Petersson Inner Product of Theta Series
PhD Defense

Nicolas SIMARD

McGill University

April 12th, 2018

L-functions at $s = 1$

It is a well-known (but fascinating) fact that many *L*-functions contain arithmetic informations in their value at $s = 1$:

1. $\zeta(s)$ at $s = 1$: Infinitely many primes
2. $L(\chi, s)$ at $s = 1$: Infinitely many primes in arithmetic progressions
3. $\zeta_F(s)$ at $s = 1$: Class number formula

L-functions at $s = 1$

It is a well-known (but fascinating) fact that many *L*-functions contain arithmetic informations in their value at $s = 1$:

1. $\zeta(s)$ at $s = 1$: Infinitely many primes
2. $L(\chi, s)$ at $s = 1$: Infinitely many primes in arithmetic progressions
3. $\zeta_F(s)$ at $s = 1$: Class number formula

Conjecture (Stark (Idea))

In general, L-functions of Artin representations have a (relatively) explicit expression involving arithmetic invariants of the number fields involved.

An observation of Stark

Let $K = \mathbb{Q}(\sqrt{-23})$ and let H be its Hilbert class field. Let

$$\psi : \mathsf{Gal}(H/K) \to \mathbb{C}^{\times} = \mathsf{GL}_1(\mathbb{C})$$

be a non-trivial one-dimensional Artin representation and let

$$\rho = \mathsf{Ind}_K^{\mathbb{Q}} \psi : \mathsf{Gal}(H/\mathbb{Q}) \to \mathsf{GL}_2(\mathbb{C})$$

be the induced representation. Then one can consider the associated Artin L-function

$$L(\psi, s) = L(\rho, s).$$

An observation of Stark

On the one hand, in accordance with his conjecture (which was known in this case), Stark shows that

$$L(\rho, 1) = \frac{2\pi}{\sqrt{23}} \log \varepsilon,$$

where ε is the real root of

$$x^3 - x - 1.$$

Note that ε generates H over K.

An observation of Stark

On the other hand, by the Deligne-Serre theorem, one has

$$L(\rho, s) = L(\theta_\psi, s),$$

where

$$\theta_\psi(q) = \eta(q)\eta(23q) = q \prod_{n \geq 1}(1 - q^n)(1 - q^{23n}) \in M_1(\Gamma_0(23), \chi_{-23}).$$

An observation of Stark

On the other hand, by the Deligne-Serre theorem, one has

$$L(\rho, s) = L(\theta_\psi, s),$$

where

$$\theta_\psi(q) = \eta(q)\eta(23q) = q \prod_{n \geq 1}(1 - q^n)(1 - q^{23n}) \in M_1(\Gamma_0(23), \chi_{-23}).$$

Then Stark proves that

$$L(\rho, 1) = \frac{2\pi}{3\sqrt{23}} \langle \theta_\psi, \theta_\psi \rangle.$$

The main motivation

It follows that

$$\langle \theta_\psi, \theta_\psi \rangle = 3 \log \varepsilon.$$

Structure of the presentation

Introduction

Petersson inner product of theta series

Algorithms

Generalizing Stark's observation

p-adic interpolation

Experimentation, invariants and observations

Conclusion

Where we are in the presentation

Introduction

Petersson inner product of theta series

Algorithms

Generalizing Stark's observation

p-adic interpolation

Experimentation, invariants and observations

Conclusion

Notation

Throughout this presentation, let

- K be an imaginary quadratic field of discriminant D with Hilbert class field H,
- h_K, w_K and Cl_K be the class number, root number and class group of K (respectively)
- ψ be a Hecke character of infinity type $(2\ell, 0)$ for some $\ell \geq 0$, i.e. a homomorphism

$$\psi : I_K \longrightarrow \mathbb{C}^\times$$

  such that $\psi((\alpha)) = \alpha^{2\ell}$ for all $\alpha \in K^\times$

- and $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ be fractional ideals of K.

