OPTI 570 LECTURE Th OCT 30

2-levels systems

Case 1: $H_0 = E_1 |1\rangle\langle 1| + E_2 |2\rangle\langle 2|$

$$H_{\{1,2\}} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$U_{\{1,2\}} = \begin{pmatrix} e^{-iE_1 t/\hbar} & 0 \\ 0 & e^{-iE_2 t/\hbar} \end{pmatrix}$$

$$|\psi(0)\rangle = \cos\frac{\theta}{2}|1\rangle + \sin\frac{\theta}{2} e^{i\varphi}|2\rangle$$

$$|\psi(t)\rangle = \cos\frac{\theta}{2} e^{-iE_1 t/\hbar}|1\rangle + \sin\frac{\theta}{2} e^{i\varphi} e^{-iE_2 t/\hbar}|2\rangle =$$

$$= \text{ph.f.} \cdot \left[\cos\frac{\theta}{2}|1\rangle + \sin\frac{\theta}{2} e^{i\varphi} e^{-i\underbrace{(E_2-E_1)/\hbar}_{\text{Bohr frequency}}\cdot t}|2\rangle\right]$$

$|\psi(0)\rangle = |1\rangle \rightarrow |\psi(t)\rangle = |1\rangle$

$= |2\rangle \rightarrow |2\rangle$

Case 2: Generic time-independent Hamiltonian

$$H = E_1 |1\rangle\langle 1| + E_2 |2\rangle\langle 2|$$

$$W = W_{21} |2\rangle\langle 1| + W_{21}^* \underbrace{|1\rangle\langle 2|}_{\text{connector}}$$

$$W = \frac{1}{2}\hbar\Omega_0 |2\rangle\langle 1| + \frac{1}{2}\hbar\Omega_0^* |1\rangle\langle 2|$$

$\Omega_0 = |\Omega_0| e^{i\varphi}$ $\quad |\Omega_0|$ - strength of W-coupling.

$$H_{\{1,2\}} = \begin{pmatrix} E_1 & \frac{1}{2}\hbar|\Omega_0| e^{-i\varphi} \\ \frac{1}{2}\hbar|\Omega_0| e^{i\varphi} & E_2 \end{pmatrix}$$

ANY time-indep Hamiltonian for 2-level

#1: $\Delta \equiv \dfrac{E_1 - E_2}{\hbar}$ "detuning"

#2: $E_m = \frac{1}{2}(E_1 + E_2)$ "mean energy"

Δ and E_m are real

$$E_1 = E_m + \frac{1}{2}\hbar\Delta$$

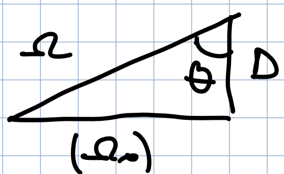
$$E_2 = E_m - \frac{1}{2}\hbar\Delta$$

$$H_{\{1,2\}} = \begin{pmatrix} E_m + \frac{1}{2}\hbar\Delta & \frac{1}{2}\hbar|\Omega_0|e^{-i\varphi} \\ \frac{1}{2}\hbar|\Omega_0|e^{i\varphi} & E_m - \frac{1}{2}\hbar\Delta \end{pmatrix} =$$

$$= \underbrace{E_m \mathbb{1}}_{\text{no phys conseq.}} + \frac{\hbar}{2}\begin{pmatrix} \Delta & |\Omega_0|e^{-i\varphi} \\ |\Omega_0|e^{i\varphi} & -\Delta \end{pmatrix}$$

#3. $\tan\theta = \frac{|\Omega_0|}{\Delta}$ $\quad 0 \le \theta \le \pi$

#4. $\Omega = \sqrt{|\Omega_0|^2 + \Delta^2}$ - real, pos #



$$\Omega = \frac{2}{\hbar}\sqrt{(E_1 - E_2)^2 + |W_{21}|^2}$$

$$H_{\{1,2\}} = E_m \mathbb{1} + \frac{\hbar\Omega}{2}\underline{\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}}$$

$$H = E_m \mathbb{1} + \frac{\hbar\Omega}{2}\vec{\sigma}_u$$

$$\sigma_u = \vec{\sigma}\cdot\hat{u} \qquad \hat{u} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{|\Omega_0|}{\Omega}\,\mathrm{Re}[e^{i\varphi}] \\ \frac{|\Omega_0|}{\Omega}\,\mathrm{Im}[e^{i\varphi}] \\ \Delta/\Omega \end{pmatrix} =$$

$$\hat{u} = \frac{1}{\Omega}\begin{pmatrix} \mathrm{Re}[\Omega_0] \\ \mathrm{Im}[\Omega_0] \\ \Delta \end{pmatrix} = \frac{1}{\hbar\Omega}\begin{pmatrix} 2\,\mathrm{Re}\{W_{21}\} \\ 2\,\mathrm{Im}\{W_{21}\} \\ (E_1 - E_2)/\hbar \end{pmatrix}$$

