# A Graph-Theoretic Resilience Analysis of the South African National Research and Education Network (SANReN)

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## Abstract

National Research and Education Networks in developing regions face persistent resilience challenges, yet planners lack a sufficient evaluation frameworks to compare networks of different sizes. This paper presents a graph-theoretic evaluation framework, developed on South African National Research and Education Network (SANReN), unifying spectral (algebraic connectivity and multiplicities of eigenvalues), core resilience (Core-Influence Strength, Core Strength), and classical centrality metrics (closeness, betweenness, degree centrality). We stress the topology with random and targeting attacks to expose fragmentation and then test three edge-addition strategies: Fiedler-Greedy, Random edge addition and the Maximize Resilience by K-Core (MRKC). We find SANReN to be fully connected, but tenuously connected with bottleneck links and redundant substructures. Fiedler-Greedy most reliably increases connectivity; Maximize Resilience K-Core heuristic strengthens periphery-to-core support but delivers modest connectivity gains. Practically, we recommend bridging bottleneck links first (Fiedler-Greedy), then consolidate periphery nodes (MRKC), whilst monitoring Core-Influence Strength and the multiplicity of the one eigenvalue to detect diminishing marginal returns.

## 1 Introduction

National Research and Education Networks connect universities, educational and research institutions within Africa and globally [17]. They face persistent challenges of performance and resilience issues, especially in developing regions of Africa [18] where they operate under tight budgets while demand for data-intensive research grows. Within South Africa, SANReN (South African National Research and Education Network) faces these same challenges. Salie et al. [17, 18] document high delays and poor throughput, in Cape Town, Durban and Port Elizabeth, resulting from structural factors of low connectivity and circuitous routing; results from active measurements of delay, hop count and throughput, Salie et al. [18] identified that Port Elizabeth experiences the highest delays due to circuitous routing, where traffic targeted for Johannesburg or Pretoria is often routed through Cape Town [18]. Salie et al. [18] illustrates the need for reinforcement in Cape Town, Durban and Port Elizabeth. However, these studies primarily focus on the symptoms rather than the structural causes. These approaches, proposed by Chavula, Salie et al. [18][17], do not quantify how the topological structure of the network drives network performance and resilience. As a result of this, they are not able to identify what to change within the network in order to improve its structural performance.

This paper addresses that gap by aiming to create a cohesive, graph-theoretic framework that integrates: Spectral measures - algebraic connectivity $a(G)$ and the eigenvalue spectrum; core-resilience metrics; and classical measures, with the goal to improve connectivity, improve resilience to fragmentation and improve redundancy within the network.

Targeted node and edge additions are implemented using techniques of Fielder-vector addition, Maximize Resilience by K-Core (MRKC) [13] and random edge addition. After each step/addition each metric ($a(G)$, $m_0$, $m_1$, $CIS$) is recorded, and the full removal suite is run and each metric at each removal fraction is recorded - this matters especially for the multiplicity of the zero eigenvalue, $m_0$: without applying removals the network remains connected ($m_0 = 1$) and the curve would be flat, but the removal-driven trajectory reveals how quickly fragmentation appears after reinforcement, allowing us to capture the behavior of fragmentation under stress and lets us compare strategies fairly. The area-under-curve (AUC) is computed as the mean of each metrics trajectory allowing us to quantify how a network's connectivity, fragmentation and redundancy improves when applying a specific reinforcement strategy, whilst ensuring the evaluation comparable for networks of different sizes: area-under-curve compresses each stepwise trajectory into a single score, letting us see which reinforcement approach delivers the best improvements in connectivity, fragmentation and redundancy across random, targeted removals and non-removals, not just in a single scenario.

This paper's research question: *To what extent can spectral graph theory and core resilience analysis quantitatively evaluate the structural resilience of SANReN topology - including node criticality, node centrality, and connectivity - and how effectively can these methods inform the design of a more resilient network topology?*

Our contributions is a evaluation framework of network structural properties, with a practical design guidance that recommends fixing structural bottlenecks using Fiedler-Greedy approach, using Maximize Resilience by K-Core (MRKC) as a periphery node consolidator whilst monitoring Core-Influence Strength, $CIS$, and the multiplicity of the one eigenvalue, $m_1$, for diminishing marginal returns. The rest of the paper details key concepts, key research, algorithms, experimental design, and a discussion that converts the results into concrete reinforcement recommendations for SANReN, along with future work.

## 2 Background and Related Work

### 2.1 Key Concepts

*2.1.1 Network Resilience.* Network resilience refers to a network being able to maintain suitable levels of connectivity and functionality when subjected to failures or targeted attacks [21]. **Global resilience** refers to a network's ability to preserve overall connectivity and function during large-scale disruptions, often evaluated through topological or performance-based measures [22]. **Structural resilience**, in contrast, captures how the network topology handles localized failures and fragmentation [24].

*2.1.2 Logical and Physical topologies.* The logical topology defines the path data takes through a network — regardless of physical connections — and highlights routing dependencies and bottlenecks [7]. The physical topology describes the physical infrastructure such as fibre cables and switches. Physical topologies enable realistic network resilience analysis as it considers geographic constraints, physical distances, and infrastructure placement — all of which affect connectivity, redundancy, and resilience to failures [6].

Sparse network topology has relatively few edges relative to the number of nodes nodes [9]. Dense network topologies contain many edges per node [9].

*2.1.3 Laplacian Matrix.* The Laplacian matrix captures overall network connectivity and is essential in assessing structural resilience [4]. The adjacency matrix $A(G)$ encodes edge connections: $a_{ij} = 1$ if nodes $i$ and $j$ are connected, and 0 otherwise, with $a_{ii} = 0$.

The **Laplacian matrix** is computed as [4]:

$$L(G) = D(G) - A(G)$$

where $D(G)$ is the diagonal **degree matrix**, with entries $d_{ii}$ equal to the degree of node $v_i$.

The **normalized Laplacian matrix** enables comparisons between topologies of different sizes and structures. It extends the basic Laplacian by adjusting for node degree, so that it can accurately compare nodes regardless of how many connections they have [4]. The normalized Laplacian matrix is defined as:

$$L_{norm}(G)(i,j) = \begin{cases} 1, & \text{if } i = j \text{ and } d_i \neq 0 \\ -\frac{1}{\sqrt{d_i d_j}}, & \text{if } i \text{ and } j \text{ are connected} \\ 0, & \text{otherwise} \end{cases}$$

where $d_i$ and $d_j$ are the degrees of nodes $i$ and $j$. This matrix is widely used in resilience analysis due to its ability to expose structural properties by analyzing the spread of the eigenvalue spectrum.

*2.1.4 Spectral Metrics.* Spectral metrics provide quantitative measures of global network resilience by applying the Laplacian matrix to assess global network resilience [20]. The model provides a complementary resilience perspective (global versus structural resilience) and can be used to simulate and evaluate failure scenarios which is useful in identifying redundancy weaknesses and informing improved topological designs [6]. They are derived from the eigenvalues of the Laplacian matrix. **Eigenvalues** provide a mathematical method of quantifying network connectivity and resilience. Let $M$ be a symmetric matrix and $I$ the identity matrix of order $n$. Eigenvalues are the roots of the characteristic polynomial:

$$\det(M - \lambda I) = 0$$

For Laplacian matrices, eigenvalues range from 0 to 2:

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

The **Eigenvalue Spectrum** is the set of eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of $M$. The overall spread and clustering of the eigenvalues in the eigenvalue spectrum can reveal how evenly resilience is distributed throughout the network. A tightly clustered spectrum typically indicates a well-balanced and resilient topology, while a widely spread distribution may expose the structural property of the network being connected in a mix of very weak regions who are poorly connected by a few or less links/edges [20].

**Algebraic Connectivity**, a(G), the second smallest eigenvalue of the Laplacian spectrum, $\lambda_2$ [15], captures the connectivity of graphs in a broader spectrum than average node degree. As networks evolve, a(G) provides a sensitive measure of how well network connectivity is maintained after failures or attacks. If $\lambda_2 > 0$, the network is connected [4]. A larger value of $\lambda_2$ indicates a more resilient network, while a smaller $\lambda_2$ suggests lower resilience and a higher risk of the network fragmentation under failure [4].

The mathematical **multiplicity** of the eigenvalue spectrum indicates the how connected the network is. Multiplicity of zero eigenvalue, $m_0$, corresponds directly to the number of connected components [4]. $m_0 = 1$ indicates that the network is fully connected, whereas multiple zeros signify fragmentation into isolated sub-graphs — a clear indication of the dependence of single long distance links between communities, cohorts of nodes, i.e., spectral bottlenecks. The multiplicity of the one eigenvalue, $m_1$ identifies duplication of nodes - they posses the same adjacency list - a global redundancy measure [4]. Therefore, monitoring how multiplicity changes after simulated failures allows us to quantify the change in fragmentation and redundancy after node removals or additions.

*2.1.5 Centrality Metrics.* Centrality metrics help identify critical nodes in a network. Betweenness centrality measures how often a node occurs on shortest paths, highlighting key connector nodes [10]. Closeness centrality illustrates how quickly a node can reach other nodes [16]. Degree centrality indicates the number of direct connections a node has to other nodes [16].

*2.1.6 Core Resilience.* The **Core Resilience Model** identifies structurally embedded nodes using $k$-core decomposition. The $k$-core decomposition reveals hierarchical layering and node importance beyond traditional centrality measures. A $k$-core is a maximal subgraph where each node has at least $k$ connections. $k$-core decomposition iteratively removes nodes with a degree less than $k$, uncovering the network's layered structure. A node's *core number* is the highest $k$ for which it remains in the k-core, reflecting its structural depth and resilience to attacks or failures. *Core strength* is the minimum number of neighbors that must be removed to reduce a node's core number - a local redundancy measure, while *core influence* measures a node's impact on the core numbers of its neighbors [13]. The *Core Influence-Strength (CIS)* metric, defined as the average core strength of the top $r\%$ most influential nodes provides a measure of network support given by the core of the network. The *Maximize Resilience by K-Core (MRKC)* algorithm enhances resilience by iteratively connecting low k-core (core number) nodes to high k-core nodes [13].

*2.1.7 Fiedler Vector.* Let G be a (connected) graph with Laplacian L = D - A. The eigenvector, $x_F$, associated with the second smallest eigenvalue is the **Fiedler Vector** [3]. The entries of the Fiedler Vector contain bottleneck structures within the network/graph: they create a clustering of nodes into communities which are useful for exposing bottleneck links between node communities [3]. The Fiedler Vector, $x_F$, highlights weak ties within the network allowing for network manipulation with the goal of improving $a(G)$ [3]. Large absolute differences in $x_f$ mark nodes lying on opposite sides of the graphs main bottleneck allowing us to identify such pairs to connect them in order to connect communities of nodes together to raise algebraic connectivity.

*2.1.8 Global and Local Redundancy.* We quantify a network's global redundancy with the Multiplicity of the One Eigenvalue, $m_1$, of the normalized Laplacian matrix. A higher $m_1$ is equivalent to more structurally duplicate neighborhoods, cohorts of nodes. In contrast, local redundancy we quantify with Core Strength and the Core-Influence Strength metrics. A larger Core Strength, Core-Influence Strength means more neighbors have to be removed in order to reduce a node's/networks-core core number. i.e., higher local redundancy.

