Constructing the p-adic zeta function via cyclotomic units

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Contents

Introduction

1 p-adic measures

In this section, we first define p-adic measures and see how they are related to Iwasawa Algebras and power series rings. We then introduce operators on them and conclude with of few results on moments of measures.

1.1 p-adic measures, distributions and Iwasawa algebras

Let $\mathfrak{G}$ be an abelian profinite group, let $\mathfrak{B}_{\mathfrak{G}}$ be the boolean algebra of compact-open subsets of $\mathfrak{G}$, let $\mathfrak{T}_{\mathfrak{G}} \subseteq \mathfrak{B}_{\mathfrak{G}}$ be the set of open subgroups of $\mathfrak{G}$ and let A be any abelian group.

Definition 1. *An A-valued distribution λ on $\mathfrak{G}$ is a finitely additive function*

$$\lambda : \mathfrak{B}_{\mathfrak{G}} \to A.$$

The set of distributions is denoted $\mathfrak{D}(\mathfrak{G}, A)$. If $A \subseteq \mathbb{C}_p$, the elements of $\mathfrak{D}(\mathfrak{G}, A)$ are called p-adic distributions.

The set $\mathfrak{D}(\mathfrak{G}, A)$ is naturally an abelian group. If A is a B-algebra for some ring B, the set $\mathfrak{D}(\mathfrak{G}, A)$ is a B-algebra under convolution product, which we won't bother to define here!

Distributions and Iwasawa algebras

If $\mathfrak{G}$ is finite, $\mathfrak{B}_{\mathfrak{G}} = \{\{g\} | g \in \mathfrak{G}\}$ and we have an isomorphism of abelian groups

$$\lambda \mapsto \sum_{g \in \mathfrak{G}} \lambda(\{g\}) g : \mathfrak{D}(\mathfrak{G}, A) \to A[\mathfrak{G}].$$

If A is a B-algebra for some ring B, so is $A[\mathfrak{G}]$ and the isomorphism is an isomorphism of B-algebras. For $\mathfrak{G}$ finite, we define

$$\Lambda(\mathfrak{G}, A) \stackrel{\text{def}}{=} A[\mathfrak{G}].$$

For $\mathfrak{G}$ not necessarily finite, we define

$$\Lambda(\mathfrak{G}, A) = \varprojlim \Lambda(\mathfrak{G}/\mathfrak{H}, A) = \varprojlim A[\mathfrak{G}/\mathfrak{H}],$$

where the limit is taken over all elements of $\mathfrak{T}_{\mathfrak{G}}$. Given $\mathfrak{H}$ in $\mathfrak{T}_{\mathfrak{G}}$, we have a natural map

$$\lambda \mapsto \lambda_{\mathfrak{H}} = \sum_{x \in \mathfrak{G}/\mathfrak{H}} c_{\mathfrak{H}}(x)x : \mathfrak{D}(\mathfrak{G}, A) \to \Lambda(\mathfrak{G}/\mathfrak{H}, A).$$

Since distributions are finitely additive, we have then a natural map

$$\mathfrak{D}(\mathfrak{G}, A) \to \Lambda(\mathfrak{G}, A),$$

which is in fact an isomorphism. In a certain sense, the elements of the Iwasawa algebra $\Lambda(\mathfrak{G}, A)$ are the generating series of distributions.

Example: If $A = \mathbb{Z}_p$, one obtains the usual Iwasawa algebra

$$\Lambda(\mathfrak{G}) \stackrel{\text{def}}{=} \Lambda(\mathfrak{G}, \mathbb{Z}_p).$$

Distributions and step functions

From now on, suppose that A is a B-algebra for some ring B.

Recall that if $s : \mathfrak{G} \to A$ is a locally constant function, also called a step function, there exists an open subgroup $\mathfrak{H}$ such that s is well defined and constant modulo $\mathfrak{H}$. Note that this subgroup $\mathfrak{H}$ is not unique. The set of step functions from $\mathfrak{G}$ to A, denoted

$$\mathsf{Step}(\mathfrak{G}, A),$$

is a B-algebra.

Let λ be an A-valued distribution on $\mathfrak{G}$, let s be a step function which is constant modulo $\mathfrak{H}$ and define

$$\int_{\mathfrak{G}} s d\lambda \stackrel{\text{def}}{=} \sum_{g \in \mathfrak{G}/\mathfrak{H}} s(g)\lambda(g).$$

This gives a well-defined B-linear map

$$\Lambda(\mathfrak{G}, A) \to \mathsf{Hom}_{B-\mathsf{mod}}(\mathsf{Step}(\mathfrak{G}, A), A).$$

For convenience, the value of any B-linear map $\lambda \in \mathsf{Hom}_{B-\mathsf{mod}}(\mathsf{Step}(\mathfrak{G}, A), A)$ at a step function $s(x)$ is denoted

$$\int_{\mathfrak{G}} s(x) d\lambda(x)$$

or simply

$$\int_{\mathfrak{G}} s d\lambda$$

when there is no risk of confusion. The B-module

$$\mathsf{Hom}_{B-\mathsf{mod}}(\mathsf{Step}(\mathfrak{G}, A), A)$$

can be equipped with a natural B-algebra structure via the convolution product which is defined as follows: for $\lambda, \mu \in \mathsf{Hom}_{B-\mathsf{mod}}(\mathsf{Step}(\mathfrak{G}, A), A)$, let $\lambda * \mu$ be defined as

$$\int_{\mathfrak{G}} s(x) d(\lambda * \mu)(x) = \int_{\mathfrak{G}} \left(\int_{\mathfrak{G}} s(x+y) d\lambda(x) \right) d\mu(y).$$

The map above is then a B-algebra homomorphism, which is in fact an isomorphism. Indeed, its inverse takes a B-linear map $\phi \in \mathrm{Hom}_{B-\mathrm{mod}}(\mathrm{Step}(\mathfrak{G}, A), A)$ to the distribution λ defined as

$$\lambda(U) = \phi(\varepsilon_U)$$

for all $U \in \mathfrak{B}_{\mathfrak{G}}$, where $\varepsilon_U \in \mathrm{Step}(\mathfrak{G}, A)$ is the characteristic function of U. This sketches the proof of the following proposition.

Proposition 1. *There is a natural B-algebra isomorphism*

$$\Lambda(\mathfrak{G}, A) \to \mathit{Hom}_{B-\mathit{mod}}(\mathit{Step}(\mathfrak{G}, A), A).$$

p-adic measures and continuous functions

From now on, suppose that A is contained in $\mathbb{C}_p$ (e.g. $A = B = \mathbb{Z}_p$). Let

$$C(\mathfrak{G}, \mathbb{C}_p)$$

be the set of continuous functions from $\mathfrak{G}$ to $\mathbb{C}_p$. This is a $\mathbb{C}_p$-Banach algebra when equipped with the sup norm

$$\| f \| = \sup_{x \in \mathfrak{G}} |f(x)|_p.$$

The set $\mathrm{Step}(\mathfrak{G}, \mathbb{C}_p)$ is dense in $C(\mathfrak{G}, \mathbb{C}_p)$.

