IOP Publishing Plasma Sources Science and Technology
Plasma Sources Sci. Technol. **18** (2009) 014004 (8pp) doi:10.1088/0963-0252/18/1/014004

The plasma–sheath boundary: its history and Langmuir's definition of the sheath edge

J E Allen[1,2,3]

[1] University College, Oxford OX1 4BH, UK
[2] OCIAM, Mathematical Institute, Oxford OX1 3LB, UK
[3] Imperial College, London SW7 2BW, UK

E-mail: john.allen@eng.ox.ac.uk

Received 3 July 2008, in final form 13 August 2008
Published 14 November 2008
Online at stacks.iop.org/PSST/18/014004

Abstract
The introduction of the terms sheath in 1923 and plasma in 1928 by Langmuir is described, followed by their use in the Tonks and Langmuir theory of the positive column at low pressures in 1929. Attention is drawn to the development of Langmuir's ideas during the period from 1923 to 1929. The well-known Bohm criterion for sheath formation, published in 1949, is shown to be closely related to the earlier work of Tonks and Langmuir. The much-used version of the Bohm criterion with the equality sign is obtained by employing the two-scale theory of the plasma and sheath, for the case where $\lambda_D/L \to 0$.

A generalized Bohm criterion is obtained by introducing the ion velocity distribution; the resulting expression can be understood by considering the propagation of ion-acoustic waves. The plasma–sheath boundary is found to be a sonic surface. Other generalizations of the Bohm criterion are given, including a mixture of positive ions, the presence of negative ions and a non-Maxwellian electron velocity distribution.

1. History of the terms plasma and sheath, with reference to ionized gases

1.1. First use of the term sheath

In 1923 Langmuir was engaged in studying the positive column of the mercury arc [1]. Analysing his experimental results he assumed that both electrons and ions were moving with random velocities, and that the associated energy was 1 eV. Langmuir then considered the case of a plane electrode immersed in the plasma. When such an electrode was electrically insulated it became negatively charged 'until the number of electrons it receives no longer exceeds the number of positive ions'. It was thought that the potential attained could not exceed 1 V, because some electrons must always reach the electrode, together with the positive ions. I shall now quote the paragraph that followed.

'Let us now assume that the plane electrode be charged to a negative potential of 100 volts. Electrons will therefore be prevented from approaching close to the electrode, whereas positive ions will be drawn towards it. There will therefore be a layer of gas near the electrode where there are positive ions but no electrons, and in this region there will therefore be a positive ion space charge. The outer edge of this *sheath* of ions will therefore have a potential of -1 and the positive ions pass this outer edge with a velocity corresponding to 2 volts'.

This was the first usage of the term 'sheath' and, at the time of writing, Langmuir was considering a region where the density of positive ions considerably exceeded that of the electrons; it was a pure ion sheath. As his research progressed Langmuir later realized that sheaths do contain some electrons near the plasma boundary. It was also found that, in these low-pressure discharges, the velocities of the positive ions were not randomly distributed but were mainly directed towards the wall of the discharge tube.

In another paper Langmuir and Mott-Smith [2] considered the sheath around a probe to extend out to a field-free plasma region. A diagram in that paper, reproduced in figure 1, illustrates a charged particle moving in a straight line, through

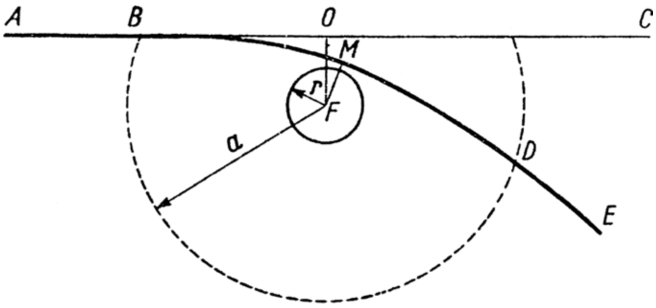

Figure 1. Diagram of the path of a charged particle passing through a sheath surrounding a cylinder or sphere, taken from an early paper by Langmuir and Mott-Smith [2]. It was later realized that the surrounding plasma was not field free, but sustained weak electric fields over considerable distances. The diagram is copied from a historical document and the lettering need not concern us here.

the plasma, until it reaches the sheath. It was later realized that a weak, but important, electric field exists in the plasma, which is quasi-neutral. The potential difference associated with this field is comparable to the temperature of the electrons, measured in volts.

1.2. First use of the term plasma

Langmuir, in a paper concerned with oscillations in ionized gases, first introduced the term 'plasma' into the subject of ionized gases [3]. In his description of ion-acoustic waves he wrote 'except near the electrodes, where there are *sheaths* containing very few electrons, the ionized gas contains ions and electrons in about equal numbers so that the resultant space charge is very small. We shall use the name *plasma* to describe this region containing balanced charges of ions and electrons'. Langmuir did not explain why he used this name; the present writer, however, is amongst those who believe that it comes from the Greek verb 'to mould'. Langmuir employed a variety of discharge tubes in his work; some of these had side arms which were filled with the glowing 'plasma', which moulded itself to the shape of the vessel.

2. The theory of Tonks and Langmuir

A most important paper is that by Tonks and Langmuir entitled 'A general theory of the plasma of an arc', published in 1929 [4]. We shall attempt to describe the essence of the work here, to be followed up later by the details of a simpler theory involving 'cold' ions.

