Petersson Inner Product of Binary Theta Series

A computational approach

Nicolas SIMARD

McGill University

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Mobius transformations

Let $\mathcal{H}$ be the Poincarre upper-half plane. Recall that $\mathrm{GL}_2(\mathbb{R})_+$ acts on $\mathcal{H}$ via Mobius transformations :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Definition

Let $N \geq 1$ and define the Hecke subgroup of level N as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | c \equiv 0 \pmod{N} \right\}.$$

Level N modular forms with characters

Definition

Let $N \geq 1$ and $k \geq 0$ be integers and let χ be a Dirichlet character mod N. A modular form of weight k, level N and character χ is a holomorphic function

$$f : \mathcal{H} \longrightarrow \mathbb{C}$$

such that

$$f(\gamma z) = \chi(d)(cz + d)^{-k} f(z)$$

for all $z \in \mathcal{H}$ and all $\gamma \in \Gamma_0(N)$, which satisfies certain growth conditions at the cusps. The $\mathbb{C}$-vector-space of such modular forms is denoted

$$M_k(\Gamma_0(N), \chi).$$

q-expansion of modular forms

Every modular form f has a Taylor (or Fourrier) expansion at infinity, called its q-expansion :

$$f(z) = \sum_{n=0}^{\infty} a_n q^n,$$

where $q = exp(2\pi iz)$. If

$$a_0(f) = 0,$$

(at all cusps) f is called a *cusp form*.

Example : weight k Eisenstein series

Let $k \geq 4$ be an even integer and define

$$G_k(z) = \sum_{m,n} \frac{1}{(mz+n)^k} \in M_k(\Gamma_0(1), 1).$$

After renormalisation, the q-expansion of G_k is

$$E_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Important non-example : weight 2 Eisenstein series

In level 1, there are no modular forms of weight 2. However, one can still define the weight 2 Eisenstein series as

$$E_2(2) = \frac{1}{8\pi\Im(z)} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n) q^n.$$

It is an example of an *almost holomorphic* modular form of level 1 and weight 2.

Spaces of modular forms

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Spaces of modular forms

- $M_k(\Gamma_0(N), \chi)$ is finite dimensional.
- For every integer $n \geq 1$, one can define a *Hecke operator* T_n (depending on k, N and χ) which acts on $M_k(\Gamma_0(N), \chi)$.
- There exists a basis of common eigenvectors for all Hecke operators T_n with $(n, N) = 1$.

Petersson inner product

Let $f, g \in S_k(\Gamma_0(N), \chi)$ be two cusp forms. The Petersson inner product of f and g is defined as

$$\langle f, g \rangle = \frac{1}{\mathrm{Vol}(\Gamma_0(N) \setminus \mathcal{H})} \int_{\Gamma_0(N)\setminus\mathcal{H}} f(x+iy)\overline{g(x+iy)}y^k \mathrm{d}\mu,$$

where

$$\mathrm{d}\mu = \frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

is the $\mathrm{SL}_2(\mathbb{R})$-invariant measure on $\mathcal{H}$. Note that the integral does not converge if neither f nor g is a cusp form.

Newforms

The space $S_k(\Gamma_0(N),\chi)$ splits naturally as

$$S_k(\Gamma_0(N),\chi) = S_k(\Gamma_0(N),\chi)^{\text{new}} \oplus S_k(\Gamma_0(N),\chi)^{\text{old}}.$$

Theorem

The space $S_k(\Gamma_0(N),\chi)^{new}$ has an orthogonal basis of eigenvectors for all Hecke operators. Elements of this basis are called newforms (after suitable normalization).

Summary

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1. The space $S_k(\Gamma_0(N), \chi)$ is a finite dimensional Hermitian inner product space, equipped with an action of Hecke operators.
2. The subspace $S_k(\Gamma_0(N), \chi)^{\text{new}}$ has distinguished elements (the newforms) which are mutually orthogonal and are eigenvectors for all Hecke operators.

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A half-integral weight theta series

Consider the function

$$\theta(z) = \sum_{x \in \mathbb{Z}} q^{x^2} = 1 + 2q + 2q^4 + O(q^5).$$

Then

$$\theta(\gamma z) = \epsilon(cz + d)^{1/2}\theta(z),$$

for all $\gamma \in \Gamma_0(4)$ and some $\epsilon_{c,d} \in \{\pm 1, \pm i\}$.

