Introduction
OO
OO

Weight One
OOO
OOOO

Higher Weight
OOO
OOOO

Spaces of Theta Series

Petersson Inner Product of Theta Series

An experimental approach

Nicolas SIMARD

McGill University

December 1st, 2017

Stark's observation

Let $K = \mathbb{Q}(\sqrt{-23})$ and let H be the HCF of K. Let

$$\psi : \mathsf{Gal}(H/K) \to \mathsf{GL}_1(\mathbb{C})$$

be a non-trivial one-dimensional Artin representation and let

$$\rho = \mathsf{Ind}_K^{\mathbb{Q}} \psi : \mathsf{Gal}(H/\mathbb{Q}) \to \mathsf{GL}_2(\mathbb{C})$$

be the induced representation.

Stark's observation (cont.)

By Deligne-Serre, one has

$$L(\rho, s) = L(\theta_\psi, s),$$

where

$$\theta_\psi(q) = \eta(q)\eta(23q) = q\prod_{n\geq 1}(1-q^n)(1-q^{23n}) \in M_1(\Gamma_0(23), \chi_{-23}).$$

Then Stark proves that

$$\langle\theta_\psi, \theta_\psi\rangle = 3\log\varepsilon,$$

where ε is the real root of

$$x^3 - x - 1.$$

Structure of the talk

Notation

Throughout this talks, let

- K be an imaginary quadratic field of discriminant D,
- H be the Hilbert class field of K,
- h_K be the class number of K,
- w_K be the number of roots of unity in K and
- Cl_K be the class group of K.

Weight one theta series

Let ψ be a class character of K, i.e. a homomorphism

$$\psi : \mathsf{Cl}_K \to \mathbb{C}^\times.$$

Then

$$\theta_\psi(q) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} \in M_1(\Gamma_0(|D|), \chi_D).$$

Introduction	Weight One	Higher Weight	Spaces of Theta Series
oo	●oo	ooo	
oo	oooo	oooo	

Weight one theta series

Let ψ be a class character of K, i.e. a homomorphism

$$\psi : \mathrm{Cl}_K \to \mathbb{C}^\times.$$

Then

$$\theta_\psi(q) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} \in M_1(\Gamma_0(|D|), \chi_D).$$

- θ_ψ is an eigenform for all Hecke operators;
- if $\psi^2 = 1$, θ_ψ is an Eisenstein series;
- if $\psi^2 \neq 1$, θ_ψ is a cusp form (in fact, a newform).

Stark's example

Let

$$K = \mathbb{Q}(\sqrt{-23})$$

and let ψ be a non-trivial class character as above. Then

$$\text{Stark's } \theta_\psi = \text{our } \theta_\psi \in M_1(\Gamma_0(23), \chi_{-23}).$$

Note that if ψ' is the other non-trivial class character, then

$$\theta_\psi = \theta_{\psi'}.$$

Introduction
○○
○○

Weight One
○○●
○○○○

Higher Weight
○○○
○○○○

Spaces of Theta Series

Petersson inner product of weight one theta series

The Petersson inner product of any two $f, g \in S_k(\Gamma_0(N), \chi)$ is defined as

$$\langle f, g \rangle = \iint_{\Gamma_0(N) \backslash \mathcal{H}} f(x + iy) \overline{g(x + iy)} y^k \mathrm{d}\mu.$$

Petersson inner product of weight one theta series

The Petersson inner product of any two $f, g \in S_k(\Gamma_0(N), \chi)$ is defined as

$$\langle f, g \rangle = \iint_{\Gamma_0(N) \backslash \mathcal{H}} f(x + iy) \overline{g(x + iy)} y^k d\mu.$$

Proposition (S.)

