# Petersson Inner Product of Theta Series

## PhD Defense

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# $L$-functions at $s = 1$

It is a well-known (but fascinating) fact that many $L$-functions contain arithmetic informations in their value at $s = 1$:

1. $\zeta(s)$ at $s = 1$: Infinitely many primes
2. $L(\chi, s)$ at $s = 1$: Infinitely many primes in arithmetic progressions
3. $\zeta_F(s)$ at $s = 1$: Class number formula

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## Conjecture (Stark (Idea))

*In general, L-functions of Artin representations have a (relatively) explicit expression involving arithmetic invariants of the number fields involved.*

# An observation of Stark

Let $K = \mathbb{Q}(\sqrt{-23})$ and let $H$ be its Hilbert class field. Let

$$\psi : \mathrm{Gal}(H/K) \to \mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$$

be a non-trivial one-dimensional Artin representation and let

$$\rho = \mathrm{Ind}_K^{\mathbb{Q}} \psi : \mathrm{Gal}(H/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$$

be the induced representation. Then one can consider the associated Artin $L$-function

$$L(\psi, s) = L(\rho, s).$$

# An observation of Stark

On the one hand, in accordance with his conjecture (which was known in this case), Stark shows that

$$L(\rho, 1) = \frac{2\pi}{\sqrt{23}} \log \varepsilon,$$

where $\varepsilon$ is the real root of

$$x^3 - x - 1.$$

Note that $\varepsilon$ generates $H$ over $K$.

# An observation of Stark

On the other hand, by the Deligne-Serre theorem, one has

$$L(\rho, s) = L(\theta_\psi, s),$$

where

$$\theta_\psi(q) = \eta(q)\eta(23q) = q\prod_{n\geq 1}(1-q^n)(1-q^{23n}) \in M_1(\Gamma_0(23), \chi_{-23}).$$

Then Stark proves that

$$L(\rho, 1) = \frac{2\pi}{3\sqrt{23}}\langle\theta_\psi, \theta_\psi\rangle.$$

# The main motivation

It follows that

$$\langle \theta_\psi, \theta_\psi \rangle = 3 \log \varepsilon.$$

# Structure of the presentation

# Where we are in the presentation

# Notation

Throughout this presentation, let

- $K$ be an imaginary quadratic field of discriminant $D$ with Hilbert class field $H$,
- $h_K, w_K$ and $\mathrm{Cl}_K$ be the class number, root number and class group of $K$ (respectively)
- $\psi$ be a Hecke character of infinity type $(2\ell, 0)$ for some $\ell \geq 0$, i.e. a homomorphism

$$\psi : I_K \longrightarrow \mathbb{C}^\times$$

  such that $\psi((\alpha)) = \alpha^{2\ell}$ for all $\alpha \in K^\times$
- and $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ be fractional ideals of $K$.

# Theta series attached to $K$

Consider

$$\left.\begin{aligned} \theta_\psi(q) &= \textstyle\sum_{\mathfrak{a}\in\mathcal{O}_K} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} \\ \theta_{\mathfrak{a}}(q) &= \textstyle\sum_{x\in\mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})} \end{aligned}\right\} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

Then

<table>
<tr><td></td><td>$\theta_\psi$</td><td>$\theta_{\mathfrak{a},\ell}$</td></tr>
<tr><td>$\ell > 0$</td><td>Newform</td><td>Cusp form</td></tr>
<tr><td rowspan="2">$\ell = 0$</td><td>$\psi^2 \neq 1$: Newform</td><td rowspan="2">Not a cusp form</td></tr>
<tr><td>$\psi^2 = 1$: (genus) Eisenstein series</td></tr>
</table>

# Some examples to keep in mind

| | $\theta_\psi$ | $\theta_{\mathfrak{a},\ell}$ |
|---|---|---|
| $\ell > 0$ | | |
| $\ell = 0$ | $q\prod_{n\geq 1}(1-q^n)(1-q^{23n})$ | $\theta_{\mathbb{Z}[i]}(q) = \sum_{x,y\in\mathbb{Z}} q^{x^2+y^2}$ |

Recall that

$$q\prod_{n\geq 1}(1-q^n)(1-q^{23n})$$

is the modular form in Stark's example.

