

# Petersson Inner Product of Binary Theta Series

A computational approach

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# Stark's idea

Coming...

# Mobius transformations

Let  $\mathcal{H}$  be the Poincaré upper-half plane. Recall that  $\mathrm{GL}_2(\mathbb{R})_+$  acts on  $\mathcal{H}$  via Mobius transformations :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

## Definition

Let  $N \geq 1$  and define the Hecke subgroup of level  $N$  as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$



# Level $N$ modular forms with characters

## Definition

*Let  $N \geq 1$  and  $k \geq 0$  be integers and let  $\chi$  be a Dirichlet character mod  $N$ . A modular form of weight  $k$ , level  $N$  and character  $\chi$  is a holomorphic function*

$$f : \mathcal{H} \longrightarrow \mathbb{C}$$

*such that*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

*for all  $z \in \mathcal{H}$  and all  $\gamma \in \Gamma_0(N)$ , which satisfies certain growth conditions at the cusps. The  $\mathbb{C}$ -vector-space of such modular forms is denoted*

$$M_k(\Gamma_0(N), \chi).$$



## $q$ -expansion of modular forms

Every modular form  $f$  has a Taylor (or Fourier) expansion at infinity, called its  $q$ -expansion :

$$f(z) = \sum_{n=0}^{\infty} a_n q^n,$$

where  $q = \exp(2\pi iz)$ . If

$$a_0(f) = 0,$$

(at all cusps)  $f$  is called a *cusp form*. The space of cusp forms is denoted

$$S_k(\Gamma_0(N), \chi).$$



## Example : weight $k$ Eisenstein series

Let  $k \geq 4$  be an even integer. Then the series

$$\sum_{m,n} \frac{1}{(mz + n)^k}$$

converges absolutely and defines a modular form in  $M_k(\mathrm{SL}_2(\mathbb{Z}))$ . After renormalization, the  $q$ -expansion of this Eisenstein series is

$$G_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$



## Important non-example : weight 2 Eisenstein series

In level 1, there are no modular forms of weight 2. However, one can still define the weight 2 Eisenstein series as

$$G_2(z) = \frac{1}{8\pi\mathfrak{I}(z)} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n)q^n.$$

It is an example of an *almost holomorphic* modular form of level 1 and weight 2.

# Finite dimensionality of spaces of modular forms

## Theorem

*The space  $M_k(\Gamma_0(N), \chi)$  is finite dimensional as a  $\mathbb{C}$ -vector-space.*

## Example

*In level  $N = 1$ , we have*

- $M_0(SL_2(\mathbb{Z})) = \mathbb{C}$ .
- $M_2(SL_2(\mathbb{Z})) = 0$ .
- $M_k(SL_2(\mathbb{Z})) = \mathbb{C}G_k$  for  $4 \leq k \leq 10$ .
- $M_{12}(SL_2(\mathbb{Z})) = \mathbb{C}G_{12} \oplus \mathbb{C}\Delta$ , where  $\Delta \in S_{12}(SL_2(\mathbb{Z}))$ .
- $\bigoplus_{k=0}^{\infty} M_k(SL_2(\mathbb{Z})) = \mathbb{C}[G_4, G_6]$ .



## Petersson inner product

Let  $f, g \in S_k(\Gamma_0(N), \chi)$  be two cusp forms. The Petersson inner product of  $f$  and  $g$  is defined as

$$\langle f, g \rangle = \int \int_{\Gamma_0(N) \backslash \mathcal{H}} f(x + iy) \overline{g(x + iy)} y^k d\mu,$$

where

$$d\mu = \frac{dx dy}{y^2}$$

is the  $SL_2(\mathbb{R})$ -invariant measure on  $\mathcal{H}$ . Note that the integral does not converge if neither  $f$  nor  $g$  is a cusp form.



