Petersson norm of theta series and derivatives of Eisenstein series

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Introduction

In these notes, we find a formula for the Petersson norm of the theta series θ_ψ attached to an imaginary quadratic field K and a Hecke character of infinity type 2ℓ. The formula is

$$\langle \theta_\psi, \theta_\psi \rangle = V_D^{-1} (|D|/4)^\ell \frac{h_K}{w_K^2} \sum_{[\mathfrak{a}] \in \mathrm{Cl}_K} \psi^2(\mathfrak{a}) \mathrm{del}_2^{2\ell-1} E_2(\mathfrak{a})$$

if $\ell > 0$ and

$$\langle \theta_\psi, \theta_\psi \rangle = -V_D^{-1} \frac{4h_K}{w_K^2} \sum_{[\mathfrak{a}] \in \mathrm{Cl}_K} \psi^2(\mathfrak{a}) \log(\Im(\tau_\mathfrak{a})^{1/2} |\eta(\tau_\mathfrak{a})|^2)$$

if $\ell = 0$ and ψ is not a genus character. Here $\partial_2^{2\ell-1}E_2$ is the non-holomorphic derivative of the non-holomorphic Eisenstein series of weight 2 and level 1, viewed as a function on lattices in the usual way, and

$$V_D = \mathrm{Vol}(\Gamma_0(|D|) \setminus \mathcal{H}).$$

In the last section, we will see that one can make sense of the first formula even for $\ell = 0$ and that it gives back exactly the second formula!

Before proving the formula, we first recall a few facts about Hecke characters, Eisenstein series and the Rankin-Selberg method. Then we introduce the theta functions θ_ψ. In the following section, we show how the Petersson norm of the θ_ψ is related to the Hecke L-function of ψ^2. Finally, we relate the Hecke L-function of ψ^2 to non-holomorphic Eisenstein series and use this relation to establish the two formulas.

If ψ is a genus character, θ_ψ is an Eisenstein series and one should use the regularized Petersson inner product. I think a similar formula holds. I will try this soon.

1 Setup and notation

Throughout, $K = \mathbb{Q}(\sqrt{D})$ denotes an imaginary quadratic field of discriminant $D < -4$ and $\mathcal{O}_K$ denotes its ring of integers.

2 Preliminaries

2.1 Hecke Grossencharacters

Let I_K be the multiplicative group of fractional ideals of K. Given an integer $\ell \geq 0$, let ψ_ℓ denote a *Hecke Grossencharacter* of conductor 1 and infinity type 2ℓ, that is a group homomorphism

$$\psi_\ell : I_K \to \mathbb{C}^\times$$

such that

$$\psi_\ell((\alpha)) = \alpha^{2\ell}, \qquad \forall \alpha \in K^\times.$$

Note that this is well-defined since $\mathcal{O}_K^\times = \{\pm 1\}$ by assumption.

Those Hecke characters are not of the form considered in the books of Miyake [Miya, Ch. 3, Sec. 3] or Iwaniec [Iwan, Ch. 12, Sec. 2]. For clarity, we call the ones they define *unitary*. Let $N : I_K \to \mathbb{Q}$ denote the norm map on ideals. Then the character

$$\psi_\ell N^{-\ell} : I_K \to \mathbb{C}^\times$$

is unitary of conductor 1 and of infinity type 2ℓ (take $u_\sigma + iv_\sigma = 2\ell$ in their definition, where $\sigma : K \hookrightarrow \mathbb{C}$ is a complex embedding).

To a Hecke character ψ (unitary or not), one attaches the Dirichlet L-series

$$L(\psi, s) = \sum_{\mathfrak{a}} \frac{\psi(\mathfrak{a})}{N(\mathfrak{a})^s},$$

which converges for s in some right-half plane in $\mathbb{C}$. Clearly, multiplying ψ with a power of the norm N^ℓ simply shifts the L-function by ℓ:

$$L(\psi, s - \ell) = L(\psi \circ N^\ell, s).$$

Define the completed L-function of $L(\psi_\ell, s)$ as

$$\Lambda(\psi_\ell, s) = |D|^{s/2}(2\pi)^{-s}\Gamma(s)L(\psi_\ell, s).$$

Theorem 1 (Hecke). *1. Λ can be analytically continued to a meromorphic function on $\mathbb{C}$ and satisfies the functional equation*

$$\Lambda(\psi_\ell, s) = w(\psi_\ell)\Lambda(\overline{\psi_\ell}, 2\ell + 1 - s),$$

where $|w(\psi_\ell)| = 1$.

2. $\Lambda(\psi_\ell, s)$ *is holomorphic on* $\mathbb{C}$*, except when* ψ_ℓ *is the trivial character (this can only happen when* $\ell = 0$*), in which case is has a pole at* $s = 0$ *and* $s = 1$*.*

3. $L(\psi_\ell, s)$ *is holomorphic on* $\mathbb{C}$*, except when* ψ_ℓ *is the trivial character, in which case it has a pole at* $s = 1$ *with residue*
$$\frac{2\pi h_K}{w_K\sqrt{|D|}},$$
where h_K *is the class number of* K *and* $w_K = 2$ *is the number of roots of unity in* K*.*

Proof. See [Miya, Ch. 3, Sec. 3]. □

2.2 Eisenstein series: holomorphic and non-holomorphic

Eisenstein series will be useful in many ways in these notes. Recall that they can be defined in essentially two (closely related) ways: as Poincare series and as sum over lattice points. The first type is used in the Rankin-Selberg method, while the second is linked to Hecke L-functions of imaginary quadratic fields. We recall a few basic facts about these series. Our main references are [Shi1, Ch.9], [Shi1, A3] and [Miya, Ch.7]

Let $N \geq 1$ and $k \geq 0$ be integers. As usual, define

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \pmod{N} \right\}$$

and for $f : \mathcal{H} \to \mathbb{C}$ a function on the upper half plane and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, define the slash-k as operator

$$(f|_k\gamma)(z) = j(\gamma, z)^{-k} f(\gamma z),$$

where $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$ in the usual way and

$$j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = cz + d.$$

Let also Γ_∞ be the stabilizer of the cusp at infinity in $\mathrm{SL}_2(\mathbb{Z})$, i.e.

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\}$$

For $(z, s) \in \mathcal{H} \times \mathbb{C}$, define the *non-holomorphic Eisenstein series of weight* k as

$$G_k(z, s) = \mathfrak{I}(z)^s \sum_{m,n} (mz + n)^{-k} |mz + n|^{-2s},$$

where the sum is over all integers m and n, not both 0. This sum converges for $\mathfrak{R}(2s) + k > 2$.

Since
$$\mathfrak{I}(\gamma z)^s = |j(\gamma, z)|^{-2s} \mathfrak{I}(z)^s,$$

the non-holomorphic Eisenstein series satisfies the following functional equation:

$$G_k(\gamma z, s) = j(\gamma, z)^k G_k(z, s).$$

In particular, k must be even.

