Assignment 8

OPTI 570 Quantum Mechanics

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Problem I

Given that $j = 1$, we have the magnetic quantum number $m \in \{1, 0, -1\}$ and $\{|z_+\rangle, |z_0\rangle, |z_-\rangle\} = \{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$. The state vector can then be spanned as:

$$|\psi\rangle = a|z_+\rangle + b|z_0\rangle + c|z_-\rangle \longrightarrow |\psi\rangle = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

a) i) In the state vector method, we want to compute $\langle\psi|J_x|\psi\rangle$. Using the eigenequation for the ladder operators:

$$J_\pm = |j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}|j, m\pm 1\rangle,$$

we have for $j = 1$

$$j_+|1,0\rangle = \hbar\sqrt{2}|1,1\rangle, \quad J_+|1,-1\rangle = \hbar\sqrt{2}|1,0\rangle, \quad J_-|1,1\rangle = \hbar\sqrt{2}|1,0\rangle, \quad J_-|1,0\rangle = \hbar\sqrt{2}|1,-1\rangle.$$

In matrix form, we have

$$J_+ = \hbar\sqrt{2}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_- = \hbar\sqrt{2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow J_x = \frac{1}{2}(J_+ + J_-) = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Its action on the state is a matrix-multiplication:

$$\langle\psi|J_x|\psi\rangle = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} a^* & b^* & c^* \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{\hbar}{\sqrt{2}}(a^*b + b^*a + b^*c + c^*b) \equiv \frac{2\hbar}{\sqrt{2}}[\mathrm{Re}\,(a^*b) + \mathrm{Re}\,(b^*c)].$$

ii) In the density operator method, the expectation we want to compute is

$$\langle J_x\rangle = \mathrm{Tr}(\rho J_x).$$

In a pure state, we have that

$$|\psi\rangle = a|z_+\rangle + b|z_0\rangle + c|z_-\rangle, \quad \rho = |\psi\rangle\langle\psi|.$$

In matrix form, the density operator is

$$\rho = \begin{bmatrix} |a|^2 & ab^* & ac^* \\ ba^* & |b|^2 & bc^* \\ ca^* & cb^* & |c|^2 \end{bmatrix}.$$

The trace then mean to sum over the diagonal of this matrix multiplied by J_x, that is, over the following elements:

$$\begin{aligned}
(\rho J_x)_{11} &= \rho_{12} J_{21} + \rho_{13} J_{31} = ab^* \frac{\hbar}{\sqrt{2}} \\
(\rho J_x)_{22} &= \rho_{21} J_{12} + \rho_{23} J_{32} = ba^* \frac{\hbar}{\sqrt{2}} + bc^* \frac{\hbar}{\sqrt{2}} \\
(\rho J_x)_{33} &= \rho_{31} J_{13} + \rho_{32} J_{23} = cb^* \frac{\hbar}{\sqrt{2}}
\end{aligned}$$

Then, the trace, and therefore the meanvalue is,

$$\langle J_x \rangle = (\rho J_x)_{11} + (\rho J_x)_{22} + (\rho J_x)_{33} = \frac{\hbar}{\sqrt{2}}[ab^* + ba^* + bc^* + cb^*] = \frac{2\hbar}{\sqrt{2}}[\mathrm{Re}\,(a^* b) + \mathrm{Re}\,(b^* c)].$$

b) For $j = 1$,

$$J^2 = J_x^2 + J_y^2 + J_z^2 = j(j+1)\hbar^2 \mathbb{1} = 2\hbar^2 \mathbb{1}.$$

Then, for any state we have that

$$\langle J^2 \rangle = \langle \psi | 2\hbar \mathbb{1} | \psi \rangle = 2\hbar^2 (|a|^2 + |b|^2 + |c|^2) = 2\hbar^2.$$

c) We know that J operator matrices for the spin-1 are:

$$J_x = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_y = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad J_z = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

