Petersson Inner Product of Theta Series

An experimental approach

Nicolas Simard

McGill University

December 1st, 2017

Stark's observation

Let $K = \mathbb{Q}(\sqrt{-23})$ and let H be the HCF of K. Let

$$\psi : \mathrm{Gal}(H/K) \to \mathrm{GL}_1(\mathbb{C})$$

be a non-trivial one-dimensional Artin representation and let

$$\rho = \mathrm{Ind}_K^{\mathbb{Q}} \psi : \mathrm{Gal}(H/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$$

be the induced representation.

Stark's observation (cont.)

By Deligne-Serre, one has

$$L(\rho, s) = L(\theta_\psi, s),$$

where

$$\theta_\psi(q) = \eta(q)\eta(23q) = q \prod_{n \geq 1} (1-q^n)(1-q^{23n}) \in M_1(\Gamma_0(23), \chi_{-23}).$$

Then Stark proves that

$$\langle \theta_\psi, \theta_\psi \rangle = 3 \log \varepsilon,$$

where ε is the real root of

$$x^3 - x - 1.$$

Structure of the talk

Notation

Throughout this talks, let

- K be an imaginary quadratic field of discriminant D,
- H be the Hilbert class field of K,
- h_K be the class number of K,
- w_K be the number of roots of unity in K and
- Cl_K be the class group of K.

Weight one theta series

Let ψ be a class character of K, i.e. a homomorphism

$$\psi : \mathrm{Cl}_K \to \mathbb{C}^\times.$$

Then

$$\theta_\psi(q) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} \in M_1(\Gamma_0(|D|), \chi_D).$$

Weight one theta series

Let ψ be a class character of K, i.e. a homomorphism

$$\psi : \mathrm{Cl}_K \to \mathbb{C}^\times.$$

Then

$$\theta_\psi(q) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} \in M_1(\Gamma_0(|D|), \chi_D).$$

- θ_ψ is an eigenform for all Hecke operators;
- if $\psi^2 = 1$, θ_ψ is an Eisenstein series;
- if $\psi^2 \neq 1$, θ_ψ is a cusp form (in fact, a newform).

Stark's example

Let

$$K = \mathbb{Q}(\sqrt{-23})$$

and let ψ be a non-trivial class character as above. Then

$$\text{Stark's } \theta_\psi = \text{our } \theta_\psi \in M_1(\Gamma_0(23), \chi_{-23}).$$

Note that if ψ' is the other non-trivial class character, then

$$\theta_\psi = \theta_{\psi'}.$$

Petersson inner product of weight one theta series

The Petersson inner product of any two $f, g \in S_k(\Gamma_0(N), \chi)$ is defined as

$$\langle f, g \rangle = \iint_{\Gamma_0(N) \backslash \mathcal{H}} f(x+iy)\overline{g(x+iy)}y^k \mathrm{d}\mu.$$

Petersson inner product of weight one theta series

The Petersson inner product of any two $f, g \in S_k(\Gamma_0(N), \chi)$ is defined as

$$\langle f, g \rangle = \iint_{\Gamma_0(N)\backslash\mathcal{H}} f(x+iy)\overline{g(x+iy)}y^k \mathrm{d}\mu.$$

Proposition (S.)

Let ψ be a class character which is not a genus character. Then

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{-h_K}{3w_K^2} \sum_{\mathcal{A}\in Cl_K} \psi^2(\mathcal{A}) \log N(\mathcal{A})^6 |\Delta(\mathcal{A})|.$$

Siegel units

Define

$$|\delta_{\mathcal{A}}| = (N(\mathfrak{a})^6 |\Delta(\mathcal{O}_K)/\Delta(\mathfrak{a}^{-1})|)^{h_K},$$

where $\mathfrak{a}$ is any ideal in the class $\mathcal{A}$. Then $|\delta_{\mathcal{A}}|$ is the absolute value of a (Siegel) unit in H.

Siegel units

Define

$$|\delta_{\mathcal{A}}| = (N(\mathfrak{a})^6 |\Delta(\mathcal{O}_K)/\Delta(\mathfrak{a}^{-1})|)^{h_K},$$

where $\mathfrak{a}$ is any ideal in the class $\mathcal{A}$. Then $|\delta_{\mathcal{A}}|$ is the absolute value of a (Siegel) unit in H.

