

Petersson Inner Product of Binary Theta Series

A computational approach

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Mobius transformations

Let \mathcal{H} be the Poincarre upper-half plane. Recall that $GL_2(\mathbb{R})_+$ acts on \mathcal{H} via Mobius transformations :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

Definition

Let $N \geq 1$ and define the Hecke subgroup of level N as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

Level N modular forms with characters

Definition

Let $N \geq 1$ and $k \geq 0$ be integers and let χ be a Dirichlet character mod N . A modular form of weight k , level N and character χ is a holomorphic function

$$f : \mathcal{H} \longrightarrow \mathbb{C}$$

such that

$$f(\gamma z) = \chi(d)(cz + d)^{-k} f(z)$$

for all $z \in \mathcal{H}$ and all $\gamma \in \Gamma_0(N)$, which satisfies certain growth conditions at the cusps. The \mathbb{C} -vector-space of such modular forms is denoted

$$M_k(\Gamma_0(N), \chi).$$

q -expansion of modular forms

Every modular form f has a Taylor (or Fourier) expansion at infinity, called its q -expansion :

$$f(z) = \sum_{n=0}^{\infty} a_n q^n,$$

where $q = \exp(2\pi iz)$. If

$$a_0(f) = 0,$$

(at all cusps) f is called a *cusp form*.

Example : weight k Eisenstein series

Let $k \geq 4$ be an even integer. Then the series

$$\sum_{m,n} \frac{1}{(mz + n)^k}$$

converges and is a modular form in $M_k(\mathrm{SL}_2(\mathbb{Z}))$. After renormalisation, the q -expansion of this Eisenstein series is

$$G_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Important non-example : weight 2 Eisenstein series

In level 1, there are no modular forms of weight 2. However, one can still define the weight 2 Eisenstein series as

$$G_2(2) = \frac{1}{8\pi\mathfrak{I}(z)} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n)q^n.$$

It is an example of an *almost holomorphic* modular form of level 1 and weight 2.

Spaces of modular forms

- $M_k(\Gamma_0(N), \chi)$ is finite dimensional.

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- For every integer $n \geq 1$, one can define a *Hecke operator* T_n (depending on k , N and χ) which acts on $M_k(\Gamma_0(N), \chi)$.

Spaces of modular forms

- $M_k(\Gamma_0(N), \chi)$ is finite dimensional.
- For every integer $n \geq 1$, one can define a *Hecke operator* T_n (depending on k , N and χ) which acts on $M_k(\Gamma_0(N), \chi)$.
- There exists a basis of common eigenvectors for all Hecke operators T_n with $(n, N) = 1$.

Petersson inner product

Let $f, g \in S_k(\Gamma_0(N), \chi)$ be two cusp forms. The Petersson inner product of f and g is defined as

$$\langle f, g \rangle = \int \int_{\Gamma_0(N) \backslash \mathcal{H}} f(x + iy) \overline{g(x + iy)} y^k d\mu,$$

where

$$d\mu = \frac{dx dy}{y^2}$$

is the $SL_2(\mathbb{R})$ -invariant measure on \mathcal{H} . Note that the integral does not converge if neither f nor g is a cusp form.

Newforms

The space $S_k(\Gamma_0(N), \chi)$ splits naturally as

$$S_k(\Gamma_0(N), \chi) = S_k(\Gamma_0(N), \chi)^{\text{new}} \oplus S_k(\Gamma_0(N), \chi)^{\text{old}}.$$

Theorem

The space $S_k(\Gamma_0(N), \chi)^{\text{new}}$ has an orthogonal basis of eigenvectors for all Hecke operators. Elements of this basis are called newforms (after suitable normalization).

Summary

1. The space $S_k(\Gamma_0(N), \chi)$ is a finite dimensional Hermitian inner product space, equipped with an action of Hecke operators.

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1. The space $S_k(\Gamma_0(N), \chi)$ is a finite dimensional Hermitian inner product space, equipped with an action of Hecke operators.
2. The subspace $S_k(\Gamma_0(N), \chi)^{\text{new}}$ has distinguished elements (the newforms) which are mutually orthogonal and are eigenvectors for all Hecke operators.

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A half-integral weight theta series

Consider the function

$$\theta(z) = \sum_{x \in \mathbb{Z}} q^{x^2} = 1 + 2q + 2q^4 + O(q^5).$$

Then

$$\theta(\gamma z) = \epsilon(cz + d)^{1/2} \theta(z),$$

for all $\gamma \in \Gamma_0(4)$ and some $\epsilon_{c,d} \in \{\pm 1, \pm i\}$.