Theta series attached to K

Consider

$$\left.\begin{array}{ll} \theta_\psi(q) & = \sum_{\mathfrak{a} \in \mathcal{O}_K} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} \\[2mm] \theta_{\mathfrak{a},\ell}(q) & = \sum_{x \in \mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})} \end{array}\right\} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

Theta series attached to K

Consider

$$\left.\begin{array}{rl} \theta_\psi(q) & = \sum_{\mathfrak{a} \in \mathcal{O}_K} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} \\[2mm] \theta_{\mathfrak{a},\ell}(q) & = \sum_{x \in \mathfrak{a}} x^{2\ell} q^{N(x)/N(\mathfrak{a})} \end{array}\right\} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D).$$

Then

	θ_ψ	$\theta_{\mathfrak{a},\ell}$
$\ell > 0$	Newform	Cusp form
$\ell = 0$	$\psi^2 \neq 1$: Newform	Not a cusp form
	$\psi^2 = 1$: (genus) Eisenstein series	

Some examples to keep in mind

	θ_ψ	$\theta_{\mathfrak{a},\ell}$
$\ell > 0$		
$\ell = 0$	$q \prod_{n \geq 1}(1 - q^n)(1 - q^{23n})$	$\theta_{\mathbb{Z}[i]}(q) = \sum_{x,y \in \mathbb{Z}} q^{x^2 + y^2}$

Recall that

$$q \prod_{n \geq 1}(1 - q^n)(1 - q^{23n})$$

is the modular form in Stark's example.

Formulas for the Petersson inner product of those theta series

Recall that the Petersson inner product of any cusp forms $f, g \in S_k(\Gamma_0(N), \chi)$ is defined as

$$\langle f, g \rangle = \iint_{\Gamma_0(N) \backslash \mathcal{H}} f(\tau) \bar{g}(\tau) \Im(\tau)^k d\mu(\tau).$$

Formulas for the Petersson inner product of those theta series

Recall that the Petersson inner product of any cusp forms $f, g \in S_k(\Gamma_0(N), \chi)$ is defined as

$$\langle f, g \rangle = \iint_{\Gamma_0(N) \backslash \mathcal{H}} f(\tau) \bar{g}(\tau) \Im(\tau)^k d\mu(\tau).$$

With minor effort, this formula can be used to compute the Petersson inner product numerically:

$$\langle f, g \rangle = \sum_{\gamma \in \Gamma_0(N) \backslash \mathsf{SL}_2(\mathbb{Z})} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\sqrt{1-x^2}}^{\infty} f(\tau) \bar{g}(\tau) y^{k-2} dy dx.$$

But this is very (very) slow and behaves badly as the level grows.

The quest for more efficient and useful formulas

Let ψ be such that θ_ψ is a cusp form. Then

1. Apply Rankin-Selberg:

$$\langle \theta_\psi, \theta_\psi \rangle = \left(\frac{\pi}{2} \frac{\phi(|D|)}{D^2} \frac{(4\pi)^{2\ell+1}}{\Gamma(2\ell+1)} \right)^{-1} L(\chi_D, 1) \operatorname{res}_{s=2\ell+1} L(\operatorname{Sym}^2 \theta_\psi, s)$$

The quest for more efficient and useful formulas

Let ψ be such that θ_ψ is a cusp form. Then

1. Apply Rankin-Selberg:

$$\langle \theta_\psi, \theta_\psi \rangle = \left(\frac{\pi}{2} \frac{\phi(|D|)}{D^2} \frac{(4\pi)^{2\ell+1}}{\Gamma(2\ell+1)} \right)^{-1} L(\chi_D, 1) \, \text{res}_{s=2\ell+1} \, L(\text{Sym}^2 \theta_\psi, s)$$

2. Isolate the residue of $L(\text{Sym}^2 \theta_\psi, s)$:

$$\text{res}_{s=2\ell+1} L(\text{Sym } 2\theta_\psi, 1, s) = \prod_{p|D} (1 - p^{-1}) L(\psi^2, 2\ell+1)$$

The quest for more efficient and useful formulas

Let ψ be such that θ_ψ is a cusp form. Then

1. Apply Rankin-Selberg:

$$\langle \theta_\psi, \theta_\psi \rangle = \left(\frac{\pi}{2} \frac{\phi(|D|)}{D^2} \frac{(4\pi)^{2\ell+1}}{\Gamma(2\ell+1)} \right)^{-1} L(\chi_D, 1) \, \mathsf{res}_{s=2\ell+1} \, L(\mathsf{Sym}^2 \theta_\psi, s)$$

