Petersson Inner Product of Theta Series

PhD Defense

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L-functions at $s = 1$

It is a well-known (but fascinating) fact that many L-functions contain arithmetic informations in their value at $s = 1$:

1. $\zeta(s)$ at $s = 1$: Infinitely many primes
2. $L(\chi, s)$ at $s = 1$: Infinitely many primes in arithmetic progressions
3. $\zeta_F(s)$ at $s = 1$: Class number formula

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Conjecture (Stark (Idea))

In general, L-functions of Artin representations have a (relatively) explicit expression involving arithmetic invariants of the number fields involved.

An observation of Stark

Let $K = \mathbb{Q}(\sqrt{-23})$ and let H be its Hilbert class field. Let

$$\psi : \mathrm{Gal}(H/K) \to \mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$$

be a non-trivial one-dimensional Artin representation and let

$$\rho = \mathrm{Ind}_K^{\mathbb{Q}} \psi : \mathrm{Gal}(H/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$$

be the induced representation. Then one can consider the associated Artin L-function

$$L(\psi, s) = L(\rho, s).$$

An observation of Stark

On the one hand, in accordance with his conjecture (which was known in this case), Stark shows that

$$L(\rho, 1) = \frac{2\pi}{\sqrt{23}} \log \varepsilon,$$

where ε is the real root of

$$x^3 - x - 1.$$

Note that ε generates H over K.

An observation of Stark

On the other hand, by the Deligne-Serre theorem, one has

$$L(\rho, s) = L(\theta_\psi, s),$$

where

$$\theta_\psi(q) = \eta(q)\eta(23q) = q\prod_{n\geq 1}(1-q^n)(1-q^{23n}) \in M_1(\Gamma_0(23), \chi_{-23}).$$

Then Stark proves that

$$L(\rho, 1) = \frac{2\pi}{3\sqrt{23}}\langle\theta_\psi, \theta_\psi\rangle.$$

The main motivation

It follows that

$$\langle \theta_\psi, \theta_\psi \rangle = 3 \log \varepsilon.$$

Structure of the presentation

Where we are in the presentation

Notation

Throughout this presentation, let

- K be an imaginary quadratic field of discriminant D with Hilbert class field H,
- h_K, w_K and Cl_K be the class number, root number and class group of K (respectively)
- ψ be a Hecke character of infinity type $(2\ell, 0)$ for some $\ell \geq 0$, i.e. a homomorphism

  $$\psi : I_K \longrightarrow \mathbb{C}^\times$$

  such that $\psi((\alpha)) = \alpha^{2\ell}$ for all $\alpha \in K^\times$
- and $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ be fractional ideals of K.

Theta series attached to K

Consider

$$\left.\begin{aligned} \theta_\psi(q) &= \textstyle\sum_{\mathfrak{a}\in\mathcal{O}_K} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} \\ \theta_\mathfrak{a}(q) &= \textstyle\sum_{x\in\mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})} \end{aligned}\right\} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

Then

	θ_ψ	$\theta_{\mathfrak{a},\ell}$
$\ell > 0$	Newform	Cusp form
$\ell = 0$	$\psi^2 \neq 1$: Newform $\psi^2 = 1$: (genus) Eisenstein series	Not a cusp form

Some examples to keep in mind

	θ_ψ	$\theta_{\mathfrak{a},\ell}$
$\ell > 0$		
$\ell = 0$	$q\prod_{n\geq 1}(1-q^n)(1-q^{23n})$	$\theta_{\mathbb{Z}[i]}(q) = \sum_{x,y\in\mathbb{Z}} q^{x^2+y^2}$

Recall that

$$q\prod_{n\geq 1}(1-q^n)(1-q^{23n})$$

is the modular form in Stark's example.

Formulas for the Petersson inner product of those theta series

Recall that the Petersson inner product of any cusp forms $f, g \in S_k(\Gamma_0(N), \chi)$ is defined as

$$\langle f, g \rangle = \iint_{\Gamma_0(N)\backslash\mathcal{H}} f(\tau)\bar{g}(\tau)\Im(\tau)^k \mathrm{d}\mu(\tau).$$

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With minor effort, this formula can be used to compute the Petersson inner product numerically:

$$\langle f, g \rangle = \sum_{\gamma\in\Gamma_0(N)\backslash\mathcal{H}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\sqrt{1-x^2}}^{\infty} f(\tau)\bar{g}(\tau) y^{k-2} \mathrm{d}y\mathrm{d}x.$$

But this is very (very) slow and behaves badly as the level grows.

