Assignment 6

OPTI 570 Quantum Mechanics

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Problem I

a) On the one hand, the action of $\tilde{a}(t)$ is

$$\begin{aligned}\tilde{a}(t)|\varphi_n\rangle &= U^\dagger(t,0)aU(t,0)|\varphi_n\rangle = U^\dagger(t,0)ae^{-i\left(N+\frac{1}{2}\right)\omega t}|\varphi_n\rangle = e^{-i\left(n+\frac{1}{2}\right)\omega t}U^\dagger(t,0)a|\varphi_n\rangle \\ &= \sqrt{n}e^{-i\left(n+\frac{1}{2}\right)\omega t}U^\dagger(t,0)|\varphi_{n-1}\rangle = \sqrt{n}e^{-i\left(n+\frac{1}{2}\right)\omega t}e^{i\left(n-\frac{1}{2}\right)\omega t}|\varphi_{n-1}\rangle = \sqrt{n}e^{-i\omega t}|\varphi_{n-1}\rangle.\end{aligned}$$

Therefore,

$$\tilde{a}(t)|\varphi_n\rangle = \sqrt{n}e^{-i\omega t}|\varphi_{n-1}\rangle = e^{-i\omega t}a|\varphi_n\rangle \longrightarrow \tilde{a}(t) = e^{-i\omega t}a.$$

On the other hand, the action of $\tilde{a}^\dagger(t)$ is

$$\begin{aligned}\tilde{a}^\dagger(t)|\varphi_n\rangle &= U^\dagger(t,0)a^\dagger U(t,0)|\varphi_n\rangle = e^{-i\left(n+\frac{1}{2}\right)\omega t}U^\dagger(t,0)a^\dagger|\varphi_n\rangle = \sqrt{n+1}e^{-i\left(n+\frac{1}{2}\right)\omega t}U^\dagger(t,0)|\varphi_{n+1}\rangle \\ &= \sqrt{n+1}e^{-i\left(n+\frac{1}{2}\right)\omega t}e^{i\left(n+\frac{3}{2}\right)\omega t}|\varphi_{n+1}\rangle.\end{aligned}$$

Consequently,

$$\tilde{a}^\dagger(t)|\varphi_n\rangle = \sqrt{n+1}e^{i\omega t}|\varphi_{n+1}\rangle = e^{i\omega t}a^\dagger|\varphi_n\rangle \longrightarrow \tilde{a}^\dagger(t) = e^{i\omega t}a^\dagger.$$

b) We can compute the operators if we stimulate them with a ket $|\varphi_n\rangle$. We make use of the $a, a^\dagger$ expression for $\tilde{X}$ and $\tilde{P}$.

$$\tilde{X}(t)|\varphi_n\rangle = \frac{\sigma}{\sqrt{2}}\left[U^\dagger a^\dagger U + U^\dagger a U\right]|\varphi_n\rangle.$$

But, we have already computed these operations in the previous part, so we will use it here:

$$\tilde{X}(t)|\varphi_n\rangle = \frac{\sigma}{\sqrt{2}}\left[e^{i\omega t}a^\dagger + e^{-i\omega t}a\right]|\varphi_n\rangle \Longrightarrow \tilde{X}(t) = \frac{\sigma}{\sqrt{2}}\left[e^{i\omega t}a^\dagger + e^{-i\omega t}a\right].$$

We can further simplified if we develop the complex exponential, obtaining:

$$\tilde{X}(t) = X\cos\omega t + \frac{P}{m\omega}\sin\omega t.$$

In the same manner, we have for $\tilde{P}$:

$$\tilde{P}(t)|\varphi_n\rangle = \frac{i\hbar}{\sqrt{2}\sigma}\left[U^\dagger a^\dagger U - U^\dagger a U\right]|\varphi_n\rangle = \frac{i\hbar}{\sqrt{2}\sigma}\left[e^{i\omega t}a^\dagger - e^{-i\omega t}a\right]|\varphi_n\rangle$$

Therefore,

$$\tilde{P}(t) = \frac{i\hbar}{\sqrt{2}\sigma}\left[e^{i\omega t}a^\dagger - e^{-i\omega t}a\right].$$

