Petersson norm of theta series and derivatives of Eisenstein series

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Introduction

In these notes, we find a formula for the Petersson norm of the theta series θ_ψ attached to an imaginary quadratic field K and a Hecke character of infinity type 2ℓ. The formula is

$$\langle\theta_\psi,\theta_\psi\rangle = V_D^{-1}(|D|/4)^\ell \frac{4h_K}{w_K^2}\sum_{[\mathfrak{a}]\in \mathrm{Cl}_K}\psi^2(\mathfrak{a})\partial_2^{2\ell-1}E_2(\mathfrak{a})$$

if $\ell>0$ and

$$\langle\theta_\psi,\theta_\psi\rangle = -V_D^{-1}\frac{4h_K}{w_K^2}\sum_{[\mathfrak{a}]\in \mathrm{Cl}_K}\psi^2(\mathfrak{a})\log(\mathfrak{I}(\tau_\mathfrak{a})^{1/2}|\eta(\tau_\mathfrak{a})|^2)$$

if $\ell=0$ and ψ is not a genus character. Here $\partial_2^{2\ell-1}E_2$ is the non-holomorphic derivative of the non-holomorphic Eisenstein series of weight 2 and level 1, viewed as a function on lattices in the usual way, and

$$V_D = \mathrm{Vol}(\Gamma_0(|D|)\setminus\mathcal{H}) = \frac{\pi}{3}|D|\prod_{p|D}(1+p^{-1}).$$

In the last section, we will see that one can make sense of the first formula even for $\ell=0$ and that it gives back exactly the second formula!

Before proving the formula, we first recall a few facts about Hecke characters, Eisenstein series and the Rankin-Selberg method. Then we introduce the theta functions θ_ψ. In the following section, we show how the Petersson norm of the θ_ψ is related to the Hecke L-function of ψ^2. Finally, we relate the Hecke L-function of ψ^2 to non-holomorphic Eisenstein series and use this relation to establish the two formulas.

If ψ is a genus character, θ_ψ is an Eisenstein series and one should use the regularized Petersson inner product. I think a similar formula holds. I will try this soon.

1 Setup and notation

Throughout, $K=\mathbb{Q}(\sqrt{D})$ denotes an imaginary quadratic field of discriminant $D<-4$ and $\mathcal{O}_K$ denotes its ring of integers.

2 Preliminaries

2.1 Hecke Grossencharacters

Let I_K be the multiplicative group of fractional ideals of K. Given an integer $\ell\geq 0$, let ψ_ℓ denote a *Hecke Grossencharacter* of conductor 1 and infinity type 2ℓ, that is a group homomorphism

$$\psi_\ell : I_K \to \mathbb{C}^\times$$

such that

$$\psi_\ell((\alpha)) = \alpha^{2\ell}, \qquad \forall \alpha\in K^\times.$$

Note that this is well-defined since $\mathcal{O}_K^\times=\{\pm 1\}$ by assumption.

Those Hecke characters are not of the form considered in the books of Miyake [Miya, Ch. 3, Sec. 3] or Iwaniec [Iwan, Ch. 12, Sec. 2]. For clarity, we call the ones they define *unitary*. Let $N: I_K\to\mathbb{Q}$ denote the norm map on ideals. Then the character

$$\psi_\ell N^{-\ell} : I_K \to \mathbb{C}^\times$$

is unitary of conductor 1 and of infinity type 2ℓ (take $u_\sigma + iv_\sigma = 2\ell$ in their definition, where $\sigma: K\hookrightarrow\mathbb{C}$ is a complex embedding).

To a Hecke character ψ (unitary or not), one attaches the Dirichlet L-series

$$L(\psi,s)=\sum_{\mathfrak{a}}\frac{\psi(\mathfrak{a})}{N(\mathfrak{a})^s},$$

which converges for s in some right-half plane in $\mathbb{C}$. Clearly, multiplying ψ with a power of the norm N^ℓ simply shifts the L-function by ℓ:

$$L(\psi, s-\ell) = L(\psi \circ N^\ell, s).$$

Define the completed L-function of $L(\psi_\ell, s)$ as

$$\Lambda(\psi_\ell, s) = |D|^{s/2}(2\pi)^{-s}\Gamma(s)L(\psi_\ell, s).$$

Theorem 1 (Hecke). *1. Λ can be analytically continued to a meromorphic function on $\mathbb{C}$ and satisfies the functional equation*

$$\Lambda(\psi_\ell, s) = w(\psi_\ell)\Lambda(\overline{\psi_\ell}, 2\ell+1-s),$$

where $|w(\psi_\ell)| = 1$.

2. *$\Lambda(\psi_\ell, s)$ is holomorphic on $\mathbb{C}$, except when ψ_ℓ is the trivial character (this can only happen when $\ell = 0$), in which case is has a pole at $s = 0$ and $s = 1$.*

3. *$L(\psi_\ell, s)$ is holomorphic on $\mathbb{C}$, except when ψ_ℓ is the trivial character, in which case it has a pole at $s = 1$ with residue*

$$\frac{2\pi h_K}{w_K\sqrt{|D|}},$$

where h_K is the class number of K and $w_K = 2$ is the number of roots of unity in K.

Proof. See [Miya, Ch. 3, Sec. 3]. □

2.2 Eisenstein series: holomorphic and non-holomorphic

Eisenstein series will be useful in many ways in these notes. Recall that they can be defined in essentially two (closely related) ways: as Poincare series and as sum over lattice points. The first type is used in the Rankin-Selberg method, while the second is linked to Hecke L-functions of imaginary quadratic fields. We recall a few basic facts about these series. Our main references are [Shi1, Ch.9], [Shi1, A3] and [Miya, Ch.7]

Let $N \geq 1$ and $k \geq 0$ be integers. As usual, define

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \pmod N\right\}$$

and for $f : \mathcal{H} \to \mathbb{C}$ a function on the upper half plane and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, define the slash-k as operator

$$(f|_k\gamma)(z) = j(\gamma, z)^{-k}f(\gamma z),$$

where $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$ in the usual way and

$$j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = cz + d.$$

Let also Γ_∞ be the stabilizer of the cusp at infinity in $\mathrm{SL}_2(\mathbb{Z})$, i.e.

$$\Gamma_\infty = \left\{\pm\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z}\right\}$$

For $(z, s) \in \mathcal{H} \times \mathbb{C}$, define the *non-holomorphic Eisenstein series of weight* k as

$$G_k(z, s) = \Im(z)^s \sum_{m,n} (mz+n)^{-k}|mz+n|^{-2s},$$

where the sum is over all integers m and n, not both 0. This sum converges for $\Re(2s) + k > 2$.

Since

$$\Im(\gamma z)^s = |j(\gamma, z)|^{-2s}\Im(z)^s,$$

the non-holomorphic Eisenstein series satisfies the following functional equation:

$$G_k(\gamma z, s) = j(\gamma, z)^k G_k(z, s).$$

In particular, k must be even.

For $k > 2$, the series converges absolutely at $s = 0$ and equals the usual Eisenstein series of weight k and level 1. For $k = 2$, it does not converge absolutely at $s = 0$. However, for $k > 0$ there is a real analytic function of $(z, s) \in \mathcal{H} \times \mathbb{C}$ which is holomorphic in s and coincides with $\Gamma(s + k)G_k(z, s)$ for $\Re(2s) + k > 2$ ([Shi1, Thm A3.5]). Therefore it still makes sense to consider $G_2(z, 0)$. Define

$$E_2(z) = 2^{-1}(2\pi i)^{-2} G_2(z, 0).$$

Then E_2 is an *almost holomorphic* modular form of weight 2 and level 1 with Fourier expansion

$$E_2(z) = \frac{1}{8\pi \Im(z)} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

which clearly has algebraic Fourier coefficients. Almost holomorphic modular forms are defined as in [Zag, Sec. 5.3][1]. In particular,

$$E_2|_2\gamma = E_2, \qquad \forall \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

Consider now the following Eisenstein series:

$$E_k^N(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \Im(z)^s|_k\gamma = \mathrm{Im}(z)^s \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} j(\gamma, z)^{-k} |j(\gamma, z)|^{-2s}.$$

This series also converges absolutely for $\Re(2s) + k > 2$ and can be analytically continued to a holomorphic function in s, except when $k = 0$, in which case $E_0^N(z, s)$ has a pole at $s = 1$ with residue

$$\mathrm{Res}_{s=1} E_0^N(z, s) = \mathrm{Vol}(\Gamma_0(N) \setminus \mathcal{H})^{-1}.$$

2.3 Rankin-Selberg method in level N

The Rankin-Selberg is well-known. We sketch it here mainly to make sure that the normalizations are correct. Our main reference is [Shi2].

Let $f(z), g(z) \in \mathcal{S}_k(\Gamma_0(N), \chi)$ be two cusp forms of weight k, level N and Nebentypus χ. Then the function

$$F(z) = f(z)\overline{g(z)}\Im(z)^k$$

if $\Gamma_0(N)$-invariant and tends to 0 rapidly as $\Im(z)$ tends to ∞, so it makes sense to define the *Petersson inner product* of f and g as

$$\langle f, g \rangle = \frac{1}{\mathrm{Vol}(\Gamma_0(N) \setminus \mathcal{H})} \int\int_{\Gamma_0(N)\setminus\mathcal{H}} F(z) d\mu(z),$$

where we integrate over a fundamental domain for the action of $\Gamma_0(N)$ on $\mathcal{H}$ and

$$d\mu(z) = \frac{dxdy}{y^2}$$

is the $\mathrm{SL}_2(\mathbb{Z})$-invariant measure on $\mathcal{H}$.

[1] Shimura calls those functions nearly holomorphic in [Shi1], but we prefer to use this term to refer to modular forms with (possibly) poles at infinity.

Now for $\Re(s)$ large enough, the series for $E_0^N(z,s)$ converges absolutely and the following manipulations are justified:

$$\begin{aligned}\iint_{\Gamma_0(N)\backslash\mathcal{H}} F(z)E_0^N(z,s)d\mu(z) &= \iint_{\Gamma_0(N)\backslash\mathcal{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} F(z)\Im(\gamma z)^s d\mu(z)\\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \iint_{\Gamma_0(N)\backslash\mathcal{H}} F(\gamma z)\Im(\gamma z)^s d\mu(z)\\ &= \iint_{\Gamma_\infty\backslash\mathcal{H}} F(z)\Im(z)^s d\mu(z)\end{aligned}$$

As a functions of s, the last integral has a residue at $s=1$. Using the value of $\mathrm{Res}_{s=1}E_0^N(z,s)$ given above, one sees that

$$\mathrm{Res}_{s=1}\iint_{\Gamma_\infty\backslash\mathcal{H}} F(z)\Im(z)^s d\mu(z) = \mathrm{Res}_{s=1}\iint_{\Gamma_0(N)\backslash\mathcal{H}} F(z)E_0^N(z,s)d\mu(z) = \langle f,g\rangle.$$

Note that it is important that $\mathrm{Res}_{s=1}E_0^N(z,s)$ does not depend on z.

