# Singular Network Elements*

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*Summary*—The properties of $n$-ports can be examined in terms of simple properties of linear vector spaces. This approach leads to a very general type of network formalism which in turn casts light on the physical realizability (or nonrealizability) of the singular *linear* network elements: the nullator (simultaneously an open and a short circuit), and the norator (the unique nonreciprocal one-port with arbitrary port voltage and current). Furthermore, a two-port (the "nullor") which combines these two elements can be shown to be a unique active building block which exhibits the extraordinary nature of the two singular one-ports, but which has other properties which make it amenable for use in practical systems.

## I. Representation of Linear Networks in Terms of Linear Spaces

IT WAS POINTED OUT in an unpublished memorandum by B. D. H. Tellegen[1] that 1) The nullator (the linear time-invariant one-port with $v = i = 0$) and the norator (the linear time-invariant one-port with $v$ and $i$ arbitrary) do not appear to be limiting cases of conventional networks in the same sense that an ideal transformer is the limiting case of a pair of coupled coils as the leakage impedance and magnetizing admittance approach zero and that 2) the two-port defined by $v_1 = i_1 = 0$ (which performs as a nullator at port 1 and a norator at port 2)[2] behaves in a more "natural" fashion particularly when interconnected with other circuit elements. The discussion which follows elaborates on these points and by considering some elementary properties of the linear vector spaces associated with the port variables, indicates the limits of behavior of these singular network elements.

In order to treat systems containing nullators and norators,[3] it is necessary to employ a network formalism which can be applied to all varieties of linear networks, since circuits containing these singular elements (and the elements themselves) may not be representable in an immittance, scattering or other conventional formalism. Let us consider the $2n$-dimensional linear space $T_{2n}$ associated with appropriately chosen port variables of a linear time-invariant $n$-port network. A useful subspace $e_1$ of $T_{2n}$ which, for example, finds application in immittance formulations contains the vectors

$$\mathbf{e}_1 = \begin{bmatrix} \mathbf{v} \\ \mathbf{i} \end{bmatrix}, \tag{1}$$

where in the present notation a boldface letter indicates a column vector, and $\mathbf{v}$ is the column array of all $n$ phasor port voltages, and $\mathbf{i}$ contains the $n$ phasor port current variables. Another subspace $e_2$ consists of the vectors

$$\mathbf{e}_2 = \begin{bmatrix} \mathbf{v} - \mathbf{i} \\ \mathbf{v} + \mathbf{i} \end{bmatrix}. \tag{2}$$

As an example of the use of $e_1$ consider an $n$ port with an impedance matrix $Z$

$$\mathbf{v} = Z\mathbf{i} \tag{3a}$$

$$Z = (z_{ij}). \tag{3b}$$

For such a network there are $n$ vectors which span the subspace $e_1$; that is, linear combinations of these $n$ vectors (a bias) give all possible vectors in $e_1$. A basis clearly consists of the $n$ linearly independent vectors

$$\mathbf{z}_1 = \begin{bmatrix} z_{11} \\ z_{21} \\ \vdots \\ z_{n1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{z}_2 = \begin{bmatrix} z_{12} \\ z_{22} \\ \vdots \\ z_{n2} \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \mathbf{z}_n = \begin{bmatrix} z_{1n} \\ \\ \vdots \\ z_{nn} \\ 0 \\ \\ \vdots \\ 1 \end{bmatrix} \tag{4}$$

and arranging these in order, and taking linear combinations of the basis vectors,

$$\begin{bmatrix} \mathbf{v} \\ \mathbf{i} \end{bmatrix} = x_1\mathbf{z}_1 + x_2\mathbf{z}_2 + \cdots + x_n\mathbf{z}_n = \begin{bmatrix} Z \\ E_n \end{bmatrix}\mathbf{x}, \tag{5}$$

where $E_n$ is the $n \times n$ identity matrix, $\mathbf{x} = (x_1, x_2, \cdots x_n)'$ and the $x_k$ are arbitrary complex multipliers (which may be frequency dependent). The prime (′) indicates matrix transpose.

