Petersson Inner Product of Binary Theta Series

A computational approach

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Motivation : Stark's remark

In *"L-functions at $s = 1$. II. Artin L-functions with Rational Characters"*, Stark makes the following remark :

An application of Theorem 1 *gives*

$$L'(0, \chi, H/\mathbb{Q}) = \log \epsilon,$$

where ϵ is the real root of

$$x^3 - x - 1 = 0$$

Actually, it is easier to note that $L(1, \chi, H/\mathbb{Q})$ is the residue at $s = 1$ of the zeta function of the real quadratic subfield of H. In any case,

$$\langle f, f \rangle = 3 \log \epsilon.$$

Level N modular forms with characters

Definition

Let $N \geq 1$ and $k \geq 0$ be integers and let χ be a Dirichlet character mod N. A modular form of weight k, level N and character χ is a holomorphic function

$$f : \mathcal{H} \longrightarrow \mathbb{C}$$

such that

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for all $z \in \mathcal{H}$ and all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | c \equiv 0 \pmod{N} \right\}$$

which satisfies certain growth conditions at the cusps.

q-expansion of modular forms

Every modular form f has a Taylor (or Fourier) expansion

$$f(z) = \sum_{n=0}^{\infty} a_n q^n,$$

where $q = exp(2\pi iz)$. We call it the q-expansion of f. If

$$a_0(f) = 0,$$

(at all cusps) f is called a *cusp form*.
The space of modular forms is denoted

$$M_k(\Gamma_0(N), \chi),$$

while the space of cusp forms is denoted

$$S_k(\Gamma_0(N), \chi).$$

Example : weight k Eisenstein series

Let $k \geq 4$ be an even integer. Then the series

$$\sum_{m,n} \frac{1}{(mz+n)^k}$$

converges absolutely and defines a modular form in $M_k(\mathrm{SL}_2(\mathbb{Z}))$. After renormalization, the q-expansion of this modular form is the Eisenstein series

$$-\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n := G_k(z),$$

where

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$

Important non-example : weight 2 Eisenstein series

In level 1, there are no modular forms of weight 2. However, one can still define the weight 2 Eisenstein series as

$$G_2(z) = \frac{1}{8\pi\Im(z)} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n) q^n.$$

It is an example of an *almost holomorphic* modular form of level 1 and weight 2.

Finite dimensionality of spaces of modular forms

Theorem

The space $M_k(\Gamma_0(N),\chi)$ is finite dimensional as a $\mathbb{C}$-vector-space.

Example

In level $N = 1$, we have

- $M_0(SL_2(\mathbb{Z})) = \mathbb{C}$.
- $M_2(SL_2(\mathbb{Z})) = 0$.
- $M_k(SL_2(\mathbb{Z})) = \mathbb{C}G_k$ *for* $4 \leq k \leq 10$.
- $M_{12}(SL_2(\mathbb{Z})) = \mathbb{C}G_{12} \oplus \mathbb{C}\Delta$, *where* $\Delta \in S_{12}(SL_2(\mathbb{Z}))$.
- $\bigoplus_{k=0}^{\infty} M_k(SL_2(\mathbb{Z})) = \mathbb{C}[G_4, G_6]$.

Petersson inner product

Let $f, g \in S_k(\Gamma_0(N), \chi)$ be two cusp forms. The Petersson inner product of f and g is defined as

$$\langle f, g \rangle = \int\int_{\Gamma_0(N)\backslash\mathcal{H}} f(x+iy)\overline{g(x+iy)}y^k \mathrm{d}\mu,$$

where

$$\mathrm{d}\mu = \frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

is the $\mathrm{SL}_2(\mathbb{R})$-invariant measure on $\mathcal{H}$. Note that the integral does not converge if both f and g are not cusp forms.

A half-integral weight theta series

Consider the function

$$\theta(z) = \sum_{x \in \mathbb{Z}} q^{x^2} = 1 + 2q + 2q^4 + O(q^5).$$

Then

$$\theta\left(\frac{az+b}{cz+d}\right) = \epsilon(cz+d)^{1/2}\theta(z),$$

whenever $4|c$, where $\epsilon_{c,d} \in \{\pm 1, \pm i\}$.