### 2.2 Taxonomy of Key Related Work

Centinkaya et al. [5] compare physical topologies and logical topologies against US interstate highway system. They computed traditional graph metrics (diameter, radius, hop count, degree) as well as the normalized Laplacian spectra - algebraic connectivity and eigenvalue spectrum. They argue that traditional graph metrics have proven inadequate for assessing resilience across networks of different sizes [4]. Research has shifted toward spectral graph theory, to evaluate connectivity, fragmentation, and structural weakness [4, 19]. Beyond SANReN-specific works, Cetinkaya et al. [4, 8] and Lee et al. [14] illustrated that physical topologies often follow transport corridors, creating vulnerable single-points of failure and bottlenecks which is useful for the application to our research.

Shatto et al. [19] also compare physical and logical topologies (two research networks and four commercial ISP networks) to transportation networks using the normalized Laplacian spectrum and eigenvalue spectrum and then evaluates these metrics using attack simulations. Shatto et al. [19] developed a framework to demonstrated how dynamic routing (re-routing of packets along alternative paths during network congestion) can improve the overall connectivity of the network if new edges are strategically placed in poorly connectivity subgraphs. They found that if you strengthen the topology by fixing low-connectivity nodes - measuring connectivity using closeness, betweenness and degree centrality - you raise algebraic connectivity, $a(G)$, reduce bottlenecks and increase path diversity. Alenazi et al. [2], provide a heuristic for designing more resilient topologies at the cheapest cost. They developed an algorithm that optimizes a topology based on algebraic connectivity and the available budget - measured using the euclidean distance between two connected nodes. Ghosh et el. [11] formulate edge-selection problem that directly maximize algebraic connectivity, showing that raising the Fiedler value tightens graph bottlenecks and improves connectivity. These results from Ghosh et al. [11] and Alenazi et al. [2] justify our use of algebraic connectivity as a primary metric for improvements across reinforcement strategies.

Albert et al. [1] show that many real-world networks are resilient to random failures, but when failures or even attacks are targeted at critical nodes the networks become significantly fragile. This research finds that random failures strike low-connected nodes leaving central nodes and connectivity intact, however when removing a careful chosen set of critical nodes the network rapidly fragments and connectivity collapses. This provides our research with an evaluation protocol that contrasts random and targeted removal suites to evaluate the effectiveness of our chosen metrics.

Laishram et al. [13] define a network's resilience as the stability of the top-$r$% of core numbers under random edge failures. Laishram et al. [13] define network resilience measures for a networks k-core structure, introduce node level network resilience measures and introduce the design of a edge-addition algorithm that increases a networks core resilience without changing core numbers, the Maximize Resilience of K-Core algorithm (MRKC). The Maximize Resilience by K-Core algorithm reinforces networks by linking low-k-core nodes to high-k-core nodes, aiming to improve resilience by connecting periphery nodes to central, anchor nodes [13]. Core resilience metrics reveal hierarchical layering within SANReN, especially in isolated regions [23, 25]. Though promising, these methods remain underused in SANReN analysis.

The eigenvalues of the normalized Laplacian Matrix and their respective multiplicities provide researchers with a heuristic of topology inference [3]. This is useful in identifying clusters, communities of nodes within a network as well as the links that connect these communities of nodes. These identified community-connecting links are critical links within the network [3] and pose risk of bottlenecks. Bertrand et al. [3] establish the second-smallest eigenvalue $\lambda_2$ and its associated eigenvector, the Fiedler Vector [12], as useful in evaluating network's connectivity [3]. Bertrand et al. [3] propose an efficient algorithm to compute the Fiedler Vector of a network and its associated algebraic connectivity. They apply this algorithm to a set of ad hoc networks, run Monte-Carlo simulations and apply random link failures with the broader goal of topology inference, not topology improvement. They find that the computation of the Fiedler Vector is useful in scaling network size and improving connectivity of a network.

## 3 Experimental Design

Prior work shows that classical graph measures (diameter, hop count, degree) are insufficient to measure and compare the resilience of networks of different sizes. Spectral analysis provides a framework to evaluate structural properties of networks of different sizes, [4, 19]. Studies that compare physical and logical topologies emphasize the network infrastructures often rely on bottleneck links that connect regional sub-networks and often concentrate flow creating single points of failure and circuitous routing. Resilience testing and therefore design of improved networks should focus on reducing the dependence on central nodes while improving local redundancy [4, 7, 14, 19].

Building on this, we evaluate SANReN and potential resilience improvement approaches using a cohesive approach leveraging spectral and core resilience analysis, whilst running removals to identify our chosen metrics ability at identifying node centrality and criticality, as well as three reinforcement strategies: 1. Fiedler Greedy that treats the Fiedler Vector as a bottleneck detector and adds edges bridging nodes with the largest absolute differences in the Fiedler Vector [3, 12]; 2. Maximize Resilience by K-Core which links low-k-core, periphery nodes to high-k-core, central nodes to strengthen the core without altering nodes' core numbers [13]; and 3. Random edge addition to act as a baseline for our study. Each removal/reinforcement step is accompanied by a full calculation of spectral and core resilience metrics and a summary statistic - area-under-curve (AUC) - is computed after each step and plotted. Plotting the area-under-curve across reinforcement steps yields comparable trajectories that reveal changes in connectivity, fragmentation, redundancy and diminishing marginal returns to connectivity.

## 3.1 Data ingestion and graph construction

The framework accepts either `.json` or `.tgf` inputs. For each file we construct a simple, undirected NetworkX graph $G = (V, E)$ upon which metrics are calculated and removals, reinforcements are applied to. Nodes are site host names and edges are unweighted, undirected links.

## 3.2 Metrics

For each $G$ we compute three metrics: *(i) Spectral metrics:* We compute the Laplacian spectrum and record (a) the algebraic connectivity $a(G) \equiv \lambda_2$, a global connectivity indicator; (b) the multiplicity of the zero eigenvalue $m_0$, equal to the number of connected components; and (c) the multiplicity near one $m_1$ (and density near 1 for the normalised Laplacian), which reflects global redundancy. *(ii) Core-resilience:* We perform a $k$-core decomposition to obtain each node's Core Number $\kappa(u)$; Core Strength (CS) using a closed-form estimator $\mathrm{CS}(u) = |\{v \in \Gamma(u) : \kappa(v) \geq \kappa(u)\}| - \kappa(u) + 1$; and Core Influence (CI) via the principal eigenvector of a sparse "support" matrix $M$ that routes unit support from lower/equal core to higher core neighbors (global load-bearing effect), and the network-level Core-Influence Strength score as the mean Core Strength among the top $f$ percentile by Core Influence. *(iii) Classical centrality metrics:* Degree, closeness, and betweenness centrality are computed for comparability.

## 3.3 Node Removal experiments

We study how each metric degrades as nodes are removed. For each network we generate a *decreasing removal order* $\pi$ under several strategies: random, core influence, degree, closeness, and betweenness. We contrast random failures to targeted removals of high-centrality nodes to show how the subjugated networks are resilient to random failures but vulnerable to targeted removals, a result observed by Albert et al. [1]. For targeted strategies, we return a descending order list, $\pi$, sorted by the chosen metric - core influence, degree, closeness and betweenness centrality - computed once on the original network. Starting from the original network, nodes are removed one by one following $\pi$, producing $G_1, G_2, \ldots$, and after each removal $t$ we recompute and record $a(G_t)$, $m_0(G_t)$, $m_1(G_t)$, and $\mathrm{CIS}(G_t)$. This allows us to identify core influences ability in identifying node criticality and centrality compared to classical centrality metrics. Removals are stopped when the number of nodes becomes less than two, $|N_t| < 2$.

## 3.4 Area-under-curve (AUC) Summary Statistic

Each reinforcement step, and the subsequent application of removal experiments to specifically evaluate resilience to fragmentation, is accompanied by calculation of each metric ($a(G)$, $m_0$, $m_1$, $CIS$) at each step yields a trajectory of each metric. We summarize this trajectory by calculating the area-under-curve (AUC): Area-under-curve is the cumulative average metric value over reinforcement steps. We plot each area-under-curve value per step on line graphs to visualize the impact of applying our given reinforcement strategies on connectivity ($a(G)$), global and local redundancy ($m_1$, $Core - Influece - Strength$, $Core - Strength$) and fragmentation ($m_0$ under removals). We define area-under-curve as (AUC):

$$\mathrm{AUC}_y\big(G_s, \mathrm{suite}\big) = \frac{1}{T+1} \sum_{t=0}^{T} y_t.$$

where $G_s$ is the graph after each $s$ reinforcement step, $y_t$ is the value of the $y$ metric after $t$ nodes have been removed according to the removal ordering given by the attack suite of random order, or by the decreasing order by core influence, betweenness, closeness and degree centrality, and $T$ is the number of removal steps. AUC is reported because: 1. Resilience is about the performance throughout a set of failures, not just a single checkpoint; 2. Area-under-curve aggregates the metrics over a the reinforcement steps capturing behavior of a specific metric of a network under a given reinforcement strategy; 3. AUC is a comparable statistic that can be used to compare networks of different sizes.

## 3.5 Topology Reinforcement Strategy

We use three edge-addition strategies (Fiedler-greedy, Maximize Resilience by K-Core Algorithm and Random addition) on each regional SANReN subgraph as well as the full SANReN graph (Isis-links) with the goal of quantifying how these additions improve connectivity, improve resilience to fragmentation, improve redundancy.

A key design choice is use of node-removal experiments to quantify the effects on fragmentation. In any connected graph the multiplicity of the one eigenvalue equals one, $m_0 = 1$, and under edge additions it remains at 1, therefore a no-removal trajectory cannot reveal whether reinforcements improves resilience to fragmentation. To measure the effects of reinforcement on fragmentation, we run full removal suites at each reinforcement step, record $m_0$ as failures accumulate and summarize the trajectory with an area-under-curve statistic. This trajectory captures: how long the network stays connected; the rate of fragmentation; and the severity of the fragmentation. This directly measures how each reinforcement strategy improves resilience to fragmentation under realistic failure simulations.

In contrast, Algebraic Connectivity, $a(G)$, Multiplicity of the One Eigenvalue, $m_1$, and Core-Influence Strength, $CIS$, respond directly to added edges without the need for simulated failures. For these metrics, we therefore report their trajectories across reinforcement steps using the area-under-curve summary statistic (as their cumulative mean). This measures how each strategy changes connectivity ($a(G)$, redundancy ($m_1$, Core Strength) and overall network support ($CIS$). At each reinforcement step we evaluate the graph in the following manner:

(1) From the current graph we add a small budget of non-edges (the full SANReN networks add ten edges and sub-networks adds one edge per step for ten steps). We then compute spectral and core resilience metrics ($a(G), m_1, CIS$) of graph before and after additions and record the change in each metric: $\Delta a(G) = a(G)_1(G_s) - a(G)_0(G_s)$. To summarize the full reinforcement path, we report a normalized area-under-curve over steps which is the cumulative mean of each metric over steps. This statistic averages performance across all additions to enable comparison across graphs of different sizes. These no-removal trajectories tell us how each strategy directly effects connectivity, redundancy and support.
(2) For the purposes of Fragmentation: After each reinforcement step we take the new graph $G_s$, and run the five node-removal patterns. Nodes chosen by the removal strategy are removed one at a time and after each removal we compute spectral and core resilience metrics: $a(G); m_0; m_1; CIS$. This produces four trajectories for each removal strategy as a function of nodes removed. The resulting area-under-curve of ($m_0$) captures the onset, rate and severity of fragmentation.
(3) We repeat this until the addition budget is exhausted or no non-edges remain, producing logs and plots for all metrics.

