

Assignment 7

OPTI 570 Quantum Mechanics

University of Arizona

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Problem I

a) The evolution operator would be of the form

$$\hat{U}_E(t) = e^{-i\hat{H}_1 t/\hbar} = e^{-i\Omega(\hat{N}^2 - 1/2)t}.$$

The checking is as follows:

$$\begin{aligned}\hat{U}_E\left(\frac{2\pi}{\Omega}\right)|\varphi_n\rangle &= e^{-i\Omega(n^2 - 1/2)\frac{2\pi}{\Omega}}|\varphi_n\rangle \\ &= e^{-i2\pi(n^2 - 1/2)}|\varphi_n\rangle \\ &= (e^{-2\pi})^{n^2} e^{i\pi}|\varphi_n\rangle \\ \hat{U}_E\left(\frac{2\pi}{\Omega}\right)|\varphi_n\rangle &= -|\varphi_n\rangle.\end{aligned}$$

b) For $\tau = \pi/2\Omega$, the evolution is

$$\begin{aligned}\hat{U}_E(\tau)|\varphi_n\rangle &= e^{-i\Omega(n^2 - 1/2)\frac{\pi}{2\Omega}}|\varphi_n\rangle \\ &= e^{-i\frac{\pi}{2}n^2} e^{i\frac{\pi}{4}}|\varphi_n\rangle \\ &= (e^{-i\frac{\pi}{2}})^{n^2} e^{i\frac{\pi}{4}}|\varphi_n\rangle \\ &= (-i)^{n^2} e^{i\frac{\pi}{4}}|\varphi_n\rangle \\ \hat{U}_E(\tau)|\varphi_n\rangle &= \begin{cases} e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, & n \text{ even} \\ e^{-i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, & n \text{ odd} \end{cases} |\varphi_n\rangle.\end{aligned}$$

c) We use the fact that in a coherent state, we can express it in terms of the energy eigenstates.

$$|\alpha_0\rangle = e^{-\frac{|\alpha_0|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} |n\rangle.$$

We have found that

$$\hat{U}_E(\tau) = \begin{cases} e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, & n \text{ even} \\ e^{-i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, & n \text{ odd} \end{cases}.$$

We then, must split the $|\alpha_0\rangle$ accordingly, in even and odd term so that the application of the evolution operator gives

$$|\psi_E(\tau)\rangle = e^{-\frac{|\alpha_0|^2}{2}} \left[e^{i\frac{\pi}{4}} S_{\text{even}} + e^{-i\frac{\pi}{4}} S_{\text{odd}} \right],$$

where

$$S_{\text{even}} = \sum_{n \text{ even}} \frac{\alpha_0^n}{\sqrt{n!}} |n\rangle, \quad \text{and} \quad S_{\text{odd}} = \sum_{n \text{ odd}} \frac{\alpha_0^n}{\sqrt{n!}} |n\rangle. \quad (1)$$

We then have that

$$\left. \begin{aligned} |\alpha_0\rangle &= e^{-\frac{|\alpha_0|^2}{2}} (S_{\text{even}} + S_{\text{odd}}) \\ |-\alpha_0\rangle &= e^{-\frac{|\alpha_0|^2}{2}} (S_{\text{even}} - S_{\text{odd}}) \end{aligned} \right\} \longrightarrow \begin{aligned} S_{\text{even}} &= \frac{1}{2} e^{\frac{|\alpha_0|^2}{2}} (|\alpha_0\rangle + |-\alpha_0\rangle) \\ S_{\text{odd}} &= \frac{1}{2} e^{\frac{|\alpha_0|^2}{2}} (|\alpha_0\rangle - |-\alpha_0\rangle) \end{aligned}.$$