Since ψ^2 is not trivial, one sees that

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{1}{3w_K^2} \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \log |\delta_{\mathcal{A}}|.$$

What about Stark's observation?

Assume $D < -4$. Then one can write

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{-h_K}{3w_K^2} \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \log N(\mathcal{A})^6 |\Delta(\mathcal{A})| = h_K \log \kappa_\psi.$$

Here,

$$\kappa_\psi = \prod_{\mathcal{A} \in \mathrm{Cl}_K} \Phi(\mathcal{A})^{-\psi^2(\mathcal{A})}$$

with

$$\Phi(\mathcal{A}) = \sqrt{N(\mathfrak{a})} |\Delta(\mathfrak{a})|^{1/12},$$

where $\mathfrak{a}$ is any ideal in the class $\mathcal{A}$.

What about Stark's observation?

Assume $D < -4$. Then one can write

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{-h_K}{3w_K^2} \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \log N(\mathcal{A})^6 |\Delta(\mathcal{A})| = h_K \log \kappa_\psi.$$

Here,

$$\kappa_\psi = \prod_{\mathcal{A} \in \mathrm{Cl}_K} \Phi(\mathcal{A})^{-\psi^2(\mathcal{A})}$$

with

$$\Phi(\mathcal{A}) = \sqrt{N(\mathfrak{a})} |\Delta(\mathfrak{a})|^{1/12},$$

where $\mathfrak{a}$ is any ideal in the class $\mathcal{A}$.

Question

Is κ_ψ a unit in H?

Script available at `https://github.com/NicolasSimard/ENT/blob/master/comps/ULcomps1.gp`

Generalizing Stark's Observation

Proposition (S.)

Let ψ be a class character such that ψ^2 is a non-trivial character with rational real part. Then κ_ψ is an algebraic integer which is a unit. Moreover, if ψ^2 is a non-trivial genus character corresponding to the factorisation $D = D_1 D_2$, with $D_1 > 0$ say, then

$$\kappa_\psi = \epsilon_{D_1}^{\frac{4h_{D_1}h_{D_2}}{w_K w_{D_2}}},$$

where ϵ_{D_1} is the fundamental unit of $\mathbb{Q}(\sqrt{D_1})$, h_{D_j} is the class number of $\mathbb{Q}(\sqrt{D_j})$ and w_{D_2} is the number of roots of unity in $\mathbb{Q}(\sqrt{D_2})$.

Stark's observation: the final word?

Note that ψ^2 has rational real part if and only if its order divides 4 or 3.

Corollary

Suppose that K has class number divisible by 2 or 3. Then there exists a class character ψ for which κ_ψ is a unit.

Stark's observation: the final word?

Note that ψ^2 has rational real part if and only if its order divides 4 or 3.

Corollary

Suppose that K has class number divisible by 2 or 3. Then there exists a class character ψ for which κ_ψ is a unit.

Question

Is the converse true?

Higher weight theta series

Let ℓ be a positive integer and let ψ be a Hecke character of infinity type $(2\ell, 0)$, i.e. a homomorphism

$$\psi : I_K \to \mathbb{C}^\times$$

such that

$$\psi(\alpha\mathcal{O}_K) = \alpha^{2\ell} \text{ for all } \alpha \in K^\times.$$

Then

$$\theta_\psi(q) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

Higher weight theta series

Let ℓ be a positive integer and let ψ be a Hecke character of infinity type $(2\ell, 0)$, i.e. a homomorphism

$$\psi : I_K \to \mathbb{C}^\times$$

such that

$$\psi(\alpha\mathcal{O}_K) = \alpha^{2\ell} \text{ for all } \alpha \in K^\times.$$

Then

$$\theta_\psi(q) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

- For $\ell = 0$, one recovers the weight one theta series introduced before;
- for $\ell > 0$, θ_ψ is a newform.

Recall

Let

$$E_2(\tau) = \frac{1}{8\pi\Im(\tau)} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n) q^n$$

be the nearly holomorphic weight 2 Eisenstein series and let

$$E_k(q) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

be the usual weight k Eisenstein series for $k \geq 4$, where $q = e^{2\pi i\tau}$.

