Notes of Optical design and instrumentation

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Preface

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Introduction to Geometrical Optics principles

Chapter 1

Applications

1.1 Thin prisms and dispersing prisms

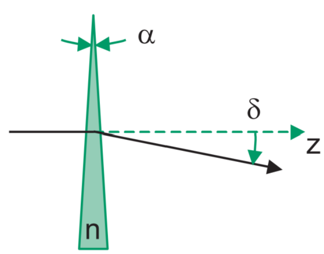
1.1.1 Thin prisms

Thin prisms introduce small angular beam deviations δ that is approximately independen of the incident angle:

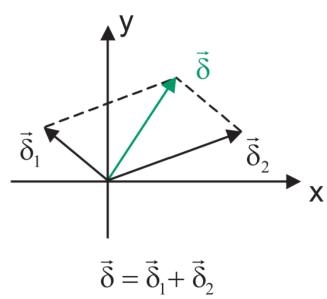
$$\delta \approx -(n-1)\alpha. \tag{1.1}$$

The deviation is measured in prism diopters. A prisms of 1 diopter deviates a beam by $1\ cm$ at $1\ m$. The beam deviation is parallel to a principal section of the prism and towards the thick end of the prism. De magnitude and direction of the deviation defines a vector perpendicular to the optical axis (xy plane). The net deviation vector for a series of thin prisms is then the vector sum of the copmonent vectors:

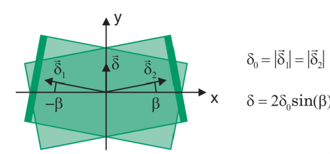
$$\boldsymbol{\delta} = \boldsymbol{\delta}_1 + \boldsymbol{\delta}_2.$$



(a) Thin prism



(b) Beam deviation



(c) Risley prism

Proof of the deviation

We tip the prism by θ so that the front face is perpendicular to the input ray (no refraction):

$$\delta = -\alpha, \delta' = \delta - \alpha \longrightarrow \begin{array}{l} n\theta = \theta' \\ -n\alpha = \delta - \alpha \\ \delta \approx -(n-1)\alpha \end{array}. \tag{1.2}$$

1.1.2 Risley prism

A **Risley prism** consists of a pair of identical, but opposing, thin prisms. The prisms are counter-rotated by $\pm\beta$ to obtain a variable net deviation in a fixed direction. The Risley prism allows the fine angular alignment for an optical system by adjusting the prism orientation β.

1.1.3 Thin prism dispersion

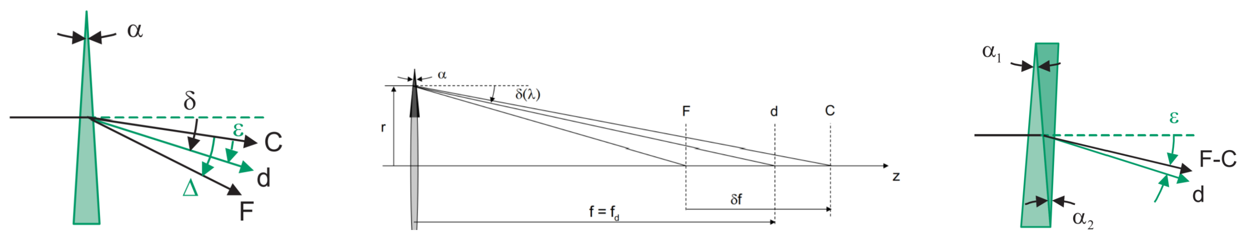
Thin prism

The **dispersion of a thin prism** Δ measures the total angular spread from C to F light, and the **secondary dispersion** ϵ gives the spread from the C to d wavelengths. The results depend on the index n_d, Abbe number ν and partial dispersion ratio P of the glass:

$$\text{Deviation} \quad \delta = -(n_d - 1)\alpha \tag{1.3}$$

$$\text{Dispersion} \quad \Delta = -(n_F - n_C)\alpha, \quad \Delta = \frac{\delta}{\nu} \tag{1.4}$$

$$\text{Secondary dispersion} \quad \varepsilon = -(n_d - n_C)\alpha, \quad \varepsilon = P\Delta = P\frac{\delta}{\nu}. \tag{1.5}$$



(a) Thin prism dispersion **(b)** Edge of a thin prism **(c)** Achromatic thin prism

How does the difference in focal length is related with the Abbe number?

$$\delta f = f_C - f_F = -\frac{r}{\delta_C} + \frac{r}{\delta_F} = -r\frac{\delta_F - \delta_C}{\delta_F \delta_C} \approx -r\frac{\delta_F - \delta_C}{\delta_d^2}.$$

$$\frac{\delta f}{f_d} = \frac{-r\frac{\delta_F - \delta_C}{\delta_d^2}}{-\frac{r}{\delta_d}} = \frac{\delta_F - \delta_C}{\delta_d} = \frac{-i(n_F - 1)\alpha + (n_C - 1)\alpha}{-(n_d - 1)\alpha} = \frac{n_F - n_C}{n_d - 1} = \frac{1}{\nu} \longrightarrow \frac{\delta f}{f_d} = \frac{1}{\nu}.$$

An inverted prism deviates a ray up and has a negative vertex angle α.