Definition 2. *A p-adic distribution $\lambda \in \mathfrak{D}(\mathfrak{G}, A)$ is called a p-adic measure if it is bounded (as a function from $\mathfrak{B}_{\mathfrak{G}}$ to $A \subseteq \mathbb{C}_p$). The set of p-adic measures is denoted $\mathfrak{M}(\mathfrak{G}, A)$.*

Note that if A is bounded, which is the case if $A = \mathbb{Z}_p$ for example, then $\mathfrak{M}(\mathfrak{G}, A) = \mathfrak{D}(\mathfrak{G}, A)$.

Proposition 2. *Let $\lambda \in \mathfrak{M}(\mathfrak{G}, \mathbb{C}_p)$ be a measure, viewed as a B-linear map*

$$\lambda : \mathit{Step}(\mathfrak{G}, \mathbb{C}_p) \to \mathbb{C}_p.$$

Then λ extends uniquely to a continuous map

$$\lambda : C(\mathfrak{G}, \mathbb{C}_p) \to \mathbb{C}_p.$$

Proof. Let λ be a p-adic measure and suppose that

$$|\lambda(U)|_p \leq M$$

for all $U \in \mathfrak{B}_{\mathfrak{G}}$ and some $M \in \mathbb{R}$. By the density of $\mathrm{Step}(\mathfrak{G}, \mathbb{C}_p)$ in $C(\mathfrak{G}, \mathbb{C}_p)$, for any $f \in C(\mathfrak{G}, \mathbb{C}_p)$ one can find a sequence of step functions $\{s_n\} \subseteq \mathrm{Step}(\mathfrak{G}, \mathbb{C}_p)$ such that

$$f(x) = \lim_{n \to \infty} s_n(x).$$

Then it is easy to see that for any integers m and n,

$$\lambda(s_n - s_m) \leq M \| s_n - s_m \| .$$

Since the sequence $\{s_n\}$ is Cauchy, so is the sequence $\{\lambda(s_n)\}$ and it makes sense to define

$$\lambda(f) = \lim_{n \to \infty} \lambda(s_n).$$

The uniqueness is clear. □

For $\lambda \in \mathrm{Hom}_{\mathrm{cont}}(C(\mathfrak{G}, \mathbb{C}_p), \mathbb{C}_p)$, define

$$\| \lambda \| = \sup_{f \in C(\mathfrak{G}, \mathbb{C}_p)} \frac{|\lambda(f)|}{\| f \|},$$

which is a finite real number by the continuity of λ. Equipped with the convolution product, this set becomes a $\mathbb{C}_p$-Banach algebra.

In the case where $A = \mathbb{Z}_p$, recall that $\mathfrak{M}(\mathfrak{G}, \mathbb{Z}_p) = \mathfrak{D}(\mathfrak{G}, \mathbb{Z}_p)$.

Proposition 3. *The image of* $\mathfrak{M}(\mathfrak{G},\mathbb{Z}_p)$ *under the injection of the previous proposition is the set of*

$$\lambda \in \mathit{Hom}_{cont}(C(\mathfrak{G},\mathbb{C}_p),\mathbb{C}_p)$$

such that

$$\parallel \lambda \parallel \leq 1 \qquad \text{and} \qquad \lambda(C(\mathfrak{G},\mathbb{Q}_p)) \subseteq \mathbb{Q}_p.$$

Proof. Let $\lambda \in \mathfrak{M}(\mathfrak{G},\mathbb{Z}_p)$ and take

$$s \in \mathrm{Step}(\mathfrak{G},\mathbb{Q}_p).$$

Writing

$$s = \sum_{x\in\mathfrak{G}/\mathfrak{H}} s(x)\varepsilon_x,$$

we see that

$$\int_{\mathfrak{G}} s d\lambda = \sum_{x\in\mathfrak{G}/\mathfrak{H}} s(x)\lambda(x) \in \mathbb{Q}_p$$

and so

$$\left|\int_{\mathfrak{G}} s d\lambda\right|_p \leq \sup_{x\in\mathfrak{G}/\mathfrak{H}} |s(x)|_p|\lambda(x)|_p \leq \parallel s \parallel .$$

From the density of $\mathrm{Step}(\mathfrak{G},\mathbb{C}_p)$ in $C(\mathfrak{G},\mathbb{C}_p)$ and the continuity of the norm function, it follows that

$$\parallel \lambda \parallel \leq 1 \qquad \text{and} \qquad \lambda(C(\mathfrak{G},\mathbb{Q}_p)) \subseteq \mathbb{Q}_p.$$

Conversely, let

$$\lambda \in \mathrm{Hom}_{\mathrm{cont}}(C(\mathfrak{G},\mathbb{C}_p),\mathbb{C}_p)$$

be such that

$$\parallel \lambda \parallel \leq 1 \qquad \text{and} \qquad \lambda(C(\mathfrak{G},\mathbb{Q}_p)) \subseteq \mathbb{Q}_p.$$

Then

$$\lambda(\varepsilon_U) \in \mathbb{Q}_p$$

for any $U \in \mathfrak{B}_{\mathfrak{G}}$ since $\varepsilon_U \in C(\mathfrak{G},\mathbb{Q}_p)$. Moreover,

$$\parallel \varepsilon_U \parallel = 1,$$

and $\parallel \lambda \parallel \leq 1$, so in fact

$$\lambda(\varepsilon_U) \in \mathbb{Z}_p.$$

This concludes the proof. □

If $\rho : \mathfrak{G} \to \mathbb{C}_p^\times$ is a continuous character, i.e. a continuous group homomorphism, and $\lambda,\mu \in \mathfrak{M}(\mathfrak{G},\mathbb{C}_p)$ then

$$\int_{\mathfrak{G}} \rho(x)d(\lambda*\mu)(x) = \int_{\mathfrak{G}} \rho(x)d\lambda(x)\int_{\mathfrak{G}} \rho(x)d\mu(x).$$

A *pseudo-measure* is an element λ of the total ring of fractions of $\Lambda(\mathfrak{G})$, i.e. a quotient $\lambda = \mu/\nu$ of elements $\Lambda(\mathfrak{G})$ where ν is not a zero divisor, with the property that

$$(g-1)\lambda \in \Lambda(\mathfrak{G})$$

for all $g \in \mathfrak{G}$ (viewed as elements of $\Lambda(\mathfrak{G})$). For any such pseudo-measure λ and any non-trivial character ρ of $\mathfrak{G}$, define

$$\int_{\mathfrak{G}} \rho(x)d\lambda(x) \stackrel{\text{def}}{=} \frac{\int_{\mathfrak{G}} \rho(x)d((g-1)\lambda)(x)}{\int_{\mathfrak{G}} \rho(x)d(g-1)(x)} = \frac{\int_{\mathfrak{G}} \rho(x)d((g-1)\lambda)(x)}{\rho(g)-1},$$

where g is any element of $\mathfrak{G}$ not in the kernel of ρ. This definition does not depend on this choice of g. Note that we used the fact that for any $g \in \mathfrak{G}$,

$$\int_{\mathfrak{G}} f dg = f(g).$$

In other words, the elements of $\mathfrak{G}$ in $\Lambda(\mathfrak{G})$ correspond to Dirac distributionsq.

