Practice Final Exam Solutions

Problem 1

$$V(x) = \frac{1}{2} m \omega^2 x^2 - \frac{\hbar^2}{md} \delta(x) = \frac{1}{2} \hbar\omega \left(\frac{x}{\sigma}\right)^2 - \left(\frac{\sigma}{d}\right) \hbar\omega\, \sigma \cdot \delta(x)$$

$\boxed{A.}$ $$H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{\hbar^2}{md}\delta(x)$$

$$\varphi_g(x) = \sqrt{\frac{1}{d}}\, e^{-|x|/d} \qquad \text{w/} \qquad E_g^0 = -\frac{\hbar^2}{2md^2}$$

A i. $W = -\lambda E_g^0 \frac{x}{d}$, $\lambda \ll 1$

$$E_g = E_g^0 + \lambda \langle \varphi_g | \hat{W} | \varphi_g \rangle$$

$$\lambda \langle \varphi_g | \hat{W} | \varphi_g \rangle = +\lambda \frac{\hbar^2}{2md^2} \cdot \frac{1}{d} \cdot \frac{1}{d} \cdot \langle e^{-|x|/d} | x | e^{-|x|/d} \rangle =$$

$$= \lambda \frac{\hbar^2}{2md^4} \int_{-\infty}^{+\infty} e^{-\frac{2|x|}{d}}\, x\, dx = 0$$ Since x is odd and $e^{-\frac{2|x|}{d}}$ is even

$$E_g = E_g^0$$

A ii. $$W = \frac{\lambda}{2} \hbar\omega \left(\frac{x}{d}\right)^2 = \frac{\cancel{d^2}}{2\sigma^2} \hbar\omega \frac{x^2}{\cancel{d^2}} = \frac{\hbar\omega}{2}\frac{x^2}{\sigma^2}$$

$$\langle \varphi_g | W | \varphi_g \rangle = \cdot \frac{1}{d} \frac{\hbar\omega}{2\sigma^2} \int_{-\infty}^{+\infty} e^{-\frac{2|x|}{d}} x^2\, dx =$$

$$= \frac{\hbar\omega}{d\sigma^2} \int_0^{\infty} e^{-\frac{2x}{d}} x^2\, dx =$$

$$= \frac{\hbar\omega}{d\sigma^2} \cdot \frac{2!}{\left(\frac{2}{d}\right)^3} = \frac{\hbar\omega}{\cancel{d}\sigma^2} \cdot \frac{2}{8} \cdot d^{\cancel{3}\,2} = \frac{1}{4}\frac{\hbar\omega}{\sigma^2} d^2$$

$$\boxed{E_A = -\frac{\hbar^2}{2md^2} + \frac{1}{4}\hbar\omega\left(\frac{d}{\sigma}\right)^2}$$

A iii $E_g^0 = -\frac{\hbar^2}{2md^2} = -\frac{\hbar^2}{m\omega} \cdot \frac{\omega}{2} \cdot \frac{1}{d^2} = \frac{\hbar\omega}{2} \cdot \left(\frac{\sigma}{d}\right)^2$

$$\cancel{2}\frac{1}{\cancel{4}} \cancel{\hbar\omega} \left(\frac{d}{\sigma}\right)^2 = \frac{\cancel{\hbar\omega}}{\cancel{2}} \left(\frac{\sigma}{d}\right)^2 \cdot \frac{1}{\cancel{100}\ 50}$$

$$\frac{d}{\sigma} = \left(\frac{1}{50}\right)^{\frac{1}{4}} = \underline{0.38}$$

$\boxed{B}$ $H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}\hbar\omega\left(\frac{x}{\sigma}\right)^2$

Bi. $W = -\frac{\sigma}{d}\hbar\omega \cdot \sigma \cdot \delta(x) = -\hbar\omega\,\delta(x)\,\frac{\sigma^2}{d}$

We know $E_0 = \frac{1}{2}\hbar\omega$ w/ $\varphi_0(x) = \left(\frac{1}{\pi\sigma^2}\right)^{\frac{1}{4}} e^{-\frac{x^2}{2\sigma^2}}$

$$\langle\varphi_0|W|\varphi_0\rangle = -\left(\frac{1}{\pi\sigma^2}\right)^{\frac{1}{2}} \cdot \hbar\omega\,\frac{\sigma^2}{d} \cdot \int_{-\infty}^{+\infty} e^{-\frac{x^2}{\sigma^2}}\,\delta(x)\,dx =$$

$$= -\left(\frac{1}{\pi\sigma^2}\right)^{\frac{1}{2}} \hbar\omega\,\frac{\sigma^2}{d} =$$

$$= -\frac{\hbar\omega}{\sqrt{\pi}}\,\frac{\sigma}{d}$$

$$E_B = \frac{1}{2}\hbar\omega - \frac{\hbar\omega}{\sqrt{\pi}}\,\frac{\sigma}{d}$$

