Problem Set 4

Problem I

a $|\psi(t)\rangle, |\phi(t)\rangle \in \mathcal{E}$ show $\frac{d}{dt}\langle\psi(t)|\varphi(t)\rangle = 0$

Postulate 6: $i\hbar \frac{d}{dt}|\varphi(t)\rangle = H(t)|\varphi(t)\rangle$

Hermitian conjugate: $-i\hbar\frac{d}{dt}\langle\psi(t)| = \langle\psi(t)|\hat{H}(t)$

since $\hat{H}^\dagger = \hat{H}$ Hermitian

Then: $\frac{d}{dt}\langle\psi(t)|\varphi(t)\rangle = \left(\frac{d}{dt}\langle\psi(t)|\right)|\varphi(t)\rangle + \langle\psi(t)|\left(\frac{d}{dt}|\varphi(t)\rangle\right) =$

$= \frac{i}{\hbar}\frac{d}{dt}\langle\psi(t)|\hat{H}(t)|\varphi(t)\rangle - \frac{i}{\hbar}\langle\psi(t)|\hat{H}(t)|\varphi(t)\rangle =$

$= 0$ $\Rightarrow$ inner product is constant in time

Lets say u is the time evolution operator. Then:

$$|\psi(t)\rangle = \hat{u}(t,t_0)|\psi(t_0)\rangle$$
$$|\varphi(t)\rangle = \hat{u}(t,t_0)|\varphi(t_0)\rangle$$

From above, we know $\langle\psi(t_0)|\varphi(t_0)\rangle = \langle\psi(t)|\varphi(t)\rangle$

$\Rightarrow \langle\psi(t)|\varphi(t)\rangle = \langle\psi(t_0)|\hat{u}^\dagger(t,t_0)\,\hat{u}(t,t_0)|\varphi(t)\rangle =$

$= \langle\psi(t_0)|\varphi(t_0)\rangle$

$\Rightarrow \hat{u}^\dagger\hat{u} = \mathbb{1}$ $\Rightarrow$ $\hat{u}$ is unitary

$\boxed{b}$ $|\psi(t_0)\rangle = |\varphi_n^i\rangle$ $\qquad \hat{H}|\varphi_n^i\rangle = E_n|\varphi_n^i\rangle$, $\hat{H}$ - time independ.

<u>Does</u> $|\psi(t)\rangle = e^{-i(E_n/\hbar)(t-t_0)}|\psi(t_0)\rangle$ solve SE?

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = i\hbar\frac{d}{dt}\left[e^{-i(E_n/\hbar)(t-t_0)}|\psi(t_0)\rangle\right] =$$

$$= i\cancel{\hbar}\cdot\left(-i\frac{E_n}{\cancel{\hbar}}\right)\cdot e^{-i(E_n/\hbar)(t-t_0)}\cdot|\varphi_n^i\rangle =$$

$$= e^{-i(E_n/\hbar)(t-t_0)}\cdot\left(E_n|\varphi_n^i\rangle\right) =$$

$$= e^{-i(E_n/\hbar)(t-t_0)}\cdot\hat{H}|\varphi_n^i\rangle =$$

$$= \hat{H}\left[e^{-i(E_n/\hbar)(t-t_0)}|\psi(t_0)\rangle\right] =$$

$$\boxed{i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle} \quad \checkmark$$

$\boxed{c}$ $|\psi(t_0)\rangle = \sum_{n,i} c_n^i|\varphi_n^i\rangle$

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = i\cancel{\hbar}\sum_{n,i} c_n^i\cdot\left[-i\left(E_n/\cancel{\hbar}\right)\right]e^{-i(E_n/\hbar)(t-t_0)}|\varphi_n^i\rangle =$$

$$= \sum_{n,i} c_n^i E_n e^{-i(E_n/\hbar)(t-t_0)}|\varphi_n^i\rangle =$$

$$= \hat{H}\sum_{n,i} c_n^i e^{-i(E_n/\hbar)(t-t_0)}|\varphi_n^i\rangle =$$

$$\boxed{i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle} \quad \checkmark$$