The Iwasawa algebra $\Lambda(\mathbb{Z}_p)$ and Mahler's transform

When $\mathfrak{G} = \mathbb{Z}_p$, one can say more about p-adic measures. This is because the $\mathbb{C}_p$-Banach algebra of continuous functions on $\mathbb{Z}_p$ has a special *Mahler basis*.

Let $e_0(x) = 1$ and define $e_n(x)$ for $n \in \mathbb{Z}_{>0}$ as

$$e_n(x) \stackrel{\text{def}}{=} \binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!}.$$

Theorem 1. *Let $f \in C(\mathbb{Z}_p, \mathbb{C}_p)$. Then there exists a unique sequence $\{a_n\}_{n\geq 0}$ of elements of $\mathbb{C}_p$ such that*

$$\lim_{n\to\infty} a_n = 0$$

and

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$$

for all x in $\mathbb{Z}_p$. This is called the Mahler expansion of f.

Proof. This is Theorem 3.3.1 in [CS]. □

Knowing that an element λ of $\Lambda(\mathbb{Z}_p)$ can be viewed as a continuous linear functional on $C(\mathbb{Z}_p, \mathbb{C}_p)$, one can form its generating function with respect to the Mahler basis:

$$\mathcal{M}(\lambda) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} T^n \int_{\mathbb{Z}_p} \binom{x}{n} d\lambda.$$

This is called the *Mahler transform* of λ. Note that

$$\mathcal{M}(\lambda) \in \mathbb{Z}_p[[T]].$$

Intuitively, the Mahler transform should determine λ (because the $e_n(x)$ form a basis of $C(\mathbb{Z}_p, \mathbb{C}_p)$). In fact, more is true.

Theorem 2. *The Mahler transform*

$$\mathcal{M} : \Lambda(\mathbb{Z}_p) \to R,$$

where

$$R \stackrel{\text{def}}{=} \mathbb{Z}_p[[T]],$$

is an isomorphism of $\mathbb{Z}_p$-algebras.

Proof. This is Theorem 3.3.3 in [CS]. □

The inverse of $\mathcal{M}$, denoted $\mathcal{Y}$ in [CS], is defined as follows: for a continuous function f with Mahler expansion

$$f(x) = \sum_{n=0}^{\infty} a_n e_n(x)$$

and for

$$g(T) = \sum_{n=0}^{\infty} b_n T^n \in \mathbb{Z}_p[[T]],$$

we define

$$\int_{\mathbb{Z}_p} f d\mathcal{Y}(g) = \sum_{n=0}^{\infty} a_n b_n.$$

Example: For any $a \in \mathbb{Z}_p$, viewed as a constant compatible sequence in $\Lambda(\mathbb{Z}_p)$, one has

$$\mathcal{M}(a) = (1+T)^a,$$

so that the power series $(1+T)^a$ correspond to the Dirac measures in $\Lambda(\mathbb{Z}_p)$.

The Iwasawa algebra $\Lambda(\mathbb{Z}_p^\times)$

Integration over $\mathfrak{G} = \mathbb{Z}_p^\times$ is closely related to integration over $\mathbb{Z}_p$. Since $\Lambda(\mathbb{Z}_p)$ has more structure, it is desirable to relate $\Lambda(\mathbb{Z}_p^\times)$ with $\Lambda(\mathbb{Z}_p)$. Since $\mathbb{Z}_p^\times$ is a subset of $\mathbb{Z}_p$, it is natural to define a map

$$\imath : \Lambda(\mathbb{Z}_p^\times) \to \Lambda(\mathbb{Z}_p)$$

as

$$\int_{\mathbb{Z}_p} f d\imath(\lambda) \overset{\text{def}}{=} \int_{\mathbb{Z}_p^\times} f|_{\mathbb{Z}_p^\times} d\lambda,$$

for all $f \in C(\mathbb{Z}_p, \mathbb{C}_p)$, where $f|_{\mathbb{Z}_p^\times} \in C(\mathbb{Z}_p^\times, \mathbb{C}_p)$ is the restriction of f to $\mathbb{Z}_p^\times$. One can check that this map is well-defined, i.e. that the functional

$$f \mapsto \int_{\mathbb{Z}_p} f d\imath(\lambda)$$

is in the image of $\Lambda(\mathbb{Z}_p)$ in $\mathrm{Hom}_{\mathrm{cont}}(C(\mathfrak{G}, \mathbb{C}_p), \mathbb{C}_p)$.

The next step is to identify the image of $\Lambda(\mathbb{Z}_p^\times)$ inside $\Lambda(\mathbb{Z}_p)$. This will be done in the next section, using the trace and restriction operators.

The Iwasawa algebras $\Lambda(\mathcal{G})$ and $\Lambda(G)$

Recall the following notation

$$\mathcal{F}_n = \mathbb{Q}(\mu_{p^{n+1}}) \qquad \text{and} \qquad F_n = \mathbb{Q}(\mu_{p^{n+1}})^+.$$

$$\mathcal{G} = \mathrm{Gal}(\mathcal{F}_\infty/\mathbb{Q}) \qquad \text{and} \qquad G = \mathrm{Gal}(F_\infty/\mathbb{Q}).$$

A generator (ζ_n) of the Tate module

$$T_p(\mu) = \varprojlim \mu_{p^{n+1}}$$

is by definition a sequence of roots of unity $\zeta_n \in \mu_{p^{n+1}}$ such that $\zeta_{n+1}^p = \zeta_n$. Fixing such a generator, we obtain an isomorphism

$$\chi : \mathcal{G} \to \mathbb{Z}_p^\times,$$

called the cyclotomic character. This indices an isomorphism

$$\tilde{\chi} : \Lambda(\mathcal{G}) \to \Lambda(\mathbb{Z}_p^\times).$$

But more is true. One can define a natural action of $\mathcal{G}$ on $\Lambda(\mathbb{Z}_p^\times)$ and $\Lambda(\mathbb{Z}_p)$ via the cyclotomic character. Then $\tilde{\chi}$ becomes a $\mathcal{G}$-isomorphism, i.e. $\tilde{\chi}$ is $\mathcal{G}$-equivariant.