2. Isolate the residue of $L(\mathsf{Sym}^2 \theta_\psi, s)$:

$$\mathsf{res}_{s=2\ell+1} L(\mathsf{Sym}\, 2\theta_\psi, 1, s) = \prod_{p|D} (1 - p^{-1}) L(\psi^2, 2\ell+1)$$

3. When $\ell > 0$, express $L(\psi^2, 2\ell+1)$ in terms of (derivatives of nearly holomorphic) Eisenstein series:

$$L(\psi^2, 2\ell+1) = \frac{4(2\pi)^{2\ell+1} \sqrt{|D|}^{2\ell-1}}{w_K \Gamma(2\ell+1)} \sum_{j=1}^{h_K} \psi^{-2}(\mathfrak{a}_j) N(\mathfrak{a}_j)^{4\ell} \delta^{2\ell-1} E_2(\bar{\mathfrak{a}}_j)$$

The most useful formulas for p-adic interpolation

	$\langle \theta_\psi, \theta_\psi \rangle$	$\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$
$\ell > 0$	$C_1 \displaystyle\sum_{\mathcal{A} \in \mathsf{Cl}_K} \psi^2(\mathcal{A}) \delta^{2\ell-1} E_2(\mathcal{A})$	
$\ell = 0$	$\psi^2 = 1$: not applicable	not applicable

The most useful formulas for *p*-adic interpolation

	$\langle \theta_\psi, \theta_\psi \rangle$	$\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$
$\ell > 0$	$C_1 \displaystyle\sum_{\mathcal{A} \in \mathsf{Cl}_K} \psi^2(\mathcal{A}) \delta^{2\ell-1} E_2(\mathcal{A})$	$C_2 \displaystyle\sum_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_{\mathfrak{c}} \mathcal{O}_K} \lambda_{\mathfrak{c}}^{2\ell} \delta^{2\ell-1} E_2(\mathfrak{c})$
$\ell = 0$	$\psi^2 = 1$: not applicable	not applicable

Using the relation

$$\theta_{\mathfrak{a},\ell} = \frac{w_K}{h_K} \sum_\psi \psi(\mathfrak{a}) \theta_\psi$$

and the orthogonality of the newforms θ_ψ.

The most useful formulas for *p*-adic interpolation

	$\langle \theta_\psi, \theta_\psi \rangle$	$\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$
$\ell > 0$	$C_1 \sum\limits_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \delta^{2\ell-1} E_2(\mathcal{A})$	$C_2 \sum\limits_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_\mathfrak{c} \mathcal{O}_K} \lambda_\mathfrak{c}^{2\ell} \delta^{2\ell-1} E_2(\mathfrak{c})$
$\ell = 0$	$C_3 \sum\limits_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \log \Phi(\mathcal{A})$ $\psi^2 = 1$: not applicable	not applicable

Here

$$\Phi^{12}(\mathcal{A}) = N(\mathcal{A})^6 |\Delta(\mathcal{A})|,$$

where

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

The most efficient formula for computations

Experimentally, one finds that the most efficient way to compute
the Petersson inner product of theta series is to compute the
q-expansion of $\delta^n E_2$ by hand:

$$
\delta^n E_2(\tau) = (-1)^n \left(\frac{1}{8\pi\Im(\tau)} - \frac{n+1}{24} \right) \frac{n!}{(4\pi\Im(\tau))^n}
$$
$$
+ \sum_{m \geq 1} \sigma(m) \left(\sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \frac{(r+2)_{n-r}}{(4\pi\Im(\tau))^{n-r}} m^r \right) q^m.
$$

Where we are in the presentation

Bridging the gap between the "explicit" formulas and the algorithms

Here are some of the things one needs to do before implementing those formulas:

- Complete the *L*-functions $L(\text{Sym}^2 \theta_\psi, s)$ and $L(\psi, s)$ and find all the information about their functional equation,

- Find a way to compute with Hecke characters,

- Find an *efficient* way to compute

$$\delta^n E_2(\mathfrak{a}),$$

- Choose the computer algebra system that allows you to do all this!