For $k > 2$, the series converges absolutely at $s = 0$ and equals the usual Eisenstein series of weight k and level 1. For $k = 2$, it does not converge absolutely at $s = 0$. However, for $k > 0$ there is a real analytic function of $(z, s) \in \mathcal{H} \times \mathbb{C}$ which is holomorphic in s and coincides with $\Gamma(s + k) G_k(z, s)$ for $\mathfrak{R}(2s) + k > 2$ ([Shi1, Thm A3.5]). Therefore it still makes sense to consider $G_2(z, 0)$. Define

$$E_2(z) = 2^{-1}(2\pi i)^{-2} G_2(z, 0).$$

Then E_2 is an *almost holomorphic* modular form of weight 2 and level 1 with Fourier expansion

$$E_2(z) = \frac{1}{8\pi\Im(z)} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

which clearly has algebraic Fourier coefficients. Almost holomorphic modular forms are defined as in [Zag, Sec. 5.3][1]. In particular,

$$E_2|_2\gamma = E_2, \qquad \forall \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

Consider now the following Eisenstein series:

$$E_k^N(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \Im(z)^s|_k\gamma = \mathrm{Im}(z)^s \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} j(\gamma,z)^{-k}|j(\gamma,z)|^{-2s}.$$

This series also converges absolutely for $\Re(2s) + k > 2$ and can be analytically continued to a holomorphic function in s, except when $k = 0$, in which case $E_0^N(z,s)$ has a pole at $s = 1$ with residue

$$\mathrm{Res}_{s=1} E_0^N(z,s) = \mathrm{Vol}(\Gamma_0(N) \setminus \mathcal{H})^{-1}.$$

2.3 Rankin-Selberg method in level N

The Rankin-Selberg is well-known. We sketch it here mainly to make sure that the normalizations are correct. Our main reference is [Shi2].

Let $f(z), g(z) \in \mathcal{S}_k(\Gamma_0(N), \chi)$ be two cusp forms of weight k, level N and Nebentypus χ. Then the function

$$F(z) = f(z)\overline{g(z)}\Im(z)^k$$

if $\Gamma_0(N)$-invariant and tends to 0 rapidly as $\Im(z)$ tends to ∞, so it makes sense to define the *Petersson inner product* of f and g as

$$\langle f, g\rangle = \frac{1}{\mathrm{Vol}(\Gamma_0(N) \setminus \mathcal{H})} \int\int_{\Gamma_0(N)\backslash\mathcal{H}} F(z) d\mu(z),$$

where we integrate over a fundamental domain for the action of $\Gamma_0(N)$ on $\mathcal{H}$ and

$$d\mu(z) = \frac{dxdy}{y^2}$$

is the $\mathrm{SL}_2(\mathbb{Z})$-invariant measure on $\mathcal{H}$.

Now for $\Re(s)$ large enough, the series for $E_0^N(z,s)$ converges absolutely and the following manipulations are justified:

$$\begin{aligned}
\int\int_{\Gamma_0(N)\backslash\mathcal{H}} F(z)E_0^N(z,s)d\mu(z) &= \int\int_{\Gamma_0(N)\backslash\mathcal{H}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} F(z)\Im(\gamma z)^s d\mu(z) \\
&= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \int\int_{\Gamma_0(N)\backslash\mathcal{H}} F(\gamma z)\Im(\gamma z)^s d\mu(z) \\
&= \int\int_{\Gamma_\infty\backslash\mathcal{H}} F(z)\Im(z)^s d\mu(z)
\end{aligned}$$

As a functions of s, the last integral has a residue at $s = 1$. Using the value of $\mathrm{Res}_{s=1} E_0^N(z,s)$ given above, one sees that

$$\mathrm{Res}_{s=1} \int\int_{\Gamma_\infty\backslash\mathcal{H}} F(z)\Im(z)^s d\mu(z) = \mathrm{Res}_{s=1} \int\int_{\Gamma_0(N)\backslash\mathcal{H}} F(z)E_0^N(z,s)d\mu(z) = \langle f, g\rangle.$$

[1] Shimura calls those functions nearly holomorphic in [Shi1], but we prefer to use this term to refer to modular forms with (possibly) poles at infinity.

Note that it is important that $\mathrm{Res}_{s=1} E_0^N(z,s)$ does not depend on z.

On the other hand, let

$$f(z) = \sum_{n=1}^{\infty} a_n q^n \qquad \text{and} \qquad g(z) = \sum_{n=1}^{\infty} b_n q^n$$

be the q-expansions of f and g. Then

$$f(z)\overline{g(z)} = \sum_{m,n=1}^{\infty} a_n \overline{b_m} e^{2\pi i n z} e^{-2\pi i m \bar{z}} = \sum_{m,n=1}^{\infty} a_n \overline{b_m} e^{2\pi i (n-m) x} e^{-2\pi (m+n) y},$$

where $z = x + iy$, so

$$\int_0^1 F(z)\Im(z)^s dx = \sum_{n=1}^{\infty} a_n \overline{b_n} e^{-4\pi n y} y^{k+s}$$

and

$$\int\int_{\Gamma_\infty\backslash\mathcal{H}} F(z)\Im(z)^s d\mu(z) = \int_0^\infty \left(\int_0^1 F(z)\Im(z)^s dx \right) \frac{dy}{y^2} = \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} \sum_{n=1}^{\infty} \frac{a_n \overline{b_n}}{n^{s+k-1}}.$$

Comparing the expressions for

$$\mathrm{Res}_{s=1} \int\int_{\Gamma_\infty\backslash\mathcal{H}} F(z)\Im(z)^s d\mu(z),$$

gives the formula

$$\langle f, g\rangle = \Gamma(k)(4\pi)^{-k} \mathrm{Res}_{s=k} D(f, g_\rho, s), \tag{1}$$

where

$$D(f,g,s) = \sum_{n=1}^{\infty} \frac{a_n b_n}{n^s}$$

and

$$g_\rho(z) = \overline{g(-\bar{z})} = \sum_{n=1}^{\infty} \overline{b_n} q^n.$$

3 Theta series attached to imaginary quadratic fields

Let $\ell \geq 0$ and $\psi = \psi_\ell$ be a Hecke character of infinity type 2ℓ. Consider the theta series

$$\theta_\psi(z) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{N(\mathfrak{a})},$$

where the sum runs over all integral ideals of $\mathcal{O}_K$. It is well known ([Iwan, Thm. 12.5]) that

$$\theta_\psi(z) \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

where χ_D is the quadratic character attached to K (i.e. the Kronecker symbol).[2]

If $\ell > 0$, θ_ψ is in fact a cusp form. If $\ell = 0$, this is also true, unless ψ is a genus character (i.e. $\psi^2 = 1$), in which case it is an Eisenstein series. In any case,

$$L(\theta_\psi, s) = L(\psi, s),$$

so the L-function of θ_ψ has an Euler product[3]. It follows that θ_ψ is a normalized (i.e. $a_1(\theta_\psi) = 1$) eigenform for all Hecke operators (see [DiSh, Thm. 5.9.2]). Moreover,

$$a_n(\theta_\psi) = \sum_{N(\mathfrak{a})=n} \psi(\mathfrak{a}),$$

[2] Note that the Hecke characters ψ_ℓ have conductor $\mathcal{O}_K$, so they are automatically primitive.

[3] One reason to choose the non-unitary Hecke characters ψ_ℓ is to have simpler formulas, like this one.

where the sum is over all integral ideals of K of norm n. It follows that

$$a_p(\theta_\psi) = \begin{cases} 0 & \text{if } \chi_D(p) = -1 \\ \psi(\mathfrak{p}) + \psi(\bar{\mathfrak{p}}) & \text{if } \chi_D(p) = 1 \text{ and } p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}} \\ \psi(\mathfrak{p}) & \text{if } \chi_D(p) = 0 \text{ and } p\mathcal{O}_K = \mathfrak{p}^2 \end{cases},$$

in accordance with the equality between the L-functions of θ_ψ and ψ.