Consider the positive column of a low-pressure discharge, the ions may be assumed to travel radially to the wall of the tube without making any collisions on the way. Referring to figure 2, the ions at any radius r are assumed to arrive from all radii z, where $0 \leqslant z \leqslant r$. If the ionization rate per unit volume is denoted by N_z and the ions start from rest and reach radius r with velocity v_z, then the ion density at r is given by

$$n_{\rm i} = \frac{1}{r}\int_0^r \frac{N_z z\,{\rm d}z}{v_z}. \qquad (2.1)$$

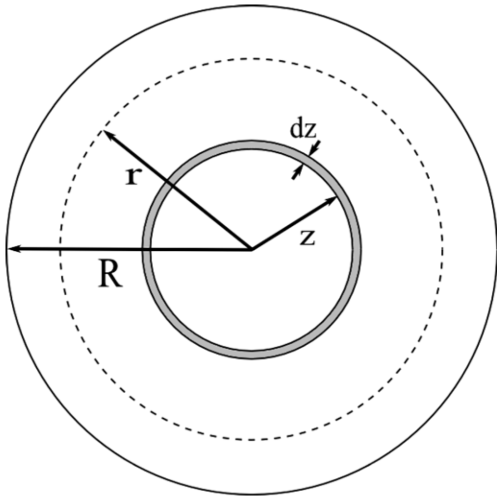

Figure 2. Diagram to illustrate the Tonks–Langmuir theory of the low-pressure positive column. The original paper also includes the plane case and the spherical case [4].

The electrons are assumed to have a Boltzmann distribution

$$n_{\rm e} = n_0 \exp(eV/kT_{\rm e}), \qquad (2.2)$$

where the potential V is measured with respect to the axis of the tube, $V < 0$.

Substituting these expressions in Poisson's equation

$$\varepsilon_0 \nabla^2 V = -(n_{\rm i} - n_{\rm e})e \qquad (2.3)$$

we obtain

$$\varepsilon_0 \nabla^2 V + \frac{e}{r}\int_0^r \frac{N_z z\,{\rm d}z}{v_z} - n_0 e \exp(eV/kT_{\rm e}) = 0. \qquad (2.4)$$

This is the complete 'plasma–sheath equation', first set out by Tonks and Langmuir [4]. The first term may be dropped where $n_{\rm e} \approx n_{\rm i}$, that is, within the plasma, and the resulting equation is known as the 'plasma equation'.

For long mean free paths and purely radial ion motion we have the following expression for v_z:

$$v_z = \left[\frac{2e(V_z - V)}{M}\right]^{1/2}. \qquad (2.5)$$

By substituting from equation (2.5) into (2.4), Tonks and Langmuir solved the plasma equation numerically for the following cases: (a) ionization rate proportional to the electron density and (b) ionization rate uniform throughout the plasma. The analysis given here refers to cylindrical geometry whereas Tonks and Langmuir also dealt with the plane one-dimensional case and the spherical case. The solutions all exhibit the same behaviour, as illustrated in figure 3. The 'plasma solution' turns around, i.e. the electric field tends to infinity at a certain radius, the potential being finite at that limiting point. The plasma solution diverges from the true 'plasma–sheath solution' as one approaches this limiting point. According to Tonks and Langmuir the plasma region ends when the Laplacian term in Poisson's equation becomes *'equal to a certain fractional part'* of either of the other two terms.

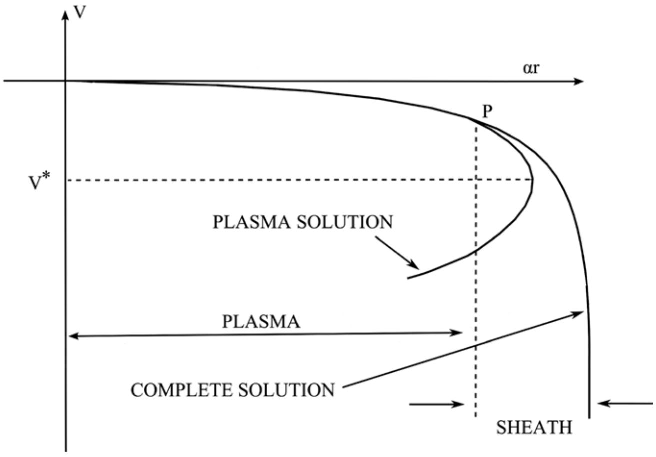

Figure 3. Diagram to illustrate the relationship of the 'plasma solution' and the 'complete solution' of the plasma–sheath integral equation [4], α is an adjustable constant. The (somewhat ill-defined) point P represents the plasma–sheath boundary. The potential at the point of infinite slope on the plasma solution is denoted by V^*. This represents the plasma–sheath boundary in the limiting case where $\lambda_D/R \to 0$.

Table 1. Values of eV^*/kT_e for three plasma geometries.

	Plane	Cylindrical	Spherical
$N_z \propto n_e$	0.854	1.155	1.418
N_z uniform	0.854	1.054	1.195

In physical terms the sheath begins where quasi-neutrality breaks down. Normalizing the potential, radius, electron and ion densities in the following way: $\eta = eV/kT_e$, $r_n = r/R$, $n_{en} = n_e/n_0$ and $n_{in} = n_i/n_0$.

Poisson's equation can be written in the form

$$\left(\frac{\lambda_D}{R}\right)^2 \nabla^2 \eta = (n_{en} - n_{in}), \tag{2.6}$$

where $\lambda_D = (\varepsilon_0 k T_e / n_0 e^2)^{1/2}$, the electron Debye distance. It is seen that quasi-neutrality breaks down at a (somewhat ill-defined) position which approaches the limiting point as the ratio of the Debye distance to the tube radius tends to zero. Table 1 gives the numerical values of the potential at this point, denoted by V^*, in units of the electron temperature, measured in volts. The results of two different ionization models are shown, ionization proportional to the electron density and uniform ionization throughout the plasma.