For a stationary state, the expectation value is constant in time. Let's calculate exp. value for an observable $\hat{A}$

$$\langle \hat{A} \rangle = \langle \psi(t) | \hat{A} | \psi(t) \rangle =$$

$$= \sum_{j,k,m,n} c_m^{j*} e^{i E_m (t-t_0)/\hbar} \langle \varphi_m^j | \hat{A} | \varphi_n^k \rangle c_n^k e^{-i E_n (t-t_0)/\hbar}$$

$$= \sum_{j,k,m,n} c_m^{j*} c_n^k e^{-i (E_n - E_m)(t-t_0)/\hbar} \langle \varphi_m^j | \hat{A} | \varphi_n^k \rangle$$

In general, $e^{-i(E_n - E_m)(t-t_0)/\hbar}$

is an oscillatory term that has sinusoidal variation over time, so these states are not stationary.

Problem II CT III 1

$$\psi(x) = N \frac{e^{ip_0x/\hbar}}{\sqrt{x^2+a^2}}$$

[a] $$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1$$

$$\Rightarrow N^2 \cdot \int_{-\infty}^{+\infty} \frac{e^{-ip_0x/\hbar}}{\sqrt{x^2+a^2}} \cdot \frac{e^{ip_0x/\hbar}}{\sqrt{x^2+a^2}} dx = 1$$

$$N^2 \cdot \int_{-\infty}^{+\infty} \frac{1}{x^2+a^2} dx = 1$$

$$\left(\frac{N}{a}\right)^2 \int_{-\infty}^{+\infty} \frac{1}{(x/a)^2+1} dx = 1 \qquad u = x/a \quad dx = a\,du$$

$$\frac{N^2}{\cancel{a^2}} \cdot \cancel{a} \cdot \int_{-\infty}^{+\infty} \frac{1}{u^2+1} du = 1$$

$$\frac{N^2}{a} \Big[\arctan(u)\Big]_{-\infty}^{\infty} = \frac{N^2}{a} \cdot \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi N^2}{a}$$

$$\Rightarrow \boxed{N = \sqrt{\frac{a}{\pi}}}$$ for N real and positive

[b] $$P\left(x \in \left[-\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right]\right) = \int_{-a/\sqrt{3}}^{a/\sqrt{3}} \psi^*(x)\,\psi(x)\,dx =$$

$$u = x/a$$

$$= \frac{a}{\pi} \cdot \int_{-a/\sqrt{3}}^{a/\sqrt{3}} \frac{1}{x^2+a^2} dx =$$

$$= \frac{a}{\pi} \cdot \frac{1}{a} \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{1}{u^2+1} du =$$

$$= \frac{1}{\pi} \cdot \Big[\arctan u\Big]_{-1/\sqrt{3}}^{1/\sqrt{3}} = \frac{1}{\pi}\left[\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right] = \boxed{\frac{1}{3}}$$

$\boxed{c}$ $\bar{p} = \langle \psi | \hat{p} | \psi \rangle = \int_{-\infty}^{+\infty} \psi^*(x) \left(-i\hbar \frac{d}{dx}\right) \psi(x)\, dx =$

$$\frac{d}{dx}\psi(x) = \sqrt{\frac{a}{\pi}}\, \frac{d}{dx}\left[\frac{e^{ip_0 x/\hbar}}{\sqrt{x^2+a^2}}\right] =$$

$$\frac{d}{dx}\left(x^2+a^2\right)^{-\frac{1}{2}} = \left(x^2+a^2\right)^{-3/2} \cdot \left(-\frac{1}{2}\right) \cdot 2x$$

$$= \sqrt{\frac{a}{\pi}} \cdot \left[i p_0/\hbar \cdot \frac{e^{ip_0 x/\hbar}}{\sqrt{x^2+a^2}} + e^{ip_0 x/\hbar} \frac{1}{\sqrt{x^2+a^2}\,(x^2+a^2)}\left(-\frac{1}{\cancel{2}}\right) \cdot \cancel{2}x \right] =$$