Substituting those in the evolution equation and rearranging:

$$|\psi_E(\tau)\rangle = \frac{1}{2} \left[(e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}}) |\alpha_0\rangle + (e^{i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}}) |-\alpha_0\rangle \right] = \frac{1}{\sqrt{2}} [|\alpha_0\rangle + i|-\alpha_0\rangle],$$

where

$$|\pm \alpha_0\rangle = e^{-\frac{|\alpha_0|^2}{2}} \sum_{n=0}^{\infty} \frac{(\pm \alpha_0)^n}{\sqrt{n!}} |n\rangle.$$

d) The transformation from the Interaction picture to the Schrodinger picture is

$$\begin{aligned} |\psi(\tau)\rangle &= \hat{U}_0(\tau) |\psi_E(\tau)\rangle, \quad \hat{U}_0(\tau) = e^{-iH_0\tau/\hbar} \\ &= e^{-i\omega\tau(\hat{N}+1/2)} |\psi_E(\tau)\rangle \\ &= \frac{1}{\sqrt{2}} e^{-i\omega\tau(\hat{N}+1/2)} [\alpha_0 + i|-\alpha_0\rangle] |n\rangle \\ &= \frac{1}{\sqrt{2}} e^{-\frac{|\alpha_0|^2}{2}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left[\alpha_0^n e^{-i\omega\tau(n+1/2)} + i(-\alpha_0)^n e^{-i\omega\tau(n+1/2)} \right] |n\rangle \\ &= \frac{1}{\sqrt{2}} e^{-i\frac{\omega}{2}\tau} e^{-\frac{|\alpha_0|^2}{2}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left[(\alpha_0 e^{-i\omega\tau})^n |n\rangle + i(-\alpha_0 e^{-i\omega\tau})^n |n\rangle \right] \\ |\psi(\tau)\rangle &= \frac{1}{\sqrt{2}} e^{-i\frac{\omega}{2}\tau} [\alpha_0 e^{-i\omega\tau} + i|-\alpha_0 e^{-i\omega\tau}\rangle]. \end{aligned}$$

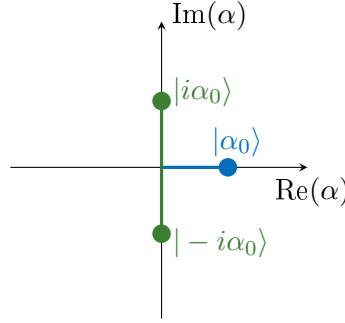
e) Evaluating with $\tau = \pi/2\omega$,

$$\begin{aligned} |\psi(\frac{\pi}{2\omega})\rangle &= \frac{1}{\sqrt{2}} e^{-i\frac{\omega}{2}\frac{\pi}{2\omega}} [\alpha_0 e^{-i\omega\frac{\pi}{2\omega}} + i|-\alpha_0 e^{-i\omega\frac{\pi}{2\omega}}\rangle] \\ &= \frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}} [\alpha_0 e^{-i\frac{\pi}{2}} + i|-\alpha_0 e^{-i\frac{\pi}{2}}\rangle] \\ |\psi(\frac{\pi}{2\omega})\rangle &= \frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}} [-i\alpha_0 + i|\alpha_0\rangle]. \end{aligned}$$

In addition, at $t = 0$ we have

$$|\psi_E(0)\rangle = |\alpha_0\rangle.$$

Then,



Problem II

The Hamiltonian in the whole range is:

$$\hat{H} = \hat{H}_0 + \hat{W} = \begin{cases} \frac{\hat{P}^2}{2m}, & t < 0 \\ \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2 \hat{X}^2, & 0 \leq t < \tau, \\ \frac{\hat{P}^2}{2m}, & t \geq 0 \end{cases} \quad \hat{W} = \frac{1}{2}m\omega^2 \hat{X}^2.$$