Theta series attached to ideals

Let K be an imaginary quadratic field of discriminant $D < -4$ and let $\mathcal{O}_K$ be its ring of integers. Fix an integer $\ell \geq 0$.
To each integral ideal $\mathfrak{a}$ of K, one can attach the following theta series :

$$\theta_{\mathfrak{a}}^{(2\ell)}(z) = \theta_{\mathfrak{a}}(z) = \sum_{x \in \mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})}.$$

Basic properties of these theta series

1. We have

$$\theta_{\mathfrak{a}} = \sum_{x \in \mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

where χ_D is the Kronecker symbol. If $\ell > 0$, then

$$\theta_{\mathfrak{a}} \in S_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

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2. If $\lambda \in K^{\times}$, then

$$\theta_{\lambda\mathfrak{a}} = \lambda^{2\ell}\theta_{\mathfrak{a}}.$$

   So there are essentially h_K theta series attached to K.

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   So there are essentially h_K theta series attached to K.
3. In general, the $\theta_{\mathfrak{a}}$ are *not* newforms.

Theta series attached to Hecke characters of K

Let I_K denote the group of fractional ideals of K.

Definition

A Hecke character ψ of K of infinity type 2ℓ (and conductor 1) is a homomorphism

$$\psi : I_K \longrightarrow \mathbb{C}^\times$$

such that

$$\psi((\alpha)) = \alpha^{2\ell}, \qquad \forall \alpha \in K^\times.$$

One defines

$$\theta_\psi = \sum_{\mathfrak{a} \subseteq \mathcal{O}_K} \psi(\mathfrak{a}) q^{N(\mathfrak{a})}.$$

Basic properties of these theta series

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$$\theta_\psi \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

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2. The θ_ψ are newforms, so
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whenever $\psi \neq \psi'$.
3. We have the identities
$$\theta_\psi = \frac{1}{w_K} \sum_{[\mathfrak{a}] \in \mathrm{Cl}_K} \psi^{-1}(\mathfrak{a}) \theta_{\mathfrak{a}} \quad \text{and} \quad \theta_{\mathfrak{a}} = \frac{w_K}{h_K} \sum_{\psi} \psi(\mathfrak{a}) \theta_\psi.$$

Stark's example

Let

$$K = \mathbb{Q}(\sqrt{-23})$$

and let ψ be a non-trivial Hecke character of infinity type 0, i.e. a non-trivial character of the class group. Then

$$\text{Stark's } f = \text{our } \theta_\psi \in M_1(\Gamma_0(23), \chi_{-23}).$$

Some questions

Keeping Stark's example in mind, we have the following questions :

- Can we find explicit formulas for the Petersson inner product of those theta series (whenever it makes sense) ?
- Can we efficiently compute it ?
- Can we use those formulas/computations to study the arithmetic properties of those quantities ?

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- Can we find explicit formulas for the Petersson inner product of those theta series (whenever it makes sense) ?
- Can we efficiently compute it ?
- Can we use those formulas/computations to study the arithmetic properties of those quantities ?

The main question is

Question

Can we p-adically interpolate those formulas for $\ell > 0$ and take the limit as $\ell \to 0$ p-adically to obtain the weight one case ?

The case $\ell = 0$

Theorem

Let θ_ψ *be a Hecke character of infinity type* 0 *and suppose that* $\psi^2 \neq 1$. *Then*

$$\langle \theta_\psi, \theta_\psi \rangle = -h_K \sum_{[\mathfrak{a}] \in Cl_K} \psi^2(\mathfrak{a}) \log(\Im(\tau_\mathfrak{a})^{1/2} |\eta(\tau_\mathfrak{a})|^2)$$

$$= h_K \log \prod_{[\mathfrak{a}] \in Cl_K} (\Im(\tau_\mathfrak{a})^{1/2} |\eta(\tau_\mathfrak{a})|^2)^{-\psi(\mathfrak{a})^2}.$$

Here,

$$\tau_\mathfrak{a} = \omega_2/\omega_1 \in \mathcal{H}$$

if $\mathfrak{a} = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \subset \mathbb{C}$ is positively oriented ideal in K and

$$\eta(z) = exp(2\pi i/24) \prod_{n=1}^{\infty} (1 - q^n).$$

Petersson norm of the θ_ψ (with $\ell > 0$)

Theorem

Let ψ be a Hecke character of K of infinity type 2ℓ, where $\ell > 0$. Then

$$\langle \theta_\psi, \theta_\psi \rangle = h_K (|D|/4)^\ell \sum_{[\mathfrak{a}] \in Cl_K} \psi^2(\mathfrak{a}) \partial^{2\ell-1} G_2(\mathfrak{a}).$$

Here,

$$\partial f = \frac{1}{2\pi i} \frac{\partial f}{\partial z} - \frac{k}{4\pi \Im(z)} f$$

is the Shimura-Maass differential operator, which preserves the graded algebra of almost holomorphic modular forms.

