

# Petersson Inner Product of Theta Series

## An experimental approach

Nicolas SIMARD

McGill University

December 1st, 2017

## Stark's observation

Let  $K = \mathbb{Q}(\sqrt{-23})$  and let  $H$  be the HCF of  $K$ . Let

$$\psi : \text{Gal}(H/K) \rightarrow \text{GL}_1(\mathbb{C})$$

be a non-trivial one-dimensional Artin representation and let

$$\rho = \text{Ind}_K^{\mathbb{Q}} \psi : \text{Gal}(H/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{C})$$

be the induced representation.

## Stark's observation (cont.)

By Deligne-Serre, one has

$$L(\rho, s) = L(\theta_\psi, s),$$

where

$$\theta_\psi(q) = \eta(q)\eta(23q) = q \prod_{n \geq 1} (1 - q^n)(1 - q^{23n}) \in M_1(\Gamma_0(23), \chi_{-23}).$$

Then Stark proves that

$$\langle \theta_\psi, \theta_\psi \rangle = 3 \log \varepsilon,$$

where  $\varepsilon$  is the real root of

$$x^3 - x - 1.$$

# Structure of the talk

## Introduction

Motivation

Structure of the Talk

## Petersson Inner Product of Weight One Theta Series

Explicit Formulas

Generalizing Stark's Observation

## Petersson Inner Product of Higher Weight Theta Series

Explicit formulas

$p$ -adic interpolation of Petersson inner product of theta series

## Spaces of Theta Series

# Notation

Throughout this talks, let

- $K$  be an imaginary quadratic field of discriminant  $D$ ,
- $H$  be the Hilbert class field of  $K$ ,
- $h_K$  be the class number of  $K$ ,
- $w_K$  be the number of roots of unity in  $K$  and
- $ClK$  be the class group of  $K$ .

## Weight one theta series

Let  $\psi$  be a class character of  $K$ , i.e. a homomorphism

$$\psi : \text{Cl}_K \rightarrow \mathbb{C}^\times.$$

Then

$$\theta_\psi(q) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} \in M_1(\Gamma_0(|D|), \chi_D).$$

Moreover,  $\theta_\psi$  is an eigenform for all Hecke operators.

If  $\psi^2 = 1$ ,  $\theta_\psi$  is an Eisenstein series.

If  $\psi^2 \neq 1$ ,  $\theta_\psi$  is a cusp form (in fact, a newform).

# Stark's example

Let

$$K = \mathbb{Q}(\sqrt{-23})$$

and let  $\psi$  be a non-trivial class character as above. Then

$$\text{Stark's } \theta_\psi = \text{our } \theta_\psi \in M_1(\Gamma_0(23), \chi_{-23}).$$

Note that if  $\psi'$  is the other non-trivial class character, then

$$\theta_\psi = \theta_{\psi'}.$$

# Petersson inner product of weight one theta series

The Petersson inner product of any two  $f, g \in S_k(\Gamma_0(N), \chi)$  is defined as

$$\langle f, g \rangle = \iint_{\Gamma_0(N) \backslash \mathcal{H}} f(x + iy) \overline{g(x + iy)} y^k d\mu.$$

Then

## Proposition (S.)

*Let  $\psi$  be a class character which is not a genus character. Then*

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{-h_K}{3w_K^2} \sum_{\mathcal{A} \in Cl_K} \psi^2(\mathcal{A}) \log N(\mathcal{A})^6 |\Delta(\mathcal{A})|.$$



## Siegel units

Let  $\mathfrak{a}$  be a fractional ideal of  $K$  and define

$$|\delta_{\mathcal{A}}| = (N(\mathfrak{a})^6 |\Delta(\mathcal{O}_K) / \Delta(\mathfrak{a}^{-1})|)^{h_K},$$

where  $\mathfrak{a}$  is any ideal in the class  $\mathcal{A}$ . Then  $|\delta_{\mathcal{A}}|$  is a unit in  $H$ . Since  $\psi^2$  is not trivial, one sees that

$$\langle \theta_{\psi}, \theta_{\psi} \rangle = \frac{1}{3w_K^2} \sum_{\mathcal{A} \in \text{Cl}_K} \psi^2(\mathcal{A}) \log |\delta_{\mathcal{A}}|,$$

where  $\{\mathfrak{a}_1, \dots, \mathfrak{a}_{h_K}\}$  is a set of class representatives for  $\text{Cl}_K$ .