# Formulas for the Petersson inner product of those theta series

Recall that the Petersson inner product of any cusp forms $f, g \in S_k(\Gamma_0(N), \chi)$ is defined as

$$\langle f, g \rangle = \iint_{\Gamma_0(N)\backslash\mathcal{H}} f(\tau)\bar{g}(\tau)\Im(\tau)^k \mathrm{d}\mu(\tau).$$

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With minor effort, this formula can be used to compute the Petersson inner product numerically:

$$\langle f, g \rangle = \sum_{\gamma\in\Gamma_0(N)\backslash\mathcal{H}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\sqrt{1-x^2}}^{\infty} f(\tau)\bar{g}(\tau)y^{k-2}\mathrm{d}y\mathrm{d}x.$$

But this is very (very) slow and behaves badly as the level grows.

# The quest for more efficient and useful formulas

Let $\psi$ be such that $\theta_\psi$ is a cusp form. Then

1. Apply Rankin-Selberg:

$$\langle \theta_\psi, \theta_\psi \rangle = \left( \frac{\pi}{2} \frac{\phi(|D|)}{D^2} \frac{(4\pi)^{2\ell+1}}{\Gamma(2\ell+1)} \right)^{-1} L(\chi_D, 1) \operatorname{res}_{s=2\ell+1} L(\operatorname{Sym}^2 \theta_\psi, s)$$

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2. Isolate the residue of $L(\operatorname{Sym}^2 \theta_\psi, s)$:

$$\operatorname{res}_{s=2\ell+1} L(\operatorname{Sym} 2\theta_\psi, 1, s) = \prod_{p|D} (1 - p^{-1}) L(\psi^2, 2\ell+1)$$

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3. When $\ell>0$, express $L(\psi^2,2\ell+1)$ in terms of (derivatives of nearly holomorphic) Eisenstein series:

$$L(\psi^2,2\ell{+}1) = \frac{4(2\pi)^{2\ell+1}\sqrt{|D|}^{2\ell-1}}{w_K\Gamma(2\ell+1)}\sum_{j=1}^{h_K}\psi^{-2}(\mathfrak{a}_j)N(\mathfrak{a}_j)^{4\ell}\delta^{2\ell-1}E_2(\bar{\mathfrak{a}}_j)$$

# The most useful formulas for $p$-adic interpolation

| | $\langle \theta_\psi, \theta_\psi \rangle$ | $\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$ |
|---|---|---|
| $\ell > 0$ | $C_1 \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \delta^{2\ell-1} E_2(\mathcal{A})$ | |
| $\ell = 0$ | $\psi^2 = 1$: not applicable | not applicable |

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Using the relation

$$\theta_{\mathfrak{a},\ell} = \frac{w_K}{h_K} \sum_\psi \psi(\mathfrak{a}) \theta_\psi$$

and the orthogonality of the newforms $\theta_\psi$.

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| $\ell = 0$ | $C_3 \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \log \Phi(\mathcal{A})$<br>$\psi^2 = 1$: not applicable | not applicable |

Here

$$\Phi^{12}(\mathcal{A}) = N(\mathcal{A})^6 |\Delta(\mathcal{A})|,$$

where

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

# Where we are in the presentation

# Bridging the gap between the "explicit" formulas and the algorithms

Here are some of the things one needs to do before implementing those formulas:

- Complete the $L$-functions $L(\mathrm{Sym}^2\,\theta_\psi, s)$ and $L(\psi, s)$ and find all the information about their functional equation,
- Find a way to compute with Hecke characters,
- Find an *efficient* way to compute

$$\delta^n E_2(\mathfrak{a}),$$

- Choose the computer algebra system that allows you to do all this!

# The most efficient formula for computations

Experimentally, one finds that the most efficient way to compute the Petersson inner product of theta series is to compute the $q$-expansion of $\delta^n E_2$ by hand:

$$\delta^n E_2(\tau) = (-1)^n \left( \frac{1}{8\pi \Im(\tau)} - \frac{n+1}{24} \right) \frac{n!}{(4\pi \Im(\tau))^n} + \sum_{m \geq 1} \sigma(m) \left( \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \frac{(r+2)_{n-r}}{(4\pi \Im(\tau))^{n-r}} m^r \right) q^m.$$

# The resulting algorithm

This leads to the following

## Theorem (S.)

*There exists a software package to compute the Petersson inner product of the theta series defined above with the following properties:*