For each $n \geq 0$, the CM field $\mathcal{F}_n$ has complex conjugation action ι_n and the fixed field of $\{1, \iota_n\}$ is F_n. This extends to a complex conjugation action ι in $\mathcal{G}$, so $\Lambda(\mathcal{G})$ is a $\mathbb{Z}_p[\mathcal{J}]$-module, where $\mathcal{J} = \{1, \iota\}$. For p odd this module decomposes naturally as

$$\Lambda(\mathcal{G}) = \Lambda(\mathcal{G})^+ \oplus \Lambda(\mathcal{G})^-,$$

where

$$\Lambda(\mathcal{G})^+ = \frac{1+\iota}{2}\Lambda(\mathcal{G}) \qquad \text{and} \qquad \Lambda(\mathcal{G})^- = \frac{1-\iota}{2}\Lambda(\mathcal{G}).$$

Finally, one has the following proposition.

Proposition 4. *The restriction to $\Lambda(\mathcal{G})^+$ of the natural surjection from $\Lambda(\mathcal{G})$ to $\Lambda(G)$ induces an isomorphism*

$$\Lambda(\mathcal{G})^+ \simeq \Lambda(G).$$

Proof. This is Lemma 4.2.1 of [CS]. □

1.2 Operators on p-adic measures

In [CS], the authors introduce a few operators in the ring $R = \mathbb{Z}_p[[T]]$. Since $\Lambda(\mathbb{Z}_p)$ is canonically isomorphic to this ring via the Mahler transform, those operators have a corresponding simple definition on the Iwasawa algebra. By combining those operators, one obtains the restriction operator, which plays an important role in the theory.

Operators on R

Let $g(T)$ be a power series in R and define the operator

$$\varphi : R \to R$$

as

$$\varphi(g)(T) = g((1+T)^p - 1).$$

This is well defined *injective* $\mathbb{Z}_p$-algebra endomorphism (see [CS, Lemma 2.2.2]).

Next, define the trace operator

$$\psi : R \to R$$

as

$$(\varphi \circ \psi)(g)(T) = \frac{1}{p} \sum_{\xi \in \mu_p} g(\xi(1+T) - 1).$$

This is a well-defined continuous $\mathbb{Z}_p$-linear endomorphism (see [CS, Proposition 2.2.3]).

The operators φ and ψ satisfy the relation

$$\psi \circ \varphi = 1_R.$$

Finally, one can introduce a derivation D on R as follows:

$$D(g)(T) \stackrel{\text{def}}{=} (1+T)g'(T) = (1+T)\frac{dg}{dT}.$$

It is enlightening to see how those operators act on power series. Suppose that $g(T)$ can be written as

$$g(T) \sum_{n=0}^{\infty} a_n (1+T)^n.$$

Then φ is simply given as

$$\varphi(g)(T) = \sum_{n=0}^{\infty} a_n (1+T)^{pn}.$$

As for ψ, a simple calculation shows that

$$\psi(g)(T) = \sum_{n=0}^{\infty} a_{np} (1+T)^n.$$

Moreover,

$$D(g)(T) = \sum_{n=0}^{\infty} n a_n (1+T)^n.$$

Thinking of $1+T$ as the parameter q at infinity on the modular curve $X(1)$, this suggests that the φ, ψ and D operators correspond formally to the V_p, U_p and $q\frac{d}{dq}$ operators on q-expansions. With that in mind, it is clear that $\psi \circ \phi$ is the identity on R.

Operators on $\Lambda(\mathbb{Z}_p)$

We now introduce the operators on p-adic measures which correspond under the Mahler transform to φ, ψ and D on R.

Let $\lambda \in \Lambda(\mathbb{Z}_p)$ be a p-adic measure on $\mathbb{Z}_p$. Then one can verify without difficulty that the $\mathbb{Z}_p$-algebra endomorphism

$$\varphi : \Lambda(\mathbb{Z}_p) \to \Lambda(\mathbb{Z}_p)$$

defined as

$$\int_{\mathbb{Z}_p} f(x) d\varphi(\lambda)(x) \stackrel{\text{def}}{=} \int_{\mathbb{Z}_p} f(px) d\lambda(x)$$

corresponds, via the Mahler transform, to the operator $\varphi : R \to R$ introduced above.

A similar calculation shows that the $\mathbb{Z}_p$-linear map

$$\psi : \Lambda(\mathbb{Z}_p) \to \Lambda(\mathbb{Z}_p)$$

defined as

$$\int_{\mathbb{Z}_p} f(x) d\psi(\lambda)(x) \stackrel{\text{def}}{=} \int_{\mathbb{Z}_p} \varepsilon_{p\mathbb{Z}_p}(x) f\left(\frac{x}{p}\right) d\lambda(x)$$

corresponds to the $\mathbb{Z}_p$-linear map $\psi : R \to R$ introduced above.

One can then see, directly or using the corresponding property on R, that

$$\psi \circ \varphi = 1_{\Lambda(\mathbb{Z}_p)}.$$

One also sees that $\varphi \circ \psi$ corresponds to "restriction to $p\mathbb{Z}_p$", since

$$\int_{\mathbb{Z}_p} f(x) d(\varphi \circ \psi)(\lambda)(x) = \int_{\mathbb{Z}_p} f(px) d\psi(\lambda)(x) = \int_{\mathbb{Z}_p} \varepsilon_{p\mathbb{Z}_p}(x) f(x) d\lambda(x).$$

Now let $f_0(x)$ be any continuous function on $\mathbb{Z}_p$ and define the measure $f_0\lambda$ as

$$\int_{\mathbb{Z}_p} f(x) d(f_0\lambda)(x) \stackrel{\text{def}}{=} \int_{\mathbb{Z}_p} f_0(x) f(x) d\lambda(x).$$

For $f_0(x) = x$, one has the relation

$$\mathcal{M}(x\lambda) = D(\mathcal{M}(\lambda)),$$

which follows formally from the identity

$$x\binom{x}{n} = (n+1)\binom{x}{n+1} + n\binom{x}{n}$$

(see the proof of Lemma 3.3.5 in [CS]). Therefore the D operator corresponds to the multiplication by x map on $\Lambda(\mathbb{Z}_p)$.

Restriction of measures from $\mathbb{Z}_p$ to $\mathbb{Z}_p^\times$

We now introduce the restriction operator. In particular, it will allow us to identify the image of $\Lambda(\mathbb{Z}_p^\times)$ inside $\Lambda(\mathbb{Z}_p)$.

