, i. e.

$$I_S = I_R e^{(\varphi_R - \varphi_S)/kT} \tag{77}$$

where φ_R, φ_S are the potentials of R and of S, and T is the temperature of the Maxwell distribution of velocities. On dividing the first of these equations by the second we obtain

$$i_S^R / I_S = e^{(\varphi_R - \varphi_S)/kT} i_R^S / I_R \tag{78}$$

The current I_S can be divided into two equal parts, of which one is composed of outwardly and the other of inwardly moving particles, and similarly with I_R. The above equation therefore is still valid if we take I_S to mean the current traversing S from the exterior to the interior, and I_R the current traversing R in the contrary sense. Finally we can remove the particles which describe closed orbits not cutting S or R since they contribute nothing to any of the currents considered. Eq. (78) now gives a relation between the solution of two problems, in the first of which the surface S emits particles with a Maxwellian distribution of velocities and a current density I_S toward an interior surface R which collects a total current i_S^R; while in the second problem the surface R emits particles with a Maxwellian distribution and current density I_R toward an exterior surface S which receives the total current i_R^S. It is to be noted that I_S, I_R are currents *per unit area* while i_R^S, i_S^R are total currents. If we let A_S, A_R stand for the areas of S and of R, f_S^R for the fraction of the total current leaving S that reaches R in the first problem, and f_R^S for the fraction of the total current leaving R that reaches S in the second problem, then Eq. (78) can be written

$$f_S^R = \frac{A_R}{A_S} e^{(\varphi_R - \varphi_S)/kT} f_R^S \tag{79}$$

By means of Eq. (79) we can deduce our Eq. (28a) relating to the case of concentric cylinders from an equation previously deduced by Schottky[12] who considered the problem, inverse to ours, of a cylinder emitting ions having a Maxwellian distribution of velocities toward an outer concentric cylinder. For a retarding potential of V volts applied between the cylinders Schottky found the result which in our notation is

$$\frac{i}{i_0} = e^{-\eta} \frac{a}{r}\left[1 - erf\sqrt{\frac{r^2\eta}{a^2 - r^2}}\right] + erf\sqrt{\frac{a^2\eta}{a^2 - r^2}} \tag{80}$$

Here i is the current collected by the outer cylinder, i_0 the saturation current from the inner cylinder, a and r the radii of the outer and inner cylinders respectively. Now a *retarding* potential in this problem is an *accelerating* one in the converse problem where the two cylinders keep their potentials but the ions are emitted by the outer one and collected by the inner one. Using Eq. (79) and letting S stand for the outer and R the inner cylinder, we find for the solution of the converse problem a result which is easily seen to be identical with Eq. (80). It is to be noted that in Schottky's problem the distribution of velocities in the space between the cylinders is not Maxwellian even though the field retards the outward moving ions. This is in agreement with our general theorem since obviously there exist interior orbits in this field.

As another example of the application of the general theorem, we may consider the characteristic of a collector which has been used by A. F. Dittmer. This consists of a plane, electrode A placed behind and close to a parallel electrode B which is pierced by a small circular hole, the front plate B shielding the back one A from the discharge so that the

[12] **Schottky, Ann. d. Physik, 44 1011 (1914).**

ions reaching the latter must pass through the hole. Let us suppose that the space in front of B is filled with ions of which those of one sign, say for definiteness the electrons, have a Maxwellian distribution of velocities and a density uniform in space. If now we make the potential of B sufficiently negative, a positive ion sheath will be formed in front of B, and the surface C bounding the sheath may be taken to be a plane parallel to B if the sheath thickness is not too small compared with the diameter of the hole. Taking the potential of C to be zero, we bring A to a positive potential sufficient to repel positive ions and allow only electrons to reach it. The electrons leaving the sheath edge C move into a retarding field, but there will be a certain surface S capping the hole, such that the component of electric force normal to the planes A, B, C vanishes at each point of S. Every electron which is able to reach this surface S finds itself in a field accelerating it toward A, as soon

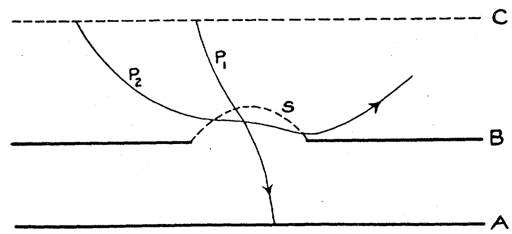

FIG. 8

as it crosses S. Most of the electrons crossing S consequently reach A, these being the ones moving along some such path as P_1 in Fig. 8. A few electrons having very high transverse velocities on reaching S will move along paths like P_2 and so fail to reach A after crossing S. But these latter electrons have such high transverse velocity on leaving C that their number, according to the Maxwellian law, is very small. Thus we may take the surface S as being effectively the "collector" determining the current to A.

The potential of this collector varies from point to point, being the same as that of B at the edge, but more positive in the middle on account of the influence of the positive potential on A. If the potentials of B and A are so adjusted that the center of S is negative, the entire surface will constitute a collector, which repels electrons. A little consideration shows that in this case there are practically no interior orbits, in the space included between C and $S+B$; that is, there are no paths leading to A and crossing S which do not originate from C. Therefore according to the general theorem the distribution of velocities among the electrons crossing S must be Maxwellian, and the distribution of space density must be governed by Boltzmann's law. This result enables us to calculate very simply the current to A if we know the shape of the surface S and the potential at each point of it. A particularly simple case occurs when the field strengths at the plate B on each side are equal in magnitude, for in this case S is plane.