Bii. $\frac{\hbar\omega}{\sqrt{\pi}}\,\frac{\sigma}{d} = \frac{1}{200}\hbar\omega$

$$\frac{\sigma}{d} = \frac{\sqrt{\pi}}{200} = 8.9 \cdot 10^{-3}$$

Biii. Second order: $\sum_{p\neq 0} \frac{|\langle\varphi_p|W|\varphi_0\rangle|^2}{\frac{1}{2}\hbar\omega - \left(\frac{1}{2}+p\right)\hbar\omega} =$

$$= \sum_{p\neq 0} \frac{\left|\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2^p p!}} \cdot \left(\frac{1}{\pi\sigma^2}\right)^{\frac{1}{2}} e^{-\frac{x^2}{\sigma^2}} H_p\left(\frac{x}{\sigma}\right)\left(-\hbar\right)\omega\,\delta(x)\,\frac{\sigma^2}{d}\right|^2}{-p\hbar\omega}$$

$$= - \cdot \frac{1}{\cancel{\hbar\omega}} \cdot \frac{1}{\pi \cancel{\sigma^2}} \cdot (\hbar\omega)^{\cancel{2}} \cdot \frac{\sigma^{\cancel{4}\,2}}{d^2} \sum_{p=1}^{\infty} \frac{1}{2^p \cdot p!} |H_p(0)|^2 \cdot \frac{1}{p} =$$

$$= - \left(\frac{\sigma}{d}\right)^2 \cdot \frac{\ln 2}{\pi} \cdot \hbar\omega$$

$$\Rightarrow \boxed{E_0 = \frac{\hbar\omega}{2} - \frac{\sigma}{d} \frac{\hbar\omega}{\sqrt{\pi}} - \left(\frac{\sigma}{d}\right)^2 \frac{\ln 2 \cdot \hbar\omega}{\pi}}$$

Problem 2

$$V(x) = \frac{1}{2} m\omega^2 [x - d \cdot f(t)]^2$$

$$W(t) = - \lambda \hbar\omega \frac{x}{\sigma} f(t) \qquad \Psi(t=-\infty) = |\varphi_0\rangle$$

a. $$f(t) = \begin{cases} 1 - \frac{|t|}{\tau} & \text{for } -\tau << t << \tau \\ 0 & \text{otherwise} \end{cases}$$

$$P_1^{(1)} = \frac{\lambda^2}{\hbar^2} \left| - \frac{\hbar\omega}{\sigma} \underbrace{\langle\varphi_2|x|\varphi_1\rangle}_{\frac{\sigma}{\sqrt{2}} \text{ from page 2 of exam}} \mathcal{F}\left(\Lambda\left(\frac{t}{\tau}\right)\right)\Big|_\omega \right|^2 =$$

$$= \frac{\lambda^2}{\cancel{\hbar^2}} \cancel{\hbar^2} \omega^2 \cdot \frac{1}{\cancel{\sigma^2}} \cdot \frac{\cancel{\sigma^2}}{2} \tau^2 \operatorname{sinc}^4\left(\frac{\omega\tau}{2}\right) =$$

$$\boxed{P_1^{(1)} = \frac{\lambda^2}{2} \tau^2 \omega^2 \operatorname{sinc}^4\left(\frac{\omega\tau}{2}\right)}$$

b. $$f(t) = \frac{1}{1 + 9t^2/\tau^2}$$

$$P_1^{(1)} = \frac{\lambda^2}{\cancel{\hbar^2}} \cdot \frac{\cancel{\hbar^2}\omega^2}{\sigma^2} \left| \underbrace{\langle 2|x|1\rangle}_{\frac{\sigma}{\sqrt{2}}} \cdot \mathcal{F}\left\{ \frac{1}{1 + 9t^2/\tau^2} \right\}\Big|_\omega \right|^2 =$$

$$= \lambda^2 \frac{\omega^2}{\cancel{\sigma^2}} \frac{\cancel{\sigma^2}}{2} \cdot \pi^2 \frac{\tau^2}{9} \omega^2 e^{-\frac{2\omega\tau}{6\pi}} =$$

$$\boxed{P_1^{(1)} = \lambda^2 \frac{\pi^2 \tau^2 \omega^2}{18} e^{-\frac{\omega\tau}{3\pi}}}$$

c.

$$\frac{P^{(1)}_{1(a)}}{P^{1}_{1(b)}} = \frac{\dfrac{\cancel{\lambda^2 \tau^2 \omega^2}}{\cancel{\hbar}} \operatorname{sinc}^4\left(\frac{\pi}{2}\right)}{\dfrac{\cancel{\lambda^2}\, \pi^2 \cancel{\tau^2 \omega^2}}{\cancel{18}\, 9} e^{-\frac{\omega\tau}{3\pi}}} = \frac{9 \cdot \operatorname{sinc}^4(\pi/2)}{\pi^2 e^{-\pi/3\pi}} = \frac{9 \cdot \left(\frac{2}{\pi}\right)^4}{\pi^2 e^{-\frac{1}{3}}} = 0.21$$

d.