We are interested in situations where $\lambda_D/R \ll 1$, where λ_D is the Debye distance and R is the radius of the tube (or other relevant dimension), otherwise there would be no plasma region. A general result, therefore, is that the ions have kinetic energies comparable to the electrons, when they arrive at the sheath edge. The velocities are *not* randomly directed, however, but are directed towards the wall with a certain velocity distribution. Tonks and Langmuir were not especially interested in the form of this distribution, but we shall return to this point later when discussing the generalized form of the Bohm criterion. The latter is satisfied at the limiting point V^* in the limit $\lambda_D/L \to 0$, where L is the characteristic length of the system (radius or width).

3. The Bohm criterion for sheath formation

The case considered by Bohm [5] was that of a sheath adjacent to a negatively charged wall which acts as a sink for the charged particles that reach it. The geometry is plane and the wall is so negative that most of the electrons are reflected back into the plasma. Three assumptions were made by Bohm: firstly that the ionization in the sheath is negligible, secondly that the electric field at the plasma edge is negligible and thirdly that the energy distribution of the ions can be neglected. In this connection we refer to 'cold ions' by which we mean that the ions, at any particular place, all have the same energy. In addition it is assumed that the electron velocity distribution is Maxwellian and that their density can be described by the Boltzmann relation

$$n_e = n_{es} \exp(eV/kT_e), \tag{3.1}$$

where V is measured from the sheath edge. The potential can be measured with respect to any suitable point, with this choice n_{es} represents the electron density at the plasma boundary. The kinetic energy which the ions possess when they enter the sheath will be denoted by eV_0. (Note that V_0 is not the potential at the plasma boundary; this has been taken as zero.) Since the ions are assumed to all have the same energy and no new ions are produced then

$$n_i = n_s[1 - (V/V_0)]^{-1/2}, \tag{3.2}$$

where n_i is the ion density at any plane in the sheath. The potential V is negative and the ion density drops as the ions are speeded up. We have employed the equation of continuity and the conservation of energy in obtaining this equation.

Poisson's equation can now be written down and takes the form

$$\varepsilon_0 \nabla^2 V = -n_s e\{(1 - V/V_0)^{-1/2} - \exp(eV/kT_e)\}, \tag{3.3}$$

where we have put $n_{es} = n_{is} = n_s$, the density at the plasma boundary. Multiplication by $\mathrm{d}V/\mathrm{d}x$ and integration with respect to x gives

$$\frac{1}{2}\varepsilon_0 \left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2 = n_s[2eV_0(1 - V/V_0)^{1/2} + kT_e \exp(eV/kT_e) + C], \tag{3.4}$$

where C is a constant of integration. This can be determined by taking the electric field at the sheath edge to be vanishingly small so that the equation can be written as

$$2n_s eV_0(1 - V/V_0)^{1/2} + n_s kT_e \exp(eV/kT_e) - \frac{1}{2}\varepsilon_0 \left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2 = 2n_s eV_0 + n_s kT_e. \tag{3.5}$$

It is of interest to note that this equation represents a *stress balance* as pointed out by Allen [6]; the electrons have a Maxwellian velocity distribution and the ions have a streaming motion. The right-hand side describes the rate of flow of momentum, carried by the ions, plus the electron pressure at the plasma boundary. The wall (or probe) experiences a force due to the impinging electrons and ions, mainly due to the ions since most of the electrons return to the plasma. In addition there is an inward electrical force on the charged wall and the net force is equal to the sum of the pressure and the flow of momentum at the plasma boundary.

For small values of V, equation (3.5) can be expanded by the binomial theorem to give

$$2n_s eV_0\left[1-\frac{1}{2}\left(\frac{V}{V_0}\right)-\frac{1}{8}\left(\frac{V}{V_0}\right)^2-\cdots\right] + n_s kT_e\left[1+\frac{eV}{kT_e}+\frac{1}{2}\left(\frac{eV}{kT_e}\right)^2+\cdots\right]-\frac{1}{2}\varepsilon_0\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2 = 2n_s eV_0 + n_s kT_e$$

and it can be seen that the zeroth and first order terms cancel leaving

$$\frac{1}{2}\varepsilon_0\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2 = \frac{n_s eV^2}{2}\left(\frac{e}{kT_e}-\frac{1}{2V_0}\right).$$

This term cannot be negative, therefore

$$eV_0 \geqslant \frac{1}{2}kT_e. \tag{3.6}$$

We can now see a connection between this work and that carried out by Tonks and Langmuir some twenty years earlier. The energies of the positive ions on entering the sheath are comparable to the electron temperature, measured in volts. Two differences can be noted, however; firstly the distribution of ion energies has been neglected and, secondly, possibly of more significance, the above version of the Bohm criterion contains an inequality sign, rather than an equality sign.