Let ψ be a class character which is not a genus character. Then

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{-h_K}{3 w_K^2} \sum_{\mathcal{A} \in Cl_K} \psi^2(\mathcal{A}) \log N(\mathcal{A})^6 |\Delta(\mathcal{A})|.$$

Siegel units

Define
$$|\delta_{\mathcal{A}}| = (N(\mathfrak{a})^6 |\Delta(\mathcal{O}_K)/\Delta(\mathfrak{a}^{-1})|)^{h_K},$$

where $\mathfrak{a}$ is any ideal in the class $\mathcal{A}$. Then $|\delta_{\mathcal{A}}|$ is the absolute value of a (Siegel) unit in H.

Siegel units

Define
$$|\delta_{\mathcal{A}}| = (N(\mathfrak{a})^6 |\Delta(\mathcal{O}_K)/\Delta(\mathfrak{a}^{-1})|)^{h_K},$$

where $\mathfrak{a}$ is any ideal in the class $\mathcal{A}$. Then $|\delta_{\mathcal{A}}|$ is the absolute value of a (Siegel) unit in H.

Since ψ^2 is not trivial, one sees that

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{1}{3w_K^2} \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \log |\delta_{\mathcal{A}}|.$$

What about Stark's observation?

Assume $D < -4$. Then one can write

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{-h_K}{3w_K^2} \sum_{\mathcal{A} \in Cl_K} \psi^2(\mathcal{A}) \log N(\mathcal{A})^6 |\Delta(\mathcal{A})| = h_K \log \kappa_\psi.$$

Here,

$$\kappa_\psi = \prod_{\mathcal{A} \in Cl_K} \Phi(\mathcal{A})^{-\psi^2(\mathcal{A})}$$

with

$$\Phi(\mathcal{A}) = \sqrt{N(\mathfrak{a})} |\Delta(\mathfrak{a})|^{1/12},$$

where $\mathfrak{a}$ is any ideal in the class $\mathcal{A}$.

Introduction	Weight One	Higher Weight	Spaces of Theta Series
oo	ooo	ooo	
oo	o●oo	oooo	

What about Stark's observation?

Assume $D < -4$. Then one can write

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{-h_K}{3w_K^2} \sum_{\mathcal{A} \in \mathsf{Cl}_K} \psi^2(\mathcal{A}) \log N(\mathcal{A})^6 |\Delta(\mathcal{A})| = h_K \log \kappa_\psi.$$

Here,

$$\kappa_\psi = \prod_{\mathcal{A} \in \mathsf{Cl}_K} \Phi(\mathcal{A})^{-\psi^2(\mathcal{A})}$$

with

$$\Phi(\mathcal{A}) = \sqrt{N(\mathfrak{a})} |\Delta(\mathfrak{a})|^{1/12},$$

where $\mathfrak{a}$ is any ideal in the class $\mathcal{A}$.

Question

Is κ_ψ a unit in H?

Script available at https://github.com/NicolasSimard/ENT/blob/master/comps/ULcomps1.gp

Introduction
○○
○○

Weight One
○○○
○○○●

Higher Weight
○○○
○○○○

Spaces of Theta Series

Generalizing Stark's Observation

Proposition (S.)

Let ψ be a class character such that ψ^2 is a non-trivial character with rational real part. Then κ_ψ is an algebraic integer which is a unit. Moreover, if ψ^2 is a non-trivial genus character corresponding to the factorisation $D = D_1 D_2$, with $D_1 > 0$ say, then

$$\kappa_\psi = \epsilon_{D_1}^{\frac{4 h_{D_1} h_{D_2}}{w_K w_{D_2}}},$$

where ϵ_{D_1} is the fundamental unit of $\mathbb{Q}(\sqrt{D_1})$, h_{D_j} is the class number of $\mathbb{Q}(\sqrt{D_j})$ and w_{D_2} is the number of roots of unity in $\mathbb{Q}(\sqrt{D_2})$.

Stark's observation: the final word?

Note that ψ^2 has rational real part if and only if its order divides 4 or 3.

Corollary

Suppose that K has class number divisible by 2 or 3. Then there exists a class character ψ for which κ_ψ is a unit.

Stark's observation: the final word?