If we square them, we have:

$$J_x^2 = \frac{\hbar^2}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad J_y^2 = \frac{\hbar^2}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad J_z^2 = \hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The expectation $\langle \psi | J_i^2 | \psi \rangle$ are then:

$$\begin{aligned}
\langle J_x^2 \rangle &= \frac{\hbar^2}{2}(|a|^2 + 2|b|^2 + |c|^2 + 2\mathrm{Re}\,(a^* c)) \\
\langle J_y^2 \rangle &= \frac{\hbar^2}{2}(|a|^2 + 2|b|^2 + |c|^2 - 2\mathrm{Re}\,(a^* c)) \\
\langle J_z^2 \rangle &= \hbar^2(|a|^2 + |b|^2) \\
\langle J^2 \rangle &= \langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle = 2\hbar(|a|^2 + |b|^2 + |c|^2) = \langle J^2 \rangle
\end{aligned}$$

d) We have from our previous parts both expectation required, so it becomes an algebra problem

$$\begin{aligned}(\Delta J_x)^2 &= \langle J_x^2\rangle - \langle J_x\rangle^2 \\ &= \frac{\hbar^2}{2}[|a|^2 + 2|b|^2 + |c|^2 + 2\mathrm{Re}\,(a^*c) - |a^*b + b^*a + b^*c + c^*b|^2] \\ (\Delta J_x)^2 &= \frac{\hbar^2}{2}[1 + |b|^2 + 2\mathrm{Re}\,(a^*c) - 4(\mathrm{Re}\,(a^*b) + \mathrm{Re}\,(b^*c))^2] \quad (|a|^2 + |b|^2 + |c|^2 = 1)\end{aligned}$$

e) If $b = 0$, then the normalization condition becomes $|a|^2 + |c|^2 = 1$, and

$$\langle J_x\rangle = 0, \quad \langle J_x^2\rangle = \frac{\hbar^2}{2}[|a|^2 + |c|^2 + 2\mathrm{Re}\,(a^*c)] = \frac{\hbar^2}{2}[1 + 2\mathrm{Re}\,(a^*c)].$$

Thus, we need to minimize the alst result, which means that $\mathrm{Re}\,(a^*c)$ must take its minimum value. If $a = |a|e^{i\alpha}$ and $c = |c|e^{i\gamma}$, then we can express the real part as $|a||c|\cos(\alpha - \gamma)$. The minimum value of the cosine is when $\alpha - \gamma = \pi$. Assumind that both magnitudes are equal (simplest scenario), we therefore have $c = -a$ and :

$$a = \frac{e^{i\phi}}{\sqrt{2}}, \quad b = 0, \quad c = -\frac{e^{i\phi}}{\sqrt{2}}.$$

Which then gives

$$\langle J_x^2\rangle = \frac{\hbar^2}{2}(1 - 1) = 0 \Longrightarrow (\Delta J_x)^2 = 0.$$

f) We can just pick an eigenstate of J_z,

$$a = 0, \quad b = e^{i\phi}, \quad c = 0.$$

Thus,

$$|\psi\rangle = |z_0\rangle \quad (\Delta J_z)^2 = 0.$$

We could also pick a non-zero value for a or c.

g) We use the generalized Heisenberg uncertainty relation to compute the required uncertainty product:

$$\Delta J_y \Delta J_z \geq \frac{1}{2}|\langle[J_y, J_z]\rangle| = \frac{1}{2}|\langle i\hbar J_x\rangle| = \frac{\hbar^2}{\sqrt{2}}|\langle \mathrm{Re}\,(a^*b) + \mathrm{Re}\,(b^*c)\rangle|.$$

We see that the uncertainty relation of two perpendicular angular momentum components J_y and J_z dependds on how much angular momentum you have along the third component x. This relations is telling us that the more precisely the J_x is known on average, the more uncertain are the perpendicular components.

Problem II

a) The expansion of $|x_+\rangle$ for $\lambda = +\hbar$ is

$$|x_+\rangle = \frac{1}{2}|z_+\rangle + \frac{1}{\sqrt{2}}|z_0\rangle + \frac{1}{2}|z_-\rangle.$$

If $|\psi(0)\rangle = |z_0\rangle$, then

$$\langle x_+|\psi(0)\rangle = \langle x_+|z_0\rangle = \frac{1}{\sqrt{2}}.$$

The probability to measure $J_x\hbar$ is therefore

$$P_x(\hbar) = |\langle x_+|z_0\rangle|^2 = \frac{1}{2}.$$

b) Now for $\lambda = 0$, the expansion is

$$|x_0\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle - |z_-\rangle).$$

The inner product $|x_0|\psi(0)\rangle$ is now zero, so the probability of measuring $J_x = 0$ is zero.

c) For $\lambda = -\hbar$,

$$|x_-\rangle = \frac{1}{2}|z_+\rangle - \frac{1}{\sqrt{2}}|z_0\rangle + \frac{1}{2}|z_-\rangle.$$