Energy eigenvalues and eigenstates

$H|E_\pm\rangle = E_\pm |E_\pm\rangle$

$E_+ = E_m + \frac{\hbar\Omega}{2}$ w/ $|E_+\rangle = \cos\frac{\theta}{2}|1\rangle + \sin\frac{\theta}{2}e^{i\varphi}|2\rangle$

$E_- = E_m - \frac{\hbar\Omega}{2}$ w/ $|E_-\rangle = \cos\frac{\theta}{2}|1\rangle - \sin\frac{\theta}{2}e^{-i\varphi}|2\rangle$

Time evolution:

$$H\{E_+, E_-\} = E_m \mathbb{1} + \frac{\hbar\Omega}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$U\{E_+, E_-\} = e^{-iE_m t/\hbar}\cdot\begin{pmatrix}e^{\frac{-i\Omega t}{2}} & 0\\ 0 & e^{\frac{i\Omega t}{2}}\end{pmatrix}$$

$$U_{\{1,2\}} = M^+ U M$$

$$U(t) = \underbrace{e^{-iE_m t/\hbar}}\ \underbrace{e^{\frac{-i\Omega t}{2}\hat{\sigma}_u}} =$$

$$= \cos\left(\frac{\Omega t}{2}\right)\mathbb{1} + i\hat{\sigma}_u \sin\left(\frac{\Omega t}{2}\right) =$$

$$U_{\{1,2\}} = \begin{pmatrix} e^{\frac{i\Omega t}{2}}\sin^2\frac{\theta}{2} + e^{\frac{-i\Omega t}{2}}\cos^2\frac{\theta}{2} & -i\sin\left(\frac{\Omega t}{2}\right)\sin\theta\, e^{-i\varphi} \\ -i\sin\left(\frac{\Omega t}{2}\right)\sin\theta\, e^{i\varphi} & e^{\frac{i\Omega t}{2}}\cos^2\frac{\theta}{2} + e^{-\frac{i\Omega t}{2}}\sin^2\frac{\theta}{2} \end{pmatrix}$$

Define: $P_{j\to k}(t)$ - transition probability

$P_{1\to1}(0) = 1$ $\qquad P_{1\to2}(0) = 0$

$P_{2\to2}(0) = 1$ $\qquad P_{2\to1}(0) = 0$

$P_{1\to2}(t)$

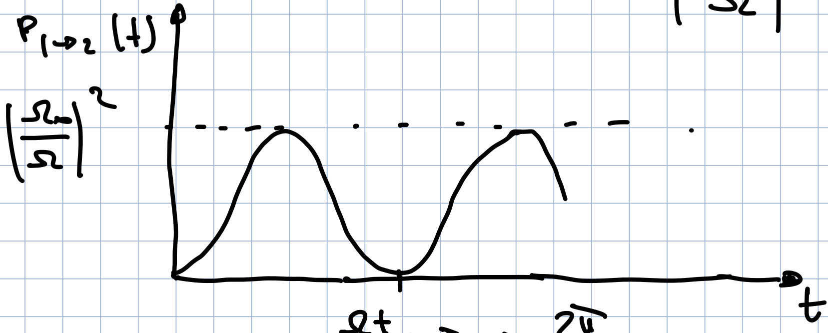
$P_{1\to1}(t)$ $\qquad P_{1\to1}(t) + P_{1\to2}(t) = 1$

$|\psi(0)\rangle = |1\rangle$, what is $P_{1\to2}(t) = ?$

$$|\psi(t)\rangle = \left(\cos^2\frac{\theta}{2}e^{-\frac{i\Omega t}{2}} + \sin^2\frac{\theta}{2}e^{\frac{i\Omega t}{2}}\right)|1\rangle -$$
$$-i\sin\theta\, e^{i\varphi}\sin\left(\frac{\Omega t}{2}\right)|2\rangle$$

$$P_2(t) = |\langle\Psi|2\rangle|^2 = \sin^2\theta \, \sin^2\left(\frac{\Omega t}{2}\right) =$$

$$= \left|\frac{\Omega_0}{\Omega}\right|^2 \sin^2\left(\frac{\Omega t}{2}\right)$$