Note that ψ^2 has rational real part if and only if its order divides 4 or 3.

Corollary

Suppose that K has class number divisible by 2 or 3. Then there exists a class character ψ for which κ_ψ is a unit.

Question

Is the converse true?

Higher weight theta series

Let ℓ be a positive integer and let ψ be a Hecke character of infinity type $(2\ell, 0)$, i.e. a homomorphism

$$\psi : I_K \to \mathbb{C}^\times$$

such that

$$\psi(\alpha \mathcal{O}_K) = \alpha^{2\ell} \text{ for all } \alpha \in K^\times.$$

Then

$$\theta_\psi(q) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

Higher weight theta series

Let ℓ be a positive integer and let ψ be a Hecke character of
infinity type $(2\ell, 0)$, i.e. a homomorphism

$$\psi : I_K \to \mathbb{C}^\times$$

such that

$$\psi(\alpha \mathcal{O}_K) = \alpha^{2\ell} \text{ for all } \alpha \in K^\times.$$

Then

$$\theta_\psi(q) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

- For $\ell = 0$, one recovers the weight one theta series introduced
  before;
- for $\ell > 0$, θ_ψ is a newform.

Recall

Let

$$E_2(\tau) = \frac{1}{8\pi\Im(\tau)} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n)q^n$$

be the nearly holomorphic weight 2 Eisenstein series and let

$$E_k(q) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$$

be the usual weight k Eisenstein series for $k \geq 4$, where $q = e^{2\pi i\tau}$.

Recall

Let

$$E_2(\tau) = \frac{1}{8\pi\Im(\tau)} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n)q^n$$

be the nearly holomorphic weight 2 Eisenstein series and let

$$E_k(q) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$$

be the usual weight k Eisenstein series for $k \geq 4$, where $q = e^{2\pi i\tau}$.
Let also δ be the Shimura-Maass differential operator, so that

$$\delta E_k = \frac{1}{2\pi i}\frac{\partial E_k}{\partial \tau} - \frac{k}{4\pi\Im(\tau)}E_k.$$

Then ∂ raises the weight by 2 and preserves the graded ring

$$\mathbb{C}[E_2, E_4, E_6]$$

of nearly holomorphic modular forms of level $SL_2(\mathbb{Z})$.

Introduction   Weight One   Higher Weight   Spaces of Theta Series
oo   ooo   ooo
oo   oooo   oooo

Petersson inner product of higher weight theta series

With the above notation, one has the following

Proposition (S.)

Let $\ell > 0$ and let ψ be a Hecke character of infinity type $(2\ell, 0)$. Then

$$\langle \theta_\psi, \theta_\psi \rangle = (|D|/4)^\ell \frac{4 h_K}{w_K^2} \sum_{\mathcal{A} \in Cl_K} \psi^2(\mathcal{A}) \delta^{2\ell - 1} E_2(\mathcal{A}).$$

Theta series attached to ideals

Let $\ell \geq 0$ and let $\mathfrak{a}$ be a fractional ideal of K and define

$$\theta_{\mathfrak{a},\ell}(q) = \sum_{x \in \mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

where χ_D is the Kronecker symbol.

Theta series attached to ideals

Let $\ell \geq 0$ and let $\mathfrak{a}$ be a fractional ideal of K and define

$$\theta_{\mathfrak{a},\ell}(q) = \sum_{x \in \mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

where χ_D is the Kronecker symbol.

- If $\ell > 0$, then

$$\theta_{\mathfrak{a},\ell} \in S_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

- For any $\ell \geq 0$, one has

$$\theta_{\mathfrak{a},\ell} = \frac{w_K}{h_K} \sum_{\psi} \psi(\mathfrak{a}) \theta_{\psi},$$

where the sum is over the h_K Hecke characters of infinity type $(2\ell, 0)$.

