$\boxed{\text{OPTI 570 LECTURE 5 Tu Sep 9}}$

Observables (continued)

If $\hat{A} = \hat{A}^\dagger$: "Hermitian"

- All eigenvalues are real.
- Can create an orthonormal basis from the eigenvectors

: "Observable"

- closure relation :

$$\boxed{\mathbb{1} = \sum_{\text{all } n} \sum_{\text{all } i=1}^{g_n} |\lambda_n^i\rangle\langle\lambda_n^i|}$$

Two observables : $\hat{A}, \hat{B}$, $[\hat{A}, \hat{B}] = 0$

↳ a common set of eigenvectors can be found.

Complete set of commuting observables (CSCO)

- one single set of common eigenvectors

ex : $\cancel{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \xleftrightarrow{\text{commute}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$$\Rightarrow \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}}_{\text{complete set :}}$$

Unitary operators

$\hat{U}$ is unitary IFF $\hat{U}$

$$\hat{U}\hat{U}^{-1} = \mathbb{1} = \hat{U}^{-1}\hat{U}$$

$$\hat{U}\hat{U}^{+} = \mathbb{1} = \hat{U}^{+}\hat{U}$$

$$\hat{U}^{+} = \hat{U}^{-1}$$

- unitary operators - preserve norm
  - preserve relationships b/w kets

$$|\psi\rangle = 2|\varphi\rangle \qquad \hat{U}|\psi\rangle = 2\hat{U}|\varphi\rangle$$

- $\hat{A} = \hat{A}^{+}$ - Hermitian

$$\hat{B} = e^{i\hat{A}} \qquad \hat{B}^{+} = e^{-i\hat{A}}$$

$$\underline{\underline{\hat{B}\hat{B}^{+}}} = e^{i\hat{A}}e^{-i\hat{A}} = e^{0} = \underline{\underline{\mathbb{1}}} \Rightarrow \boxed{B - \text{unitary}}$$

$$= e^{i\hat{A} - i\hat{A}}$$

$$[\hat{A}, \hat{A}] = 0$$

Q: Is $\hat{P}_{\psi} = |\psi\rangle\langle\psi|$ unitary?

check: $\hat{P}_{\psi}\hat{P}^{+} = |\psi\rangle\langle\psi|\left(|\psi\rangle\langle\psi|\right)^{+} =$

$$= |\psi\rangle\underbrace{\langle\psi|\psi\rangle}_{1}\langle\psi| =$$

$$= |\psi\rangle\langle\psi|$$

$$\hat{P}_{\psi}\hat{P}^{+} = \hat{P} \neq \mathbb{1} \Rightarrow \underline{\text{not unitary}}$$

Functions of operators   PSET 3   FG 10

$f(y)$ is a function, analytic, continuous, smooth (cont. diff.)
w/ Taylor expansion around $y = 0$

$$f(y) = \sum_{n=0}^{\infty}\left(\underbrace{\frac{1}{n!}\frac{d^n f(y)}{dy^n}\bigg|_{y=0}}_{c_n}\right) y^n$$

$$= c_0 + c_1 y + c_2 y^2 + c_3 y^3 + \dots$$

$$f(\hat{A}) = c_0 + c_1 \underline{\underline{\hat{A}}} + c_2 \underline{\underline{\hat{A}^2}} + c_3 \underline{\underline{\hat{A}^3}} + \dots$$

ex: $$f(y) = e^y = \sum_{n=1}^{\infty} \frac{y^n}{n!} = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \dots$$

$$e^{\hat{A}} = \mathbb{1} + \hat{A} + \frac{\hat{A}^2}{2} + \frac{\hat{A}^3}{6} + \dots$$

In pset 3: $$e^{iy} = \cos y + i \sin y = 1 + iy + \frac{1}{2}(iy)^2 + \frac{1}{6}(iy^3) + \dots$$

$$e^{i\hat{A}} = \cos \hat{A} + i \sin \hat{A}$$

IF. $\hat{A}|\lambda_n\rangle = \lambda_n|\lambda_n\rangle$, then $F(\hat{A})|\lambda_n\rangle = F(\lambda_n)|\lambda_n\rangle$

ex: $$\underbrace{e^{i\hat{H}t/\hbar}}_{F(\hat{H})}|u_n\rangle = \underbrace{e^{iE_nt/\hbar}}_{F(E_n)}|u_n\rangle$$

time evolution operator

$[\hat{A}, \hat{B}] = 0$ then $[F(\hat{A}), F(\hat{B})] = 0$

ex: $$[\hat{A}, e^{\hat{B}}] = [\hat{A}, \mathbb{1} + \hat{B} + \frac{\hat{B}^2}{2} + \dots] =$$

$$= \underbrace{[\hat{A}, \mathbb{1}]}_{0} + \underbrace{[\hat{A}, \hat{B}]}_{0} + \underbrace{[\hat{A}, \frac{\hat{B}^2}{2}]}_{0} + \dots =$$

$$= 0$$

- if $[\hat{A}, \hat{B}] = \hat{C}$, $[\hat{A}, \hat{C}] = 0$, $[\hat{B}, \hat{C}] = 0$ then

#1 $$[\hat{A}, F(\hat{B})] = [\hat{A}, \hat{B}] \frac{dF(\hat{B})}{d\hat{B}}$$

$$[F(\hat{A}), \hat{B}] = [\hat{A}, \hat{B}] \frac{dF(\hat{A})}{d\hat{A}}$$

$\frac{dF(y)}{dy}$ and then Taylor expand.