# 4 Programmatic Logic Algorithms

This section explains the exact algorithms used in our experiments. The code is made up of modular, small, and reusable components each performing a specific task. The pseudocode of each algorithm is included so that readers can re-implement these algorithms.

## 4.1 K–core and Core Number.

The $k$–core of a graph is the largest subgraph in which every node has degree at least $k$ within that subgraph - degree denoting the number of connections a particular node has. By peeling $k$=1, 2, ... we obtain hierarchical layering. A node's Core Number $\kappa(u)$ is the deepest layer it belongs to. Intuitively, $\kappa(u)$ is a measure of how well embedded node $u$ is in the network's core structure.

## 4.2 Core Strength (CS) - Local Redundancy

For a node $u$, **core strength answers:** *how many of $u$'s neighbors could fail before $u$'s core number would decrease?* High Core Strength means $u$ has a high local redundancy among its peers who are as strong or stronger than node $u$ [13].

**How we compute it.** Let $\Gamma(u)$ be $u$'s neighbors and $\kappa(\cdot)$ be their core numbers. Count $u$'s neighbors that are at least as strong as $u$:

$$N_{\geq}(u) \;=\; \left|\{\, v \in \Gamma(u) \;:\; \kappa(v) \geq \kappa(u) \,\}\right|.$$

Then we use the closed-form estimator

$$\mathrm{CS}(u) \;=\; N_{\geq}(u) \;-\; \kappa(u) \;+\; 1,$$

which captures the smallest local loss that can force $u$ out of the $\kappa(u)$–core. In code (`compute_core_strength`), we compute $\kappa(\cdot)$ once (linear in edges), then, for each $u$, count neighbors with $\kappa(v) \geq \kappa(u)$ and apply the formula - overall order being $O(|E|)$.

To remain in the $\kappa(u)$–core, $u$ needs *at least* $\kappa(u)$ neighbors within that core. Only neighbors with $\kappa(v) \geq \kappa(u)$ can still be present when that core is formed; $\mathrm{CS}(u)$ is, therefore, the difference between "strong neighbors available" and the threshold $\kappa(u)$.

## 4.3 Core Influence (CI): Global Support - Connectivity Measure

Core influence asks: how much does node $u$ support other nodes with *lower* core numbers? Nodes with high Core Influence are global "load-bearing" points, i.e., they provide a central connection between many nodes within the network: removing them tends to make many weaker nodes lose their core membership increasing fragmentation of the network [13].

We build a sparse "influence" matrix $M$ from the graph and core numbers (function compute_core_influence()):

- For each undirected edge $\{u, v\}$, we add a directed arc from the lower or equal core to the higher core. If $\kappa(u) \leq \kappa(v)$ we add $u \to v$ with weight $\frac{1}{d_{\geq}(v)}$, where $d_{\geq}(v)$ is the number of neighbors of $v$ whose core number is at least $\kappa(v)$.

This weighting means each higher-core node $v$ *shares* one unit of dependence evenly across the set of neighbors that are strong enough to support $v$. The principal right eigenvector $r$ of $M$ (dominant eigenvector; we compute it with a sparse eigensolver) is then taken as the stationary "support credit" each node accumulates. We normalise $r$ to unit length and use its nonnegative entries as $\mathrm{CI}(u)$.

Influence flows upwards: lower/equal-core nodes can support higher-core nodes, but not vice-versa. By normalising by $d_{\geq}(v)$ we avoid over-crediting high-degree nodes; the eigenvector aggregates direct and indirect support (supporters of supporters), capturing the global nature of influence. In practice, this yields a robust ranking of nodes whose removal would cause many weaker nodes to slip in core number.

## 4.4 Core Influence–Strength (CIS) - Network-Level Summary

**CIS asks a network-level question:** *among the most influential nodes, how much local redundancy do they have?* If topology influencers are also locally well-supported, the whole core is resilient; if they are fragile, the core can collapse quickly.

**How we compute it.** Given Core Influence and Core Strength for all nodes, pick a top percentile $r$ of most influential nodes (e.g., $r = 0.9$ selects the top 10% by Core Influence):

$$S_r \;=\; \{\, u : \; \mathrm{CI}(u) \text{ is in the top } 100(1-r)\% \,\}.$$

Then

$$\mathrm{CIS}_r \;=\; \frac{1}{|S_r|} \sum_{u \in S_r} \mathrm{CS}(u).$$

In code (`compute_CIS`), we take a percentile threshold on CI, form $S_r$, and average their Core Strength. Larger $\mathrm{CIS}_r$ means the influential core has more redundancy and is thus more resilient.

Circuitous routing and poor connectivity arise when global cohesion is low. **Core Influence** highlights those load-bearing connectors; **Core Strength** tells us whether they have local redundancy;

**Core-Influence Strength** summarizes the core's overall robustness. In our experiments, we compare removal strategies ranked by Core Influence and reinforce the topology (e.g., by adding edges between Fiedler-separated regions or anchoring low-Core Strength influencers) and attempt to show improved algebraic connectivity (higher AUC of $a(G)$), reduced fragmentation and improved redundancy (lower AUC of $m_0/m_1$).

### 4.5 Fiedler Vector

Fiedler-Greedy chooses a non-edge $(u, v)$ whose addition increases algebraic connectivity the most by directly, hence the greedy approach, bridging spectral bottlenecks: adding edges that connects two nodes whose $\lambda_2$ have the largest difference in the Fiedler Vector. Algorithm Steps:

(1) Compute the Fiedler Vector ($x_f$) - the eigenvector associated with the second smallest eigenvalue $\lambda_2$.
(2) For every non-edge, compute a score: $s(u, v) = |(f_u - f_v)^2|$.
(3) Return the non-edge with the highest score - the largest absolute difference. i.e., the score between nodes who lie on opposite sides of the Fiedler Vector's 'main bottleneck'.

Connecting these nodes, with the largest absolute difference, creates alternative paths bypassing the main spectral bottleneck, which tend to greedily increase algebraic connectivity.

### 4.6 Maximize Resilience by K-Core Heuristic

The goal of this reinforcement algorithm is to pull the lowest k-core nodes to the high k-core nodes by adding edges that connect these low k-core nodes to high k-core anchors. This tends to reduce fragmentation risk at the periphery of the network, shorten paths from leaf nodes to core nodes and increase algebraic connectivity. We use a heuristic of the Maximize Resilience by K-Core algorithm proposed by Laishram et al. [13]. Algorithm Steps:

(1) Compute all node's core number $k(v)$.
(2) Identify two sets:
  (a) Vulnerable set $V_{\min} = u : k(v) = min_u k(v)$
  (b) Anchor set $V_{\max} = v : k(v) = max_v k(v)$
(3) Sort $V_{\min}$ in ascending order and $V_{\max}$ in descending order of core number.
(4) Scan the Cartesian product of $V_{\min}$ x $V_{\max}$ and return the first non-edge $(u, v)$ (skipping duplicates).

This returns a candidate edge together with a tuple containing their core numbers $k(v), k(u)$, connecting periphery nodes to core nodes to create short alternative paths into the core of the network. This process is repeated until no such non-edge pair exists or the budget of the reinforcement is reached.

### 4.7 Random Edge Addition

The goal is to add new non-edges at random. Algorithm steps:

- Repeatedly sample two distinct nodes $u, v$ at random.
- If $(u, v)$ is a non-edge, add it and record the addition.
- Continue until the budget is reached.

This acts as a useful baseline to contrast against the Fiedler-Greedy and Maximize Resilience by K-Core algorithms.

## 5 Materials

### 5.1 Datasets

We analyze the full SANReN national backbone and regional subgraphs, each file is an edge list for each of the communities of nodes within SANReN. All inputs were provided by SANReN and normalized as simple, undirected and unweighted graphs. Nodes represent sites and edges represent links between sites. During pre-processing we standardize node labels between inputted graph. Isis-links is the full SANReN backbone. Regional SANReN subgraphs (`.tgf` files) include: `cpt.tgf` for Cape Town; `jnb.tgf` for Johannesburg; `pta.tgf` for Pretoria; `bfn.tgf` for Bloemfontein; `dur.tgf` for Durban; `els.tgf` for East London; `pzb.tgf` for Port Elizabeth; and `vdp.tgf` for Vanderbijlpark.

### 5.2 Software Frameworks

All graph computations, plotting, reinforcement and attack strategies were implemented in Python using: NetowrkX for graph construction, graph operations (edge additions, removals), computation of classical centrality measures, k-core computations, constructing the Laplacian matrix from which the $a(G)$, $m_0$, $m_1$ are computed and derived; Numpy for linear algebra, counting eigenvalue multiplicities, and calculating the Core-Influence Strength metric; Pandas for data handling of reinforcement, removal, area-under-curve summaries using pandas Dataframes and for efficient exporting to CSV's for result storage; and Matplotlib for plotting of the effects of the reinforcement strategies on $a(G)$, $m_0$, $m_1$, and CIS under each removal ordering as well as trajectories of $a(G)$, $m_0$, $m_1$, and CIS under removals and non-removals.

## 6 Results

### 6.1 Core Resilience Results

Across SANReN and its regional subgraphs, Core-Influence Strength (*CIS*) provides a summary metric of core resilience metrics (core number, strength, influence) and highlights how much support the core provides to the rest of the graph. Core-Influence Strength indicates how much local redundancy the most influential, critical and central nodes have. Larger *CIS* indicates a strong core with strong connections to the rest of the network providing a high-load bearing core and high local redundancy. In Table 1, Durban (5.667), Johannesburg (5.5) and Cape Town (5.286) sit at the top of the CIS rankings and exhibit strong cores who redistribute load effectively. The Pretoria and full SANReN graph (isis-links) are mid tier and offer moderate support to the rest of the network. The smallest scores occur in Vanderbijlpark, Bloemfontein and Port Elizabeth indicating shallow cores with limited support due to limited reach to the rest of the network.

**Table 1: Core-Influence Strength (CIS) per graph**

| Name | CIS |
|---|---|
| Bloemfontein | 2.000 |
| Cape Town | 5.286 |
| Durban | 5.667 |
| East London | 3.000 |
| Johannesburg | 5.500 |
| Pretoria | 4.143 |
| Port Elizabeth | 2.500 |
| Vaderbijlpark | 1.000 |
| Isis-links | 4.207 |

## 6.2 Spectral Metrics

The full SANReN graph, as seen in Table 2, possesses a low algebraic connectivity ($a(G) = 0.02468$), $m_0 = 1$ and $m_1 = 23$, indicating that this network is connected ($m_0$), and it possesses many redundant sub-components ($m_1$). This indicates that the backbone network of the full SANReN graph is tenuously connected. Subgraphs of Pretoria and Cape Town also exhibit low connectivity. In contrast, sparser networks such as Port Elizabeth, Vanderbijlpark, and East London are more spectrally connected. All graphs have a $m_0 = 1$ so they are connected with no disconnected subgraphs. Redundancy of dense graphs such as the Isis-links, Cape Town and Johannesburg is high, indicated by the $m_1$. This indicates many similar node communities within these network.

