

# Constructing the p-adic zeta function via cyclotomic units

Nicolas Simard

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## Introduction

### 1 p-adic measures

In this section, we first define p-adic measures and see how they are related to Iwasawa Algebras and power series rings. We then introduce operators on them and conclude with a few results on moments of measures.

#### 1.1 p-adic measures, distributions and Iwasawa algebras

Let  $\mathcal{G}$  be an abelian profinite group, let  $\mathfrak{B}_{\mathcal{G}}$  be the boolean algebra of open subsets of  $\mathcal{G}$ , let  $\mathfrak{T}_{\mathcal{G}} \subseteq \mathfrak{B}_{\mathcal{G}}$  be the set of open subgroups of  $\mathcal{G}$  and let  $A$  be any abelian group.

**Definition 1.** An  $A$ -valued distribution  $\lambda$  on  $\mathcal{G}$  is a finitely additive function

$$\lambda : \mathfrak{T}_{\mathcal{G}} \rightarrow A.$$

The set of distributions is denoted  $\mathcal{D}(\mathcal{G}, A)$ . If  $A \subseteq \mathbb{C}_p$ , the elements of  $\mathcal{D}(\mathcal{G}, A)$  are called p-adic distributions.

The set  $\mathcal{D}(\mathcal{G}, A)$  is naturally an abelian group. If  $A$  is a  $B$ -algebra for some ring  $B$ , the set  $\mathcal{D}(\mathcal{G}, A)$  is a  $B$ -algebra under convolution product, which we won't bother to define here!

### 1.1.1 Distributions and Iwasawa algebras

If  $\mathfrak{G}$  is finite,  $\mathfrak{B}_{\mathfrak{G}} = \{\{g\} | g \in \mathfrak{G}\}$  and we have an isomorphism of abelian groups

$$\lambda \mapsto \sum_{g \in \mathfrak{G}} \lambda(\{g\})g : \mathfrak{D}(\mathfrak{G}, A) \rightarrow A[\mathfrak{G}].$$

If  $A$  is a  $B$ -algebra for some ring  $B$ , so is  $A[\mathfrak{G}]$  and the isomorphism is an isomorphism of  $B$ -algebras. For  $\mathfrak{G}$  finite, we define

$$\Lambda(\mathfrak{G}, A) := A[\mathfrak{G}]$$

and call it an Iwasawa algebra.

For  $\mathfrak{G}$  not necessarily finite, we define

$$\Lambda(\mathfrak{G}, A) = \varprojlim \Lambda(\mathfrak{G}/\mathfrak{H}, A) = \varprojlim A[\mathfrak{G}/\mathfrak{H}],$$

where the limit is taken over all elements of  $\mathfrak{T}_{\mathfrak{G}}$ . Given  $\mathfrak{H}$  in  $\mathfrak{T}_{\mathfrak{G}}$ , we have a natural map

$$\lambda \mapsto \sum_{g \bmod \mathfrak{H}} \lambda(g)(g + \mathfrak{H}) : \mathfrak{D}(\mathfrak{G}, A) \rightarrow \Lambda(\mathfrak{G}/\mathfrak{H}, A).$$

Since distributions are finitely additive, we have then a natural map

$$\mathfrak{D}(\mathfrak{G}, A) \rightarrow \Lambda(\mathfrak{G}, A),$$

which is in fact an isomorphism. In a certain sense, the elements of the Iwasawa algebra  $\Lambda(\mathfrak{G}, A)$  are like the generating series of distributions.

**Example:** If  $A = \mathbb{Z}_p$ , one obtains the usual Iwasawa algebra

$$\Lambda(\mathfrak{G}) := \Lambda(\mathfrak{G}, \mathbb{Z}_p).$$

### 1.1.2 Distributions and step functions

From now on, suppose that  $A$  is a  $B$ -algebra for some ring  $B$ .

Recall that if  $s : \mathfrak{G} \rightarrow A$  is a locally constant function, also called a step function, there exists an open subgroup  $\mathfrak{H}$  such that  $s$  is well defined on  $\mathfrak{G}/\mathfrak{H}$ , i.e.

$$s(x) = \sum_{g \in \mathfrak{G}/\mathfrak{H}} s(g)\varepsilon_g(x),$$

where  $\varepsilon_g(x)$  is the characteristic function of the (open) coset  $g \in \mathfrak{G}/\mathfrak{H}$ . Note that such a representation of  $s$  as a finite linear combination of characteristic functions is not unique. The set of step functions from  $\mathfrak{G}$  to  $A$  is denoted

$$\text{Step}(\mathfrak{G}, A).$$

If  $A$  is a  $B$ -algebra,  $\text{Step}(\mathfrak{G}, A)$  is a  $B$ -algebra under the pointwise addition and multiplication of functions.

Let  $\lambda$  be an  $A$ -valued distribution on  $\mathfrak{G}$ , let  $s$  be a step function which is well-defined  $\bmod \mathfrak{H}$  as above and define

$$\int_{\mathfrak{G}} s d\lambda := \sum_{g \in \mathfrak{G}/\mathfrak{H}} s(g)\lambda(g).$$

This gives a well-defined  $B$ -linear map

$$\Lambda(\mathfrak{G}, A) \rightarrow \text{Hom}_{B\text{-alg}}(\text{Step}(\mathfrak{G}, A), A).$$

For convenience, the value of an element  $\lambda \in \text{Hom}_{B\text{-alg}}(\text{Step}(\mathfrak{G}, A), A)$  at a step function  $s(x)$  is denoted

$$\int_{\mathfrak{G}} s(x) d\lambda(x)$$

or simply

$$\int_{\mathfrak{G}} s d\lambda$$

when there is no risk of confusion. The  $B$ -module

$$\text{Hom}_{B\text{-alg}}(\text{Step}(\mathfrak{G}, A), A)$$

can be equipped with a natural  $B$ -algebra structure via the convolution product which is defined as follows: for  $\lambda, \mu \in \text{Hom}_{B\text{-alg}}(\text{Step}(\mathfrak{G}, A), A)$ , let  $\lambda * \mu$  be defined as

$$\int_{\mathfrak{G}} f d\lambda * \mu = \int_{\mathfrak{G}} (f(x+y) d\lambda(x)) d\mu(y).$$

The map above is then a  $B$ -algebra homomorphism. In fact, it is an isomorphism. Indeed, its inverse takes a homomorphism  $\lambda \in \text{Hom}_{B\text{-alg}}(\text{Step}(\mathfrak{G}, A), A)$  to the distribution  $\lambda$  defined as

$$\lambda(U) = g(\varepsilon_U).$$

This sketches the proof of the following proposition.

**Proposition 1.** *There is a natural  $B$ -algebra isomorphism*

$$\Lambda(\mathfrak{G}, A) \rightarrow \text{Hom}_{B\text{-alg}}(\text{Step}(\mathfrak{G}, A), A).$$

### 1.1.3 $p$ -adic measures and continuous functions

From now on, suppose that  $A$  is contained in  $\mathbb{C}_p$  (e.g.  $A = B = \mathbb{Z}_p$ ). Let

$$C(\mathfrak{G}, \mathbb{C}_p)$$

be the set of continuous functions from  $\mathfrak{G}$  to  $\mathbb{C}_p$ . This is a  $\mathbb{C}_p$ -Banach when equipped with the norm

$$\|f\| = \sup_{x \in \mathfrak{G}} |f(x)|_p.$$

The set  $\text{Step}(\mathfrak{G}, \mathbb{C}_p)$  is dense in  $C(\mathfrak{G}, \mathbb{C}_p)$ .