Deviations and dispersions adds.

$$\delta = \sum_i \delta_i, \quad \Delta = \sum_i \Delta_i, \quad \varepsilon = \sum_i \varepsilon_i. \tag{1.6}$$

Achromatic thin prism

An **achromatic thin prism** or **achromatic wedge** provides deviation without dispersion. Opposite prisms made from two different glasses (n_{d1}, ν_1, P_1) and (n_{d2}, ν_2, P_2) are combined to force the dispersion between the F and C wavelengths to be zero. A deviation of δ is maintained for d light:

$$\text{Achromatic relations} \quad \boxed{\frac{\alpha_1}{\delta} = \frac{1}{\nu_2 - \nu_1}\frac{\nu_1}{n_{d1} - 1}, \quad \frac{\alpha_2}{\delta} = -\frac{1}{\nu_2 - \nu_1}\frac{\nu_2}{n_{d2} - 1}}. \tag{1.7}$$

Proof of the above relations

We force the dispersion of F and C to be zero:

$$\Delta = \Delta_1 + \Delta_2 = \frac{\delta_1}{\nu_1} + \frac{\delta_2}{\nu_2} = 0 \longrightarrow \delta_2 = -\frac{\nu_2}{\nu_1}\delta_1.$$

A deviation δ for d light is maintained:

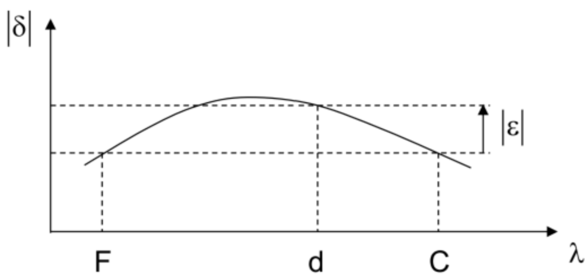
$$\delta = \delta_1 + \delta_2 = \delta_1 - \frac{\nu_2}{\nu_1}\delta_1 = (\nu_1 - \nu_2)\frac{\delta_1}{\nu_1} = -(\nu_1 - \nu_2)\frac{(n_{d1} - 1)\alpha_1}{\nu_1}.$$

Doing α_1/δ and α_2/δ yields the above results.

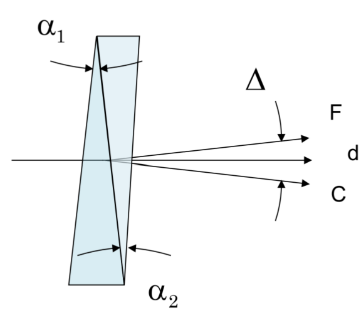
The high-dispersion prism is inverted to obtain an opposing deviation. While the F and C wavelegths are corrected, a residual secondary dispersion remains. For most glass pairs, d light will be bent more than the F and C wavelengths:

$$\text{Secondary dispersion (11)} \qquad \frac{\varepsilon}{\delta} = \frac{P_2 - P_1}{\nu_2 - \nu_1} = \frac{\Delta P}{\Delta \nu}. \tag{1.8}$$

For most glases, $\frac{\varepsilon}{\delta} > 0$. The shape of the curve of $|\delta| - \lambda$ is concave and the maxium dispersion does not occur at d ligth. The achromatic thin prism has about 40 less secondary dispersion compared to a simple thin prism.



(a) Secondary disperion curve



(b) Direct vision prism

Direct vision prism

A **direct vision prism** uses opposing prisms to provide dispersion without deviation of the d light. We first set the total dispersion to zero: $\delta = \delta_1 + \delta_2 = 0$. Then,

$$\Delta = \Delta_1 + \Delta_2 = \frac{\delta_1}{\nu_1} + \frac{\delta_2}{\nu_2} = -\frac{\nu_1 - \nu_2}{\nu_1\nu_2}\delta_1 \longrightarrow \frac{\alpha_1}{\Delta} = \left(\frac{\nu_1\nu_2}{\nu_1 - \nu_2}\right)\left(\frac{1}{n_{s1} - 1}\right) \wedge \frac{\alpha_2}{\Delta} = -\left(\frac{\nu_1\nu_2}{\nu_1 - \nu_2}\right)\left(\frac{1}{n_{d2} - 1}\right).$$

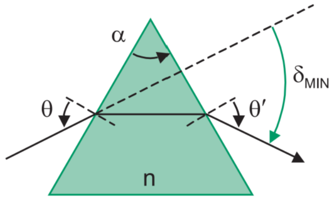
1.1.4 Dispersing prism

The total deviation δ in the **dispersion prism** is the sum of the deviations at the two surfaces:

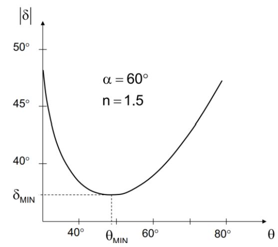
$$\text{Total deviation} \qquad \delta = \alpha - \sin^{-1}[\sqrt{n^2 - \sin^2\theta}\sin\alpha - \cos\alpha\sin\theta] - \theta.$$

There is a minimum deviation angle δ_{min}, at which the ray path through the prism is symetric $\theta' = -\theta$. The ray is bent an equal amount at each surface. The deviation is negative for the orientation of the prism in the figure. The **angle of minimum deviation** is

$$\delta_{\text{min}} = \alpha - 2\sin^{-1}[n\sin(\alpha/2)]. \tag{1.9}$$



(a) Dispersing prism

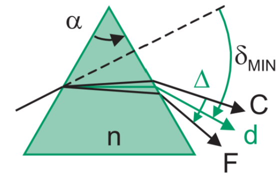


(b) Deviation in θ

For α = 60°

n	δ_{MIN}
1.3	–21.1°
1.4	–28.9°
1.5	–37.2°
1.6	–46.3°
1.7	–56.4°
1.8	–68.3°
2.0	–120°

(c) Table for δ_{min}



(d) Dispersion of the prism

The following table shows δ_{min} for several n.