# A half-integral weight theta series

Consider the function

$$\theta(z) = \sum_{x \in \mathbb{Z}} q^{x^2} = 1 + 2q + 2q^4 + O(q^5).$$

Then

$$\theta(\gamma z) = \epsilon(cz + d)^{1/2} \theta(z),$$

for all  $\gamma \in \Gamma_0(4)$  and some  $\epsilon_{c,d} \in \{\pm 1, \pm i\}$ .

# Theta series attached to ideals

Let  $K$  be an imaginary quadratic field of discriminant  $D < -4$  and let  $\mathcal{O}_K$  be its ring of integers. Fix an integer  $\ell \geq 0$ . To each integral ideal  $\mathfrak{a}$  of  $K$ , one can attach the following theta series :

$$\theta_{\mathfrak{a}}^{(2\ell)}(z) = \theta_{\mathfrak{a}}(z) = \sum_{x \in \mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})}.$$

# Basic properties of these theta series

1. We have

$$\theta_{\mathfrak{a}} = \sum_{x \in \mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

where  $\chi_D$  is the Kronecker symbol. If  $\ell \neq 0$ , then

$$\theta_{\mathfrak{a}} \in S_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

2. If  $\lambda \in K^\times$ , then

$$\theta_{\lambda\mathfrak{a}} = \lambda^{2\ell} \theta_{\mathfrak{a}}.$$

So there are essentially  $h_K$  theta series attached to  $K$ .

3. In general, the  $\theta_{\mathfrak{a}}$  are *not* newforms.



# Theta series attached to Hecke characters of $K$

Let  $I_K$  denote the group of fractionnal ideals of  $K$ . A Hecke character  $\psi$  of  $K$  of infinity type  $2\ell$  (and conductor 1) is a homomorphism

$$\psi : I_K \longrightarrow \mathbb{C}^\times$$

such that

$$\psi((\alpha)) = \alpha^{2\ell}, \quad \forall \alpha \in K^\times.$$

One can define

$$\theta_\psi = \sum_{\mathfrak{a} \subseteq \mathcal{O}_K} \psi(\mathfrak{a}) q^{N(\mathfrak{a})}.$$

# Basic properties of these theta series

1. We have

$$\theta_\psi \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

where  $\chi_D$  is the Kronecker symbol. If  $\psi^2 \neq 1$ , then

$$\theta_\psi \in S_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

2. The  $\theta_\psi$  are newforms.

3. We have the identities

$$\theta_\psi = \frac{1}{w_K} \sum_{[\alpha] \in \text{Cl}_K} \psi^{-1}(\alpha) \theta_\alpha \quad \text{and} \quad \theta_\alpha = \frac{w_K}{h_K} \sum_{\psi} \psi(\alpha) \theta_\psi.$$

## Some questions

- Can we efficiently compute the Petersson inner product of theta series (whenever it makes sense) ?
- Can we find explicit formulas for it ?
- Can we use those formulas/computations to study the arithmetic properties of those quantities ?
- What about the  $p$ -adic properties of these quantities ?



## Petersson norm of the $\theta_\psi$ (with $\ell > 0$ )

### Theorem

Let  $\psi$  be a Hecke character of  $K$  of infinity type  $2\ell$ , where  $\ell > 0$ .  
Then

$$\langle \theta_\psi, \theta_\psi \rangle = h_K(|D|/4)^\ell \sum_{[\alpha] \in Cl_K} \psi^2(\alpha) \partial^{2\ell-1} G_2(\alpha).$$

Here,

$$\partial f = \frac{1}{2\pi i} \frac{\partial f}{\partial z} - \frac{k}{4\pi \Im(z)} f$$

is the Shimura-Mass differential operator, which preserves the graded algebra of almost holomorphic modular forms.