**Definition 2.** A  $p$ -adic distribution  $\lambda \in \mathfrak{D}(\mathfrak{G}, A)$  is called a  $p$ -adic measure if it is bounded (as a function from  $\mathfrak{T}_{\mathfrak{G}}$  to  $A \subseteq \mathbb{C}_p$ ). The set of  $p$ -adic measures is denoted  $\mathfrak{M}(\mathfrak{G}, A)$ .

Note that if  $A$  is bounded, which is the case if  $A = \mathbb{Z}_p$  for example, then  $\mathfrak{M}(\mathfrak{G}, A) = \mathfrak{D}(\mathfrak{G}, A)$ . The importance of introducing measures is the following proposition.

**Proposition 2.** *Let  $\lambda \in \mathfrak{M}(\mathfrak{G}, \mathbb{C}_p)$  be viewed as an element of*

$$\text{Hom}_{B\text{-alg}}(\text{Step}(\mathfrak{G}, \mathbb{C}_p), \mathbb{C}_p)$$

*via the above isomorphism. Then  $\lambda$  extends uniquely to a continuous map*

$$\lambda : C(\mathfrak{G}, \mathbb{C}_p) \rightarrow \mathbb{C}_p.$$

*Proof.* Let  $\lambda$  be a  $p$ -adic measure and suppose that

$$|\lambda(U)|_p \leq M$$

for all  $U \in \mathfrak{B}_{\mathfrak{G}}$  and some  $M \in \mathbb{R}$ . By the density of  $\text{Step}(\mathfrak{G}, \mathbb{C}_p)$  in  $C(\mathfrak{G}, \mathbb{C}_p)$ , for any  $f \in C(\mathfrak{G}, \mathbb{C}_p)$  one can find a sequence of step functions  $s_n \in \text{Step}(\mathfrak{G}, \mathbb{C}_p)$  such that

$$f(x) = \lim_{n \rightarrow \infty} s_n(x).$$

Then it is easy to see that for any integers  $m$  and  $n$ ,

$$\lambda(s_n - s_m) \leq M \|s_n - s_m\|.$$

Since the sequence  $\{s_n\}$  is Cauchy, so is the sequence  $\{\lambda(s_n)\}$  and it makes sense to define

$$\lambda(f) = \lim_{n \rightarrow \infty} \lambda(s_n).$$

The uniqueness is clear. □

For  $\lambda \in \text{Hom}_{\text{cont}}(C(\mathfrak{G}, \mathbb{C}_p), \mathbb{C}_p)$ , define

$$\|\lambda\| = \sup_{f \in C(\mathfrak{G}, \mathbb{C}_p)} \frac{|\lambda(f)|}{\|f\|},$$

which is a finite real number by the continuity of  $\lambda$ . Equipped with the convolution product, this set becomes a  $\mathbb{C}_p$ -Banach algebra.

In the case where  $A = \mathbb{Z}_p$ , we have  $\mathfrak{M}(\mathfrak{G}, \mathbb{Z}_p) = \mathfrak{D}(\mathfrak{G}, \mathbb{Z}_p)$  and we have the following proposition.

**Proposition 3.** *The image of  $\mathfrak{M}(\mathfrak{G}, \mathbb{Z}_p)$  under the injection of the previous proposition is the set of*

$$\lambda \in \text{Hom}_{\text{cont}}(C(\mathfrak{G}, \mathbb{C}_p), \mathbb{C}_p)$$

such that

$$\|\lambda\| \leq 1 \quad \text{and} \quad \lambda(C(\mathfrak{G}, \mathbb{Q}_p)) \subseteq \mathbb{Q}_p.$$

*Proof.* Let  $\lambda \in \mathfrak{M}(\mathfrak{G}, \mathbb{Z}_p)$  and take

$$s \in \text{Step}(\mathfrak{G}, \mathbb{Q}_p).$$

Writing

$$s = \sum_{g \in \mathfrak{G}/\text{Hfrak}} s(g) \varepsilon_g,$$

we see that

$$\int_{\mathfrak{G}} s d\lambda = \sum_{g \in \mathfrak{G}/\text{Hfrak}} s(g) \lambda(g) \in \mathbb{Q}_p$$

and so

$$\left\| \int_{\mathfrak{G}} s d\lambda \right\|_p \leq \sup_{g \in \mathfrak{G}/\text{Hfrak}} |s(g)|_p |\lambda(g)|_p \leq \|s\|.$$

From the density of  $\text{Step}(\mathfrak{G}, \mathbb{C}_p)$  in  $C(\mathfrak{G}, \mathbb{C}_p)$  and the continuity of the norm function, it follows that

$$\|\lambda\| \leq 1 \quad \lambda(C(\mathfrak{G}, \mathbb{Q}_p)) \subseteq \mathbb{Q}_p.$$

Conversely, let

$$\lambda \in \text{Hom}_{\text{cont}}(C(\mathfrak{G}, \mathbb{C}_p), \mathbb{C}_p)$$

be such that

$$\|\lambda\| \leq 1 \quad \text{and} \quad \lambda(C(\mathfrak{G}, \mathbb{Q}_p)) \subseteq \mathbb{Q}_p.$$

Then

$$\lambda(\varepsilon_U) \in \mathbb{Q}_p$$

for any  $U \in \mathfrak{B}_{\mathfrak{G}}$  since  $\varepsilon_U \in C(\mathfrak{G}, \mathbb{Q}_p)$ . Moreover,

$$\|\varepsilon_U\| = 1,$$

and  $\|\lambda\| \leq 1$ , so in fact

$$\lambda(\varepsilon_U) \in \mathbb{Z}_p.$$

This concludes the proof. □

If  $\rho : \mathfrak{G} \rightarrow \mathbb{C}_p^\times$  is a character, i.e. a group homomorphism, and  $\lambda, \mu \in \mathfrak{M}(\mathfrak{G}, \mathbb{C}_p)$  then

$$\int_{\mathfrak{G}} \rho d\lambda * \mu = \int_{\mathfrak{G}} \rho d\lambda \int_{\mathfrak{G}} \rho d\mu.$$

A *pseudo-measure* is an element  $\lambda$  of the total ring of fractions of  $\Lambda(\mathfrak{G})$ , i.e. a quotient  $\lambda = \mu/\nu$  of elements  $\Lambda(\mathfrak{G}, \mathbb{Z}_p)$  where  $\nu$  is not a zero divisor, with the property that

$$(g - 1)\lambda \in \Lambda(\mathfrak{G}).$$

For any pseudo-measure  $\lambda$  as above and any non-trivial character  $\rho$  of  $\mathfrak{G}$ , define

$$\int_{\mathfrak{G}} \rho d\lambda := \frac{\int_{\mathfrak{G}} \rho d\mu}{\int_{\mathfrak{G}} \rho d(g - 1)} = \frac{\int_{\mathfrak{G}} \rho d\mu}{\rho(g) - 1},$$

where  $g$  is any element of  $\mathfrak{G}$  not in the kernel of  $\rho$ . This definition does not depend on this choice of  $g$ . Note that we used the fact that for any  $g \in \mathfrak{G}$ ,

$$\int_{\mathfrak{G}} f dg = f(g).$$

In other words, the elements of  $\mathfrak{G}$  correspond to Dirac measures.