And,

$$\tilde{P}(t) = P\cos\omega t - m\omega X\sin\omega t.$$

They oscillate in time with the operators of position an momentum acting on kets, the coefficients of each trigonometric function resemble to the classical motion of HO.

c) To show that $U^\dagger|x\rangle$ is an eigenvector of P with a given eigenvalue, we must satisfy the eigenequation of P. Using the $\tilde{P}$ found in the previous part, we have:

$$PU^\dagger(\frac{\pi}{2\omega},0)|x\rangle = U^\dagger UPU^\dagger|x\rangle = U^\dagger\tilde{P}^\dagger(\pi/2\omega)|x\rangle = U^\dagger[P\cos\pi/2 + m\omega X\sin\pi/2]|x\rangle$$
$$PU^\dagger(\frac{\pi}{2\omega},0)|x\rangle = U^\dagger[m\omega X|x\rangle] = m\omega x U^\dagger(\frac{\pi}{2\omega},0)|x\rangle.$$

In consequence, $U^\dagger(\pi/2\omega,0)|x\rangle$ is an eigenvector of P with eigenvalue $m\omega x$. Similarly, we have that $U^\dagger(\pi/2\omega,0)|p\rangle$ is an eigenvector of X with eigenvalue $-p/m\omega$:

$$XU^\dagger(\frac{\pi}{2\omega},0)|p\rangle = -\frac{p}{m\omega}U^\dagger(\frac{\pi}{2\omega})|p\rangle.$$

d) It turnt out that when $t_q = q\pi/2\omega$ evolves, the operators $\tilde{X}$ and $\tilde{P}$ also evolve in a way that in some cases we have purely the action of X while in others the action of P. Assuming that $\psi(x,0)$ is normalized, all the other evolutions will remains normalized as the evolutino operator is unitary. Then,

$$\psi(x,t_q) = \langle x|U(t_q)|\psi(0)\rangle, \quad t_q = \frac{q\pi}{2\omega}.$$

For the $tildeP(t_1)$ we have:

$$\tilde{P}(t_1) = U(t_1)PU^\dagger(t_1) = m\omega X.$$

Multiplying to the left by $\langle x|$ and to the right by $U(t_1)$ yields:

$$(\langle x|U(t_1))P = m\omega x\langle x|U(t_1) = c\langle p = m\omega x|.$$

Then,

$$\psi(x,t_1) = \langle x|U(t_1)|\psi(0)\rangle = C\langle p = m\omega x|\psi(0)\rangle = c\tilde{\psi}(m\omega x),$$

where $\tilde{\psi}$ is te Fourier transform evaluated at $p = m\omega x$. To find c, we impose normalization:

$$|c|^2\int_{-\infty}^{\infty}|\tilde{\psi}(m\omega x)|^2 dx = \frac{|c|^2}{m\omega}\int_{-\infty}^{\infty}|\tilde{\psi(p)}|^2\, dp = 1 \longrightarrow |c| = \sqrt{m\omega}.$$

The phase of c can be found by taking the global factor of the evolution operator:

$$U(t_1) = e^{iHt/\hbar} = e^{-i\omega t/2} e^{-ia^\dagger at} \longrightarrow \angle c|_{t=t_1} = e^{-i\pi/4}.$$

So that:

$$\psi(x, t_1) = e^{-i\pi/4}\sqrt{m\omega}\tilde{\psi}(m\omega x).$$

When we evolve to t_2, there is another 90° that performs a Fourier transform over $\psi(t_1)$. By property of this transformation, two successive applications return the original function with the coordinates inverted:

$$\psi(x, t_1) = \psi(-x, 0).$$

Then, for t_3 we obtain $\psi(t_1)$ with the negative obtained above:

$$\psi(x, t_3) = \sqrt{m\omega}\tilde{\psi}(-m\omega x).$$

And another t_4 will return the original function:

$$\psi(x, t_4) = \psi(-(-x), 0) = \psi(x, 0).$$

This can be done indefinitely.

e) We select $\varphi_n(x)$ the energy eigenstate of the HO. We saw that

$$\psi(x, \frac{\pi}{2\omega}) = e^{-i\pi/4}\sqrt{m\omega}\tilde{\psi}(m\omega x).$$