The measurement of the index depends only on δ_{min} and the prism apex angle α:

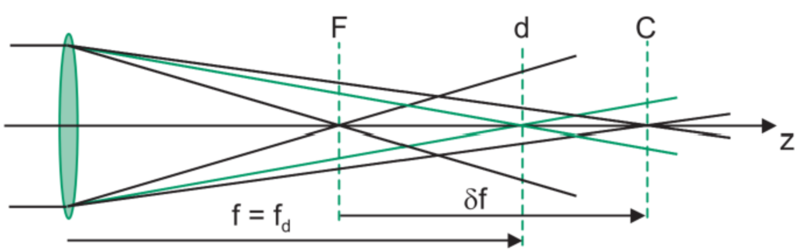
$$n = \frac{\sin \dfrac{\alpha - \delta_{\text{min}}}{2}}{\sin(\alpha/2)}. \tag{1.10}$$

Prism spectrometers can obtain accuracies of one part in 10^6.

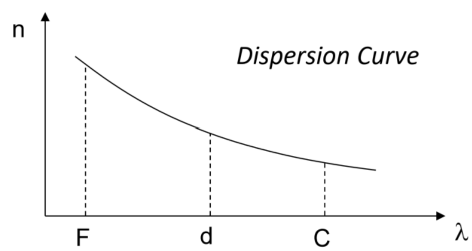
1.2 Chromatic effects

1.2.1 Chromatic aberration

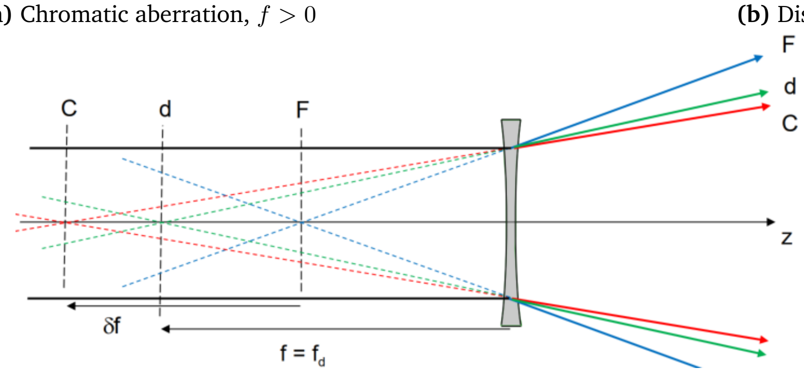
Axial chromatic aberration or **axial color** is a variation of the system focal lenfgth with wavelenegth. This aberration derives from the dispersion of the glass as the index changes with wavelength $n(\lambda)$.



(a) Chromatic aberration, $f > 0$



(b) Disperison curve



(c) Chromatic aberration, $f < 0$

Because of the higher index for F light, blue light is bent more and therefore the blue focus is closest to the lens.

How much does the focal length change for the F and C wavelengths?

We look at the difference in power between these two wavelengths:

$$\delta\phi = \phi_F - \phi_C = (n_F - 1)(C_1 - C_2) - (n_C - 1)(C_1 - C_2) = (n_F - n_C)(C_1 - C_2)$$
$$\delta\phi = \underbrace{\frac{n_F - n_C}{n_d - 1}}_{1/\nu} \underbrace{(n_d - 1)(C_1 - C_2)}_{\phi_d} = \frac{\phi_d}{\nu}.$$

Similarly, for the focal length:

$$\delta f = f_C - f_F = \frac{1}{\phi_C} - \frac{1}{\phi_F} = \frac{\phi_F - \phi_C}{\phi_C \phi_F} = \frac{\delta\phi}{\phi_C \phi_F} \approx \frac{\delta\phi}{\phi_d^2} = \frac{\phi_d}{\nu \phi_d^2} = \frac{f_d}{\nu}.$$

The foci of F, d and C are not evenly spaced due to the shape of the dispersion curve. The relative order of the foci is reversed for a negative lens.

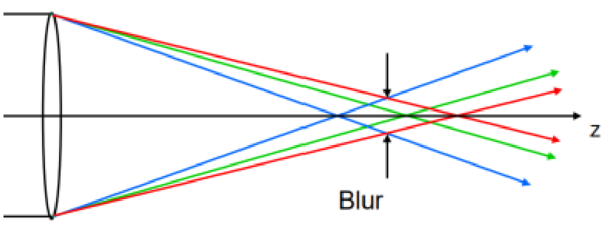
+ lens chromatic aberration $$\delta f_{CF} = f_C - f_F, \quad \delta\phi_{FC} = \phi_F - \phi_C, \quad \frac{\delta f_{CF}}{f_d} = \frac{\delta\phi_{FC}}{\phi_d} = \frac{1}{\nu} \tag{1.11}$$

- lens chromatic aberration $$\text{same, with } f_d < 0, \quad \phi_d < 0, \quad \delta f_{CF} < 0, \quad \delta\phi_{FC} < 0\,. \tag{1.12}$$

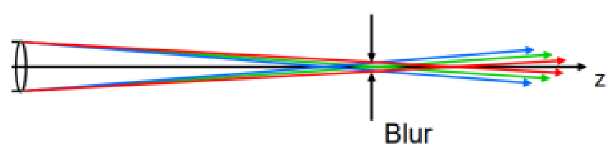
1.2.2 Type of chromatic aberrations

Longitudinal chromatic aberration

The blur associated wit hthe chromatic aberration of the objective lens limits the performance of an objective. To reduce the blut, a small diameter objective lens is required. The blur is then proportional to the lens diameter.