The projection of $|\psi(0)\rangle$ onto $|x_-\rangle$ is $1/\sqrt{2}$, so that

$$P_x(-\hbar) = |\langle x_-|z_0\rangle|^2 = \frac{1}{2}.$$

d) Since $|\psi(0)\rangle = |z_0\rangle$, it is evident that the probability of getting $|z_+\rangle$ is not possible. So,

$$P_z(\hbar) = |\langle z_+|z_0\rangle|^2 = 0.$$

e) After the measurement, the state will evolve and live between the eigenstate of J_x with certain probabilities:

$$\rho = \frac{1}{2}|x_+\rangle\langle x_+| + \frac{1}{2}|x_-\rangle\langle x_-| \quad (P_x(0) = 0).$$

This is a mixture of the eigenstate with the coefficient being the probability. If now we take a measurement with J_z, we will have, for a given angular momentum m

$$P_z(m\hbar) = \mathrm{Tr}(|z_m\rangle\langle z_m|\rho) = \frac{1}{2}\left[|\langle zm|x_+\rangle|^2 + |\langle zm|x_-\rangle|^2\right].$$

In $|x_+\rangle$ and $|x_-\rangle$ we substitute their expression in terms of the eigenstate of J_z and evaluate for $m \in \{\hbar, 0, -\hbar\}$:

$$P_z(\hbar) = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4}, \quad P_z(0) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2}, \quad P_z(-\hbar) = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4}.$$

f) If the measurement is uncertain, then we can express it as a mixed state. However, if we know the result then it collapses into a single eigenstate of J_x, that is, a pure state. We can see for the statistical mixture that the trace of the density operator squared is less than 1, reinforcing this fact: $\mathrm{Tr}(\rho^2) = 1/2 < 1$.

g) This is the same we have computed previouslty, its matrix form is

$$\rho^{(x)} = \frac{1}{2}|x_+\rangle\langle x_+| + \frac{1}{2}|x_-\rangle\langle x_-| = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

h) The transformation matrix is needed to pass from x to z, so we use the above definition in terms of x to express in terms of z and the formulation given in the field guide.

$$\begin{aligned} |x_+\rangle &= \frac{1}{2}|z_+\rangle + \frac{1}{\sqrt{2}}|z_0\rangle + \frac{1}{2}|z_-\rangle \\ |x_0\rangle &= \frac{1}{\sqrt{2}}|z_+\rangle - \frac{1}{\sqrt{2}}|z_-\rangle \\ |x_-\rangle &= \frac{1}{2}|z_+\rangle - \frac{1}{\sqrt{2}}|z_0\rangle + \frac{1}{2}|z_-\rangle \end{aligned}$$

We concatenate them to construct the transformation matrix:

$$M = \begin{bmatrix} | & | & | \\ |x_+\rangle & |x_0\rangle & |x_-\rangle \\ | & | & | \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}.$$

We use it to transform the state vector and the density operator as:

$$|\psi\rangle_z = M|\psi\rangle_x, \quad \text{and} \quad \rho^{(z)} = M\rho^{(x)}M^\dagger.$$

This is exactly the same process shown in the field guide, because these elements on M correspond to the inner product $M_{ij} = \langle z_i|x_j\rangle$.

i) The specific matrix $\rho^{(z)}$ is

$$\rho^{(z)} = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4}. \end{bmatrix}$$

Using it in the expectation values yields

$$\begin{aligned} \langle J_z\rangle &= \mathrm{Tr}(\rho^{(z)}J_z) = \hbar\mathrm{Tr}\left(\begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4}. \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right) = \hbar\left(\frac{1}{4} - \frac{1}{4}\right) = 0 \\ \langle J_x\rangle &= \mathrm{Tr}(\rho^{(z)}J_x) = \frac{\hbar}{\sqrt{2}}\mathrm{Tr}\left(\begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4}. \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right) = 0. \end{aligned}$$

j) They should match the answer from part e as the transformation matrix shouldnt change the meaning of them.

$$P_z(\hbar) = \mathrm{Tr}(\rho^{(z)}P_+) = \mathrm{Tr}\left(\begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4}. \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \mathrm{Tr}\left(\begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 0 & 0 \\ 1/4 & 0 & 0 \end{bmatrix}\right) = \frac{1}{4}$$

$$P_z(0) = \mathrm{Tr}(\rho^{(z)}P_0) = \mathrm{Tr}\left(\begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4}. \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \frac{1}{2}$$

$$P_z(-\hbar) = \mathrm{Tr}(\rho^{(z)}P_-) = \mathrm{Tr}\left(\begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4}. \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 & 1/4 \\ 0 & 0 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}\right) = \frac{1}{4}$$

We see that these probabilities coincide with part e). The P_m are the projectors into the respective m eigenstate.