The evolution operator is $U_0(t) = e^{-iH_0 t/\hbar} = e^{-i\hat{P}^2 t/2m\hbar}$. The effective Hamiltonian in terms of the Schrodinger picture position and momentum operators is:

$$\begin{aligned} H_E &= U_0^\dagger(t, 0) H_1 U_0(t, 0) = e^{i\frac{\hat{P}^2 t}{2m\hbar}} \left[\frac{1}{2}m\omega^2 \hat{X}^2 \right] e^{-i\hat{P}^2 t/2m\hbar} = \frac{1}{2}m\omega^2 e^{i\hat{P}^2 t/2m\hbar} \hat{X}^2 e^{-i\hat{P}^2 t/2m\hbar} \\ &\stackrel{(a)}{=} \frac{1}{2}m\omega^2 \left[e^{i\hat{P}^2 t/2m\hbar} \hat{X} e^{-i\hat{P}^2 t/2m\hbar} \right]^2. \end{aligned}$$

In (a), we used the property. We can see the term inside the brackets as the product ABC of operators, where we would like to switch the position of \hat{X} with the right exponential, that why we use

$$ABC = A[B, C] + ACB.$$

The commutator $[B, C]$ is

$$[B, C] = [\hat{X}, e^{-i\frac{\hat{P}^2 t}{2m\hbar}}] = i\hbar \partial_{\hat{P}} e^{-i\frac{\hat{P}^2 t}{2m\hbar}} = i\hbar \frac{-i2\hat{P}t}{2m\hbar} e^{-i\frac{\hat{P}^2 t}{2m\hbar}} = \frac{\hat{P}t}{m} e^{-i\frac{\hat{P}^2 t}{2m\hbar}}.$$

Then, substituting this commutator in the above relation

$$e^{i\frac{\hat{P}^2 t}{2m\hbar}} \hat{X} e^{-i\frac{\hat{P}^2 t}{2m\hbar}} = e^{i\frac{\hat{P}^2 t}{2m\hbar}} \frac{\hat{P}t}{m} e^{-i\frac{\hat{P}^2 t}{2m\hbar}} + e^{i\frac{\hat{P}^2 t}{2m\hbar}} e^{-i\frac{\hat{P}^2 t}{2m\hbar}} \hat{X} = \hat{X} + \frac{\hat{P}t}{m}.$$

Finally,

$$H_E = \frac{1}{2}m\omega^2 \left[\hat{X} + \frac{\hat{P}t}{m} \right]^2.$$

Problem III

The Hamiltonian is

$$H = \begin{cases} H_0, & t < 0 \\ H_0 + W(t), & 0 \leq t < \tau = \frac{4\pi}{\omega} \\ H_0, & t > \tau \end{cases}.$$

a) sa

$$|\psi_I(t)\rangle = U_0^\dagger(\tau, 0)|\psi_S(t)\rangle = e^{-i4\pi(\hat{N}+1/2)}|\psi_S(t)\rangle = e^{-i4\pi n}e^{-i2\pi}|\psi_S(t)\rangle = |\psi_S(t)\rangle.$$

b) The effective Hamiltonian is:

$$\begin{aligned} H_E &= U_0^\dagger W(t) U_0 \\ &= \frac{i\hbar\Omega}{2} \left[e^{i\omega(\hat{N}+1/2)t} (\hat{a}^2 e^{i2\omega t} - (\hat{a}^\dagger)^2 e^{-i2\omega t}) e^{-i\omega(\hat{N}+1/2)t} \right] \\ &= \frac{i\hbar\Omega}{2} \left\{ e^{i2\omega t} [e^{i\omega(\hat{N}+1/2)t} \hat{a} e^{-i\omega(\hat{N}+1/2)t}]^2 - e^{-i2\omega t} [e^{i\omega(\hat{N}+1/2)t} \hat{a}^\dagger e^{-i\omega(\hat{N}+1/2)t}]^2 \right\} \\ H_E &= \frac{i\hbar\Omega}{2} \left\{ \hat{a}^2 - (\hat{a}^\dagger)^2 \right\}. \end{aligned}$$

c) We use the expression of the \hat{a} operators in terms of \hat{X} and \hat{P} :