On the other hand, let

$$f(z) = \sum_{n=1}^\infty a_n q^n \qquad \text{and} \qquad g(z) = \sum_{n=1}^\infty b_n q^n$$

be the q-expansions of f and g. Then

$$f(z)\overline{g(z)} = \sum_{m,n=1}^\infty a_n\overline{b_m}e^{2\pi i n z}e^{-2\pi i m\bar{z}} = \sum_{m,n=1}^\infty a_n\overline{b_m}e^{2\pi i(n-m)x}e^{-2\pi(m+n)y},$$

where $z = x+iy$, so

$$\int_0^1 F(z)\Im(z)^s dx = \sum_{n=1}^\infty a_n\overline{b_n}e^{-4\pi n y}y^{k+s}$$

and

$$\iint_{\Gamma_\infty\backslash\mathcal{H}} F(z)\Im(z)^s d\mu(z) = \int_0^\infty\left(\int_0^1 F(z)\Im(z)^s dx\right)\frac{dy}{y^2} = \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}}\sum_{n=1}^\infty \frac{a_n\overline{b_n}}{n^{s+k-1}}.$$

Comparing the expressions for

$$\mathrm{Res}_{s=1}\iint_{\Gamma_\infty\backslash\mathcal{H}} F(z)\Im(z)^s d\mu(z),$$

gives the formula

$$\langle f,g\rangle = \Gamma(k)(4\pi)^{-k}\mathrm{Res}_{s=k}D(f,g_\rho,s), \tag{1}$$

where

$$D(f,g,s) = \sum_{n=1}^\infty \frac{a_n b_n}{n^s}$$

and

$$g_\rho(z) = \overline{g(-\bar{z})} = \sum_{n=1}^\infty \overline{b_n}q^n.$$

3 Theta series attached to imaginary quadratic fields

Let $\ell \geq 0$ and $\psi = \psi_\ell$ be a Hecke character of infinity type 2ℓ. Consider the theta series

$$\theta_\psi(z) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{N(\mathfrak{a})},$$

where the sum runs over all integral ideals of $\mathcal{O}_K$. It is well known ([Iwan, Thm. 12.5]) that

$$\theta_\psi(z) \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

where χ_D is the quadratic character attached to K (i.e. the Kronecker symbol).[2]

If $\ell > 0$, θ_ψ is in fact a cusp form. If $\ell = 0$, this is also true, unless ψ is a genus character (i.e. $\psi^2 = 1$), in which case it is an Eisenstein series. In any case,

$$L(\theta_\psi, s) = L(\psi, s),$$

so the L-function of θ_ψ has an Euler product[3]. It follows that θ_ψ is a normalized (i.e. $a_1(\theta_\psi) = 1$) eigenform for all Hecke operators (see [DiSh, Thm. 5.9.2]). Moreover,

$$a_n(\theta_\psi) = \sum_{N(\mathfrak{a})=n} \psi(\mathfrak{a}),$$

where the sum is over all integral ideals of K of norm n. It follows that

$$a_p(\theta_\psi) = \begin{cases} 0 & \text{if } \chi_D(p) = -1 \\ \psi(\mathfrak{p}) + \psi(\bar{\mathfrak{p}}) & \text{if } \chi_D(p) = 1 \text{ and } p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}} \\ \psi(\mathfrak{p}) & \text{if } \chi_D(p) = 0 \text{ and } p\mathcal{O}_K = \mathfrak{p}^2 \end{cases},$$

in accordance with the equality between the L-functions of θ_ψ and ψ.

Using the fact that the adjoint of the Hecke operators T_p acting on $S_{2\ell+1}(\Gamma_0(|D|), \chi_D)$ with respect to the Petersson inner product is

$$T_p^* = \overline{\chi_D}(p) T_p$$

for all p not dividing D (see [DiSh, Thm. 5.5.3]), one sees that

$$a_p(\theta_\psi) = \chi_D(p) \overline{a_p(\theta_\psi)}$$

for all p not dividing D, whenever θ_ψ is a cusp form.

Lemma 1.

$$a_n(\theta_\psi) \in \mathbb{R}$$

whenever θ_ψ is a cusp form.

Proof. By the multiplicativity property of the $a_n(\theta_\psi)$, it suffices to prove the result for $n = p^k$ =a prime power. Recall that

$$a_{p^{k+1}}(\theta_\psi) = a_p(\theta_\psi) a_{p^k}(\theta_\psi) - \chi_D(p) p^{2\ell} a_{p^{k-1}}(\theta_\psi),$$

for all $k \geq 1$.

If p is inert in K, $a_p(\theta_\psi) = 0$ and so $a_{p^k}(\theta_\psi) = 0$ for all $k \geq 0$.

If p splits in K, $a_p(\theta_\psi) = \chi_D(p)\overline{a_p(\theta_\psi)} = \overline{a_p(\theta_\psi)}$, so $a_p(\theta_\psi) \in \mathbb{R}$ and the claim follows from the recursive formula.

Finally if p ramifies, say $p\mathcal{O}_K = \mathfrak{p}^2$, then $a_p(\theta_\psi) = \pm p^\ell$ since

$$p^{2\ell} = \psi((p)) = \psi(\mathfrak{p}^2) = \psi(\mathfrak{p})^2$$

and the claim follows again from the recursive formula. □

[2]Note that the Hecke characters ψ_ℓ have conductor $\mathcal{O}_K$, so they are automatically primitive.

[3]One reason to choose the non-unitary Hecke characters ψ_ℓ is to have simpler formulas, like this one.

4 The Petersson norm of θ_ψ

In this section, suppose θ_ψ is a cusp form, i.e. $\psi^2 \neq 1$. We will prove that the Petersson norm of θ_ψ is

$$\langle \theta_\psi, \theta_\psi \rangle = V_D^{-1} \frac{4h_k}{w_k} \sqrt{|D|} \frac{\Gamma(2\ell+1)}{(4\pi)^{2\ell+1}} L(\psi^2, 2\ell+1), \tag{2}$$

where $V_D = \mathrm{Vol}(\Gamma_0(|D|) \setminus \mathcal{H})$, as before.

Note that if $\psi^2 = 1$, $\ell = 0$ and so $L(\psi^2, s)$ has a pole at $s = 1$.

For each prime p, the L-function of θ_ψ has Euler factor at p equal to

$$1 - a_p(\theta_\psi)p^{-s} + \chi_D(p)p^{2l-2s} = (1 - \alpha_p p^{-s})(1 - \beta_p p^{-s}),$$

where we set $\beta_p = 0$ if $p|D$. One can then define the symmetric square L-function of θ_ψ as

$$L(\mathrm{Sym}^2\theta_\psi, s) = \prod_p ((1 - \alpha_p^2 p^{-s})(1 - \alpha_p \beta_p p^{-s})(1 - \beta_p^2 p^{-s}))^{-1}$$

for $\Re(s)$ large enough. This L-function can be analytically continued to a meromorphic funtion on the whole complex plane, with (possibly) poles at $s = 2\ell$ and $s = 2\ell + 1$ (see [Shi2, Thm. 2]).

Using the description of $a_p(\theta_\psi)$ given in the previous section, one sees that

$$\{\alpha_p, \beta_p\} = \begin{cases} \{\pm p^\ell, \mp p^\ell\} & \text{if } \chi_D(p) = -1 \\ \{\psi(\mathfrak{p}), \psi(\bar{\mathfrak{p}})\} & \text{if } \chi_D(p) = 1 \text{ and } p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}} \\ \{\psi(\mathfrak{p}), 0\} & \text{if } \chi_D(p) = 0 \text{ and } p\mathcal{O}_K = \mathfrak{p}^2 \end{cases}.$$

The proof of formula 2 relies on the Rankin-Selberg method:

$$\langle \theta_\psi, \theta_\psi \rangle = (4\pi)^{-2\ell-1}\Gamma(2\ell+1)\mathrm{Res}_{s=2\ell+1} D(\theta_\psi, \theta_\psi, s),$$

where we used the fact that θ_ψ has real Fourrier coefficients (Lemma 1). Before proving the formula, we mention the following Lemma of Shimura (see [Shi3, Ch.3, Lem.1]).

Lemma 2. *Suppose we have formally*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p ((1 - \alpha_p p^{-s})(1 - \beta_p p^{-s}))^{-1},$$

$$\sum_{n=1}^{\infty} \frac{b_n}{n^s} = \prod_p ((1 - \alpha'_p p^{-s})(1 - \beta'_p p^{-s}))^{-1}.$$

Then

$$\sum_{n=1}^{\infty} \frac{a_n b_n}{n^s} = \prod_p (1 - \alpha_p\beta_p\alpha'_p\beta'_p p^{-2s})((1 - \alpha_p\alpha'_p p^{-s})(1 - \alpha_p\beta'_p p^{-s})(1 - \beta_p\alpha'_p p^{-s})(1 - \beta_p\beta'_p p^{-s}))^{-1}.$$

The first step in the proof is the following.

Lemma 3. *For all s, one has*

$$\zeta_D(2s - 4\ell)D(\theta_\psi, \theta_\psi, s) = L(\mathit{Sym}^2\theta_\psi, s)L(\chi_D, s - 2\ell),$$

where $\zeta_D(s)$ is the usual Riemann zeta function with the Euler factors at $p|D$ removed and $L(\chi_D, s)$ is the Dirichlet L-function attached to χ_D.

Proof. The idea is to compare the Euler factors at each prime on each side for $\Re(s)$ large enough, using Shimura's lemma.

For p split or inert, the Euler factor on the left simplifies to

$$(1-p^{4\ell-2s})^{-1}(1-p^{4\ell-2s})((1-\alpha_p^2p^{-s})(1-\alpha_p\beta_pp^{-s})(1-\beta_p^2p^{-s}))^{-1}(1-\chi_D(p)p^{2\ell-s})^{-1},$$

while the one on the right is

$$((1-\alpha_p^2p^{-s})(1-\alpha_p\beta_pp^{-s})(1-\beta_p^2p^{-s}))^{-1}(1-\chi_D(p)p^{2\ell-s})^{-1}.$$

If p ramifies, $\beta_p=0$ and $\chi_D(p)=0$. Then the Euler factor on the left is

$$(1-p^{2\ell-s})^{-1},$$

which is also equal to the one on the right. □

The last step is to relate $L(\mathrm{Sym}^2\theta_\psi,s)$ to $L(\psi^2,s)$.