Eq. (5) can be written

$$\begin{bmatrix} \mathbf{v} \\ \mathbf{i} \end{bmatrix} = Q\mathbf{x}, \tag{6}$$

where $Q$ is a matrix whose columns are the chosen basis vectors and (6) (referred to as the $Q$ *formalism*)[4] is in

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[1] Private communication to the author.

[2] B. H. D. Tellegen, "La recherche pour une serie complete d'elements de circuit ideaux non-lineaires," *Rendiconti Del Seminario Matematico e Fisico di Milano*, vol. 25, pp. 134–144; April, 1954.

[3] H. J. Carlin and D. C. Youla, "Network synthesis with negative resistors," Proc. IRE, vol. 49, pp. 907–920; May, 1961.

[4] Another type of description which is also applicable to any linear network is the $AB$ formalism, $[A, B] \begin{bmatrix} \mathbf{v} \\ \mathbf{i} \end{bmatrix} = 0$ (see V. Belevitch, "Four dimensional transformations of 4-pole matrices with applications to the synthesis of $n$ ports," IRE Trans. on Circuit Theory, vol. CT-3, pp. 105–111; June, 1956. The $AB$ and $Q$ formalisms may be deduced from each other.

the form of the most general formalism for linear networks, which in the special case considered in (5) yields $\mathbf{i} = \mathbf{x}$, $\mathbf{v} = Z\mathbf{x} = Z\mathbf{i}$. Note that the basis vectors may be assembled in $Q$ in any order, but as in (5), a specific arrangement is chosen for convenience.

When a scattering matrix exists, $S = (s_{ij})$,

$$\mathbf{b} = S\mathbf{a} \tag{7}$$

where $\mathbf{b}$ and $\mathbf{a}$ are the reflected and incident port variables, respectively, the $Q$ formalism in $e_2$ gives

$$\begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix} = \begin{bmatrix} \mathbf{v} - \mathbf{i} \\ \mathbf{v} + \mathbf{i} \end{bmatrix} = \begin{bmatrix} S \\ E_n \end{bmatrix} \mathbf{x}, \tag{8}$$

where, since $\mathbf{b} = \mathbf{v} - \mathbf{i}$, $\mathbf{a} = \mathbf{v} + \mathbf{i}$, unit port normalization numbers are implied. This is a matter of convenience and results in no loss of generality. The $n$-basis vectors in $e_2$ for the $n$-port network with a scattering description are obviously

$$\begin{aligned} \mathbf{s}_1 &= [s_{11}, s_{21}, \cdots, s_{n1}, 1, 0, \cdots, 0]' \\ \mathbf{s}_2 &= [s_{12}, s_{22}, \cdots, s_{n2}, 0, 1, \cdots, 0]' \\ &\vdots \\ \mathbf{s}_n &= [s_{1n}, s_{2n}, \cdots, s_{nn}, 0, 0, \cdots, 1]'. \end{aligned} \tag{9}$$

Linear passive networks always possess a scattering description.[5] Thus, $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_n$ is always a basis in $e_2$ for any such passive network,[6] and clearly the dimensionality of $e_2$ for passive networks must be $n$. On the other hand, many passive networks[7] do not possess impedance formalisms or admittance formalisms so that the general prescription for a basis in $e_1$ can not be given *a priori* in the simple manner specified by (9). Nevertheless, since $e_1$ and $e_2$ are both in $T_{2n}$, and further $\mathbf{e}_2$ goes into $\mathbf{e}_1$ by the nonsingluar transformation

$$\mathbf{e}_1 = \begin{bmatrix} \frac{1}{2}E_n & \frac{1}{2}E_n \\ -\frac{1}{2}E_n & \frac{1}{2}E_n \end{bmatrix} \mathbf{e}_2, \tag{10}$$

it follows that the dimensionality of $e_1$ is the same as that of $e_2$. The two subspaces are, in fact, isomorphic. Thus, in $e_1$, one can always choose $n$-independent variables among the $2n$-current-voltage port variables of a passive network but the specific choice is dictated by the particular network being treated. In $e_2$ on the other hand, $\mathbf{a} = \mathbf{v} + \mathbf{i}$ can always be chosen as the independent set of variables.