$$\begin{aligned} \hat{H}_E &= \frac{i\hbar\Omega}{2} \left\{ \frac{1}{2} \left(\frac{\hat{X}}{\sigma} + i \frac{\hat{P}\sigma}{\hbar} \right)^2 - \frac{1}{2} \left(\frac{\hat{X}}{\sigma} - i \frac{\hat{P}\sigma}{\hbar} \right)^2 \right\} \\ &= \frac{i\hbar\Omega}{4} \left\{ \frac{\hat{X}^2}{\sigma^2} + \frac{i}{\hbar} (\hat{X}\hat{P} + \hat{P}\hat{X}) - \frac{\hat{P}^2\sigma^2}{\hbar^2} - \left[\frac{\hat{X}^2}{\sigma^2} - \frac{i}{\hbar} (\hat{X}\hat{P} + \hat{P}\hat{X}) - \frac{\hat{P}^2\sigma^2}{\hbar^2} \right] \right\} \\ &= -\frac{\Omega}{2} (\hat{X}\hat{P} + \hat{P}\hat{X}) \\ \hat{H}_E &= -\frac{\Omega}{2} \{\hat{X}, \hat{P}\}, \quad \{\cdot\} = \text{anti-commutator}. \end{aligned}$$

d) We use the effective Hamiltonian of part b):

$$\hat{U}_E(\tau) = e^{-\frac{i\tau}{\hbar} \frac{i\hbar\Omega}{2} (\hat{a}^2 - (\hat{a}^\dagger)^2)} = e^{\frac{\Omega\tau}{2} (\hat{a}^2 - (\hat{a}^\dagger)^2)} = e^{\frac{b}{2} (\hat{a}^2 - (\hat{a}^\dagger)^2)}.$$

e) In this case, we compare the definition the above operator with the new definition, and we look at the exponent of each function:

$$e^{-\hat{B}} = \hat{U}_E(\tau) = e^{\frac{b}{2} (\hat{a}^2 - (\hat{a}^\dagger)^2)} \implies \hat{B} = -\frac{b}{2} (\hat{a}^2 - (\hat{a}^\dagger)^2).$$

As a verification, the adjoint is:

$$\hat{B}^\dagger = -\frac{b}{2} ((\hat{a}^\dagger)^2 - \hat{a}^2) = \frac{b}{2} (\hat{a}^2 - (\hat{a}^\dagger)^2) = -\hat{B}.$$

So \hat{B} is anti-Hermitian.

f) asgasg

$$\begin{aligned} [\hat{B}, \hat{a}] &= -\frac{b}{2}\{[\hat{a}^2, \hat{a}] - [(\hat{a}^\dagger)^2, \hat{a}]\} = \frac{b}{2}[(\hat{a}^\dagger)^2, \hat{a}] = \frac{b}{2}\{\hat{a}^\dagger[\hat{a}^\dagger, \hat{a}] + [\hat{a}^\dagger, \hat{a}]\hat{a}^\dagger\} = -b\hat{a}^\dagger \\ [\hat{B}, \hat{a}^\dagger] &= -\frac{b}{2}\{[\hat{a}^2, \hat{a}^\dagger] - [(\hat{a}^\dagger)^2, \hat{a}^\dagger]\} = -\frac{b}{2}[\hat{a}^2, \hat{a}^\dagger] = -\frac{b}{2}\{\hat{a}[\hat{a}, \hat{a}^\dagger] + [\hat{a}, \hat{a}^\dagger]\hat{a}\} = -b\hat{a}. \end{aligned}$$

g) We now from the previous part the commutators, and also

$$\hat{Q}(b) = e^{-\hat{B}} = e^{\frac{b}{2}(\hat{a}^2 - (\hat{a}^\dagger)^2)}, \quad \hat{Q}^\dagger(b) = e^{\hat{B}} = e^{-\frac{b}{2}(\hat{a}^2 - (\hat{a}^\dagger)^2)}, \quad \hat{Q}(b)\hat{Q}^\dagger(b) = \mathbf{1}.$$