Lemma 4. *For all s, one has*

$$L(\mathit{Sym}^2\theta_\psi,s)=L(\psi^2,s)\zeta_D(s-2\ell).$$

Proof. Again, it suffices to compare the euler factors on both sides for $\Re(s)$ large enough.

If p is inert, the Euler factor on the left is

$$((1-p^{2\ell-s})(1+p^{2\ell-s})(1-p^{2\ell-s}))^{-1},$$

while the one on the right is

$$(1-\psi^2((p))p^{-2s})^{-1}(1-p^{2\ell-s})^{-1}=(1-p^{4\ell-2s})^{-1}(1-p^{2\ell-s})^{-1}.$$

If p splits as $p\mathcal{O}_K=\mathfrak{p}\bar{\mathfrak{p}}$, the Euler factor on the left is

$$(1-\psi^2(\mathfrak{p})p^{-s})(1-\psi(\mathfrak{p})\psi(\bar{\mathfrak{p}})p^{-s})(1-\psi^2(\bar{\mathfrak{p}})p^{-s}))^{-1}=((1-\psi^2(\mathfrak{p})p^{-s})(1-\psi^2(\bar{\mathfrak{p}})p^{-s}))^{-1}(1-p^{2\ell-s})^{-1},$$

which is clearly equal to the one on the right.

The case p ramified is similar. □

Putting those two lemmas together gives

$$\zeta_D(2s-4\ell)D(\theta_\psi,\theta_\psi,s)=L(\chi_D,s-2\ell)\zeta_D(s-2\ell)L(\psi^2,s).$$

By taking residues on both sides of this equation at $s=2\ell+1$ and using the fact that $L(\psi^2,s)$ is analytic at $2\ell+1$,

$$\mathrm{Res}_{s=2\ell+1}\zeta_D(s-2\ell)=\prod_{p|D}(1-p^{-1})\mathrm{Res}_{s=1}\zeta(s)=\prod_{p|D}(1-p^{-1})$$

and

$$\zeta_D(2)=\prod_{p|D}(1-p^{-2})\zeta(2),$$

we get

$$\langle\theta_\psi,\theta_\psi\rangle=\zeta(2)^{-1}\frac{\Gamma(2\ell+1)}{(4\pi)^{2\ell+1}}L(\chi_D,1)\prod_{p|D}(1+p^{-1})^{-1}L(\psi^2,2\ell+1).$$

Using the Dirichlet class number formula for $L(\chi_D,1)$ gives Formula 2.

5 Special values of Hecke L-functions and Eisenstein series

In this section, we first relate $L(\psi^2,s)$ to non-holomorphic Eisenstein series. Then we use this relation to express the special value of $L(\psi^2,s)$ at $2\ell+1$ in terms of derivatives of E_2 evaluated at CM points when $\ell>0$. The case $\ell=0$ is different and must be treated separately.

Throughout this section, fix a Hecke character ψ of K of infinity type 2ℓ.

5.1 Hecke L-functions and non-holomorphic Eisenstein series

Recall that is f is a $|_k\gamma$-invariant function for all γ in $\mathrm{SL}_2(\mathbb{Z})$, then one can define a weight k homogeneous function F on the space of (positively) oriented lattices in $\mathbb{C}$ as

$$F(\omega_1\mathbb{Z}+\omega_2\mathbb{Z}) = \omega_2^{-k} f(\omega_1/\omega_2).$$

Recall that an oriented lattice is a lattice $\mathfrak{a}$ equipped with a $\mathbb{Z}$-basis $[\omega_1, \omega_2]$, where the order of the basis elements is important. If $\Im(\omega_1/\omega_2) > 0$, $\mathfrak{a}$ is called positively oriented. If the $\mathbb{Z}$-basis $[\omega_1, \omega_2]$ is not positively oriented, the basis $[\omega_2, \omega_1]$ is, so that any lattice $\mathfrak{a}$ can be positively oriented. The point $\omega_1/\omega_2 \in \mathcal{H}$ attached to a positively oriented basis of $\mathfrak{a}$ will sometimes be denoted $\tau_{\mathfrak{a}}$. Note that we do not make any holomorphy assumptions on f.

Recall that he non-holomorphic Eisenstein series $G_k(z,s)$ of weight k is defined as

$$G_k(z,s) = \Im(z)^s \sum_{m,n} (mz+n)^{-k}|mz+n|^{-2s},$$

where the sum runs over all integers m and n not both 0. If $\mathfrak{a}$ is any fractional $\mathcal{O}_K$-ideal with oriented basis $[\omega_1, \omega_2]$, define

$$G_k(\mathfrak{a}, s) = \omega_2^{-k}\left(\frac{\sqrt{|D|}N(\mathfrak{a})}{2}\right)^{-s} G_k(\omega_1/\omega_2, s),$$

where D is the discriminant of K. To see that this definition makes sense, first note that

$$\Im(\omega_1/\omega_2) = |\omega_2|^{-2}\left(\frac{\sqrt{|D|}N(\mathfrak{a})}{2}\right).$$

Then

$$G_k(\mathfrak{a}, s) = \sum_{m,n} (m\omega_1+n\omega_2)^{-k}|m\omega_1+n\omega_2|^{-2s},$$

so that $G_k(\mathfrak{a}, 0)$ is the usual weight k Eisenstein series on lattices for $k > 2$. Moreover,

$$G_k(\mu\mathfrak{a}, s) = \mu^{-k}|\mu|^{-2s}G_k(\mathfrak{a}, s)$$

for any $\mu \in K^\times$.

Consider now the following partial Hecke L-function

$$L^{(2\ell)}(\mathfrak{a}, s) = \sum_{\lambda\in\mathfrak{a}-0} \frac{\overline{\lambda}^{2\ell}}{|\lambda|^{2s}}.$$

The first basic relation between Eisenstein series and Hecke L-functions is based on the following

Proposition 1. *Let ψ be a Hecke character of infinity type 2ℓ as above. Then*

$$L(\psi, s) = \frac{1}{w_K}\sum_{[\mathfrak{a}]\in Cl_K} \frac{\psi(\mathfrak{a})}{N(\mathfrak{a})^{2\ell-s}} L^{(2\ell)}(\mathfrak{a}, s),$$

where the sum runs over (any choice of) representatives of the ideal class group of K.

Proof. The fact that the sum does not depend on the choice of representatives of Cl_K follows from the fact that

$$L^{(2\ell)}(\mu\mathfrak{a}, s) = \overline{\mu}^{-2\ell}|\mu|^{-2s}L^{(2\ell)}(\mathfrak{a}, s).$$

To prove formula, first write

$$L(\psi, s) = \sum_{[\mathfrak{a}]\in \mathrm{Cl}_K}\sum_{\mathfrak{c}\in[\mathfrak{a}]} \frac{\psi(\mathfrak{c})}{N(\mathfrak{c})^s},$$

where the inner sum runs over the integral ideals $\mathfrak{c}$ in the class of $\mathfrak{a}$. Now fix $\mathfrak{b} \in [\mathfrak{a}]^{-1}$ such that $1 \in \mathfrak{b}$. Then $\mathfrak{c} \in [\mathfrak{a}]$ with $\mathfrak{c} \subseteq \mathcal{O}_K$ if and only if $\mathfrak{c}\mathfrak{b} = \lambda\mathcal{O}_K$ with $\lambda \in \mathfrak{b}$. Note that λ is unique up to an element of $\mathcal{O}_K^\times$ and $N(\mathfrak{c}) = N(\lambda)N(\mathfrak{b})^{-1}$. It follows that

$$\sum_{\mathfrak{c}\in[\mathfrak{a}]} \frac{\psi(\mathfrak{c})}{N(\mathfrak{c})^s} = \frac{1}{w_K}\frac{N(\mathfrak{b})^s}{\psi(\mathfrak{b})}\sum_{\lambda\in\mathfrak{b}}\frac{\lambda^{2\ell}}{|\lambda|^{2s}}.$$

Since $\mathfrak{a}\bar{\mathfrak{a}} = N(\mathfrak{a})\mathcal{O}_K$, one can take $\mathfrak{b} = \bar{\mathfrak{a}}N(\mathfrak{a})^{-1}$ (which contains 1) and then a short computation shows that the previous formula becomes

$$\sum_{\mathfrak{c}\in[\mathfrak{a}]} \frac{\psi(\mathfrak{c})}{N(\mathfrak{c})^s} = \frac{1}{w_K}\frac{N(\mathfrak{b})^s}{\psi(\mathfrak{b})}\sum_{\lambda\in\mathfrak{b}}\frac{\lambda^{2\ell}}{|\lambda|^{2s}} = \frac{1}{w_K}\frac{\psi(\mathfrak{a})}{N(\mathfrak{a})^{2\ell-s}}\sum_{\lambda\in\mathfrak{a}}\frac{\bar{\lambda}^{2\ell}}{|\lambda|^{2s}} = \frac{1}{w_K}\frac{\psi(\mathfrak{a})}{N(\mathfrak{a})^{2\ell-s}}L^{(2\ell)}(\mathfrak{a},s).$$

□

Since

$$L^{(2\ell)}(\mathfrak{a},s) = G_{2\ell}(\mathfrak{a}, s-2\ell),$$

we obtain

Corollary 1. *Let ψ be a Hecke character of infinity type 2ℓ as above. Then*

$$L(\psi,s) = \frac{1}{w_K}\sum_{[\mathfrak{a}]\in Cl_K}\frac{\psi(\mathfrak{a})}{N(\mathfrak{a})^{2\ell-s}}G_{2\ell}(\mathfrak{a},s-2\ell) = \frac{1}{w_K}\left(\frac{2}{\sqrt{|D|}}\right)^{s-2\ell}\sum_{[\mathfrak{a}]=[[\omega_1,\omega_2]]}\frac{\psi(\mathfrak{a})}{\omega_2^{2\ell}}G_{2\ell}(\omega_1/\omega_2, s-2\ell),$$

where the first sum runs over (any choice of) representatives of the ideal class group of K and in the second one, $[\omega_1,\omega_2]$ is a positively oriented basis of $\mathfrak{a}$.

5.2 The case $\ell = 0$: kronecker limit formula

When $\ell = 0$, Corollary 1 applied to ψ^2 (of infinity type 4ℓ) gives

$$L(\psi^2,s) = \frac{1}{w_K}\left(\frac{2}{\sqrt{|D|}}\right)^{s}\sum_{[\mathfrak{a}]=[[\omega_1,\omega_2]]}\psi^2(\mathfrak{a})G_0(\omega_1/\omega_2,s). \tag{3}$$

Recall that we are interested in the value of $L(\psi^2,s)$ at $s = 2\ell+1 = 1$. Since the non-holomorphic Eisenstein series of weight 0 has a pole at $s = 1$, we need to look at the next term in the Taylor expansion around $s = 1$.

