OPTI 570 LECTURE Tu OCT 28

Last time:

- spin-$\frac{1}{2}$ particle in cons.$^{\text{uniform}}$ magnetic field, along $\hat{z}$, mag. B_0

$\omega_0 = -\gamma B_0$

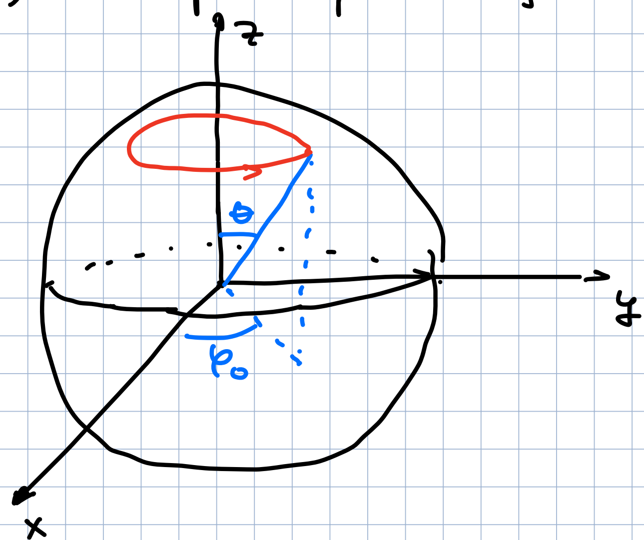
- $|\psi(0)\rangle = |+\rangle_u \qquad \hat{u} = \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix}$

Found

$$|\psi(t)\rangle = e^{-i\omega_0 t/2}\left[\cos\frac{\theta}{2}\,|+\rangle_z + \sin\frac{\theta}{2}\,e^{i\varphi(t)}\,|-\rangle_z\right]$$

$$\varphi(t) = \varphi_0 + \omega_0 t$$

$|\psi(t)\rangle \to$ spin points along direction $\begin{pmatrix} \sin\theta \cos(\varphi+\omega_0 t) \\ \sin\theta \sin(\varphi+\omega_0 t) \\ \cos\theta \end{pmatrix}$



"Bloch Vector": $\langle\psi|\vec{\sigma}|\psi\rangle = (\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle)$

$$\langle\vec{S}\rangle = \frac{\hbar}{2}\langle\vec{\sigma}\rangle$$

$$|\psi(t)\rangle = \cos\frac{\theta}{2}\,|+\rangle_z + \sin\frac{\theta}{2}\,e^{i\varphi(t)}\,|-\rangle_z$$

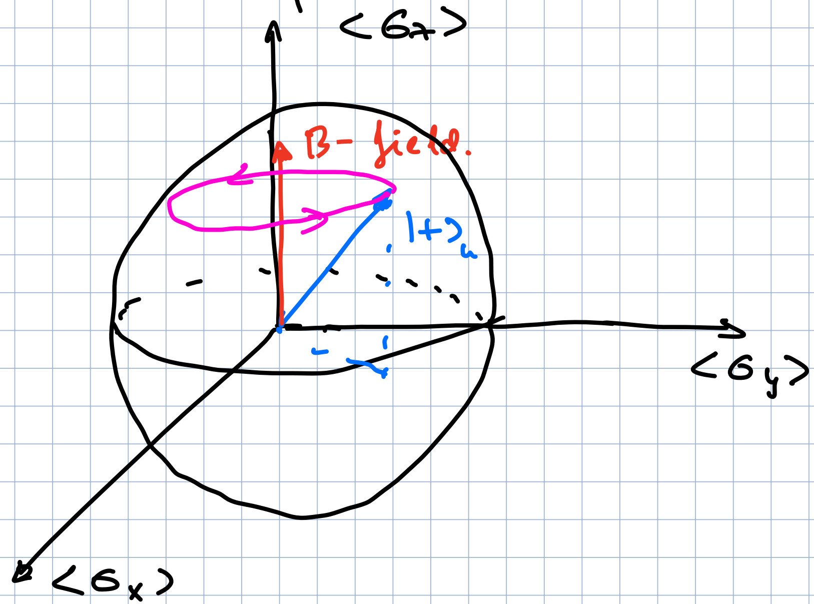
$$\langle\sigma_x\rangle(t) = \left(\cos\frac{\theta}{2} \quad \sin\frac{\theta}{2}\,e^{-i\varphi(t)}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\,e^{i\varphi(t)} \end{pmatrix} =$$

$$= \ldots$$

$$= \sin\theta\,\cos(\varphi(0) + \omega_0 t)$$

$$\langle \sigma_y \rangle (t) = \sin\theta \, \sin(\varphi(0) + \omega_0 t)$$
$$\langle \sigma_z \rangle (t) = \cos\theta$$

$$\langle \vec{\sigma} \rangle (t) = \begin{pmatrix} \sin\theta \, \cos(\varphi(0) + \omega_0 t) \\ \sin\theta \, \sin(\varphi(0) + \omega_0 t) \\ \cos\theta \end{pmatrix}$$