- *It is fast (relative to the definition),*
- *It supports arbitrary precision (no coefficients stored, no database involved),*
- *User friendly (easy to download, help functions, well commented source code),*

## Proof.

See the calculations at the end of the thesis! □

# Where we are in the presentation

# What about Stark's observation?

Using the above formula when $\ell = 0$, one has

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{-h_K}{3w_K^2} \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \log N(\mathcal{A})^6 |\Delta(\mathcal{A})| = h_K \log \kappa_\psi,$$

where

$$\kappa_\psi = \prod_{\mathcal{A} \in \mathrm{Cl}_K} \Phi(\mathcal{A})^{-\psi^2(\mathcal{A})},$$

with

$$\Phi(\mathcal{A}) = \sqrt{N(\mathfrak{a})} |\Delta(\mathfrak{a})|^{1/12}$$

as before, where now $\mathfrak{a}$ is any ideal in the class $\mathcal{A}$.

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## Question

*Is $\kappa_\psi$ a unit in $H$?*

# Generalizing Stark's Observation

## Proposition (S.)

*Let $\psi$ be a class character such that $\psi^2$ is a non-trivial character with rational real part (equivalently, the character of $\mathrm{Ind}_K^{\mathbb{Q}} \psi$ is rational). Then $\kappa_\psi$ is an algebraic integer which is a unit. Moreover, if $\psi^2$ is a non-trivial genus character corresponding to the factorisation $D = D_1 D_2$, with $D_1 > 0$ say, then*

$$\kappa_\psi = \epsilon_{D_1}^{\frac{4h_{D_1}h_{D_2}}{w_K w_{D_2}}},$$

*where $\epsilon_{D_1}$ is the fundamental unit of $\mathbb{Q}(\sqrt{D_1})$, $h_{D_j}$ is the class number of $\mathbb{Q}(\sqrt{D_j})$ and $w_{D_2}$ is the number of roots of unity in $\mathbb{Q}(\sqrt{D_2})$.*

# A corollary and some examples

It follows that

### Corollary

*If $K$ has class number divisible by 2 or 3, there exists a class character $\psi$ for which $\kappa_\psi$ is a unit.*

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Some examples:

- $K = \mathbb{Q}(\sqrt{-23})$ ($h_K = 3$): $\kappa_\psi$ is a unit and numerically, $\kappa_\psi = \varepsilon$,
- $K = \mathbb{Q}(\sqrt{-39})$ ($h_K = 4$): $\kappa_\psi = \epsilon_{13}^{\frac{1}{3}}$ is a unit, but not in $H$,
- $K = \mathbb{Q}(\sqrt{-47})$ ($h_K = 5$): $\kappa_\psi$ doesn't seem to be a unit for any $\psi$.

# Where we are in the presentation

# Two objectives

Recall that

| | $\langle \theta_\psi, \theta_\psi \rangle$ | $\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$ |
|---|---|---|
| $\ell > 0$ | $C_1 \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \delta^{2\ell-1} E_2(\mathcal{A})$ | $C_2 \sum_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_{\mathfrak{c}} \mathcal{O}_K} \lambda_{\mathfrak{c}}^{2\ell} \delta^{2\ell-1} E_2(\mathfrak{c})$ |
| $\ell = 0$ | $C_3 \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \log \Phi(\mathcal{A})$ <br> $\psi^2 = 1$: not applicable | not applicable |

Two objectives for the $p$-adic interpolation

1. Show that the quantities $\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$ can be $p$-adically interpolated for $\ell > 0$ (under certain restrictions),
2. Evaluate the $p$-adic analytic function obtained at $\ell = 0$.

# $p$-adic interpolation of Petersson inner product of theta series: setup

Suppose that $D$ is prime and let $p > 3$ be a prime $\neq 2, 3$ which splits in $K$, say $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$.
Let $\mathfrak{a}$ and $\mathfrak{b}$ be two fractional ideals of $K$ which are such that

$$\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \mathcal{O}_K$$

and fix an isomorphism

$$\mathbb{Q}_p/\mathbb{Z}_p \to \bigcup_{n\geq 1} \bar{\mathfrak{p}}^{-n}\mathfrak{c}/\mathfrak{c}.$$

Let also

$$\mathcal{W} = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$$

denote the $p$-adic weight space.