Theta series attached to ideals

Let K be an imaginary quadratic field of discriminant $D < -4$ and let $\mathcal{O}_K$ be its ring of integers. Fix an integer $\ell \geq 0$. To each integral ideal $\mathfrak{a}$ of K, one can attach the following theta series :

$$\theta_{\mathfrak{a}}^{(2\ell)} = \theta_{\mathfrak{a}} = \sum_{x \in \mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})}.$$

Basic properties of these theta series

1. We have

$$\theta_{\mathfrak{a}} = \sum_{x \in \mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

   where χ_D is the Kronecker symbol. If $\ell \neq 0$, then

$$\theta_{\mathfrak{a}} \in S_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

2. If $\lambda \in K^\times$, then

$$\theta_{\lambda\mathfrak{a}} = \lambda^{2\ell} \theta_{\mathfrak{a}}.$$

   So there are essentially h_D theta series attached to K.
3. In general, the $\theta_{\mathfrak{a}}$ are *not* newforms.

Theta series attached to Hecke characters of K

Let I_K denote the group of fractionnal ideals of K. A Hecke character ψ of K of infinity type 2ℓ (and conductor 1) is a homomorphism

$$\psi : I_K \longrightarrow \mathbb{C}^\times$$

such that

$$\psi((\alpha)) = \alpha^{2\ell}, \qquad \forall \alpha \in K^\times.$$

One can define

$$\theta_\psi = \sum_{\mathfrak{a} \subseteq \mathcal{O}_K} \psi(\mathfrak{a}) q^{N(\mathfrak{a})}.$$

Basic properties of these theta series

1. We have
$$\theta_\psi M_{2\ell+1}(\Gamma_0(|D|),\chi_D),$$
where χ_D is the Kronecker symbol. If $\psi^2 \neq 1$, then
$$\theta_\psi \in S_{2\ell+1}(\Gamma_0(|D|),\chi_D).$$
2. The θ_ψ are newforms.
3. We have the identities
$$\theta_\psi = \frac{1}{w_K}\sum_{[\mathfrak{a}]\in \mathrm{Cl}_K} \psi^{-1}(\mathfrak{a})\theta_{\mathfrak{a}} \quad \text{and} \quad \theta_{\mathfrak{a}} = \frac{w_K}{h_K}\sum_{\psi}\psi(\mathfrak{a})\theta_\psi.$$

Some questions

- Can we efficiently compute the Petersson inner product of theta series (whenever it makes sense) ?
- Can we find explicit formulas for it ?
- Can we use those formulas/computations to study the arithmetic properties of those quantities ?
- What about the p-adic properties of these quantities ?

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Petersson norm of the θ_ψ (with $\ell > 0$)

Theorem

Let ψ be a Hecke character of K of infinity type 2ℓ, where $\ell > 0$. Then

$$\langle \theta_\psi, \theta_\psi \rangle = V_D^{-1} (|D|/4)^\ell \frac{4h_K}{w_K^2} \sum_{[\mathfrak{a}] \in Cl_K} \psi^2(\mathfrak{a}) \partial^{2\ell-1} E_2(\mathfrak{a}),$$

where

$$V_D = Vol(\Gamma_0(|D|) \setminus \mathcal{H}).$$

Here,

$$\partial f = \frac{1}{2\pi i} \frac{\partial f}{\partial z} - \frac{k}{4\pi \Im(z)} f$$

is the Shimura-Mass diffential operator, which preserves the graded algebra of almost holomorphic modular forms.

Petersson inner product of the theta series $\theta_{\mathfrak{a}}$

Theorem

Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of K and suppose $\ell > 0$. Then

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle = C_K^{(2\ell)} N(\mathfrak{b})^{2\ell} \sum_{\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}} \mathcal{O}_K} \lambda_{\mathfrak{c}}^{2\ell} \partial^{2\ell-1} E_2(\mathfrak{c}),$$

where

$$C_K^{(2\ell)} = 4 V_D^{-1} (|D|/4)^{\ell}.$$

A few direct consequences of the formula

Corollary

For $\ell > 0$,

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle = 0$$

whenever $\mathfrak{a}$ and $\mathfrak{b}$ are not in the same genus (i.e. the classes of $\mathfrak{a}$ and $\mathfrak{b}$ are distinct in the genus group Cl_K / Cl_K^2).

Corollary

For $\ell > 0$,

$$\langle \theta_{\mathfrak{ac}}, \theta_{\mathfrak{bc}} \rangle = N(\mathfrak{bc})^{2\ell} \langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{b}} \rangle.$$

Arithmetic consequences

Let

$$\Omega_K = \frac{1}{\sqrt{4\pi|D|}} \left(\prod_{j=1}^{|D|-1} \Gamma\left(\frac{j}{|D|}\right) \right)^{w_K/4h_k}$$

be the Chowla-Selberg period attached to K.