The much employed version of the so-called Bohm criterion has an equality sign. In this case the kinetic energy of the ions, on leaving the plasma and entering the sheath, is $(kT_e/2)$. Perhaps the first use of the equality sign was by Allen and Thonemann [7], who were carrying out experiments on current limitation in low-pressure mercury arcs. These authors recognized that $\mathrm{d}n_e/\mathrm{d}V = \mathrm{d}n_i/\mathrm{d}V$ at the plasma–sheath boundary, at least to a high degree of approximation, and this led to the version of the Bohm criterion with an equality sign. More generally we can write $\mathrm{d}\rho/\mathrm{d}V = 0$, i.e. the first derivative of the space charge w.r.t. potential vanishes at the plasma–sheath boundary. The equality form of the Bohm criterion can be justified by considering the limiting case when $\lambda_D/R \to 0$. Strictly speaking it applies to this case only. But when λ_D/R is small it gives excellent estimates of the potential, the electron (and ion) density, and the velocity of the positive ions at the plasma boundary.

If the equality form of the Bohm criterion is adopted, and the resulting value of eV_0, i.e. $eV_0 = \frac{1}{2}kT_e$, is substituted into equation (3.5), a further integration is then required in order to obtain the potential distribution in the sheath. This cannot be done analytically, but can be readily calculated numerically and the result is shown in figure 4. The unit of distance is the Debye distance and the example shown is for a wall potential of $3kT_e/e$. The form of the curve does not depend on this potential, provided that it is negative enough to reflect most of the electrons back into the plasma, so that the Boltzmann relation is valid. The potential is measured from the plasma boundary, which is approached asymptotically from the sheath side.

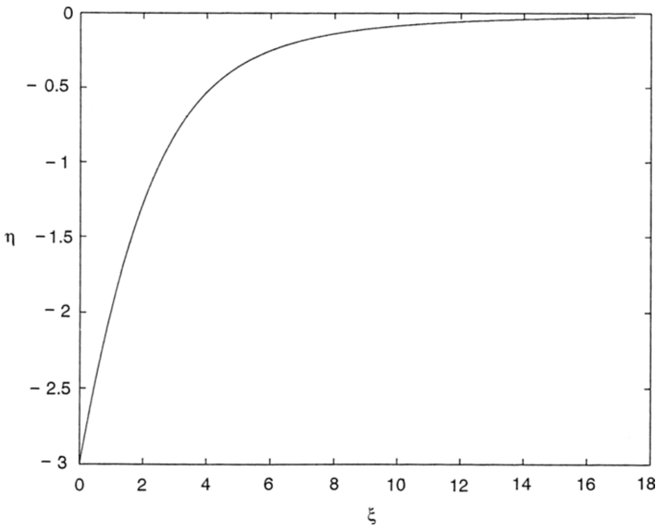

Figure 4. An example of the potential distribution in a sheath using the equality version of the Bohm criterion, i.e. $eV_0 = kT_e/2$. We can recall that Bohm obtained the relation with an inequality sign [5], rather than the equality sign employed here.

Some quotations from Bohm's work have been given by Godyak and Sternberg [8] and will be reproduced here for discussion. 'But near a highly negative electrode, such as a wall or probe, very few electrons have enough energy to penetrate. Here the ions are mostly positive, moving under the influence of their own space charge. Because the ion density is usually quite high, the entire potential drop between electrode and plasma can be sustained *in a very small positive ion sheath, usually less than a millimetre in thickness*.'

'Within the plasma region a very gradual change in potential takes place. Although there is no precise point at which the sheath begins, there is a transition region in which the plasma region, characterized by the stability of the neutral state of zero field, is in a short distance replaced by *the sheath region, characterized by negligible electron density*.'

Reading these quotations it is clear that a sheath entirely composed of positive ions is envisaged. We must pay greater attention, however, to the mathematics than to the written word. Bohm's theoretical analysis certainly deals with a space charge region which contains both positive ions and electrons. Just inside the sheath the respective densities are almost equal. Further clarification comes by studying the two-scale theory, as we shall see below. What is meant in this paper by 'two-scale theory' is the limiting case where $\lambda_D/L \to 0$, where L is the characteristic length of the system. Finite values of λ_D/L will be considered elsewhere in this Langmuir Festschrift, by Riemann.

4. The work of Caruso and Cavaliere

In an important but little-known paper, Caruso and Cavaliere [9] made a rigorous study of the limiting case where $\lambda_D/L \to 0$; in this case the transition from the plasma to

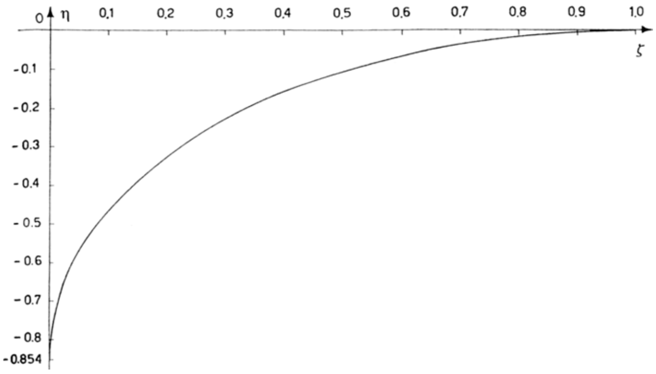

Figure 5. The plasma solution for the Langmuir model, for the plane case, computed by Caruso and Cavaliere [9].