## II. Singular Behavior of Nullators, Norators

We now consider linear active networks in which neither immittance nor scattering descriptions need exist and hence we confine our representations to the $Q$ formalism since this applies to *any* linear network. The simplest case is a one-port. For passive one-ports, the dimensionality of $e_1$ is one, and in general any one-port (not necessarily passive) of dimensionality unity has a basis in $e_1$ of

$$Q = \mathbf{z} = \begin{bmatrix} z \\ 1 \end{bmatrix} \tag{11a}$$

or

$$Q = \mathbf{y} = \begin{bmatrix} 1 \\ y \end{bmatrix}, \tag{11b}$$

where $z$ and $y$ are, respectively, the impedance or admittance of the one-port in question. As indicated in (11a) and (11b), $\mathbf{z}$ and $\mathbf{y}$ are also the $Q$ matrices of the respective one-ports.

There is however, no need to assume unit dimensionality, (*i.e.*, one independent variable) for a one-port and indeed, the nullator with $v = i = 0$ defines a subspace $e_1$ which is the nullspace of $T_2$. That is, for a nullator the subspace $e_1$ is of dimensionality zero and the basis is

$$\mathbf{f} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{12}$$

On the other hand, the norator[8] (the linear one-port with $v$ and $i$ arbitrary) represents the other extreme in which $e_1$ is of dimensionality 2 and the basis vectors can be chosen as

$$\mathbf{f}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{f}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{13}$$

so that the $Q$ representation for the nullator ($Q_0$) and norator ($Q_\infty$) are, respectively,

$$\begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} x = Q_0 x \tag{14a}$$

$$\begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} = Q_\infty \mathbf{x}. \tag{14b}$$

It is now immediately obvious that *the nullator and norator cannot be derived by a limiting process from any "normal" network,* where by a normal network for a

[5] D. C. Youla, H. J. Carlin, and L. J. Castriota, "Bounded real scattering matrices and the foundations of linear passive network theory," IRE Trans. on Circuit Theory, vol. CT-6, pp. 102–124; March, 1959.

[6] Strictly speaking, this statement only applies to "normal" networks as defined in Secs. II and III. The nullator has no scattering description, has passive terminal behavior, but is not "normal." It may be shown, however, that the nullator cannot be realized with passive elements.

[7] Passivity implies $Re \sum_{k=1}^{n} i_k^* v_k \geq 0$ for an $n$-port. In terms of $Q$, passivity requires that

$$\tilde{Q} \begin{bmatrix} 0 & E_n \\ E_n & 0 \end{bmatrix} Q$$

be a positive definite or semidefinite hermetian matrix. The ($\sim$) means conjugate transpose or adjoint.

[8] The norator does not obey Lorentz reciprocity relations and hence is a nonreciprocal *one-port*. In terms of $Q$, reciprocity requires that

$$Q' \begin{bmatrix} 0 & -E_n \\ E_n & 0 \end{bmatrix} Q = 0;$$

$Q_\infty$ does not satisfy this requirement.

one-port, we shall mean a structure which defines a subspace $e_1$ of dimensionality 1. Eqs. (11) define all normal one-ports, and any limiting process on the components of these networks corresponds to taking the resultant limit on $z$ or $y$. Such a process cannot yield (14a) or (14b) out of (11); that is, the rows of (11) cannot *independently* approach the elements in the rows of (14a) or (14b) by any manipulation of the physical constants of a normal one-port. This is also not possible in any other admissible basis for (11). Hence the nullator and norator are singular elements. We therefore expect that any equivalent circuit which represents these elements has infinite sensitivity. That is if some circuit element in the equivalent structure for a nullator or norator is changed slightly, the terminal performance will no longer be similar to that of the nullator or norator. Further, if nullators and norators are used in combination with other conventional circuit elements, the over-all behavior of the complete network at any prescribed set of accessible ports is completely altered if the circuit parameters used to construct the nullator and norator elements deviate in any way from the exact values required.