$$f(t) = \begin{cases} -\sin(2\omega t) & \text{for } 0 \le t \le \frac{\pi}{\omega} \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda b_1^{(1)}(t) = \frac{1}{i\cancel{\hbar}} \int_{t=0}^{\frac{\pi}{\omega}} e^{i\omega t} \left[\frac{\lambda \cdot \cancel{\hbar}\omega \cdot}{\cancel{\sigma}} \cdot \frac{\cancel{\sigma}}{\sqrt{2}} \cdot \left[-\sin(2\omega t)\right]\right] dt =$$

$$= \frac{i\lambda}{\sqrt{2}} \quad \omega \int_{t=0}^{\pi/\omega} e^{i\omega t} \sin(2\omega t)\, dt =$$

$$u = \omega t \qquad du = \omega\, dt$$

$$= \frac{i\lambda}{\sqrt{2}} \quad \cancel{\omega} \cdot \frac{1}{\cancel{\omega}} \int_{u=0}^{\pi} e^{iu} \sin(2u)\, du =$$

$$= i\lambda \, \frac{2\sqrt{2}}{3}$$

$$\boxed{P^{(1)}_1 = \frac{8}{9} \lambda^2}$$

e.

$$\lambda b_1^{(1)}(t) = \frac{1}{i\cancel{\hbar}} \int_{t=0}^{\frac{\pi}{\omega}} e^{i\omega t} \left[\frac{\lambda \cdot \cancel{\hbar}\omega \cdot}{\cancel{\sigma}} \cdot \frac{\cancel{\sigma}}{\sqrt{2}} \cdot \left[\sin^2(\omega t)\right]\right] dt =$$

$$= \frac{-i\omega\lambda}{\sqrt{2}} \int_{t=0}^{\pi/\omega} e^{i\omega t} \sin^2(\omega t)\, dt =$$

$$u = \omega t \qquad du = \omega\, dt$$

$$= -\frac{i\cancel{\omega}\lambda}{\sqrt{2}} \frac{1}{\cancel{\omega}} \int_{t=0}^{\pi} e^{iu} \sin^2 u\, du =$$

$$= -\frac{i\lambda}{\sqrt{2}} \frac{4}{3} i = -\frac{4\lambda}{3\sqrt{2}}$$

$$\boxed{P^{(1)}_1 = \frac{8}{9} \lambda^2}$$

Problem 3

a. to first order:

$$V(x) = \frac{1}{2} m \omega^2 \left[x^2 - 2x\,d \cdot f_x(t) + y^2 - 2y\,d\,f_y(t)\right]$$

$$\Rightarrow \quad W = -m\omega^2 d\left[x f_x(t) + y f_y(t)\right] = -\lambda \frac{\hbar\omega}{\sigma}\left[x f_x(t) + y f_y(t)\right]$$

in $\{|0,0\rangle, |\Psi_+\rangle, |\Psi_-\rangle \dots\}$ repres:

$$X = \frac{\sigma}{2}\begin{pmatrix} 0 & 1 & 1 & \dots \\ 1 & 0 & 0 & \\ 1 & 0 & 0 & \\ \vdots & & & \end{pmatrix} \qquad Y = \frac{\sigma}{2}\begin{pmatrix} 0 & i & -i & \dots \\ -i & 0 & 0 & \\ i & 0 & 0 & \\ \vdots & & & \end{pmatrix}$$

$$\lambda\, b_+^{(1)}(t) = \frac{1}{i\hbar}\left(-\lambda\frac{\hbar\omega}{\sigma}\right)\left[\int_0^{\pi/\omega} dt\, e^{i\omega t}\frac{\sigma}{2}\left[-\sin(2\omega t)\right] + \right.$$

$$\left. + \int_0^{\pi/\omega} dt\, e^{i\omega t}\frac{\sigma}{2} i\left[\sin^2(\omega t)\right] \cdot i \right] =$$

$$= \frac{i\lambda\omega}{\cancel{\sigma}} \cdot \frac{\cancel{\sigma}}{2\sqrt{2}} \cdot \left(\frac{4}{3\cancel{\omega}} - \frac{4}{3\cancel{\omega}}\right) =$$

$$= 0$$

$$\boxed{P_+^{(1)}(t) = 0}$$

$$\lambda\, b_-^{(1)}(t) = \frac{1}{i\hbar}\left(-\lambda\frac{\hbar\omega}{\sigma}\right)\left[\int_0^{\pi/\omega} dt\, e^{i\omega t}\frac{\sigma}{2}\left[-\sin(2\omega t)\right] + \right.$$

$$\left. + \int_0^{\pi/\omega} dt\, e^{i\omega t} - i\frac{\sigma}{2}\left[-i\sin^2(\omega t)\right) \right] =$$

$$= \frac{i\lambda\cancel{\omega}}{\cancel{\sigma}} \cdot \frac{\cancel{\sigma}}{2}\left(-\frac{4}{3\cancel{\omega}} - \frac{4}{3\cancel{\omega}}\right) =$$

$$= -i\lambda\frac{4}{3} \qquad \Rightarrow \boxed{P_-^{(1)}(t) = \frac{16}{9}\lambda^2 = \frac{16}{9}\left(\frac{d}{\sigma}\right)^2}$$

b. The two states correspond to left and right-handed angular momentum, so only one of them gets "excited" by the clockwise motion of the trap.