**Table 2: Spectral Results per Graph**

| Name | Algebraic Connectivity | Mult. of $m_1$ | Density near $m_1$ | Mult. of $m_0$ |
|---|---|---|---|---|
| Isis-links | 0.02468 | 23 | 0.08214 | 1 |
| Bloemfontein | 0.38732 | 8 | 0.47059 | 1 |
| Vanderbijlpark | 0.70293 | 3 | 0.37500 | 1 |
| Port Elizabeth | 0.77547 | 4 | 0.36364 | 1 |
| Pretoria | 0.05663 | 4 | 0.06557 | 1 |
| Johannesburg | 0.22360 | 8 | 0.25000 | 1 |
| East London | 0.53404 | 5 | 0.35714 | 1 |
| Durban | 0.21591 | 5 | 0.18519 | 1 |
| Cape Town | 0.07631 | 12 | 0.17391 | 1 |

## 6.3 Node and Edge Removals

Targeted removals, by core-influence, degree, closeness and betweenness centrality, decreases algebraic connectivity, redundancy as well causing early increases in fragmentation and followed by rapid declines in fragmentation (Figures 1, 2, 3). The initial increase in fragmentation is intuitive: removing critical nodes early increases the number of unique connected components (a community of nodes who are heavily dependent on a central, critical node now become two isolated communities when this central node is removed). This causes rapid increases in fragmentation early on. After many removals the graph becomes a pile of singletons and removing further nodes erases whole components, causing the count of the zero eigenvalue to fall. Appendix B shows results of removals on the full SANReN backbone network.

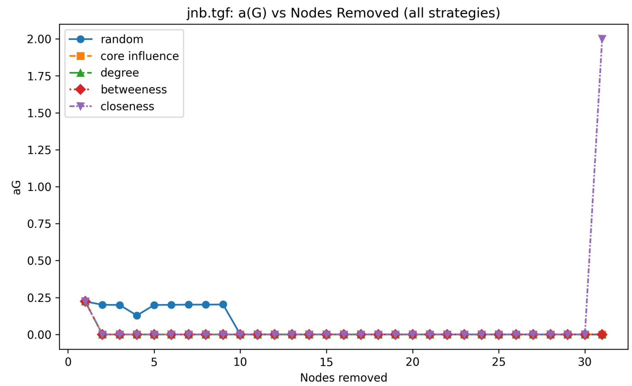


**Figure 1: Removal Effects on $a(G)$ of Johannesburg**

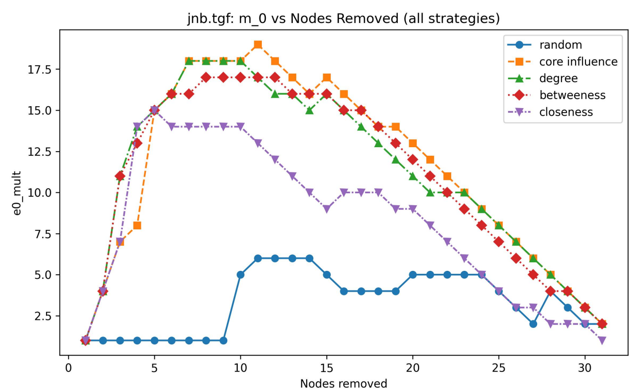


**Figure 2: Removal Effects on $m_0$ of Johannesburg**

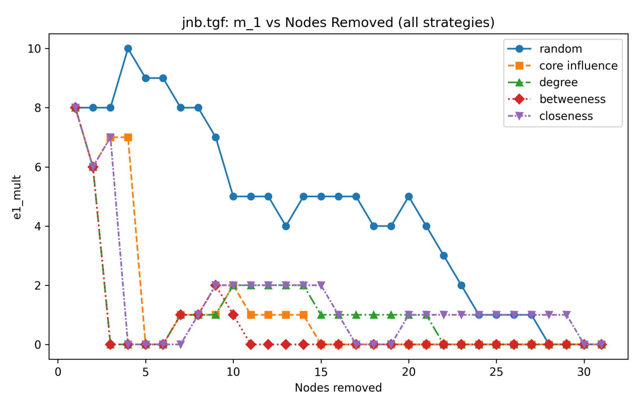


**Figure 3: Removal Effects on $m_0$ of Johannesburg**

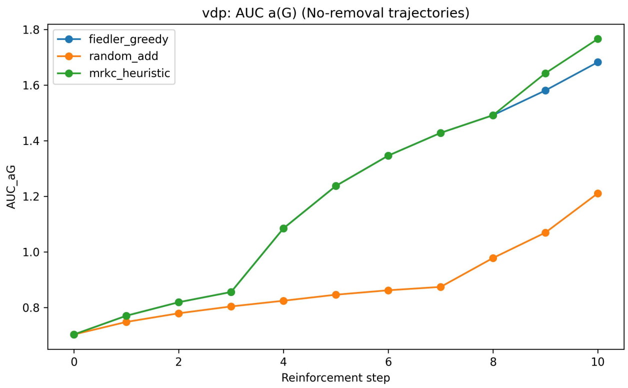


**Figure 4: Change in Algebraic Connectivity per Reinforcement Strategy on Vanderbijlpark Network**

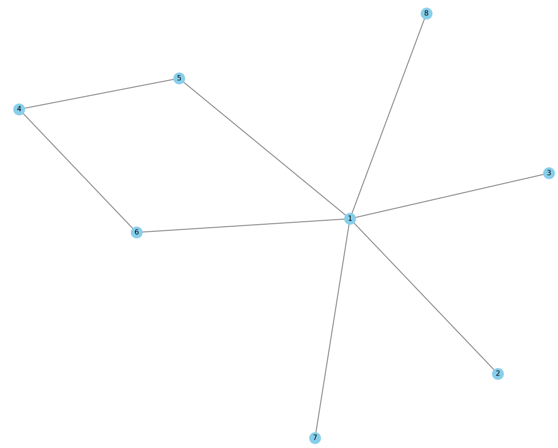

**Figure 5: Vanderbijlpark Network**

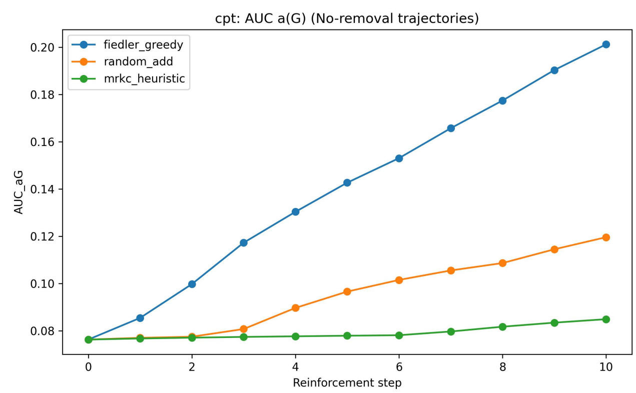


**Figure 6: Change in Algebraic Connectivity per Reinforcement Strategy on Cape Town Network**

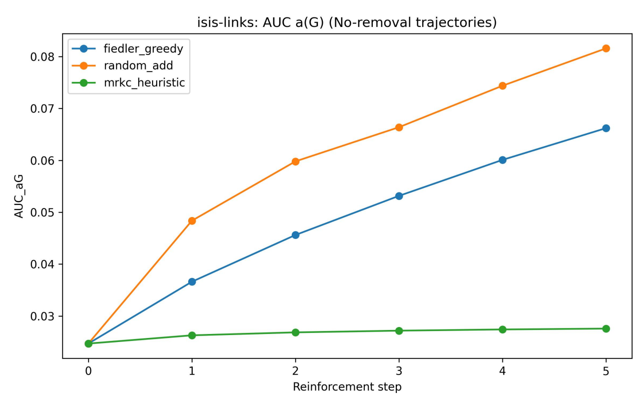


**Figure 7: Change in Algebraic Connectivity per Reinforcement Strategy on Isis-links**

## 6.4 Topology Reinforcements

The Fiedler-Greedy reinforcement approach increased average algebraic connectivity from 0.3 to 1.7 (466% increase) in the smallest subnetwork within SANReN (Figures 5, 4). The Maximize Resilience by K-Core heuristic outperforms the Fiedler approach in the Vanderbijlpark graph - achieving the highest average increase in algebraic connectivity (0.3 - 1.75). In contrast, in the larger subgraphs of SANReN specifically Cape Town (Figure 6), Bloemfontein (Figure 11b), Durban (Figure 11d), Johannesburg (Figure 11f), and Pretoria (Figure 11h) networks, the Fiedler approach significantly outperforms Maximize Resilience by K-Core and random edge addition in improving a networks algebraic connectivity, $a(G)$ over non-removal reinforcement strategies. Interestingly the connectivity results from reinforcements of the full SANReN graph, Isis-links, show that the Random edge addition strategy outperformed both Fiedler and Maximize Resilience by K-Core (Figure 7). Overall the Maximize Resilience by K-Core heuristic performed significantly the worst, seeing minimal gains under reinforcement steps and for larger more dense graphs whilst the Fiedler approach outperformed both other strategies in all the SANReN subgraphs, (Figure 11, Appendix C).

Across graphs and reinforcement strategies, we observe a consistent downtrend in global redundancy. This effect is clearest in sparser graphs (Figure 10) where added edges tend to differentiate once similar neighborhoods. In larger denser graphs (Figure 8, 9) the effect is less, where equivalent neighborhoods remain even after several additions. The Fiedler approach tends to keep global redundancy stable throughout most denser graphs (Figure 13a 13c, 13g) and tends to outperform the other two strategies in keeping global redundancy stable. In larger networks the Maximize Resilience by K-Core approach outperforms Fiedler and random in improving the network's core support (Figures 14c, 14f, 14g) and increasing local redundancy (Figures 15i, 15g, 15d, 15h). While Fiedler reliably increases local redundancy and random edge addition generally performs the worst in improving core support and local redundancy. By wiring low-k-core nodes to high-k-core nods the Maximize Resilience by K-Core reinforcement strategy raises local redundancy (Core Strength, Core-Influence Strength). The gains are largest in smaller, sparser graphs. In larger, denser graphs Core-Influence Strength (global support) improves but plateaus - indicating diminishing marginal returns to support. Because the Fiedler-Greedy approach targets spectral bottlenecks rather than anchors, it lifts Core-Influence Strength, but gains are modest and often stagnate in denser graphs. Random addition usually increase support (*CIS*), and on the Isis-links network it outperforms the other two reinforcement strategies by diversifying support instead on focusing on specific areas.

