Petersson Inner Product of Theta Series

PhD Defense

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L-functions at $s = 1$

It is a well-known (but fascinating) fact that many L-functions contain arithmetic informations in their value at $s = 1$:

1. $\zeta(s)$ at $s = 1$: Infinitely many primes
2. $L(\chi, s)$ at $s = 1$: Infinitely many primes in arithmetic progressions
3. $\zeta_F(s)$ at $s = 1$: Class number formula

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Conjecture (Stark (Idea))

In general, L-functions of Artin representations have a (relatively) explicit expression involving arithmetic invariants of the number fields involved.

An observation of Stark

Let $K = \mathbb{Q}(\sqrt{-23})$ and let H be its Hilbert class field. Let

$$\psi : \mathrm{Gal}(H/K) \to \mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$$

be a non-trivial one-dimensional Artin representation and let

$$\rho = \mathrm{Ind}_K^{\mathbb{Q}} \psi : \mathrm{Gal}(H/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$$

be the induced representation. Then one can consider the associated Artin L-function

$$L(\psi, s) = L(\rho, s).$$

An observation of Stark

On the one hand, in accordance with his conjecture (which was known in this case), Stark shows that

$$L(\rho, 1) = \frac{2\pi}{\sqrt{23}} \log \varepsilon,$$

where ε is the real root of

$$x^3 - x - 1.$$

Note that ε generates H over K.

An observation of Stark

On the other hand, by the Deligne-Serre theorem, one has

$$L(\rho, s) = L(\theta_\psi, s),$$

where

$$\theta_\psi(q) = \eta(q)\eta(23q) = q\prod_{n\geq 1}(1-q^n)(1-q^{23n}) \in M_1(\Gamma_0(23), \chi_{-23}).$$

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Then Stark proves that

$$L(\rho, 1) = \frac{2\pi}{3\sqrt{23}} \langle \theta_\psi, \theta_\psi \rangle.$$

The main motivation

It follows that

$$\langle \theta_\psi, \theta_\psi \rangle = 3 \log \varepsilon.$$

Structure of the presentation

Where we are in the presentation

Notation

Throughout this presentation, let

- K be an imaginary quadratic field of discriminant D with Hilbert class field H,
- h_K, w_K and Cl_K be the class number, root number and class group of K (respectively)
- ψ be a Hecke character of infinity type $(2\ell, 0)$ for some $\ell \geq 0$, i.e. a homomorphism

$$\psi : I_K \longrightarrow \mathbb{C}^\times$$

such that $\psi((\alpha)) = \alpha^{2\ell}$ for all $\alpha \in K^\times$

- and $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ be fractional ideals of K.

Theta series attached to K

Consider

$$\left.\begin{aligned} \theta_\psi(q) &= \textstyle\sum_{\mathfrak{a}\in\mathcal{O}_K} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} \\ \theta_{\mathfrak{a},\ell}(q) &= \textstyle\sum_{x\in\mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})} \end{aligned}\right\} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

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Then

<table>
<tr><th></th><th>θ_ψ</th><th>$\theta_{\mathfrak{a},\ell}$</th></tr>
<tr><td>$\ell > 0$</td><td>Newform</td><td>Cusp form</td></tr>
<tr><td rowspan="2">$\ell = 0$</td><td>$\psi^2 \neq 1$: Newform</td><td rowspan="2">Not a cusp form</td></tr>
<tr><td>$\psi^2 = 1$: (genus) Eisenstein series</td></tr>
</table>

Some examples to keep in mind

	θ_ψ	$\theta_{\mathfrak{a},\ell}$
$\ell > 0$		
$\ell = 0$	$q\prod_{n\geq 1}(1-q^n)(1-q^{23n})$	$\theta_{\mathbb{Z}[i]}(q) = \sum_{x,y\in\mathbb{Z}} q^{x^2+y^2}$

Recall that

$$q\prod_{n\geq 1}(1-q^n)(1-q^{23n})$$

is the modular form in Stark's example.

Formulas for the Petersson inner product of those theta series

Recall that the Petersson inner product of any cusp forms $f, g \in S_k(\Gamma_0(N), \chi)$ is defined as

$$\langle f, g \rangle = \iint_{\Gamma_0(N)\backslash\mathcal{H}} f(\tau)\bar{g}(\tau)\Im(\tau)^k \mathrm{d}\mu(\tau).$$

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With minor effort, this formula can be used to compute the Petersson inner product numerically:

$$\langle f, g \rangle = \sum_{\gamma \in \Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\sqrt{1-x^2}}^{\infty} f(\tau)\bar{g}(\tau) y^{k-2} \mathrm{d}y\mathrm{d}x.$$

But this is very (very) slow and behaves badly as the level grows.