# Petersson inner product of the theta series $\theta_a$

## Theorem

Let  $\mathfrak{a}$  and  $\mathfrak{b}$  be ideals of  $K$  and suppose  $\ell > 0$ . Then

$$\langle \theta_a, \theta_b \rangle = C_K^{(2\ell)} N(\mathfrak{b})^{2\ell} \sum_{\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^2 = \lambda_{\mathfrak{c}} \mathcal{O}_K} \lambda_{\mathfrak{c}}^{2\ell} \partial^{2\ell-1} G_2(\mathfrak{c}),$$

where

$$C_K^{(2\ell)} = 4(|D|/4)^\ell.$$

## A few direct consequences of the formula

### Corollary

For  $\ell > 0$ ,

$$\langle \theta_a, \theta_b \rangle = 0$$

whenever  $a$  and  $b$  are not in the same genus (i.e. the classes of  $a$  and  $b$  are distinct in the genus group  $Cl_K/Cl_K^2$ ).

### Corollary

For  $\ell > 0$ ,

$$\langle \theta_{ac}, \theta_{bc} \rangle = N(\mathfrak{bc})^{2\ell} \langle \theta_a, \theta_b \rangle.$$



## Arithmetic consequences

Let

$$\Omega_K = \frac{1}{\sqrt{4\pi|D|}} \left( \prod_{j=1}^{|D|-1} \Gamma\left(\frac{j}{|D|}\right)^{\chi_D(j)} \right)^{w_K/4h_K}$$

be the Chowla-Selberg period attached to  $K$ .

### Corollary

For  $\ell > 0$ , the complex numbers

$$\frac{\langle \theta_\psi, \theta_\psi \rangle}{\Omega_K^{4\ell}} \quad \text{and} \quad \frac{\langle \theta_a, \theta_b \rangle}{\Omega_K^{4\ell}}$$

are algebraic.



## The case $\ell = 0$

If  $\ell = 0$ , the modular form  $\theta_{\mathfrak{a}}$  is not a cusp form. But for  $\theta_{\psi}$ , we have the following

### Theorem

*Let  $\theta_{\psi}$  be a Hecke character of infinity type 0 and suppose that  $\psi^2 \neq 1$ . Then*

$$\langle \theta_{\psi}, \theta_{\psi} \rangle = -h_K \sum_{[\mathfrak{a}] \in Cl_K} \psi^2(\mathfrak{a}) \log(\mathfrak{I}(\tau_{\mathfrak{a}})^{1/2} |\eta(\tau_{\mathfrak{a}})|^2),$$

*where  $\tau_{\mathfrak{a}} \in \mathcal{H}$  is the complex root attached to  $\mathfrak{a}$  and*

$$\eta(z) = \exp(2\pi i/24) \prod_{n=1}^{\infty} (1 - q^n).$$



## Formally obtaining the case $\ell = 0$ from the case $\ell > 0$

Strictly speaking, the formula

$$\langle \theta_\psi, \theta_\psi \rangle = h_K(|D|/4)^\ell \sum_{[\mathfrak{a}] \in \text{Cl}_K} \psi^2(\mathfrak{a}) \partial^{2\ell-1} G_2(\mathfrak{a}).$$

does not make sense for  $\ell = 0$ , since the expression

$$\partial^{-1} G_2$$

is not well-defined. However, we observe that

$$\partial_0 \log(\Im(z)^{1/2} |\eta(z)|^2) = -G_2(z),$$

so

$$"\partial^{-1} G_2 = -\log(\Im(z)^{1/2} |\eta(z)|^2)"$$

and we *formally* obtain the case  $\ell = 0$  from the case  $\ell > 0$ .

## Computing $\partial^n G_2$

To compute

$$\partial^n G_2$$

we have the following formulas :

$$\partial G_2 = \frac{5}{6} G_4 - 2 G_2^2 \quad \partial G_4 = \frac{7}{10} G_6 - 8 G_2 G_4 \quad \partial G_6 = \frac{400}{7} G_4^2 - 12 G_2 G_6.$$

For example,

$$\partial^3 G_2 = -48 G_2^4 + 120 G_4 G_2^2 - 14 G_6 G_2 + 25 G_4^2.$$

# Class number 1

In this case,

$$\theta_{\mathcal{O}_K} = \theta_{\psi_0}$$

and we only need to compute

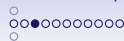
$$\langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell} \in \overline{\mathbb{Q}}.$$