To the application of the formula provided, we compute the following commutators:

$$[\hat{B}, \hat{a}] = -b\hat{a}^\dagger, \quad [\hat{B}, [\hat{B}, \hat{a}]] = [\hat{B}, -b\hat{a}^\dagger] = b^2\hat{a}, \quad [\hat{B}, [\hat{B}, [\hat{B}, \hat{a}]]] = -b^3\hat{a}^\dagger, \quad \dots$$

Then,

$$\begin{aligned} \hat{Q}^\dagger(b)\hat{a}\hat{Q}(b) &= \hat{a} + (-b\hat{a}^\dagger) + \frac{1}{2!}[\hat{B}, [\hat{B}, \hat{a}]] + \frac{1}{3!}[\hat{B}, [\hat{B}, [\hat{B}, \hat{a}]]] + \dots \\ &= \hat{a} - b\hat{a}^\dagger + \frac{b^2}{2!}\hat{a} - \frac{b^3}{3!}\hat{a}^\dagger + \dots \\ &= \left(1 + \frac{b^2}{2!} + \dots\right)\hat{a} - \left(b - \frac{b^3}{3!}\right)\hat{a}^\dagger \\ \hat{Q}^\dagger(b)\hat{a}\hat{Q}(b) &= \cosh(b)\hat{a} - \sinh(b)\hat{a}^\dagger, \end{aligned}$$

and,

$$[\hat{Q}^\dagger(b)\hat{a}\hat{Q}(b)]^\dagger = \hat{Q}^\dagger(b)\hat{a}^\dagger\hat{Q}(b) = \cosh(b)\hat{a}^\dagger - \sinh(b)\hat{a}.$$

h) We use both result from above

$$\begin{aligned} \hat{Q}^\dagger(b)\hat{X}\hat{Q}(b) &= \frac{\sigma}{\sqrt{2}}\hat{Q}^\dagger(b)(\hat{a}^\dagger + \hat{a})\hat{Q}(b) = \frac{\sigma}{\sqrt{2}}\left[\hat{Q}^\dagger(b)\hat{a}^\dagger\hat{Q}(b) + \hat{Q}^\dagger(b)\hat{a}\hat{Q}(b)\right] \\ &= \frac{\sigma}{\sqrt{2}}\left[\cosh(b)\hat{a}^\dagger - \sinh(b)\hat{a} + \cosh(b)\hat{a} - \sinh(b)\hat{a}^\dagger\right] \\ &= \frac{\sigma}{\sqrt{2}}[(\hat{a}^\dagger + \hat{a})(\cosh(b) - \sinh(b))] \\ \hat{Q}^\dagger(b)\hat{X}\hat{Q}(b) &= e^{-b}\hat{X}. \end{aligned}$$

For the momentum operator, we have similarly

$$\begin{aligned} \hat{Q}^\dagger(b)\hat{P}\hat{Q}(b) &= \frac{i\hbar}{\sqrt{2}\sigma}\hat{Q}^\dagger(b)(\hat{a}^\dagger - \hat{a})\hat{Q}(b) = \frac{i\hbar}{\sqrt{2}\sigma}\left[\hat{Q}^\dagger(b)\hat{a}^\dagger\hat{Q}(b) - \hat{Q}^\dagger(b)\hat{a}\hat{Q}(b)\right] \\ &= \frac{i\hbar}{\sqrt{2}\sigma}\left[\cosh(b)\hat{a}^\dagger - \sinh(b)\hat{a} - \cosh(b)\hat{a} + \sinh(b)\hat{a}^\dagger\right] \\ &= \frac{i\hbar}{\sqrt{2}\sigma}[(\hat{a}^\dagger + \hat{a})(\cosh(b) + \sinh(b))] \\ \hat{Q}^\dagger(b)\hat{P}\hat{Q}(b) &= e^b\hat{P}. \end{aligned}$$