#### 1.1.4 The Iwasawa algebra $\Lambda(\mathbb{Z}_p)$ and Mahler's transform

When  $\mathfrak{G} = \mathbb{Z}_p$ , one can say more about  $p$ -adic measures. This is because the  $\mathbb{C}_p$ -Banach algebra of continuous functions on  $\mathbb{Z}_p$  has a special *Mahler basis*.

For  $n \in \mathbb{Z}_{\geq 0}$ , define

$$e_n(x) := \binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!}.$$

**Theorem 1.** *Let  $f \in C(\mathbb{Z}_p, \mathbb{C}_p)$ . Then there exists a unique sequence  $\{a_n\}_{n \geq 0}$  of elements of  $\mathbb{C}_p$  such that*

$$\lim_{n \rightarrow \infty} a_n = 0$$

and

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}.$$

*This is called the Mahler expansion of  $f$ .*

*Proof.* This is Theorem 3.3.1 in [CS]. □

Knowing that elements of  $\Lambda(\mathbb{Z}_p)$  can be viewed as continuous linear functional on  $C(\mathbb{Z}_p, \mathbb{C}_p)$ , one could form their generating function with respect to the Mahler basis:

$$\mathcal{M}(\lambda) := \sum_{n=0}^{\infty} T^n \int_{\mathbb{Z}_p} \binom{x}{n} d\lambda.$$

This is called the *Mahler transform* of  $\lambda$ . Note that

$$\mathcal{M}(\lambda) \in \mathbb{Z}_p[[T]].$$

Intuitively, the Mahler transform should determine  $\lambda$  (because the  $e_n(x)$  form a basis of  $C(\mathbb{Z}_p, \mathbb{C}_p)$ ). In fact, much more is true.

**Theorem 2.** *The Mahler transform*

$$\mathcal{M} : \Lambda(\mathbb{Z}_p) \rightarrow \mathbb{Z}_p[[T]]$$

*is an isomorphism of  $\mathbb{Z}_p$ -algebras.*

*Proof.* This is Theorem 3.3.3 in [CS]. □

The inverse of  $\mathcal{M}$ , denoted  $\mathcal{Y}$  in [CS], is defined as follows. If a continuous function  $f$  has Mahler expansion

$$f(x) = \sum_{n=0}^{\infty} a_n e_n(x)$$

and

$$g(T) = \sum_{n=0}^{\infty} b_n T^n \in \mathbb{Z}_p[[T]],$$

we define

$$\int_{\mathbb{Z}_p} f d\mathcal{Y}(g) = \sum_{n=0}^{\infty} a_n b_n.$$

For convenience, we sometimes denote  $\mathcal{Y}(g)$  by  $\lambda_g$ .

**Example:** For any  $a \in \mathbb{Z}_p$ , viewed as a constant compatible sequence in  $\Lambda(\mathbb{Z}_p)$ , one has

$$\mathcal{M}(a) = (1 + T)^a,$$

so that the power series  $(1 + T)^a$  corresponds to the Dirac measures in  $\Lambda(\mathbb{Z}_p)$ .

### 1.1.5 The Iwasawa algebra $\Lambda(\mathbb{Z}_p^\times)$

Integration over  $\mathfrak{G} = \mathbb{Z}_p^\times$  is closely related to integration over  $\mathbb{Z}_p$ . Since  $\Lambda(\mathbb{Z}_p)$  has more structure, it is desirable to relate  $\Lambda(\mathbb{Z}_p^\times)$  to  $\Lambda(\mathbb{Z}_p)$ . Since  $\mathbb{Z}_p^\times$  is a subset of  $\mathbb{Z}_p$ , it is natural to define a map

$$\iota : \Lambda(\mathbb{Z}_p^\times) \rightarrow \Lambda(\mathbb{Z}_p)$$

as

$$\int_{\mathbb{Z}_p} f d\iota(\lambda) := \int_{\mathbb{Z}_p^\times} f|_{\mathbb{Z}_p^\times} d\lambda,$$

for all  $f \in C(\mathbb{Z}_p, \mathbb{C}_p)$ , where  $f|_{\mathbb{Z}_p^\times} \in C(\mathbb{Z}_p^\times, \mathbb{C}_p)$  is the restriction of  $f$  to  $\mathbb{Z}_p^\times$ . One can check that this map is well-defined, i.e. that the functional

$$f \mapsto \int_{\mathbb{Z}_p} f d\iota(\lambda)$$

is in the image of  $\Lambda(\mathbb{Z}_p)$  in  $\lambda \in \text{Hom}_{\text{cont}}(C(\mathfrak{G}, \mathbb{C}_p), \mathbb{C}_p)$ .

The next step is to identify the image of  $\Lambda(\mathbb{Z}_p^\times)$  inside  $\Lambda(\mathbb{Z}_p)$ . This will be done in the next section, after we introduce the trace and restriction operators.

### 1.1.6 The Iwasawa algebras $\Lambda(\mathcal{G})$ and $\Lambda(G)$

Recall the following notation

$$\mathcal{F}_n = \mathbb{Q}(\mu_{p^{n+1}}) \quad \text{and} \quad F_n = \mathbb{Q}(\mu_{p^{n+1}})^+.$$

$$\mathcal{G} = \text{Gal}(\mathcal{F}_\infty/\mathbb{Q}) \quad \text{and} \quad G = \text{Gal}(F_\infty/\mathbb{Q}).$$

A generator  $(\zeta_n)$  for the Tate module  $T_p(\mu)$  is a sequence of  $\zeta_n \in \mu_{p^{n+1}}$  such that  $\zeta_{n+1}^p = \zeta_n$ . Fixing such a generator, we obtain an isomorphism

$$\chi : \mathcal{G} \rightarrow \mathbb{Z}_p^\times,$$

called the cyclotomic character. This induces an isomorphism

$$\tilde{\chi} : \Lambda(\mathcal{G}) \rightarrow \Lambda(\mathbb{Z}_p^\times).$$

But more is true. One can define a natural action of  $\mathcal{G}$  on  $\Lambda(\mathbb{Z}_p^\times)$  and  $\Lambda(\mathbb{Z}_p)$  via the cyclotomic character. Then  $\tilde{\chi}$  becomes a  $\mathcal{G}$ -isomorphism, i.e.  $\tilde{\chi}(\mathcal{G})$  is  $\mathcal{G}$ -equivariant.