Petersson inner product of the theta series $\theta_{\mathfrak{a}}$

Corollary

Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of K and suppose $\ell > 0$. Then

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle = C_K^{(2\ell)} N(\mathfrak{b})^{2\ell} \sum_{\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}} \mathcal{O}_K} \lambda_{\mathfrak{c}}^{2\ell} \partial^{2\ell-1} G_2(\mathfrak{c}),$$

where

$$C_K^{(2\ell)} = 4(|D|/4)^{\ell}.$$

A few direct consequences of the formula

Corollary

For $\ell > 0$,

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle = 0$$

whenever $\mathfrak{a}$ *and* $\mathfrak{b}$ *are not in the same genus (i.e. the classes of* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are distinct in the genus group* Cl_K/Cl_K^2*).*

Corollary

For $\ell > 0$,

$$\langle \theta_{\mathfrak{ac}}, \theta_{\mathfrak{bc}} \rangle = N(\mathfrak{bc})^{2\ell} \langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle.$$

An arithmetic consequence

Let

$$\Omega_K = \frac{1}{\sqrt{4\pi|D|}} \left(\prod_{j=1}^{|D|-1} \Gamma\left(\frac{j}{|D|}\right)^{\chi_D(j)} \right)^{w_K/4h_k}$$

be the Chowla-Selberg period attached to K.

Corollary

For $\ell > 0$, the complex numbers

$$\frac{\langle \theta_\psi, \theta_\psi \rangle}{\Omega_K^{4\ell}} \quad \textit{and} \quad \frac{\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle}{\Omega_K^{4\ell}}$$

are algebraic.

Formally obtaining the case $\ell = 0$ from the case $\ell > 0$

Strictly speaking, the formula

$$\langle \theta_\psi, \theta_\psi \rangle = h_K (|D|/4)^\ell \sum_{[\mathfrak{a}] \in \mathrm{Cl}_K} \psi^2(\mathfrak{a}) \partial^{2\ell-1} G_2(\mathfrak{a}).$$

does not make sense for $\ell = 0$, since the expression

$$\partial^{-1} G_2$$

is not well-defined.

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$$\langle \theta_\psi, \theta_\psi \rangle = h_K(|D|/4)^\ell \sum_{[\mathfrak{a}]\in \mathrm{Cl}_K} \psi^2(\mathfrak{a})\partial^{2\ell-1}G_2(\mathfrak{a}).$$

does not make sense for $\ell = 0$, since the expression

$$\partial^{-1}G_2$$

is not well-defined. However, we observe that

$$\partial_0 \log(\Im(z)^{1/2}|\eta(z)|^2) = -G_2(z),$$

so

$$"\partial^{-1}G_2(z) = -\log(\Im(z)^{1/2}|\eta(z)|^2)"$$

and we *formally* obtain the case $\ell = 0$ from the case $\ell > 0$.

Computing $\partial^n G_2$

To compute

$$\partial^n G_2$$

we have the following formulas :

$$\partial G_2 = \frac{5}{6} G_4 - 2G_2^2 \quad \partial G_4 = \frac{7}{10} G_6 - 8 G_2 G_4 \quad \partial G_6 = \frac{400}{7} G_4^2 - 12 G_2 G_6.$$

For example,

$$\partial^3 G_2 = -48 G_2^4 + 120 G_4 G_2^2 - 14 G_6 G_2 + 25 G_4^2.$$

$K = \mathbb{Q}(\sqrt{-23})$ (class number 3, one genus) : $\ell = 0$

For ψ a non-trivial Hecke character of infinity type 0, the explicit formula in case $\ell = 0$ gives

$$\langle f, f \rangle = \langle \theta_\psi, \theta_\psi \rangle = 3 \log \epsilon,$$

where

$$\epsilon = \prod_{[\mathfrak{a}] \in \mathrm{Cl}_K} \left(\Im(\tau_\mathfrak{a})^{1/2} |\eta(\tau_\mathfrak{a})|^2 \right)^{-\psi(\mathfrak{a})^2}$$

is the real root of

$$x^3 - x - 1$$

and generates the Hilbert class field of K.