Using the fact that the adjoint of the Hecke operators T_p acting on $S_{2\ell+1}(\Gamma_0(|D|), \chi_D)$ with respect to the Petersson inner product is

$$T_p^* = \overline{\chi_D}(p) T_p$$

for all p not dividing D (see [DiSh, Thm. 5.5.3]), one sees that

$$a_p(\theta_\psi) = \chi_D(p)\overline{a_p(\theta_\psi)}$$

for all p not dividing D, whenever θ_ψ is a cusp form.

Lemma 1.

$$a_n(\theta_\psi) \in \mathbb{R}$$

whenever θ_ψ is a cusp form.

Proof. By the multiplicativity property of the $a_n(\theta_\psi)$, it suffices to prove the result for $n = p^k$ =a prime power. Recall that

$$a_{p^{k+1}}(\theta_\psi) = a_p(\theta_\psi) a_{p^k}(\theta_\psi) - \chi_D(p) p^{2\ell} a_{p^{k-1}}(\theta_\psi),$$

for all $k \geq 1$.

If p is inert in K, $a_p(\theta_\psi) = 0$ and so $a_{p^k}(\theta_\psi) = 0$ for all $k \geq 0$.

If p splits in K, $a_p(\theta_\psi) = \chi_D(p)\overline{a_p(\theta_\psi)} = \overline{a_p(\theta_\psi)}$, so $\overline{a_p(\theta_\psi)} \in \mathbb{R}$ and the claim follows from the recursive formula.

Finally if p ramifies, say $p\mathcal{O}_K = \mathfrak{p}^2$, then $a_p(\theta_\psi) = \pm p^\ell$ since

$$p^{2\ell} = \psi((p)) = \psi(\mathfrak{p}^2) = \psi(\mathfrak{p})^2$$

and the claim follows again from the recursive formula. ☐

4 The Petersson norm of θ_ψ

In this section, suppose θ_ψ is a cusp form, i.e. $\psi^2 \neq 1$. We will prove that the Petersson norm of θ_ψ is

$$\langle \theta_\psi, \theta_\psi \rangle = \zeta(2)^{-1} \frac{\Gamma(2\ell+1)}{(4\pi)^{2\ell+1}} L(\chi_D, 1) \prod_{p|D} (1 + p^{-1})^{-1} L(\psi^2, 2\ell+1) \tag{2}$$

Note that if $\psi^2 = 1$, $\ell = 0$ and so $L(\psi^2, s)$ has a pole at $s = 1$.

For each prime p, the L-function of θ_ψ has Euler factor at p equal to

$$1 - a_p(\theta_\psi) p^{-s} + \chi_D(p) p^{2l-2s} = (1 - \alpha_p p^{-s})(1 - \beta_p p^{-s}),$$

where we set $\beta_p = 0$ if $p|D$. One can then define the symmetric square L-function of θ_ψ as

$$L(\mathrm{Sym}^2 \theta_\psi, s) = \prod_p ((1 - \alpha_p^2 p^{-s})(1 - \alpha_p \beta_p p^{-s})(1 - \beta_p^2 p^{-s}))^{-1}$$

for $\mathfrak{R}(s)$ large enough. This L-function can be analytically continued to a meromorphic funtion on the whole complex plane, with (possibly) poles at $s = 2\ell$ and $s = 2\ell + 1$ (see [Shi2, Thm. 2]).

Using the description of $a_p(\theta_\psi)$ given in the previous section, one sees that

$$\{\alpha_p, \beta_p\} = \begin{cases} \{\pm p^\ell, \mp p^\ell\} & \text{if } \chi_D(p) = -1 \\ \{\psi(\mathfrak{p}), \psi(\bar{\mathfrak{p}})\} & \text{if } \chi_D(p) = 1 \text{ and } p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}} \\ \{\psi(\mathfrak{p}), 0\} & \text{if } \chi_D(p) = 0 \text{ and } p\mathcal{O}_K = \mathfrak{p}^2 \end{cases}.$$

The proof of formula 2 relies on the Rankin-Selberg method:

$$\langle \theta_\psi, \theta_\psi \rangle = (4\pi)^{-2\ell-1}\Gamma(2\ell+1)\mathrm{Res}_{s=2\ell+1} D(\theta_\psi, \theta_\psi, s),$$

where we used the fact that θ_ψ has real Fourrier coefficients (Lemma 1). Before proving the formula, we mention the following Lemma of Shimura (see [Shi3, Ch.3, Lem.1]).

Lemma 2. *Suppose we have formally*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p ((1-\alpha_p p^{-s})(1-\beta_p p^{-s}))^{-1},$$

$$\sum_{n=1}^{\infty} \frac{b_n}{n^s} = \prod_p ((1-\alpha'_p p^{-s})(1-\beta'_p p^{-s}))^{-1}.$$

Then

$$\sum_{n=1}^{\infty} \frac{a_n b_n}{n^s} = \prod_p (1-\alpha_p\beta_p\alpha'_p\beta'_p p^{-2s})((1-\alpha_p\alpha'_p p^{-s})(1-\alpha_p\beta'_p p^{-s})(1-\beta_p\alpha'_p p^{-s})(1-\beta_p\beta'_p p^{-s}))^{-1}.$$

The first step in the proof is the following.

Lemma 3. *For all s, one has*

$$\zeta_D(2s-4\ell)D(\theta_\psi, \theta_\psi, s) = L(\mathit{Sym}^2\theta_\psi, s)L(\chi_D, s-2\ell),$$

where $\zeta_D(s)$ is the usual Riemann zeta function with the Euler factors at $p|D$ removed and $L(\chi_D, s)$ is the Dirichlet L-function attached to χ_D.

Proof. The idea is to compare the Euler factors at each prime on each side for $\mathfrak{R}(s)$ large enough, using Shimura's lemma.

For p split or inert, the Euler factor on the left simplifies to

$$(1-p^{4\ell-2s})^{-1}(1-p^{4\ell-2s})((1-\alpha_p^2 p^{-s})(1-\alpha_p\beta_p p^{-s})(1-\beta_p^2 p^{-s}))^{-1}(1-\chi_D(p)p^{2\ell-s})^{-1},$$

while the one on the right is

$$((1-\alpha_p^2 p^{-s})(1-\alpha_p\beta_p p^{-s})(1-\beta_p^2 p^{-s}))^{-1}(1-\chi_D(p)p^{2\ell-s})^{-1}.$$

If p ramifies, $\beta_p = 0$ and $\chi_D(p) = 0$. Then the Euler factor on the left is

$$(1-p^{2\ell-s})^{-1},$$

which is also equal to the one on the right. ∎

The last step is to relate $L(\mathrm{Sym}^2\theta_\psi, s)$ to $L(\psi^2, s)$.

Lemma 4. *For all s, one has*

$$L(\mathit{Sym}^2\theta_\psi, s) = L(\psi^2, s)\zeta_D(s-2\ell).$$

Proof. Again, it suffices to compare the euler factors on both sides for $\Re(s)$ large enough.