<u>Summary:</u> * 1-1 correspondance b/w any $|\psi\rangle \in \mathcal{E}_{1/2}$ and a specific direction in space $|\psi\rangle = |+\rangle_u$

* also corresp. - to specific $\langle \vec{\sigma} \rangle$

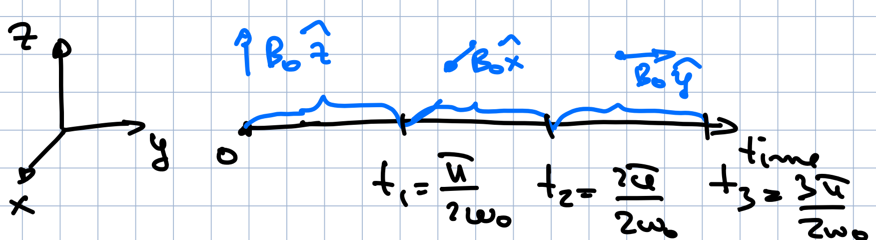
\- to specific $\langle \vec{S} \rangle$

* State spaces w/ spin $\frac{1}{2}$ - 2D space

Example Problem

$|\psi(0)\rangle = |+\rangle_x$ $\qquad \omega_0 = -\gamma B_0$ (const) Let $\gamma < 0$

$\omega_0 > 0$



$|\psi(t_3)\rangle = ?$

Method 1

$$U = e^{-iH\Delta t/\hbar} = e^{-i\frac{1}{2}\hbar\omega\,\sigma_z\cdot\Delta t/\hbar}$$

$$|\psi(t_3)\rangle = e^{-\frac{i}{2}\frac{\pi}{2}\hat{\sigma}_y}\; e^{-\frac{i}{2}\frac{\pi}{2}\hat{\sigma}_x}\; \underbrace{e^{-\frac{i}{2}\frac{\pi}{2}\hat{\sigma}_z}\;|+\rangle_x}_{|\psi(t_1)\rangle}$$

$$|\psi(t_1)\rangle = e^{-\frac{i\pi}{4}\hat{\sigma}_z}\;|+\rangle_x =$$

$$= e^{-\frac{i\pi}{4}\hat{\sigma}_z}\left[\frac{1}{\sqrt{2}}|+\rangle_z + \frac{1}{\sqrt{2}}|-\rangle_z\right] =$$

$$= \frac{1}{\sqrt{2}}\left[e^{-\frac{i\pi}{4}}|+\rangle_z + e^{\frac{i\pi}{4}}|-\rangle_z\right] =$$

$$= e^{-\frac{i\pi}{4}}\left[\frac{1}{\sqrt{2}}|+\rangle_z + \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}}|-\rangle_z\right] =$$

$$= \underbrace{e^{-\frac{i\pi}{4}}}_{\text{neglect}}\;\underbrace{\left[\frac{1}{\sqrt{2}}|+\rangle_z + \frac{i}{\sqrt{2}}|-\rangle_z\right]}_{|+\rangle_y} =$$

$$=$$

$$|\psi(t_2)\rangle = e^{-i\frac{\pi}{4}\hat{\sigma}_x}\left[\frac{1}{\sqrt{2}}|+\rangle_z + \frac{i}{\sqrt{2}}|-\rangle_z\right] =$$

$$= e^{-i\frac{\pi}{4}\hat{\sigma}_x}\left[\frac{1}{2}|+\rangle_x + \frac{1}{2}|-\rangle_x + \frac{i}{2}|+\rangle_x - \frac{i}{2}|-\rangle_x\right] =$$

$$= e^{-\frac{i\pi}{4}\hat{\sigma}_x}\left[\frac{1+i}{2}|+\rangle_x + \frac{1-i}{2}|-\rangle_x\right] =$$