The quest for more efficient and useful formulas

Let ψ be such that θ_ψ is a cusp form. Then

1. Apply Rankin-Selberg:

$$\langle \theta_\psi, \theta_\psi \rangle = \left(\frac{\pi}{2} \frac{\phi(|D|)}{D^2} \frac{(4\pi)^{2\ell+1}}{\Gamma(2\ell+1)} \right)^{-1} L(\chi_D, 1) \operatorname{res}_{s=2\ell+1} L(\operatorname{Sym}^2 \theta_\psi, s)$$

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2. Isolate the residue of $L(\mathrm{Sym}^2 \theta_\psi, s)$:

$$\mathrm{res}_{s=2\ell+1} L(\mathrm{Sym}\, 2\theta_\psi, 1, s) = \prod_{p|D} (1 - p^{-1}) L(\psi^2, 2\ell+1)$$

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3. When $\ell > 0$, express $L(\psi^2, 2\ell + 1)$ in terms of (derivatives of nearly holomorphic) Eisenstein series:

$$L(\psi^2, 2\ell{+}1) = \frac{4(2\pi)^{2\ell+1} \sqrt{|D|}^{2\ell-1}}{w_K \Gamma(2\ell + 1)} \sum_{j=1}^{h_K} \psi^{-2}(\mathfrak{a}_j) N(\mathfrak{a}_j)^{4\ell} \delta^{2\ell-1} E_2(\bar{\mathfrak{a}}_j)$$

The most useful formulas for p-adic interpolation

	$\langle\theta_\psi, \theta_\psi\rangle$	$\langle\theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell}\rangle$
$\ell > 0$	$C_1 \sum_{\mathcal{A}\in \mathrm{Cl}_K} \psi^2(\mathcal{A})\delta^{2\ell-1}E_2(\mathcal{A})$	
$\ell = 0$	$\psi^2 = 1$: not applicable	not applicable

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Using the relation

$$\theta_{\mathfrak{a},\ell} = \frac{w_K}{h_K} \sum_{\psi} \psi(\mathfrak{a}) \theta_\psi$$

and the orthogonality of the newforms θ_ψ.

The most useful formulas for p-adic interpolation

	$\langle \theta_\psi, \theta_\psi \rangle$	$\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$
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$\ell = 0$	$C_3 \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \log \Phi(\mathcal{A})$ $\psi^2 = 1$: not applicable	not applicable

Here

$$\Phi^{12}(\mathcal{A}) = N(\mathcal{A})^6 |\Delta(\mathcal{A})|,$$

where

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

The most efficient formula for computations

Experimentally, one finds that the most efficient way to compute the Petersson inner product of theta series is to compute the q-expansion of $\delta^n E_2$ by hand:

$$\begin{aligned}\delta^n E_2(\tau) =& (-1)^n \left(\frac{1}{8\pi\Im(\tau)} - \frac{n+1}{24}\right)\frac{n!}{(4\pi\Im(\tau))^n} \\ &+ \sum_{m\geq 1}\sigma(m)\left(\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\frac{(r+2)_{n-r}}{(4\pi\Im(\tau))^{n-r}}m^r\right)q^m.\end{aligned}$$

Where we are in the presentation

Bridging the gap between the "explicit" formulas and the algorithms

Here are some of the things one needs to do before implementing those formulas:

- Complete the L-functions $L(\mathrm{Sym}^2 \theta_\psi, s)$ and $L(\psi, s)$ and find all the information about their functional equation,
- Find a way to compute with Hecke characters,
- Find an *efficient* way to compute

$$\delta^n E_2(\mathfrak{a}),$$

- Choose the computer algebra system that allows you to do all this!

The resulting algorithm

This leads to the following

Theorem (S.)

There exists a software package in PARI/GP to compute the Petersson inner product of the theta series defined above with the following properties:

- *It is fast (relative to the definition),*
- *It supports arbitrary precision (no coefficients stored, no database involved),*
- *User friendly (easy to download, help functions, well commented source code).*

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Proof.

See the calculations in Part 3 of the thesis and look at the source code online! □

Where we are in the presentation

What about Stark's observation?

Using the above formula when $\ell = 0$, one has

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{-h_K}{3w_K^2} \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \log N(\mathcal{A})^6 |\Delta(\mathcal{A})| = h_K \log \kappa_\psi,$$

where

$$\kappa_\psi = \prod_{\mathcal{A} \in \mathrm{Cl}_K} \Phi(\mathcal{A})^{-\psi^2(\mathcal{A})},$$

with

$$\Phi(\mathcal{A}) = \sqrt{N(\mathfrak{a})} |\Delta(\mathfrak{a})|^{1/12}$$

as before, where now $\mathfrak{a}$ is any ideal in the class $\mathcal{A}$.

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Question

Is κ_ψ a unit in H?