A stationary state only evolves in its global phase factor, so:

$$\psi(x, \frac{\pi}{2\omega}) = e^{-iE_n(\pi/2\omega)/\hbar}\varphi_n(x) = e^{-i(n+1/2)\pi/2}\varphi_n(x) = e^{-i\pi/4}e^{-in\pi/2}\varphi_n(x).$$

Equating this result with the previous one:

$$\begin{aligned} e^{-i\pi/4}e^{-in\pi/2}\varphi_n(x) &= e^{-i\pi/4}\sqrt{m\omega}\tilde{\psi}(m\omega x) \\ \varphi_n(x) &= i^n\sqrt{m\omega}\tilde{\varphi}_n(m\omega x). \end{aligned}$$

f) i) This corresponds to a plane wave. For $t_1 = \pi/2\omega$ we have its Fourier transform, which is a delta function shifted proportional to t_1. Then, we recover the plane wave with an additinoal exponential term, and so on.
ii) Im not sure about this one, I know its Fourier transform but its difficult to describe it. It has a singularity proportional to ρ.
iii) This is a rect function, so at t_1 we obtain a sinc function and so on.
iv) This is a Gaussian function, so its Fourier transform remains Gaussian.

Problem II

a) We know that $\alpha_0 = |\psi(0)\rangle$ is a coherent state, so $a|\alpha\rangle = \alpha|\alpha\rangle$. Then, replcaing the definitions of X and P in terms of $a, a^\dagger$ and doing algebra we find:

$$\langle\alpha_0|P|\alpha_0\rangle = \frac{i\hbar}{\sqrt{2}\sigma}\langle\alpha_0|(a^\dagger - a)|\alpha_0\rangle = \frac{i\hbar}{\sqrt{2}\sigma}\left[\langle\alpha_0|a^\dagger|\alpha_0\rangle - \langle\alpha_0|a|\alpha_0\rangle\right] = \frac{i\hbar}{\sqrt{2}\sigma}\left[\alpha_0^* - \alpha_0\right] = -\frac{i2\hbar}{\sqrt{2}\sigma}\mathrm{Im}\left(\alpha_0\right).$$

In the same manner,

$$\langle\alpha_0|X|\alpha_0\rangle = \frac{\sigma}{\sqrt{2}}\langle\alpha_0|(a^\dagger + a)|\alpha_0\rangle = \frac{\sigma}{\sqrt{2}}\left[\langle\alpha_0|a^\dagger|\alpha_0\rangle + \langle\alpha_0 a|\alpha_0\rangle\right] = \frac{\sigma}{\sqrt{2}}[\alpha_0^* + \alpha_0] = \frac{2\sigma}{\sqrt{2}}\mathrm{Re}\,(\alpha_0)\,.$$

Then,

$$\alpha_0 = \frac{1}{\sqrt{2}}\left[\frac{\langle X\rangle(0)}{\sigma} + i\frac{\sigma\langle P\rangle(0)}{\hbar}\right].$$

These mean values are the ones we have computed.

b) Due to the momentum kick, we know that

$$\alpha_0 = \frac{ip_o\sigma}{\sqrt{2}\hbar}.$$

Then, doing the same as in the previous part, we have:

$$\langle\alpha_0|H|\alpha_0\rangle = \hbar\omega\langle\alpha_0|(a^\dagger a + 1/2)|\alpha_0\rangle = \hbar\omega\left[|\alpha_0|^2 + \frac{1}{2}\right] = \frac{p_0^2\sigma^2\omega}{2\hbar} + \frac{\hbar\omega}{2}.$$

But, the oscillator length is given by:

$$\sigma = \sqrt{\frac{\hbar}{m\omega}},$$

so

$$\langle\alpha_0|H|\alpha_0\rangle = \frac{p_0^2}{2m} + \frac{\hbar\omega}{2}.$$

c) The eigenvalue is $\alpha(t)$:

$$\alpha(t) = \frac{ip_0\sigma}{\sqrt{2}\hbar}e^{-i\omega t}.$$

This is because the state vector remains a coherent state over time, and under that condition the eigenequation of a gives you α_0 when acting on such a vector.