A simple example will make this clear. Fig. 1(a) shows an equivalent circuit for a nullator in terms of a circulator (designed for unit resistive terminations), a positive and a negative unit resistor.[3] Since the incident voltage at a matched, negative- resistive termination is zero, and the reflected voltage at a matched, positive-resistive termination is zero, coupling this with the properties of a circulator the input scattering variables must be zero, $a = b = 0$. Hence by (10), $v = i = 0$. Now suppose the resistors deviate from the prescribed values and are $R_1$ and $R_2$, respectively, then at the input with

$$b = \frac{R_1 - 1}{R_1 + 1} \cdot \frac{R_2 - 1}{R_2 + 1}\, a$$

we have

$$z = \frac{v}{i} = \frac{a + b}{a - b} = \frac{1 + R_1 R_2}{R_1 + R_2}. \tag{15}$$

Referring to (15), it is clear that $\lim_{\substack{R_1 \to -1 \\ R_2 \to 1}} z$ is completely undefined, *i.e.*, depending on the manner in which $R_1$ and $R_2$ approach their final values $\mp 1$, $z$ may range anywhere from $0 \leq z \leq \infty$. On the other hand, no matter how close the resistors approach $\mp 1$, provided they do not simultaneously take on these values, $z$ will be well defined and except in the limit, Fig. 1(a) represents a normal network but one which in no sense operates like a nullator. The performance of the circuit as a nullator only occurs as a singular instance, not derivable by a limiting process, when $R_1 = -1$, $R_2 = 1$. A similar discussion applies to the norator equivalent of Fig. 1(b).

Now, consider the equivalent circuit for a gyrator shown in Fig. 2(a) which is constructed entirely of two-terminal branches including one nullator and one norator.[3] If we visualize the circuit of Fig. 1(a) replacing the nullator, and that of Fig. 1(b) replacing the norator, then it is

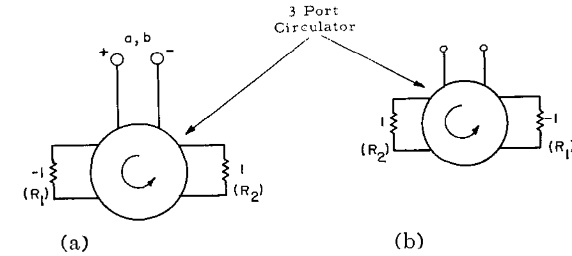


Fig. 1—(a) Nullator equivalent. (b) Norator equivalent.

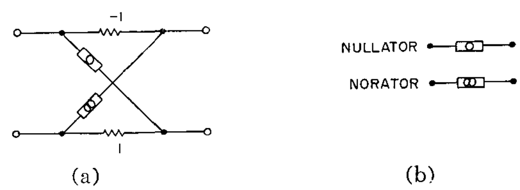


Fig. 2—Equivalent circuit for gyrator.

clear that if $R_1$ and $R_2$ deviate in the least from their prescribed values then the nullator and norator elements in Fig. 2 behave like ordinary one-port impedances, and the circuit of Fig. 2, which at the singular point precisely obeys the nonreciprocal gyrator equations, no longer can have even a semblance of nonreciprocal behavior.

To conclude, it is well to point out that despite the singular behavior of nullators and norators (actually because of it) these are necessary elements for the complete representation of all possible linear time-invariant $n$-port network systems. One, in fact, may state that any subspace $e_i$ in $T_{2n}$ of dimensionality $0 \leq r \leq 2n$, whose elements are rational functions of a complex frequency variable has a $2n$-port network representation which employs conventional two-terminal passive network elements ($R$, $L$, $C$) and nullators and norators.[3]