Problem III

a) By looking the table, the function asked to integrate is:

$$F(\theta,\phi) = (Y_0^0)^* Y_1^0 Y_1^1 = -\frac{3}{4\pi\sqrt{8\pi}}\cos\theta\sin\theta e^{i\phi}.$$

Its integration is:

$$\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\ \sin\theta d\theta d\phi = -\frac{3}{4\pi\sqrt{8\pi}}\int_0^{2\pi}\int_0^{\pi}\cos\theta\sin\theta e^{i\phi}\ \sin\theta d\theta d\phi$$

$$= \int_0^{\pi}\cos\theta\sin^2\theta\left[\underbrace{\int_0^{2\pi} e^{i\phi}\ d\phi}_{0}\right]d\theta$$

$$\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\ \sin\theta d\theta d\phi = 0.$$

b) Integration of the function yields

$$\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\ \sin\theta d\theta d\phi = \frac{3}{8\pi\sqrt{4\pi}}\int_0^{2\pi}\int_0^{\pi}\sin^2\theta e^{i2\phi}\ \sin\theta d\theta d\phi$$

$$= \frac{3}{8\pi\sqrt{4\pi}}\int_0^{\pi}\sin^3\theta\left[\underbrace{\int_0^{2\pi} e^{i2\phi}\ d\phi}_{0}\right]\ d\theta$$

$$\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\ \sin\theta d\theta d\phi = 0.$$

c) Here, we have

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\ \sin\theta d\theta d\phi &= -\frac{3}{8\pi\sqrt{4\pi}}\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\ \sin\theta\ d\theta d\phi \\
&= -\frac{3}{4\sqrt{4\pi}}\int_0^{\pi}\sin^3\theta\ d\theta \\
&= -\frac{3}{4\sqrt{4\pi}}\frac{4}{3} \\
\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\ \sin\theta d\theta d\phi &= -\frac{1}{\sqrt{4\pi}}.
\end{aligned}$$

d) In this case,

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\ \sin\theta d\theta d\phi &= \frac{3}{4\pi\sqrt{4\pi}}\int_0^{2\pi}\int_0^{\pi}\cos^2\theta\ \sin\theta d\theta d\phi \\
&= \frac{3}{2\sqrt{4\pi}}\int_0^{\pi}\cos^2\theta\sin\theta\ d\theta \\
&= \frac{3}{2\sqrt{4\pi}}\frac{2}{3} \\
&= \frac{1}{\sqrt{4\pi}}.
\end{aligned}$$

e) The integration is as follows

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\ \sin\theta d\theta d\phi &= \frac{3\sqrt{15}}{16\pi^{3/2}}\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\cos^2\theta\ \sin\theta d\theta d\phi \\
&= \frac{3\sqrt{15}}{8\sqrt{\pi}}\int_0^{\pi}\sin^3\theta\cos^2\theta\ d\theta \\
&= \frac{3\sqrt{15}}{8\sqrt{\pi}}\frac{4}{15} \quad \text{(integration by parts, } u=\cos\theta,\ du=-\sin\theta d\theta) \\
\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\ \sin\theta d\theta d\phi &= \frac{\sqrt{15}}{10\sqrt{\pi}}.
\end{aligned}$$

f) As each spherical harmonic is a separable function, we can treat them as the product of a θ-dependent function and a ϕ-dependent function:

$$F(\theta,\phi) = g(\theta)h(\phi), \quad \text{where} \quad h(\phi) = e^{-im_1\phi}e^{im_2\phi}e^{m_3\phi}.$$

Because of the hint, we perform the integration only along ϕ:

$$\int_0^{2\pi} h(\phi)\ d\phi = \int_0^{2\pi} e^{i(-m_1+m_2+m_3)\phi}\ d\phi = 2\pi\delta_{-m_1+m_2+m_3,0}.$$

The Kronecker function is non-zero only when the indicex match, that is,

$$m_1 = m_2 + m_3.$$

g) If $l_2 = 1$ means that $m_2 \in \{-1, 0, +1\}$. In addition, using the Δm_{13} we have

$$\begin{aligned} m_1 &= m_2 + m_3 \\ m_1 - m_3 &= m_2 \\ \Delta m_{13} &= m_2. \end{aligned}$$

Therefore,

$$\Delta m_{13} \in \{-1, 0, +1\}.$$

In this problem, all the integrals were raised and then if needed was used integral calculator to facilitate the result.