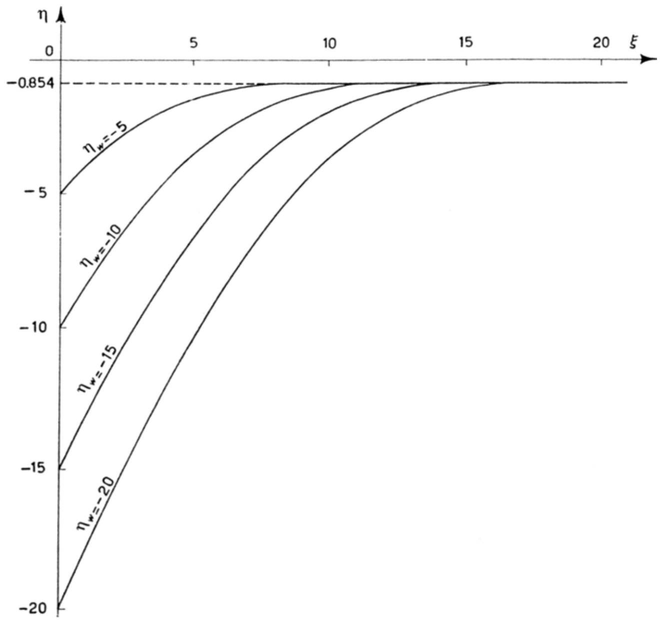

Figure 6. The potential distribution in the sheath, calculated by Caruso and Cavaliere for different values of the wall potential [9]. The unit of distance is the electron Debye distance, and the curves are all identical in form.

the sheath takes place precisely at the singular point V^* of the potential distribution. This is a two-scale theory and Caruso and Cavalier were able to calculate not only the potential distribution in the plasma, already found by Tonks and Langmuir, but also the distribution in the sheath (for the plane case). These are shown in figures 5 and 6. This two-scale theory, for the case where $\lambda_D/L \to 0$, although a mathematical concept which does not correspond precisely to the physical situation, is an extremely useful one. One can obtain excellent estimates of the following quantities at the sheath edge: the electric potential, the electron (and ion) density and the ion energy distribution. This limiting case, where $\lambda_D/L \to 0$, yields a sheath criterion which can be compared with, and indeed identified with, the Bohm criterion. Two different points have to be made, however; firstly, it corresponds to the equality version of the Bohm criterion. Secondly, the criterion is more sophisticated than that of Bohm in that the ion energy distribution is taken into account, as discussed later in this paper. The paper of Caruso and Cavaliere, which is readily accessible, will not be reproduced here, but a simpler model to illustrate the two-scale concept will be presented below.

5. The 'cold ion' fluid model

A simplified model of the plasma in which *cold ions* are considered was introduced by Woods [10] and further employed by Kino and Shaw [11]. In this model the ions at any particular point in space all have the same velocity; this condition is imposed on the ions, the pressure tensor being discarded. Relatively simple differential equations are obtained using this approach. Woods considered the cylindrical case only, since he was interested in ion-acoustic waves in the positive column. Kino and Shaw studied both planar and cylindrical geometries and found results remarkably close to those found by Tonks and Langmuir with their more elaborate model.

Initially this technique was used to study the plasma region only, but it can also be employed to obtain a simplified version of the two-scale model, involving both plasma and sheath. Here we shall follow a succinct presentation by Riemann [12], following a recent controversy about the Bohm criterion [8]. The plane case will be considered.

The electron density was assumed to follow the Boltzmann relation

$$n_e = n_0 \exp(eV/kT_e). \tag{5.1}$$

The continuity equation, assuming ionization proportional to the electron density, is

$$\frac{d}{dx}(n_i v) = z n_e. \tag{5.2}$$

The momentum equation for the positive ions takes the form

$$Mv\frac{dv}{dx} + e\frac{dV}{dx} + (Mzvn_e)/n_i = 0. \tag{5.3}$$

The only one of Maxwell's equations relevant to the present problem is Poisson's equation:

$$\varepsilon_0 \frac{d^2V}{dx^2} = -e(n_i - n_e). \tag{5.4}$$

An ionization term appears in the momentum equation because the newly produced ions are given the momentum corresponding to the main ion flow. Using this stratagem the principle of conservation of momentum is not violated in this model.

The meaning of the symbols should be apparent. It is convenient to measure the potential with respect to the distance where the quasi-neutral 'plasma solution' breaks down. The density at this point will also be used as the unit for normalization of the electron and ion densities.

The physical problem has two associated characteristic lengths; the electron Debye distance $(\varepsilon_0 kT_e/n_0 e^2)^{1/2}$ and the ionization length $L = c_s/z$, where c_s is the ion-acoustic velocity. Normalizing the equations in two different ways, corresponding to these different length scales we obtain the following two sets of equations, where

$$\varepsilon = \lambda_D/L, \quad \varsigma = x/L, \quad \xi = x/\lambda_D.$$

Using the ionization length

$$\frac{d}{d\zeta}(nu) = e^{\eta}, \tag{5.5}$$

$$u\frac{du}{d\zeta} = -\frac{d\eta}{d\zeta} - (e^{\eta}u)/n, \tag{5.6}$$

$$\varepsilon^2\frac{d^2\eta}{d\zeta^2} = -(n - e^{\eta}) \tag{5.7}$$

and using the Debye distance

$$\frac{d}{d\xi}(nu) = \varepsilon e^{\eta}, \tag{5.8}$$

$$u\frac{du}{d\xi} = -\frac{d\eta}{d\xi} - (\varepsilon e^{\eta}u)/n, \tag{5.9}$$

$$\frac{d^2\eta}{d\xi^2} = -(n - e^{\eta}). \tag{5.10}$$

As ε tends to zero the first set of equations yields the 'plasma solution', obtained by Kino and Shaw [11]. Woods [10] had previously considered the cylindrical case, in connection with a study of experimental results on the propagation of ion-acoustic waves. In the plane case the results are

$$x = 2\tan^{-1} u - u, \qquad \eta = \ln 2 - \ln(1 + u^2). \tag{5.11}$$

The second set of equations, using the Debye distance as the unit of distance, is identical to the set of equations used by Bohm [5], and reproduced above. The velocity of the positive ions, on entering the sheath region from the plasma region, is equal to unity in normalized units, so that the Bohm criterion with the equality sign is obtained. It is evident that the sheath cannot be considered in isolation; both the plasma and the sheath must be considered.