## III. Singular Two-Ports—The "Nullor"

In the case of a one-port, any given network can be described in terms of a space $e_1$ which has dimensionality of either 0 (nullator), 1 (normal one-port) or 2 (norator). In the case of a two-port, four of the five possible dimensionalities of $e_i$, 0, 1, 3, 4 can be realized, respectively, by two nullators, one normal one-port and a nullator, one normal one-port and a norator, and two norators. These then present no essentially new features beyond those already discussed in Section II. The case $r = 2$, however, possesses special interest. There are really two situations which should be considered. The first of these may be regarded as "normal" for a two-port. A normal $n$-port is one in which behavior at each port is that of a normal one-port when all the other ports are terminated in arbitrary normal one-ports. This means that in a normal two-port (or $n$-port), there are two ($n$ for an $n$-port) independent variables and one of these may be associated with each port. Note that the variables *need* not be chosen in this fashion but normality implies that they can be so chosen. If the $Z$ matrix exists, the two independent variables are $i_1$ and $i_2$. An ideal transformer (which has neither $Z$ nor $Y$) is normal in that $v_1$, $i_2$ may be chosen as the two independent variables. In the first instance (*i.e.* $Z$ exists), a basis is

$$\mathbf{z}_1 = [z_{11}, z_{21}, 1, 0]', \qquad \mathbf{z}_2 = [z_{12}, z_{22}, 0, 1]'$$

and for the $1 : n$ ideal transformer a basis is

$$\mathbf{f}_1 = [1, n, 0, 0]', \qquad \mathbf{f}_2 = [0, 0, -n, 1]'.$$

Thus the network with a $Z$ matrix has

$$Q = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{16}$$

and the transformer yields

$$Q = \begin{bmatrix} 1 & 0 \\ n & 0 \\ 0 & -n \\ 0 & 1 \end{bmatrix}. \tag{17}$$

This definition of normality can be given in the $Q$ formalism if $Q$ is partitioned into two $n \times n$ submatrices

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}.$$

Then in a "normal" $n$-port the rank of $Q$ is $n$, and correspondingly numbered row vectors in $Q_1$ and $Q_2$ may not be simultaneously zero. In the one-port, one need only state that the rank is 1 as discussed in Section II. As the word implies, a normal network has the behavior we usually regard as conventional. Clearly, all netwroks possessing $Z$, $Y$ or $S$ descriptions are normal, and therefore all passive networks are normal. (However, see footnote[6].)

In the case of a two-port associated with a space $e_1$ of dimensionality $r = 2$, there is only one non-normal network. This corresponds to the two-port structure $v_1 = i_1 = 0$ (the case of a two-port with $v_2 = i_2 = 0$ is just a renumbering of ports), and the basis vectors may be chosen as $[0, 1, 0, 0]'$ and $[0, 0, 0, 1]'$. Thus

$$Q = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{18}$$

The normal property is clearly violated (row 1 of $Q_1$ and row 1 of $Q_2$ are both zero), although as in (16) and (17) for normal two-ports, the rank of $Q$ is 2. Eq. (18) defines a unique two-port (the only one with two independent variables, but which is nevertheless non-normal) which presents a nullator at port 1 and a norator at port 2, and with apologies to the reader is herewith dubbed a "nullor" and is illustrated in Fig. 3(a). The definition of normality used here emphasizes the unconventional

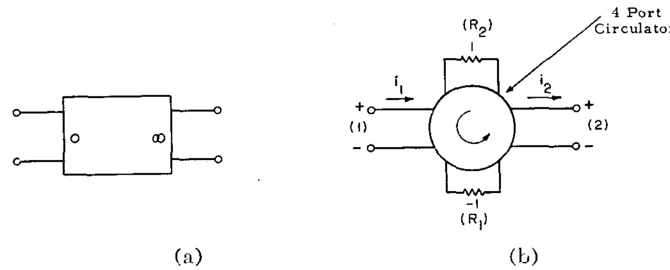


Fig. 3—The nullor. (a) Schematic. (b) Circuit representation.

features of the nullor. However, in view of footnote[10], the simplest measure of pathologic character for an $n$ port is *just* the dimensionality of $\mathbf{v}$, $\mathbf{i}$ space from $n$.