(a) Axial longitudinal chromatic aberration



(b) Small diameter to reduce aberration

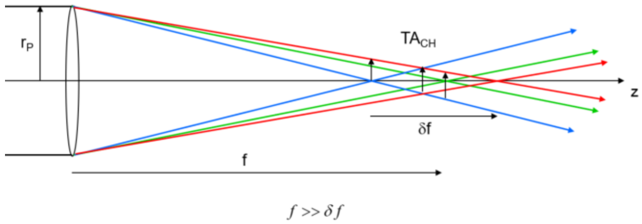
Transverse axial chromatic aberration

Transverse axial chromatic aberration measures the image blur size due to axial chromatic aberration. It depends only on the glass and the pupil radius r_P (stop at the lens):

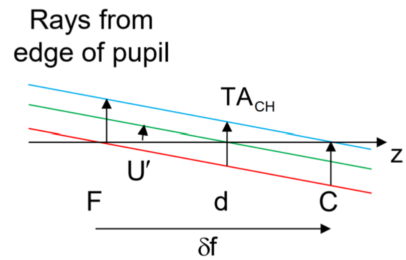
Transverse axial chromatic aberration (10) $$\mathrm{TA}_{CH} = \frac{r_P}{\nu}. \tag{1.13}$$

Lateral chromatic aberration

Lateral chromatic aberration or **lateral color** is caused by dispersion of the chief ray. The edge of the lens behaves like a thin prism. Off-axis image points will exhibit a radial color smear. The blur length



(a) Transverse axial chromatic aberration



(b) Color swaped, should be blue first

increases linearly with the image height. Each color has a different lateral magnification.

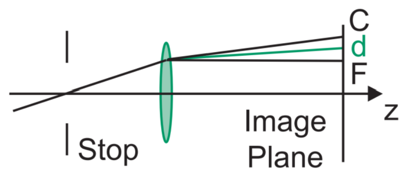
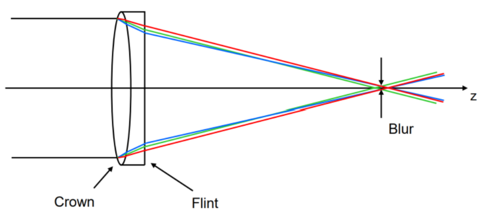


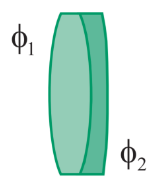
Figure 1.4 Lateral chromatic aberration

1.2.3 Achromatic doublet

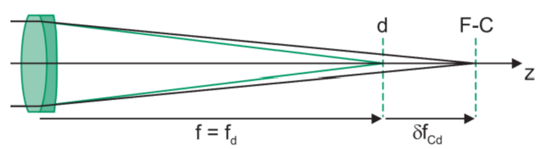
The thin lens **achromatic doublet** corrects longtiduinal chromatic aberration by combining a positive element with a negatvie one. Two different glasses (ν_1, P_1) and (ν_2, P_2) are used.



(a) Achromatic doublet



(b) Composition



(c) Secondary aberration

How do we design the individual powers of the achromatic doublet?

Red and blue light are made to focus at the same location $(\Delta f = 0)$:

$$\phi = \phi_1 + \phi_2 \Longrightarrow \delta\phi_{FC} = \delta\phi_{FC1} + \delta\phi_{FC2} = \frac{\phi_1}{\nu_1} + \frac{\phi_2}{\nu_2} = \phi_F - \phi_C = 0 \longrightarrow \frac{\phi_1}{\nu_1} = -\frac{\phi_2}{\nu_2}.$$

$$\phi = \phi_2 - \frac{\nu_1}{\nu_2}\phi_2 = \frac{\nu_2 - \nu_1}{\nu_2}\phi_2 \longrightarrow= \frac{\phi_2}{\phi} = -\frac{\nu_2}{\nu_1 - \nu_2} \wedge \frac{\phi_1}{\phi} = \frac{\nu_1}{\nu_1 - \nu_2}.$$

All we have done is to force the axial focus for F and C light. However, the d line can focus at a different location. This is known as **secondary chromatic aberration**.

1.2.4 Secondary chromatic aberration

We have ensured the F and C lines are at the same longitudinal distance, but the d line havea different focal position.

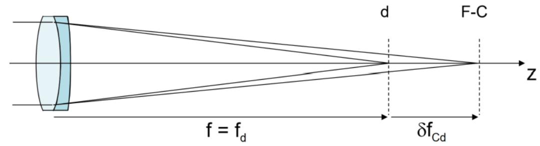


Figure 1.6 Secondary chromatic aberration.