A corollary and some examples

Some examples:

- $K = \mathbb{Q}(\sqrt{-23})$ ($h_K = 3$): κ_ψ is a unit and numerically, $\kappa_\psi = \varepsilon$,
- $K = \mathbb{Q}(\sqrt{-39})$ ($h_K = 4$): $\kappa_\psi = \epsilon_{13}^{\frac{1}{3}}$ is a unit, but not in H,
- $K = \mathbb{Q}(\sqrt{-47})$ ($h_K = 5$): κ_ψ doesn't seem to be a unit for any ψ.

Generalizing Stark's Observation

Proposition (S.)

Let ψ be a class character such that ψ^2 is a non-trivial character with rational real part (equivalently, the character of $\mathrm{Ind}_K^{\mathbb{Q}}\psi$ is rational). Then κ_ψ is an algebraic integer which is a unit. Moreover, if ψ^2 is a non-trivial genus character corresponding to the factorisation $D = D_1 D_2$, with $D_1 > 0$ say, then

$$\kappa_\psi = \epsilon_{D_1}^{\frac{4h_{D_1}h_{D_2}}{w_K w_{D_2}}},$$

where ϵ_{D_1} is the fundamental unit of $\mathbb{Q}(\sqrt{D_1})$, h_{D_j} is the class number of $\mathbb{Q}(\sqrt{D_j})$ and w_{D_2} is the number of roots of unity in $\mathbb{Q}(\sqrt{D_2})$.

A corollary

It follows from this proposition that

Corollary

If K has class number divisible by 2 or 3, there exists a class character ψ for which κ_ψ is a unit.

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Question

Is the converse true?

Where we are in the presentation

Two objectives

Recall that

	$\langle \theta_\psi, \theta_\psi \rangle$	$\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$
$\ell > 0$	$C_1 \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \delta^{2\ell-1} E_2(\mathcal{A})$	$C_2 \sum_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_\mathfrak{c} \mathcal{O}_K} \lambda_\mathfrak{c}^{2\ell} \delta^{2\ell-1} E_2(\mathfrak{c})$
$\ell = 0$	$C_3 \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \log \Phi(\mathcal{A})$ $\psi^2 = 1$: not applicable	not applicable

Two objectives for the *p*-adic interpolation

1. Show that the quantities $\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$ can be *p*-adically interpolated for $\ell > 0$ (under certain restrictions),
2. Evaluate the *p*-adic analytic function obtained at $\ell = 0$.

p-adic interpolation of Petersson inner product of theta series: setup

Suppose that:

1. D is prime,
2. p is a prime $\neq 2, 3$ which splits in K (say $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$),
3. $\mathfrak{a}$ and $\mathfrak{b}$ are two fractional ideals of K such that

$$\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \mathcal{O}_K.$$

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$$\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \mathcal{O}_K.$$

Moreover, fix an isomorphism

$$\mathbb{Q}_p/\mathbb{Z}_p \to \bigcup_{n \geq 1} \bar{\mathfrak{p}}^{-n}\mathfrak{c}/\mathfrak{c}$$

and let

$$\mathcal{W} = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$$

denote the p-adic weight space.

p-adic interpolation of Petersson inner product of theta series: result

Under the above assumptions, one has the following

Theorem (S.)

There exists a p-adic analytic function

$$F : \mathcal{W} \to \mathbb{C}_p$$

with the property that

$$F(\ell) = (\mathrm{Frob}_{\mathfrak{p}}^{-1} - p^{2\ell-1})(\mathrm{Frob}_{\mathfrak{p}}^{-1} - p^{2\ell}) \left(\frac{\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle}{((2\pi i)^{-1}\Omega_{\mathbb{C}}(\mathfrak{c}))^{4\ell}} \right) \text{ for all } \ell > 0$$

where $\mathrm{Frob}_{\mathfrak{p}} = \left(\frac{H/K}{\mathfrak{p}}\right)$ *is the Artin symbol.*

Evaluation of F at $\ell = 0$

Let

$$g_0(q) = \frac{\Delta(q)}{\Delta(q^p)}$$

and let

$$g_0^{(p)}(q) = \frac{g_0(q^p)}{g_0^p(q)} = \frac{\Delta^{p+1}(q^p)}{\Delta^p(q)\Delta(q^{p^2})}.$$

Then $\log_p g_0^{(p)}$ is a p-adic modular form and one has the following

Theorem (S. (loose form))

The following equality holds in $\mathbb{C}_p$:

$$F(0) = -\frac{1}{6p} \log_p g_0^{(p)}(\mathfrak{c}).$$

Interpretation of the above theorem (in the current setup)

Again, recall that

<table>
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Using the formula for $\langle\theta_\psi,\theta_\psi\rangle$ even if $\psi^2=1$, one has