The remaining ones are computed easily with the formular used in past exercise:

$$\begin{aligned} \hat{Q}^\dagger(b)\hat{X}^2\hat{Q}(b) &= [\hat{Q}^\dagger(b)\hat{X}\hat{Q}(b)]^2 = e^{-2b}\hat{X}^2 \\ \hat{Q}^\dagger(b)\hat{P}^2\hat{Q}(b) &= [\hat{Q}^\dagger(b)\hat{P}\hat{Q}(b)]^2 = e^{2b}\hat{P}^2. \end{aligned}$$

i) The ground state of QHO is:

$$\langle 0|\hat{X}|0\rangle = 0, \quad \langle 0|\hat{P}|0\rangle = 0, \quad \langle 0|\hat{X}^2|0\rangle = \frac{\sigma^2}{2}, \quad \langle 0|\hat{P}^2|0\rangle = \frac{\hbar^2}{2\sigma^2}.$$

Using the above results, and the definition of the state $|\varphi\rangle = \hat{Q}(b)|0\rangle$

$$\begin{aligned} \langle \varphi|\hat{X}|\varphi\rangle &= e^{-b}\langle 0|\hat{X}|0\rangle = 0 \\ \langle \varphi|\hat{P}|\varphi\rangle &= e^b\langle 0|\hat{P}|0\rangle = 0 \\ \langle \varphi|\hat{X}^2|\varphi\rangle &= e^{-2b}\langle 0|\hat{X}^2|0\rangle = e^{-2b}\frac{\sigma^2}{2} \\ \langle \varphi|\hat{P}^2|\varphi\rangle &= e^{2b}\langle 0|\hat{P}^2|0\rangle = e^{2b}\frac{\hbar^2}{2\sigma^2}. \end{aligned}$$

j) The uncertainty is therefore

$$\left. \begin{aligned} \Delta\hat{X} &= \sqrt{\langle \hat{X}^2\rangle} = e^{-b}\frac{\sigma}{\sqrt{2}} \\ \Delta\hat{P} &= \sqrt{\langle \hat{P}^2\rangle} = e^b\frac{\hbar}{\sqrt{2}\sigma} \end{aligned} \right\} \Delta\hat{X}\Delta\hat{P} = \frac{\hbar}{2}.$$

k) The wavefunction in the $\{|x\rangle\}$ representation is $\varphi(x) = \langle x|\varphi\rangle$. We need to know the action of t

$$\hat{X}(\hat{Q}(b)|x\rangle) = \hat{Q}(b)\hat{Q}^\dagger(b)\hat{X}\hat{Q}(b)|x\rangle = \hat{Q}(b)e^{-b}\hat{X}|x\rangle = xe^{-b}(\hat{Q}(b)|x\rangle).$$

Therefore,

$$\hat{X}(\hat{Q}(b)|x\rangle) = xe^{-b}(\hat{Q}(b)|x\rangle) \implies \hat{Q}(b)|x\rangle = C|e^{-b}x\rangle.$$

The eigenstate $|e^{-b}x\rangle$ is proportional by the factor C . We find the coefficient c

$$\langle x'|x\rangle = \langle x|\hat{Q}^\dagger(b)\hat{Q}(b)|x\rangle = |c|^2\langle e^{-b}x'|e^{-b}x\rangle = |c|^2\delta[e^{-b}(x' - x)] = |c|^2e^b\delta(x' - x) = 1.$$

The coefficient is:

$$|c|^2e^b = 1 \longrightarrow c = e^{-b/2}.$$

Because the expression for the ground state is a gaussian of the form:

$$\psi_0(x) = \left(\frac{1}{\pi\sigma^2}\right)^{1/4} e^{-\frac{x^2}{2\sigma^2}},$$

we construct our function as:

$$\varphi(x) = \langle x|\hat{Q}(b)|0\rangle = e^{-b/2}\langle e^bx|0\rangle = C\psi_0(e^bx) = e^{-b/2}\left(\frac{1}{\pi\sigma^2}\right)^{1/4} e^{b/2}e^{-\frac{e^{2b}x^2}{2\sigma^2}} = \left(\frac{1}{\pi\gamma^2}\right)^{1/4} e^{-\frac{x^2}{2\gamma^2}},$$

with $\gamma = \sigma e^{-b}$.