Fiedler-Greedy reinforcement decreases the $m_0$ causing the graphs to fragment less as potential edges are added. This is especially noticeable in the East London, Bloemfontein and Isis-links graphs (Figures 20, 21, 17). Sparse graphs (East London, Vanderbijlpark) experience the largest decline in fragmentation, dense cores (Cape Town) see limited gains, and hub-centric graphs (Pretoria) experience negative effects to fragmentation due to focus of Maximize Resilience by K-Core to add edges to critical nodes. The Maximize Resilience by K-Core approach leads to creating super-hubs around central nodes worsening resilience of fragmentation. The Fiedler-Greedy approach is useful in decreasing the effect of fragmentation on a sparse network, but when networks are dense and hub-centric the Fiedler-Greedy approach improves resilience fragmentation the best, Appendix C.

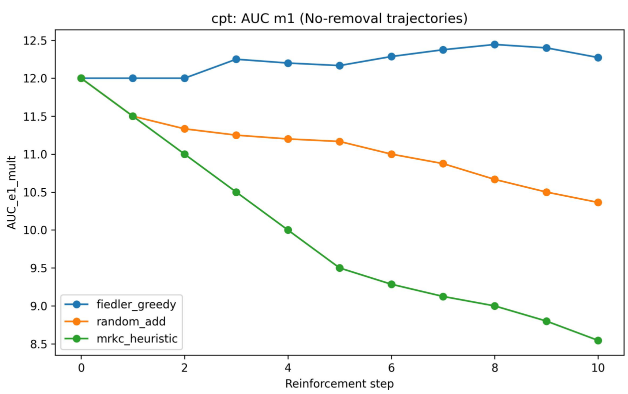


**Figure 8: Change in Multiplicity of One Eigenvalue per Reinforcement Strategy for Cape Town**

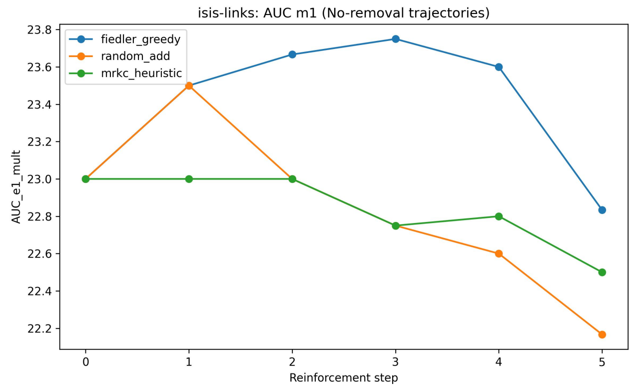


**Figure 9: Change in Multiplicity of One Eigenvalue per Reinforcement Strategy for Cape Town**

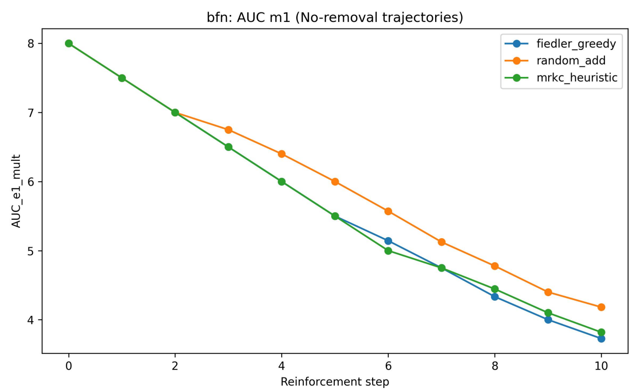


**Figure 10: Change in Multiplicity of One Eigenvalue per Reinforcement Strategy for Bloemfontein**

## 7 Discussion

Under targeted removals by core influence, $a(G)$ degrades, *CIS* degrades, and fragmentation, $m_0$, increases in a similar fashion to classical measures. Interpreting the results from these removals we are able to deduce that core influence is a good measure of node criticality and centrality because it is able to identify critical and important nodes, that when removed cause similar fashions of degradation in spectral and core metrics ($a(G)$, $m_0$, $m_1$, $CIS$) when compared to classical measures.

At baseline, SANReN and its subgraphs exhibit a range of connectivity scores, with the larger more dense graphs showing lesser connectivity than those of the smaller sparser networks. Several metros possess strong cores (high CIS) but still exhibit bottlenecks (low $a(G)$), e.g., Isis-links, while others have thin cores (low CIS) despite having decent connectivity, e.g., Vanderbijlpark. This presents the need for effective reinforcement to raise connectivity, keep $m_0$ at 1 under stress conditions by adding cross-community edges and improve alternative paths within the network by improving $m_1$.

This study set out to test whether spectral graph theory and core resilience analysis can be used cohesively to test the structural properties, specifically structural weaknesses, and guide practical reinforcements. At baseline, SANReN is fully connected, $m_0 = 1$, exhibits poor connectivity, low $a(G)$, and possesses many redundant twin structures. This pattern indicates to us that SANReN hinges on a few overloaded bottlenecks while many neighbors look 'equivalent'. Regional subgraphs exhibit different characteristics: subgraphs such as Cape Town, Durban, Johannesburg, and Pretoria have dense cores (high CIS) are are poorly connected (low $a(G)$), whilst others (Vanderbijlpark, Port Elizabeth, East London) are sparse networks who are highly connected but rely on one to two nodes as their core.

The reinforcement results show us that: connectivity improves the most when edges are added that bridge spectral bottlenecks; fragmentation resistance is best understood by observing how $m_0$ changes under removal suites; and redundancy should be considered in two facets - global redundancy ($m_1$) and local redundancy (Core Strength, Core-Influence Strength). The Fiedler-Greedy reinforcement approach raises resilience to fragmentation and algebraic connectivity ($a(G)$) the most reliably across graphs by wiring across spectral bottlenecks, which is exactly where Fiedler identifies bottlenecks. In smaller, sparse networks these improvements show up early reflecting the creation of new cross-community routes. In denser graphs these gains are apparent but gains plateau once major bottlenecks are bridged. Maximize Resilience by K-Core improves algebraic connectivity in smaller networks, but in larger more dense networks it tends to re-attach many periphery (low-k-core) nodes to the same anchors (high-k-core), causing flow to be centralized and yielding small connectivity gains. Therefore, this core prioritization is not effective in improving overall algebraic connectivity. Maximize Resilience by K-Core is the most effective at improving local redundancy and resilience to fragmentation by strengthening weak periphery nodes who fragment first under failures. Therefore, Maximize Resilience by K-Core should be used as a late consolidator of a network.

Fiedler-Greedy and Maximize Resilience by K-Core reinforcement tends to decrease global redundancy, $m_1$, by creating fewer duplicate neighborhoods, while increasing local redundancy (higher Core Strength, Core-Influence Strength). Higher Core Strength and Core-Influence Strength indicates that reinforcements create stronger load-bearing support for the rest of the network. Together with the rise in algebraic connectivity the net effect of reinforcements by Fiedler-Greedy and Maximize Resilience by K-Core is positive: fewer fragile duplicates, stronger core and better global connectivity whilst improving local redundancy of nodes. These reinforcements create a network that is better connected less prone to fragmentation and better able to tolerate random failures. In practice, watching the slope of CIS and $m_1$ gives us, and future work, an indicative measure of diminishing marginal returns indicating additional resilience yield would be worth less than the cost of adding such an edge.

## 8 Conclusion

This work shows that spectral metrics ($a(G)$, $m_0$, $m_1$) and core resilience metrics (core influence and CIS) form a diagnoses framework for SANReN. Together they diagnose a connected but weakly connected and fragmented network exhibiting many redundant substructures, which are able to directly inform the need for reinforcements. Fiedler-Greedy is very useful in bridging spectral bottlenecks improving resilience to fragmentation, local redundancy and connectivity in both sparse and dense networks. Maximize Resilience by K-Core is useful in tidying up/consolidating the periphery improving local redundancy and resilience to fragmentation. Random addition is a good baseline that helps diversify central, critical nodes within large networks. Therefore, this combined spectral-core framework answers the research question because it establishes both a evaluation framework of SANReN's resilience, structural properties and guides a combined reinforcement approach that mixes Fiedler, Maximize Resilience by K-Core and random edge addition.

The implication of this study is to plan the reinforcement in phases: 1. Bridge communities of nodes to reduce spectral bottlenecks using Fiedler; 2. Connect minimum core nodes with high core nodes using Maximize Resilience by K-Core to tidy up and harden the new corridors created by Fiedler - by connecting periphery nodes to central ones, whilst limiting the per node connections to prevent the creation of super-hubs; 3. Track *CIS* and $m_1$ to detect saturation identified by diminishing marginal returns of these metrics. Together this framework improves connectivity, resilience to fragmentation and redundancy. Future work should incorporate weights and traffic, as well as dynamic re-ranking of $\pi$. The present approach provides a clear road map from measurement of structural properties and bottlenecks to intervention through reinforcements using a combined approach.

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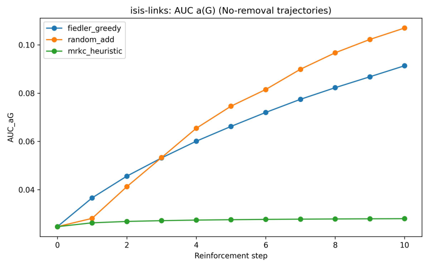
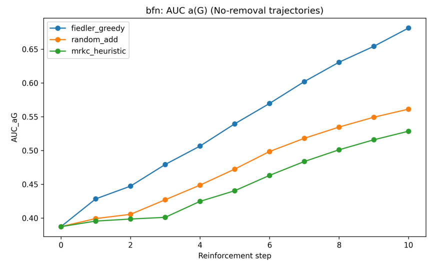
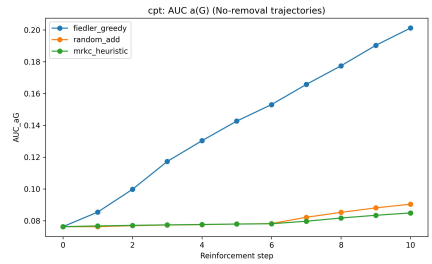
# A Appendix A: Area Under Curve by Strategy and Reinforcement Step for Bloemfontein



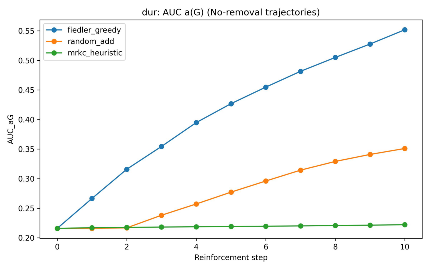

(a) isis-links: AUC $a(G)$ (no removals)



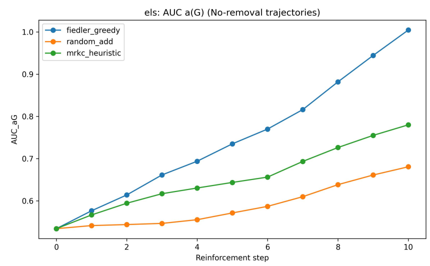

(b) bfn: AUC $a(G)$ (no removals)