#2 $$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} + \hat{B}} \cdot \underline{e^{1/2[\hat{A}, \hat{B}]}}$$ BCH formula ex: $[\hat{x}, \hat{p}] = i\hbar$, $[\hat{x}, i\hbar] = 0$, $[\hat{p}, i\hbar] = 0$

Q: $[\hat{x}, \hat{p}^3] = [\hat{x}, \hat{p}] \frac{d}{d\hat{p}} (\hat{p}^3) = i\hbar \, 3 \cdot \hat{p}^2$

Q: $[e^{-ia\hat{x}/\hbar}, \hat{p}] = [\hat{x}, \hat{p}] \frac{d}{d\hat{x}} (e^{-ia\hat{x}/\hbar}) =$

$= i\cancel{\hbar} (-ia/\cancel{\hbar}) \, e^{-ia\hat{x}/\hbar} =$

$= a \, e^{-ia\hat{x}/\hbar}$

Functions of matrices

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

$F(M)$ · Taylor expansion

$$\neq \begin{pmatrix} F(m_{11}) & F(m_{12}) \\ F(m_{21}) & F(m_{22}) \end{pmatrix}$$

ex: $\cos \begin{pmatrix} \pi & 0 \\ 0 & -\pi \end{pmatrix} \neq \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

However: if M is diagonal $M = \begin{pmatrix} m_{11} & & 0 \\ & m_{22} & \\ 0 & & m_{33} \\ & & & \ddots \end{pmatrix}$

then: $F(M) = \begin{pmatrix} F(m_{11}) & & 0 \\ & F(m_{22}) & \\ 0 & & F(m_{33}) \\ & & & \ddots \end{pmatrix}$

Continuous bases representations: Position and Momentum

Let $\mathcal{E}$ - state space isomorphic w/ a function space $\tilde{\mathcal{F}}_x$

ex: 1D QHO

$\mathcal{E}$		$\mathcal{F}_x$
$\vert\psi\rangle$	$\Leftrightarrow$	$\psi(x)$
$\vert\varphi\rangle$	$\longleftrightarrow$	$\varphi(x)$

$$\boxed{\text{inner product} \quad \langle\varphi|\psi\rangle \quad \longleftrightarrow \quad \int_{-\infty}^{\infty} dx\, \varphi^*(x)\, \psi(x)} \qquad \underline{\text{CT (II-A)}}$$

isomorphic.

$$\langle\psi|\psi\rangle = 1 \quad \text{then} \quad \int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = 1$$

<u>bases</u>: · 2 bases in $\underline{\mathcal{F}_x}$

$\mathcal{F}_x$		$\mathcal{E}$
$\{\delta(x-x')\}$ for all real positions x'	$\Leftrightarrow$	$\{\vert x'\rangle\}$
$\left\{\underbrace{\frac{1}{\sqrt{2\pi\hbar}}\, e^{ip'x/\hbar}}_{\text{plane wave}}\right\}$	$\Leftrightarrow$	$\{\vert p'\rangle\}$

<u>basis representation</u>:

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp'\, \underline{c_{p'}}\, e^{ip'x/\hbar}$$

$$\boxed{\begin{aligned} \langle x''|x'\rangle &= \int_{\text{all } x} dx\, \delta(x-x'')\,\delta(x-x') = \delta(\underbrace{x''-x'}_{0}) \\ \langle p''|p'\rangle &= \delta(p''-p') \end{aligned}}$$

General orthonormality conditions

Closure relations

$$\mathbb{1} = \int_{-\infty}^{+\infty} dx\, |x\rangle\langle x| = \int_{-\infty}^{\infty} dp\, |p\rangle\langle p|$$

ex: $\langle\psi|\psi\rangle = \langle\psi|\hat{\mathbb{1}}|\psi\rangle =$

$$= \langle\psi| \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|)\, |\psi\rangle =$$

$$= \int_{-\infty}^{\infty} \langle\psi|x\rangle\langle x|\psi\rangle\, dx$$

$$= \int_{-\infty}^{\infty} |\psi(x)|^2 dx \quad \checkmark$$

$$\langle x'|\psi\rangle = \int_{-\infty}^{\infty} \delta^*(x-x')\,\psi(x) = \psi(x')$$

$$\boxed{\langle x|\psi\rangle = \psi(x)}$$

$$\langle p|\psi\rangle = \bar{\psi}(p)$$

$|\psi\rangle$ in $\{|p\rangle\}$ representation. $\Rightarrow \bar{\psi}(p)$

Q: $\langle p|x'\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ipx/\hbar}\, \delta(x-x')\, dx =$

$$\langle p|x'\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{-i\,px'/\hbar}$$

$$\bar{\psi}(p) = \langle p|\psi\rangle$$

$$= \langle p|\mathbb{1}_x|\psi\rangle =$$

$$= \langle p| \int_{-\infty}^{+\infty} |x\rangle\langle x|\psi\rangle\, dx =$$

$$= \int_{-\infty}^{+\infty} \langle p|x\rangle\langle x|\psi\rangle dx =$$

$$\boxed{\tilde{\varphi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, \varphi(x)\, dx}$$

$\Leftarrow$ Fourier transform of $\varphi(x)$