Petersson inner product of theta series attached to ideals

Using the above relation between the two set of theta series, one has the following

Corollary

Let $\ell > 0$ and let $\mathfrak{a}$ and $\mathfrak{b}$ be two fractional ideals of K. Then

$$\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle = 4(|D|/4)^{\ell} \sum_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K} \lambda_{\mathfrak{c}}^{2\ell} \delta^{2\ell-1} E_2(\mathfrak{c}),$$

where the sum is over a set of representatives $\mathfrak{c}$ of ideals classes in Cl_K such that $\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K$.

Algebrizing the Petersson inner product

Recall that

$$\delta^n E_2 \in \mathbb{C}[E_2, E_4, E_6].$$

There are two ways to algebrize CM values of modular forms.

Algebrizing the Petersson inner product

Recall that

$$\delta^n E_2 \in \mathbb{C}[E_2, E_4, E_6].$$

There are two ways to algebrize CM values of modular forms.

Proposition (Chowla-Selberg period)

Let

$$\Omega_K = \frac{1}{\sqrt{4\pi|D|}} \left(\prod_{n=1}^{|D|-1} \Gamma\left(\frac{n}{|D|}\right)^{\chi_D(n)} \right)^{w_K/(4h_K)}$$

and let $\mathfrak{c}$ be a fractional ideal of K. Then

$$E_k(\mathfrak{c}) \in \Omega_K^k \bar{\mathbb{Q}}$$

for $k = 2, 4$ and 6.

Script available at https://github.com/NicolasSimard/ENT/blob/master/comps/ULcomps2.gp

Algebrizing the Petersson inner product

Recall that

$$\delta^n E_2 \in \mathbb{C}[E_2, E_4, E_6].$$

There are two ways to algebrize CM values of modular forms.

Proposition (CM theory)

Let $\Omega_{\mathfrak{c}} \in \mathbb{C}^{\times}$ be such that the elliptic curve

$$\mathbb{C}/\Omega_{\mathfrak{c}}\mathfrak{c}$$

is defined over H. Then

$$E_k(\mathfrak{c}) \in (2\pi i \Omega_{\mathfrak{c}})^{-k} H$$

for $k = 2, 4$ and 6.

Script available at https://github.com/NicolasSimard/ENT/blob/master/comps/ULcomps2.gp

p-adic interpolation of Petersson inner product of theta series

Suppose that D is prime and let p be a prime $\neq 2, 3$ which splits in K, say $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be two fractional ideals of K which are such that

$$\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \mathcal{O}_K$$

and fix an isomorphism

$$\mathbb{Q}_p/\mathbb{Z}_p \to \bigcup_{n \geq 1} \bar{\mathfrak{p}}^{-n}\mathfrak{c}/\mathfrak{c}.$$

Let also

$$\mathcal{W} = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^{\times}, \mathbb{Z}_p^{\times})$$

denote the p-adic weight space.

Introduction
○○
○○

Weight One
○○○
○○○○

Higher Weight
○○○
○○○●

Spaces of Theta Series

p-adic interpolation of Petersson inner product of theta series

With the notation above, one has the following

Theorem (S.)

There exists a p-adic analytic function

$$F : \mathcal{W} \to \mathbb{C}_p$$

with the property that

$$F(\ell) = (\mathrm{Frob}_{\mathfrak{p}}^{-1} - p^{2\ell-1})(\mathrm{Frob}_{\mathfrak{p}}^{-1} - p^{2\ell})\left(\frac{\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell}\rangle}{(2\pi i \Omega_{\mathfrak{c}})^{-4\ell}}\right) \text{ for all } \ell > 0,$$

where $\mathrm{Frob}_{\mathfrak{p}} = \left(\frac{H/K}{\mathfrak{p}}\right)$ *is the Artin symbol.*

Introduction
○○
○○

Weight One
○○○
○○○○

Higher Weight
○○○
○○○○

Spaces of Theta Series

Thank you!

Presentation and notes available at:
https://github.com/NicolasSimard/Notes
Code available at : https://github.com/NicolasSimard/ENT
Or from my webpage: http://www.math.mcgill.ca/nsimard/