l) Knowing that

$$\hat{H}_0 = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2, \quad \langle\hat{X}^2\rangle_\varphi = e^{-2b}\frac{\sigma^2}{2}, \quad \langle\hat{P}^2\rangle_\varphi = e^{2b}\frac{\hbar^2}{2\sigma^2}.$$

The mean value of the Hamiltonian can be expressed in terms of the mean values of the position and momentum operators:

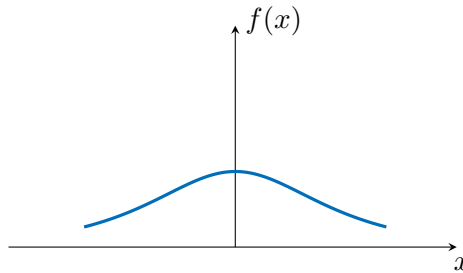
$$\langle H_0 \rangle_\varphi = \frac{1}{2m}e^{2b}\frac{\hbar^2}{2\sigma^2} + \frac{1}{2}m\omega^2e^{-2b}\frac{\sigma^2}{2} = \frac{\hbar\omega}{2}\cosh(2b),$$

where we substituted $\sigma = \sqrt{\hbar/m\omega}$ to simplify further the expression. For $b = 0$, we see that $\langle H_0 \rangle_\varphi = \hbar\omega/2$ the ground state energy level.

m) The operator $\hat{Q}(p)$ changes the uncertainty of the quadratures increasing one and reducing the other respectively so that the uncertainty product is maintained.

Problem IV

a) We plot the function $\text{sech}(x)$ to verify its parity. We can see that it is **even**.



This fact will facilitate us when computing ΔX , as we must integrate over $|\phi(x)|^2$ which therefore, is also even. We then have,

$$\begin{aligned} \langle X \rangle &= \int_{-\infty}^{\infty} x|\phi(x)|^2 dx = \frac{1}{2\beta} \int_{-\infty}^{\infty} x \text{sech}(x/\beta) dx = 0 \\ \langle X^2 \rangle &= \int_{-\infty}^{\infty} x^2|\phi(x)|^2 dx = \frac{1}{2\beta} \int_{-\infty}^{\infty} x^2 \text{sech}(x/\beta) dx = \frac{\beta^2}{2} \int_{-\infty}^{\infty} u^2 \text{sech}^2(u) du = \frac{\pi^2\beta^2}{12}. \end{aligned}$$

The X uncertainty is

$$\Delta X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \frac{\pi\beta}{2\sqrt{3}}.$$

Similarly, for the Fourier transform we have:

$$\begin{aligned} \langle P \rangle &= \int_{-\infty}^{\infty} p|\hat{\phi}(p)|^2 dp = \frac{\pi\beta}{4\hbar} \int_{-\infty}^{\infty} p \text{sech}^2\left(\frac{\pi\beta p}{2\hbar}\right) dp = 0 \\ \langle P^2 \rangle &= \int_{-\infty}^{\infty} p^2|\hat{\phi}(p)|^2 dp = \frac{\pi\beta}{4\hbar} \int_{-\infty}^{\infty} p^2 \text{sech}^2\left(\frac{\pi\beta p}{2\hbar}\right) dp = \frac{2\hbar^2}{\pi^2\beta^2} \int_{-\infty}^{\infty} u^2 \text{sech}^2(u) du = \frac{\hbar^2}{\beta^2 3}. \end{aligned}$$

Thus

$$\Delta P = \sqrt{\langle P^2 \rangle - \langle P \rangle^2} = \frac{\hbar}{\beta\sqrt{3}}.$$