$$= i p_0/\hbar \cdot \psi(x) - \frac{x}{x^2+a^2}\psi(x)$$

$$\Rightarrow \bar{p} = -i\hbar \left[\int_{-\infty}^{+\infty} \psi^*(x)\, i p_0/\hbar\, \psi(x)\, dx - \int_{-\infty}^{+\infty} \psi^*(x) \frac{x}{x^2+a^2}\psi(x)\, dx \right] =$$

$$= p_0 \cdot \underbrace{\int_{-\infty}^{+\infty} |\psi(x)|^2 dx}_{1} + \underbrace{i\hbar \int_{-\infty}^{+\infty} \underbrace{|\psi(x)|^2}_{\text{even}} \underbrace{x}_{\text{odd}} \cdot \underbrace{\frac{1}{x^2+a^2}}_{\text{even}} dx}_{=0}$$

$$\boxed{\bar{p} = p_0}$$

Problem III CT III 12

$$|\psi(0)\rangle = a_1|\varphi_1\rangle + a_2|\varphi_2\rangle + a_3|\varphi_3\rangle + a_4|\varphi_4\rangle$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

a) $E_1 = \dfrac{\pi^2\hbar^2}{2ma^2}$ $E_2 = 2\dfrac{\pi^2\hbar^2}{ma^2}$ $E_3 = \dfrac{9}{2}\dfrac{\pi^2\hbar^2}{ma^2}$

$\uparrow$ $\dfrac{3\pi^2\hbar^2}{ma^2}$

$$\Rightarrow \boxed{P\left(E < \frac{3\pi\hbar^2}{ma^2}\right) = |a_1|^2 + |a_2|^2}$$

d) If $E = \dfrac{8\pi^2\hbar^2}{ma^2} = \dfrac{16\pi^2\hbar^2}{2ma^2} = E_4$

Then $|\psi(t > t_{meas})\rangle = |\varphi_4\rangle$ after the measurement.

Energy measurements after that will return E_4.

Problem IV CT III 14 $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$

$$H = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} ; \quad A = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} ; \quad B = b \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$$

a) Energies are eigenvalues of the Hamiltonian so either $\hbar\omega_0$ or $2\hbar\omega_0$

$\hbar\omega_0$ w/ $P(\hbar\omega_0) = \langle\psi(0)|u_1\rangle\langle u_1|\psi(0)\rangle = \frac{1}{2}$

$2\hbar\omega_0$ w/ $P(2\hbar\omega_0) = \langle\psi(0)|\left(|u_2\rangle\langle u_2| + |u_3\rangle\langle u_3|\right)|\psi(0)\rangle =$
$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$$\overline{H} = \langle\psi(0)|\hat{H}|\psi(0)\rangle = \hbar\omega_0 \cdot P(\hbar\omega_0) + 2\hbar\omega_0 \cdot P(2\hbar\omega_0) =$$
$$= \frac{1}{2}\hbar\omega_0 + 2\hbar\omega_0 \cdot \frac{1}{2} =$$
$$\boxed{\overline{H} = \frac{3}{2}\hbar\omega_0}$$

$$\Delta H = \sqrt{\langle H^2\rangle - \langle H\rangle^2}$$

$$\hat{H}^2 = (\hbar\omega_0)^2 \cdot \left(|u_1\rangle\langle u_1| + 2|u_2\rangle\langle u_2| + 2|u_3\rangle\langle u_3|\right)^2 =$$
$$= (\hbar\omega_0)^2 \cdot \left(|u_1\rangle\langle u_1| + 4|u_2\rangle\langle u_2| + 4|u_3\rangle\langle u_3|\right)$$

$$\langle H^2\rangle = \langle\psi(0)|\hat{H}^2|\psi(0)\rangle$$
$$= (\hbar\omega_0)^2 \cdot \left(\frac{1}{2} + 4\cdot\frac{1}{2}\right) =$$
$$= \frac{5}{2}(\hbar\omega_0)^2$$