$\frac{\Omega t}{2} = \pi \to \frac{2\pi}{\Omega}$

" Rabi Oscillations "

Ω_0: " Bare Rabi Frequency "

Ω: " Rabi Frequency "

" Generalized Rabi Frequency "

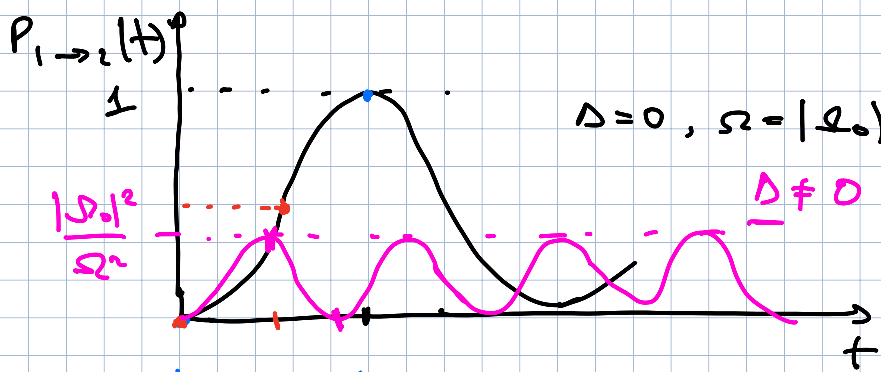
units:

$2\pi \cdot \longrightarrow \frac{\text{rad}}{s}$

Δ: " Detuning " $\qquad \Omega = \sqrt{\Omega_0^2 + \Delta^2}$

<u>Rabi oscillations $\Delta = 0$</u> $\to \tan\theta = \infty, \quad \theta = \frac{\pi}{2}$

$H = W$



$\Delta = 0, \ \Omega = |\Omega_0|$

$\Delta \neq 0 \Rightarrow \boxed{\Omega = \sqrt{|\Omega_0|^2 + \Delta^2}}$

$\pi/|\Omega_0| \to \pi$-pulse

$\frac{\pi}{2|\Omega_0|} \to \frac{\pi}{2}$ - pulse

$$\frac{1}{\sqrt{2}}|E_1\rangle + \frac{1}{\sqrt{2}} \underline{\quad} |E_2\rangle$$

$\Delta = 0$ – " special angle phase "

<u>Off-resonance</u>:

2-level dynamics on the Bloch sphere

$|\psi(t)\rangle = a_1(t)|1\rangle + a_2(t)|2\rangle$

$|\psi(t)\rangle_{\{1,2\}} = \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}$

Bloch Vector:

$\langle \sigma_z \rangle = \langle \psi | \hat{\sigma}_z | \psi \rangle = (a_1^* \; a_2^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = |a_1|^2 - |a_2|^2$

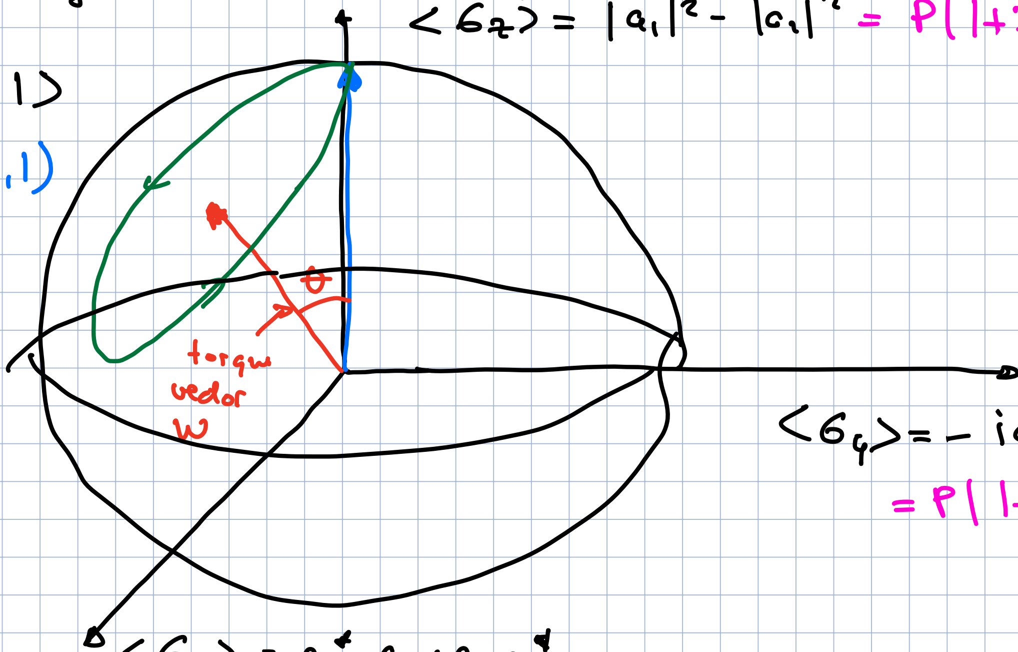
$\langle \sigma_x \rangle = a_1^* a_2 + a_1 a_2^*$