Theorem 2 (Kronecker Limit Formula)**.** *Define the eta-function as*

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q = e^{2\pi i z}$ and let

$$G_0(z,s) = \Im(z)^s\sum_{m,n}|mz+n|^{-2s}$$

be the non-holomorphic Eisenstein series of weight 0. Then

$$G_0(z,s) = \pi\left(\frac{1}{s-1} + C(z) + O(s-1)\right),$$

where

$$C(z) = 2\gamma - \log 4 - 2\log(\Im(z)^{1/2}|\eta(z)|^2)$$

($\gamma =$Euler's constant).

Proof. See [Cohe, Thm. 10.4.6]. Note that our definition of $G_0(z,s)$ differs from Cohen's by a factor of 1/2. □

When ψ^2 is the trivial character, formula 3 is nothing else but the well-known decomposition of the Dedekind zeta function of K into a sum of Epstein zeta functions. Comparing the residues gives the class number formula for imaginary quadratic fields:

$$\mathrm{Res}_{s=1}\zeta_K(s) = L(\chi_D, 1) = \frac{2\pi h_K}{w_K\sqrt{|D|}}$$

and comparing the constant terms gives the Chowla-Selberg formula.

When ψ^2 is not trivial, the function $L(\psi^2, s)$ is analytic at $s = 1$[4] and has value

$$L(\psi^2, 1) = -\frac{4\pi}{w_K\sqrt{|D|}}\sum_{[\mathfrak{a}]\in \mathrm{Cl}_K}\psi^2(\mathfrak{a})\log(\Im(\tau_{\mathfrak{a}})^{1/2}|\eta(\tau_{\mathfrak{a}})|^2).$$

Putting this in formula 2, we get

$$\langle\theta_\psi, \theta_\psi\rangle = -V_D^{-1}\frac{4h_K}{w_K^2}\sum_{[\mathfrak{a}]\in \mathrm{Cl}_K}\psi^2(\mathfrak{a})\log(\Im(\tau_{\mathfrak{a}})^{1/2}|\eta(\tau_{\mathfrak{a}})|^2). \tag{4}$$

Note that factoring out the volume helps understanding the algebraic properties of the quantity on the right. This formula tells us that normalizing the Petersson inner product by dividing by the volume, as we did, artificially introduces transcendental numbers in the Petersson norm. We will come back to this point after we treat the case $\ell > 0$.

5.3 The case $\ell > 0$: derivative of almost holomorphic Eisenstein series

Define as usual the following differential operators on real analytic functions on the upper half-plane

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

For any integer k and congruence subgroup Γ, let $\widehat{M}_k(\Gamma)$ be the space of *almost holomorphic modular forms* of weight k and level Γ. An element of this space is a $|_k\gamma$-invariant function for all $\gamma \in \Gamma$, but instead of being holomorphic on $\mathcal{H}$, it is a polynomial in $1/\Im(z)$ with holomorphic coefficients satisfying some growth condition at infinity. The simplest example (and the only one we need) of almost holomorphic modular form is $E_2 \in \widehat{M}_2(\mathrm{SL}_2(\mathbb{Z}))$.

If $f \in \widehat{M}_k(\Gamma)$ is an almost holomorphic modular form, the operator ∂_k defined as

$$\partial_k f = \frac{1}{2\pi i}\frac{\partial f}{\partial z} - \frac{k}{4\pi\Im(z)}f$$

takes f to an element of $\widehat{M}_{k+2}(\Gamma)$. To simplify the notation, define

$$\partial_k^n = \partial_{k+2n-2}\circ\cdots\circ\partial_{k+2}\circ\partial_k.$$

The following lemma is the starting point of our investigation.

Lemma 5. *Let $G_k(z,s)$ be the non-holomorphic Eisenstein series of weight k defined in section 2.2. Then*

$$\partial_k^n G_k(z,s) = (-4\pi)^{-n}\frac{\Gamma(k+s+n)}{\Gamma(s+k)}G_{k+2n}(z, s-n)$$

[4] Note again the importance of the fact that the residue of the non-holomorphic Eisenstein series at $s = 1$ does not depend on z.

Proof. This is [Shi1, Formula 9.12] with $N = 1$ and $p = q = 0$. Note also that our ∂_k is Shimura's D_k (we follows Zagier's notation). $\square$

This leads to the following

Corollary 2. *Let ψ be a Hecke character of infinity type $2\ell > 2$ as above and let m be an integer such that $\ell + 1 \leq m \leq 2\ell$. Then*

$$L(\psi, m) = \frac{1}{w_K}(-4\pi)^{2\ell-m}\frac{\Gamma(2m-2\ell)}{\Gamma(m)}\left(\frac{\sqrt{|D|}}{2}\right)^{2\ell-m}\sum_{[\mathfrak{a}]=[[\omega_1,\omega_2]]}\frac{\psi(\mathfrak{a})}{\omega_2^{2\ell}}\partial^{2\ell-m}G_{2m-2\ell}(\omega_1/\omega_2, 0),$$

where as usual the sum runs over positively oriented basis of representatives of the ideal class group of K.

Proof. Using the Lemma above with $n = 2\ell - m \geq 0$ and $k = 2m - 2\ell \geq 2$, we see that

$$G_{2\ell}(z, s+m-2\ell) = (-4\pi)^{2\ell-m}\frac{\Gamma(s+2m-2\ell)}{\Gamma(s+m)}\partial^{2\ell-m}G_{2m-2\ell}(z,s).$$

Putting this in the formula of Corollary 1 (evaluated at $s+m$), we see that

$$L(\psi, s+m) = \frac{1}{w_K}(-4\pi)^{2\ell-m}\frac{\Gamma(s+2m-2\ell)}{\Gamma(s+m)}\left(\frac{\sqrt{|D|}}{2}\right)^{-(s+m-2\ell)}\sum_{[\mathfrak{a}]=[[\omega_1,\omega_2]]}\frac{\psi(\mathfrak{a})}{\omega_2^{2\ell}}\partial^{2\ell-m}G_{2m-2\ell}(z,s),$$

$\square$

Using the fact that

$$2^{-1}(2\pi i)^{-k}\Gamma(k)G_k(z,0) = E_k(z),$$

for all $k \geq 2$, where

$$E_2(z) = \frac{1}{8\pi\Im(z)} - \frac{1}{24} + \sum_{n=1}^{\infty}\sigma_1(n)q^n \tag{5}$$

and

$$E_k = -\frac{B_k}{2k} + \sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n \tag{6}$$

is the usual holomorphic Eisenstein series for $k \geq 4$ (see [Shi1, Sec 9.2]), one sees that the previous Corollary relates certain special values of the Hecke L-function attached to ψ to the derivatives of the usual Eisenstein series. Indeed, the formula becomes

$$L(\psi, m) = (-1)^{\ell}\sqrt{|D|}^{2\ell-m}\frac{(2\pi)^m}{\Gamma(m)}\sum_{[\mathfrak{a}]\in \mathrm{Cl}_K}\psi(\mathfrak{a})\partial^{2\ell-m}E_{2m-2\ell}(\mathfrak{a}). \tag{7}$$

Applying this formula to ψ^2 with $m = 2\ell+1$, we get

$$L(\psi^2, 2\ell+1) = \sqrt{|D|}^{2\ell-1}\frac{(2\pi)^{2\ell+1}}{\Gamma(2\ell+1)}\sum_{[\mathfrak{a}]=[[\omega_1,\omega_2]]}\frac{\psi^2(\mathfrak{a})}{\omega_2^{4\ell}}\partial^{2\ell-1}E_2(\omega_1/\omega_2).$$

Using this value of $L(\psi^2, 2\ell+1)$ in formula 2 and simplifying finally gives

$$\langle\theta_\psi, \theta_\psi\rangle = V_D^{-1}(|D|/4)^{\ell}\frac{4h_K}{w_K^2}\sum_{[\mathfrak{a}]=[[\omega_1,\omega_2]]}\frac{\psi^2(\mathfrak{a})}{\omega_2^{4\ell}}\partial^{2\ell-1}E_2(\omega_1/\omega_2). \tag{8}$$

Note that this can also be written in homogeneous form as

$$\langle\theta_\psi, \theta_\psi\rangle = V_D^{-1}(|D|/4)^{\ell}\frac{4h_K}{w_K^2}\sum_{[\mathfrak{a}]\in \mathrm{Cl}_K}\psi^2(\mathfrak{a})\partial^{2\ell-1}E_2(\mathfrak{a}).$$

Corollary 3. *For $\ell > 0$,*

$$V_D \langle \theta_\psi, \theta_\psi \rangle = \alpha \Omega_K^{4\ell},$$

where α is an algebraic number and Ω_K is the Chowla-Selberg period attached to K and depends only on K.

Proof. From the Corollary of Proposition 27 in [Zag], it follows that

$$\partial_2^{2\ell-1} E_2(\tau)$$

is an algebraic multiple of $\Omega_K^{2+2(2\ell-1)} = \Omega_K^{4\ell}$, whenever $\tau \in K \cap \mathcal{H}$ is a CM point. The Corollary follows from the fact that the values of the Hecke characters ψ_ℓ and all the other quantities in formula 8 are algebraic. $\square$

5.4 The case $\ell = 0$ revisited

Strictly speaking, formula 8 does not make sense for $\ell = 0$. However, it is natural to define ∂_2^{-1} as a weight 0 "modular form" f such that

$$\partial_0 f(z) = E_2(z).$$

We claim that

$$\partial_0 \log(\Im(z)^{1/2} |\eta(z)|^2) = -E_2(z),$$

where

$$\partial_0 = \frac{1}{2\pi i} \frac{\partial}{\partial z}.$$

This follows from the well known fact (see [Zag, Prop. 7]) that

$$\frac{1}{2\pi i} \frac{\partial}{\partial z} \log \Delta(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

and the identity

$$\Delta(z) = \eta(z)^{24}.$$

Indeed, since

$$\log |\Delta(z)| = \Re(\log \Delta(z)),$$

this implies

$$\frac{\partial}{\partial z} \log |\Delta(z)| = \frac{1}{2} \frac{\partial}{\partial z} \log \Delta(z)$$

(recall that $\frac{\partial \bar{f}}{\partial z} = \overline{\frac{\partial f}{\partial \bar{z}}} = 0$ if $f(z)$ is holomorphic).