If p is inert, the Euler factor on the left is

$$((1-p^{2\ell-s})(1+p^{2\ell-s})(1-p^{2\ell-s}))^{-1},$$

while the one on the right is

$$(1-\psi^2((p))p^{-2s})^{-1}(1-p^{2\ell-s})^{-1}=(1-p^{4\ell-2s})^{-1}(1-p^{2\ell-s})^{-1}.$$

If p splits as $p\mathcal{O}_K=\mathfrak{p}\bar{\mathfrak{p}}$, the Euler factor on the left is

$$(1-\psi^2(\mathfrak{p})p^{-s})(1-\psi(\mathfrak{p})\psi(\bar{\mathfrak{p}})p^{-s})(1-\psi^2(\bar{\mathfrak{p}})p^{-s}))^{-1}=((1-\psi^2(\mathfrak{p})p^{-s})(1-\psi^2(\bar{\mathfrak{p}})p^{-s}))^{-1}(1-p^{2\ell-s})^{-1},$$

which is clearly equal to the one on the right.

The case p ramified is similar. ☐

Putting those two lemmas together gives

$$\zeta_D(2s-4\ell)D(\theta_\psi,\theta_\psi,s)=L(\chi_D,s-2\ell)\zeta_D(s-2\ell)L(\psi^2,s).$$

Formula 2 then follows by taking residues on both sides of this equation at $s=2\ell+1$, using the fact that $L(\psi^2,s)$ is analytic at $2\ell+1$,

$$\mathrm{Res}_{s=2\ell+1}\zeta_D(s-2\ell)=\prod_{p|D}(1-p^{-1})\mathrm{Res}_{s=1}\zeta(s)=\prod_{p|D}(1-p^{-1})$$

and

$$\zeta_D(2)=\prod_{p|D}(1-p^{-2})\zeta(2).$$

5 Special values of Hecke L-functions and Eisenstein series

In this section, we first relate $L(\psi^2,s)$ to non-holomorphic Eisenstein series. Then we use this relation to express the special value of $L(\psi^2,s)$ at $2\ell+1$ in terms of derivatives of E_2 evaluated at CM points when $\ell>0$. The case $\ell=0$ is different and must be treated separately.

Throughout this section, fix a Hecke character ψ of K of infinity type 2ℓ.

5.1 Hecke L-functions and non-holomorphic Eisenstein series

Recall that is f is a $|_k\gamma$-invariant function for all γ in $\mathrm{SL}_2(\mathbb{Z})$, then one can define a weight k homogeneous function F on the space of (positively) oriented lattices in $\mathbb{C}$ as

$$F(\omega_1\mathbb{Z}+\omega_2\mathbb{Z})=\omega_2^{-k}f(\omega_1/\omega_2).$$

Recall that an oriented lattice is a lattice $\mathfrak{a}$ equipped with a $\mathbb{Z}$-basis $[\omega_1,\omega_2]$, where the order of the basis elements is important. If $\Im(\omega_1/\omega_2)>0$, $\mathfrak{a}$ is called positively oriented. If the $\mathbb{Z}$-basis $[\omega_1,\omega_2]$ is not positively oriented, the basis $[\omega_2,\omega_1]$ is, so that any lattice $\mathfrak{a}$ can be positively oriented. The point $\omega_1/\omega_2\in\mathcal{H}$ attached to a positively oriented basis of $\mathfrak{a}$ will sometimes be denoted $\tau_\mathfrak{a}$. Note that we do not make any holomorphy assumptions on f.

Recall that he non-holomorphic Eisenstein series $G_k(z,s)$ of weight k is defined as

$$G_k(z,s)=\Im(z)^s\sum_{m,n}(mz+n)^{-k}|mz+n|^{-2s},$$

where the sum runs over all integers m and n not both 0. If $\mathfrak{a}$ is any fractional $\mathcal{O}_K$-ideal with oriented basis $[\omega_1,\omega_2]$, define

$$G_k(\mathfrak{a},s)=\omega_2^{-k}\left(\frac{\sqrt{|D|}N(\mathfrak{a})}{2}\right)^{-s}G_k(\omega_1/\omega_2,s),$$

where D is the discriminant of K. To see that this definition makes sense, first note that

$$\Im(\omega_1/\omega_2) = |\omega_2|^{-2}\left(\frac{\sqrt{|D|}N(\mathfrak{a})}{2}\right).$$

Then

$$G_k(\mathfrak{a}, s) = \sum_{m,n}(m\omega_1 + n\omega_2)^{-k}|m\omega_1 + n\omega_2|^{-2s},$$

so that $G_k(\mathfrak{a}, 0)$ is the usual weight k Eisenstein series on lattices for $k > 2$. Moreover,

$$G_k(\mu\mathfrak{a}, s) = \mu^{-k}|\mu|^{-2s}G_k(\mathfrak{a}, s)$$

for any $\mu \in K^\times$.

Consider now the following partial Hecke L-function

$$L^{(2\ell)}(\mathfrak{a}, s) = \sum_{\lambda\in\mathfrak{a}-0}\frac{\overline{\lambda}^{2\ell}}{|\lambda|^{2s}}.$$

The first basic relation between Eisenstein series and Hecke L-functions is based on the following

Proposition 1. *Let ψ be a Hecke character of infinity type 2ℓ as above. Then*

$$L(\psi, s) = \frac{1}{w_K}\sum_{[\mathfrak{a}]\in Cl_K}\frac{\psi(\mathfrak{a})}{N(\mathfrak{a})^{2\ell-s}}L^{(2\ell)}(\mathfrak{a}, s),$$

where the sum runs over (any choice of) representatives of the ideal class group of K.

Proof. The fact that the sum does not depend on the choice of representatives of Cl_K follows from the fact that

$$L^{(2\ell)}(\mu\mathfrak{a}, s) = \overline{\mu}^{-2\ell}|\mu|^{-2s}L^{(2\ell)}(\mathfrak{a}, s).$$

To prove formula, first write

$$L(\psi, s) = \sum_{[\mathfrak{a}]\in Cl_K}\sum_{\mathfrak{c}\in[\mathfrak{a}]}\frac{\psi(\mathfrak{c})}{N(\mathfrak{c})^s},$$

where the inner sum runs over the integral ideals $\mathfrak{c}$ in the class of $\mathfrak{a}$. Now fix $\mathfrak{b} \in [\mathfrak{a}]^{-1}$ such that $1 \in \mathfrak{b}$. Then $\mathfrak{c} \in [\mathfrak{a}]$ with $\mathfrak{c} \subseteq \mathcal{O}_K$ if and only if $\mathfrak{c}\mathfrak{b} = \lambda\mathcal{O}_K$ with $\lambda \in \mathfrak{b}$. Note that λ is unique up to an element of $\mathcal{O}_K^\times$ and $N(\mathfrak{c}) = N(\lambda)N(\mathfrak{b})^{-1}$. It follows that

$$\sum_{\mathfrak{c}\in[\mathfrak{a}]}\frac{\psi(\mathfrak{c})}{N(\mathfrak{c})^s} = \frac{1}{w_K}\frac{N(\mathfrak{b})^s}{\psi(\mathfrak{b})}\sum_{\lambda\in\mathfrak{b}}\frac{\lambda^{2\ell}}{|\lambda|^{2s}}.$$