For each  $n \geq 0$ , the CM field  $\mathcal{F}_n$  has complex conjugation action  $\iota_n$  and the fixed field of  $\{1, \iota_n\}$  is  $F_n$ . This extends to a complex conjugation action  $\iota$  in  $\mathcal{G}$  which fixes  $G$ . This makes  $\Lambda(\mathcal{G})$  into a  $\mathbb{Z}_p[\mathcal{J}]$ -module, where  $\mathcal{J} = \{1, \iota\}$ . For  $p$  odd decomposes naturally as

$$\Lambda(\mathcal{G}) = \Lambda(\mathcal{G})^+ \oplus \Lambda(\mathcal{G})^-,$$

where

$$\Lambda(\mathcal{G})^+ = \frac{1+\iota}{2}\Lambda(\mathcal{G}) \quad \text{and} \quad \Lambda(\mathcal{G})^- = \frac{1-\iota}{2}\Lambda(\mathcal{G}).$$

Finally, one has the following proposition.

**Proposition 4.** *The restriction to  $\Lambda(\mathcal{G})^+$  of the natural surjection from  $\Lambda(\mathcal{G})$  to  $\Lambda(G)$  induces an isomorphism*

$$\Lambda(\mathcal{G})^+ \simeq \Lambda(G).$$

*Proof.* This is Lemma 4.2.1 of [CS]. □

## 1.2 Operators on $p$ -adic measures

In [CS], the authors introduce a few operators in the ring  $R = \mathbb{Z}_p[[T]]$ . Since  $\Lambda(\mathbb{Z}_p)$  is canonically isomorphic to this ring via the Mahler transform, those operators have a corresponding simple definition on the Iwasawa algebra. By combining those operators, one obtains the restriction operator, which has a natural interpretation.

### 1.2.1 Operators on $R$

Let  $g(T)$  be a power series in  $R$  and define the operator

$$\varphi : R \rightarrow R$$

as

$$\varphi(g)(T) = g((1+T)^p - 1).$$

This is well defined *injective*  $\mathbb{Z}_p$ -algebra endomorphism (see [CS, Lemma 2.2.2]).

One can then define trace operator

$$\psi : R \rightarrow R$$

as

$$(\varphi \circ \psi)(g)(T) = \frac{1}{p} \sum_{\xi \in \mu_p} g(\xi(1+T) - 1).$$

This is a well-defined continuous  $\mathbb{Z}_p$ -linear map from  $R$  to itself (see [CS, Proposition 2.2.3]). Moreover,

$$\psi \circ \varphi = 1_R.$$

Finally, one can introduce a derivation  $D$  on  $R$  as follows:

$$D(g)(T) = (1+T) \frac{dg}{dT}.$$

It is enlightening to interpret those operators in a different way. Suppose that  $g(T)$  can be written as

$$g(T) = \sum_{n=0}^{\infty} a_n (1+T)^n.$$

Then  $\varphi$  is simply given as

$$\varphi(g)(T) = \sum_{n=0}^{\infty} a_n (1+T)^{pn}.$$

As for  $\psi$ , as simple calculation shows that

$$\psi(g)(T) = \sum_{n=0}^{\infty} a_{np}(1+T)^n.$$

Moreover,

$$D(g)(T) = \sum_{n=0}^{\infty} na_n(1+T)^n.$$

Letting  $q = 1 + T$ , this proves that the  $\varphi$ ,  $\psi$  and  $D$  operators correspond formally to the  $V_p$ ,  $U_p$  and  $q \frac{d}{dq}$  operators on  $q$ -expansions of level 1 modular forms. With that in mind, it is clear that  $\psi \circ \phi$  is the identity on  $R$ .

### 1.2.2 Operators on $\Lambda(\mathbb{Z}_p)$

We now introduce the operators on  $p$ -adic measures, i.e. elements of the Iwasawa algebra  $\Lambda(\mathbb{Z}_p)$ , which correspond to  $\varphi$ ,  $\psi$  and  $D$  on  $R$ .

Let  $\lambda \in \Lambda(\mathbb{Z}_p)$  be a  $p$ -adic measure on  $\mathbb{Z}_p$ . Then one can verify without difficulty that the  $\mathbb{Z}_p$ -algebra endomorphism

$$\varphi : \Lambda(\mathbb{Z}_p) \rightarrow \Lambda(\mathbb{Z}_p)$$

defined as

$$\int_{\mathbb{Z}_p} f(x) d\varphi(\lambda)(x) := \int_{\mathbb{Z}_p} f(px) d\lambda(x)$$

corresponds, via the Mahler transform, to the operator  $\varphi : R \rightarrow R$  introduced above.

A similar calculation shows that the  $\mathbb{Z}_p$ -linear map

$$\psi : \Lambda(\mathbb{Z}_p) \rightarrow \Lambda(\mathbb{Z}_p)$$

defined as

$$\int_{\mathbb{Z}_p} f(x) d\psi(\lambda)(x) := \int_{\mathbb{Z}_p} \varepsilon_{p\mathbb{Z}_p}(x) f\left(\frac{x}{p}\right) d\lambda(x)$$

corresponds to the  $\mathbb{Z}_p$ -linear map  $\psi : R \rightarrow R$  introduced above.

One can then see, directly or using the corresponding property on  $R$ , that

$$\psi \circ \varphi = 1_{\Lambda(\mathbb{Z}_p)}.$$

One also sees that  $\varphi \circ \psi$  corresponds to "restriction on  $p\mathbb{Z}_p$ ", since

$$\int_{\mathbb{Z}_p} f(x) d(\varphi \circ \psi)(\lambda)(x) = \int_{\mathbb{Z}_p} f(px) d\psi(\lambda)(x) = \int_{\mathbb{Z}_p} \varepsilon_{p\mathbb{Z}_p}(x) f(x) d\lambda(x).$$

Now let  $f_0(x)$  be any continuous function on  $\mathbb{Z}_p$  and define the measure  $f_0\lambda$  as

$$\int_{\mathbb{Z}_p} f(x) d(f_0\lambda)(x) = \int_{\mathbb{Z}_p} f_0(x) f(x) d\lambda(x).$$

For  $f_0(x) = x$ , one has the relation

$$\mathcal{M}(x\lambda) = D(\mathcal{M}(\lambda)),$$

which follows formally from the identity

$$x \binom{x}{n} = (n+1) \binom{x}{n+1} + n \binom{x}{n}$$

(see the proof of Lemma 3.3.5 in [CS]). Therefore the  $D$  operator corresponds to the multiplication by  $x$  map on  $\Lambda(\mathbb{Z}_p)$ .



### 1.2.3 Restriction of measures from $\mathbb{Z}_p$ to $\mathbb{Z}_p^\times$

We now introduce the restriction operator. In particular, this will allow us to identify the image of  $\Lambda(\mathbb{Z}_p^\times)$  inside  $\Lambda(\mathbb{Z}_p)$ .