The equality

$$\partial_0 \log(\Im(z)^{1/2} |\eta(z)|^2) = -E_2(z)$$

implies that formula 8 also makes sense for $\ell = 0$ and gives back exactly formula 4. Note also that $\log(\Im(z)^{1/2} |\eta(z)|^2)$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant, as desired. However, I don't think $\log(\Im(z)^{1/2} |\eta(z)|^2)$ is almost holomorphic.

6 Theta functions attached to ideals in imaginary quadratic fields

In this section we define theta series attached to ideals in imaginary quadratic fields and certain spherical polynomials and see how these theta functions are relate to the theta functions θ_ψ.

Throughout this section, fix an integer $\ell \geq 0$.

Let $\mathfrak{a}$ be a fractional ideal of K and define the theta function attached to $\mathfrak{a}$ (and ℓ) as

$$\theta_{\mathfrak{a}}^{(2\ell)} = \theta_{\mathfrak{a}}(z) = \sum_{\lambda \in \mathfrak{a}} \lambda^{2\ell} q^{N(\lambda)/N(\mathfrak{a})},$$

where we define $0^0 = 1$ in case $\ell = 0$. Then we have the following

Proposition 2. *The function $\theta_{\mathfrak{a}}$ is a modular form of weight $2\ell+1$, level $\Gamma_0(|D|)$ and Nebentypus χ_D. Moreover, it is a cusp form if $\ell > 0$.*

Proof. This is well-know, but tedious to prove! A good reference for that is [Iwan, Thm. 10.9]. The point is that the function $\lambda \mapsto \lambda^{2\ell}$ is a spherical polynomial for the binary quadratic form $N(\lambda)/N(\mathfrak{a})$. $\square$

If ψ is a Hecke character of infinity type 2ℓ, the theta function θ_ψ decomposes as follows:

$$\theta_\psi = \frac{1}{w_K} \sum_{[\mathfrak{a}] \in \mathrm{Cl}_K} \psi(\mathfrak{a})^{-1} \theta_{\mathfrak{a}}, \tag{9}$$

where the sum runs over representatives of the class group. Note that $\theta_{\mathfrak{a}}$ depends on the choice of $\mathfrak{a}$ in $[\mathfrak{a}]$, since

$$\theta_{\mu\mathfrak{a}} = \mu^{2\ell}\theta_{\mathfrak{a}}$$

for any $\mu \in K^\times$, but the sum is still independent of this choice. To prove formula 9, one uses the same trick as in the proof of Proposition 1.

Note that the L-function attached to $\theta_{\mathfrak{a}}^{(2\ell)}$ is the partial Hecke L-function $L^{(2\ell)}(\bar{\mathfrak{a}}, s)$ introduced before.

Our next goal is to write $\theta_{\mathfrak{a}}$ in terms of the θ_ψ. For this, the following Lemma is useful.

Lemma 6. *Fix an integer $\ell \geq 0$ and let $\mathfrak{c}$ be a fractional ideal of K. Then*

$$\sum_\psi \psi(\mathfrak{c}) = \begin{cases} 0 & \text{if } [\mathfrak{c}] \neq [\mathcal{O}_K] \\ \lambda^{2\ell} h_K & \text{if } \mathfrak{c} = \lambda\mathcal{O}_K \end{cases},$$

where the sum runs over all Hecke characters of K of infinity type 2ℓ.

Proof. Fix a Hecke character χ of infinity type 2ℓ. Then

$$\sum_\psi \psi\chi^{-1}(\mathfrak{c}) = \begin{cases} 0 & \text{if } [\mathfrak{c}] \neq [\mathcal{O}_K] \\ h_k & \text{if } \mathfrak{c} = \lambda\mathcal{O}_K \end{cases},$$

by the orthogonality relations of finite abelian group characters, since $\psi\chi^{-1}$ is a character of Cl_K. The claim follows by multiplying both sides by $\chi(\mathfrak{c})$ since $\chi(\lambda\mathcal{O}_K) = \lambda^{2\ell}$. $\square$

This leads to the following

Proposition 3. *With $\theta_{\mathfrak{a}}$ defined as above,*

$$\theta_{\mathfrak{a}} = \frac{w_K}{h_K} \sum_\psi \psi(\mathfrak{a})\theta_\psi,$$

where the sum runs over all Hecke characters of infinity type 2ℓ.

Proof. This follows formally from the previous Lemma and the expression for θ_ψ in terms of the $\theta_{\mathfrak{a}}$. $\square$

Using the orthogonality of the θ_ψ under the Petersson inner product when $\ell > 0$, one can compute $\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle$ in terms of the Petersson norm of the θ_ψ. When $\ell = 0$, the θ_ψ are not always cusp forms and we have not found a way to compute (or even define) the Petersson norm of all the θ_ψ. However, we still have the following

Proposition 4. *Let $\ell > 0$ and let $\theta_{\mathfrak{a}}$ and $\theta_{\mathfrak{b}}$ be defined as above. Then*

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle = C_K N(\mathfrak{b})^{2\ell} \sum_{\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K} \lambda_{\mathfrak{c}}^{2\ell} \partial^{2\ell-1} E_2(\mathfrak{c}),$$

where the sum runs over all ideal classes $[\mathfrak{c}] \in \mathrm{Cl}_K$ such that $\mathfrak{c}^2\mathfrak{a}\mathfrak{b}^{-1} = \lambda_{\mathfrak{c}}\mathcal{O}_K$ for some $\lambda_{\mathfrak{c}} \in K$ and

$$C_K = 4V_D^{-1}(|D|/4)^\ell.$$

In particular, $\theta_{\mathfrak{a}}$ and $\theta_{\mathfrak{b}}$ are orthogonal if $\mathfrak{a}$ and $\mathfrak{b}$ are not in the same genus.

Proof. First, we compute

$$\begin{aligned}
\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle &= \frac{w_K^2}{h_K^2} \sum_{\psi,\chi} \psi(\mathfrak{a})\overline{\chi}(\mathfrak{b}) \langle \theta_\psi, \theta_\chi \rangle \\
&= \frac{w_K^2}{h_K^2} \sum_{\psi} \psi(\mathfrak{a})\overline{\psi}(\mathfrak{b}) \langle \theta_\psi, \theta_\psi \rangle \\
&= \frac{w_K^2}{h_K^2} \sum_{\psi} \psi(\mathfrak{a}) N(\mathfrak{b})^{2\ell} \psi^{-1}(\mathfrak{b}) \langle \theta_\psi, \theta_\psi \rangle \\
&= \frac{C_K}{h_K} N(\mathfrak{b})^{2\ell} \sum_{\psi,[\mathfrak{c}]} \psi(\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2) \partial^{2\ell-1} E_2(\mathfrak{c}),
\end{aligned}$$

where the first sum is a double sum over all Hecke characters of infinity type 2ℓ and we used the orthogonality of the newforms θ_ψ in the second equality.

Summing the last sum over ψ first and using Lemma 6, we see that

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle = 0$$

if for all $[\mathfrak{c}] \in \mathrm{Cl}_K$, $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2$ is not principal, i.e. $\mathfrak{a}$ and $\mathfrak{b}$ are not in the same genus. Otherwise, if $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K$ for some $\lambda_{\mathfrak{c}} \in K$, then

$$\sum_{\psi} \psi(\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2) = \lambda_{\mathfrak{c}}^{2\ell} h_K$$

and the last line of the above computation becomes

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle = C_K N(\mathfrak{b})^{2\ell} \sum_{\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}} \lambda_{\mathfrak{c}}^{2\ell} \partial^{2\ell-1} E_2(\mathfrak{c}).$$

□

The folowing corollary can be seen as a way to "factor" inner products of theta series.

Corollary 4. *Fix $\ell > 0$. Then*

$$\langle \theta_{\mathfrak{a}\mathfrak{c}}, \theta_{\mathfrak{c}\mathfrak{b}} \rangle = N(\mathfrak{b}\mathfrak{c})^{2\ell} \langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle.$$

A consequence of this corollary is that the Petersson norm of the theta series attached to a quadratic ideal of K depends only on its norm and the parameter ℓ.

Corollary 5. *Fix $\ell > 0$. Then*

$$V_D \langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle = \alpha \Omega_K^{4\ell},$$

where α is some algebraic number and Ω_K is the Chowla-Selberg period attached to K.

7 An efficient algorithm to compute the Petersson inner product of binary theta series

Formula 4 can be used to numerically evaluate the Petersson inner product of theta series attached to imaginary quadratic fields in an efficient way (1000 decimals in a few seconds!). To implement this formula, one should be able to find the derivatives of E_2, to evaluate them at lattices and find, for fixed ideals $\mathfrak{a}$ and $\mathfrak{b}$, all ideal classes $\mathfrak{c}$ such that $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K$. We talk about those problems in the next section and then we give a pseudo-algorithm to solve our initial problem of computing $\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle$.[5]

[5] A PARI/GP implementation of this algorithm is available on https://github.com/NicolasSimard/ENT.

7.1 Towards the algorithm

7.1.1 Derivatives of almost holomorphic modular forms

First, recall that the ring of almost holomorphic of level 1 is isomorphic as a $\mathbb{C}$-algebra to

$$\mathbb{C}[E_2, E_4, E_6].$$

It follows that in order to compute $\partial^n E_2$, it suffices to know $\partial E_2, \partial E_4$ and ∂E_6. For this, we have the following

Proposition 5. *Let E_2, E_4 and E_6 be the almost holomorphic modular forms defined by equation 5 and 6 and let*

$$\partial_k = \frac{1}{2\pi i}\frac{\partial}{\partial z} - \frac{k}{4\pi\Im(z)}$$

. Then

$$\partial E_2 = \frac{5}{6}E_4 - 2E_2^2, \qquad \partial E_4 = \frac{7}{10}E_6 - 8E_2E_4, \qquad \partial E_6 = \frac{400}{7}E_4^2 - 12E_2E_6.$$

Proof. This is in [Shi1, Sec 9.2], plus the fact that

$$120E_4^2 = E_8.$$

□

Add recursive formula to express E_K as a polynomial in E_4 and E_6.

7.1.2 Evaluating Eisenstein series at CM points

By the above, the problem reduces to evaluating E_2, E_4 and E_6 at lattices. Generally, the Fourier expansions of these Eisenstein series converge very quickly. However, we have some freedom in choosing the lattice at which we evaluate them. As the following example shows, one should really take advantage of that.