d) We use the expresion of a in terms of X and P:

$$[a, X] = \left[\frac{1}{\sqrt{2}}\left\{\frac{X}{\sigma} + i\frac{\sigma}{\hbar}P\right\}, X\right] = i\frac{\sigma}{\sqrt{2}\hbar}[P, X] = i\frac{\sigma}{\sqrt{2}\hbar}(-i\hbar) = \frac{\sigma}{\sqrt{2}}.$$

For the other commutator, we can use the commutator of a function property:

$$[a, T(p_0)] = [a, e^{ip_0X/\hbar}] = [a, X]\frac{d(e^{ip_oX/\hbar})}{dX} = \frac{\sigma}{\sqrt{2}}\frac{ip_0}{\hbar}e^{ip_0X/\hbar} = \alpha_0 T(p_0).$$

Using the definition of the commutator and the result above:

$$aT(p_0) - T(p_0)a = \alpha_0 T(p_0) \longrightarrow aT(p_0) = (a + \alpha_0)T(p_0).$$

e) Yes, it remains a coherent state. The only adding is a global phase factor. The eigenvalue at $t = t_1$ is $\alpha(t_1) = \alpha_0 e^{-i\omega t_1}$.

$$a(T(p_0)|\psi(t_1)\rangle) = T(p_0)(a + \alpha_0)|\psi(t_1)\rangle = (\alpha(t_1) + \alpha_0)T(p_0)|\psi(t_1)\rangle.$$

so that

$$\alpha_2 = \alpha(t_1) + \alpha_0 = \alpha_0(e^{-i\omega t_1} + 1).$$

f) We do the same as before, but with the new α.

$$\langle\alpha_2|H|\alpha_2\rangle = \hbar\omega\left[|\alpha_0(1 + e^{-i\omega t_1})|^2 + 1/2\right] = \hbar\omega|\alpha_0|^2(2 + 2\cos\omega t_1) + \frac{\hbar\omega}{2} = \hbar\omega\frac{2p_0^2\sigma^2}{2\hbar^2}(1 + \cos\omega t_1) + \frac{\hbar\omega}{2}$$
$$\langle\alpha_2|H|\alpha_2\rangle = \frac{p_0^2}{m}(1 + \cos\omega t_1) + \frac{\hbar\omega}{2}.$$

g) This means that the mean value $\langle H\rangle$ before and after the second kick must be equal:

$$\langle H\rangle_{\text{before}} = \frac{p_0^2}{2m} + \frac{\hbar\omega}{2}, \quad \text{and} \quad \langle H\rangle_{\text{after}} = \frac{p_0^2}{m}(1 + \cos\omega t_1) + \frac{\hbar\omega}{2}.$$

We set both to be equal and solve for t_1:

$$\begin{aligned}\frac{p_0^2}{m}(1 + \cos\omega t_1) + \frac{\hbar\omega}{2} &= \frac{p_0^2}{2m} + \frac{\hbar\omega}{2}\\ \cos\omega t_1 &= -\frac{1}{2}\\ t_1 &\in \left\{\frac{2\pi}{3\omega}, \frac{4\pi}{3\omega}\right\}.\end{aligned}$$

h) Each time evolution given by $t_1 = 2\pi/3\omega$ rotates a third of the full cycle, so with three we complete a full period. We know that a kick adds α, while a time evolution rotates along the phase-space. The sequence is

$$Q = T(p_0)U(t_1)T(p_0)U(t_1)T(p_0).$$

Problem III

We consider the system as a tensor product of each dimension.

a) The changes in position can be represented in the following way:

$$|\alpha_x\rangle = T(m\delta_v)|\varphi_0\rangle, \quad |\alpha_y\rangle = |\varphi_0\rangle, \quad |\alpha_z\rangle = S(1\ mm)|\varphi_0\rangle.$$

At $t = 0^+$, we have

$$\alpha_x(0) = \frac{1}{\sqrt{2}}\frac{i\sigma m\delta_V}{\hbar}, \quad \alpha_y(0) = 0, \quad \alpha_z(0) = \frac{1}{\sqrt{2}}\frac{1\ mm}{\sigma}.$$