# Class number 1 case

Computation of  $\langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell}$ :

		$\ell$	
		1	2
$D$	-7	$2^2 3$	$-2^2$
	-8	$-2$	$-2^2 5$
	-11	$-2^2$	$-2^3 5$
	-19	$-2^2 3^{-1} 13$	$-2^3 7 1$
	-43	$-2^3 3^{-1} 107$	$-2^4 5 6 4 7$
	-67	$-2^2 3^{-1} 7^2 31$	$-2^3 5 \cdot 86629$
	-163	$-2^3 3^{-1} 150473$	$-2^4 11 \cdot 461681471$





## Class number 2

In this case,  $K$  has two genera. If  $\mathfrak{a}$  is a representative of the non-trivial class in  $\text{Cl}_K$ , we have

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathcal{O}_K} \rangle = \langle \theta_{\mathcal{O}_K}, \theta_{\mathfrak{a}} \rangle = 0$$

and

$$\langle \theta_{\mathfrak{a}}, \theta_{\mathfrak{a}} \rangle = N(\mathfrak{a})^{2\ell} \langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle,$$

so it suffices to compute the quantity

$$\langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell} \in \overline{\mathbb{Q}}.$$



## Class number 2

As in the class number 1 case, the quantity

$$\langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell}$$

is an integer, except for  $\ell = 1$  and  $D = -91, -403$  and  $-427$ .



$$K = \mathbb{Q}(\sqrt{-23}) \text{ (class number 3, one genus)}$$

In  $K$ , the prime 2 splits as

$$2\mathcal{O}_K = \mathfrak{p}_2\bar{\mathfrak{p}}_2$$

and

$$\text{Cl}_K = \{1, [\mathfrak{p}_2], [\bar{\mathfrak{p}}_2]\}.$$

Moreover, we have  $\langle \theta_{\bar{\mathfrak{p}}_2}, \theta_{\mathcal{O}_K} \rangle = \overline{\langle \theta_{\mathfrak{p}_2}, \theta_{\mathcal{O}_K} \rangle}$ , so we only care about

$$\langle \theta_{\mathfrak{p}_2}, \theta_{\mathcal{O}_K} \rangle \quad \text{and} \quad \langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle.$$



$$K = \mathbb{Q}(\sqrt{-23}) \text{ (class number 3, one genus)}$$

Consider the algebraic number

$$a(\ell) = \langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell}.$$

For  $\ell = 1, 2$  and  $4$ , we find that  $a(\ell)^3$  is a root of a monic cubic polynomial and generates the Hilbert class field over  $K$ .

### Example

$a(1)$  is a root of the polynomial

$$x^9 - 2816x^6 - 905216x^3 - 89915392.$$



$$K = \mathbb{Q}(\sqrt{-23}) \text{ (class number 3, one genus)}$$

Consider the algebraic number

$$a(\ell) = \langle \theta_{\mathcal{O}_K}, \theta_{\mathcal{O}_K} \rangle / \Omega_K^{4\ell}.$$

For  $\ell = 3, 6$  and  $9$ , we find that  $a(\ell)$  is a root of a cubic polynomial and generates the Hilbert class field over  $K$ .

### Example

$a(3)$  is a root of

$$x^3 - 6740x^2 - 169034720x - 1027491892288.$$



$$K = \mathbb{Q}(\sqrt{-23}) \text{ (class number 3, one genus)}$$

A few computations of the Gramm matrix for this basis.