Corollary

For $\ell > 0$, the complex numbers

$$\frac{V_D\langle \theta_\psi, \theta_\psi \rangle}{\Omega_K^{4\ell}} \quad \textit{and} \quad \frac{V_D\langle \theta_\mathfrak{a}, \theta_\mathfrak{b} \rangle}{\Omega_K^{4\ell}}$$

are algebraic.

If $\ell = 0$, the modular form $\theta_{\mathfrak{a}}$ is not a cusp form. But for θ_ψ, we have the following

Theorem

Let θ_ψ be a Hecke character of infinity type 0 and suppose that $\psi^2 \neq 1$. Then

$$\langle \theta_\psi, \theta_\psi \rangle = -V_D^{-1} \frac{4h_K}{w_K^2} \sum_{[\mathfrak{a}] \in Cl_K} \psi^2(\mathfrak{a}) \log(\Im(\tau_{\mathfrak{a}})^{1/2} |\eta(\tau_{\mathfrak{a}})|^2),$$

where $\tau_{\mathfrak{a}} \in \mathcal{H}$ is the complex root attached to $\mathfrak{a}$ and

$$\eta(z) = exp(2\pi i/24) \prod_{n=1}^{\infty} (1 - q^n)$$

is the standard eta-function.

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First step : compute $\partial^n E_2$

We have the following formulas :

$$\partial E_2 = \frac{5}{6}E_4 - 2E_2^2 \quad \partial E_4 = \frac{7}{10}E_6 - 8E_2E_4 \quad \partial E_6 = \frac{400}{7}E_4^2 - 12E_2E_6.$$

For example,

$$\partial^3 E_2 = -48E_2^4 + 120E_4E_2^2 - 14E_6E_2 + 25E_4^2.$$

Second step : Evaluate Hecke characters

The idea is simple : let $\mathfrak{a}$ be a fractional ideal of K and suppose

$$\mathfrak{a}^e = \lambda \mathcal{O}_K.$$

Then

$$\psi(\mathfrak{a})^e = \psi(\mathfrak{a}^e) = \psi((\lambda)) = \lambda^{2\ell},$$

so $\psi(\mathfrak{a})$ is determined (up to a e-root of unity).

Other second step

Given ideals $\mathfrak{a}$ and $\mathfrak{b}$, can we efficiently find all classes $[\mathfrak{c}]$ such that

$$\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K,$$

if any ? If we have representatives $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_d\}$ of $\mathrm{Cl}_K[2]$, it suffices to find one such $\mathfrak{c}_0$. Then the other solutions to the equation are

$$\mathfrak{c}_0\mathfrak{a}_i$$

for $i = 1, \ldots, d$.

Class number 1

In this case,

$$\theta_{\mathcal{O}_K} = \theta_{\psi_0}$$

and we only need to compute

$$V_D \langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell} \in \overline{\mathbb{Q}}.$$

Class number 1 case

Computation of $V_D\langle\theta_{\mathcal{O}_K},\theta_{\mathcal{O}_K}\rangle/\Omega_K^{4\ell}$:

		ℓ = 1	ℓ = 2
D	-7	$2^2 3$	-2^2
	-8	-2	$-2^2 5$
	-11	-2^2	$-2^3 5$
	-19	$-2^2 3^{-1} 13$	$-2^3 71$
	-43	$-2^3 3^{-1} 107$	$-2^4 5647$
	-67	$-2^2 3^{-1} 7^2 31$	$-2^3 5 \cdot 86629$
	-163	$-2^3 3^{-1} 150473$	$-2^4 11 \cdot 461681471$

Class number 2

In this case, K has two genera. If $\mathfrak{a}$ is a representative of the non-trivial class in Cl_K, we have

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathcal{O}_K} \rangle = \langle \theta_{\mathcal{O}_K}, \theta_{\mathfrak{a}} \rangle = 0$$

and

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{a}} \rangle = N(\mathfrak{a})^{2\ell} \langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle,$$

so it suffices to compute the quantity

$$V_D \langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell} \in \overline{\mathbb{Q}}.$$

Class number 2

As in the class number 1 case, the quantity

$$V_D\langle\theta_{\mathcal{O}_K},\theta_{\mathcal{O}_K}\rangle/\Omega_K^{4\ell}$$

is an integer, except for $\ell = 1$ and $D = -91, -403$ and -427.

Class number 3

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