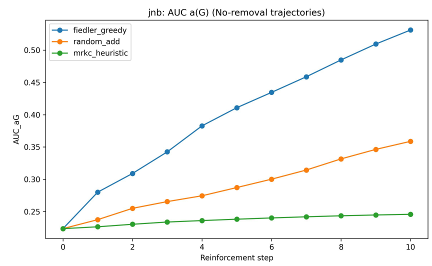

(c) cpt: AUC $a(G)$ (no removals)



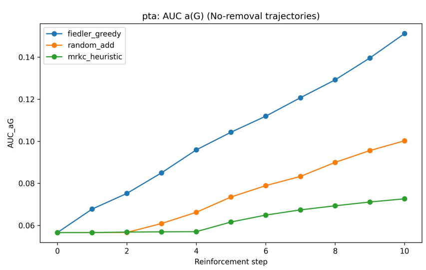

(d) dur: AUC $a(G)$ (no removals)



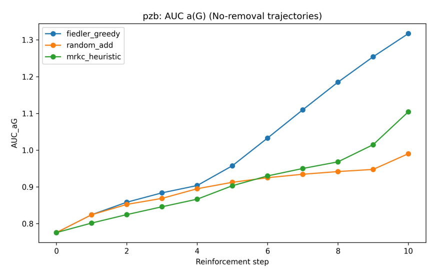

(e) els: AUC $a(G)$ (no removals)



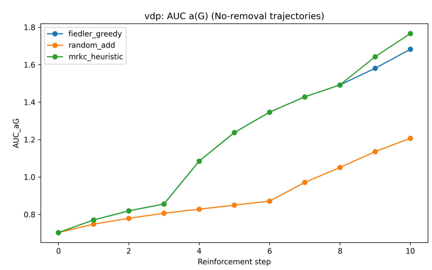

(f) jnb: AUC $a(G)$ (no removals)




(g) pta: AUC $a(G)$ (no removals)

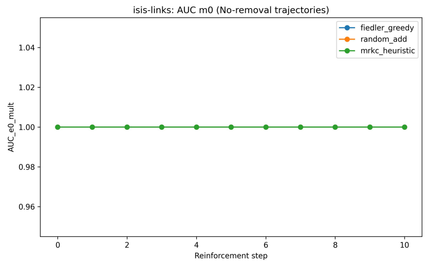



(h) pzb: AUC $a(G)$ (no removals)

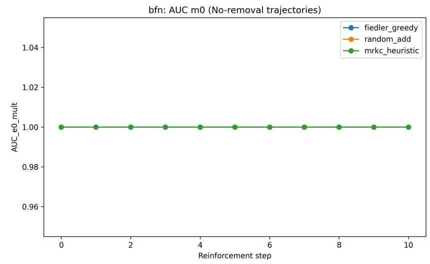



(i) vdp: AUC $a(G)$ (no removals)

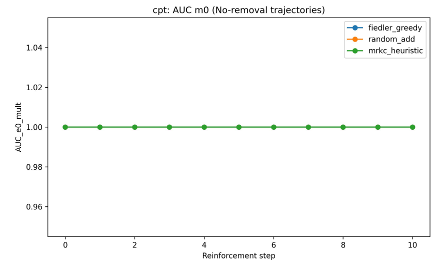
**Figure 11: No Removals - Cumulative Mean (AUC) of Algebraic Connectivity Across All Graphs for All Reinforcement Strategies**



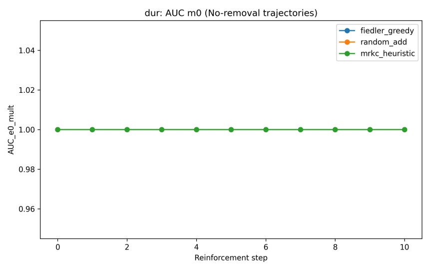

(a) isis-links: AUC $m_0$ (no removals)



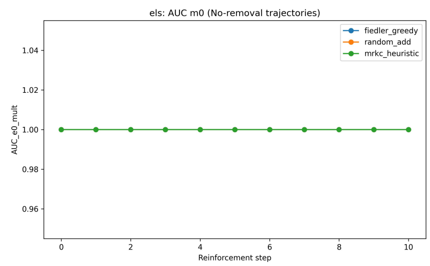

(b) bfn: AUC $m_0$ (no removals)



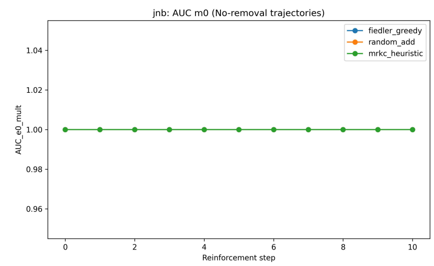

(c) cpt: AUC $m_0$ (no removals)



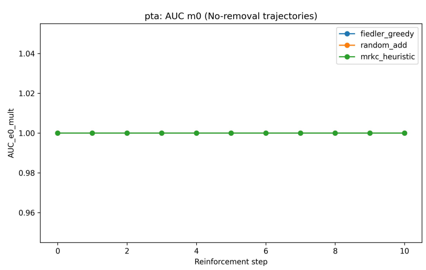

(d) dur: AUC $m_0$ (no removals)



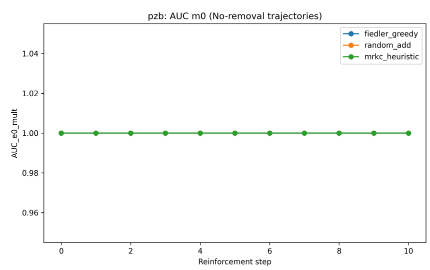

(e) els: AUC $m_0$ (no removals)



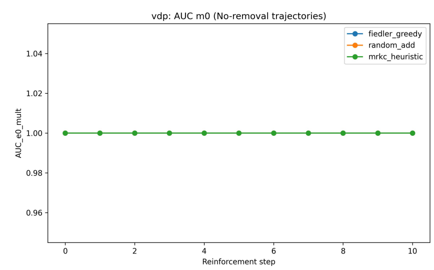

(f) jnb: AUC $m_0$ (no removals)




(g) pta: AUC $m_0$ (no removals)

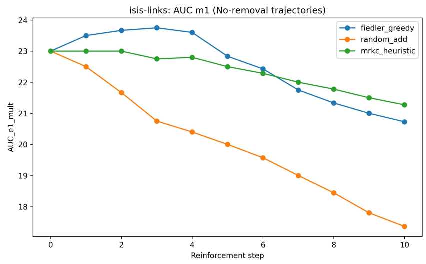



(h) pzb: AUC $m_0$ (no removals)

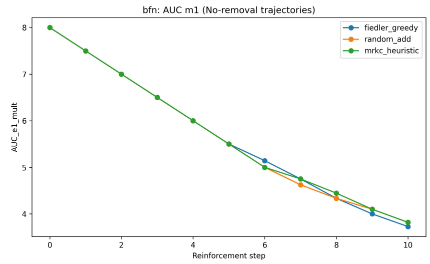



(i) vdp: AUC $m_0$ (no removals)

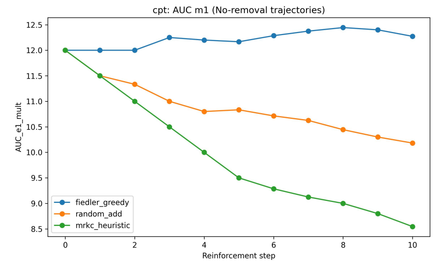
**Figure 12: No Removals - Cumulative Mean (AUC) of Multiplicity of Zero Eigenvalue Across All Graphs for All Reinforcement Strategies**



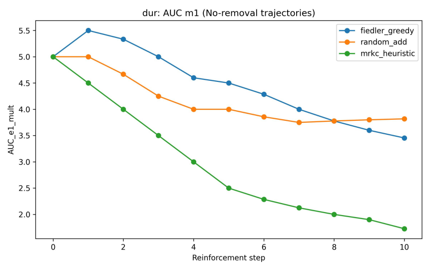

**(a) isis-links: AUC $m_1$ (no removals)**



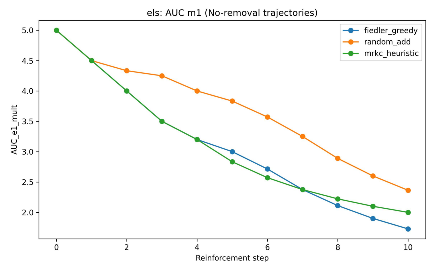

**(b) bfn: AUC $m_1$ (no removals)**



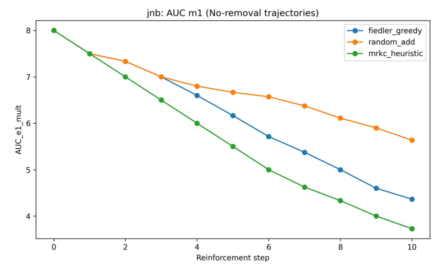

**(c) cpt: AUC $m_1$ (no removals)**



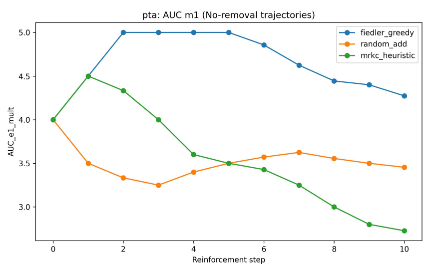

**(d) dur: AUC $m_1$ (no removals)**



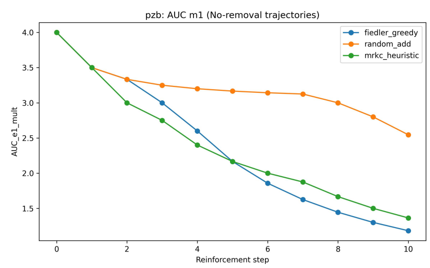

**(e) els: AUC $m_1$ (no removals)**



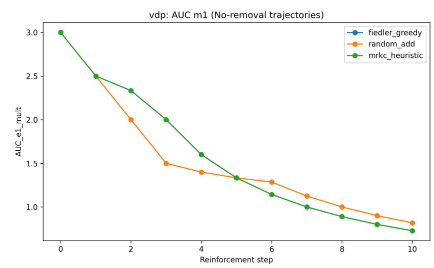

**(f) jnb: AUC $m_1$ (no removals)**




**(g) pta: AUC $m_1$ (no removals)**

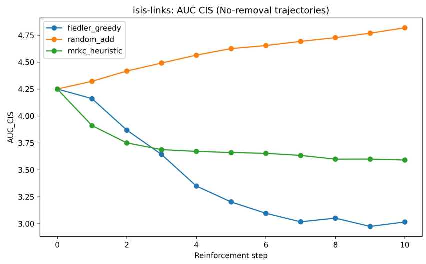



**(h) pzb: AUC $m_1$ (no removals)**

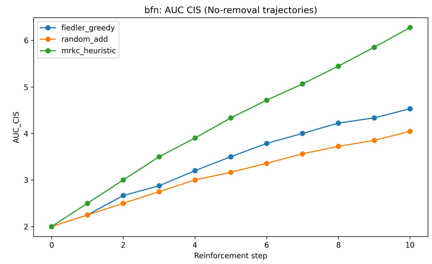



**(i) vdp: AUC $m_1$ (no removals)**

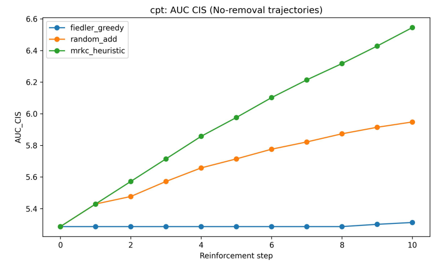
**Figure 13: No Removals - Cumulative Mean (AUC) of Multiplicity of One Eigenvalue Across All Graphs for All Reinforcement Strategies**