The quest for more efficient and useful formulas

Let ψ be such that θ_ψ is a cusp form. Then

1. Apply Rankin-Selberg:

$$\langle \theta_\psi, \theta_\psi \rangle = \left(\frac{\pi}{2} \frac{\phi(|D|)}{D^2} \frac{(4\pi)^{2\ell+1}}{\Gamma(2\ell+1)} \right)^{-1} L(\chi_D, 1) \operatorname{res}_{s=2\ell+1} L(\operatorname{Sym}^2 \theta_\psi, s)$$

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2. Isolate the residue of $L(\mathrm{Sym}^2 \theta_\psi, s)$:

$$\mathrm{res}_{s=2\ell+1} L(\mathrm{Sym}\, 2\theta_\psi, 1, s) = \prod_{p|D} (1 - p^{-1}) L(\psi^2, 2\ell + 1)$$

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3. When $\ell > 0$, express $L(\psi^2, 2\ell + 1)$ in terms of (derivatives of nearly holomorphic) Eisenstein series:

$$L(\psi^2, 2\ell{+}1) = \frac{4(2\pi)^{2\ell+1} \sqrt{|D|}^{2\ell-1}}{w_K \Gamma(2\ell+1)} \sum_{j=1}^{h_K} \psi^{-2}(\mathfrak{a}_j) N(\mathfrak{a}_j)^{4\ell} \delta^{2\ell-1} E_2(\bar{\mathfrak{a}}_j)$$

The most useful formulas for p-adic interpolation

	$\langle \theta_\psi, \theta_\psi \rangle$	$\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$
$\ell > 0$	$C_1 \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \delta^{2\ell-1} E_2(\mathcal{A})$	
$\ell = 0$	$\psi^2 = 1$: not applicable	not applicable

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$\ell > 0$	$C_1 \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \delta^{2\ell-1} E_2(\mathcal{A})$	$C_2 \sum_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_\mathfrak{c} \mathcal{O}_K} \lambda_\mathfrak{c}^{2\ell} \delta^{2\ell-1} E_2(\mathfrak{c})$
$\ell = 0$	$\psi^2 = 1$: not applicable	not applicable

Using the relation

$$\theta_{\mathfrak{a},\ell} = \frac{w_K}{h_K} \sum_{\psi} \psi(\mathfrak{a}) \theta_\psi$$

and the orthogonality of the newforms θ_ψ.

The most useful formulas for p-adic interpolation

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$\ell > 0$	$C_1 \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \delta^{2\ell-1} E_2(\mathcal{A})$	$C_2 \sum_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_\mathfrak{c} \mathcal{O}_K} \lambda_\mathfrak{c}^{2\ell} \delta^{2\ell-1} E_2(\mathfrak{c})$
$\ell = 0$	$C_3 \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \log \Phi(\mathcal{A})$	not applicable
	$\psi^2 = 1$: not applicable	

Here

$$\Phi(\mathcal{A}) = N(\mathcal{A})^6 |\Delta(\mathcal{A})|,$$

where

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

Where we are in the presentation

Bridging the gap between the "explicit" formulas and the algorithms

Here are some of the things one needs to do before implementing those formulas:

- Complete the L-functions $L(\mathrm{Sym}^2\,\theta_\psi, s)$ and $L(\psi, s)$ and find all the information about their functional equation,
- Find a way to compute with Hecke characters,
- Find an *efficient* way to compute

$$\delta^n E_2(\mathfrak{a}),$$

- Choose the computer algebra system that allows you to do all this!

The most efficient formula for computations

Experimentally, one finds that the most efficient way to compute the Petersson inner product of theta series is to compute the q-expansion of $\delta^n E_2$ by hand:

$$\delta^n E_2(\tau) = (-1)^n \left(\frac{1}{8\pi \Im(\tau)} - \frac{n+1}{24} \right) \frac{n!}{(4\pi\Im(\tau))^n} + \sum_{m \geq 1} \sigma(m) \left(\sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \frac{(r+2)_{n-r}}{(4\pi\Im(\tau))^{n-r}} m^r \right) q^m.$$

The resulting algorithm

This leads to the following

Theorem (S.)

There exists a software package to compute the Petersson inner product of the theta series defined above with the following properties:

- *It is fast (relative to the definition),*
- *It supports arbitrary precision (no coefficients stored, no database involved),*
- *User friendly (easy to download, help functions, well commented source code),*

Proof.

See the calculations at the end of the thesis! □

Where we are in the presentation

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