$\langle \sigma_y \rangle = -i a_1^* a_2 + i a_1 a_2^*$

$\langle \sigma_z \rangle = |a_1|^2 - |a_2|^2 = P(|+\rangle_z) - P(|-\rangle_z)$

$|\psi(0)\rangle = |1\rangle$

$\langle \vec{\sigma} \rangle (0) = (0,0,1)$



$\langle \sigma_y \rangle = -i a_1^* a_2 + i a_1 a_2^*$

$= P(|+\rangle_y) - P(|-\rangle_y)$

$\langle \sigma_x \rangle = a_1^* a_2 + a_1 a_2^*$

$= P(|+\rangle_x) - P(|-\rangle_x)$

$\langle \sigma_z \rangle = |a_1|^2 - |a_2|^2 = P(1) - P(2)$

$P(1) = \frac{1}{2}(1 + \langle \sigma_z \rangle)$

$P(2) = \frac{1}{2}(1 - \langle \sigma_z \rangle)$

<u>Bloch sphere</u> – tells you the <u>exact</u> state, not just probabilities

Example: Spin $\frac{1}{2}$ particle in a magnetic field $\gamma < 0$

$$\bar{B} = (0, 0, B_z)$$

$$H = \frac{1}{2}\hbar \omega_z \sigma_z, \quad \omega_z = -\gamma B_z > 0$$

at $t = 0$, $\quad \bar{B} = (0, B_y, B_z) \quad t > 0$

$$W = \frac{1}{2}\hbar \omega_z \hat{\sigma}_y, \quad \omega_y = -\gamma B_y > 0$$

$t > 0$, $\quad \hat{H} = \frac{1}{2}\hbar(\omega_y \hat{\sigma}_y + \omega_z \hat{\sigma}_z)$

$$H\{|\pm\rangle_z\} = \frac{\hbar}{2}\begin{pmatrix} \omega_z & -i\omega_y \\ i\omega_y & -\omega_z \end{pmatrix}$$

$$= E_m \mathbb{1} + \frac{\hbar}{2}\begin{pmatrix} \Delta & |\Omega_0| e^{-i\varphi} \\ |\Omega_0| e^{i\varphi} & -\Delta \end{pmatrix}$$

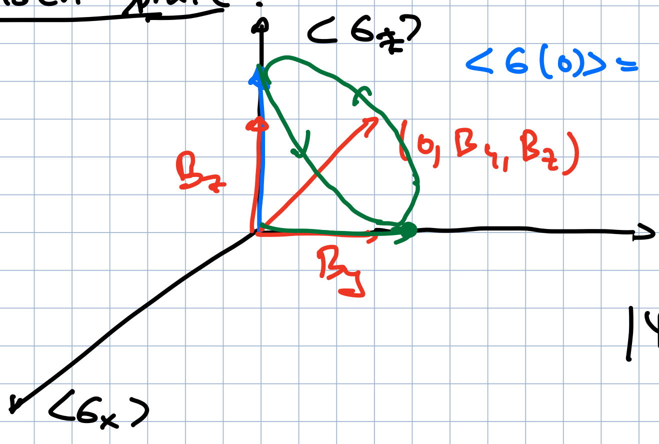
$\Delta = \omega_z$, $|\Omega_0| = \omega_y$, $\varphi = \frac{\pi}{2}$, $E_m = 0$

$\Omega = \sqrt{\omega_y^2 + \omega_z^2} \qquad \tan\theta = \frac{|\Omega_0|}{\Delta} = \frac{\omega_y}{\omega_z} = \frac{B_y}{B_z}$

$$P_{|+\rangle_z} \Rightarrow P_{|-\rangle_z}(t) = \frac{|\Omega_0|^2}{\Omega^2}\sin^2\left(\frac{\Omega t}{2}\right) =$$

$$= \frac{B_y^2}{B_y^2 + B_z^2} \cdot \sin^2\left(\frac{t}{2}\sqrt{\omega_y^2 + \omega_z^2}\right)$$

Bloch sphere:



$$|\psi(\frac{t}{\Omega})\rangle = |+\rangle_y$$