The resulting algorithm

This leads to the following

Theorem (S.)

There exists a software package in PARI/GP to compute the Petersson inner product of the theta series defined above with the following properties:

- *It is fast (relative to the definition),*
- *It supports arbitrary precision (no coefficients stored, no database involved),*
- *User friendly (easy to download, help functions, well commented source code).*

The resulting algorithm

This leads to the following

Theorem (S.)

There exists a software package in PARI/GP to compute the Petersson inner product of the theta series defined above with the following properties:

- *It is fast (relative to the definition),*
- *It supports arbitrary precision (no coefficients stored, no database involved),*
- *User friendly (easy to download, help functions, well commented source code).*

Proof.

See the calculations in Part 3 of the thesis and look at the source code online! □

Where we are in the presentation

What about Stark's observation?

Using the above formula when $\ell = 0$, one has

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{-h_K}{3w_K^2} \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \log N(\mathcal{A})^6 |\Delta(\mathcal{A})| = h_K \log \kappa_\psi,$$

where

$$\kappa_\psi = \prod_{\mathcal{A} \in \mathrm{Cl}_K} \Phi(\mathcal{A})^{-\psi^2(\mathcal{A})},$$

with

$$\Phi(\mathcal{A}) = \sqrt{N(\mathfrak{a})} |\Delta(\mathfrak{a})|^{1/12}$$

as before, where now $\mathfrak{a}$ is any ideal in the class $\mathcal{A}$.

What about Stark's observation?

Using the above formula when $\ell = 0$, one has

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{-h_K}{3w_K^2} \sum_{\mathcal{A} \in \mathsf{Cl}_K} \psi^2(\mathcal{A}) \log N(\mathcal{A})^6 |\Delta(\mathcal{A})| = h_K \log \kappa_\psi,$$

where

$$\kappa_\psi = \prod_{\mathcal{A} \in \mathsf{Cl}_K} \Phi(\mathcal{A})^{-\psi^2(\mathcal{A})},$$

with

$$\Phi(\mathcal{A}) = \sqrt{N(\mathfrak{a})}|\Delta(\mathfrak{a})|^{1/12}$$

as before, where now $\mathfrak{a}$ is any ideal in the class $\mathcal{A}$.

Question
Is κ_ψ a unit in H?

A corollary and some examples

Some examples:

- $K = \mathbb{Q}(\sqrt{-23})$ ($h_K = 3$): κ_ψ is a unit and numerically, $\kappa_\psi = \varepsilon$,

- $K = \mathbb{Q}(\sqrt{-39})$ ($h_K = 4$): $\kappa_\psi = \epsilon_{13}^{\frac{1}{3}}$ is a unit, but not in H,

- $K = \mathbb{Q}(\sqrt{-47})$ ($h_K = 5$): κ_ψ doesn't seem to be a unit for any ψ.

Generalizing Stark's Observation

Proposition (S.)

Let ψ be a class character such that ψ^2 is a non-trivial character with rational real part (equivalently, the character of $\mathrm{Ind}_K^{\mathbb{Q}} \psi$ is rational). Then κ_ψ is an algebraic integer which is a unit. Moreover, if ψ^2 is a non-trivial genus character corresponding to the factorisation $D = D_1 D_2$, with $D_1 > 0$ say, then

$$\kappa_\psi = \epsilon_{D_1}^{\frac{4 h_{D_1} h_{D_2}}{w_K w_{D_2}}},$$

where ϵ_{D_1} is the fundamental unit of $\mathbb{Q}(\sqrt{D_1})$, h_{D_j} is the class number of $\mathbb{Q}(\sqrt{D_j})$ and w_{D_2} is the number of roots of unity in $\mathbb{Q}(\sqrt{D_2})$.

A corollary

It follows from this proposition that

Corollary

If K has class number divisible by 2 or 3, there exists a class character ψ for which κ_ψ is a unit.

A corollary

It follows from this proposition that

Corollary

If K has class number divisible by 2 or 3, there exists a class character ψ for which κ_ψ is a unit.