$$\Rightarrow \Delta H = \sqrt{\frac{5}{2}(\hbar\omega_0)^2 - \left(\frac{3}{2}\right)^2(\hbar\omega_0)^2} = \sqrt{\frac{1}{4}(\hbar\omega_0)^2} =$$

$$\boxed{\Delta H = \frac{1}{2}\hbar\omega_0}$$

$\boxed{b}$ Measure A at $t=0$, possible results eigenvalues of A

$$A - \lambda \mathbb{I} = 0$$

$$\begin{vmatrix} a-\lambda & 0 & 0 \\ 0 & -\lambda & a \\ 0 & a & -\lambda \end{vmatrix} = 0 = (a-\lambda)\lambda^2 - (a-\lambda)a^2 =$$

$$= \underbrace{(a-\lambda)}_{\lambda = a} \underbrace{(\lambda^2 - a^2)}_{\lambda = \pm a}$$

<u>eigenvectors</u>:

<u>$\lambda = a$</u> · 2-degenerate

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = a \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_3 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\Rightarrow |a_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |a_2\rangle = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

<u>$\lambda = -a$</u> · single degenerate

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = -a \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_3 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_1 \\ -v_2 \\ -v_3 \end{pmatrix}$$

$$\Rightarrow v_1 = 0, \quad v_3 = -v_2$$

$$\Rightarrow |a_3\rangle = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$P(a) = \langle \psi(0) | \Big[|a_1\rangle\langle a_1| + |a_2\rangle\langle a_2| \Big] | \psi(0) \rangle$$

$$= \langle \psi(0) | \Big[|u_1\rangle\langle u_1| + \tfrac{1}{2}\big(|u_2\rangle + |u_3\rangle\big)\big(\langle u_2| + \langle u_3|\big) \Big] | \psi(0) \rangle$$

$$= \langle \psi(0) | \Big[|u_1\rangle\langle u_1| + \tfrac{1}{2} |u_2\rangle\langle u_2| + \tfrac{1}{2} |u_3\rangle\langle u_3| + \tfrac{1}{2} |u_2\rangle\langle u_3| + \tfrac{1}{2} |u_3\rangle\langle u_2| \Big] | \psi(0) \rangle$$

$$= \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1.$$

$$P(-a) = \langle \psi(0) | |a_3\rangle\langle a_3| \psi(0)\rangle =$$

$$= \langle \psi(0) | \frac{1}{2} \left(|u_2\rangle - |u_3\rangle \right) \left(\langle u_2| - \langle u_3| \right) | \psi(0)\rangle =$$

$$= \frac{1}{2} \langle \psi(0) | \left(|u_2\rangle\langle u_2| + |u_3\rangle\langle u_3| - |u_2\rangle\langle u_3| - |u_3\rangle\langle u_2| \right) | \psi(0)\rangle =$$

$$= \frac{1}{2} \left(\frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4} \right) = 0.$$

Therefore, we will measure a with probability of 1.

$|\psi(0)\rangle = \frac{1}{\sqrt{2}} |a_1\rangle + \frac{1}{\sqrt{2}} |a_2\rangle$ already in subspace $\{|a_1\rangle, |a_2\rangle\}$

so the system will still be in $|\psi(0)\rangle$ after measurement

c Using time evolution:

$$\boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega_0 t} |u_1\rangle + \frac{1}{2} e^{-i2\omega_0 t} \left(|u_2\rangle + |u_3\rangle \right)}$$

d $\langle A(t)\rangle = \langle \psi(t) | A | \psi(t)\rangle =$

$$= a \cdot \left(\frac{1}{\sqrt{2}} e^{i\omega_0 t} \quad \frac{1}{2} e^{i2\omega_0 t} \quad \frac{1}{2} e^{i2\omega_0 t} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\omega_0 t} \\ 1/2\, e^{-i2\omega_0 t} \\ 1/2\, e^{-i2\omega_0 t} \end{pmatrix} =$$