Recall that the operator  $\delta : \mathcal{R} \rightarrow \mathcal{R}$  is defined in section 3.4 of [CS] as

$$\delta(g)(T) = g(T) - \varphi \circ \psi(g)(T) = (1 - \varphi \circ \psi)(g)(T).$$

We define the restriction operator as

$$\text{Res}_{\mathbb{Z}_p^\times} := 1 - \varphi \circ \psi.$$

It is not so clear why this operator on power series should be viewed as a restriction operator. However, on measures we have

$$\begin{aligned} \int_{\mathbb{Z}_p} f(x) d\text{Res}_{\mathbb{Z}_p^\times}(\lambda)(x) &= \int_{\mathbb{Z}_p} f(x) d\lambda(x) - \int_{\mathbb{Z}_p} f(x) d(\varphi \circ \psi)(\lambda)(x) \\ &= \int_{\mathbb{Z}_p} f(x) d\lambda(x) - \int_{\mathbb{Z}_p} \varepsilon_{p\mathbb{Z}_p}(x) f(x) d\lambda(x) \\ &= \int_{\mathbb{Z}_p} (1 - \varepsilon_{p\mathbb{Z}_p}(x)) f(x) d\lambda(x) \\ &= \int_{\mathbb{Z}_p} \varepsilon_{\mathbb{Z}_p^\times}(x) f(x) d\lambda(x) \end{aligned}$$

which justifies the notation. Note that the operator  $\text{Res}_{\mathbb{Z}_p^\times}$  on measures is denoted  $\#$  in [CS].

The operator  $\text{Res}_{\mathbb{Z}_p^\times}$  is a projection, i.e.  $\text{Res}_{\mathbb{Z}_p^\times} \circ \text{Res}_{\mathbb{Z}_p^\times} = \text{Res}_{\mathbb{Z}_p^\times}$ . A formal computation shows that

$$\text{Res}_{\mathbb{Z}_p^\times} g(T) = g(T) \Leftrightarrow \psi(g)(T) = 0 \Leftrightarrow g \in \mathcal{R}^{\psi=0},$$

where

$$\mathcal{R}^{\psi=0} = \{g \in \mathcal{R} \mid \psi(g) = 0\}.$$

**Proposition 5.** *The image of  $\Lambda(\mathbb{Z}_p^\times)$  in  $\Lambda(\mathbb{Z}_p)$  under the injection  $\iota$  is the set of measures fixed by the  $\text{Res}_{\mathbb{Z}_p^\times}$  operator.*

*Proof.* This follows from Lemma 3.4.1 and Lemma 3.4.2 in [CS]. □

This proposition means that the restriction of  $p$ -adic measures on  $\mathbb{Z}_p$  can be viewed as  $p$ -adic measures on  $\mathbb{Z}_p^\times$  times. It also implies that the following diagram

$$\begin{array}{ccc} \Lambda(\mathbb{Z}_p) & \xrightarrow{\mathcal{M}} & \mathcal{R} \\ \uparrow \iota & & \uparrow \\ \Lambda(\mathbb{Z}_p^\times) & \xrightarrow{\mathcal{M} \circ \iota} & \mathcal{R}^{\psi=0} \end{array}$$

is commutative.

Using the analogy between  $\varphi \leftrightarrow V_p$  and  $\psi \leftrightarrow U_p$  discussed above, we see that the restriction operator looks like the  $p$ -stabilisation operator on modular forms.

## 1.3 Moments of $p$ -adic measures

The special values of the zeta function will be obtained by computing the moments of a pseudo-measure on  $\Lambda(\mathcal{G})$ . We collect here a few results that help us compute those moments later.

First, it follows directly from the results of the previous section that

$$\int_{\mathbb{Z}_p} x^k d\lambda(x) = \int_{\mathbb{Z}_p} d((x^k \lambda)(x)) = \mathcal{M}(x^k \lambda)(0)$$

and since

$$\mathcal{M}(x\lambda) = D\mathcal{M}(\lambda)$$

we have

$$\int_{\mathbb{Z}_p} x^k d\lambda(x) = D^k \mathcal{M}(\lambda)(0). \quad (1)$$

Second, one would like to have a relation between

$$\int_{\mathbb{Z}_p} x^k d\lambda(x) \quad \text{and} \quad \int_{\mathbb{Z}_p} x^k d(\text{Res}_{\mathbb{Z}_p^\times} \lambda)(x).$$

To have a simple relation, *suppose*  $\psi(\lambda) = \lambda$ . We compute

$$\begin{aligned} \int_{\mathbb{Z}_p} x^k d\text{Res}_{\mathbb{Z}_p^\times}(\lambda)(x) &= \int_{\mathbb{Z}_p} x^k d(1 - \varphi \circ \psi)(\lambda)(x) \\ &= \int_{\mathbb{Z}_p} x^k d(1 - \varphi)(\lambda)(x) && \text{since } \psi(\lambda) = \lambda \\ &= \int_{\mathbb{Z}_p} x^k d\lambda(x) - \int_{\mathbb{Z}_p} x^k d\varphi(\lambda)(x) \\ &= \int_{\mathbb{Z}_p} x^k d\lambda(x) - \int_{\mathbb{Z}_p} (px)^k d\psi(\lambda)(x) \\ &= (1 - p^k) \int_{\mathbb{Z}_p} x^k d\lambda(x). \end{aligned}$$

In brief,

$$\int_{\mathbb{Z}_p} x^k d\text{Res}_{\mathbb{Z}_p^\times}(\lambda)(x) = (1 - p^k) \int_{\mathbb{Z}_p} x^k d\lambda(x). \quad (2)$$

Note that this is consistent with our observation that the restriction operator can be thought of as a  $p$ -stabilisation operator, since multiplication by  $1 - p^k$  corresponds to  $p$ -stabilisation on  $L$ -functions (i.e. removing the euler factors at  $p$ ).

Finally, moments determine measures on  $\mathbb{Z}_p^\times$ .

**Proposition 6.** *Let  $\lambda \in \Lambda(\mathcal{G})$  be a measure. If*

$$\int_{\mathcal{G}} x^k d\lambda(x) \quad \text{for } k = 1, 3, 5, \dots,$$

*then  $\lambda \in \Lambda(\mathcal{G})^+$ . Similarly, if*

$$\int_{\mathcal{G}} x^k d\lambda(x) \quad \text{for } k = 2, 4, 6, \dots,$$

*then  $\lambda \in \Lambda(\mathcal{G})^-$ . In particular,*

$$\int_{\mathcal{G}} x^k d\lambda(x) \quad \text{for all } k > 0,$$

*then  $\lambda = 0$ . Similar statements are true for pseudo-measures.*

*Proof.* This is Lemma 4.4.2 and Corollary 4.2.3 of [CS]. □

## 1.4 Summary

1. For a general abelian group, we have

$$\mathfrak{D}(\mathfrak{G}, A) \simeq \Lambda(\mathfrak{G}, A)$$

canonically as abelian groups. When  $A$  is a  $B$ -algebra, this is an isomorphism of  $B$ -algebras.

2. When  $A$  is a  $B$ -algebra, we have

$$\Lambda(\mathfrak{G}, A) \simeq \text{Hom}_{B\text{-alg}}(\text{Step}(\mathfrak{G}, A), A),$$

canonically as  $B$ -algebras, where the product on the right set is the convolution product.