Since $\mathfrak{a}\bar{\mathfrak{a}} = N(\mathfrak{a})\mathcal{O}_K$, one can take $\mathfrak{b} = \bar{\mathfrak{a}}N(\mathfrak{a})^{-1}$ (which contains 1) and then a short computation shows that the previous formula becomes

$$\sum_{\mathfrak{c}\in[\mathfrak{a}]}\frac{\psi(\mathfrak{c})}{N(\mathfrak{c})^s} = \frac{1}{w_K}\frac{N(\mathfrak{b})^s}{\psi(\mathfrak{b})}\sum_{\lambda\in\mathfrak{b}}\frac{\lambda^{2\ell}}{|\lambda|^{2s}} = \frac{1}{w_K}\frac{\psi(\mathfrak{a})}{N(\mathfrak{a})^{2\ell-s}}\sum_{\lambda\in\mathfrak{a}}\frac{\bar{\lambda}^{2\ell}}{|\lambda|^{2s}} = \frac{1}{w_K}\frac{\psi(\mathfrak{a})}{N(\mathfrak{a})^{2\ell-s}}L^{(2\ell)}(\mathfrak{a}, s).$$

□

Since

$$L^{(2\ell)}(\mathfrak{a}, s) = G_{2\ell}(\mathfrak{a}, s - 2\ell),$$

we obtain

Corollary 1. *Let ψ be a Hecke character of infinity type 2ℓ as above. Then*

$$L(\psi, s) = \frac{1}{w_K}\sum_{[\mathfrak{a}]\in Cl_K}\frac{\psi(\mathfrak{a})}{N(\mathfrak{a})^{2\ell-s}}G_{2\ell}(\mathfrak{a}, s - 2\ell) = \frac{1}{w_K}\left(\frac{2}{\sqrt{|D|}}\right)^{s-2\ell}\sum_{[\mathfrak{a}]=[[\omega_1,\omega_2]]}\frac{\psi(\mathfrak{a})}{\omega_2^{2\ell}}G_{2\ell}(\omega_1/\omega_2, s - 2\ell),$$

where the first sum runs over (any choice of) representatives of the ideal class group of K *and in the second one, $[\omega_1, \omega_2]$ is a positively oriented basis of $\mathfrak{a}$.*

5.2 The case $\ell = 0$: kronecker limit formula

When $\ell = 0$, Corollary 1 applied to ψ^2 (of infinity type 4ℓ) gives

$$L(\psi^2, s) = \frac{1}{w_K}\left(\frac{2}{\sqrt{|D|}}\right)^s \sum_{[\mathfrak{a}]=[[\omega_1,\omega_2]]} \psi^2(\mathfrak{a}) G_0(\omega_1/\omega_2, s). \tag{3}$$

Recall that we are interested in the value of $L(\psi^2, s)$ at $s = 2\ell+1 = 1$. Since the non-holomorphic Eisenstein series of weight 0 has a pole at $s = 1$, we need to look at the next term in the Taylor expansion around $s = 1$.

Theorem 2 (Kronecker Limit Formula). *Define the eta-function as*

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q = e^{2\pi i z}$ *and let*

$$G_0(z, s) = \Im(z)^s \sum_{m,n} |mz + n|^{-2s}$$

be the non-holomorphic Eisenstein series of weight 0. *Then*

$$G_0(z, s) = \pi\left(\frac{1}{s-1} + C(z) + O(s-1)\right),$$

where

$$C(z) = 2\gamma - \log 4 - 2\log(\Im(z)^{1/2}|\eta(z)|^2)$$

(γ =Euler's constant).

Proof. See [Cohe, Thm. 10.4.6]. Note that our definition of $G_0(z, s)$ differs from Cohen's by a factor of $1/2$. □

When ψ^2 is the trivial character, formula 3 is nothing else but the well-known decomposition of the Dedekind zeta function of K into a sum of Epstein zeta functions. Comparing the residues gives the class number formula for imaginary quadratic fields:

$$\mathrm{Res}_{s=1}\zeta_K(s) = L(\chi_D, 1) = \frac{2\pi h_K}{w_K\sqrt{|D|}}$$

and comparing the constant terms gives the Chowla-Selberg formula.

When ψ^2 is not trivial, the function $L(\psi^2, s)$ is analytic at $s = 1$[4] and has value

$$L(\psi^2, 1) = -\frac{4\pi}{w_K\sqrt{|D|}} \sum_{[\mathfrak{a}]\in \mathrm{Cl}_K} \psi^2(\mathfrak{a}) \log(\Im(\tau_\mathfrak{a})^{1/2}|\eta(\tau_\mathfrak{a})|^2).$$

Putting this in formula 2, we get

$$\langle\theta_\psi, \theta_\psi\rangle = -\frac{6}{\pi^2}\frac{1}{4\pi}\frac{2\pi h_K}{w_K\sqrt{|D|}}\prod_{p|D}(1+p^{-1})^{-1}\frac{4\pi}{w_K\sqrt{|D|}}\sum_{[\mathfrak{a}]\in \mathrm{Cl}_K} \psi^2(\mathfrak{a}) \log(\Im(\tau_\mathfrak{a})^{1/2}|\eta(\tau_\mathfrak{a})|^2)$$

which simplifies to

$$\langle\theta_\psi, \theta_\psi\rangle = -V_D^{-1}\frac{4h_K}{w_K^2} \sum_{[\mathfrak{a}]\in \mathrm{Cl}_K} \psi^2(\mathfrak{a}) \log(\Im(\tau_\mathfrak{a})^{1/2}|\eta(\tau_\mathfrak{a})|^2), \tag{4}$$

[4] Note again the importance of the fact that the residue of the non-holomorphic Eisenstein series at $s = 1$ does not depend on z.

where V_D is defined as

$$V_D = \text{Vol}(\Gamma_0(|D|) \setminus \mathcal{H}) = \text{Vol}(SL_2(\mathbb{Z}) \setminus \mathcal{H})[SL_2(\mathbb{Z}) : \Gamma_0(|D|)] = \frac{\pi}{3}|D| \prod_{p|D}(1+p^{-1}).$$

Note that factoring out the volume helps understanding the algebraic properties of the quantity on the right. This formula tells us that normalizing the Petersson inner product by dividing by the volume, as we did, artificially introduces transcendental numbers in the Petersson norm. We will come back to this point after we treat the case $\ell > 0$.

5.3 The case $\ell > 0$: derivative of almost holomorphic Eisenstein series

Define as usual the following differential operators on real analytic functions on the upper half-plane

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

For any integer k and congruence subgroup Γ, let $\widehat{M}_k(\Gamma)$ be the space of *almost holomorphic modular forms* of weight k and level Γ. An element of this space is a $|_k\gamma$-invariant function for all $\gamma \in \Gamma$, but instead of being holomorphic on $\mathcal{H}$, it is a polynomial in $1/\Im(z)$ with holomorphic coefficients satisfying some growth condition at infinity. The simplest example (and the only one we need) of almost holomorphic modular form is $E_2 \in \widehat{M}_2(SL_2(\mathbb{Z}))$.

If $f \in \widehat{M}_k(\Gamma)$ is an almost holomorphic modular form, the operator ∂_k defined as

$$\partial_k f = \frac{1}{2\pi i}\frac{\partial f}{\partial z} - \frac{k}{4\pi\Im(z)} f$$

takes f to an element of $\widehat{M}_{k+2}(\Gamma)$. To simplify the notation, define

$$\partial_k^n = \partial_{k+2n-2} \circ \cdots \circ \partial_{k+2} \circ \partial_k.$$

The following lemma is the starting point of our investigation.