$$= a \left(\frac{1}{\sqrt{2}} e^{i\omega_0 t} \quad \frac{1}{2} e^{i2\omega_0 t} \quad \frac{1}{2} e^{i2\omega_0 t} \right) \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\omega_0 t} \\ 1/2\, e^{-i2\omega_0 t} \\ 1/2\, e^{-i2\omega_0 t} \end{pmatrix} =$$

$$= a \cdot \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{4} \right) =$$

$$\boxed{\langle A(t)\rangle = a}$$

Comments: we expect A is time independent. Check that $[H, A] = 0$

$$[H,A] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} = 0$$

$$\boxed{[H,A] = 0} \checkmark$$

$$\langle B\rangle = \langle \psi(t)|B|\psi(t)\rangle =$$

$$= b\left(\frac{e^{i\omega_0 t}}{\sqrt{2}} \quad \frac{e^{i2\omega_0 t}}{2} \quad \frac{e^{i2\omega_0 t}}{2}\right)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{e^{-i\omega_0 t}}{\sqrt{2}} \\ \frac{e^{-i2\omega_0 t}}{2} \\ \frac{e^{-i2\omega_0 t}}{2} \end{pmatrix} =$$

$$= b\left(\frac{e^{i\omega_0 t}}{\sqrt{2}} \quad \frac{e^{i2\omega_0 t}}{2} \quad \frac{e^{i2\omega_0 t}}{2}\right)\begin{pmatrix} \frac{e^{-i2\omega_0 t}}{2} \\ \frac{e^{-i\omega_0 t}}{\sqrt{2}} \\ \frac{e^{-i2\omega_0 t}}{2} \end{pmatrix} =$$

$$= b\left(\frac{e^{-i\omega_0 t}}{2\sqrt{2}} + \frac{e^{i\omega_0 t}}{2\sqrt{2}} + \frac{1}{4}\right)$$

$$\boxed{\langle B\rangle = b\left[\frac{\cos(\omega_0 t)}{2\sqrt{2}} + \frac{1}{4}\right]}$$

Comments: seems that B is time dependent

$$[H,B] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{[H,B] \neq 0} \checkmark$$

$\boxed{e}$ $\langle A \rangle (t)$ will give same results $(a, -a)$

<u>Measuring B:</u> we expect time dependence

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (1-\lambda)(\lambda^2 - 1) = 0$$

$$\lambda = \begin{cases} b & - 2\times \text{deg} \\ -b & - 1\times \text{deg} \end{cases}$$

- <u>b</u>

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \Rightarrow \underset{|b_1\rangle}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}, \underset{|b_2\rangle}{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}}$$

- <u>-b</u>

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_1 \\ -v_2 \\ -v_3 \end{pmatrix} \Rightarrow \underset{|b_3\rangle}{\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}}$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega_0 t} |u_1\rangle + \frac{1}{2} e^{-i2\omega_0 t} \left(|u_2\rangle + |u_3\rangle \right)$$

$$P(b) = \langle \psi(t) | \left[|b_1\rangle\langle b_1| + |b_2\rangle\langle b_2| \right] | \psi(t)\rangle =$$

$$= \langle \psi(t) | \left[|u_3\rangle\langle u_3| + \frac{1}{2} \left(|u_1\rangle + |u_2\rangle \right)\left(\langle u_1| + \langle u_2| \right) \right] | \psi(t)\rangle =$$

$$= \frac{1}{2} \cdot \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{2\sqrt{2}} e^{-i\omega_0 t} + \frac{1}{2\sqrt{2}} e^{i\omega_0 t} \right) =$$

$$\boxed{P(b) = \frac{5}{8} + \frac{1}{2\sqrt{2}} \cos(\omega_0 t)}$$

$$P(-b) = \langle \psi(t) | b_3\rangle\langle b_3 | \psi(t)\rangle =$$

$$= \frac{1}{2} \left[\frac{1}{2} + \frac{1}{4} - \frac{1}{2\sqrt{2}} e^{-i\omega_0 t} + e^{i\omega_0 t} \right] =$$