Effect of reflection and of secondary electron emission on the characteristics of collectors. We come now to the consideration of the question as to how the equations of Section II relating to the characteristics of collectors must be modified when either reflection or secondary electron emission occurs at the collector. In the first place, when the potential of the collector is such as to accelerate ions of a particular class, any of these ions which may be reflected at the collector surface will eventually be drawn back to the surface, since they leave it with less velocity than they had on striking it.* Thus the volt ampere characteristic of a collector can only be affected by reflection when the *field at the collector surface* is such as to repel the ions considered. In accordance with our convention with regard to the sign of voltages, this means that the equations of Section II will be in error on account of reflection only in case V is negative.

Although the reflection coefficient of a surface for electrons is usually assumed to be a function of the angle of incidence of the electrons at the surface, there is reason to believe that for a surface carefully cleaned by heating or ion bombardment the dependence upon the angle of incidence disappears.[13] Even with this simplification the calculation of the characteristics of a collector on which reflection takes place will be in general very complicated, since the reflection coefficient still depends upon the velocity with which the electrons reach the collector. The carrying out of the calculation is scarcely justified, since the form of the reflection function is known only approximately and for a few different materials, and since in any event it is known that a collector may change its reflection coefficient for electrons very greatly in the course of a single series of experiments through the deposition of a film on its sur-

* Exceptions to this statement may occur in certain cases which are illustrated by the following example. Let a plane collector be divided into two parts A and B placed close together, the current to each part being measured separately. Suppose that ions are being drawn to the collectors by a rather large accelerating voltage, so that they arrive at A or B in a direction practically normal to the surface. If an ion is reflected from a point near the edge of A, whether it returns to A or to B depends largely on the transverse component of velocity which it has after reflection so that a certain fraction only of the current reflected from A will return to A, the rest going to B. Thus the ratio of the currents to A and to B will depend to a certain extent upon their relative reflecting powers.

[13] Cf. C. Tingwalt, Zeits. f. Physik **34**, 280 (1925).

face. We shall content ourselves with pointing out in a general way the effect of reflection in the cases which we have treated.

The effect of electron reflection on the collector characteristic depends upon the velocity distribution of the electrons. In certain cases the result of the reflection can be found quite simply. For instance, in case the distribution is of type A or B (where the electrons all have equal energies) the electrons arriving at the collector surface will all have equal velocities. If now the reflection coefficient is independent of the angle of incidence but is known as a function of the velocity, it is plain that the actual characteristics of the collector can be found by multiplying the current calculated on the basis of no reflection by $1-R(v)$, where v is the velocity with which electrons arrive at the collector as determined by its potential and $R(v)$ is the corresponding reflection coefficient.

Another distribution giving simple results is the Maxwellian one. We have seen that in this kind of distribution a collector of any shape whatever gives a characteristic for retarding potential which is such that if the logarithm of the current is plotted against the voltage a straight line is obtained. Furthermore the distribution of velocities among the electrons arriving at the collector is the same whatever the collector potential may be. Thus the *percentage reflection is the same for every value of the collector voltage*, and consequently the effect of reflection will be simply to displace downward the straight line of the semilogarithmic plot. The slope of this line will however still correspond to the temperature of the distribution. The effect of the reflection will only become apparent when the collector passes through the space potential when there should be a sudden increase of the current due to the recapture of the reflected electrons.

Secondary emission of electrons due to electron bombardment is in general indistinguishable from electron reflection, and its effects upon the characteristics of a collector will be the same as those of reflection. Emission of electrons due to the bombardment of *positive* ions however would have the effect of increasing the total current to a collector whose potential is such as to attract positive ions and keep off all electrons from the discharge (i. e. a highly negative potential). This increase would be very noticeable, since the positive ion current to the collector in this range usually increases only very slowly as the collector is made more negative. Thus the study of the characteristics of a collector under these conditions would show immediately whether or not the emission of electrons occur. We may note for instance that the characteristics of a large plane collector in a mercury arc discharge at low pressure (8 bars) show that the current of secondary electrons due to the bombardment of mercury positive ions of 1000 volts velocity cannot be greater than 5 percent of the total current to the collector,[14] this being approximately the experimental error in determining the current.

We have said that the equations developed in this article are valid even when reflection occurs in the cases where the ions are being accelerated toward the collector. This is true in the sense that the equations give the correct value of the current as a function of the voltage *and of the dimensions of the sheath*, when these dimensions are a factor. But the effect of reflection will be to make the dimensions of the sheath different from what they would be in the absence of reflections, so that in this manner the value of the current may be affected indirectly. If an accurate theory giving the dimensions of the sheath calculated from the space charge equation were available, a comparison between the calculated and observed dimensions would show whether or not reflection was taking place.

RESEARCH LABORATORY,
GENERAL ELECTRIC COMPANY, SCHENECTADY, N. Y.
July 6, 1926.

[14] Langmuir and Mott-Smith, G. E. Rev. **27**, 545 (1924).