Lemma 5. *Let $G_k(z,s)$ be the non-holomorphic Eisenstein series of weight k defined in section 2.2. Then*

$$\partial_k^n G_k(z,s) = (-4\pi)^{-n}\frac{\Gamma(k+s+n)}{\Gamma(s+k)} G_{k+2n}(z, s-n)$$

Proof. This is [Shi1, Formula 9.12] with $N = 1$ and $p = q = 0$. Note also that our ∂_k is Shimura's D_k (we follows Zagier's notation). □

This leads to the following

Corollary 2. *Let ψ be a Hecke character of infinity type $2\ell > 2$ as above and let m be an integer such that $\ell + 1 \leq m \leq 2\ell$. Then*

$$L(\psi, m) = \frac{1}{w_K}(-4\pi)^{2\ell-m}\frac{\Gamma(2m-2\ell)}{\Gamma(m)}\left(\frac{\sqrt{|D|}}{2}\right)^{2\ell-m} \sum_{[\mathfrak{a}]=[[\omega_1,\omega_2]]} \frac{\psi(\mathfrak{a})}{\omega_2^{2\ell}} \partial^{2\ell-m} G_{2m-2\ell}(\omega_1/\omega_2, 0),$$

where as usual the sum runs over positively oriented basis of representatives of the ideal class group of K.

Proof. Using the Lemma above with $n = 2\ell - m \geq 0$ and $k = 2m - 2\ell \geq 2$, we see that

$$G_{2\ell}(z, s+m-2\ell) = (-4\pi)^{2\ell-m}\frac{\Gamma(s+2m-2\ell)}{\Gamma(s+m)}\partial^{2\ell-m} G_{2m-2\ell}(z,s).$$

Putting this in the formula of Corollary 1 (evaluated at $s+m$), we see that

$$L(\psi, s+m) = \frac{1}{w_K}(-4\pi)^{2\ell-m}\frac{\Gamma(s+2m-2\ell)}{\Gamma(s+m)}\left(\frac{\sqrt{|D|}}{2}\right)^{-(s+m-2\ell)} \sum_{[\mathfrak{a}]=[[\omega_1,\omega_2]]} \frac{\psi(\mathfrak{a})}{\omega_2^{2\ell}} \partial^{2\ell-m} G_{2m-2\ell}(z,s),$$

□

Using the fact that

$$2^{-1}(2\pi i)^{-k}\Gamma(k)G_k(z,0) = E_k(z),$$

for all $k \geq 2$, where

$$E_2(z) = \frac{1}{8\pi\Im(z)} - \frac{1}{24} + \sum_{n=1}^{\infty}\sigma_1(n)q^n \tag{5}$$

and

$$E_k = -\frac{B_k}{2k} + \sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n \tag{6}$$

is the usual holomorphic Eisenstein series for $k \geq 4$ (see [Shi1, Sec 9.2]), one sees that the previous Corollary relates certain special values of the Hecke L-function attached to ψ to the derivatives of the usual Eisenstein series.

Applying the previous Corollary to ψ^2 with $m = 2\ell + 1$, we get

$$L(\psi^2, 2\ell+1) = \frac{1}{w_K}\frac{(-4\pi)^{2\ell-1}}{\Gamma(2\ell+1)}\left(\frac{\sqrt{|D|}}{2}\right)^{2\ell-1}\sum_{[\mathfrak{a}]=[[\omega_1,\omega_2]]}\frac{\psi^2(\mathfrak{a})}{\omega_2^{4\ell}}\partial^{2\ell-1}G_2(\omega_1/\omega_2, 0).$$

Using this value of $L(\psi^2, 2\ell+1)$ in formula 2 and simplifying finally gives

$$\langle\theta_\psi, \theta_\psi\rangle = V_D^{-1}\frac{|D|^\ell}{2^{2\ell}}\frac{4h_K}{w_K^2}\frac{1}{2(2\pi i)^2}\sum_{[\mathfrak{a}]=[[\omega_1,\omega_2]]}\frac{\psi^2(\mathfrak{a})}{\omega_2^{4\ell}}\partial^{2\ell-1}G_2(\omega_1/\omega_2, 0),$$

which can be rewritten as

$$\langle\theta_\psi, \theta_\psi\rangle = V_D^{-1}(|D|/4)^\ell\frac{4h_K}{w_K^2}\sum_{[\mathfrak{a}]=[[\omega_1,\omega_2]]}\frac{\psi^2(\mathfrak{a})}{\omega_2^{4\ell}}\partial^{2\ell-1}E_2(\omega_1/\omega_2). \tag{7}$$

Note that this can also be written in homogeneous form as

$$\langle\theta_\psi, \theta_\psi\rangle = V_D^{-1}(|D|/4)^\ell\frac{4h_K}{w_K^2}\sum_{[\mathfrak{a}]\in Cl_K}\psi^2(\mathfrak{a})\partial^{2\ell-1}E_2(\mathfrak{a}).$$

Corollary 3. *For $\ell > 0$,*

$$Vol(\Gamma_0(|D|)\setminus\mathcal{H})\langle\theta_\psi, \theta_\psi\rangle = \alpha\Omega_K^{4\ell},$$

where α is an algebraic number and Ω_K is the Chowla-Selberg period attached to K and depends only on K.

Proof. From the Corollary of Proposition 27 in [Zag], it follows that

$$\partial_2^{2\ell-1}E_2(\tau)$$

is an algebraic multiple of $\Omega_K^{2+2(2\ell-1)} = \Omega_K^{4\ell}$, whenever $\tau \in K\cap\mathcal{H}$ is a CM point. The Corollary follows from the fact that the values of the Hecke characters ψ_ℓ and all the other quantities in formula 7 are algebraic. □

5.4 The case $\ell = 0$ revisited

Strictly speaking, formula 7 does not make sense for $\ell = 0$. However, it is natural to define ∂_2^{-1} as a weight 0 "modular form" f such that

$$\partial_0 f(z) = E_2(z).$$

We claim that

$$\partial_0\log(\Im(z)^{1/2}|\eta(z)|^2) = -E_2(z),$$

where

$$\partial_0 = \frac{1}{2\pi i}\frac{\partial}{\partial z}.$$

This follows from the well known fact (see [Zag, Prop. 7]) that

$$\frac{1}{2\pi i}\frac{\partial}{\partial z}\log\Delta(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$$

and the identity

$$\Delta(z) = \eta(z)^{24}.$$

Indeed, since

$$\log|\Delta(z)| = \Re(\log\Delta(z)),$$

this implies

$$\frac{\partial}{\partial z}\log|\Delta(z)| = \frac{1}{2}\frac{\partial}{\partial z}\log\Delta(z)$$

(recall that $\frac{\partial \bar{f}}{\partial z} = \overline{\frac{\partial f}{\partial \bar{z}}} = 0$ if $f(z)$ is holomorphic).

The equality

$$\partial_0 \log(\Im(z)^{1/2}|\eta(z)|^2) = -E_2(z)$$

implies that formula 7 also makes sense for $\ell = 0$ and gives back exactly formula 4. Note also that $\log(\Im(z)^{1/2}|\eta(z)|^2)$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant, as desired. However, I don't think $\log(\Im(z)^{1/2}|\eta(z)|^2)$ is almost holomorphic.

6 Theta functions attached to ideals in imaginary quadratic fields

In this section we define theta series attached to ideals in imaginary quadratic fields and certain spherical polynomials and see how these theta functions are relate to the theta functions θ_ψ.

Throughout this section, fix an integer $\ell \geq 0$.