6. The generalized sheath criterion

The generalized sheath criterion can be readily derived by introducing a distribution of ion energies. By considering the sheath only, as had Bohm [5], but with this one change, the following result was obtained by Harrison and Thompson [13]:

$$\frac{1}{2}M\left\langle\frac{1}{v^2}\right\rangle^{-1} \geqslant \frac{1}{2}kT_e, \tag{6.1}$$

where the factor of $\frac{1}{2}$ has been included for easy comparison with equation (3.6).

A more precise statement can be made if we consider the two-scale theory for the case where $\lambda_D/L \to 0$, due to Caruso and Cavaliere [9], and illustrated in detail above for the simple 'cold ion' case. There is no longer any uncertainty as to the position of the plasma boundary since the 'plasma solution' is valid as far as the singularity indicated by V^* in figure 3. The generalized Bohm criterion can now be written as

$$\frac{1}{2}M\left\langle\frac{1}{v^2}\right\rangle^{-1} = \frac{1}{2}kT_e. \tag{6.2}$$

It was pointed out by Allen [14] that this criterion is valid for the Tonks–Langmuir theory of the positive column [4] and for the spherical probe theory of Bohm *et al* [15]. It is of interest to note that the latter authors do in fact derive a 'plasma solution', unlike the more well-known paper of Bohm [5].

The physical interpretation of the generalized Bohm criterion is not immediately clear. Allen has given a physical interpretation of the criterion by considering the propagation of ion-acoustic waves in the plasma [14]; the equation describing the propagation of longitudinal waves is

$$\omega_{pe}^2\int_c \frac{df_e/dv}{(v-\omega/k)}\,dv + \omega_{pi}^2\int_c \frac{df_i/dv}{(v-\omega/k)}\,dv = k^2, \tag{6.3}$$

where the path of integration is the Landau contour,

$$\int_{-\infty}^{+\infty} f_e\,dv = 1, \qquad \int_{-\infty}^{+\infty} f_i\,dv = 1$$

and the other symbols have their usual meaning. For small wave-numbers the rhs can be neglected, which means that quasi-neutrality holds true. The equation has been derived from Poisson's equation and the terms on the lhs represent perturbations of the electron and ion density respectively. Assuming a Maxwellian distribution for the electrons, and using the fact that the phase velocity is much less than the electron thermal velocity, it can be shown that the generalized sheath criterion corresponds to the case where $\omega/k = 0$. Thus we conclude that the flow at the plasma boundary is 'sonic', the plasma boundary is a Mach surface. This concept had previously been developed for the case of monoenergetic ions by Stangeby and Allen [16]. An ion-acoustic wave is unable to travel back into the plasma from the plasma boundary since its velocity in the laboratory frame is reduced to zero.

We can note in passing that the generalization can be extended to the case of a non-Maxwellian electron velocity distribution. Equation (6.2) is then replaced by the following:

$$\frac{1}{M}\left\langle\frac{1}{v^2}\right\rangle = -\frac{1}{m}\int_{-\infty}^{+\infty}\frac{1}{v}\frac{df_e}{dv}\,dv. \tag{6.4}$$

7. Generalization of the Bohm criterion to the case of a mixture of different ions

The extension of the Bohm criterion to the case of a mixture of different positive ions is straightforward and was derived by Cooke in a DPhil Thesis [17]. The motivation in that case was of a study of mercury discharges in which molecular mercury ions were formed by associative ionization as a result of strong irradiation with mercury lines [18], thus a mixture of atomic and molecular ions was obtained. A convenient way to obtain a general result is to consider once again the propagation of longitudinal waves of long wavelength, i.e. ion-acoustic waves, in the plasma and simply allow for different ion populations.

An equation similar to equation (6.2), but allowing for different masses and ionic charges yields

$$\frac{n_1 Z_1^2}{n_e v_1^2 M_1} + \frac{n_2 Z_2^2}{n_e v_2^2 M_2} = \frac{1}{kT_e}. \tag{7.1}$$

A special case is that where the following relationships hold

$$\tfrac{1}{2} M_1 v_1^2 = \tfrac{1}{2} Z_1 k T_e \qquad \text{and} \qquad \tfrac{1}{2} M_2 v_2^2 = \tfrac{1}{2} Z_2 k T_e, \tag{7.2}$$

i.e. each species of ion attains its own 'Bohm velocity' but these results *are not valid in general*. This generalized form of the Bohm criterion, equation (7.1), is less useful than that for a single ion species. We cannot use it in a purely sheath model. When any particular plasma model is developed it is found that the criterion holds at the plasma boundary. A plasma model is necessary to determine the ion densities and velocities appearing on the lhs of the equation.