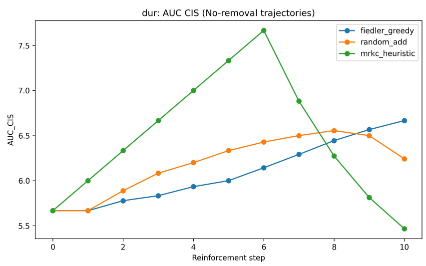

**(a) isis-links: AUC CIS (no removals)**



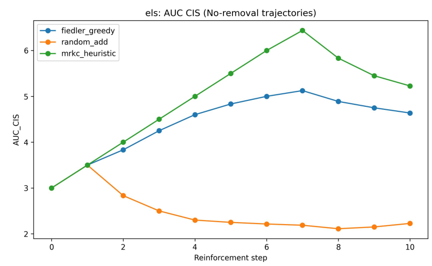

**(b) bfn: AUC CIS (no removals)**



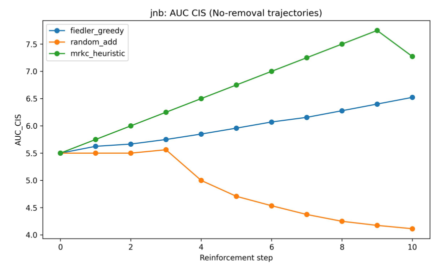

**(c) cpt: AUC CIS (no removals)**



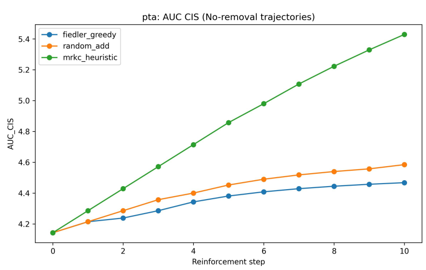

**(d) dur: AUC CIS (no removals)**



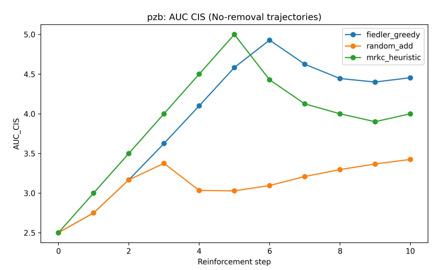

**(e) els: AUC CIS (no removals)**



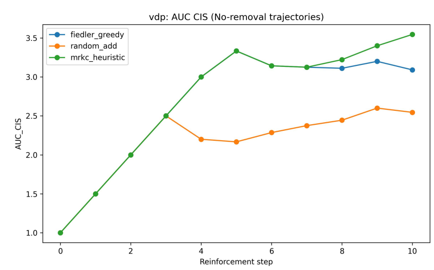

**(f) jnb: AUC CIS (no removals)**




**(g) pta: AUC CIS (no removals)**

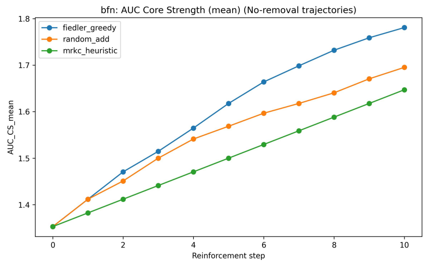



**(h) pzb: AUC CIS (no removals)**

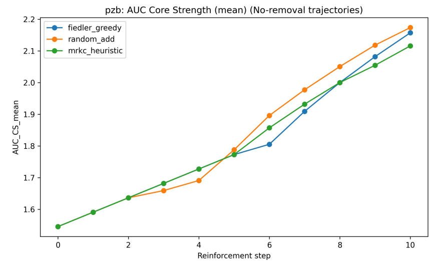



**(i) vdp: AUC CIS (no removals)**

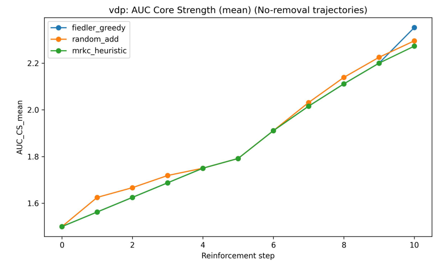
**Figure 14: No Removals - Cumulative Mean (AUC) of Core-Influence Strength Across All Graphs for All Reinforcement Strategies**



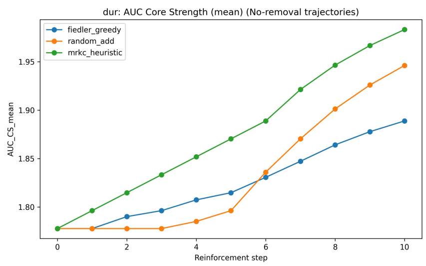

(a) Core Strength over Reinforcements for Bloemfontein



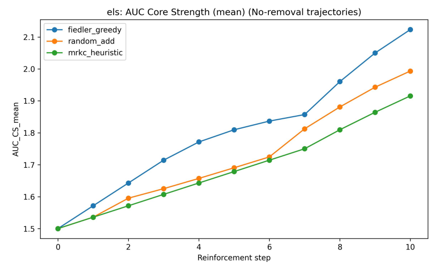

(b) Core Strength over Reinforcements for Bloemfontein



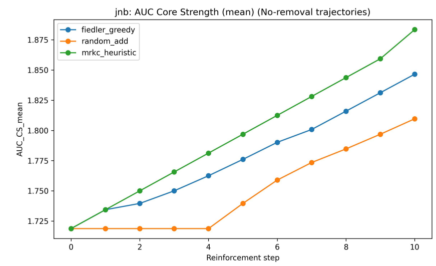

(c) Core Strength over Reinforcements for Bloemfontein



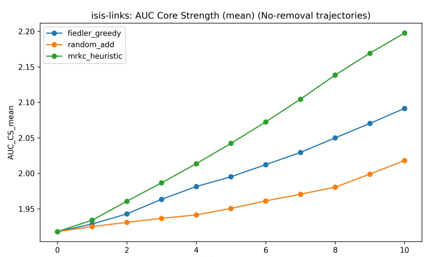

(d) Core Strength over Reinforcements for Bloemfontein



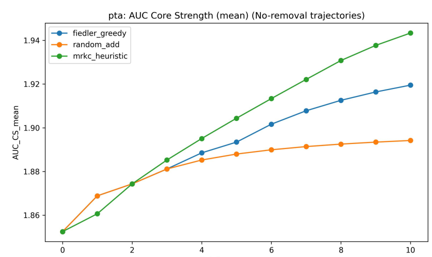

(e) Core Strength over Reinforcements for Bloemfontein



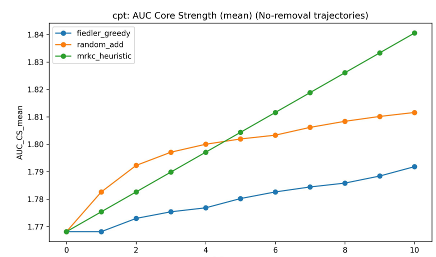

(f) Core Strength over Reinforcements for Bloemfontein




(g) Core Strength over Reinforcements for Bloemfontein




(h) Core Strength over Reinforcements for Bloemfontein

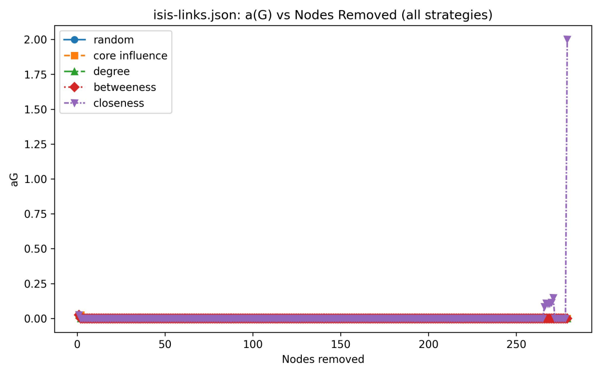


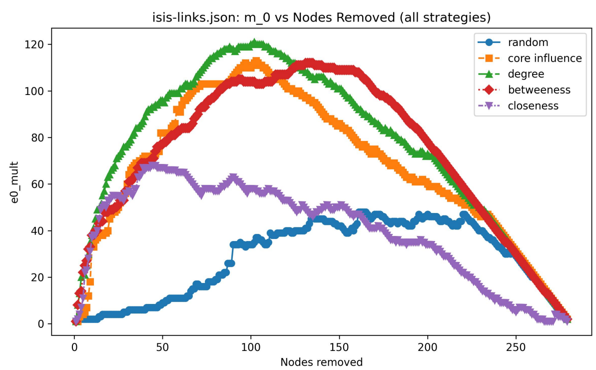

(i) Core Strength over Reinforcements for Bloemfontein

**Figure 15: No Removals - Cumulative Mean (AUC) of Core Strength Across All Graphs for All Reinforcement Strategies**

# B Appendix B: SANReN Removal Trajectories

Figures show metric trajectories (algebraic connectivity $a(G)$, and multiplicities $m_0$ and $m_1$) under node removals for the full SANReN network.