# What about Stark's observation?

One can write

$$\langle \theta_\psi, \theta_\psi \rangle = h_K \log \kappa_\psi,$$

where

$$\kappa_\psi = \prod_{j=1}^{h_K} \Phi(\mathfrak{a}_j)^{-\psi^2(\mathfrak{a}_j)}$$

with

$$\Phi(\mathfrak{a}) = \sqrt{N(\mathfrak{a})} |\Delta(\mathfrak{a})|^{1/12}.$$

## Question

*Is  $\kappa_\psi$  a unit in  $H$ ?*

Calcs in class nbr 3, 4, 5, 6.

# Generalizing Stark's Observation

## Proposition (S.)

Let  $\psi$  be a class character such that  $\psi^2$  is a non-trivial character with rational real part. Then  $\kappa_\psi$  is an algebraic integer which is a unit. Moreover, if  $\psi^2$  is a non-trivial genus character corresponding to the factorisation  $D = D_1 D_2$ , with  $D_1 > 0$  say, then

$$\kappa_\psi = \epsilon_{D_1}^{\frac{4h_{D_1} h_{D_2}}{w_K w_{D_2}}},$$

where  $\epsilon_{D_1}$  is the fundamental unit of  $\mathbb{Q}(\sqrt{D_1})$ ,  $h_{D_j}$  is the class number of  $\mathbb{Q}(\sqrt{D_j})$  and  $w_{D_2}$  is the number of roots of unity in  $\mathbb{Q}(\sqrt{D_2})$ .

# Examples

If  $K = \mathbb{Q}(\sqrt{-23})$ , the Proposition implies that  $\kappa_\psi$  is a unit. But is it in the Hilbert class field?

# Examples

If  $K = \mathbb{Q}(\sqrt{-23})$ , the Proposition implies that  $\kappa_\psi$  is a unit. But is it in the Hilbert class field?

If  $K = \mathbb{Q}(\sqrt{-39})$ , the Proposition implies

$$\kappa_\psi = \epsilon_{13}^{\frac{1}{3}},$$

which is *not* in the Hilbert class field.

## Stark's observation: the final word?

Note that  $\psi^2$  has rational real part if and only if its order divides 4 or 3.

### Corollary

*Suppose that  $K$  has class number divisible by 2 or 3. Then there exists a class character  $\psi$  such that*

$$\kappa_\psi$$

*is a unit.*

### Question

*Is the converse true?*

## Higher weight theta series

Let  $\ell$  be a positive integer and let  $\psi$  be a Hecke character of infinity type  $(2\ell, 0)$ , i.e. a homomorphism

$$\psi : I_K \rightarrow \mathbb{C}^\times$$

such that

$$\psi(\alpha \mathcal{O}_K) = \alpha^{2\ell} \text{ for all } \alpha \in K^\times.$$

Then

$$\theta_\psi(q) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

If  $\ell = 0$ , one recovers the weight one theta series introduced before. If  $\ell > 0$ , then  $\theta_\psi$  is a newform.

## Some notation

Let

$$E_2(\tau) = \frac{1}{8\pi\Im(\tau)} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n)q^n$$

be the nearly holomorphic weight 2 Eisenstein series and let

$$E_k(q) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$$

be the usual weight  $k$  Eisenstein series for  $k \geq 4$ , where  $q = e^{2\pi i\tau}$ .  
Let also  $\delta$  be the Shimura-Maass differential operator, so that

$$\delta E_k = \frac{1}{2\pi i} \frac{\partial E_k}{\partial \tau} - \frac{k}{4\pi\Im(\tau)} E_k.$$

Then  $\partial$  preserves the graded ring

$$\mathbb{C}[E_2, E_4, E_6]$$

of nearly holomorphic modular forms of level  $\mathrm{SL}_2(\mathbb{Z})$ .



# Petersson inner product of higher weight theta series

With the above notation, one has the following

## Proposition (S.)

*Let  $\ell > 0$  and let  $\psi$  be a Hecke character of infinity type  $(2\ell, 0)$ .  
Then*

$$\langle \theta_\psi, \theta_\psi \rangle = (|D|/4)^\ell \frac{4h_K}{w_K^2} \sum_{\mathcal{A} \in Cl_K} \psi^2(\mathcal{A}) \delta^{2\ell-1} E_2(\mathcal{A}).$$

# Theta series attached to ideals

Let  $\ell \geq 0$  and let  $\mathfrak{a}$  be a fractional ideal of  $K$ . Then

$$\theta_{\mathfrak{a},\ell}(q) = \sum_{x \in \mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

where  $\chi_D$  is the Kronecker symbol. If  $\ell > 0$ , then

$$\theta_{\mathfrak{a},\ell} \in S_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

