Petersson Inner Product of Binary Theta Series

A computational approach

Nicolas SIMARD

McGill University

October 8th, 2016

Motivation : Stark's remark

In *"L-functions at $s = 1$. II. Artin L-functions with Rational Characters"*, Stark makes the following remark :

An application of Theorem 1 gives

$$L'(0, \chi, H/\mathbb{Q}) = \log \epsilon,$$

where ϵ is the real root of

$$x^3 - x - 1 = 0$$

Actually, it is easier to note that $L(1, \chi, H/\mathbb{Q})$ is the residue at $s = 1$ of the zeta function of the real quadratic subfield of H. In any case,

$$\langle f, f \rangle = 3 \log \epsilon.$$

Eisenstein series

Let $k \geq 2$ be an even integer. Define

$$G_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

for $k \geq 4$ and

$$G_2(z) = \frac{1}{8\pi\Im(z)} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n) q^n,$$

where

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$

Petersson inner product

Let $f, g \in S_k(\Gamma_0(N), \chi)$ be two cusp forms. The Petersson inner product of f and g is defined as

$$\langle f, g \rangle = \int\int_{\Gamma_0(N)\backslash\mathcal{H}} f(x+iy)\overline{g(x+iy)}y^k \mathrm{d}\mu,$$

where

$$\mathrm{d}\mu = \frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

is the $\mathrm{SL}_2(\mathbb{R})$-invariant measure on $\mathcal{H}$. Note that the integral does not converge if both f and g are not cusp forms.

Theta series attached to ideals

Let K be an imaginary quadratic field of discriminant $D < -4$ and let $\mathcal{O}_K$ be its ring of integers. Fix an integer $\ell \geq 0$.
To each integral ideal $\mathfrak{a}$ of K, one can attach the following theta series :

$$\theta_{\mathfrak{a}}^{(2\ell)}(z) = \sum_{x \in \mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

where χ_D is the Kronecker symbol. If $\ell > 0$, then

$$\theta_{\mathfrak{a}} \in S_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

Theta series attached to Hecke characters of K

Let I_K denote the group of fractional ideals of K and let ψ be a Hecke character of infinity type 2ℓ, i.e. a homomorphism

$$\psi : I_K \longrightarrow \mathbb{C}^\times$$

such that

$$\psi((\alpha)) = \alpha^{2\ell}, \qquad \forall \alpha \in K^\times.$$

Then one defines

$$\theta_\psi = \sum_{\mathfrak{a} \subseteq \mathcal{O}_K} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

where χ_D is the Kronecker symbol. If $\psi^2 \neq 1$, then

$$\theta_\psi \in S_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

Stark's example

Let

$$K = \mathbb{Q}(\sqrt{-23})$$

and let ψ be a non-trivial Hecke character of infinity type 0, i.e. a non-trivial character of the class group. Then

$$\text{Stark's } f = \text{our } \theta_\psi \in M_1(\Gamma_0(23), \chi_{-23}).$$

Some questions

Keeping Stark's example in mind, we have the following questions :

- Can we find explicit formulas for the Petersson inner product of those theta series (whenever it makes sense) ?
- Can we efficiently compute it ?
- Can we use those formulas/computations to study the arithmetic properties of those quantities ?

Some questions

Keeping Stark's example in mind, we have the following questions :

- Can we find explicit formulas for the Petersson inner product of those theta series (whenever it makes sense) ?
- Can we efficiently compute it ?
- Can we use those formulas/computations to study the arithmetic properties of those quantities ?

The main question is

Question

Can we p-adically interpolate those formulas for $\ell > 0$ and take the limit as $\ell \to 0$ p-adically to obtain the weight one case ?

The case $\ell = 0$

Theorem

Let ψ be a Hecke character of infinity type 0 *which is not a genus character. Then*

$$\langle \theta_\psi, \theta_\psi \rangle = -h_K \sum_{[\mathfrak{a}] \in Cl_K} \psi^2(\mathfrak{a}) \log(N(\mathfrak{a})^{1/2} |\eta(\mathfrak{a})|^2)$$

$$= h_K \log \prod_{[\mathfrak{a}] \in Cl_K} (N(\mathfrak{a})^{1/2} |\eta(\mathfrak{a})|^2)^{-\psi(\mathfrak{a})^2}.$$

Here,

$$\eta(z) = exp(2\pi i/24) \prod_{n=1}^{\infty} (1 - q^n).$$

Petersson norm of the θ_ψ (with $\ell > 0$)