Take $K = \mathbb{Q}(\sqrt{-26})$. Then $D = -104$, and Cl_K is cyclic of order 6, generated by any prime above 5. In fact, since

$$N(109 - 12\sqrt{-26}) = 5^6,$$
$$\mathfrak{p}_5^6 = \lambda\mathcal{O}_K,$$

where $\lambda = 109 + 12\sqrt{-26}$ and $\mathfrak{p}_5$ is one of the two primes above 5 (chosen so that the equation holds). Using PARI/GP, we find $\mathbb{Z}$-basis for $\mathfrak{p}_5^4$ and $\mathfrak{p}_5^{-2}$:

$$\mathfrak{p}_5^4 = [625, 43 + \sqrt{-26}], \qquad \mathfrak{p}_5^{-2} = [1, (7 + \sqrt{-26})/25].$$

From the equality $\mathfrak{p}_5^4 = \lambda\mathfrak{p}_5^{-2}$, we deduce that

$$\partial^n E_2(\mathfrak{p}_5^4) = \lambda^{-(2+2n)}\partial^n E_2(\mathfrak{p}_5^{-2})$$

and using the above $\mathbb{Z}$-basis, we have

$$625^{-(2+2n)}\partial^n E_2((43 + \sqrt{-26})/625) = \lambda^{-(2+2n)}\partial^n E_2((7 + \sqrt{-26})/25).$$

For $n = 1$ and working with 500 digits of precision, the left-hand side of the equation takes about 30 times more time to evaluate than the right-hand side (which takes around 1sec to evaluate on my desktop computer)! This proves that the running time of the algorithm depends in a crucial way on the choice of class representatives in a given ideal class.

The reason for the large difference in computation time in the above example is of course that the imaginary part of $(43 + \sqrt{-26})/625$ is smaller than the imaginary part of $(7 + \sqrt{-26})/25$. Using the correspondence between ideal classes and equivalence classes of positive definite integral binary quadratic forms, we see that the ideal corresponding to the quadratic form $[a, b, c]$ has $\mathbb{Z}$-basis

$$[a, (-b + \sqrt{D})/2]$$

and the imaginary part of the corresponding point in the upper-half plane is

$$\frac{\sqrt{|D|}}{2a}.$$

For fixed D, our problem is then to minimize a. It turns out that in a given class, the quadratic form with minimal a is the unique reduced quadratic form in that class. Moreover, for any reduced form, one has the following upper bound for a

$$a \leq \sqrt{|D|}/3,$$

which leads to a lower bound on the imaginary part of to corresponding point in the upper-half plane. This proves the following

Proposition 6. *Let K be an imaginary quadratic field and let $\mathcal{C}$ be an ideal class in Cl_K. Then there exists an explicit positively oriented ideal $[\omega_1, \omega_2]$ such that*

$$\frac{3}{2} \leq \Im(\omega_1/\omega_2).$$

This discussion leads to the following simple algorithm to evaluate $\partial^n E_2$ at an ideal in an imaginary quadratic field: find the class to which this ideal belongs and use the CM point corresponding to the reduced form in that class to evaluate $\partial^n E_2$. The lower bound above is a kind of guarantee on the speed of this algorithm.

7.1.3 Ambiguous classes

The last problem in computing $\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle$ is to find all ideal classes $\mathfrak{c}$ such that $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K$ (and find $\lambda_{\mathfrak{c}}$ too). Given a set of generators for Cl_K, it is easy to determine if the class $\mathfrak{a}\mathfrak{b}^{-1}$ is a square in Cl_K and to find a class $\mathfrak{c}_0$ such that

$$\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}_0^2 = \lambda\mathcal{O}_K.$$

Indeed, write $\mathfrak{a}\mathfrak{b}^{-1}$ in term of those generators and check that only even powers of the generators occur. The following proposition completes the task.

Proposition 7. *Let $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}_0^2 = \lambda\mathcal{O}_K$ for some ideal $\mathfrak{c}_0$. Let*

$$\{\mathfrak{a}_1, \ldots, \mathfrak{a}_g\}$$

be representatives of $Cl_K[2]$ and define α_i for $1 \leq i \leq g$ as

$$\mathfrak{a}_i^2 = \alpha_i\mathcal{O}_K.$$

Then

$$\sum_{\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2=\lambda_{\mathfrak{c}}\mathcal{O}_K} \lambda_{\mathfrak{c}}^{2\ell}\partial^{2\ell-1}E_2(\mathfrak{c}) = \lambda^{2\ell}\sum_{i=1}^{g}\alpha_i^{2\ell}\partial^{2\ell-1}E_2(\mathfrak{c}_0\mathfrak{a}_i).$$

Proof. It suffices to note that any ideal $\mathfrak{c}$ such that

$$\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K$$

is equivalent to $\mathfrak{c}_0\mathfrak{a}_i$ for some i and that

$$\mathfrak{a}\mathfrak{b}^{-1}(\mathfrak{c}_0\mathfrak{a}_i)^2 = \lambda\alpha_i\mathcal{O}_K.$$

□

The 2-torsion classes in Cl_K are also known as the ambiguous classes of K. They are easy to compute using the theory of binary quadratic forms[6] and there are exactly g of them, where g is the number of genera in K (i.e. $g = |Cl_K/Cl_K^2|$).

[6] Indeed, the inverse of the positive definite binary quadratic form $[a, b, c]$ is $[a, -b, c]$ and forcing these forms to be equivalent puts big restrictions on a, b and c.

7.2 A pseudo algorithm

At this point, the problem is purely computational. Our goal is to compute $\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle$, for fixed quadratic field K of discriminant D and varying $\mathfrak{a}, \mathfrak{b}$ and ℓ. To do so, first define an initializing function which takes a fundamental discriminant D as input and returns a list L(D) of length 4 of the form

$$L(D) = [K, \mathcal{R}, \mathcal{A}, M],$$

where

- K is the quadratic field of discriminant D;
- $\mathcal{R}$ is a list of representatives of Cl_K corresponding to reduced quadratic forms;
- $\mathcal{A}$ is a list of representatives of $\mathrm{Cl}_K[2]$ (the $\mathfrak{a}_i$s) together with the α_is such that $\mathfrak{a}_i = \alpha_i \mathcal{O}_K$.
- M is a $3 \times h_K$ matrix where $M[k, d] = E_{2k}(f_d)$ for $1 \le k \le 3$ and $f_d \in \mathcal{R}$.

The most time consuming part when computing this vector is of course to compute M, but this is done as efficiently as possible by our choice of representatives of Cl_K.

Now given L(D), it is a simple exercise to compute

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle.$$

A systematic way of doing this is to follow these steps:

1. Determine if $\mathfrak{a}\mathfrak{b}^{-1}$ is a square. If it is not, return 0. Otherwise, find $\mathfrak{c}_0$ and λ such that
$$\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}_0^2 = \lambda \mathcal{O}_K.$$
2. Express $\partial^{2\ell-1}E_2$ as a polynomial on E_2, E_4 and E_6.
3. Compute $\partial^{2\ell-1}E_2(\mathfrak{c}_0\mathfrak{a}_i)$ for all $\mathfrak{a}_i \in \mathcal{A}$(using M).
4. Compute
$$\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle = C_K N(\mathfrak{b})^{2\ell}\lambda^{2\ell} \sum_{i=1}^{g} \alpha_i^{2\ell} \partial^{2\ell-1}E_2(\mathfrak{c}_0\mathfrak{a}_i).$$

Each step is very quick, given the previously computed data L(D). Note that to evaluate $\partial^{2\ell-1}E_2(\mathfrak{c}_0\mathfrak{a}_i)$ in step 3., one must find $f_d \in \mathcal{R}$ and $\mu \in K^\times$ such that

$$\mathfrak{c}_0\mathfrak{a}_i = \mu f_d$$

and so

$$\partial^{2\ell-1}E_2(\mathfrak{c}_0\mathfrak{a}_i) = \mu^{-4\ell}\partial^{2\ell-1}E_2(f_d),$$

which is easy to compute using M and step 2..

8 Numerical examples

In this section, we give numerical examples of computations using the above formulas when $\ell > 0$. Our goal is to compute the determinant of the matrix

$$M_{\mathcal{R}}(\ell) = (V_D \langle \theta_{\mathfrak{a}_i}^{(2\ell)}, \theta_{\mathfrak{a}_j}^{(2\ell)} \rangle)_{1 \le i,j \le h_K},$$

where $\mathcal{R} = \{\mathfrak{a}_1, \dots, \mathfrak{a}_h\}$ is a set of representatives of the class group of K. Of course, $M_{\mathcal{R}}(\ell)$ depends on $\mathcal{R}$. In fact, if $\mathcal{R}'$ is the set of representatives obtained from $\mathcal{R}$ by changing one of the $\mathfrak{a}_i$ by $\mu\mathfrak{a}_i$ for some $\mu \in K^\times$, one sees immediately that

$$\det M_{\mathcal{R}'}(\ell) = N(\mu)^{2\ell} \det M_{\mathcal{R}}(\ell).$$

Note that by Proposition 4, $M_{\mathcal{R}}(\ell)$ is a block diagonal matrix (after reordering the ideals of $\mathcal{R}$, if necessary).

When $\ell > 0$, the determinant of $M_{\mathcal{R}}$ is explicitly related to the determinant of the diagonal matrix

$$M_K(\ell) = \begin{pmatrix} V_D\langle\theta_{\psi_1},\theta_{\psi_1}\rangle & & & \\ & V_D\langle\theta_{\psi_2},\theta_{\psi_2}\rangle & & \\ & & \ddots & \\ & & & V_D\langle\theta_{\psi_h},\theta_{\psi_h}\rangle \end{pmatrix},$$

where the ψ_i are the Hecke characters of K of infinity type 2ℓ. Note that $M_K(\ell)$ is canonically attached to K and ℓ and that its determinant is a product of special values of Hecke L-functions by Formula 2.

Both matrices have transcendental entries. However, it is possible to explicitly normalize the entries to make them algebraic, as was proved in Corollaries 3 and 5. In the computations that follow, we normalize using the Chowla-Selberg attached to K, defined here as

$$\Omega_K = \frac{1}{\sqrt{4\pi|D|}}\left(\prod_{j=1}^{|D|-1}\Gamma(j/|D|)^{\chi_D(j)}\right)^{w_K/(4h_K)}.$$

8.1 Class number 1

If K has class number 1, there is only one theta series and

$$\theta_{\mathcal{O}_K} = \theta_{\psi_0},$$

where ψ_0 is the only Hecke character of infinity type 2ℓ. In the following table, we find numerically the algebraic number

$$V_D\langle\theta_{\mathcal{O}_K},\theta_{\mathcal{O}_K}\rangle/\Omega_K^{4\ell},$$

for all imaginary quadratic fields of class number one and for $1 \le \ell \le 4$.