It must not be supposed that a nullor has no properties other than those of an isolated single nullator and norator. These singular one-port elements cannot have terminal behavior which is the limit of any normal one-port, but it will now be shown that because of the possibility of internal coupling between ports, the nullor can be determined as the limit of a normal two-port. Consider Fig. 3(b) which shows an equivalent circuit for a nullor.[3] When $R_1 \neq -1$ this network has a scattering matrix normalized to unity (the polarity of $i_2$ is here presumed opposite to that shown in the figure)

$$S = \begin{bmatrix} 0 & \dfrac{R_1 - 1}{R_1 + 1} \\ \dfrac{R_2 - 1}{R_2 + 1} & 0 \end{bmatrix} \tag{19}$$

so that except in the limit ($R_1 = -1$) the structure shown is normal, since $S$ exists.

Now, even though the network is normal, we may choose a different set of basis vectors other than the pair which selects one independent variable per port. Thus in an impedance transfer ($ABCD$) description, for the polarities shown in Fig. 3(b), the basis vectors may be chosen as

$$\boldsymbol{\alpha}_1 = [A, 1, C, 0]'$$

$$\boldsymbol{\alpha}_2 = [B, 0, D, 1]'$$

so that

$$\begin{bmatrix} v_1 \\ v_2 \\ i_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ 1 & 0 \\ C & D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Q\mathbf{x}; \quad x_1 = v_2, \quad x_2 = i_2. \tag{20}$$

The values of $A$, $B$, $C$, $D$ may be readily calculated as

$$\begin{aligned} A &= \frac{\epsilon_1\epsilon_2 + \epsilon_1 - \epsilon_2}{(\epsilon_1 - 2)(\epsilon_2 + 2)} \qquad & C &= \frac{\epsilon_1 + \epsilon_2}{(\epsilon_1 - 2)(\epsilon_2 + 2)} \\ B &= \frac{\epsilon_1 + \epsilon_2}{(\epsilon_1 - 2)(\epsilon_2 + 2)} \qquad & D &= -\frac{\epsilon_1 + \epsilon_2}{(\epsilon_1 - 2)(\epsilon_2 + 2)}, \end{aligned} \tag{21}$$

where $\epsilon_1 = R_1 + 1$, $\epsilon_2 = R_2 - 1$. Except for specific values of $\epsilon_1$, $\epsilon_2$, (*e.g.*, $\epsilon_1 = -\epsilon_2$) $AD - BC \neq 0$, and the first and third rows of $Q$ in (20) are linearly independent when $\epsilon_1$, $\epsilon_2$ are independent variables.

As $R_1$, $R_2$ approach their limiting values $\mp 1$; $\epsilon_1 \to 0$, $\epsilon_2 \to 0$ and clearly $\lim A = \lim B = \lim C = \lim D = 0$. Since the first and third rows are independent, they can be made to approach zero independently of each other as the network constants (corresponding to $\epsilon_1$ and $\epsilon_2$) are adjusted. In the limit, the $Q$ matrix of the normal network defined by (20) is given by

$$\lim_{\substack{R_1 \to -1 \\ R_2 \to 1}} Q = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{22}$$

which is precisely that of the nullor as shown in (18). Thus when $R_1$ and $R_2$ independently approach their final values in any arbitrary manner, the circuit of Fig. 3(b) in the limit becomes the singular two-port element, the nullor.

It is significant to note that the limiting behavior of the nullor occurs even when the parameters of the circulator, as well as the terminating resistors $R_1$ and $R_2$, are allowed to deviate from their ideal values. Thus, one may assume finite isolation and insertion loss between circulator ports and by a process similar to that already described, the result given in (22) is verified as the isolation between nominally decoupled ports becomes infinite. Furthermore, the performance of the nullor is independent of the insertion loss between coupled ports in Fig. 3(b) so that, for example, a low-frequency nullor could be constructed utilizing a resistive hall plate as the circulator element.