Theorem

Let ψ be a Hecke character of K of infinity type 2ℓ, where $\ell > 0$. Then

$$\langle \theta_\psi, \theta_\psi \rangle = h_K (|D|/4)^\ell \sum_{[\mathfrak{a}] \in Cl_K} \psi^2(\mathfrak{a}) \partial^{2\ell-1} G_2(\mathfrak{a}).$$

Here,

$$\partial f = \frac{1}{2\pi i} \frac{\partial f}{\partial z} - \frac{k}{4\pi \Im(z)} f$$

is the Shimura-Maass differential operator, which preserves the graded algebra of almost holomorphic modular forms.

Petersson inner product of the theta series $\theta_{\mathfrak{a}}$

Corollary

Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of K and suppose $\ell > 0$. Then

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle = C_K^{(2\ell)} N(\mathfrak{b})^{2\ell} \sum_{\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K} \lambda_{\mathfrak{c}}^{2\ell} \partial^{2\ell-1} G_2(\mathfrak{c}),$$

where

$$C_K^{(2\ell)} = 4(|D|/4)^{\ell}.$$

Formally obtaining the case $\ell = 0$ from the case $\ell > 0$

Strictly speaking, the formula

$$\langle \theta_\psi, \theta_\psi \rangle = h_K (|D|/4)^\ell \sum_{[\mathfrak{a}] \in \mathrm{Cl}_K} \psi^2(\mathfrak{a}) \partial^{2\ell-1} G_2(\mathfrak{a}).$$

does not make sense for $\ell = 0$, since the expression

$$\partial^{-1} G_2$$

is not well-defined.

Formally obtaining the case $\ell = 0$ from the case $\ell > 0$

Strictly speaking, the formula

$$\langle \theta_\psi, \theta_\psi \rangle = h_K(|D|/4)^\ell \sum_{[\mathfrak{a}] \in \mathrm{Cl}_K} \psi^2(\mathfrak{a}) \partial^{2\ell-1} G_2(\mathfrak{a}).$$

does not make sense for $\ell = 0$, since the expression

$$\partial^{-1} G_2$$

is not well-defined. However, we observe that

$$\partial_0 \log(\Im(z)^{1/2} |\eta(z)|^2) = -G_2(z),$$

so

$$"\partial^{-1} G_2(z) = -\log(\Im(z)^{1/2} |\eta(z)|^2)"$$

and we *formally* obtain the case $\ell = 0$ from the case $\ell > 0$.

$K = \mathbb{Q}(\sqrt{-23})$ (class number 3, one genus) : $\ell = 0$

For ψ a non-trivial Hecke character of infinity type 0, the explicit formula in case $\ell = 0$ gives

$$\langle f, f \rangle = \langle \theta_\psi, \theta_\psi \rangle = 3 \log \epsilon,$$

where

$$\epsilon = \prod_{[\mathfrak{a}] \in \mathrm{Cl}_K} \left(N(\mathfrak{a})^{1/2} |\eta(\mathfrak{a})|^2 \right)^{-\psi(\mathfrak{a})^2}$$

is the real root of

$$x^3 - x - 1$$

and generates the Hilbert class field of K.

Class field theory

Theorem

Let D be a prime discriminant and let H be the Hilbert class field of $K = \mathbb{Q}(\sqrt{D})$. Then

$$\prod_{\psi \neq 1} \langle \theta_\psi, \theta_\psi \rangle = \frac{2}{w_H} h_K^{h_k - 2} h_H \operatorname{reg} H,$$

where $w_H = |\mathcal{O}_H^\times|$, h_H is the class number of H and $\operatorname{reg} H$ is the regulator of H.

Algebraic part of the Petersson inner product for $\ell > 0$

Let

$$\Omega_K = \frac{1}{\sqrt{4\pi|D|}}\left(\prod_{j=1}^{|D|-1}\Gamma\left(\frac{j}{|D|}\right)^{\chi_D(j)}\right)^{w_K/4h_k}$$

be the Chowla-Selberg period attached to K.

Corollary

For $\ell > 0$, the complex numbers

$$\frac{\langle\theta_\psi,\theta_\psi\rangle}{\Omega_K^{4\ell}} \quad \textit{and} \quad \frac{\langle\theta_{\mathfrak{a}},\theta_{\mathfrak{b}}\rangle}{\Omega_K^{4\ell}}$$

are algebraic.