Theta series attached to ideals

Let K be an imaginary quadratic field of discriminant $D < -4$ and let \mathcal{O}_K be its ring of integers. Fix an integer $\ell \geq 0$. To each integral ideal \mathfrak{a} of K , one can attach the following theta series :

$$\theta_{\mathfrak{a}}^{(2\ell)}(z) = \theta_{\mathfrak{a}}(z) = \sum_{x \in \mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})}.$$

Basic properties of these theta series

1. We have

$$\theta_{\mathfrak{a}} = \sum_{x \in \mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

where χ_D is the Kronecker symbol. If $\ell \neq 0$, then

$$\theta_{\mathfrak{a}} \in S_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

2. If $\lambda \in K^\times$, then

$$\theta_{\lambda\mathfrak{a}} = \lambda^{2\ell} \theta_{\mathfrak{a}}.$$

So there are essentially h_D theta series attached to K .

3. In general, the $\theta_{\mathfrak{a}}$ are *not* newforms.

Theta series attached to Hecke characters of K

Let I_K denote the group of fractionnal ideals of K . A Hecke character ψ of K of infinity type 2ℓ (and conductor 1) is a homomorphism

$$\psi : I_K \longrightarrow \mathbb{C}^\times$$

such that

$$\psi((\alpha)) = \alpha^{2\ell}, \quad \forall \alpha \in K^\times.$$

One can define

$$\theta_\psi = \sum_{\mathfrak{a} \subseteq \mathcal{O}_K} \psi(\mathfrak{a}) q^{N(\mathfrak{a})}.$$

Basic properties of these theta series

1. We have

$$\theta_\psi \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

where χ_D is the Kronecker symbol. If $\psi^2 \neq 1$, then

$$\theta_\psi \in S_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

2. The θ_ψ are newforms.

3. We have the identities

$$\theta_\psi = \frac{1}{w_K} \sum_{[\mathfrak{a}] \in \text{Cl}_K} \psi^{-1}(\mathfrak{a}) \theta_{\mathfrak{a}} \quad \text{and} \quad \theta_{\mathfrak{a}} = \frac{w_K}{h_K} \sum_{\psi} \psi(\mathfrak{a}) \theta_\psi.$$

Some questions

- Can we efficiently compute the Petersson inner product of theta series (whenever it makes sense) ?
- Can we find explicit formulas for it ?
- Can we use those formulas/computations to study the arithmetic properties of those quantities ?
- What about the p -adic properties of these quantities ?

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Petersson norm of the θ_ψ (with $\ell > 0$)

Theorem

Let ψ be a Hecke character of K of infinity type 2ℓ , where $\ell > 0$.
Then

$$\langle \theta_\psi, \theta_\psi \rangle = (|D|/4)^\ell \frac{4h_K}{w_K^2} \sum_{[\mathfrak{a}] \in Cl_K} \psi^2(\mathfrak{a}) \partial^{2\ell-1} G_2(\mathfrak{a}).$$

Here,

$$\partial f = \frac{1}{2\pi i} \frac{\partial f}{\partial z} - \frac{k}{4\pi \Im(z)} f$$

is the Shimura-Mass differential operator, which preserves the graded algebra of almost holomorphic modular forms.

Petersson inner product of the theta series θ_a

Theorem

Let \mathfrak{a} and \mathfrak{b} be ideals of K and suppose $\ell > 0$. Then

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle = C_K^{(2\ell)} N(\mathfrak{b})^{2\ell} \sum_{\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}} \mathcal{O}_K} \lambda_{\mathfrak{c}}^{2\ell} \partial^{2\ell-1} G_2(\mathfrak{c}),$$

where

$$C_K^{(2\ell)} = 4(|D|/4)^{\ell}.$$

A few direct consequences of the formula

Corollary

For $\ell > 0$,

$$\langle \theta_a, \theta_b \rangle = 0$$

whenever a and b are not in the same genus (i.e. the classes of a and b are distinct in the genus group Cl_K / Cl_K^2).

Corollary

For $\ell > 0$,

$$\langle \theta_{ac}, \theta_{bc} \rangle = N(\mathfrak{bc})^{2\ell} \langle \theta_a, \theta_b \rangle.$$

Arithmetic consequences

Let

$$\Omega_K = \frac{1}{\sqrt{4\pi|D|}} \left(\prod_{j=1}^{|D|-1} \Gamma\left(\frac{j}{|D|}\right)^{\chi_D(j)} \right)^{w_K/4h_K}$$

be the Chowla-Selberg period attached to K .

Corollary

For $\ell > 0$, the complex numbers

$$\frac{\langle \theta_\psi, \theta_\psi \rangle}{\Omega_K^{4\ell}} \quad \text{and} \quad \frac{\langle \theta_a, \theta_b \rangle}{\Omega_K^{4\ell}}$$

are algebraic.