First, we have:

$$\delta\phi_{dC} = \delta\phi_{dC1} + \delta\phi_{dC2}$$

We derive each term:

$$\delta\phi_{dC1} = (n_{d1} - 1)(C_1 - C_2) - (n_{c1} - 1)(C_1 - C_2) = (n_{d1} - n_{C1})(C_1 - C_2)$$
$$\delta\phi_{dC1} = \frac{(n_{d1} - n_{C1})}{(n_{F1} - n_{C1})}(n_{F1} - n_{C1})(C_1 - C_2) = P_1\delta\phi_{FC1}.$$

Similarly,

$$\delta\phi_{dC1} = (n_{d1} - n_{C1})(C_1 - C_2), \quad \delta\phi_{FC1} = (n_{F1} - n_{C1})(C_1 - C_2), \quad P_1 = \frac{n_{d1} - n_{C1}}{n_{F1} - n_{C1}}, \quad P_2 = \frac{n_{d2} - n_{C2}}{n_{F2} - n_{C2}}.$$

We go back to the initial equation:

$$\begin{aligned}\delta\phi_{dC} &= \delta\phi_{dC1} + \delta\phi_{dC2}\\ &= P_1\delta\phi_{FC1} + P_2\delta\phi_{FC2}\\ &= P_1\frac{\phi_1}{\nu_1} + P_2\frac{\phi_2}{\nu_2}\\ &= \frac{P_1 - P_2}{\nu_1 - \nu_2}\phi \quad \text{Achromat eq: } \frac{\phi_1}{\nu_1} = -\frac{\phi_2}{\nu_2}\\ \delta\phi_{dC} &= \frac{\Delta P}{\Delta\nu}\phi\end{aligned}$$

In order to obtain zero secondary achromatic aberration with a doublet, the partial dispersion ratio P of the two glasses must be zero:

$$\Delta P = 0 \Longrightarrow \delta f_{Cd} = \frac{\Delta P}{\Delta\nu} f = 0. \tag{1.14}$$

To correct chromatic aberration at additional wavelength, more than two glasses are used.

For a singlet and doublet, we have:

$$\delta f_{Cf} \approx \frac{f}{50} \quad \text{vs} \quad \delta f_{Cd} \approx \frac{f}{2200}. \tag{1.15}$$

Ejemplo 1.1 **Design of an achromatic doublet**

Design a $160\ mm$ focal length thin-lens ahcromatic doublet using the following glases. Provide the focal lengths and indices of refraction of the two thin lenses.

Glass 1: Fused Silica, 458678, Glass 2: SF6, 805254.

Solution

$$\text{Glass 1: } n_1 = 1458,\ \nu_1 = 67.8, \quad \text{Glass 2: } n_2 = 1.805,\ \nu_2 = 25.4.$$

$$\frac{1}{f_2} = -\frac{\nu_2}{\nu_1 - \nu_2}\frac{1}{f} = -\frac{25.4}{67.8 - 25.4}\frac{1}{160} = -0.00374\ mm^{-1} \longrightarrow f_2 = -267.380\ mm$$
$$\frac{1}{f_1} = \frac{\nu_1}{\nu_1 - \nu_2}\frac{1}{f} = \frac{67.8}{67.8 - 25.4}\frac{1}{160} = 0.0010\ mm^{-1} \longrightarrow f_1 = 100\ mm.$$

1.3 Illumination systems

1.3.1 Illumination systems and types

The illumination system provides light for the optical system. Important considrations are the amount of light, the uniformity, and the angular spread of the light as seen by the object.

A **projector** is the general term for an imaging system that also provudes the illumination for the object.

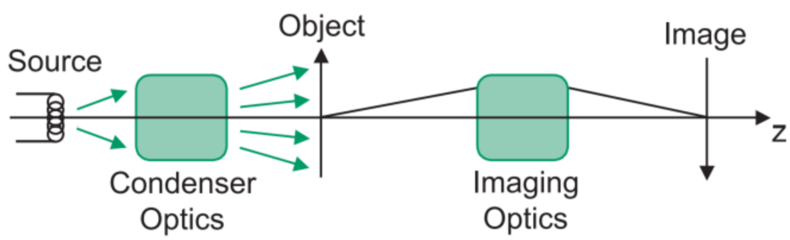


Figure 1.1 Projector as the illumination system.

There are three basic classifications of illumination systems:

- **Diffuse illumination** Light with a large angular spread is incident on the object. There is no attempt to image the source into the imaging system. It provides uniform illumination but is light inefficient.

  No source coupling

- **Specular illumination** The light source is imagesd by the condenser optics into the EP of the imaging optics. As it is good light eficient, is used for most optical systems with an integral light source.

  Source to pupil coupling

- **Critical illumination** The light source is imaged directly onto the object.

  Source to object coupling

1.3.2 Specular illumination

Source is coupled into the EP of the imaging system.

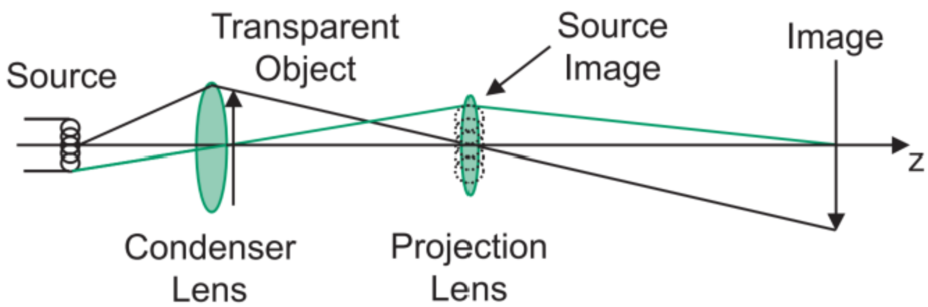


Figure 1.2 Projection condenser system for specular illumination.

Projection condenser system

The most common example is the **projection condenser system**.

In the projection condenser system, the source is optical conjugated (imaged) at the EP of the projection lens (imaging system).