$K = \mathbb{Q}(\sqrt{-23})$ (class number 3, one genus) : $\ell > 0$

In K, the prime 2 splits as

$$2\mathcal{O}_K = \mathfrak{p}_2\bar{\mathfrak{p}}_2$$

and

$$\mathrm{Cl}_K = \{1, [\mathfrak{p}_2], [\bar{\mathfrak{p}}_2]\}.$$

Moreover, we have $\langle \theta_{\bar{\mathfrak{p}}_2}, \theta_{\mathcal{O}_K} \rangle = \overline{\langle \theta_{\mathfrak{p}_2}, \theta_{\mathcal{O}_K} \rangle}$. We will focus on

$$\langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell}.$$

$K = \mathbb{Q}(\sqrt{-23})$ (class number 3, one genus) : $\ell > 0$

Consider the algebraic number

$$a(\ell) = \langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell}.$$

For $\ell = 1, 2, 4$ and 5, we find that $a(\ell)^3$ is a root of a monic cubic polynomial and generates the Hilbert class field over K.

Example

$a(1)$ is a root of the polynomial

$$x^9 - 2816x^6 - 905216x^3 - 89915392.$$

$K = \mathbb{Q}(\sqrt{-23})$ (class number 3, one genus) : $\ell > 0$

Consider the algebraic number

$$a(\ell) = \langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell}.$$

For $\ell = 3, 6$ and 9, we find that $a(\ell)$ is a root of a cubic polynomial and generates the Hilbert class field over K.

Example

$a(3)$ is a root of

$$x^3 - 6740x^2 - 169034720x - 1027491892288.$$

$K = \mathbb{Q}(\sqrt{-23})$ (class number 3, one genus) : $\ell > 0$

A few computations of the Gramm matrix for this basis.

ℓ	$\det(\langle \theta_{\mathfrak{a}_i}^{(2\ell)}, \theta_{\mathfrak{a}_j}^{(2\ell)} \rangle)_{\mathfrak{a}_i,\mathfrak{a}_j \in \mathrm{Cl}_K} / (\Omega_K^{4\ell})^3$
1	$-2^{10}23$
2	$-2^{14}19 \cdot 23 \cdot 619$
3	$-2^{18}5^2 11 \cdot 23 \cdot 337 \cdot 27299$
4	$-2^{22}7^2 23 \cdot 163 \cdot 2113 \cdot 117741979$
5	$-2^{26}5^3 23 \cdot 229 \cdot 23761 \cdot 808991 \cdot 20338663$
6	$-2^{30}5^2 11^2 13 \cdot 19 \cdot 23 \cdot 67^2 101 \cdot 868697 \cdot 505912247899$

$K = \mathbb{Q}(\sqrt{-23})$ (class number 3, one genus) : $\ell > 0$

Consider now the algebraic number

$$N(\psi, \ell) = \langle \theta_\psi, \theta_\psi \rangle / \Omega_K^{4\ell}$$

For $\ell = 1, 2, 4$ and 5, the numbers $N(\psi_i, \ell)$, for $0 \leq i \leq 2$, are distinct and their cube are the three real roots of a monic cubic polynomial.

Example

The numbers $N(\psi_i, 1)^3$, for $0 \leq i \leq 2$, are the three roots of the irreducible polynomial

$$x^3 - 6966x^2 + 11569230x - 239483061.$$

$K = \mathbb{Q}(\sqrt{-23})$ (class number 3, one genus) : $\ell > 0$

Consider now the algebraic number

$$N(\psi, \ell) = \langle \theta_\psi, \theta_\psi \rangle / \Omega_K^{4\ell}$$

When $\ell = 3, 6$ and 9, for one of the characters, say ψ_0, the algebraic number $N(\psi_0, \ell)$ is an *integer*. For the two others, we find that their cube are the roots of a monic quadratic polynomial.

Example

We have

$$N(\psi_0, 3) = 5055 = 3 \cdot 5 \cdot 337$$

and $N(\psi_1, 3)^3$ *and* $N(\psi_2, 3)^3$ *are the roots of*

$$x^2 - 16287872873193x + 3002197924865107829684587 5.$$

$K = \mathbb{Q}(\sqrt{-23})$ (class number 3, one genus) : $\ell > 0$

A few computations of the Gramm matrix for this basis.