## IV. Interconnection of Nullors with Other Circuits

Various types of network performance can be realized by interconnecting nullors with conventional elements. In order to study such circuits, it is useful to have three- and four-terminal equivalent circuits for the singular two-port. These are shown in Fig. 4(a) and 4(b).[9] The gyrator of Fig. 2 may be realized with a four-terminal nullor as in Fig. 5(a). It is of interest to note that if the nullor circuit of Fig. 4(b) is used here, the resistors cancel out and one is immediately left with the gyrator. Fig. 5(b) shows the equivalent circuit of a triode using a three-terminal nullor.

The major result to be shown in this section is that the nullor has finite sensitivity. That is, if a network of interconnected circuit elements, including nullors, acts as a normal $n$-port with well-defined terminal behavior, then if the parameters of the nullor structures deviate slightly from their prescribed values, the terminal performance of the over-all $n$-port also deviates only slightly from its prescribed final value. This, of course, is radically different from the situation if one-port nullators and norators are used. The result stated is readily proved as follows.

[9] It is readily shown that the circuit of Fig. 4(b) is the minimum realization of a nullor, *i.e.*, one gyrator, one positive and one negative resistor. The circuit of Fig. 4(a) can be verified by noting that it consists of two outer-connected three-port circulators forming a four-port circulator terminated as in Fig. 3(b).

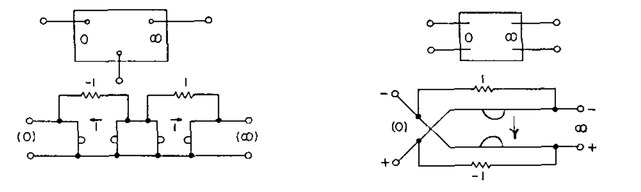


Fig. 4—Nullor equivalent circuits. (a) Three terminal nullor equivalent. (b) Four terminal nullor equivalent.

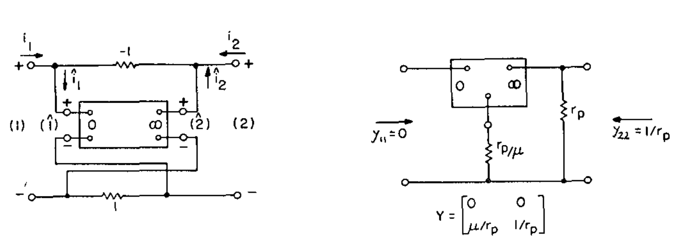


Fig. 5—Circuits with nullors. (a) Gyrator equivalent. (b) Triode equivalent.

First, suppose that one nullor is imbedded in an $n$-port. Let this be represented by considering an $(n+2)$-port, $N$. Ports 1 through $n$ are the final accessible ports and the nullor is connected across the remaining two ports $\hat{1}$, $\hat{2}$. Then with

$$\mathbf{i} = [i_1, i_2, \cdots, i_n]' \tag{23a}$$

$$\mathbf{v} = [v_1, v_2, \cdots, v_n]' \tag{23b}$$

$$\mathbf{u}_1 = [\hat{v}_1, \hat{i}_1] \tag{23c}$$

$$\mathbf{u}_2 = [\hat{v}_2, \hat{i}_2], \tag{23d}$$

we assume a representation between the port voltages and currents of $N$ of the form

$$\begin{aligned} \mathbf{i} &= F_{11}\mathbf{v} + F_{12}\mathbf{u}_2 \\ \mathbf{u}_1 &= F_{21}\mathbf{v} + F_{22}\mathbf{u}_2. \end{aligned} \tag{24}$$

The $F_{ij}$ are submatrices with $F_{11}$, $n \times n$; $F_{12}$, $n \times 2$; $F_{21}$, $2 \times n$; $F_{22}$, $2 \times 2$.

When the nullor is connected to ports $_1$, $_2$, it imposes the constraint

$$\mathbf{u}_1 = \mathfrak{A}\mathbf{u}_2, \tag{25}$$

where $\mathfrak{A}$ is the $ABCD$ matrix of the nullor, for example, as in (21). This applies when the parameters of the nullor differ from their final values and also in the limit as the two-port takes on the exact terminal characteristics of the nullor.