Let $\mathfrak{a}$ be a fractional ideal of K and define the theta function attached to $\mathfrak{a}$ (and ℓ) as

$$\theta_{\mathfrak{a}}^{(2\ell)} = \theta_{\mathfrak{a}}(z) = \sum_{\lambda\in\mathfrak{a}} \lambda^{2\ell} q^{N(\lambda)/N(\mathfrak{a})},$$

where we define $0^0 = 1$ in case $\ell = 0$. Then we have the following

Proposition 2. *The function $\theta_{\mathfrak{a}}$ is a modular form of weight $2\ell + 1$, level $\Gamma_0(|D|)$ and Nebentypus χ_D. Moreover, it is a cusp form if $\ell > 0$.*

Proof. This is well-know, but tedious to prove! A good reference for that is [Iwan, Thm. 10.9]. The point is that the function $\lambda \mapsto \lambda^{2\ell}$ is a spherical polynomial for the binary quadratic form $N(\lambda)/N(\mathfrak{a})$. □

If ψ is a Hecke character of infinity type 2ℓ, the theta function θ_ψ decomposes as follows:

$$\theta_\psi = \frac{1}{w_K}\sum_{[\mathfrak{a}]\in \mathrm{Cl}_K} \psi(\mathfrak{a})^{-1}\theta_{\mathfrak{a}}, \tag{8}$$

where the sum runs over representatives of the class group. Note that $\theta_{\mathfrak{a}}$ depends on the choice of $\mathfrak{a}$ in $[\mathfrak{a}]$, since

$$\theta_{\mu\mathfrak{a}} = \mu^{2\ell}\theta_{\mathfrak{a}}$$

for any $\mu \in K^\times$, but the sum is still independent of this choice. To prove formula 8, one uses the same trick as in the proof of Proposition 1.

Note that the L-function attached to $\theta_{\mathfrak{a}}^{(2\ell)}$ is the partial Hecke L-function $L^{(2\ell)}(\bar{\mathfrak{a}}, s)$ introduced before.

Our next goal is to write $\theta_{\mathfrak{a}}$ in terms of the θ_ψ. For this, the following Lemma is useful.

Lemma 6. *Fix an integer $\ell \geq 0$ and let $\mathfrak{c}$ be a fractional ideal of K. Then*

$$\sum_{\psi} \psi(\mathfrak{c}) = \begin{cases} 0 & \text{if } [\mathfrak{c}] \neq [\mathcal{O}_K] \\ \lambda^{2\ell} h_K & \text{if } \mathfrak{c} = \lambda \mathcal{O}_K \end{cases},$$

where the sum runs over all Hecke characters of K of infinity type 2ℓ.

Proof. Fix a Hecke character χ of infinity type 2ℓ. Then

$$\sum_{\psi} \psi\chi^{-1}(\mathfrak{c}) = \begin{cases} 0 & \text{if } [\mathfrak{c}] \neq [\mathcal{O}_K] \\ h_k & \text{if } \mathfrak{c} = \lambda \mathcal{O}_K \end{cases},$$

by the orthogonality relations of finite abelian group characters, since $\psi\chi^{-1}$ is a character of Cl_K. The claim follows by multiplying both sides by $\chi(\mathfrak{c})$ since $\chi(\lambda\mathcal{O}_K) = \lambda^{2\ell}$. □

This leads to the following

Proposition 3. *With $\theta_\mathfrak{a}$ defined as above,*

$$\theta_\mathfrak{a} = \frac{w_K}{h_K} \sum_{\psi} \psi(\mathfrak{a})\theta_\psi,$$

where the sum runs over all Hecke characters of infinity type 2ℓ.

Proof. This follows formally from the previous Lemma and the expression for θ_ψ in terms of the $\theta_\mathfrak{a}$. □

Using the orthogonality of the θ_ψ under the Petersson inner product when $\ell > 0$, one can compute $\langle \theta_\mathfrak{a}, \theta_\mathfrak{b} \rangle$ in terms of the Petersson norm of the θ_ψ. When $\ell = 0$, the θ_ψ are not always cusp forms and we have not found a way to compute (or even define) the Petersson norm of all the θ_ψ. However, we still have the following

Proposition 4. *Let $\ell > 0$ and let $\theta_\mathfrak{a}$ and $\theta_\mathfrak{b}$ be defined as above. Then*

$$\langle \theta_\mathfrak{a}, \theta_\mathfrak{b} \rangle = C_K N(\mathfrak{b})^{2\ell} \sum_{\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_\mathfrak{c}\mathcal{O}_K} \lambda_\mathfrak{c}^{2\ell} \partial^{2\ell-1} E_2(\mathfrak{c}),$$

where the sum runs over all ideal classes $[\mathfrak{c}] \in Cl_K$ such that $\mathfrak{c}^2\mathfrak{a}\mathfrak{b}^{-1} = \lambda_\mathfrak{c}\mathcal{O}_K$ for some $\lambda_\mathfrak{c} \in K$ and

$$C_K = 4V_D^{-1}(|D|/4)^\ell.$$

In particular, $\theta_\mathfrak{a}$ and $\theta_\mathfrak{b}$ are orthogonal if $\mathfrak{a}$ and $\mathfrak{b}$ are not in the same genus.

Proof. First, we compute

$$\begin{aligned}
\langle \theta_\mathfrak{a}, \theta_\mathfrak{b} \rangle &= \frac{w_K^2}{h_K^2} \sum_{\psi,\chi} \psi(\mathfrak{a})\overline{\chi}(\mathfrak{b}) \langle \theta_\psi, \theta_\chi \rangle \\
&= \frac{w_K^2}{h_K^2} \sum_{\psi} \psi(\mathfrak{a})\overline{\psi}(\mathfrak{b}) \langle \theta_\psi, \theta_\psi \rangle \\
&= \frac{w_K^2}{h_K^2} \sum_{\psi} \psi(\mathfrak{a}) N(\mathfrak{b})^{2\ell} \psi^{-1}(\mathfrak{b}) \langle \theta_\psi, \theta_\psi \rangle \\
&= \frac{C_K}{h_K} N(\mathfrak{b})^{2\ell} \sum_{\psi,[\mathfrak{c}]} \psi(\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2) \partial^{2\ell-1} E_2(\mathfrak{c}),
\end{aligned}$$

where the first sum is a double sum over all Hecke characters of infinity type 2ℓ and we used the orthogonality of the newforms θ_ψ in the second equality.

Summing the last sum over ψ first and using Lemma 6, we see that

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle = 0$$

if for all $[\mathfrak{c}] \in \mathrm{Cl}_K$, $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2$ is not principal, i.e. $\mathfrak{a}$ and $\mathfrak{b}$ are not in the same genus. Otherwise, if $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K$ for some $\lambda_{\mathfrak{c}} \in K$, then

$$\sum_{\psi} \psi(\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2) = \lambda_{\mathfrak{c}}^{2\ell} h_K$$

and the last line of the above computation becomes

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle = C_K N(\mathfrak{b})^{2\ell} \sum_{\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}} \lambda_{\mathfrak{c}}^{2\ell} \partial^{2\ell-1} E_2(\mathfrak{c}).$$

□

Corollary 4. *Fix $\ell > 0$ and let $\theta_{\mathfrak{a}}$ and $\theta_{\mathfrak{b}}$ be defined as above. Then*

$$\mathit{Vol}(\Gamma_0(|D|) \setminus \mathcal{H}) \langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle = \alpha \Omega_K^{4\ell},$$

where α is some algebraic number and Ω_K is the Chowla-Selberg period attached to K.