Recall

Let

$$E_2(\tau) = \frac{1}{8\pi\Im(\tau)} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n) q^n$$

be the nearly holomorphic weight 2 Eisenstein series and let

$$E_k(q) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

be the usual weight k Eisenstein series for $k \geq 4$, where $q = e^{2\pi i \tau}$. Let also δ be the Shimura-Maass differential operator, so that

$$\delta E_k = \frac{1}{2\pi i} \frac{\partial E_k}{\partial \tau} - \frac{k}{4\pi\Im(\tau)} E_k.$$

Then ∂ raises the weight by 2 and preserves the graded ring

$$\mathbb{C}[E_2, E_4, E_6]$$

of nearly holomorphic modular forms of level $\mathrm{SL}_2(\mathbb{Z})$.

Petersson inner product of higher weight theta series

With the above notation, one has the following

Proposition (S.)

Let $\ell > 0$ and let ψ be a Hecke character of infinity type $(2\ell, 0)$. Then

$$\langle \theta_\psi, \theta_\psi \rangle = (|D|/4)^\ell \frac{4h_K}{w_K^2} \sum_{\mathcal{A} \in Cl_K} \psi^2(\mathcal{A}) \delta^{2\ell-1} E_2(\mathcal{A}).$$

Theta series attached to ideals

Let $\ell \geq 0$ and let $\mathfrak{a}$ be a fractional ideal of K and define

$$\theta_{\mathfrak{a},\ell}(q) = \sum_{x \in \mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

where χ_D is the Kronecker symbol.

Theta series attached to ideals

Let $\ell \geq 0$ and let $\mathfrak{a}$ be a fractional ideal of K and define

$$\theta_{\mathfrak{a},\ell}(q) = \sum_{x \in \mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

where χ_D is the Kronecker symbol.

- If $\ell > 0$, then
$$\theta_{\mathfrak{a},\ell} \in S_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$
- For any $\ell \geq 0$, one has
$$\theta_{\mathfrak{a},\ell} = \frac{w_K}{h_K} \sum_{\psi} \psi(\mathfrak{a}) \theta_{\psi},$$
where the sum is over the h_K Hecke characters of infinity type $(2\ell, 0)$.

Petersson inner product of theta series attached to ideals

Using the above relation between the two set of theta series, one has the following

Corollary

Let $\ell > 0$ and let $\mathfrak{a}$ and $\mathfrak{b}$ be two fractional ideals of K. Then

$$\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle = 4(|D|/4)^{\ell} \sum_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K} \lambda_{\mathfrak{c}}^{2\ell} \delta^{2\ell-1} E_2(\mathfrak{c}),$$

where the sum is over a set of representatives $\mathfrak{c}$ of ideals classes in Cl_K such that $\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K$.

Algebrizing the Petersson inner product

Recall that

$$\delta^n E_2 \in \mathbb{C}[E_2, E_4, E_6].$$

There are two ways to algebrize CM values of modular forms.

Algebrizing the Petersson inner product

Recall that

$$\delta^n E_2 \in \mathbb{C}[E_2, E_4, E_6].$$

There are two ways to algebrize CM values of modular forms.

Proposition (Chowla-Selberg period)

Let

$$\Omega_K = \frac{1}{\sqrt{4\pi|D|}} \left(\prod_{n=1}^{|D|-1} \Gamma\left(\frac{n}{|D|}\right)^{\chi_D(n)} \right)^{w_K/(4h_K)}$$

and let $\mathfrak{c}$ be a fractional ideal of K. Then

$$E_k(\mathfrak{c}) \in \Omega_K^k \bar{\mathbb{Q}}$$

for $k = 2, 4$ and 6.

Script available at `https://github.com/NicolasSimard/ENT/blob/master/comps/ULcomps2.gp`

Algebrizing the Petersson inner product

Recall that

$$\delta^n E_2 \in \mathbb{C}[E_2, E_4, E_6].$$

There are two ways to algebrize CM values of modular forms.

Proposition (CM theory)

Let $\Omega_{\mathfrak{c}} \in \mathbb{C}^\times$ be such that the elliptic curve

$$\mathbb{C}/\Omega_{\mathfrak{c}}\mathfrak{c}$$

is defined over H. Then

$$E_k(\mathfrak{c}) \in (2\pi i \Omega_{\mathfrak{c}})^{-k} H$$

for $k = 2, 4$ and 6.