3. When  $A$  is contained in  $\mathbb{C}_p$ , any  $p$ -adic measure  $\lambda$  extends uniquely to a continuous map

$$\lambda : C(\mathfrak{G}, \mathbb{C}_p) \rightarrow \mathbb{C}_p.$$

4. When  $A = \mathbb{Z}_p$ , the image of  $\mathfrak{M}(\mathfrak{G}, \mathbb{Z}_p)$  under the above injection is the set of

$$\lambda \in \text{Hom}_{\text{cont}}(C(\mathfrak{G}, \mathbb{C}_p), \mathbb{C}_p)$$

such that

$$\|\lambda\| \leq 1 \quad \text{and} \quad \lambda(C(\mathfrak{G}, \mathbb{Q}_p)) \subseteq \mathbb{Q}_p.$$

5. When  $A = \mathbb{Z}_p$  and  $\mathfrak{G} = \mathbb{Z}_p$ , the Mahler transform  $\mathcal{M}$  establishes a  $\mathbb{Z}_p$ -algebra isomorphism

$$\mathcal{M} : \Lambda(\mathbb{Z}_p) \rightarrow \mathbb{Z}_p[[T]].$$

6. When  $A = \mathbb{Z}_p$  and  $\mathfrak{G} = \mathbb{Z}_p^\times$ , we have a natural injection

$$\iota : \Lambda(\mathbb{Z}_p^\times) \rightarrow \Lambda(\mathbb{Z}_p).$$

7. When  $A = \mathbb{Z}_p$  and  $\mathfrak{G} = \mathcal{G}$ , we have a  $\mathcal{G}$ -isomorphism  $\Lambda(\mathcal{G}) \approx \Lambda(\mathbb{Z}_p^\times)$ . Moreover,  $\Lambda(\mathcal{G})$  can be canonically identified as a  $\mathbb{Z}_p$ -submodule of  $\Lambda(\mathcal{G})$ .

Here are all the maps we introduced for  $\mathbb{Z}_p$ -valued measures:

$$\Lambda(\mathcal{G}) \simeq \Lambda(\mathcal{G})^+ \hookrightarrow \Lambda(\mathcal{G}) \xrightarrow{\tilde{\chi}} \Lambda(\mathbb{Z}_p^\times) \xrightarrow{\iota} \Lambda(\mathbb{Z}_p) \simeq \mathfrak{D}(\mathbb{Z}_p, \mathbb{Z}_p) = \mathfrak{M}(\mathbb{Z}_p, \mathbb{Z}_p) \hookrightarrow \text{Hom}_{\text{cont}}(C(\mathfrak{G}, \mathbb{C}_p), \mathbb{C}_p).$$

## 2 $p$ -adic measure attached to compatible systems of local units

As we know, the  $p$ -adic zeta function is associated with the cyclotomic units. Those units come in compatible systems, i.e. they are elements of

$$\mathcal{U}_\infty = \varprojlim \mathcal{U}_n,$$

where  $\mathcal{U}_n$  is the group of local units in  $\mathcal{K}_n = \mathbb{Q}_p(\mu_{p^{n+1}})$ . The first step in building this pseudo-measure is to pass from units to power series via the Coleman power series. Then one uses the map

$$\mathcal{L} : W \rightarrow R^{\Psi=0}$$

to get a power series which corresponds, under the inverse Mahler transform, to a measure on  $\mathcal{G}$ . As we will see, applying the map  $\mathcal{L}$  is essentially like taking the log and then restricting it.

## 2.1 The map $\tilde{\mathcal{L}} : \mathcal{U}_\infty \rightarrow \Lambda(\mathcal{G})$

The main technical tool to pass from units to p-adic measures is the Coleman power series attached to compatible systems of units.

**Theorem 3.** *For each  $u = (u_n) \in \mathcal{U}_\infty$ , there exists a unique power series  $f_u(T) \in R$  such that  $f_u(\pi_n) = u_n$ , where*

$$\pi_n = \zeta_n - 1$$

*for some generator  $(\zeta_n)$  of the Tate module  $T_p(\mu)$ .*

*Proof.* This is Theorem 2.1.2 in [CS], which is proved in Chapter 2. □

Recall that one can define a norm operator  $\mathcal{N} : R \rightarrow R$  as

$$(\varphi \circ \mathcal{N})(g)(T) = \prod_{\xi \in \mu_p} g(\xi(1+T) - 1).$$

The image of  $\mathcal{U}_\infty$  under the map  $u \mapsto f_u$  of the Theorem is

$$W = \{g \in R^\times \mid \mathcal{N}(g) = g\}.$$

See [CS, Corollary 2.3.7]. This gives an isomorphism

$$\begin{array}{c} \mathcal{U}_\infty \\ \downarrow \text{C.P.S.} \wr \\ W \end{array}$$

which also respects the action of  $\mathcal{G}$  on both sides (recall that  $g \in \mathcal{G}$  acts on  $R$  by sending  $T$  to  $(1+T)^{x(g)} - 1$ ).

The map

$$\mathcal{L} : W \rightarrow R^{\Psi=0}$$

is defined as

$$\mathcal{L}(g)(T) = \frac{1}{p} \log \left( \frac{g(T)^p}{\varphi(g)(T)} \right)$$

in Lemma 2.5.1.<sup>1</sup> One can think of this map as the restriction of the logarithm of power series in  $W$ . Indeed, we *formally* have

$$\begin{aligned} \mathcal{L}(g)(T) &= \frac{1}{p} \log \left( \frac{g(T)^p}{\varphi(g)(T)} \right) && \text{(by definition)} \\ &= \log g(T) - \frac{1}{p} \log \varphi(g)(T) && \text{(formally)} \\ &= \log g(T) - \frac{1}{p} \log (\varphi \circ \mathcal{N})(g)(T) && \text{(since } g \in W) \\ &= \log g(T) - \frac{1}{p} \log \prod_{\xi \in \mu_p} g(\xi(1+T) - 1) && \text{(by definition of } \varphi \circ \mathcal{N}) \\ &= \log g(T) - \frac{1}{p} \sum_{\xi \in \mu_p} \log g(\xi(1+T) - 1) && \text{(formally)} \\ &= \log g(T) - (\varphi \circ \psi)(\log g(T)) && \text{(by definition of } \varphi \circ \psi) \\ &= (\text{Res}_{\mathbb{Z}_p^\times} \log)(g)(T) && \text{(by definition of } \text{Res}_{\mathbb{Z}_p^\times}) \end{aligned}$$

so that we could define a map

$$\text{Res}_{\mathbb{Z}_p^\times} \log := \mathcal{L}.$$

---

<sup>1</sup>Actually, the map is defined on  $R^\times$ , not just  $W$ , but the image of  $\mathcal{L}$  lies in  $R^{\Psi=0}$ .

Note that this is just notation, since the logarithm map is not necessarily well-defined on all  $W$ .