Question

Is the converse true?

Where we are in the presentation

Introduction

Petersson inner product of theta series

Algorithms

Generalizing Stark's observation

p-adic interpolation

Experimentation, invariants and observations

Conclusion

Two objectives

Recall that

	$\langle \theta_\psi, \theta_\psi \rangle$	$\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$
$\ell > 0$	$C_1 \sum\limits_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \delta^{2\ell-1} E_2(\mathcal{A})$	$C_2 \sum\limits_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_{\mathfrak{c}} \mathcal{O}_K} \lambda_{\mathfrak{c}}^{2\ell} \delta^{2\ell-1} E_2(\mathfrak{c})$
$\ell = 0$	$C_3 \sum\limits_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \log \Phi(\mathcal{A})$	not applicable
	$\psi^2 = 1$: not applicable	

Two objectives for the *p*-adic interpolation

1. Show that the quantities $\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$ can be *p*-adically interpolated for $\ell > 0$ (under certain restrictions),

2. Evaluate the *p*-adic analytic function obtained at $\ell = 0$.

p-adic interpolation of Petersson inner product of theta series: setup

Suppose that:

1. D is prime,
2. p is a prime $\neq 2, 3$ which splits in K (say $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$),
3. $\mathfrak{a}$ and $\mathfrak{b}$ are two fractional ideals of K such that

$$\mathfrak{a}\overline{\mathfrak{b}}\mathfrak{c}^2 = \mathcal{O}_K.$$

p-adic interpolation of Petersson inner product of theta series: setup

Suppose that:

1. D is prime,
2. p is a prime $\neq 2, 3$ which splits in K (say $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$),
3. $\mathfrak{a}$ and $\mathfrak{b}$ are two fractional ideals of K such that

$$\mathfrak{a}\overline{\mathfrak{b}}\mathfrak{c}^2 = \mathcal{O}_K.$$

Moreover, fix an isomorphism

$$\mathbb{Q}_p/\mathbb{Z}_p \to \bigcup_{n \geq 1} \bar{\mathfrak{p}}^{-n}\mathfrak{c}/\mathfrak{c}$$

and let

$$\mathcal{W} = \mathsf{Hom}_{\mathsf{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$$

denote the *p*-adic weight space.

p-adic interpolation of Petersson inner product of theta series: result

Under the above assumptions, one has the following

Theorem (S.)

There exists a p-adic analytic function

$$F : \mathcal{W} \to \mathbb{C}_p$$

with the property that

$$F(\ell) = (\mathrm{Frob}_{\mathfrak{p}}^{-1} - p^{2\ell-1})(\mathrm{Frob}_{\mathfrak{p}}^{-1} - p^{2\ell}) \left(\frac{\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle}{((2\pi i)^{-1}\Omega_{\mathbb{C}}(\mathfrak{c}))^{4\ell}} \right) \text{ for all } \ell > 0$$

where $\mathrm{Frob}_{\mathfrak{p}} = \left(\frac{H/K}{\mathfrak{p}} \right)$ *is the Artin symbol.*

Evaluation of F at $\ell = 0$

Let

$$g_0(q) = \frac{\Delta(q)}{\Delta(q^p)}$$

and let

$$g_0^{(p)}(q) = \frac{g_0(q^p)}{g_0^p(q)} = \frac{\Delta^{p+1}(q^p)}{\Delta^p(q)\Delta(q^{p^2})}.$$

Then $\log_p g_0^{(p)}$ is a p-adic modular form and one has the following

Theorem (S. (loose form))

The following equality holds in $\mathbb{C}_p$:

$$F(0) = -\frac{1}{6p} \log_p g_0^{(p)}(\mathfrak{c}).$$

Interpretation of the above theorem (in the current setup)