Recall that the operator $\delta : R \to R$ is defined in section 3.4 of [CS] as

$$\delta(g)(T) = g(T) - \varphi \circ \psi(g)(T) = (1 - \varphi \circ \psi)(g)(T).$$

We define the restriction operator as

$$\mathrm{Res}_{\mathbb{Z}_p^\times} \stackrel{\text{def}}{=} 1 - \varphi \circ \psi.$$

It is not so clear why this operator on power series should be viewed as a restriction operator. However, on measures we have

$$\begin{aligned}
\int_{\mathbb{Z}_p} f(x) d\mathrm{Res}_{\mathbb{Z}_p^\times}(\lambda)(x) &= \int_{\mathbb{Z}_p} f(x) d\lambda(x) - \int_{\mathbb{Z}_p} f(x) d(\varphi \circ \psi)(\lambda)(x) \\
&= \int_{\mathbb{Z}_p} f(x) d\lambda(x) - \int_{\mathbb{Z}_p} \varepsilon_{p\mathbb{Z}_p}(x) f(x) d\lambda(x) \\
&= \int_{\mathbb{Z}_p} (1 - \varepsilon_{p\mathbb{Z}_p}(x)) f(x) d\lambda(x) \\
&= \int_{\mathbb{Z}_p} \varepsilon_{\mathbb{Z}_p^\times}(x) f(x) d\lambda(x).
\end{aligned}$$

Note that the operator $\mathrm{Res}_{\mathbb{Z}_p^\times}$ on measures is denoted $\#$ in [CS].

The operator $\mathrm{Res}_{\mathbb{Z}_p^\times}$ is a projection, i.e. $\mathrm{Res}_{\mathbb{Z}_p^\times} \circ \mathrm{Res}_{\mathbb{Z}_p^\times} = \mathrm{Res}_{\mathbb{Z}_p^\times}$. A formal computation shows that

$$g(T) \in \mathrm{Im}\mathrm{Res}_{\mathbb{Z}_p^\times} \Leftrightarrow \mathrm{Res}_{\mathbb{Z}_p^\times} g(T) = g(T) \Leftrightarrow \psi(g)(T) = 0 \Leftrightarrow g \in R^{\psi=0},$$

where

$$R^{\psi=0} = \{g \in R | \psi(g) = 0\}.$$

Proposition 5. *The image of $\Lambda(\mathbb{Z}_p^\times)$ in $\Lambda(\mathbb{Z}_p)$ under the injection $\imath$ is the image of the restriction operator $\mathrm{Res}_{\mathbb{Z}_p^\times}$.*

Proof. This follows from Lemma 3.4.1 and Lemma 3.4.2 in [CS]. □

This proposition proves that the restriction of p-adic measures on $\mathbb{Z}_p$ can be viewed as p-adic measures on $\mathbb{Z}_p^\times$. It also implies that the following diagram

$$\begin{tikzcd}
\Lambda(\mathbb{Z}_p) \arrow[r, "\mathcal{M}"] & R \\
\Lambda(\mathbb{Z}_p^\times) \arrow[u, "\imath"] \arrow[r, "\mathcal{M}\circ\imath"] & R^{\psi=0} \arrow[u]
\end{tikzcd}$$

is commutative.

Using the analogy between $\varphi \leftrightarrow V_p$ and $\psi \leftrightarrow U_p$ discussed above, we see that the restriction operator looks like the p-stabilisation operator on modular forms.

1.3 Moments of p-adic measures

The special values of the zeta function will be obtained by computing the moments of a pseudo-measure on $\Lambda(\mathcal{G})$. We collect here a few results that will help us compute those moments later.

First, it follows directly from the results of the previous section that

$$\int_{\mathbb{Z}_p} x^k d\lambda(x) = \int_{\mathbb{Z}_p} d(x^k\lambda)(x) = \int_{\mathbb{Z}_p} e_0(x) d(x^k\lambda)(x) = \mathcal{M}(x^k\lambda)(0)$$

and since

$$\mathcal{M}(x\lambda) = D\mathcal{M}(\lambda)$$

we have

$$\int_{\mathbb{Z}_p} x^k d\lambda(x) = D^k \mathcal{M}(\lambda)(0). \tag{1}$$

Second, one would like to have a relation between

$$\int_{\mathbb{Z}_p} x^k d\lambda(x) \qquad \text{and} \qquad \int_{\mathbb{Z}_p} x^k d(\mathrm{Res}_{\mathbb{Z}_p^\times}\lambda)(x).$$

To have a simple relation, *suppose* $\psi(\lambda) = \lambda$. We compute

$$\begin{aligned}
\int_{\mathbb{Z}_p} x^k d\mathrm{Res}_{\mathbb{Z}_p^\times}(\lambda)(x) &= \int_{\mathbb{Z}_p} x^k d(1-\varphi\circ\psi)(\lambda)(x) \\
&= \int_{\mathbb{Z}_p} x^k d(1-\varphi)(\lambda)(x) && \text{since } \psi(\lambda)=\lambda \\
&= \int_{\mathbb{Z}_p} x^k d\lambda(x) - \int_{\mathbb{Z}_p} x^k d\varphi(\lambda)(x) \\
&= \int_{\mathbb{Z}_p} x^k d\lambda(x) - \int_{\mathbb{Z}_p} (px)^k d\lambda(x) \\
&= (1-p^k)\int_{\mathbb{Z}_p} x^k d\lambda(x).
\end{aligned}$$

In brief,

$$\int_{\mathbb{Z}_p} x^k d\mathrm{Res}_{\mathbb{Z}_p^\times}(\lambda)(x) = (1-p^k)\int_{\mathbb{Z}_p} x^k d\lambda(x). \tag{2}$$

Note that this is consistent with our observation that the restriction operator can be thought of as a p-stabilisation operator, since multiplication by $1-p^k$ corresponds to p-stabilisation on L-functions (i.e. removing the euler factors at p).

Finally, moments determine measures on $\mathbb{Z}_p^\times$.

Proposition 6. *Let $\lambda \in \Lambda(\mathcal{G})$ be a measure. If*

$$\int_{\mathcal{G}} \chi^k(g)d\lambda(g) = 0 \qquad \textit{for } k = 1,3,5,\dots,$$

then $\lambda \in \Lambda(\mathcal{G})^+$. Similarly, if

$$\int_{\mathcal{G}} \chi^k(g)d\lambda(g) = 0 \qquad \textit{for } k = 2,4,6,\dots,$$

then $\lambda \in \Lambda(\mathcal{G})^-$. In particular,

$$\int_{\mathcal{G}} \chi^k(g)d\lambda(g) = 0 \qquad \textit{for all } k > 0,$$

then $\lambda = 0$. The same is true for pseudo-measures.