$K = \mathbb{Q}(\sqrt{-23})$ (class number 3, one genus) : $\ell > 0$

In K, the prime 2 splits as

$$2\mathcal{O}_K = \mathfrak{p}_2\bar{\mathfrak{p}}_2$$

and

$$\mathrm{Cl}_K = \{1, [\mathfrak{p}_2], [\bar{\mathfrak{p}}_2]\}.$$

Moreover, we have $\langle \theta_{\bar{\mathfrak{p}}_2}, \theta_{\mathcal{O}_K} \rangle = \overline{\langle \theta_{\mathfrak{p}_2}, \theta_{\mathcal{O}_K} \rangle}$. We will focus on

$$\langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell}.$$

$K = \mathbb{Q}(\sqrt{-23})$ (class number 3, one genus) : $\ell > 0$

Consider the algebraic number

$$a(\ell) = \langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell}.$$

For $\ell = 1, 2, 4$ and 5, we find that $a(\ell)^3$ is a root of a monic cubic polynomial and generates the Hilbert class field over K.

Example

$a(1)$ is a root of the polynomial

$$x^9 - 2816x^6 - 905216x^3 - 89915392.$$

$K = \mathbb{Q}(\sqrt{-23})$ (class number 3, one genus) : $\ell > 0$

Consider the algebraic number

$$a(\ell) = \langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell}.$$

For $\ell = 3, 6$ and 9, we find that $a(\ell)$ is a root of a cubic polynomial and generates the Hilbert class field over K.

Example

$a(3)$ is a root of

$$x^3 - 6740x^2 - 169034720x - 1027491892288.$$

$K = \mathbb{Q}(\sqrt{-23})$ (class number 3, one genus) : $\ell > 0$

A few computations of the Gramm matrix for this basis.

ℓ	$\det(\langle \theta_{\mathfrak{a}_i}^{(2\ell)}, \theta_{\mathfrak{a}_j}^{(2\ell)} \rangle)_{\mathfrak{a}_i,\mathfrak{a}_j \in \mathrm{Cl}_K} / (\Omega_K^{4\ell})^3$
1	$-2^{10}23$
2	$-2^{14}19 \cdot 23 \cdot 619$
3	$-2^{18}5^{2}11 \cdot 23 \cdot 337 \cdot 27299$
4	$-2^{22}7^{2}23 \cdot 163 \cdot 2113 \cdot 117741979$
5	$-2^{26}5^{3}23 \cdot 229 \cdot 23761 \cdot 808991 \cdot 20338663$
6	$-2^{30}5^{2}11^{2}13 \cdot 19 \cdot 23 \cdot 67^{2}101 \cdot 868697 \cdot 505912247899$

$K = \mathbb{Q}(\sqrt{-23})$ (class number 3, one genus) : $\ell > 0$

Consider now the algebraic number

$$N(\psi, \ell) = \langle \theta_\psi, \theta_\psi \rangle / \Omega_K^{4\ell}$$

For $\ell = 1, 2, 4$ and 5, the numbers $N(\psi_i, \ell)$, for $0 \leq i \leq 2$, are distinct and their cube are the three real roots of a monic cubic polynomial.

Example

The numbers $N(\psi_i, 1)^3$, for $0 \leq i \leq 2$, are the three roots of the irreducible polynomial

$$x^3 - 6966x^2 + 11569230x - 239483061.$$

$K = \mathbb{Q}(\sqrt{-23})$ (class number 3, one genus) : $\ell > 0$

Consider now the algebraic number

$$N(\psi, \ell) = \langle \theta_\psi, \theta_\psi \rangle / \Omega_K^{4\ell}$$

When $\ell = 3, 6$ and 9, for one of the characters, say ψ_0, the algebraic number $N(\psi_0, \ell)$ is an *integer*. For the two others, we find that their cube are the roots of a monic quadratic polynomial.