If (25) is substituted in (24) and $\mathbf{u}_1$, $\mathbf{u}_2$ eliminated the result is

$$\mathbf{i} = [F_{11} + F_{12}(\mathfrak{A} - F_{22})^{-1}F_{21}]\mathbf{v} \tag{26a}$$

which applies whenever the representation of (24) exists and if $(\mathcal{A} - F_{22})$ is nonsingular. Under these conditions, the admittance matrix of $N$ is

$$Y = F_{11} + F_{12}(\mathcal{A} - F_{22})^{-1}F_{21}. \tag{26b}$$

Since $\lim \mathcal{A} = 0$, we can approach as closely as we wish the final terminal performance of $N$ given by

$$\lim_{A,B,C,D\to 0} Y = Y_0 = F_{11} - F_{12}F_{22}^{-1}F_{21} \tag{27}$$

when $F_{22}$ is nonsingular, by allowing the parameters of the nullor to approach their limiting values as closely as we wish. In a manner of speaking, we may say that an approximate nullor gives approximate terminal performance of the over-all network in which it is imbedded.

The result just deduced may be simply extended to a network with any number of nullors by representing $N$ with $n$-input ports and $2k$ additional (imbedding) ports for the connection of $k$ nullors. We then define for the $j$th pair of imbedding ports

$$\mathbf{u}_1^{(j)} = \begin{bmatrix} \hat{v}_1^{(j)} \\ \hat{\imath}_1^{(j)} \end{bmatrix}, \qquad \mathbf{u}_2^{(j)} = \begin{bmatrix} \hat{v}_2^{(j)} \\ \hat{\imath}_2^{(j)} \end{bmatrix}$$

and

$$\mathbf{u}_1 = [\mathbf{u}_1^{(1)\prime}, \cdots, \mathbf{u}_1^{(k)\prime}]'$$
$$\mathbf{u}_2 = [\mathbf{u}_2^{(1)\prime}, \cdots, \mathbf{u}_2^{(k)\prime}]'.$$

Further, we let

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & \mathcal{A}_k \end{bmatrix}$$

with

$$\mathbf{u}_1^{(j)} = \mathcal{A}_j \mathbf{u}_2^{(j)}.$$

The manipulations of (24) which lead to the desired result proceed exactly as before except one merely notes that the dimensions of $F_{ij}$ are now: $F_{11}$, $n \times n$; $F_{12}$, $n \times 2k$; $F_{21}$, $2k \times n$; $F_{22}$, $2k \times 2k$.[10]

[10] It also follows that any $n$ port whose $\mathbf{v}$, $\mathbf{i}$ space (*i.e.*, el) is of dimensionality $n$ (whether normal or not) can be transformed into a normal network by imbedding in a normal network.

As an example of the process described, consider the gyrator representation of Fig. 5(a). For this specific system shown, (24) becomes

$$\mathbf{i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{v} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{u}_2$$
$$\mathbf{u}_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\mathbf{v} + \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}\mathbf{u}_2$$

Then referring to (26b)

$$Y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} A-1 & B \\ C & D-1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$
$$= \begin{bmatrix} \delta_1 & 1+\delta_2 \\ -1+\delta_3 & \delta_4 \end{bmatrix}$$

where[11] $\delta_1, \delta_2, \delta_3, \delta_4 \to 0$ as $A, B, C, D, \to 0$ and since $A$, $B$, $C$, $D$ can be made arbitrarily small by letting $R_1 \to -1$, $R_2 \to 1$ in Fig. 3(b), $Y$ can approach as closely as desired to

$$\lim_{A,B,C,D\to 0} Y = Y_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

which is the admittance matrix of the gyrator.

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[11] The specific values of the $\delta_k$, with $\delta = AD - BC$, and $\Delta = (A-1)(D-1) - BC$, are

$$\delta_1 = \frac{A+B-C-D}{\Delta}, \quad \delta_2 = \frac{A-B-C+D-2\delta}{\Delta},$$
$$\delta_3 = \frac{-(A+B+C+D)+2\delta}{\Delta}, \quad \delta_4 = \frac{-A+B-C+D}{\Delta}.$$