At this point, we have the following diagram of maps

$$\begin{array}{ccc} \mathcal{U}_\infty & & \\ \text{C.P.S.} \downarrow \wr & \text{Res}_{\mathbb{Z}_p^\times} \log & \\ W & \xrightarrow{\quad} & \mathbb{R}^{\Psi=0} \end{array}$$

Using the isomorphism  $\Lambda(\mathcal{G}) \simeq \mathbb{R}^{\Psi=0}$  of the previous section, we can lift the map  $\mathcal{L}$  to a map  $\tilde{\mathcal{L}} : \mathcal{U}_\infty \rightarrow \Lambda(\mathcal{G})$ , which we denote  $\widetilde{\text{Res}_{\mathbb{Z}_p^\times} \log}$ :

$$\begin{array}{ccc} \mathcal{U}_\infty & \xrightarrow{\widetilde{\text{Res}_{\mathbb{Z}_p^\times} \log}} & \Lambda(\mathcal{G}) \\ \text{C.P.S.} \downarrow \wr & & \downarrow \tilde{\mathcal{M}} \\ W & \xrightarrow{\text{Res}_{\mathbb{Z}_p^\times} \log} & \mathbb{R}^{\Psi=0} \end{array}$$

## 2.2 Moments of p-adic measures obtained via $\tilde{\mathcal{L}}$

The moments of the p-adic measures obtained via  $\tilde{\mathcal{L}}$  are related to the so-called higher logarithm derivative map. More precisely, we prove Proposition 3.5.2 of [CS] in this section, i.e. that

$$\int_{\mathcal{G}} \chi(g)^k d\tilde{\mathcal{L}}(u) = \delta_k(u),$$

where

$$\delta_k(u) = \left( D^{k-1} \left( (1+T) \frac{f'_u(T)}{f_u(T)} \right) \right)_{T=0}.$$

The map  $\delta_k(u)$  is called the higher logarithmic derivative map.

First, recall that the map

$$\Delta(g)(T) = (1+T) \frac{g'(T)}{g(T)}$$

takes  $W$  to  $\mathbb{R}^{\Psi=1} = \{g \in \mathbb{R} \mid \psi(g) = g\}$  (this is Lemma 2.4.5 in [CS]). Applying the operator  $1 - \varphi$  (denoted  $\theta$  in [CS]), which is just the restriction operator, since

$$1 - \varphi = 1 - \varphi \circ \psi = \text{Res}_{\mathbb{Z}_p^\times}$$

on  $\mathbb{R}^{\Psi=1}$ , we fall in  $\mathbb{R}^{\Psi=0}$ . In the proof Theorem 2.5.2 of [CS], we learn that the diagram

$$\begin{array}{ccc} W & \xrightarrow{\mathcal{L}} & \mathbb{R}^{\Psi=0} \\ \Delta \downarrow & & \downarrow D \\ \mathbb{R}^{\Psi=1} & \xrightarrow{\theta} & \mathbb{R}^{\Psi=0} \end{array}$$

Using our notation, this is just saying that the  $D$  and  $\text{Res}_{\mathbb{Z}_p^\times}$  operators commute:

$$D \circ \text{Res}_{\mathbb{Z}_p^\times} \log = \text{Res}_{\mathbb{Z}_p^\times} \circ D \log.$$

Altogether, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{U}_\infty & \xrightarrow{\widetilde{\text{Res}_{\mathbb{Z}_p^\times} \log}} & \Lambda(\mathcal{G}) \\ \text{C.P.S.} \downarrow \wr & & \downarrow \tilde{\mathcal{M}} \\ W & \xrightarrow{\text{Res}_{\mathbb{Z}_p^\times} \log} & \mathbb{R}^{\Psi=0} \\ D \log \downarrow & & \downarrow D \\ \mathbb{R}^{\Psi=1} & \xrightarrow{\text{Res}_{\mathbb{Z}_p^\times}} & \mathbb{R}^{\Psi=0} \end{array}$$

Using this diagram, we now compute the moments:

$$\begin{aligned}
\int_{\mathcal{G}} \chi(g)^k d\tilde{\mathcal{L}}(u) &= \int_{\mathcal{G}} \chi(g)^k d\widetilde{\text{Res}_{\mathbb{Z}_p^\times} \log(u)} && \text{(notation)} \\
&= \int_{\mathbb{Z}_p} \chi^k d\mathcal{Y}(\text{Res}_{\mathbb{Z}_p^\times} \log f_u) && \text{(commutativity of top square)} \\
&= \int_{\mathbb{Z}_p} \chi^{k-1} d\chi \mathcal{Y}(\text{Res}_{\mathbb{Z}_p^\times} \log f_u) \\
&= \int_{\mathbb{Z}_p} \chi^{k-1} d\mathcal{Y}(D \circ \text{Res}_{\mathbb{Z}_p^\times} \log f_u) && \text{(formula 1)} \\
&= \int_{\mathbb{Z}_p} \chi^{k-1} d\mathcal{Y}(\text{Res}_{\mathbb{Z}_p^\times} \circ D \log f_u) && \text{(commutativity of bottom square)} \\
&= \int_{\mathbb{Z}_p} \chi^{k-1} d\text{Res}_{\mathbb{Z}_p^\times} \mathcal{Y}(D \log f_u) \\
&= (1 - p^{k-1}) \int_{\mathbb{Z}_p} \chi^{k-1} d\mathcal{Y}(D \log f_u) && \text{(formula 2, since } D \log f_u \in R^{\psi=1}) \\
&= (1 - p^{k-1}) (D^{k-1} (D \log f_u))_{T=0} && \text{(formula 1)} \\
&= (1 - p^{k-1}) \delta_k(u) && \text{(by definition of } \delta_k(u))
\end{aligned}$$

### 2.3 Measures attached to generators of the Tate module $T_p(\mu)$

As one can see, any generator  $(\zeta_n)$  of the Tate module  $T_p(\mu)$  is a norm compatible sequence of units, hence can be viewed as an element of  $\mathcal{U}_\infty$ . Can we obtain interesting measure from those elements? Unfortunately, no. To see this, note that since by definition  $\pi_n = \zeta_n - 1$ , it is clear that the Coleman power series of  $(\zeta_n) \in \mathcal{U}_\infty$  is simply  $1 + T$ . But then

$$\mathcal{L}((\tilde{\zeta}_n)) = \mathcal{L}(1 + T) = \frac{1}{p} \log \left( \frac{(1 + T)^p}{(1 + (1 + T)^p - 1)} \right) = \frac{1}{p} \log 1 = 0,$$

so the corresponding measure on  $\mathcal{G}$  is the zero measure! In fact, one has the following *Fundamental Exact Sequence* of  $\mathcal{G}$ -modules

$$0 \longrightarrow \mu_{p-1} \times T_p(\mu) \longrightarrow \mathcal{U}_\infty \xrightarrow{\tilde{\mathcal{L}}} \Lambda(\mathcal{G}) \xrightarrow{\beta} T_p(\mu) \longrightarrow 0,$$

where the map  $\beta$  sends  $\lambda$  to  $(\zeta_n)^{\int_{\mathcal{G}} \chi d\lambda}$  (see [CS, Theorem 3.5.1]). Note this sequence is the main ingredient in the proof of Iwasawa's theorem (Theorem 4.4.1 in [CS]).