Example

We have

$$N(\psi_0, 3) = 5055 = 3 \cdot 5 \cdot 337$$

and $N(\psi_1, 3)^3$ *and* $N(\psi_2, 3)^3$ *are the roots of*

$$x^2 - 16287872873193x + 3002197924865107829684587.$$

$K = \mathbb{Q}(\sqrt{-23})$ (class number 3, one genus) : $\ell > 0$

A few computations of the Gramm matrix for this basis.

ℓ	$\det(\langle \theta_{\psi_i}, \theta_{\psi_j} \rangle)_{1 \le i,j \le 3} / (\Omega_K^{4\ell})^3$
1	$-3^3 23$
2	$-3^3 19 \cdot 23 \cdot 619$
3	$-3^3 5^2 11 \cdot 23 \cdot 337 \cdot 27299$
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Class number 1 case

In this case,

$$\theta_{\mathcal{O}_K} = \theta_{\psi_0}$$

and we only need to compute

$$\langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell} \in \overline{\mathbb{Q}}.$$

Class number 1 case

Computation of $\langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell}$:

		ℓ = 1	ℓ = 2
D	-7	$2^2 3$	-2^2
	-8	-2	$-2^2 5$
	-11	-2^2	$-2^3 5$
	-19	$-2^2 3^{-1} 13$	$-2^3 71$
	-43	$-2^3 3^{-1} 107$	$-2^4 5647$
	-67	$-2^2 3^{-1} 7^2 31$	$-2^3 5 \cdot 86629$
	-163	$-2^3 3^{-1} 150473$	$-2^4 11 \cdot 461681471$

Example of computation for $K = \mathbb{Q}(\sqrt{-23})$: $N(\psi_0, 3)$

```
parisize = 4000000, primelimit = 500000
(13:14) gp > \r Thetapip.gp ;
(13:14) gp > \r ../lfunc/Qhc.gp
(13:15) gp > \r ../lfunc/qhlfun-eisender.gp ;
(13:15) gp > \p 500
   realprecision = 500 significant digits
(13:15) gp > #
   timer = 1 (on)
(13:15) gp > K=bnfinit(x^2+23); OmK=CSperiod(-23);
time = 130 ms.
(13:15) gp > data = pipinit(K);
time = 267 ms.
(13:15) gp > pnorm(data,[[0],[6,0]])/OmK^12
time = 3 ms.
%252 = 5055.00000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000091366 + 1.485403444 E-496*I
```

Main steps in the proof (case $\ell > 0$)

1. Use Rankin-Selberg to prove that

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{4h_k}{w_k}\sqrt{|D|}\frac{\Gamma(2\ell+1)}{(4\pi)^{2\ell+1}} L(\psi^2, 2\ell+1).$$

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2. Relate Hecke L-series of imaginary quadratic fields to real-analytic Eisenstein series :

$$L(\psi^2, 2\ell+1) = \frac{1}{w_K} \sum_{[\mathfrak{a}] \in \mathrm{Cl}_K} \frac{\psi^2(\mathfrak{a})}{N(\mathfrak{a})^{4\ell-s}} G_{4\ell}(\mathfrak{a}, 1-2\ell).$$

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3. Replace real-analytic Eisenstein series by derivatives of Eisenstein series :

$$\partial^{2\ell-1} G_2(z) = (-4\pi)^{1-2\ell}\frac{\Gamma(s+2\ell+1)}{\Gamma(s+2)} G_{4\ell}(z, 1-2\ell).$$

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4. Find $\langle \theta_\mathfrak{a}, \theta_\mathfrak{b} \rangle$ using $\langle \theta_\psi, \theta_\psi \rangle$.

What we would like to know

1. Can we explain what we observed in the computations ?
2. Can we say something about the Petersson inner product of non-cuspidal weight one theta series ?

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1. Can we explain what we observed in the computations ?
2. Can we say something about the Petersson inner product of non-cuspidal weight one theta series ?

But again, the main question remains

Question

Can we p-adically interpolate the formulas for $\ell > 0$ and take the limit as $\ell \to 0$ p-adically to obtain the weight one case ?

Thank you !

Presentation available at : `https://github.com/NicolasSimard/Notes/tree/master/PANTS%20XXVI`

Code available at :

`https://github.com/NicolasSimard/ENT`

Notes available at :`https://github.com/NicolasSimard/Notes/tree/master/Theta%20Norm`