Again, recall that

	$\langle \theta_\psi, \theta_\psi \rangle$	$\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$
$\ell > 0$	$C_1 \displaystyle\sum_{\mathcal{A} \in \mathsf{Cl}_K} \psi^2(\mathcal{A}) \delta^{2\ell-1} E_2(\mathcal{A})$	$C_2 \displaystyle\sum_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_\mathfrak{c} \mathcal{O}_K} \lambda_\mathfrak{c}^{2\ell} \delta^{2\ell-1} E_2(\mathfrak{c})$
$\ell = 0$	$C_3 \displaystyle\sum_{\mathcal{A} \in \mathsf{Cl}_K} \psi^2(\mathcal{A}) \log \Phi(\mathcal{A})$	not applicable
	$\psi^2 = 1$: not applicable	

Interpretation of the above theorem (in the current setup)

Using the formula for $\langle \theta_\psi, \theta_\psi \rangle$ even if $\psi^2 = 1$, one has

	$\langle \theta_\psi, \theta_\psi \rangle$	$\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$
$\ell > 0$	$C_1 \displaystyle\sum_{\mathcal{A} \in \mathsf{Cl}_K} \psi^2(\mathcal{A}) \delta^{2\ell-1} E_2(\mathcal{A})$	$4\delta^{2\ell-1} E_2(\mathfrak{c})$
$\ell = 0$	$C_3 \displaystyle\sum_{\mathcal{A} \in \mathsf{Cl}_K} \psi^2(\mathcal{A}) \log \Phi(\mathcal{A})$	$-\frac{1}{3} \log(N(\mathfrak{c})^6 |\Delta(\mathfrak{c})|)$

Interpretation of the above theorem (in the current setup)

Using the formula for $\langle \theta_\psi, \theta_\psi \rangle$ even if $\psi^2 = 1$, one has

	$\langle \theta_\psi, \theta_\psi \rangle$	$\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$
$\ell > 0$	$C_1 \displaystyle\sum_{\mathcal{A} \in \mathsf{Cl}_K} \psi^2(\mathcal{A}) \delta^{2\ell-1} E_2(\mathcal{A})$	$4\delta^{2\ell-1} E_2(\mathfrak{c})$
$\ell = 0$	$C_3 \displaystyle\sum_{\mathcal{A} \in \mathsf{Cl}_K} \psi^2(\mathcal{A}) \log \Phi(\mathcal{A})$	$-\frac{1}{3} \log(N(\mathfrak{c})^6 |\Delta(\mathfrak{c})|)$

Using the above result, a *formal* computation gives

$$F(0) = -\frac{1}{6p} \log_p \frac{\Delta^{p+1}(q^p)}{\Delta^p(q)\Delta(q^{p^2})}$$

$$= -\frac{1}{6}(\mathrm{Frob}_\mathfrak{p}^{-1} - p^{-1})(\mathrm{Frob}_\mathfrak{p}^{-1} - 1) \log_p \Delta(\mathfrak{c}).$$

Where we are in the presentation

Experimenting with Petersson norm of theta series

Consider the algebraic number

$$N(\psi) = \frac{\langle \theta_\psi, \theta_\psi \rangle}{\Omega_K^{4\ell}} \qquad \text{for} \ell > 0,$$

where Ω_K is the Chowla-Selberg period.

Experimenting with Petersson norm of theta series

Consider the algebraic number

$$N(\psi) = \frac{\langle \theta_\psi, \theta_\psi \rangle}{\Omega_K^{4\ell}} \qquad \text{for} \ell > 0,$$

where Ω_K is the Chowla-Selberg period.
When ℓ is divisible by h_K, it appears numerically that

$$N(\psi_0) \in \mathbb{Z}$$

for a unique Hecke character ψ_0.

Experimenting with Petersson norm of theta series

Consider the algebraic number

$$N(\psi) = \frac{\langle \theta_\psi, \theta_\psi \rangle}{\Omega_K^{4\ell}} \qquad \text{for} \ell > 0,$$

where Ω_K is the Chowla-Selberg period.
When ℓ is divisible by h_K, it appears numerically that

$$N(\psi_0) \in \mathbb{Z}$$

for a unique Hecke character ψ_0.
This ψ_0 is the Hecke character

$$\psi_0(\mathfrak{a}) = \alpha^{2\ell/h_K},$$

where $\mathfrak{a}^{h_K} = (\alpha)$.