Problem IV

a) We have a spin-1/2 system, where the spin operators are defined as $S_i = (\hbar/2)\sigma_i$. Measuring with S_y and getting the eigenvalue $\hbar/2$ means that

$$S_y|y\rangle = \frac{\hbar}{2}|y\rangle.$$

The two eigenvectors from it allows to span the $|\pm y\rangle$ as follows:

$$|\pm y\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm i|-\rangle).$$

After the measurement, we will have:

$$|\psi(0)\rangle = |+y\rangle = \frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle).$$

b) The Hamiltonian in this case is

$$H(t) = \omega_0(t)S_z = \frac{\hbar}{2}\omega_0(t)\sigma_z, \quad 0 \leq t \leq T.$$

Applying the evolution operator to $|\psi(0)\rangle$ allows to have the state at time t. Recall that the eigenvalues of σ_z are ± 1. In addition, as the B-field behaves as a ramp function we have

$$\omega_0(t) = \frac{\omega_0}{T}t, \quad 0 \leq t \leq T.$$

Another property is that the Hamiltonian is always proportional to S_z at any arbitrary time, which is the same as

$$[H(t), H(t')] = 0, \quad \forall t, t'.$$

This makes us express the time evolution as:

$$U(t,0) = e^{-\frac{i}{\hbar}\int_0^t H(t')\,dt'} = e^{-i\frac{i}{\hbar}\int_0^t \omega_0(t')S_z\,dt'} = e^{-\frac{i}{\hbar}S_z\frac{\omega_0}{2T}t^2}.$$

Application of $U(t,0)$ to the initial state yields

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{2}\frac{\omega_0}{2T}t^2}|+\rangle + ie^{+\frac{i}{2}\frac{\omega_0}{2T}t^2}|-\rangle\right) = \frac{1}{\sqrt{2}}(e^{i\theta(t)}|+\rangle + ie^{-i\theta(t)}|-\rangle), \quad \theta(t) = -\frac{\omega_0}{4T}t^2.$$

This is a rotation around z by the angle $\theta(t)$ on the Bloch sphere.

c) In the previous part we expressed the expansion of $|\pm y\rangle$, so that the probabilities are computed as the projection of $|\psi(t)\rangle$ onto them:

$$P_y(\hbar/2) = |\langle +y|\psi\rangle|^2 = \left|\frac{e^{i\theta}+e^{-i\theta}}{2}\right|^2 = \cos^\theta$$
$$P_y(-\hbar/2) = |\langle -y|\psi\rangle|^2 = \left|\frac{e^{i\theta}-e^{-i\theta}}{2}\right|^2 = \sin^2\theta.$$

At $t > T$, we have $\theta(t) = \omega_0 T/4$ and:

$$P_y(\hbar/2) = \cos^2\frac{\omega_0 T}{4}, \quad P_y(-\hbar/2) = \sin^2\frac{\omega_0 T}{4}.$$

To be certain of getting $\hbar/2$, the cosine term must be unity, meaning that we must have:

$$\begin{aligned}\cos^2\frac{\omega_0 T}{4} &= 1\\ \cos\frac{\omega_0 T}{4} &= 1/\cos^{-1}(\cdot)\\ \frac{\omega_0 T}{4} &= \pi n, \quad n \in \mathbb{Z}\\ \omega_0 T &= 4\pi n, \quad n \in \mathbb{Z}.\end{aligned}$$

Similarly, to be sure of getting $-\hbar/2$ we must have the sine term to be one, and therefore:

$$\begin{aligned}\sin^2\frac{\omega_0 T}{4} &= 1\\ \sin\frac{\omega_0 T}{4} &= 1/\sin^{-1}(\cdot)\\ \frac{\omega_0 T}{4} &= \frac{\pi(2n+1)}{2}, \quad n \in \mathbb{Z}\\ \omega_0 T &= 2\pi(2n+1), \quad n \in \mathbb{Z}.\end{aligned}$$