The proof this sequence is exact follows essentially from the exactness of the sequence

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow R^{\psi=1} \xrightarrow{\theta=\text{Res}_{\mathbb{Z}_p^\times}} R^{\psi=0} \xrightarrow{\text{ev}_T=0} \mathbb{Z}_p \longrightarrow 0,$$

which is proved in [CS, Lemma 2.4.3]. Indeed, the exactness of the sequence

$$0 \longrightarrow \mu_{p-1} \longrightarrow W \xrightarrow{D \log} R^{\psi=1} \longrightarrow 0 \tag{3}$$

allows us to lift this sequence to obtain the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{A} & \longrightarrow & W & \xrightarrow{\text{Res}_{\mathbb{Z}_p^\times} \log} & R^{\psi=0} \xrightarrow{\alpha} \mathbb{Z}_p \longrightarrow 0 \\
& & & & \downarrow D \log & & \downarrow D \\
0 & \longrightarrow & \mathbb{Z}_p & \longrightarrow & R^{\psi=1} & \xrightarrow{\text{Res}_{\mathbb{Z}_p^\times}} & R^{\psi=0} \xrightarrow{\text{ev}_T=0} \mathbb{Z}_p \longrightarrow 0,
\end{array}$$

where

$$A = \{\xi(1+T)^a \mid \xi \in \mu_{p-1}, a \in \mathbb{Z}_p\}$$

and  $\alpha(g) = (Dg)(0)$  (see [CS, Theorem 2.5.2]). The bottom row can be simply seen as the additive version of the top row. Finally, using the isomorphisms

$$\begin{array}{ccc} \mathcal{U}_\infty & & \Lambda(\mathcal{G}) \\ \text{C.P.S.} \downarrow \wr & & \downarrow \wr \tilde{\mathcal{M}} \\ W & & R^{\Psi=0} \end{array}$$

one can further lift this sequence to obtain

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mu_{p-1} \times T_p(\mu) & \longrightarrow & \mathcal{U}_\infty & \xrightarrow{\tilde{L}} & \Lambda(\mathcal{G}) \xrightarrow{\beta} T_p(\mu) \longrightarrow 0 \\ & & & & \downarrow \text{C.P.S.} \wr & & \downarrow \wr \tilde{\mathcal{M}} \\ 0 & \longrightarrow & A & \longrightarrow & W & \xrightarrow{\text{Res}_{\mathbb{Z}_p^\times} \log} & R^{\Psi=0} \xrightarrow{\alpha} \mathbb{Z}_p \longrightarrow 0 \\ & & & & \downarrow D \log & & \downarrow D \\ 0 & \longrightarrow & \mathbb{Z}_p & \longrightarrow & R^{\Psi=1} & \xrightarrow{\text{Res}_{\mathbb{Z}_p^\times}} & R^{\Psi=0} \xrightarrow{\text{ev}_T=0} \mathbb{Z}_p \longrightarrow 0. \end{array}$$

In brief, one needs to work a little bit harder to obtain non-trivial measures on  $\mathcal{G}$ . This is where the cyclotomic units come in!

### 3 p-zeta function via cyclotomic units

At this point, the construction of the p-adic zeta function is almost a formality! First, recall that

$$\mathcal{K}_n = \mathbb{Q}_p(\mu_{p^{n+1}})$$

and that  $\mathcal{U}_n$  is the set of units in  $\mathcal{K}_n$  ( $n \geq 0$ ). As usual, fix a generator ( $\zeta_n$ ) of the Tate module  $T_p(\mu)$ , so that

$$\zeta_n \in \mathcal{U}_n \quad \text{and} \quad \zeta_n^p = \zeta_{n-1} \text{ for } n \geq 1.$$

For integers  $a$  and  $b$ , define

$$c_n(a, b) \frac{\zeta_n^{-a/2} - \zeta_n^{a/2}}{\zeta_n^{-b/2} - \zeta_n^{b/2}} = \zeta_n^{(a-b)/2} \frac{\zeta_n^{-a} - 1}{\zeta_n^{-b} - 1}.$$

If  $a$  and  $b$  are not divisible by  $p$ , the  $c_n(a, b)$  is a unit in  $\mathcal{F}_n$ , hence in  $\mathcal{K}_n$  (this follows from [Mi, Proposition 6.2(c)]). We claim that the sequence

$$c(a, b) := (c_n(a, b))$$

is a norm compatible sequence of local units, i.e.

$$c(a, b) \in \mathcal{U}_\infty.$$

To see this, it suffices to note that

$$N_{n, n-1}(\zeta_n) = \zeta_{n-1},$$

where  $N_{n, n-1} : \mathcal{K}_n \rightarrow \mathcal{K}_{n-1}$  is the norm map. This follows from the fact that  $\zeta_n$  is a root of

$$X^p - \zeta_{n-1},$$

which is irreducible over  $\mathcal{K}_{n-1}$  since

$$(X+1)^p - \zeta_{n-1}$$

is Eisenstein at the prime  $\pi_{n-1}$ . Those elements  $c(a, b)$  are the key to defining the  $p$ -adic zeta function.

Let

$$\lambda(a, b) := \widetilde{\text{Res}_{\mathbb{Z}_p^\times} \log(c(a, b))} = \tilde{\mathcal{L}}(c(a, b)).$$

Then

$$\int_{\mathcal{G}} \chi(g)^k d\lambda(a, b) = (1 - p^{k-1}) \delta_k(c(a, b)).$$

The computation of  $\delta_k(c(a, b))$  is done in [CS, Proposition 2.6.3]:

$$\delta_k(c(a, b)) = \begin{cases} 0 & \text{if } k = 1, 3, 5, \dots \\ (b^k - a^k) \zeta(1 - k) & \text{if } k = 2, 4, 6, \dots \end{cases}.$$

The  $p$ -adic measure  $\lambda(a, b)$  is almost the  $p$ -adic zeta function, except the it interpolates the  $p$ -stabilized values of  $\zeta$  at the negative integers *times the factor*  $b^k - a^k$ . To cancel this factor, take  $a$  different from  $b$  and define

$$\theta(a, b) = \sigma_b - \sigma_a \in \Lambda(\mathcal{G})$$

such that

$$\chi(\theta(a, b)) = b^k - a^k \in \mathbb{Z}_p^\times.$$

Then

$$\tilde{\zeta}_p = \frac{\lambda(a, b)}{\theta(a, b)}$$

is a pseudo-measure on  $\mathcal{G}$  which is independent of the choice of  $a$  and  $b$ . More importantly,

$$\int_{\mathcal{G}} \chi(g)^k d\tilde{\zeta}_p = \begin{cases} 0 & \text{if } k = 1, 3, 5, \dots \\ \zeta(1 - k) & \text{if } k = 2, 4, 6, \dots \end{cases}.$$

(see [CS, Proposition 4.2.4] for more details). Note that since pseudo-measures are determined by their moments,  $\tilde{\zeta}_p$  is the unique pseudo-measure which interpolates the critical values of the Riemann zeta function.

Finally, since the odd moments of  $\tilde{\zeta}_p$  are zero,

$$\tilde{\zeta}_p \in \Lambda(\mathcal{G})^+.$$

Letting  $\zeta_p$  denote the image of  $\tilde{\zeta}_p$  under the identification  $\Lambda(\mathcal{G})^+ \simeq \Lambda(G)$ , we have the following theorem.

**Theorem 4.** *There exists a unique pseudo-measure  $\zeta_p$  on  $G$  such that*

$$\int_G \chi(g)^k d\zeta_p = (1 - p^{k-1}) \zeta(1 - k)$$

for all even integers  $k \geq 2$ .

## References

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