# $p$-adic interpolation of Petersson inner product of theta series: result

With the notation above, one has the following

**Theorem (S.)**

*There exists a $p$-adic analytic function*

$$F : \mathcal{W} \to \mathbb{C}_p$$

*with the property that*

$$F(\ell) = (\mathrm{Frob}_{\mathfrak{p}}^{-1} - p^{2\ell-1})(\mathrm{Frob}_{\mathfrak{p}}^{-1} - p^{2\ell}) \left( \frac{\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle}{((2\pi i)^{-1}\Omega_{\mathbb{C}}(\mathfrak{c}))^{4\ell}} \right) \quad \textit{for all}$$

*where* $\mathrm{Frob}_{\mathfrak{p}} = \left(\frac{H/K}{\mathfrak{p}}\right)$ *is the Artin symbol.*

# Evaluation of $F$ at $\ell = 0$

Let

$$g_0(q) = \frac{\Delta(q)}{\Delta(q^p)}$$

and let

$$g_0^{(p)}(q) = \frac{g_0(q^p)}{g_0^p(q)} = \frac{\Delta^{p+1}(q^p)}{\Delta^p(q)\Delta(q^{p^2})}.$$

Then $\log_p g_0^{(p)}$ is a $p$-adic modular form and one has the following

**Theorem (S. (loose form))**

*The following equality holds in $\mathbb{C}_p$:*

$$F(0) = -\frac{1}{6p}\log_p g_0^{(p)}(\mathfrak{c}).$$

# Interpretation of the above theorem (in the current setup)

Using the formula for $\langle \theta_\psi, \theta_\psi \rangle$ even if $\psi^2 = 1$, one has

| | $\langle \theta_\psi, \theta_\psi \rangle$ | $\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$ |
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| $\ell > 0$ | $C_1 \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \delta^{2\ell-1} E_2(\mathcal{A})$ | $4\delta^{2\ell-1} E_2(\mathfrak{c})$ |
| $\ell = 0$ | $C_3 \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \log \Phi(\mathcal{A})$ | $-\frac{1}{3} \log(N(\mathfrak{c})^6 \lvert \Delta(\mathfrak{c}) \rvert)$ |

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On the other had, a *formal* computation gives

$$F(0)=-\frac{1}{6p}\log_p\frac{\Delta^{p+1}(q^p)}{\Delta^p(q)\Delta(q^{p^2})}$$
$$=-\frac{1}{6}(\mathrm{Frob}_{\mathfrak{p}}^{-1}-p^{-1})(\mathrm{Frob}_{\mathfrak{p}}^{-1}-1)\log_p\Delta(\mathfrak{c})$$

# Where we are in the presentation

# Experimenting with Petersson norm of theta series

Consider the algebraic number

$$N(\psi, \ell) = \frac{\langle \theta_\psi, \theta_\psi \rangle}{\Omega_K^{4\ell}} \qquad \text{for} \ell > 0,$$

where $\Omega_K$ is the Chowla-Selberg period.

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$$N(\psi_0, \ell) \in \mathbb{Z}$$

for a unique Hecke character $\psi_0$.

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for a unique Hecke character $\psi_0$.
This $\psi_0$ is the Hecke character

$$\psi_0((\alpha)) = \alpha^{2\ell/h_K}.$$

# Computing the Gram matrix

Let

$$\mathrm{Gram}(f_1, \ldots, f_d) = \det(\langle f_i, f_j \rangle)_{1 \le i,j \le d}$$

be the determinant of the Gram matrix of the Petersson inner product for a basis $\{f_1, \ldots, f_d\}$ in a vector space.

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be the determinant of the Gram matrix of the Petersson inner product for a basis $\{f_1, \ldots, f_d\}$ in a vector space. Then one has the following

### Proposition

*Let $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_{h_K}\}$ be a set of representatives of $Cl_K$ and let $\{\psi_1, \ldots, \psi_{h_K}\}$ be the Hecke characters of $K$ of infinity type $(2\ell, 0)$. Then*

$$\mathrm{Gram}(\theta_{\mathfrak{a}_1,\ell}, \ldots, \theta_{\mathfrak{a}_{h_K},\ell}) = \left(\frac{w_K^2}{h_K}\right)^{h_K} \left(\prod_{i=1}^{h_k} N(\mathfrak{a}_i)\right) \mathrm{Gram}(\theta_{\psi_1}, \ldots, \theta_{\psi_{h_K}}).$$

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