Proof. This is Lemma 4.4.2 and Corollary 4.2.3 of [CS]. □

2 p-adic measure attached to compatible systems of local units

As we know, the p-adic zeta function is associated with the cyclotomic units. Those units come in compatible systems, i.e. they are elements of

$$\mathcal{U}_\infty = \varprojlim \mathcal{U}_n,$$

where $\mathcal{U}_n$ is the group of local units in $\mathcal{K}_n = \mathbb{Q}_p(\mu_{p^{n+1}})$. The first step in building the pseudo-measure attached to zeta is to pass from units to power series via the Coleman power series. Then one uses the map

$$\mathcal{L} : W \to R^{\psi=0}$$

to get a power series which corresponds, under the inverse Mahler transform, to a measure on $\mathcal{G}$. As we will see, applying the map $\mathcal{L}$ is essentially like taking the $\log$ of the power series and then restricting it to $\mathbb{Z}_p^\times$.

2.1 The map $\tilde{\mathcal{L}} : \mathcal{U}_\infty \to \Lambda(\mathcal{G})$

We begin by introducing the Coleman power series of local units. First, recall that if (ζ_n) is a generator of the Tate module $T_p(\mu)$, then

$$\pi_n = \zeta_n - 1$$

is a uniformizer for $\mathcal{K}_n$.

Theorem 3. *For each $u = (u_n) \in \mathcal{U}_\infty$, there exists a unique power series $f_u(T) \in R$ such that $f_u(\pi_n) = u_n$ for all $n \geq 0$.*

Proof. This is Theorem 2.1.2 in [CS], which is proved in Chapter 2. □

Recall that one can define the norm operator

$$\mathcal{N} : R \to R$$

as

$$(\varphi \circ \mathcal{N})(g)(T) = \prod_{\xi \in \mu_p} g(\xi(1+T) - 1).$$

The image of $\mathcal{U}_\infty$ under the map $u \mapsto f_u$ of the Theorem is

$$W = \{g \in R^\times | \mathcal{N}(g) = g\}.$$

See [CS, Corollary 2.3.7]. This gives an isomorphism

$$\begin{tikzcd} \mathcal{U}_\infty \arrow[d, "\text{C.P.S.}"', "\wr"] \\ W \end{tikzcd}$$

which is also equivariant under the action of $\mathcal{G}$ on both sides (recall that $g \in \mathcal{G}$ acts on R by sending T to $(1+T)^{\chi(g)} - 1$).

The map

$$\mathcal{L} : W \to R^{\psi=0}$$

is defined as

$$\mathcal{L}(g)(T) = \frac{1}{p} \log \left(\frac{g(T)^p}{\varphi(g)(T)} \right)$$

in Lemma 2.5.1 of [CS].[1] One can think of this map as the restriction of the logarithm of power series in W. Indeed, we *formally* have

$$\begin{aligned}
\mathcal{L}(g)(T) &= \frac{1}{p} \log \left(\frac{g(T)^p}{\varphi(g)(T)} \right) && \text{(by definition)} \\
&= \log g(T) - \frac{1}{p} \log \varphi(g)(T) && \text{(formally)} \\
&= \log g(T) - \frac{1}{p} \log (\varphi \circ \mathcal{N})(g)(T) && \text{(since } g \in W\text{)} \\
&= \log g(T) - \frac{1}{p} \log \prod_{\xi \in \mu_p} g(\xi(1+T) - 1) && \text{(by definition of } \varphi \circ \mathcal{N}\text{)} \\
&= \log g(T) - \frac{1}{p} \sum_{\xi \in \mu_p} \log g(\xi(1+T) - 1) && \text{(formally)} \\
&= \log g(T) - (\varphi \circ \psi)(\log g(T)) && \text{(by definition of } \varphi \circ \psi\text{)} \\
&= (\mathrm{Res}_{\mathbb{Z}_p^\times} \log)(g)(T)) && \text{(by definition of } \mathrm{Res}_{\mathbb{Z}_p^\times}\text{)}
\end{aligned}$$

[1] Actually, the map is defined on $R^\times$, not just W, but the image of $\mathcal{L}$ lies in $R^{\psi=0}$.

so that we could define a map

$$\mathrm{Res}_{\mathbb{Z}_p^\times} \log \stackrel{\text{def}}{=} \mathcal{L}.$$

Note that this is just notation, since the logarithm map is not necessarily well-defined on all W. Note also that the fact that the image of $\mathcal{L}$ is contained in $R^{\psi=0}$ is consistent with the fact that the restriction operator takes R to $R^{\psi=0}$.

At this point, we have the following diagram of maps

$$\begin{tikzcd} \mathcal{U}_\infty \arrow[d, "\text{C.P.S.}"', "\wr"] & \\ W \arrow[r, "\mathrm{Res}_{\mathbb{Z}_p^\times} \log"] & R^{\psi=0} \end{tikzcd}$$

Using the isomorphism $\Lambda(\mathcal{G}) \stackrel{\tilde{\mathcal{M}}}{\to} R^{\psi=0}$ of the previous section, we can lift the map $\mathcal{L}$ to a map $\tilde{\mathcal{L}} : \mathcal{U}_\infty \to \Lambda(\mathcal{G})$, which we denote $\widetilde{\mathrm{Res}_{\mathbb{Z}_p^\times}} \log$:

$$\begin{tikzcd} \mathcal{U}_\infty \arrow[r, "\widetilde{\mathrm{Res}_{\mathbb{Z}_p^\times}} \log"] \arrow[d, "\text{C.P.S.}"', "\wr"] & \Lambda(\mathcal{G}) \arrow[d, "\tilde{\mathcal{M}}", "\wr"'] \\ W \arrow[r, "\mathrm{Res}_{\mathbb{Z}_p^\times} \log"] & R^{\psi=0} \end{tikzcd}$$

2.2 Moments of p-adic measures obtained via $\tilde{\mathcal{L}}$

The moments of the p-adic measures obtained via $\tilde{\mathcal{L}}$ are related to the so-called higher logarithm derivative map. More precisely, we prove Proposition 3.5.2 of [CS] in this section, i.e. that

$$\int_{\mathcal{G}} \chi(g)^k d\tilde{\mathcal{L}}(u) = \delta_k(u),$$

where

$$\delta_k(u) = \left(D^{k-1} \left((1+T) \frac{f_u'(T)}{f_u(T)} \right) \right)_{T=0}.$$

The map $\delta_k(u)$ is called the higher logarithmic derivative map.