		ℓ			
		1	2	3	4
D	-7	$2^2 3$	-2^2	$-2^2 17$	$-2^2 7 \cdot 191$
	-8	-2	$-2^2 5$	$-2^4 23$	$-2^5 181$
	-11	-2^2	$-2^3 5$	$-2^2 139$	$-2^9 5^3$
	-19	$-2^2 3^{-1} 13$	$-2^3 71$	$-2^2 11 \cdot 29^2$	$-2^8 14753$
	-43	$-2^3 3^{-1} 107$	$-2^4 5647$	$-2^2 16876283$	$-2^8 23 \cdot 15431881$
	-67	$-2^2 3^{-1} 7^2 31$	$-2^3 5 \cdot 86629$	$-2^2 3547447667$	$-2^{10} 281 \cdot 3529 \cdot 105607$
	-163	$-2^3 3^{-1} 150473$	$-2^4 11 \cdot 461681471$	$-2^2 127 \cdot 659 \cdot 119633471311$	$-2^8 13^2 53 \cdot 383 \cdot 2729 \cdot 15275296963$

Note that the entries are rational integers (and even integers most of the time).

8.2 Class number 2

If K has class number 2, there are 2 genera and each of them contains a single class. If $\mathfrak{a}$ is a representative of the non-trivial ideal class of K, one sees using formula 4 that

$$\langle\theta_{\mathfrak{a}},\theta_{\mathcal{O}_K}\rangle = \langle\theta_{\mathcal{O}_K},\theta_{\mathfrak{a}}\rangle = 0$$

and

$$\langle\theta_{\mathfrak{a}},\theta_{\mathfrak{a}}\rangle = N(\mathfrak{a})^{2\ell}\langle\theta_{\mathcal{O}_K},\theta_{\mathcal{O}_K}\rangle,$$

so

$$\det M_{\{\mathcal{O}_K,\mathfrak{a}\}}(\ell) = N(\mathfrak{a})^{2\ell}\langle\theta_{\mathcal{O}_K},\theta_{\mathcal{O}_K}\rangle^2.$$

Therefore, it suffices to analyse the numbers $\langle\theta_{\mathcal{O}_K},\theta_{\mathcal{O}_K}\rangle$. In the following table, we find numerically the algebraic number

$$V_D\langle\theta_{\mathcal{O}_K},\theta_{\mathcal{O}_K}\rangle/\Omega_K^{4\ell},$$

for all imaginary quadratic fields of class number two and for $1 \le \ell \le 4$.

		ℓ			
		1	2	3	4
D	-15	-2^2	$-2^2 3\cdot 13$	$-2^2 3\cdot 5\cdot 53$	$-2^2 3^2 5\cdot 11\cdot 73$
	-20	-2^4	$-2^3 37$	$-2^7 5\cdot 43$	$-2^6 5\cdot 10657$
	-24	$-2^2 7$	$-2^3 3\cdot 47$	$-2^5 3\cdot 23\cdot 37$	$-2^6 3^2 7\cdot 3163$
	-35	$-2^2 3^2$	$-2^3 3\cdot 199$	$-2^3 3\cdot 5\cdot 3301$	$-2^8 3^4 5\cdot 7\cdot 229$
	-40	$-2^2 29$	$-2^3 37\cdot 41$	$-2^5 3^2 5\cdot 2143$	$-2^6 5\cdot 11\cdot 304867$
	-51	$-2^2 43$	$-2^3 3\cdot 5\cdot 181$	$-2^3 3\cdot 386489$	$-2^8 3^2 5\cdot 11\cdot 29\cdot 1979$
	-52	$-2^4 17$	$-2^3 6421$	$-2^7 3\cdot 53597$	$-2^6 1613\cdot 181913$
	-88	$-2^2 7\cdot 73$	$-2^3 23\cdot 31\cdot 373$	$-2^5 3^3 47\cdot 109\cdot 1217$	$-2^6 5003\cdot 82114223$
	-91	$-2^7 3^{-1} 19$	$-2^5 139\cdot 157$	$-2^3 71\cdot 79\cdot 24859$	$-2^8 7\cdot 23\cdot 57233807$
	-115	$-2^2 3\cdot 197$	$-2^3 31\cdot 11657$	$-2^3 3^2 5\cdot 17\cdot 31\cdot 65449$	$-2^9 5\cdot 29744878249$
	-123	$-2^4 5\cdot 59$	$-2^6 3\cdot 7\cdot 29\cdot 269$	$-2^3 3\cdot 7\cdot 19\cdot 31\cdot 599\cdot 877$	$-2^8 3^2 5\cdot 23\cdot 2018719939$
	-148	$-2^4 11\cdot 139$	$-2^3 101\cdot 421\cdot 653$	$-2^7 3\cdot 12612115157$	$-2^6 16658933\cdot 180376241$
	-187	$-2^2 7\cdot 547$	$-2^3 20086217$	$-2^3 3^3 23\cdot 533745103$	$-2^{11} 7\cdot 59\cdot 119478576781$
	-232	$-2^2 3^2 9677$	$-2^3 2447\cdot 1773907$	$-2^5 3^3 9718885998641$	$-2^6 43\cdot 1368715394403766639$
	-235	$-2^2 16619$	$-2^3 29\cdot 6766423$	$-2^3 3^2 5\cdot 200329\cdot 1210103$	$-2^9 5\cdot 3617\cdot 1212552488207$
	-267	$-2^2 17\cdot 53\cdot 79$	$-2^3 3\cdot 17\cdot 29\cdot 2069213$	$-2^3 3\cdot 79231\cdot 2668717679$	$-2^8 3^2 199\cdot 4141371112096921$
	-403	$-2^2 3^{-1} 431\cdot 1789$	$-2^3 137\cdot 322181789$	$-2^3 33547\cdot 1222350596561$	$-2^8 783588203\cdot 1859251547159$
	-427	$-2^2 3^{-1} 5\cdot 19\cdot 23\cdot 647$	$-2^3 2437\cdot 48695077$	$-2^3 51449\cdot 913573\cdot 3081919$	$-2^8 5\cdot 7\cdot 272407\cdot 1278942841515113$

Note again that these quantities are rational integers (and integers most of the time).

For the θ_ψ, we see that

$$\langle\theta_{\psi_1},\theta_{\psi_1}\rangle = \langle\theta_{\psi_2},\theta_{\psi_2}\rangle,$$

where ψ_1 and ψ_2 are the two Hecke characters of K of infinity type 2ℓ, since $\psi_1^2 = \psi_2^2$. It also turns out that $\langle\theta_{\psi_1},\theta_{\psi_1}\rangle$ and $\langle\theta_{\mathcal{O}_K},\theta_{\mathcal{O}_K}\rangle$ are essentially equal (up to powers of 2).

8.3 Idoneal numbers

		ℓ			
		1	2	3	4
	-84	-2^9	$-2^4 3 \cdot 2897$	$-2^{13} 3 \cdot 3877$	$-2^7 3^2 7 \cdot 7282459$
	-120	$-2^3 233$	$-2^4 3 \cdot 103 \cdot 257$	$-2^6 3 \cdot 5 \cdot 7 \cdot 359 \cdot 769$	$-2^7 3^2 5 \cdot 31 \cdot 30659543$
	-132	$-2^4 151$	$-2^4 3 \cdot 13^2 233$	$-2^7 3 \cdot 11941247$	$-2^7 3^2 12365291437$
	-168	$-2^3 13 \cdot 61$	$-2^4 3 \cdot 227 \cdot 1093$	$-2^6 3 \cdot 113 \cdot 2216989$	$-2^7 3^2 7 \cdot 51546898267$
	-228	$-2^4 5 \cdot 283$	$-2^4 3 \cdot 163 \cdot 14699$	$-2^7 3 \cdot 773 \cdot 5097683$	$-2^7 3^2 5 \cdot 3389 \cdot 2048278621$
	-280	$-2^3 23 \cdot 211$	$-2^4 11 \cdot 2047063$	$-2^6 3^2 5 \cdot 11 \cdot 9011 \cdot 26759$	$-2^7 5 \cdot 7 \cdot 112583 \cdot 569016817$
	-312	$-2^3 31 \cdot 421$	$-2^4 3 \cdot 11 \cdot 71 \cdot 57251$	$-2^6 3 \cdot 554176930991$	$-2^7 3^2 41 \cdot 1433519 \cdot 133798411$
D	-340	$-2^5 29 \cdot 97$	$-2^4 105209333$	$-2^8 3^3 5 \cdot 7 \cdot 377853659$	$-2^7 5 \cdot 23 \cdot 59 \cdot 7948500647621$
	-372	$-2^6 13 \cdot 17 \cdot 19$	$-2^4 3 \cdot 5 \cdot 17 \cdot 463 \cdot 6563$	$-2^9 3 \cdot 43 \cdot 8783 \cdot 2336771$	$-2^7 3^2 19 \cdot 59 \cdot 103 \cdot 887 \cdot 2671 \cdot 962131$
	-408	$-2^3 7 \cdot 31 \cdot 263$	$-2^4 3 \cdot 722719007$	$-2^6 3 \cdot 398557 \cdot 84903367$	$-2^7 3^2 43 \cdot 83 \cdot 514247587049069$
	-420	$-2^8 151$	$-2^5 3 \cdot 47 \cdot 49417$	$-2^{11} 3 \cdot 5 \cdot 19 \cdot 409 \cdot 14221$	$-2^8 3^2 5 \cdot 7 \cdot 43 \cdot 21735127133$
	-520	$-2^3 7 \cdot 16519$	$-2^4 107 \cdot 83439599$	$-2^6 3 \cdot 5 \cdot 151 \cdot 3517 \cdot 99178571$	$-2^7 5 \cdot 241 \cdot 31815617 \cdot 7280136961$
	-532	$-2^9 5 \cdot 313$	$-2^4 83 \cdot 84815009$	$-2^{13} 3 \cdot 43 \cdot 40813878811$	$-2^7 5 \cdot 7 \cdot 521 \cdot 9580507980739999$
	-660	$-2^6 4019$	$-2^5 3 \cdot 84955769$	$-2^9 3 \cdot 5 \cdot 769 \cdot 3079 \cdot 29129$	$-2^8 3^2 5 \cdot 23^2 12826651596377$
	-708	$-2^4 211 \cdot 5233$	$-2^4 3 \cdot 14083 \cdot 55570667$	$-2^7 3 \cdot 38281 \cdot 13122545866403$	$-2^7 3^2 631 \cdot 112237 \cdot 22318536285190567$
	-760	$-2^3 3^2 148331$	$-2^4 7 \cdot 137 \cdot 986380123$	$-2^6 3^2 5 \cdot 17958574802156873$	$-2^7 5 \cdot 19793 \cdot 53777 \cdot 1053071 \cdot 442405567$
	-840	$-2^4 179 \cdot 347$	$-2^5 3 \cdot 61 \cdot 1597 \cdot 10103$	$-2^7 3 \cdot 5 \cdot 11^2 90223100377$	$-2^8 3^2 5 \cdot 7 \cdot 661 \cdot 709 \cdot 1511 \cdot 155380321$
	-1012	$-2^7 3 \cdot 47 \cdot 2473$	$-2^4 16504437324451$	$-2^{10} 3^2 7^2 13 \cdot 4463 \cdot 145619278193$	$-2^7 1663 \cdot 93287 \cdot 115469 \cdot 37218419688193$
	-1092	$-2^5 5 \cdot 17359$	$-2^5 3 \cdot 9721 \cdot 768881$	$-2^8 3 \cdot 167 \cdot 12647 \cdot 264316363$	$-2^8 3^2 5 \cdot 7 \cdot 59 \cdot 241 \cdot 423292626320989$
	-1320	$-2^4 47 \cdot 16069$	$-2^5 3 \cdot 47 \cdot 367 \cdot 6613879$	$-2^7 3 \cdot 5 \cdot 13 \cdot 6874687 \cdot 139706417$	$-2^8 3^2 5 \cdot 103 \cdot 1867 \cdot 6737 \cdot 1468799 \cdot 4281731$
	-1380	$-2^7 7 \cdot 16349$	$-2^5 3 \cdot 13 \cdot 97 \cdot 487 \cdot 287117$	$-2^{10} 3 \cdot 5 \cdot 31 \cdot 395027 \cdot 228192919$	$-2^8 3^2 5 \cdot 18597324231281857131113$
	-1428	$-2^{11} 79 \cdot 83$	$-2^5 3 \cdot 47 \cdot 8527 \cdot 382999$	$-2^{15} 3 \cdot 348685527772061$	$-2^8 3^2 7 \cdot 11 \cdot 344171 \cdot 2964701350076467$
	-1540	$-2^5 3 \cdot 59 \cdot 1747$	$-2^5 1289 \cdot 184546987$	$-2^8 3^4 5 \cdot 13 \cdot 23 \cdot 421 \cdot 1169291867$	$-2^8 5 \cdot 7 \cdot 631 \cdot 16369 \cdot 39779 \cdot 14329084171$
	-1848	$-2^4 37 \cdot 53 \cdot 2689$	$-2^5 3 \cdot 4820737472711$	$-2^7 3 \cdot 19 \cdot 659 \cdot 1693 \cdot$	$-2^8 3^2 7 \cdot 14447 \cdot$