Script available at `https://github.com/NicolasSimard/ENT/blob/master/comps/ULcomps2.gp`

p-adic interpolation of Petersson inner product of theta series

Suppose that D is prime and let p be a prime $\neq 2, 3$ which splits in K, say $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$.
Let $\mathfrak{a}$ and $\mathfrak{b}$ be two fractional ideals of K which are such that

$$\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \mathcal{O}_K$$

and fix an isomorphism

$$\mathbb{Q}_p/\mathbb{Z}_p \to \bigcup_{n\geq 1} \bar{\mathfrak{p}}^{-n}\mathfrak{c}/\mathfrak{c}.$$

Let also

$$\mathcal{W} = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$$

denote the p-adic weight space.

p-adic interpolation of Petersson inner product of theta series

With the notation above, one has the following

Theorem (S.)

There exists a p-adic analytic function

$$F : \mathcal{W} \to \mathbb{C}_p$$

with the property that

$$F(\ell) = (\mathrm{Frob}_{\mathfrak{p}}^{-1} - p^{2\ell-1})(\mathrm{Frob}_{\mathfrak{p}}^{-1} - p^{2\ell}) \left(\frac{\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle}{(2\pi i \Omega_{\mathfrak{c}})^{-4\ell}} \right) \text{ for all } \ell > 0,$$

where $\mathrm{Frob}_{\mathfrak{p}} = \left(\frac{H/K}{\mathfrak{p}} \right)$ *is the Artin symbol.*

"Petersson inner product" of weight one theta series

Recall that

$$\theta_{\mathfrak{a},0} \in M_1(\Gamma_0(|D|), \chi_D).$$

Using the relation

$$\theta_\psi(q) = \frac{1}{w_K} \sum_{j=1}^{h_K} \psi^{-1}(\mathfrak{a}_j)\theta_{\mathfrak{a}_j,\ell}(q)$$

and the explicit formulas for $\langle \theta_\psi, \theta_\psi \rangle$, one has *formally*

$$\langle \theta_{\mathfrak{a},0}, \theta_{\mathfrak{b},0} \rangle = \frac{-1}{3} \log(N(\mathfrak{c})^6 |\Delta(\mathfrak{c})|)$$

when D is prime.

Value of F outside the range of interpolation

With the same notation as before, one has the following

Theorem (S.)

Let $g_0^{(p)}$ be the p-adic modular form with q-expansion

$$g_0^{(p)}(q) = \frac{\Delta(q^p)^{p+1}}{\Delta(q)^p \Delta(q^{p^2})}.$$

Then

$$F(0) = \frac{-1}{6p} \log_p g_0^{(p)}(P_{\mathfrak{c}}),$$

where $P_{\mathfrak{c}}$ is a trivialized CM Elliptic curve attached to $\mathfrak{c}$.

A formal computation

Formally, one sees that

$$\begin{aligned} F(0) &= \frac{-1}{6p} \log_p g_0^{(p)}(P_{\mathfrak{c}}) \\ &= \frac{-1}{6}(\mathrm{Frob}_{\mathfrak{p}}^{-1} - p^{-1})(\mathrm{Frob}_{\mathfrak{p}}^{-1} - 1) \log_p \Delta(\mathfrak{c}) \end{aligned}$$

A formal computation

Formally, one sees that

$$\begin{aligned} F(0) &= \frac{-1}{6p}\log_p g_0^{(p)}(P_{\mathfrak{c}}) \\ &= \frac{-1}{6}(\mathrm{Frob}_{\mathfrak{p}}^{-1} - p^{-1})(\mathrm{Frob}_{\mathfrak{p}}^{-1} - 1)\log_p \Delta(\mathfrak{c}) \end{aligned}$$

Compare with

$$F(\ell) = (\mathrm{Frob}_{\mathfrak{p}}^{-1} - p^{2\ell-1})(\mathrm{Frob}_{\mathfrak{p}}^{-1} - p^{2\ell})\left(\frac{\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell}\rangle}{(2\pi i \Omega_{\mathfrak{c}})^{-4\ell}}\right)$$

at $\ell = 0$ with the formal expression

$$\langle \theta_{\mathfrak{a},0}, \theta_{\mathfrak{b},0}\rangle = \frac{-1}{3}\log(N(\mathfrak{c})^6|\Delta(\mathfrak{c})|).$$

Thank you!

Presentation and notes available at:
https://github.com/NicolasSimard/Notes
Code available at : https://github.com/NicolasSimard/ENT
Or from my webpage: http://www.math.mcgill.ca/nsimard/