7 An efficient algorithm to compute the Petersson inner product of binary theta series

Formula 4 can be used to numerically evaluate the Petersson inner product of theta series attached to imaginary quadratic fields in an efficient way (1000 decimals in a few seconds!). To implement this formula, one should be able to find the derivatives of E_2, to evaluate them at lattices and find, for fixed ideals $\mathfrak{a}$ and $\mathfrak{b}$, all ideal classes $\mathfrak{c}$ such that $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K$. We talk about those problems in the next section and then we give a pseudo-algorithm to solve our initial problem of computing $\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle$.[5]

7.1 Towards the algorithm

7.1.1 Derivatives of almost holomorphic modular forms

First, recall that the ring of almost holomorphic of level 1 is isomorphic as a $\mathbb{C}$-algebra to

$$\mathbb{C}[E_2, E_4, E_6].$$

It follows that in order to compute $\partial^n E_2$, it suffices to know $\partial E_2, \partial E_4$ and ∂E_6. For this, we have the following

Proposition 5. *Let E_2, E_4 and E_6 be the almost holomorphic modular forms defined by equation 5 and 6 and let*

$$\partial_k = \frac{1}{2\pi i}\frac{\partial}{\partial z} - \frac{k}{4\pi \Im(z)}$$

. Then

$$\partial E_2 = \frac{5}{6}E_4 - 2E_2^2, \qquad \partial E_4 = \frac{7}{10}E_6 - 8E_2E_4, \qquad \partial E_6 = \frac{400}{7}E_4^2 - 12E_2E_6.$$

Proof. This is in [Shi1, Sec 9.2], plus the fact that

$$120E_4^2 = E_8.$$

□

[5] A PARI/GP implementation of this algorithm is available on https://github.com/NicolasSimard/ENT.

7.1.2 Evaluating Eisenstein series at CM points

By the above, the problem reduces to evaluating E_2, E_4 and E_6 at lattices. Generally, the Fourier expansions of these Eisenstein series converge very quickly. However, we have some freedom in choosing the lattice at which we evaluate them. As the following example shows, one should really take advantage of that.

Take $K = \mathbb{Q}(\sqrt{-26})$. Then $D = -104$, and Cl_K is cyclic of order 6, generated by any prime above 5. In fact, since

$$N(109 - 12\sqrt{-26}) = 5^6,$$

$$\mathfrak{p}_5^6 = \lambda \mathcal{O}_K,$$

where $\lambda = 109 + 12\sqrt{-26}$ and $\mathfrak{p}_5$ is one of the two primes above 5 (chosen so that the equation holds). Using PARI/GP, we find $\mathbb{Z}$-basis for $\mathfrak{p}_5^4$ and $\mathfrak{p}_5^{-2}$:

$$\mathfrak{p}_5^4 = [625, 43 + \sqrt{-26}], \qquad \mathfrak{p}_5^{-2} = [1, (7 + \sqrt{-26})/25].$$

From the equality $\mathfrak{p}_5^4 = \lambda \mathfrak{p}_5^{-2}$, we deduce that

$$\partial^n E_2(\mathfrak{p}_5^4) = \lambda^{-(2+2n)} \partial^n E_2(\mathfrak{p}_5^{-2})$$

and using the above $\mathbb{Z}$-basis, we have

$$625^{-(2+2n)} \partial^n E_2((43 + \sqrt{-26})/625) = \lambda^{-(2+2n)} \partial^n E_2((7 + \sqrt{-26})/25).$$

For $n = 1$ and working with 500 digits of precision, the left-hand side of the equation takes about 30 times more time to evaluate than the right-hand side (which takes around 1sec to evaluate on my desktop computer)! This proves that the running time of the algorithm depends in a crucial way on the choice of class representatives in a given ideal class.

The reason for the large difference in computation time in the above example is of course that the imaginary part of $(43 + \sqrt{-26})/625$ is smaller than the imaginary part of $(7 + \sqrt{-26})/25$. Using the correspondence between ideal classes and equivalence classes of positive definite integral binary quadratic forms, we see that the ideal corresponding to the quadratic form $[a, b, c]$ has $\mathbb{Z}$-basis

$$[a, (-b + \sqrt{D})/2]$$

and the imaginary part of the corresponding point in the upper-half plane is

$$\frac{\sqrt{|D|}}{2a}.$$

For fixed D, our problem is then to minimize a. It turns out that in a given class, the quadratic form with minimal a is the unique reduced quadratic form in that class. Moreover, for any reduced form, one has the following upper bound for a

$$a \leq \sqrt{|D|/3},$$

which leads to a lower bound on the imaginary part of to corresponding point in the upper-half plane. This proves the following

Proposition 6. *Let K be an imaginary quadratic field and let $\mathcal{C}$ be an ideal class in Cl_K. Then there exists an explicit positively oriented ideal $[\omega_1, \omega_2]$ such that*

$$\frac{3}{2} \leq \Im(\omega_1/\omega_2).$$

This discussion leads to the following simple algorithm to evaluate $\partial^n E_2$ at an ideal in an imaginary quadratic field: find the class to which this ideal belongs and use the CM point corresponding to the reduced form in that class to evaluate $\partial^n E_2$. The lower bound above is a kind of guarantee on the speed of this algorithm.

7.1.3 Ambiguous classes

The last problem in computing $\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle$ is to find all ideal classes $\mathfrak{c}$ such that $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K$ (and find $\lambda_{\mathfrak{c}}$ too). Given a set of generators for Cl_K, it is easy to determine if the class $\mathfrak{a}\mathfrak{b}^{-1}$ is a square in Cl_K and to find a class $\mathfrak{c}_0$ such that

$$\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}_0^2 = \lambda\mathcal{O}_K.$$

Indeed, write $\mathfrak{a}\mathfrak{b}^{-1}$ in term of those generators and check that only even powers of the generators occur. The following proposition completes the task.

Proposition 7. *Let $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}_0^2 = \lambda\mathcal{O}_K$ for some ideal $\mathfrak{c}_0$. Let*

$$\{\mathfrak{a}_1, \ldots, \mathfrak{a}_g\}$$

be representatives of $Cl_K[2]$ and define α_i for $1 \leq i \leq g$ as

$$\mathfrak{a}_i^2 = \alpha_i\mathcal{O}_K.$$

Then

$$\sum_{\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K} \lambda_{\mathfrak{c}}^{2\ell}\partial^{2\ell-1}E_2(\mathfrak{c}) = \lambda^{2\ell}\sum_{i=1}^{g}\alpha_i^{2\ell}\partial^{2\ell-1}E_2(\mathfrak{c}_0\mathfrak{a}_i).$$

Proof. It suffices to note that any ideal $\mathfrak{c}$ such that

$$\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K$$

is equivalent to $\mathfrak{c}_0\mathfrak{a}_i$ for some i and that

$$\mathfrak{a}\mathfrak{b}^{-1}(\mathfrak{c}_0\mathfrak{a}_i)^2 = \lambda\alpha_i\mathcal{O}_K.$$

□

The 2-torsion classes in Cl_K are also known as the ambiguous classes of K. They are easy to compute using the theory of binary quadratic forms[6] and there are exactly g of them, where g is the number of genera in K (i.e. $g = |\mathrm{Cl}_K/\mathrm{Cl}_K^2|$).

7.2 The algorithm

8 Computing some special values of Hecke L-functions

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[6] Indeed, the inverse of the positive definite binary quadratic form $[a, b, c]$ is $[a, -b, c]$ and forcing these forms to be equivalent puts big restrictions on a, b and c.

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