The uncertainty product is

$$\Delta X \Delta P = \frac{\pi\beta}{2\sqrt{3}} \frac{\hbar}{\beta\sqrt{3}} = \frac{\hbar\pi}{6}.$$

b) The evolution in $\pi/2\omega$ gives a well-known quantity, a scaled Fourier transform of the wavefunction.

$$\Phi(x, \frac{\pi}{2\omega}) = U(\frac{\pi}{2\omega}, 0)\Phi(x, 0) = e^{-i\pi/4} \sqrt{\frac{\hbar}{\sigma^2}} \mathcal{F}\{\Phi(x, 0)\} \Big|_{p=\hbar x/\sigma^2}$$

We can see that the function to be computed its Fourier transform is spatially shifted by x_0 so we could directly use the respective property of Fourier transform of a shifter function:

$$\mathcal{F}\{\Phi(x, 0)\} = \hat{\Phi}(p, 0) \implies \mathcal{F}\{\Phi(x - x_0, 0)\} = e^{-ipx_0/\hbar} \hat{\Phi}(p, 0).$$

So,

$$\Phi(x, \frac{\pi}{2\omega}) = -e^{-i\pi/4} \sqrt{\frac{\hbar}{\sigma^2}} \left[e^{-ipx_0/\hbar} \hat{\Phi}(p, 0) \right] \Big|_{p=\hbar x/\sigma^2} = -\sqrt{\frac{\pi\beta}{4\sigma^2}} e^{-i\pi/4} e^{-i\frac{x x_0}{\sigma^2}} \operatorname{sech}\left(\frac{\pi\beta x}{2\sigma^2}\right).$$

c) To maintain the width $\Delta X = \frac{\pi\beta}{2\sqrt{3}}$, we compute ΔX for $\Phi(0, \pi/2\omega)$ and equate it to the uncertainty at $t = 0$:

$$\left. \begin{aligned} \langle X \rangle &= 0 \\ \langle X^2 \rangle &= \frac{\pi\beta}{4\sigma^2} \int_{-\infty}^{\infty} x^2 \operatorname{sech}^2\left(\frac{\pi\beta x}{2\sigma^2}\right) dx = \frac{\sigma^4}{3\beta^2} \end{aligned} \right\} \Delta X = \sqrt{\langle X^2 \rangle} = \frac{\sigma^2}{\sqrt{3}\beta}.$$

Equating it with the uncertainty of the wavefunction at $t = 0$:

$$\frac{\pi\beta}{2\sqrt{3}} = \frac{\sigma^2}{\sqrt{3}\beta} \longrightarrow \beta = \sqrt{\frac{2\sigma^2}{\pi}}.$$

Problem V

We handle the problem by approximating the potential given with its second-order Taylor expansion:

$$V(x) = -V_0 - \frac{V_0}{2b^2} x^2 + O(x^3) + \dots = \frac{1}{2} m\omega^2 x^2.$$

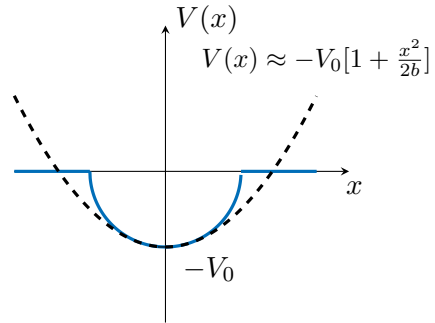
The figure below represents the behavior of this approximation versus the real potential.

Comparing the quadratic term of the expansion with the QHO yields the following frequency:

$$\omega = \sqrt{\frac{V_0}{mb^2}}.$$

The energy levels in the QHO is:

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right), \quad n = 0, 1, 2, \dots$$



So, in this case they will be shifted

$$E'_n = -V_0 + E_n = -V_0 + \hbar\omega \left(n + \frac{1}{2} \right), \quad n = 0, 1, 2, \dots$$

The ground and first excited state energy eigenvalues are:

$$E_0 = -V_0 + \frac{1}{2}\hbar\omega, \quad E_1 = -V_0 + \frac{3}{2}\hbar\omega.$$