If $\ell = 0$, the modular form θ_{α} is not a cusp form. But for θ_{ψ} , we have the following

Theorem

Let θ_{ψ} be a Hecke character of infinity type 0 and suppose that $\psi^2 \neq 1$. Then

$$\langle \theta_{\psi}, \theta_{\psi} \rangle = -\frac{4h_K}{w_K^2} \sum_{[\alpha] \in Cl_K} \psi^2(\alpha) \log(\Im(\tau_{\alpha})^{1/2} |\eta(\tau_{\alpha})|^2),$$

where $\tau_{\alpha} \in \mathcal{H}$ is the complex root attached to α and

$$\eta(z) = \exp(2\pi i/24) \prod_{n=1}^{\infty} (1 - q^n)$$

is the standard eta-function.

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Compute $\partial^n G_2$

We have the following formulas :

$$\partial G_2 = \frac{5}{6} G_4 - 2 G_2^2 \quad \partial G_4 = \frac{7}{10} G_6 - 8 G_2 G_4 \quad \partial G_6 = \frac{400}{7} G_4^2 - 12 G_2 G_6.$$

For example,

$$\partial^3 G_2 = -48 G_2^4 + 120 G_4 G_2^2 - 14 G_6 G_2 + 25 G_4^2.$$

Evaluate Hecke characters

The idea is simple : let \mathfrak{a} be a fractional ideal of K and suppose

$$\mathfrak{a}^e = \lambda \mathcal{O}_K.$$

Then

$$\psi(\mathfrak{a})^e = \psi(\mathfrak{a}^e) = \psi((\lambda)) = \lambda^{2\ell},$$

so $\psi(\mathfrak{a})$ is determined (up to a e -root of unity).

Find ideals \mathfrak{c} such that $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K$

Given ideals \mathfrak{a} and \mathfrak{b} , can we efficiently find all classes $[\mathfrak{c}]$ such that

$$\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K,$$

if any ? If we have representatives $\{\mathfrak{a}_1, \dots, \mathfrak{a}_d\}$ of $\text{Cl}_K[2]$, it suffices to find one such \mathfrak{c}_0 . Then the other solutions to the equation are

$$\mathfrak{c}_0\mathfrak{a}_i$$

for $i = 1, \dots, d$.

Class number 1

In this case,

$$\theta_{\mathcal{O}_K} = \theta_{\psi_0}$$

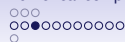
and we only need to compute

$$\langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell} \in \overline{\mathbb{Q}}.$$

Class number 1 case

Computation of $\langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell}$:

		ℓ	
		1	2
D	-7	$2^2 3$	-2^2
	-8	-2	$-2^2 5$
	-11	-2^2	$-2^3 5$
	-19	$-2^2 3^{-1} 13$	$-2^3 7 1$
	-43	$-2^3 3^{-1} 107$	$-2^4 5 6 47$
	-67	$-2^2 3^{-1} 7^2 31$	$-2^3 5 \cdot 86629$
	-163	$-2^3 3^{-1} 150473$	$-2^4 11 \cdot 461681471$



Class number 2

In this case, K has two genera. If \mathfrak{a} is a representative of the non-trivial class in Cl_K , we have

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathcal{O}_K} \rangle = \langle \theta_{\mathcal{O}_K}, \theta_{\mathfrak{a}} \rangle = 0$$

and

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{a}} \rangle = N(\mathfrak{a})^{2\ell} \langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle,$$

so it suffices to compute the quantity

$$\langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell} \in \overline{\mathbb{Q}}.$$

Class number 2

As in the class number 1 case, the quantity

$$\langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell}$$

is an integer, except for $\ell = 1$ and $D = -91, -403$ and -427 .

$$K = \mathbb{Q}(\sqrt{-23}) \text{ (class number 3, one genus)}$$

In K , the prime 2 splits as

$$2\mathcal{O}_K = \mathfrak{p}_2 \bar{\mathfrak{p}}_2$$

and

$$\text{Cl}_K = \{1, [\mathfrak{p}_2], [\bar{\mathfrak{p}}_2]\}.$$

Moreover, we have $\langle \theta_{\bar{\mathfrak{p}}_2}, \theta_{\mathcal{O}_K} \rangle = \overline{\langle \theta_{\mathfrak{p}_2}, \theta_{\mathcal{O}_K} \rangle}$, so we only care about

$$\langle \theta_{\mathfrak{p}_2}, \theta_{\mathcal{O}_K} \rangle \quad \text{and} \quad \langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle.$$

$$K = \mathbb{Q}(\sqrt{-23}) \text{ (class number 3, one genus)}$$

Consider the algebraic number

$$a(\ell) = \langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell}.$$

For $\ell = 1, 2$ and 4 , we find that $a(\ell)^3$ is a root of a monic cubic polynomial and generates the Hilbert class field over K .