- The condenser lens serves as a field lens, bending the chief ray of the iamging system back into the projection lens.
- The condenser lens should be as fast as possible ($f/\#_W$ faster than $f/1$ on the source side).
- The projection lens diameter must be larger than the size of the source image.
- The marginal ray of the condenser system becomes the chief ray of the imaging system and vicev-ersa.

Without a condenser lens, the light colletion angle α is limited by the projection lens. The amount of source energy that is collected and used is defined by the solid angle given by the angular size of the projection lens.

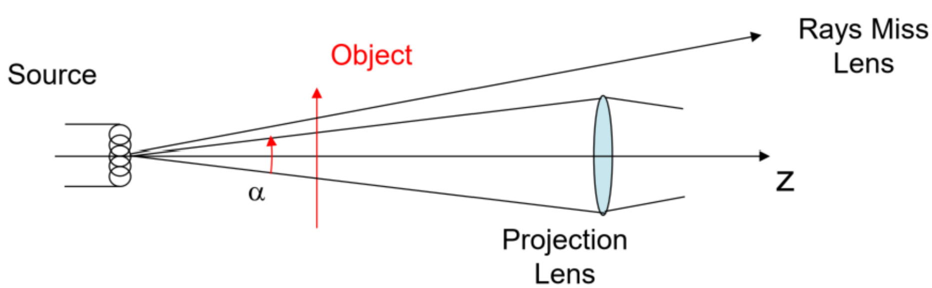
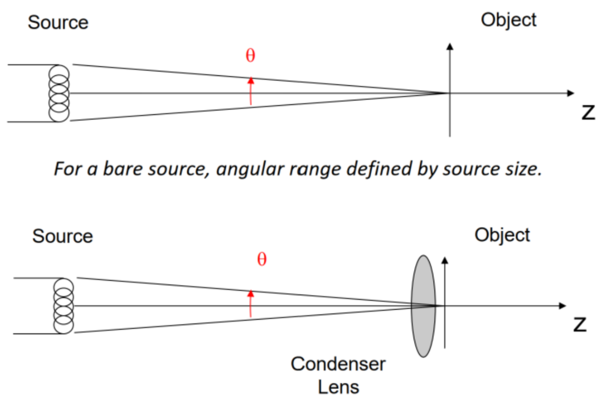


Figure 1.3 Without the condenser lens, the collection angle is limited by the projection lens.

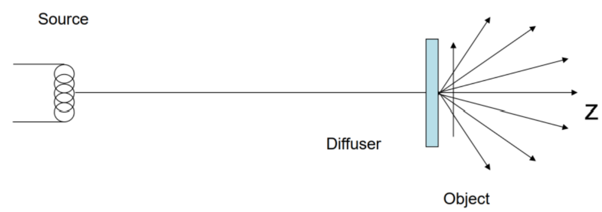
The inclusion of the condenser makes that each point on the source illuminates all points on the object, resulting in a uniform illumination. The collection angle α is limited by the condenser lens size.

Apparent source size

With a bare source, the apparent source size (how large appears to be at the observation point) is limited by the angular size of the source, regardless if there is a condenser lens. However, the inclusion of a **diffuser** makes that all angles are presents due to scattering; the source no appears to be large.



(a) Bare source



(b) Diffuser source

Projection condenser design

- **What drives the choise in projection lens?**
  We image the transparent object. How big is the screen? Where is it? What kind of magnification do we need?
- **What drives the physical size of the condenser lens?**
  The object size and any separation between the condenser lens and object. Pick the fastest condenser lens you can.
- **What drives the position of the source?**
  The image of the source must be at the projection lens.
- **Must be the projection lens diameter be larger than the source image size?**
  Yes, thisdetermines the $f/\#$ of the projection lens.

Kohler illumination

The **Kohler illumination** is an example of specular illumination, often used in microscopes.

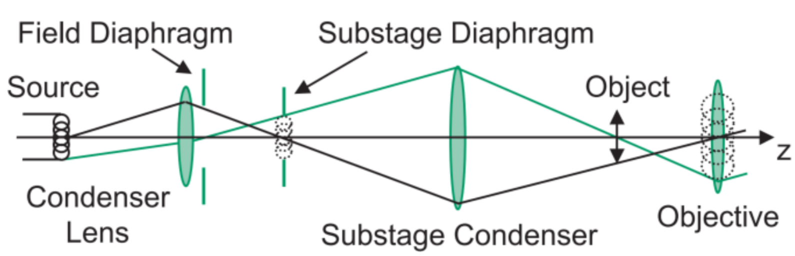


Figure 1.5 Kohler illumination system.

The source produces an intermediate image at the substage diaphragm, and an image at the objective. Similarly, the object is at the field diaphargm and generates an intermediate image after the substage condenser. The **subtage diaphragm** (at source image) allows the overall light level to be varied, and the **field diaphragm** (at object image) changes the amount of the object that is illuminated.

1.3.3 Critical illumination

Critical illumination images the light directly onto the object. While it is light efficient, is not very used as the image is modulated by the source structure. This requires to use a very uniform source. The FOV of this system is typically small

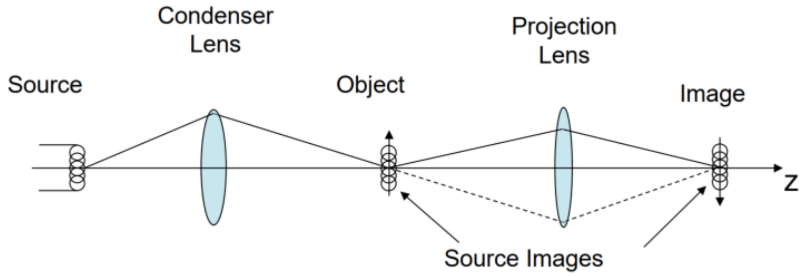


Figure 1.6 Critical illumination system.

