Petersson Inner Product of Binary Theta Series

A computational approach

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Mobius transformations

Let $\mathcal{H}$ be the Poincarre upper-half plane. Recall that $\mathrm{GL}_2(\mathbb{R})_+$ acts on $\mathcal{H}$ via Mobius transformations :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Definition

Let $N \geq 1$ and define the Hecke subgroup of level N as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | c \equiv 0 \pmod{N} \right\}.$$

Level N modular forms with characters

Definition

Let $N \geq 1$ and $k \geq 0$ be integers and let χ be a Dirichlet character mod N. A modular form of weight k, level N and character χ is a holomorphic function

$$f : \mathcal{H} \longrightarrow \mathbb{C}$$

such that

$$f(\gamma z) = \chi(d)(cz + d)^{-k} f(z)$$

for all $z \in \mathcal{H}$ and all $\gamma \in \Gamma_0(N)$, which satisfies certain growth conditions at the cusps. The $\mathbb{C}$-vector-space of such modular forms is denoted

$$M_k(\Gamma_0(N), \chi).$$

q-expansion of modular forms

Every modular form f has a Taylor (or Fourrier) expansion at infinity, called its q-expansion :

$$f(z) = \sum_{n=0}^{\infty} a_n q^n,$$

where $q = exp(2\pi iz)$. If

$$a_0(f) = 0,$$

f is called a *cusp form*.

Example : weight k Eisenstein series

Let $k \geq 4$ be an even integer and define

$$G_k(z) = \sum_{m,n} \frac{1}{(mz+n)^k} \in M_k(\Gamma_0(1), 1).$$

After renormalisation, the q-expansion of G_k is

$$E_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Important non-example : weight 2 Eisenstein series

In level 1, there are no modular forms of weight 2. However, one can still define the weight 2 Eisenstein series as

$$E_2(2) = \frac{1}{8\pi\Im(z)} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n) q^n.$$

It is an example of an *almost holomorphic* modular form of level 1 and weight 2.

Spaces of modular forms

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Spaces of modular forms

- $M_k(\Gamma_0(N), \chi)$ is finite dimensional.
- For every integer $n \geq 1$, one can define a *Hecke operator* T_n (depending on k, N and χ) which acts on $M_k(\Gamma_0(N), \chi)$.
- There exists a basis of common eigenvectors for all Hecke operators T_n with $(n, N) = 1$.

Petersson inner product

Let $f, g \in S_k(\Gamma_0(N), \chi)$ be two cusp forms. The Petersson inner product of f and g is defined as

$$\langle f, g \rangle = \frac{1}{\mathrm{Vol}(\Gamma_0(N) \setminus \mathcal{H})} \int_{\Gamma_0(N)\setminus\mathcal{H}} f(x+iy)\overline{g(x+iy)}y^k \mathrm{d}\mu,$$

where

$$\mathrm{d}\mu = \frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

is the $\mathrm{SL}_2(\mathbb{R})$-invariant measure on $\mathcal{H}$. Note that the intergal does not converge if neither f nor g is a cusp form.

Newforms

The space $S_k(\Gamma_0(N), \chi)$ splits naturally as

$$S_k(\Gamma_0(N), \chi) = S_k(\Gamma_0(N), \chi)^{\text{new}} \oplus S_k(\Gamma_0(N), \chi)^{\text{old}}.$$

Theorem

The space $S_k(\Gamma_0(N), \chi)^{new}$ has an orthogonal basis of eigenvectors for all Hecke operators. Elements of this basis are called newforms (after suitable normalization).

Summary

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2. The subspace $S_k(\Gamma_0(N), \chi)^{\text{new}}$ has distinguished elements (the newforms) which are mutually orthogonal and are eigenvectors for all Hecke operators.

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A half-integral weight theta series

Consider the function

$$\theta(z) = \sum_{x \in \mathbb{Z}} q^{x^2} = 1 + 2q + 2q^4 + O(q^5).$$

Then

$$\theta(\gamma z) = \epsilon(cz + d)^{1/2}\theta(z),$$

for all $\gamma \in \Gamma_0(4)$ and some $\epsilon_{c,d} \in \{\pm 1, \pm i\}$.

The setup for this talk