Example

$a(1)$ is a root of the polynomial

$$x^9 - 2816x^6 - 905216x^3 - 89915392.$$

$$K = \mathbb{Q}(\sqrt{-23}) \text{ (class number 3, one genus)}$$

Consider the algebraic number

$$a(\ell) = \langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell}.$$

For $\ell = 3, 6$ and 9 , we find that $a(\ell)$ is a root of a cubic polynomial and generates the Hilbert class field over K .

Example

$a(3)$ is a root of

$$x^3 - 6740x^2 - 169034720x - 1027491892288.$$

$$K = \mathbb{Q}(\sqrt{-23}) \text{ (class number 3, one genus)}$$

A few computations of the Gramm matrix for this basis.

ℓ	$\det(\langle \theta_{a_i}^{(2\ell)}, \theta_{a_j}^{(2\ell)} \rangle)_{a_i, a_j \in \text{Cl}_K} / (\Omega_K^{4\ell})^3$
1	$-2^{10} 23$
2	$-2^{14} 19 \cdot 23 \cdot 619$
3	$-2^{18} 5^2 11 \cdot 23 \cdot 337 \cdot 27299$
4	$-2^{22} 7^2 23 \cdot 163 \cdot 2113 \cdot 117741979$
5	$-2^{26} 5^3 23 \cdot 229 \cdot 23761 \cdot 808991 \cdot 20338663$
6	$-2^{30} 5^2 11^2 13 \cdot 19 \cdot 23 \cdot 67^2 101 \cdot 868697 \cdot 505912247899$

$$K = \mathbb{Q}(\sqrt{-23}) \text{ (class number 3, one genus)}$$

Consider now the algebraic number

$$N(\psi, \ell) = \langle \theta_\psi, \theta_\psi \rangle / \Omega_K^{4\ell}$$

For $\ell = 1, 2, 4$ and 5 , the numbers $N(\psi_i, \ell)$, for $0 \leq i \leq 2$, are distinct and their cube are the three real roots of a monic cubic polynomial.

Example

The numbers $N(\psi_i, 1)^3$, for $0 \leq i \leq 2$, are the three roots of the irreducible polynomial

$$x^3 - 6966x^2 + 11569230x - 239483061.$$

$$K = \mathbb{Q}(\sqrt{-23}) \text{ (class number 3, one genus)}$$

Consider now the algebraic number

$$N(\psi, \ell) = \langle \theta_\psi, \theta_\psi \rangle / \Omega_K^{4\ell}$$

For $\ell = 3, 6$ and 9 , one of the characters, say ψ_0 , the algebraic number $N(\psi_0, \ell)$ is an *integer*. For the two others, we find that their cube are the roots of a monic quadratic polynomial.

Example

We have

$$N(\psi_0, 3) = 5055 = 3 \cdot 5 \cdot 337$$

and $N(\psi_1, 3)^3$ and $N(\psi_2, 3)^3$ are the roots of

$$x^2 - 16287872873193x + 30021979248651078296845875.$$

$$K = \mathbb{Q}(\sqrt{-23}) \text{ (class number 3, one genus)}$$

A few computations of the Gramm matrix for this basis.

ℓ	$\det(\langle \theta_{\psi_i}, \theta_{\psi_j} \rangle)_{1 \leq i, j \leq 3} / (\Omega_K^{4\ell})^3$
1	$-3^3 23$
2	$-3^3 19 \cdot 23 \cdot 619$
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Main steps in the proof (case $\ell > 0$)

1. Use Rankin-Selberg to prove that

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{4h_k}{w_k} \sqrt{|D|} \frac{\Gamma(2\ell + 1)}{(4\pi)^{2\ell+1}} L(\psi^2, 2\ell + 1).$$

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2. Relate Hecke L-series to non-holomorphic Eisenstein series :

$$L(\psi^2, 2\ell + 1) = \frac{1}{w_K} \sum_{[\mathfrak{a}] \in \text{Cl}_K} \frac{\psi^2(\mathfrak{a})}{N(\mathfrak{a})^{4\ell-s}} G_{4\ell}(\mathfrak{a}, 1 - 2\ell).$$

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3. Replace non-holomorphic Eisenstein series by derivatives of Eisenstein series :

$$\partial^{2\ell-1} G_2(z) = (-4\pi)^{1-2\ell} \frac{\Gamma(s + 2\ell + 1)}{\Gamma(s + 2)} G_{4\ell}(z, 1 - 2\ell).$$

Main steps in the proof (case $\ell > 0$)

1. Use Rankin-Selberg to prove that

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$$\partial^{2\ell-1} G_2(z) = (-4\pi)^{1-2\ell} \frac{\Gamma(s + 2\ell + 1)}{\Gamma(s + 2)} G_{4\ell}(z, 1 - 2\ell).$$

4. Find $\langle \theta_a, \theta_b \rangle$ using $\langle \theta_\psi, \theta_\psi \rangle$.

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