8. Generalization of the Bohm criterion to the case of electronegative plasmas

A situation of considerable interest is that of electronegative plasmas, since these are much employed in plasma processing. An early paper on this subject was by Boyd and Thompson [19], but these authors did not discuss the multi-valued functions which are obtained from the mathematical model. The subject was studied by Braithwaite and Allen [20] who attacked the problem in three different ways: (a) by considering sonic flow at the sheath edge, (b) by using the standard approach from the sheath model and (c) by studying a spherical probe. The same result was obtained from all three approaches. Let us consider the sonic flow approach. The assumption was made that the negative ions and the electrons both satisfied the Boltzmann relation. A condition for the validity of this expression is that the thermal velocity must be much greater than the phase velocity of the wave. If, however, at a plasma boundary the phase velocity of the wave in the laboratory frame of reference is zero, then this criterion is automatically satisfied. Braithwaite and Allen employed the result for the speed of sound in a plasma with two electron temperatures. This could be extended to the case of negative ions, because the masses of the negative particles do not enter into the equation. The sound velocity in this case is given by Wickens and Allen [21] as

$$c_s = [(n_- + n_e)kT_- T_e/(n_- T_e + n_e T_-)]^{1/2}. \tag{8.1}$$

By considering the simple case of a spherical probe the authors obtained the following equation for the required potential difference, in terms of the ratio of the negative-ion density to electron density in the bulk plasma:

$$\alpha_0 \exp[-\eta_s(\gamma - 1)] = (1 - 2\eta_s)/(2\gamma\eta_s - 1). \tag{8.2}$$

Figure 7 shows plots of this (normalized) potential difference η_s as a function of the ratio of the negative-ion density to electron density in the bulk plasma α_0. Multi-valued functions are to be seen and examination of equation (8.2) shows that

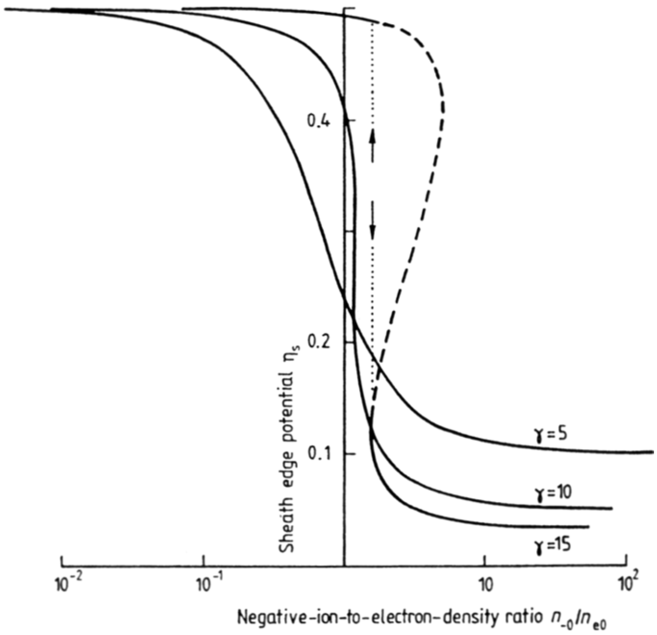

Figure 7. Plot of the normalized potential at the sheath edge against the ratio of the negative-ion density to electron density in the bulk plasma for three values of γ, the ratio of electron temperature to negative-ion temperature. The arrowed dotted line indicates the necessary jump in the solution when γ exceeds the critical value $(5 + \sqrt{24}) \approx 9.9$ [20].

these are found when $\gamma \geqslant (5 + \sqrt{24}) \approx 9.9$, where γ is the ratio of electron temperature to negative-ion temperature. The required solution is that corresponding to the smallest value of the sheath edge potential.

This generalization of the Bohm criterion, as that of the mixture of different positive ions, is less useful than that of a single ion species. The criterion indicates a breakdown of quasi-neutrality, but it does *not* guarantee an overall monotonic sheath potential distribution as far as the wall (or probe). The Bohm criterion is associated with a local happening. One result of this fact is that it cannot be used (in general) to calculate the ion flux leaving the discharge. In some situations an outer plasma may develop, after a space-charge region, with the formation of a further plasma–sheath boundary adjacent to the tube wall. The Bohm criterion would then be applicable at more than one position in the discharge, as discussed by Franklin [22]. Further discussion, with reference to other work in this field, can be found in a paper by Allen [23].

9. The Bohm criterion in the presence of radio-frequency fields

The present paper is concerned with steady state plasmas, and not situations involving transients. In view of their considerable industrial importance, however, we must briefly comment on radio-frequency plasmas, and enquire whether the Bohm criterion is still valid. According to papers by Riemann [24] and Allen and Skorik [25], who studied the propagation of longitudinal waves, the Bohm criterion is still valid and the *longitudinal* electric fields are largely screened out of the plasma. Capacitively coupled radio-frequency plasmas, however, are rather complex entities and it may well be that further work is required [22]. The electron velocity distribution differs from the Maxwellian when stochastic heating is operative, so that the Boltzmann relation may not be sufficiently accurate for application in this case. Electrons gain energy, on the average, when they dive into the sheath and then return to the plasma. There is a departure from thermal equilibrium for the electrons.

10. Summary and conclusions

The introduction and the use of terms *plasma* and *sheath* by Irving Langmuir has been described. Not surprisingly Langmuir's ideas became clearer as his work on gas discharge plasmas progressed. The paper with Lewi Tonks should be taken to represent Langmuir's view of what is meant by *plasma* and *sheath* and what is meant by the *sheath edge*. The latter is situated where quasi-neutrality breaks down. This takes place near a specific point which is the limit of the 'plasma solution' as $\lambda_D/L \to 0$. Much use can be made of this limit, which gives excellent estimates of a number of plasma parameters when λ_D/L is small.