## Theta series attached to ideals

Let  $\ell \geq 0$  and let  $\mathfrak{a}$  be a fractional ideal of  $K$ . Then

$$\theta_{\mathfrak{a},\ell}(q) = \sum_{x \in \mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

where  $\chi_D$  is the Kronecker symbol. If  $\ell > 0$ , then

$$\theta_{\mathfrak{a},\ell} \in S_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

Moreover,

$$\theta_{\psi}(q) = \frac{1}{w_K} \sum_{j=1}^{h_K} \psi^{-1}(\mathfrak{a}_j) \theta_{\mathfrak{a}_j,\ell}(q),$$

where  $\{\mathfrak{a}_1, \dots, \mathfrak{a}_{h_K}\}$  is a set of class representatives of  $\text{Cl}_K$ .

# Petersson inner product of theta series attached to ideals

Using the above relation between the two set f theta series, one has the following

## Corollary

*Let  $\ell > 0$  and let  $\mathfrak{a}$  and  $\mathfrak{b}$  be two fractional ideals of  $K$ . Then*

$$\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle = 4(|D|/4)^\ell \sum_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K} \lambda_{\mathfrak{c}}^{2\ell} \delta^{2\ell-1} E_2(\mathfrak{c}),$$

*where the sum is over a set of representatives  $\mathfrak{c}$  of ideals classes in  $Cl_K$  such that  $\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K$ .*

# Algebraizing the Petersson inner product

Recall that

$$\delta^n E_2 \in \mathbb{C}[E_2, E_4, E_6].$$

To  $p$ -adically interpolate, one first needs to algebraize the Petersson inner product.

## Proposition (Chowla-Selberg period)

Let

$$\Omega_K = \frac{1}{\sqrt{4\pi|D|}} \left( \prod_{n=1}^{|D|-1} \Gamma\left(\frac{n}{|D|}\right)^{\chi_D(n)} \right)^{w_K/(4h_K)}$$

and let  $\mathfrak{c}$  be a fractional ideal of  $K$ . Then

$$E_k(\mathfrak{c}) \in \Omega_K \bar{\mathbb{Q}}$$

for  $k = 2, 4$  and  $6$ .

# Algebraizing the Petersson inner product

Recall that

$$\delta^n E_2 \in \mathbb{C}[E_2, E_4, E_6].$$

To  $p$ -adically interpolate, one first needs to algebraize the Petersson inner product.

## Proposition (CM theory)

Let  $\Omega_{\mathfrak{c}} \in \mathbb{C}^\times$  be such that the elliptic curve

$$\mathbb{C}/\Omega_{\mathfrak{c}}\mathfrak{c}$$

is defined over  $H$ . Then

$$E_k(\mathfrak{c}) \in (2\pi i \Omega_{\mathfrak{c}})^{-k} H$$

for  $k = 2, 4$  and  $6$ .

# $p$ -adic interpolation of Petersson inner product of theta series

Suppose that  $D$  is prime and let  $p$  be a prime  $\neq 2, 3$  which splits in  $K$ , say  $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$ .

Let  $\mathfrak{a}$  and  $\mathfrak{b}$  be two fractional ideals of  $K$  which are such that

$$\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \mathcal{O}_K$$

and fix an isomorphism

$$\mathbb{Q}_p/\mathbb{Z}_p \rightarrow \bigcup_{n \geq 1} \bar{\mathfrak{p}}^{-n}\mathfrak{c}/\mathfrak{c}.$$

Let also

$$\mathcal{W} = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$$

denote the  $p$ -adic weight space.

# $p$ -adic interpolation of Petersson inner product of theta series

With the notation above, one has the following

## Theorem (S.)

*There exists a  $p$ -adic analytic function*

$$F : \mathcal{W} \rightarrow \mathbb{C}_p$$

*with the property that*

$$F(\ell) = (\text{Frob}_p^{-1} - p^{2\ell-1})(\text{Frob}_p^{-1} - p^{2\ell}) \left( \frac{\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle}{(2\pi i \Omega_c)^{-4\ell}} \right) \text{ for all } \ell > 0,$$

*where  $\text{Frob}_p = \left( \frac{H/K}{p} \right)$  is the Artin symbol.*



# Thank you!

Presentation and notes available at:

<https://github.com/NicolasSimard/Notes>

Code available at : <https://github.com/NicolasSimard/ENT>

Or from my webpage: <http://www.math.mcgill.ca/nsimard/>