First, recall that the map

$$\Delta(g)(T) = (1+T)\frac{g'(T)}{g(T)}$$

takes W to $R^{\psi=1} = \{g \in R | \psi(g) = g\}$ (this is Lemma 2.4.5 in [CS]). We define

$$D \log \stackrel{\text{def}}{=} \Delta.$$

Then, by applying the operator $1 - \varphi$ (denoted θ in [CS]), which is just the restriction operator since

$$1 - \varphi = 1 - \varphi \circ \psi = \mathrm{Res}_{\mathbb{Z}_p^\times}$$

on $R^{\psi=1}$, we fall in $\mathbb{R}^{\psi=0}$.

At this point, we have two maps from W to $R^{\psi=0}$, namely $\mathcal{L}$ (or $\mathrm{Res}_{\mathbb{Z}_p^\times} \log$) and $\theta \circ \Delta$ (or $\mathrm{Res}_{\mathbb{Z}_p^\times} \circ D \log$). In the proof Theorem 2.5.2 of [CS], we learn that the two maps are related in the following way

$$\begin{tikzcd} W \arrow[r, "\mathcal{L}"] \arrow[d, "\Delta"'] & R^{\psi=0} \arrow[d, "D"] \\ R^{\psi=1} \arrow[r, "\theta"] & R^{\psi=0} \end{tikzcd}.$$

Using our notation, this is just saying that the D and $\mathsf{Res}_{\mathbb{Z}_p^\times}$ operators commute:

$$\mathrm{D} \circ \mathsf{Res}_{\mathbb{Z}_p^\times} \log = \mathsf{Res}_{\mathbb{Z}_p^\times} \circ \mathrm{D} \log .$$

Altogether, we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{U}_\infty \arrow[r, "\widetilde{\mathsf{Res}_{\mathbb{Z}_p^\times} \log}"] \arrow[d, "\wr", "\text{C.P.S.}"'] & \Lambda(\mathcal{G}) \arrow[d, "\tilde{\mathcal{M}}", "\wr"'] \\
W \arrow[r, "\mathsf{Res}_{\mathbb{Z}_p^\times} \log"] \arrow[d, "\mathrm{D}\log"'] & R^{\psi=0} \arrow[d, "\mathrm{D}"] \\
R^{\psi=1} \arrow[r, "\mathsf{Res}_{\mathbb{Z}_p^\times}"] & R^{\psi=0}
\end{tikzcd}$$

Using this diagram, we now compute the moments of $\tilde{\mathcal{L}}(u)$:

$$\begin{aligned}
\int_{\mathcal{G}} \chi(g)^k d\tilde{\mathcal{L}}(u) &= \int_{\mathcal{G}} \chi(g)^k d\widetilde{\mathsf{Res}_{\mathbb{Z}_p^\times} \log}(u) && \text{(notation)} \\
&= \int_{\mathbb{Z}_p} x^k d\mathcal{Y}(\mathsf{Res}_{\mathbb{Z}_p^\times} \log f_u) && \text{(commutativity of top square)} \\
&= \int_{\mathbb{Z}_p} x^{k-1} dx\mathcal{Y}(\mathsf{Res}_{\mathbb{Z}_p^\times} \log f_u) \\
&= \int_{\mathbb{Z}_p} x^{k-1} d\mathcal{Y}(\mathrm{D} \circ \mathsf{Res}_{\mathbb{Z}_p^\times} \log f_u) && \text{(formula 1)} \\
&= \int_{\mathbb{Z}_p} x^{k-1} d\mathcal{Y}(\mathsf{Res}_{\mathbb{Z}_p^\times} \circ \mathrm{D} \log f_u) && \text{(commutativity of bottom square)} \\
&= \int_{\mathbb{Z}_p} x^{k-1} d\mathsf{Res}_{\mathbb{Z}_p^\times} \mathcal{Y}(\mathrm{D} \log f_u) \\
&= (1-p^{k-1}) \int_{\mathbb{Z}_p} x^{k-1} d\mathcal{Y}(\mathrm{D} \log f_u) && \text{(formula 2, since } \mathrm{D}\log f_u \in R^{\psi=1}\text{)} \\
&= (1-p^{k-1})(\mathrm{D}^{k-1}(\mathrm{D} \log f_u))_{T=0} && \text{(formula 1)} \\
&= (1-p^{k-1})\delta_k(u) && \text{(by definition of } \delta_k(u)\text{)}
\end{aligned}$$

2.3 Measures attached to generators of the Tate module $T_p(\mu)$

Any generator (ζ_n) of the Tate module $T_p(\mu)$ is a norm compatible sequence of local units, hence can be viewed as an element of $\mathcal{U}_\infty$. To see this, first recall that

$$\mathcal{K}_n = \mathbb{Q}_p(\mu_{p^{n+1}})$$

and that $\mathcal{U}_n$ is the set of units in $\mathcal{K}_n$ $(n \geq 0)$. We need to prove that

$$N_{n,n-1}(\zeta_n) = \zeta_{n-1},$$

where $N_{n,n-1} : \mathcal{K}_n \to \mathcal{K}_{n-1}$ is the norm map. But this follows from the fact that by definition ζ_n is a root of

$$X^p - \zeta_{n-1} \in \mathcal{K}_n[X],$$

which is irreducible since

$$(X+1)^p - \zeta_{n-1}$$

is Eisenstein at the prime π_{n-1}.

Can we obtain interesting measure from those elements? Unfortunately, no. Indeed, it is clear that the Coleman power series of $(\zeta_n) \in \mathcal{U}_\infty$ is simply

$$f_{(\zeta_n)}(T) = 1 + T,$$

so that

$$\tilde{\mathcal{L}}((\zeta_n)) = \mathcal{L}(1+T) = \frac{1}{p} \log \left(\frac{(1+T)^p}{(1+(1+T)^p - 1)} \right) = \frac{1}{p} \log 1 = 0.$$

This proves that the p-adic measure on $\mathcal{G}$ corresponding to (ζ_n) is the zero measure! In fact, one has the following *Fundamental Exact Sequence* of $\mathcal{G}$-modules

$$\begin{tikzcd} 0 \arrow[r] & \mu_{p-1} \times T_p(\mu) \arrow[r] & \mathcal{U}_\infty \arrow[r, "\tilde{\mathcal{L}}"] & \Lambda(\mathcal{G}) \arrow[r, "\beta"] & T_p(\mu) \arrow[r] & 0 \end{tikzcd},$$

where the map β sends λ to $(\zeta_n)^{\int_{\mathcal{G}} \chi d\lambda}$ (see [CS, Theorem 3.5.1]). Note this sequence is the main ingredient in the proof of Iwasawa's theorem (Theorem 4.4.1 in [CS]).