$\ell$	$\det(\langle \theta_{a_i}^{(2\ell)}, \theta_{a_j}^{(2\ell)} \rangle)_{a_i, a_j \in \text{Cl}_K} / (\Omega_K^{4\ell})^3$
1	$-2^{10} 23$
2	$-2^{14} 19 \cdot 23 \cdot 619$
3	$-2^{18} 5^2 11 \cdot 23 \cdot 337 \cdot 27299$
4	$-2^{22} 7^2 23 \cdot 163 \cdot 2113 \cdot 117741979$
5	$-2^{26} 5^3 23 \cdot 229 \cdot 23761 \cdot 808991 \cdot 20338663$
6	$-2^{30} 5^2 11^2 13 \cdot 19 \cdot 23 \cdot 67^2 101 \cdot 868697 \cdot 505912247899$

$$K = \mathbb{Q}(\sqrt{-23}) \text{ (class number 3, one genus)}$$

Consider now the algebraic number

$$N(\psi, \ell) = \langle \theta_\psi, \theta_\psi \rangle / \Omega_K^{4\ell}$$

For  $\ell = 1, 2, 4$  and  $5$ , the numbers  $N(\psi_i, \ell)$ , for  $0 \leq i \leq 2$ , are distinct and their cube are the three real roots of a monic cubic polynomial.

### Example

*The numbers  $N(\psi_i, 1)^3$ , for  $0 \leq i \leq 2$ , are the three roots of the irreducible polynomial*

$$x^3 - 6966x^2 + 11569230x - 239483061.$$

$$K = \mathbb{Q}(\sqrt{-23}) \text{ (class number 3, one genus)}$$

Consider now the algebraic number

$$N(\psi, \ell) = \langle \theta_\psi, \theta_\psi \rangle / \Omega_K^{4\ell}$$

For  $\ell = 3, 6$  and  $9$ , one of the characters, say  $\psi_0$ , the algebraic number  $N(\psi_0, \ell)$  is an *integer*. For the two others, we find that their cube are the roots of a monic quadratic polynomial.

### Example

We have

$$N(\psi_0, 3) = 5055 = 3 \cdot 5 \cdot 337$$

and  $N(\psi_1, 3)^3$  and  $N(\psi_2, 3)^3$  are the roots of

$$x^2 - 16287872873193x + 30021979248651078296845875.$$





$$K = \mathbb{Q}(\sqrt{-23}) \text{ (class number 3, one genus)}$$

A few computations of the Gramm matrix for this basis.

$\ell$	$\det(\langle \theta_{\psi_i}, \theta_{\psi_j} \rangle)_{1 \leq i, j \leq 3} / (\Omega_K^{4\ell})^3$
1	$-3^3 23$
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# Main steps in the proof (case $\ell > 0$ )

1. Use Rankin-Selberg to prove that

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{4h_k}{w_k} \sqrt{|D|} \frac{\Gamma(2\ell + 1)}{(4\pi)^{2\ell+1}} L(\psi^2, 2\ell + 1).$$

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2. Relate Hecke L-series to non-holomorphic Eisenstein series :

$$L(\psi^2, 2\ell + 1) = \frac{1}{w_K} \sum_{[\mathfrak{a}] \in \text{Cl}_K} \frac{\psi^2(\mathfrak{a})}{N(\mathfrak{a})^{4\ell-s}} G_{4\ell}(\mathfrak{a}, 1 - 2\ell).$$



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3. Replace non-holomorphic Eisenstein series by derivatives of Eisenstein series :

$$\partial^{2\ell-1} G_2(z) = (-4\pi)^{1-2\ell} \frac{\Gamma(s + 2\ell + 1)}{\Gamma(s + 2)} G_{4\ell}(z, 1 - 2\ell).$$

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4. Find  $\langle \theta_a, \theta_b \rangle$  using  $\langle \theta_\psi, \theta_\psi \rangle$ .



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## What we would like to know

1. Can we say something about the Petersson inner product of non-cuspidal weight one theta series ?
2. Can we explain what can be observed from the computations ?
3. What are the  $p$ -adic properties of those quantities as  $\ell$  varies ? In particular, does the case  $\ell > 0$  tend to the case  $\ell = 0$   $p$ -adically ?

# Thank you !

**Code available at :**

<https://github.com/NicolasSimard/ENT>

**Notes available at :**

<https://github.com/NicolasSimard/Notes>