The relationship between the well-known Bohm criterion and the earlier work of Tonks and Langmuir is explained. The equality form of the Bohm criterion is closely related to the Langmuir limit when $\lambda_D/L \to 0$. It is pointed out that the Bohm criterion was derived mathematically and that his description in the text is not in accord with his theoretical presentation.

Reference is made to a rigorous two-scale theory, due to Caruso and Cavaliere, that provides solutions for both the plasma and sheath regions in the limit $\lambda_D/L \to 0$. A simpler version of the two-scale theory is presented here to illustrate the concept.

Other definitions of the sheath edge can be found in the literature, but the writer suggests that to have a variety of definitions is not helpful. Some authors wish to consider a sheath composed of positive ions only, but in many practical cases the electron density does not become negligible until very near the wall of the discharge tube, and so there is no positive ion sheath. If high negative voltages are applied to an electrode, such a positive ion sheath is produced; the Child–Langmuir law then (and only then) becomes a good approximation.

The study of the transition between plasma and sheath for finite values of λ_D/L is an extensive subject not considered in this paper, but is left to other contributors to this commemorative publication. The effect of collisions is not discussed in this paper, but it can be shown that the Bohm criterion is still valid if $\lambda_D/L \to 0$, where L is now taken to be the ion mean free path. Another topic not discussed is that of the plasma–sheath boundary in a magnetic field; this subject is being actively studied at present, the Bohm criterion still holds, but for the *normal component* of the ion velocity at the sheath edge. The Boltzmann relation cannot always be taken as valid in the presence of a magnetic field. This paper does not deal with transient cases, where the version of the Bohm criterion with the inequality sign may be applicable. An exception is made for the important case of radio-frequency plasmas, where it has been claimed that the Bohm criterion is still valid. Further information on these and other topics can be found in the comprehensive review papers by Riemann [26, 27] and Franklin [22].

References

[1] Langmuir I 1923 Positive ion currents in the positive column of the mercury arc *Gen. Electr. Rev.* **26** 731

[2] Langmuir I and Mott-Smith H 1924 Studies of electric discharges in gases at low pressures *Gen. Electr. Rev.* **27** 449

[3] Langmuir I 1928 Oscillations in ionized gases 1928 *Proc. Natl Acad. Sci.* **14** 627

[4] Tonks L and Langmuir I 1929 A general theory of the plasma of an arc 1929 *Phys. Rev.* **34** 876

[5] Bohm D 1949 *The Characteristics of Electrical Discharges in Magnetic fields* ed A Guthrie and R K Wakerling (New York: McGraw–Hill) chapter 3

[6] Allen J E 1974 Probe measurements *Plasma Physics* ed B E Keen (London and Bristol: Institute of Physics Publishing)

[7] Allen J E and Thonemann P C 1954 Current limitation in the low-pressure mercury arc 1954 *Proc. Phys. Soc.* B **67** 768

[8] Godyak V and Sternberg N 2002 On the consistency of the collisionless sheath model *Phys. Plasmas* **9** 4427

[9] Caruso A and Cavaliere A 1962 The structure of the collisionless plasma-sheath transition *Nuovo Cimento* **26** 1389

[10] Woods L C 1965 Density waves in low-pressure plasma columns *J. Fluid Mech.* **23** 315

[11] Kino G S and Shaw E K 1966 Two-dimensional low-pressure discharge theory *Phys. Fluids* **9** 587

[12] Riemann K-U 2003 Comment on 'On the consistency of the collisionless sheath model' *Phys. Plasmas* **10** 3432

[13] Harrison E R and Thompson W B 1959 The low pressure plane symmetric discharge *Proc. Phys. Soc.* **74** 145

[14] Allen J E 1976 A note on the generalized sheath criterion *J. Phys. D: Appl. Phys.* **9** 2331

[15] Bohm D, Burhop E H S and Massey H S W 1949 *The Characteristics of Electrical Discharges in Magnetic fields* ed A Guthrie and R K Wakerling (New York: McGraw–Hill) chapter 2

[16] Stangeby P C and Allen J E 1970 Plasma boundary as a Mach surface *J. Phys. A: Gen. Phys.* **3** 304

[17] Cooke M J 1980 *DPhil Thesis* University of Oxford

[18] Johnson P C, Cooke M J and Allen J E 1978 The formation of a plasma by strong irradiation of mercury vapour with mercury lines *J. Phys. D: Appl. Phys.* **11** 1877

[19] Boyd R L F and Thompson J B 1959 The operation of Langmuir probes in electronegative plasmas 1959 *Proc. R. Soc.* A **252** 102

[20] Braithwaite N St J and Allen J E 1988 Boundaries and probes in electronegative plasmas *J. Phys. D: Appl. Phys.* **21** 1733

[21] Wickens L M and Allen J E 1979 Free expansion of a plasma with two electron temperatures *J. Plasma Phys.* **22** 167

[22] Franklin R N 2003 The plasma-sheath boundary region *J. Phys. D: Appl. Phys.* **36** R309

[23] Allen J E 2004 A note on the Bohm criterion for electronegative gases *Plasma Sources Sci. Technol.* **13** 48

[24] Riemann K-U 1992 The validity of Bohm's sheath criterion in rf-discharges *Phys. Fluids* B **4** 2693

[25] Allen J E and Skorik M A 1993 The Bohm criterion in the presence of radio-frequency fields *J. Plasma Phys.* **50** 243

[26] Riemann K-U 1991 The Bohm criterion and sheath formation *J. Phys. D: Appl. Phys.* **24** 493

[27] Riemann K-U 2000 Theory of the plasma-sheath transition *J. Tech. Phys.* **41** 89