8.4 Example with class number 3: $D = -23$

In this seciton, we consider the case $K = \mathbb{Q}(\sqrt{-23})$, which has cyclic class group of order 3 generated by one of the primes above 2, say $\mathfrak{p}_2$.

Using Corolary 4 and the structure of the class group, we see that one only needs to compute

$$\langle \theta_{\mathfrak{p}_2^i}, \theta_{\mathcal{O}_K} \rangle$$

for $0 \leq i \leq 2$ and since $\mathfrak{p}_2^2 \sim \bar{\mathfrak{p}}2$ in the class group and $\langle \theta_{\bar{\mathfrak{p}}_2}, \theta_{\mathcal{O}_K} \rangle = \overline{\langle \theta_{\mathfrak{p}_2}, \theta_{\mathcal{O}_K} \rangle}$, we only care about $\langle \theta_{\mathfrak{p}_2}, \theta_{\mathcal{O}_K} \rangle$ and $\langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle$.

We first look at the algebraic number

$$a(\ell) = V_D \langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell}.$$

For $\ell = 1, 2$ and 4, we find that $a(\ell)^3$ is a root of a monic degree 3 polynomial and generates the Hilbert class field over K. For example, we find that $a(1)$ is a root of the polynomial

$$x^9 - 2816x^6 - 905216x^3 - 89915392$$

and using the command `polredbest` in PARI/GP we find that

$$x^3 - 2816x^2 - 905216x - 89915392$$

generates the Hilbert class field of K (which is generated by the roots of $x^3 - x - 1$). When $l = 3, 6$ and 9, we find that $a(\ell)$ is a root of a cubic polynomial and gererates the Hilbert class field over K. For example, $a(3)$ is a root of

$$x^3 - 6740x^2 - 169034720x - 1027491892288.$$

Note that there seems to be a dependance on ℓ modulo 3. Also, all these quantities are algebraic integers.

We now look at the algebraic number

$$b(\ell) = V_D \langle \theta_{\mathfrak{p}_2}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell}.$$

For $\ell = 1, 2$ and 4, $b(\ell)^3$ is a root of a monic degree 6 polynomial. For example, $b(1)$ is a root of

$$x^{18} + 19712x^{15} + 579022848x^{12} - 1231996846080x^9 + 2076139946246144x^6 - 2333695837768515584x^3 + 2119375970242045935$$

For $\ell = 3, 6$ and 9, $b(\ell)$ is a root of a degree 6 polynomial. It is also the case that all the degree 6 polynomials encountered when dealing with $b(\ell)$ generate the same number field as the polynomial

$$x^6 - 3x^5 + 5x^4 - 5x^3 + 5x^2 - 3x + 1.$$

So far, I didn't figure out where this polynomial comes from!

With this data, it is possible to compute the matrix $M_{\{\mathcal{O}_K, \mathfrak{p}_2, \bar{\mathfrak{p}_2}\}}$ and its determinant for small values of ℓ. We find:

ℓ	$\det M_{\{\mathcal{O}_K, \mathfrak{p}_2, \bar{\mathfrak{p}_2}\}}(\ell)/\Omega_K^{12\ell}$
1	$-2^{10}23$
2	$-2^{14}19 \cdot 23 \cdot 619$
3	$-2^{18}5^2 11 \cdot 23 \cdot 337 \cdot 27299$
4	$-2^{22}7^2 23 \cdot 163 \cdot 2113 \cdot 117741979$
5	$-2^{26}5^3 23 \cdot 229 \cdot 23761 \cdot 808991 \cdot 20338663$
6	$-2^{30}5^2 11^2 13 \cdot 19 \cdot 23 \cdot 67^2 101 \cdot 868697 \cdot 505912247899$
7	$-2^{34}7^3 23 \cdot 14139407 \cdot 8653254414564166163204458 73$
8	$-2^{38}5^6 7^2 23 \cdot 6071587776583422106365009838251744 4617$
9	$-2^{42}5^2 11 \cdot 17^2 23 \cdot 31 \cdot 53 \cdot 181 \cdot 1879 \cdot 2861 \cdot 319129 \cdot 620671 \cdot 12513856379 \cdot 245047645005307$
10	$-2^{46}5^3 19^2 23 \cdot 40322967540986794792203928785469127547464962765409763 7757$

It is surprising at first that although the entries of the matrix $M_{\{\mathcal{O}_K,\mathfrak{p}_2,\bar{\mathfrak{p}_2}\}}$ are algebraic of high degree (up to 18) and belong to different fields, the determinant is rational.

Next, we consider the algebraic quantities

$$N(\psi,\ell) = V_D\langle\theta_\psi,\theta_\psi\rangle/\Omega_K^{4\ell}.$$

In this section, we only present the results. For more details on how to compute these quantities, in particular how to numerically evaluate the L-function attached to Hecke characters of quadratic fields, see the next chapter. For fixed ℓ, let ψ_0,ψ_1 and ψ_2 be the Hecke characters of infinity type 2ℓ.

For $\ell = 1,2,4$ and 5, the numbers $N(\psi_i,\ell)$, for $0 \le i \le 2$, are distinct and their cube are the three real roots of a monic cubic polynomial. For example, the numbers $N(\psi_i,1)^3$, for $0 \le i \le 2$, are the three roots of the irreducible polynomial

$$x^3 - 6966x^2 + 11569230x - 239483061.$$

Note that the constant term is $621^3 = (3^3 23)^3$, a perfect cube. It follows that the determinant of $M_K(\ell)$ is 621, an integer.

For $\ell = 3,6$ and 9, the situation is slightly different. For one of the characters, suppose it is ψ_0, the algebraic number $N(\psi_0,\ell)$ is an *integer*. For the two others, we find that their cube are the roots of a monic quadratic polynomial. For example,

$$N(\psi_0,3) = 5055 = 3\cdot 5\cdot 337$$

and $N(\psi_1,3)^3$ and $N(\psi_2,3)^3$ are the roots of

$$x^2 - 16287872873193x + 30021979248651078296845875.$$

Again, the constant term of this quadratic polynomial is a perfect cube.

In the following table, we compute the determinant of the matrix $M_K(\ell)$ for $1 \le \ell \le 10$.

ℓ	$\det M_K(\ell)/\Omega_K^{12\ell}$
1	$-3^3 23$
2	$-3^3 19\cdot 23\cdot 619$
3	$-3^3 5^2 11\cdot 23\cdot 337\cdot 27299$
4	$-3^3 7^2 23\cdot 163\cdot 2113\cdot 117741979$
5	$-3^3 5^3 23\cdot 229\cdot 23761\cdot 808991\cdot 20338663$
6	$-3^3 5^2 11^2 13\cdot 19\cdot 23\cdot 67^2 101\cdot 868697\cdot 505912247899$
7	$-3^3 7^3 23\cdot 14139407\cdot 8653254414564166163204458 73$
8	$-3^3 5^6 7^2 23\cdot 607158777765834221063650098382517444617$
9	$-3^3 5^2 11\cdot 17^2 23\cdot 31\cdot 53\cdot 181\cdot 1879\cdot 2861\cdot 319129\cdot 620671\cdot 12513856379\cdot 245047645005307$
10	$-3^3 5^3 19^2 23\cdot 403229675409867947922039287854691275474649627654097637757$

As one can see, the entries of the last two tables are very similar, but they differ by powers of 2 and 3. This is explained in part by our choice of representatives $\{\mathcal{O}_K,\mathfrak{p}_2,\bar{\mathfrak{p}_2}\}$ for the class group, since $\mathfrak{p}_2$ has norm 2 (recall that the matrix $M_{\mathcal{R}}(\ell)$ depends on the choice of representatives for the class group). The matrix $M_K(\ell)$ on the other hand depends only on K and ℓ and so it makes sense to say that its determinant is an integer. It also seems that the Petersson norms of the θ_ψ exhibit a simpler structure, since their cube are the roots of a cubic polynomial when ℓ is not divisible by 3. The fact that only one of the norms is an integral multiple $\Omega_K^{4\ell}$ when 3 divides ℓ seems mysterious for the moment.

8.5 $D = -104$

8.6 $D = -2660$

9 Computing some special values of Hecke L-functions

Use formula for $D = -23$ and infinity type 2. Show how formula 7 fits into Deligne's conjectures (see Watkins sec. 5.3.2).

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