1.3.4 Diffuse illumination

Diffuse illumination provides light with a large angular spread onto the object. There is no attempt to image the source into the imaging system. It is usually achieved by the insertion of a diffuser into the system. This makes a very uniform illumination but with a light inefficiency.

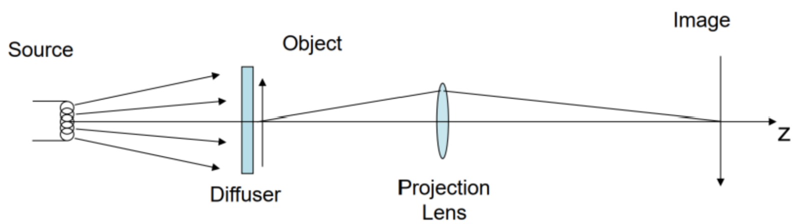


Figure 1.7 Diffuse illumination system.

The diffuser increases the apparent source size, resulting in greater uniformity of illumination. This greater range of illumination angles also provides **scratch suppresion** that wll hide phase errors on the object, such as scratch or defect in the usbstrate of the object transparency. Diffusers are commonly made of ground glass or opal glass.

1.3.5 Integrating sphere

Works through the ideas of diffure illumination. Inside the hollow sphere is coated with a highly-reflective diffuse white coating. Light directed into the entry port undergoes many random refletions before escaping through the exit port. Output light is extremely uniform with a brightness that is independent of viewgin angle, and a very good approximation to a Lambertian source.

The system is in general light inefficient, and the two ports are usually at 90° to prevent the direct viewing of the source and the first source reflection.

1.3.6 Source mirrors

Concave source

Placing a **concave mirror** behind the source can increase the light level in the projection system. The source is placed at the center of curvature of the mirror. This creates an intermediate image at the source, which becomes the object for the condenser lens. An improvement of less than a factor of two is obtained.

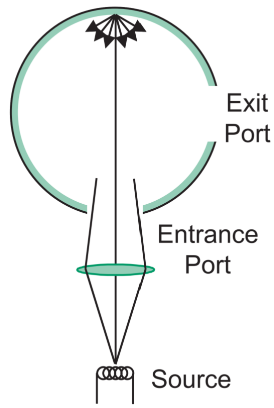
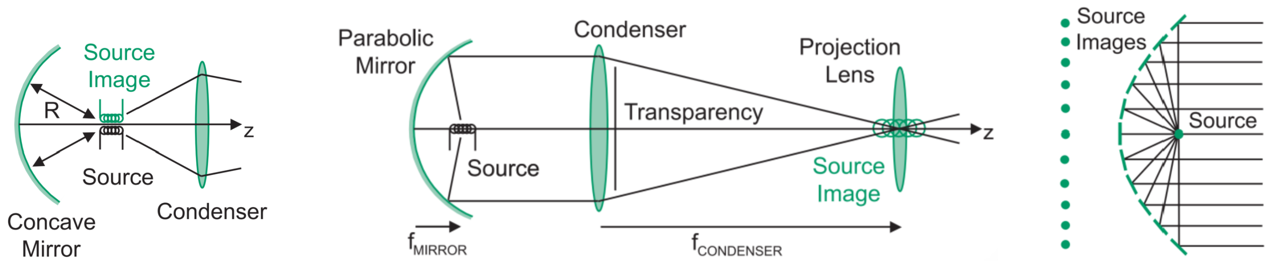


Figure 1.8 Integrating sphere.



(a) Concave mirror **(b)** Parabolic reflector **(c)** Faceted reflector

Parabolic reflector

Dramatic increases in illumination level occur by placing the source at the focus of the concave mirror. The source image occurs at infinity. The solid angle of the mirror can be more than $2\pi sr$, and the amount of light intercepted and reflected by the mirror can exceed the light directly collected by the condenser by a factor of 10 or more. $f/\#$ of the condenser lens does not influence the light collection efficiency.

The design of these systems typically ignore the forward light through the condenser. The mirror shape is usually parabolic.

Faceted reflectors

To provide a greated level of diffuseness, the surface of the parabola can be segmented into small flat mirrors. A virtual source is formed behind each facet. The details of the **faceted prabolic reflector** are complicated, but for design purposes it can be modeled as an extended source located at or near the concave mirror. The mirror aperture defines the extent of the extended source.

The number, size and tilt of the facets are designed so that uniform illumination is achieved at the object plane. The total view is limited bt the overalll aperture of the reflector. This overall aperture is imaged into the projection lens as well as an image of the source, but this one contians much less light.

Elliptical reflector

An elliptical reflector can be used to focus the source into a small aperture. The source is placed at one focus of the ellipse, and a real image is formed at the other focus.

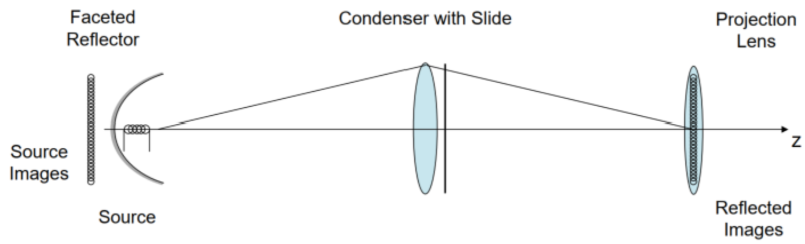


Figure 1.10 System with the facet reflector.