The maximum displacements in z is $A_z = 1\ mm$, whereas for x we know that $p_x(0) = m\omega\delta_v$ so:

$$A_x(0) = \frac{\delta_v}{m\omega} = \frac{\delta_v}{100}.$$

b) In this case, $A_x = A_z$ and

$$\frac{\delta_v}{\omega} = A_z = 1\ mm \longrightarrow \delta_v = (1\ mm)\omega = 100\ mm/s = 0.1\ m/s.$$

c) The probability follows a Poisson distribution. First, we have that

$$\alpha_x = \frac{i}{\sqrt{2}}\sqrt{\frac{m}{\hbar\omega}}\delta_v, \quad \alpha_z = \frac{1\ mm\sigma}{\sqrt{2}}.$$

The square of them corresponds to the Poisson mean and we will take the floor of that value as the most likelikood:

$$|\alpha_x|^2 = \frac{1}{2}\frac{m\delta_v^2}{\hbar\omega} = \frac{1}{2}\frac{(1.5\times10^{-25})(0.1)^2}{(1.054571817\times10^{-34})(100)} = 71118.91.$$

Therefore,

$$|n_x = 71118\rangle, \quad |n_z = 71118\rangle, \quad |n_y = 0\rangle.$$

d) They both are zero:

$$\langle X\rangle_{\text{after}} = \langle P\rangle_{\text{after}} = 0.$$

Problem IV

Part 1.

a) We can treat the problem as a harmonic oscillator. The potential makes a force of the form

$$F_r = -\partial_r V(r) = -\frac{GM_E m}{R_E^3} r = -m\omega^3 r, \quad \omega = \sqrt{\frac{GM_E}{R_E^3}}.$$

Using the information provided, we can obtain the period as:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_E^3}{GM_E}} = 5085.24\ s \equiv 84.75\ min.$$

For the case of the velocity, we know that because we are ignoring other forces the system must conserves the energy and it will be distributed in potential and kinetic energy. So,

$$-\frac{3}{2}\frac{GM_E m}{R_E} = \frac{1}{2}mv_c^2 - \frac{3}{2}\frac{GM_E m}{R_E} \longrightarrow v_c = \sqrt{\frac{GM_E}{R_E}} = 7907.67\ m/s \equiv 28.467\ km/h.$$

Part 2.

b) For the potential $r < R$, we can take Gauss law to give a structure and then froce throught the whole space to be something continuous. However, I will compare it with the form of the graviational force to deduce that:

$$V_C(r) = -\frac{3}{2}\frac{k_e e^2}{R} + \frac{k_e e^2}{2R^3}r^2$$

Now, the force can be derived similarly and equated to the clasical harmonic oscillator:

$$F = -\partial_r V_C(r) = -\frac{k_e e^2}{R^3}r = -m_p\omega^2 r \longrightarrow \omega = \sqrt{\frac{k_e e^2}{m_p R^3}} = 2.945\times10^{15}\ rad/s.$$

c) The oscillator length σ is:

$$\sigma = \sqrt{\frac{\hbar}{m_p \omega}} = 4.588 \times 10^{-12}\ m.$$

d) The step of the energy is $\Delta E = \hbar\omega$, and this value is also hc/λ so equating them and solving for λ:

$$\lambda = \frac{hc}{\hbar\omega} = \frac{2\pi c}{\omega} = 640\ nm.$$

e) Using the general potential we have derived evaluated at $r = R$ yields

$$V_C(R) = -\frac{k_e e^2}{R}.$$

At that position, there is no kinetic energy as it is at rest. Equating it with the total energy of an arbitrary position provides:

$$\frac{1}{2} m_p v^2 + V_C(r) = -\frac{k_e e^2}{R} \longrightarrow v = \sqrt{\frac{k_e e^2}{m_p R^3}(R^2 - r^2)}.$$

At the origin, we have $r = 0$ so

$$v = \sqrt{\frac{k_e e^2}{m_p R}} = 7.4 \times 10^4\ m/s \equiv 2.66 \times 10^5\ km/h.$$

It is about 10 times faster that under gravitational potential.