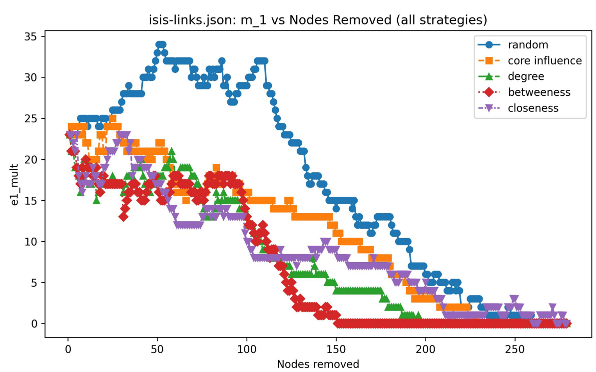




**Figure 16: Isis-links Removal Trajectories**

# C Appendix C: Multiplicity of the Zero Eigenvalue under Targeted Removals

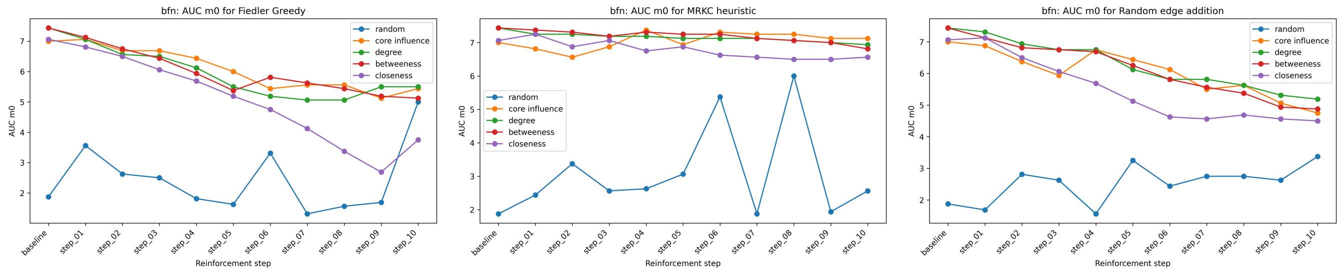



**Figure 17: Bloemfontein Multiplicity of Zero Eigenvalue Across Removals per Reinforcement Steps**

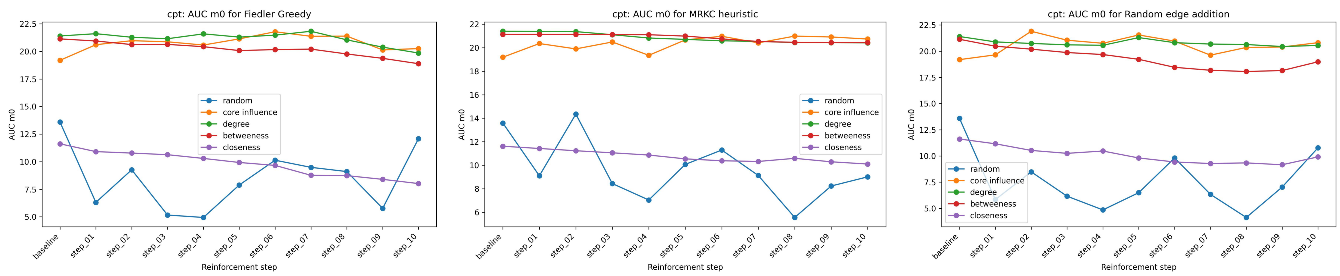



**Figure 18: Cape Town Multiplicity of Zero Eigenvalue Across Removals per Reinforcement Steps**

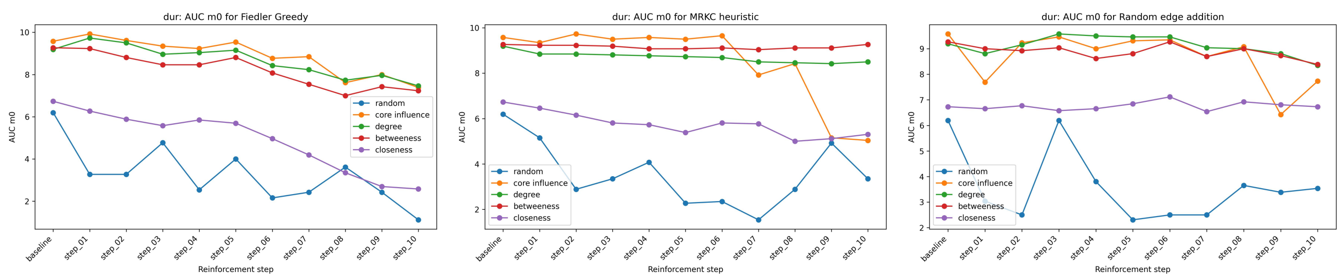



**Figure 19: Durban Multiplicity of Zero Eigenvalue Across Removals per Reinforcement Steps**

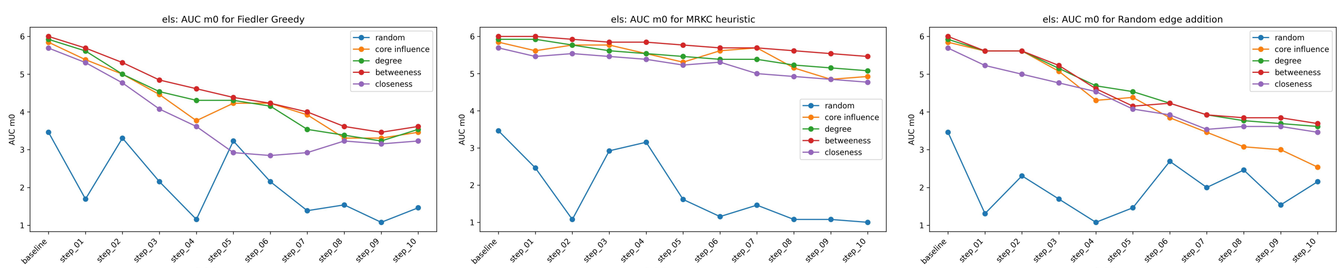



**Figure 20: East London Multiplicity of Zero Eigenvalue Across Removals per Reinforcement Steps**

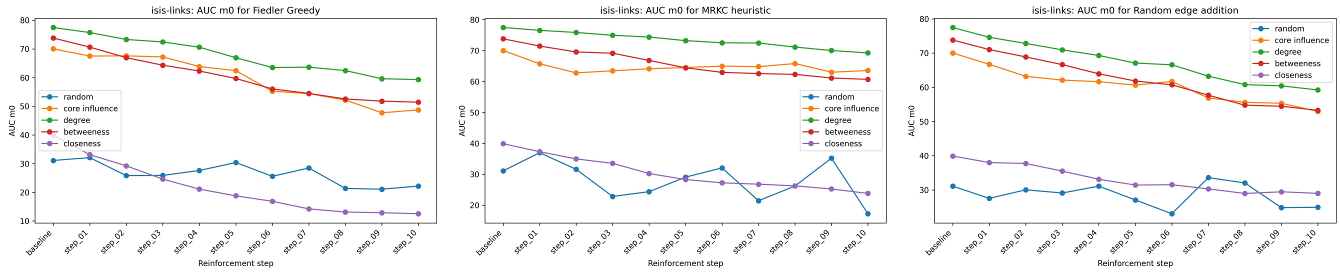



**Figure 21: Isis-links Multiplicity of Zero Eigenvalue Across Removals per Reinforcement Steps**

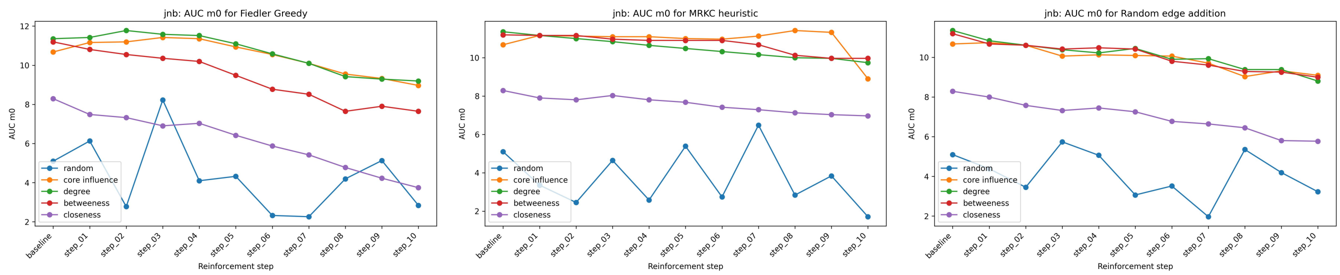



**Figure 22: Johannesburg Multiplicity of Zero Eigenvalue Across Removals per Reinforcement Steps**

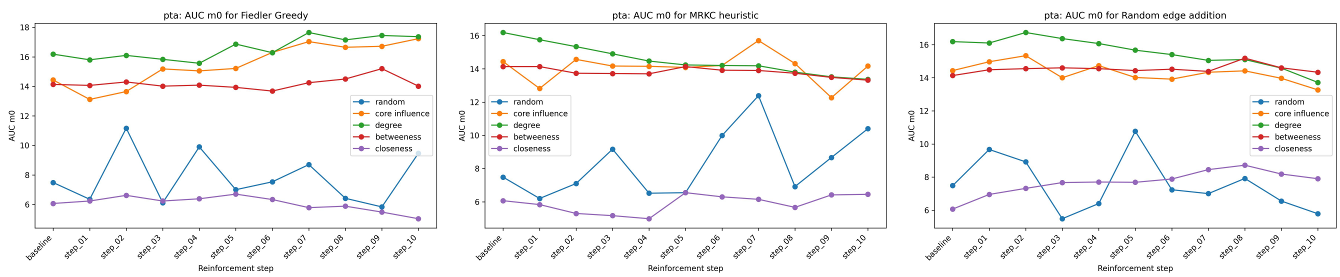



**Figure 23: Pretoria Multiplicity of Zero Eigenvalue Across Removals per Reinforcement Steps**

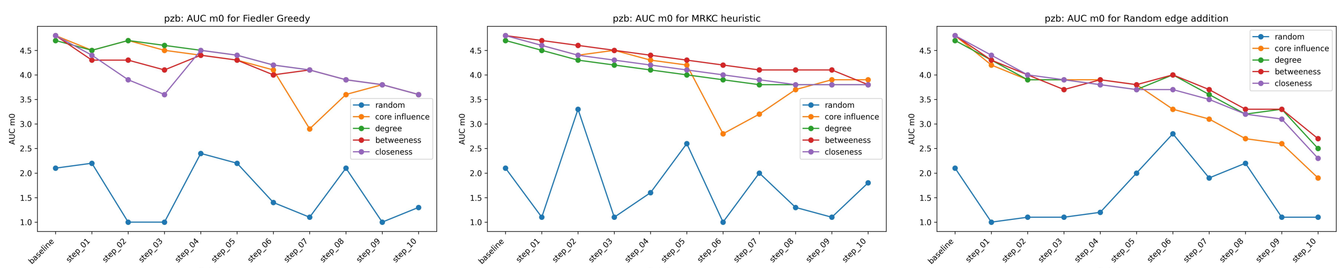



**Figure 24: Port Elizabeth Multiplicity of Zero Eigenvalue Across Removals per Reinforcement Steps**

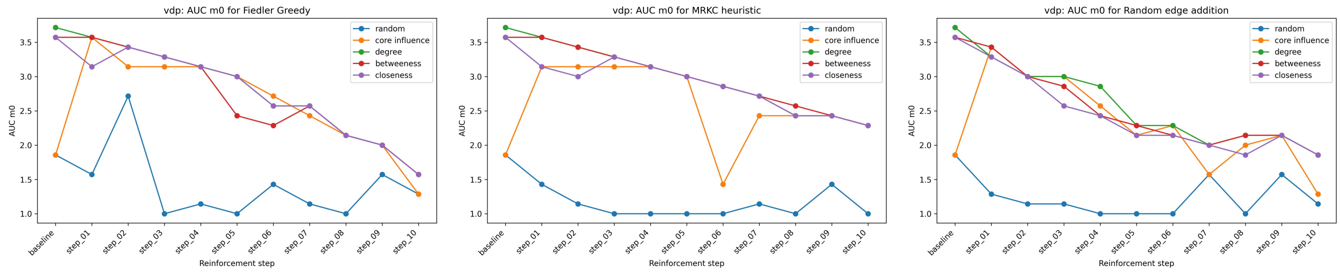



**Figure 25: Vanderbijlpark Multiplicity of Zero Eigenvalue Across Removals per Reinforcement Steps**