	$\langle\theta_\psi,\theta_\psi\rangle$	$\langle\theta_{\mathfrak{a},\ell},\theta_{\mathfrak{b},\ell}\rangle$
$\ell>0$	$C_1\sum_{\mathcal{A}\in \mathrm{Cl}_K}\psi^2(\mathcal{A})\delta^{2\ell-1}E_2(\mathcal{A})$	$4\delta^{2\ell-1}E_2(\mathfrak{c})$
$\ell=0$	$C_3\sum_{\mathcal{A}\in \mathrm{Cl}_K}\psi^2(\mathcal{A})\log\Phi(\mathcal{A})$	$-\frac{1}{3}\log(N(\mathfrak{c})^6\lvert\Delta(\mathfrak{c})\rvert)$

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$\ell = 0$	$C_3 \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \log \Phi(\mathcal{A})$	$-\frac{1}{3} \log(N(\mathfrak{c})^6 \lvert \Delta(\mathfrak{c}) \rvert)$

Using the above result, a *formal* computation gives

$$F(0) = -\frac{1}{6p} \log_p \frac{\Delta^{p+1}(q^p)}{\Delta^p(q)\Delta(q^{p^2})}$$
$$= -\frac{1}{6}(\mathrm{Frob}_{\mathfrak{p}}^{-1} - p^{-1})(\mathrm{Frob}_{\mathfrak{p}}^{-1} - 1) \log_p \Delta(\mathfrak{c}).$$

Where we are in the presentation

Experimenting with Petersson norm of theta series

Consider the algebraic number

$$N(\psi) = \frac{\langle \theta_\psi, \theta_\psi \rangle}{\Omega_K^{4\ell}} \qquad \text{for} \ell > 0,$$

where Ω_K is the Chowla-Selberg period.

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When ℓ is divisible by h_K, it appears numerically that

$$N(\psi_0) \in \mathbb{Z}$$

for a unique Hecke character ψ_0.

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$$N(\psi_0) \in \mathbb{Z}$$

for a unique Hecke character ψ_0.
This ψ_0 is the Hecke character

$$\psi_0(\mathfrak{a}) = \alpha^{2\ell/h_K},$$

where $\mathfrak{a}^{h_K} = (\alpha)$.

Computing the Gram matrix in the space of theta series

Let

$$\mathrm{Gram}(f_1, \ldots, f_d) = \det(\langle f_i, f_j \rangle)_{1 \le i,j \le d}$$

be the determinant of the Gram matrix of the Petersson inner product for a basis $\{f_1, \ldots, f_d\}$ in a vector space.

Computing the Gram matrix in the space of theta series

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Proposition

Let $\{\mathfrak{a}_1, \dots, \mathfrak{a}_{h_K}\}$ be a set of representatives of Cl_K and let $\{\psi_1, \dots, \psi_{h_K}\}$ be the Hecke characters of K of infinity type $(2\ell, 0)$. Then

$$\mathrm{Gram}(\theta_{\mathfrak{a}_1,\ell}, \dots, \theta_{\mathfrak{a}_{h_K},\ell}) = \left(\frac{w_K^2}{h_K}\right)^{h_K} \left(\prod_{i=1}^{h_k} N(\mathfrak{a}_i)\right) \mathrm{Gram}(\theta_{\psi_1}, \dots, \theta_{\psi_{h_K}}).$$

Computing the Gram matrix in the space of theta series

Numerically, it appears that

$$A(K,\ell) = \prod_{i=1}^{h_K} N(\psi_i) = \frac{\mathrm{Gram}(\theta_{\psi_1},\dots,\theta_{\psi_{h_K}})}{\Omega_K^{4h_k\ell}}$$

is integral (except in some cases when $\ell = 1$):

D	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$
-7	$\frac{1}{3}$	1	17	1337
-8	$\frac{1}{2}$	5	92	1448
-11	1	10	139	16000
-15	4	6084	2528100	5222952900
-19	$\frac{13}{3}$	142	9251	944192
-20	64	21904	189337600	2907434214400
-23	621	7303581	1571089526325	1233974294487401229

A basis of normalized theta series

Suppose that $D < -4$. For any ideal class $\mathcal{A} = [\mathfrak{a}] \in \mathrm{Cl}_K$ and any $\ell > 0$, define

$$\theta_{\mathcal{A},\ell} = \frac{\theta_{\mathfrak{a},\ell}}{E_2(\mathfrak{a}^{-1})^{\ell}}.$$

Then by CM theory

$$\langle \theta_{\mathcal{A}_1,\ell}, \theta_{\mathcal{A}_2,\ell} \rangle \in H.$$

Define

$$C(K,\ell) = \mathrm{Gram}(\theta_{\mathcal{A}_1,\ell}, \ldots, \theta_{\mathcal{A}_{h_K},\ell}).$$

Where we are in the presentation