The proof that this sequence is exact essentially follows from the exactness of the sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_p \arrow[r] & R^{\psi=1} \arrow[r, "\theta = \mathrm{Res}_{\mathbb{Z}_p^\times}"] & R^{\psi=0} \arrow[r, "\mathrm{ev}_{T=0}"] & \mathbb{Z}_p \arrow[r] & 0 \end{tikzcd} \tag{3}$$

of Lemma 2.4.3 [CS]. Indeed, the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mu_{p-1} \arrow[r] & W \arrow[r, "D \log"] & R^{\psi=1} \arrow[r] & 0 \end{tikzcd},$$

which is obtained by combining Lemma 2.4.5 and Theorem 2.4.6 of [CS], allows us to lift the sequence 3 to obtain the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & W \arrow[r, "\mathrm{Res}_{\mathbb{Z}_p^\times} \log"] \arrow[d, "D \log"'] & R^{\psi=0} \arrow[r, "\alpha"] \arrow[d, "D"] & \mathbb{Z}_p \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{Z}_p \arrow[r] & R^{\psi=1} \arrow[r, "\mathrm{Res}_{\mathbb{Z}_p^\times}"] & R^{\psi=0} \arrow[r, "\mathrm{ev}_{T=0}"] & \mathbb{Z}_p \arrow[r] & 0, \end{tikzcd}$$

where

$$A = \{\xi(1+T)^a | \xi \in \mu_{p-1}, a \in \mathbb{Z}_p\}$$

and $\alpha(g) = (Dg)(0)$ (this is proved in [CS, Theorem 2.5.2]). The bottom row can simply be thought of as the additive version of the top row. Finally, using the isomorphisms

$$\begin{tikzcd} \mathcal{U}_\infty \arrow[d, "\mathrm{C.P.S.}"', "\wr"] & \Lambda(\mathcal{G}) \arrow[d, "\wr"', "\tilde{\mathcal{M}}"] \\ W & R^{\psi=0} \end{tikzcd}$$

one can further lift this sequence to obtain

$$\begin{tikzcd} 0 \arrow[r] & \mu_{p-1} \times T_p(\mu) \arrow[r] & \mathcal{U}_\infty \arrow[r, "\tilde{\mathcal{L}}"] \arrow[d, "\mathrm{C.P.S.}"', "\wr"] & \Lambda(\mathcal{G}) \arrow[r, "\beta"] \arrow[d, "\wr"', "\tilde{\mathcal{M}}"] & T_p(\mu) \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r] & W \arrow[r, "\mathrm{Res}_{\mathbb{Z}_p^\times} \log"] \arrow[d, "D \log"'] & R^{\psi=0} \arrow[r, "\alpha"] \arrow[d, "D"] & \mathbb{Z}_p \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{Z}_p \arrow[r] & R^{\psi=1} \arrow[r, "\mathrm{Res}_{\mathbb{Z}_p^\times}"] & R^{\psi=0} \arrow[r, "\mathrm{ev}_{T=0}"] & \mathbb{Z}_p \arrow[r] & 0. \end{tikzcd}$$

In brief, one needs to work a little bit harder to obtain non-trivial measures on $\mathcal{G}$. This is where the cyclotomic units come in.

3 p-zeta function via cyclotomic units

At this point, the construction of the p-adic zeta function is almost a formality! For integers a and b, define

$$c_n(a,b) \stackrel{\text{def}}{=} \frac{\zeta_n^{-a/2} - \zeta_n^{a/2}}{\zeta_n^{-b/2} - \zeta_n^{b/2}} = \zeta_n^{(a-b)/2} \frac{\zeta_n^{-a} - 1}{\zeta_n^{-b} - 1}.$$

If a and b are not divisible by p, $c_n(a,b)$ is a unit in $\mathcal{F}_n$, hence in $\mathcal{K}_n$ (this follows from [Mi, Proposition 6.2(c)]). The sequence

$$c(a,b) \stackrel{\text{def}}{=} (c_n(a,b))$$

is a norm compatible sequence of local units, i.e.

$$c(a,b) \in \mathcal{U}_\infty.$$

Those elements $c(a,b)$ are the key to defining the p-adic zeta function.

For integers a and b not divisible by p, let

$$\lambda(a,b) \stackrel{\text{def}}{=} \widetilde{\mathrm{Res}_{\mathbb{Z}_p^\times}} \log(c(a,b)) = \tilde{\mathcal{L}}(c(a,b)).$$

Then

$$\int_{\mathcal{G}} \chi(g)^k d\lambda(a,b) = (1-p^{k-1})\delta_k(c(a,b)).$$

The computation of $\delta_k(c(a,b))$ is done in [CS, Proposition 2.6.3] and gives

$$\delta_k(c(a,b)) = \begin{cases} 0 & \text{if } k = 1,3,5,\dots \\ (b^k - a^k)\zeta(1-k) & \text{if } k = 2,4,6,\dots \end{cases}.$$

The p-adic measure $\lambda(a,b)$ is almost the p-adic zeta function, except the it interpolates the p-stabilized values of ζ at the negative integers *times the factor* $b^k - a^k$. To cancel this factor, take a different from b and define

$$\theta(a,b) = \sigma_b - \sigma_a \in \Lambda(\mathcal{G})$$

such that

$$\chi(\theta(a,b)) = b^k - a^k \in \mathbb{Z}_p^\times.$$

Then

$$\tilde{\zeta}_p = \frac{\lambda(a,b)}{\theta(a,b)}$$

is a pseudo-measure on $\mathcal{G}$ which is independent of the choice of a and b and interpolates the critical values of ζ:

$$\int_{\mathcal{G}} \chi(g)^k d\tilde{\zeta}_p = \begin{cases} 0 & \text{if } k = 1,3,5,\dots \\ (1-p^{k-1})\zeta(1-k) & \text{if } k = 2,4,6,\dots \end{cases}$$

(see [CS, Proposition 4.2.4] for more details). Note that since pseudo-measures are determined by their moments, $\tilde{\zeta}_p$ is the unique pseudo-measure which interpolates the critical values of the Riemann zeta function.

Finally, since the odd moments of $\tilde{\zeta}_p$ are zero,

$$\tilde{\zeta}_p \in \Lambda(\mathcal{G})^+.$$

Letting ζ_p denote the image of $\tilde{\zeta}_p$ under the identification $\Lambda(\mathcal{G})^+ \simeq \Lambda(G)$, we have the following theorem.

Theorem 4. *There exists a unique pseudo-measure ζ_p on G such that*

$$\int_G \chi(g)^k d\zeta_p = (1-p^{k-1})\zeta(1-k)$$

for all even integers $k \geq 2$.

References

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[Mi] *Milne, J., Algebraic Number Theory (v3.04)*, 2012, Available at `www.jmilne.org/math`.