$$\boxed{P(-b) = \frac{3}{8} - \frac{1}{2\sqrt{2}} \cos(\omega_0 t)}$$

check:

$$\langle B \rangle = b \cdot P(b) - b \cdot P(-b) =$$

$$= b \cdot \left(\frac{5}{8} - \frac{3}{8} \right) + b \cdot \frac{1}{\sqrt{2}} \cos(\omega_0 t) =$$

$$= \frac{b}{4} + \frac{b}{\sqrt{2}} \cos(\omega_0 t)$$ ✓ same as before

Problem V

(a) **Ehrenfest theorem**: $\frac{d}{dt}\langle p\rangle = -\langle V'(x)\rangle$

free particle $\Rightarrow$ $V'(x)=0$ $\Rightarrow$ $\boxed{\frac{d}{dt}\langle p\rangle = 0}$ $p(t)=p(0)$

$$\frac{d}{dt}\langle x\rangle = \frac{1}{m}\langle p\rangle \text{ const}$$

$$\langle x\rangle(t) = \frac{1}{m}\langle p\rangle(0)\cdot t + \langle x\rangle(0)$$

So $\langle x\rangle$ is a linear function of time

(b) $$\frac{d}{dt}\langle x^2\rangle = \frac{1}{i\hbar}\langle [x^2, H^2]\rangle + \langle \frac{dx^2}{dt}\rangle =$$

$$= \frac{1}{i\hbar}\cdot\frac{1}{2m}\langle [x^2, p^2]\rangle$$

$[x^2, p^2] = xxpp - ppxx =$ $\qquad [x,p] = xp - px = i\hbar$

$= x(i\hbar + px)p - p(xp - i\hbar)x =$

$= i\hbar(xp + px) + xpxp - pxpx =$

$= (i\hbar)(xp + px) + (i\hbar + px)(i\hbar + px) - pxpx =$

$= (i\hbar)(xp + px) + (i\hbar)^2 + 2i\hbar px =$

$= (i\hbar)(xp + px) + (i\hbar)\cdot(px + xp) =$

$= 2i\hbar(xp + px)$

$$\Rightarrow \boxed{\frac{d}{dt}\langle x^2\rangle = \frac{1}{i\hbar}\langle 2i\hbar(xp + px)/2m\rangle = \frac{1}{m}\langle xp + px\rangle}$$

$$\frac{d}{dt}\langle xp + px\rangle = \frac{1}{i\hbar}\langle [xp + px, \frac{p^2}{2m}]\rangle = \frac{1}{2mi\hbar}\langle [xp, p^2] + [px, p^2]\rangle$$

$$= \frac{1}{2mi\hbar}\langle (xp^2 - p^2x)p + p(xp^2 - p^2x)\rangle =$$

$$= \frac{1}{2mi\hbar} \langle [x, p^2] p + p [x, p^2] \rangle =$$

$$= \frac{1}{\cancel{2}mi\hbar} \langle \underbrace{[x, p]}_{\cancel{i\hbar}} (\cancel{2}p) p + p \underbrace{[x, p]}_{\cancel{i\hbar}} (\cancel{2}p) \rangle =$$

$$\boxed{\frac{d}{dt} \langle xp + px \rangle = \frac{2}{m} \langle p^2 \rangle}$$

$$\frac{d}{dt} \langle p^2 \rangle = \frac{1}{i\hbar} \langle [p^2, \frac{p^2}{2m}] \rangle = 0 \Rightarrow \langle p^2 \rangle \text{ also no time dependence}$$

<u>integrating</u>:

$$\boxed{\langle x^2 \rangle (t) = \frac{\langle p^2 (t=0) \rangle t^2}{m^2} + \frac{\langle xp + px \, (t=0) \rangle t}{m} + \langle x^2 (t=0) \rangle}$$

$$\boxed{\langle xp + px \rangle (t) = \frac{2 \langle p^2 (t=0) \rangle \cdot t}{m} + \langle xp + px \rangle (t=0)}$$