Computing the Gram matrix in the space of theta series

Let

$$\mathrm{Gram}(f_1, \ldots, f_d) = \det(\langle f_i, f_j \rangle)_{1 \leq i,j \leq d}$$

be the determinant of the Gram matrix of the Petersson inner product for a basis $\{f_1, \ldots, f_d\}$ in a vector space.

Computing the Gram matrix in the space of theta series

Let

$$\text{Gram}(f_1, \ldots, f_d) = \det(\langle f_i, f_j \rangle)_{1 \le i, j \le d}$$

be the determinant of the Gram matrix of the Petersson inner product for a basis $\{f_1, \ldots, f_d\}$ in a vector space. Then one has the following

Proposition

Let $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_{h_K}\}$ be a set of representatives of Cl_K and let $\{\psi_1, \ldots, \psi_{h_K}\}$ be the Hecke characters of K of infinity type $(2\ell, 0)$. Then

$$\text{Gram}(\theta_{\mathfrak{a}_1, \ell}, \ldots, \theta_{\mathfrak{a}_{h_K}, \ell}) = \left(\frac{w_K^2}{h_K} \right)^{h_K} \left(\prod_{i=1}^{h_k} N(\mathfrak{a}_i) \right) \text{Gram}(\theta_{\psi_1}, \ldots, \theta_{\psi_{h_K}}).$$

Computing the Gram matrix in the space of theta series

Numerically, it appears that

$$A(K, \ell) = \prod_{i=1}^{h_K} N(\psi_i) = \frac{\mathsf{Gram}(\theta_{\psi_1}, \ldots, \theta_{\psi_{h_K}})}{\Omega_K^{4h_k \ell}}$$

is integral (except in some cases when $\ell = 1$):

D	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
-7	$\frac{1}{3}$	1	17	1337
-8	$\frac{1}{2}$	5	92	1448
-11	1	10	139	16000
-15	4	6084	2528100	5222952900
-19	$\frac{13}{3}$	142	9251	944192
-20	64	21904	189337600	2907434214400
-23	621	7303581	1571089526325	1233974294487401229

A basis of normalized theta series

Suppose that $D < -4$. For any ideal class $\mathcal{A} = [\mathfrak{a}] \in \mathrm{Cl}_K$ and any $\ell > 0$, define

$$\theta_{\mathcal{A},\ell} = \frac{\theta_{\mathfrak{a},\ell}}{E_2(\mathfrak{a}^{-1})^\ell}.$$

Then by CM theory

$$\langle \theta_{\mathcal{A}_1,\ell}, \theta_{\mathcal{A}_2,\ell} \rangle \in H.$$

Define

$$C(K,\ell) = \mathrm{Gram}(\theta_{\mathcal{A}_1,\ell}, \ldots, \theta_{\mathcal{A}_{h_K},\ell}).$$

Relation to the other invariants

After running some experiments, one notices that numerically,

$$\prod_{i=1}^{h_K}(12\sqrt{|D|}E_2(\mathfrak{a}_i)N(\mathfrak{a}_i)) = n(K)\Omega_K^{2h_K}$$

for some *integer* $n(K)$.

Relation to the other invariants

After running some experiments, one notices that numerically,

$$\prod_{i=1}^{h_K}(12\sqrt{|D|}E_2(\mathfrak{a}_i)N(\mathfrak{a}_i)) = n(K)\Omega_K^{2h_K}$$

for some *integer $n(K)$*.
It follows that $B(K,\ell)$ and $C(K,\ell)$ only differ by $n(K)$, up to an explicit quantity.

Where we are in the presentation

Introduction

Petersson inner product of theta series

Algorithms

Generalizing Stark's observation

p-adic interpolation

Experimentation, invariants and observations

Conclusion

Take home

The Petersson inner product of theta series

1. Can be efficiently computed,

Take home

The Petersson inner product of theta series

1. Can be efficiently computed,
2. Can be p-adically interpolated,

Take home

The Petersson inner product of theta series

1. Can be efficiently computed,
2. Can be *p*-adically interpolated,
3. Contains interesting information in weight 1 and greater.

Thank you!

Presentation available at:
https://github.com/NicolasSimard/Notes
Code available at : https://github.com/NicolasSimard/ENT