ℓ	$\det(\langle \theta_{\psi_i}, \theta_{\psi_j} \rangle)_{1 \le i,j \le 3} / (\Omega_K^{4\ell})^3$
1	$-3^3 23$
2	$-3^3 19 \cdot 23 \cdot 619$
3	$-3^3 5^2 11 \cdot 23 \cdot 337 \cdot 27299$
4	$-3^3 7^2 23 \cdot 163 \cdot 2113 \cdot 117741979$
5	$-3^3 5^3 23 \cdot 229 \cdot 23761 \cdot 808991 \cdot 20338663$
6	$-3^3 5^2 11^2 13 \cdot 19 \cdot 23 \cdot 67^2 101 \cdot 868697 \cdot 505912247899$

Main steps in the proof (case $\ell > 0$)

1. Use the Rankin-Selberg to prove that

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{4h_k}{w_k}\sqrt{|D|}\frac{\Gamma(2\ell+1)}{(4\pi)^{2\ell+1}} L(\psi^2, 2\ell+1).$$

Main steps in the proof (case $\ell > 0$)

1. Use the Rankin-Selberg to prove that

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{4h_k}{w_k}\sqrt{|D|}\frac{\Gamma(2\ell+1)}{(4\pi)^{2\ell+1}} L(\psi^2, 2\ell+1).$$

2. Relate Hecke L-series of imaginary quadratic fields to real-analytic Eisenstein series :

$$L(\psi^2, 2\ell+1) = \frac{1}{w_K} \sum_{[\mathfrak{a}]\in \mathrm{Cl}_K} \frac{\psi^2(\mathfrak{a})}{N(\mathfrak{a})^{4\ell-s}} G_{4\ell}(\mathfrak{a}, 1-2\ell).$$

Main steps in the proof (case $\ell > 0$)

1. Use the Rankin-Selberg to prove that

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{4h_k}{w_k}\sqrt{|D|}\frac{\Gamma(2\ell+1)}{(4\pi)^{2\ell+1}} L(\psi^2, 2\ell+1).$$

2. Relate Hecke L-series of imaginary quadratic fields to real-analytic Eisenstein series :

$$L(\psi^2, 2\ell+1) = \frac{1}{w_K}\sum_{[\mathfrak{a}]\in \mathrm{Cl}_K} \frac{\psi^2(\mathfrak{a})}{N(\mathfrak{a})^{4\ell-s}} G_{4\ell}(\mathfrak{a}, 1-2\ell).$$

3. Replace real-analytic Eisenstein series by derivatives of Eisenstein series :

$$\partial^{2\ell-1} G_2(z) = (-4\pi)^{1-2\ell}\frac{\Gamma(s+2\ell+1)}{\Gamma(s+2)} G_{4\ell}(z, 1-2\ell).$$

Main steps in the proof (case $\ell > 0$)

1. Use the Rankin-Selberg to prove that

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{4h_k}{w_k} \sqrt{|D|} \frac{\Gamma(2\ell+1)}{(4\pi)^{2\ell+1}} L(\psi^2, 2\ell+1).$$

2. Relate Hecke L-series of imaginary quadratic fields to real-analytic Eisenstein series :

$$L(\psi^2, 2\ell+1) = \frac{1}{w_K} \sum_{[\mathfrak{a}] \in \mathrm{Cl}_K} \frac{\psi^2(\mathfrak{a})}{N(\mathfrak{a})^{4\ell-s}} G_{4\ell}(\mathfrak{a}, 1-2\ell).$$

3. Replace real-analytic Eisenstein series by derivatives of Eisenstein series :

$$\partial^{2\ell-1} G_2(z) = (-4\pi)^{1-2\ell} \frac{\Gamma(s+2\ell+1)}{\Gamma(s+2)} G_{4\ell}(z, 1-2\ell).$$

4. Find $\langle \theta_\mathfrak{a}, \theta_\mathfrak{b} \rangle$ using $\langle \theta_\psi, \theta_\psi \rangle$.

What we would like to know

1. Can we explain what we observed in the computations ?
2. Can we say something about the Petersson inner product of non-cuspidal weight one theta series ?

What we would like to know

1. Can we explain what we observed in the computations ?
2. Can we say something about the Petersson inner product of non-cuspidal weight one theta series ?

But again, the main question remains

Question

Can we p-adically interpolate the formulas for $\ell > 0$ *and take the limit as* $\ell \to 0$ *p-adically to obtain the weight one case ?*

Thank you !

Presentation and notes available at :
`https://github.com/NicolasSimard/Notes`
Code available at :
`https://github.com/NicolasSimard/ENT`