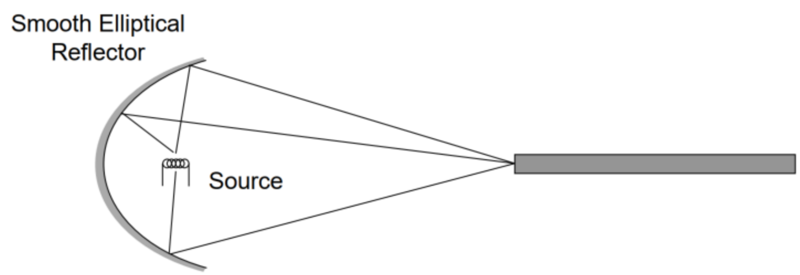
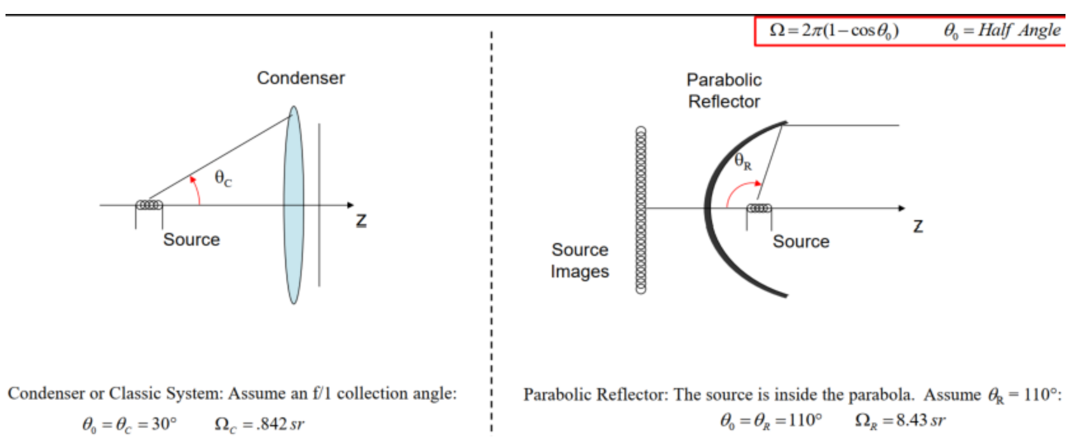


Figure 1.11 Elliptical reflector.

Light collection efficiency

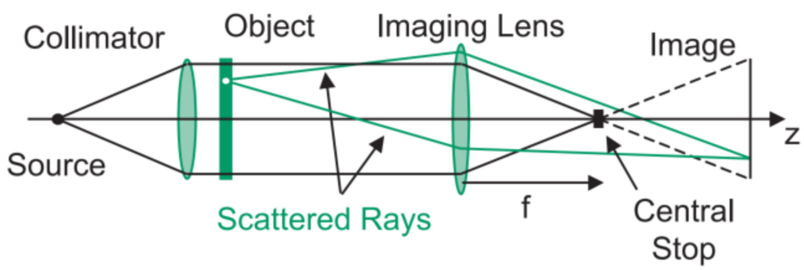


1.3.7 Schlieren and dark field systems

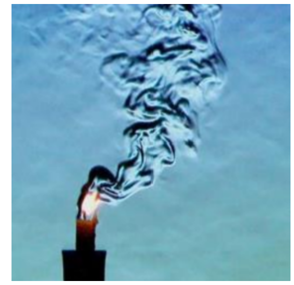
Specular or narrow angle illumination can be used to identify features on an object.

Schlieren system

In a **schlieren system**, light from a small source is collimated before passing through the object plane. The image of the source is blocked by an opaque disk or knife edge. With no object present, the image appears black. When the object is inserted, any feature will scatter some light past the obscuration. These localized areas n the object will appear bright in the image.



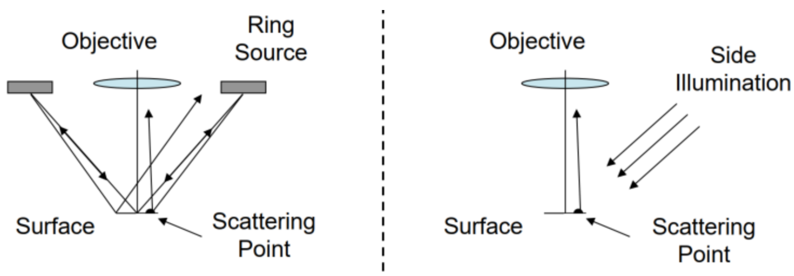
(a) Schlieren system



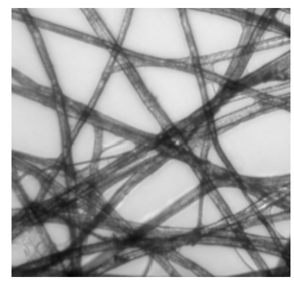
(b) Schlieren example

Dark field system

Dark field illumination is a variation of this technique using directional lightning. The light source is placed to the side of the objective lens, or in a ring around the lens. If the object is perfectly smooth (a mirror), a specular reflection within the FOV misses the objective, and the image is dark. Features on the surface will scatter light into the objective and appear bright in the image.



(a) Dark field system



(b) Dark field example

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