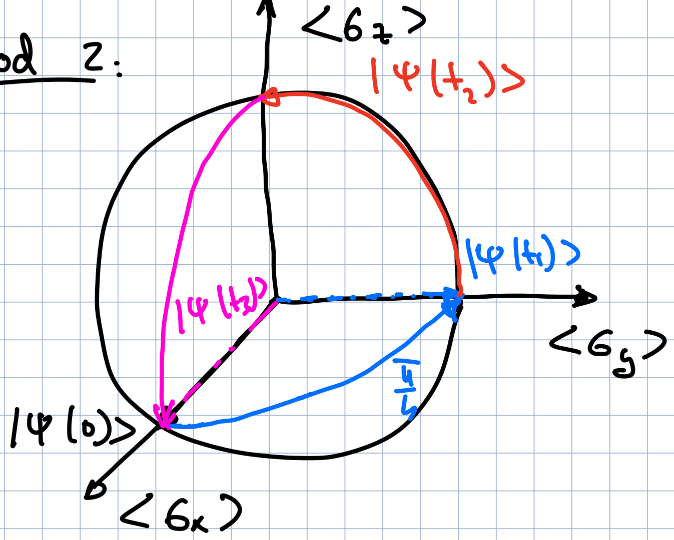
$$= \frac{1}{\sqrt{2}}|+\rangle_x + \frac{1}{\sqrt{2}}|-\rangle_x =$$

$$= |+\rangle_z$$

$$|\psi(t_3)\rangle = e^{-i\frac{\pi}{4}\hat{\sigma}_y}\;|+\rangle_z =$$

$$= e^{-i\frac{\pi}{4}}|+\rangle_x$$

Method 2:



$|\psi(0)\rangle = |+\rangle_x \rightarrow \langle \vec{\sigma} \rangle = (1,0,0)$

$|\psi(t_1)\rangle = |+\rangle_y \rightarrow \langle \vec{\sigma} \rangle = (0,1,0)$

$|\psi(t_2)\rangle = |+\rangle_z \rightarrow \langle \vec{\sigma} \rangle = (0,0,1)$

$|\psi(t_3)\rangle = |+\rangle_x$

$\langle \vec{\sigma} \rangle = (1,0,0)$

2-level system

- state $\mathcal{E}_{2D}$
- $|\psi\rangle \in \mathcal{E}_{2D}$

$$H_0 = E_1 |1\rangle\langle 1| + E_2 |2\rangle\langle 2|$$

$$H_0|1\rangle = E_1|1\rangle$$

$$H_0|2\rangle = E_2|2\rangle$$

Any $|\psi\rangle \in \mathcal{E}_{2D}$

$$|\psi\rangle = \cos\frac{\theta}{2}|1\rangle + \sin\frac{\theta}{2} e^{i\varphi}|2\rangle$$

for $\theta, \varphi \in \mathbb{R}$

Bloch vector:

$$\langle \sigma_x \rangle = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} e^{-i\varphi} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\varphi} \end{pmatrix}$$

$$= \dots$$

$$\langle \vec{\sigma} \rangle = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

Example 2-level system

Given: $H(t)$, $|\psi(t=0)\rangle$

Find: $|\psi(t)\rangle$

Case 1: $H_0 = E_1 |1\rangle\langle 1| + E_2 |2\rangle\langle 2|$

$$H = H_0 + \varepsilon \mathbb{1} =$$
$$= (E_1 + \varepsilon)|1\rangle\langle 1| + (E_2+\varepsilon)|2\rangle\langle 2|$$

$$H_{\{1,2\}} = \begin{pmatrix} E_1 + \varepsilon & 0 \\ 0 & \varepsilon_2 + \varepsilon \end{pmatrix}$$

$$U_{\{1,2\}} = e^{i\varepsilon t/\hbar} \begin{pmatrix} e^{-iE_1 t/\hbar} & 0 \\ 0 & e^{-iE_2 t/\hbar} \end{pmatrix}$$

$$|\psi(0)\rangle = \cos\frac{\theta}{2}|1\rangle + \sin\frac{\theta}{2} e^{i\varphi}|2\rangle$$

$$|\psi(t)\rangle = e^{-i\varepsilon t/\hbar}\left[\cos\frac{\theta}{2} e^{-iE_1 t/\hbar}|1\rangle + \sin\frac{\theta}{2} e^{i\varphi} e^{-iE_2 t/\hbar}|2\rangle\right]$$

$$= e^{-i\varepsilon t/\hbar} e^{-iE_1 t/\hbar}\left[\cos\frac{\theta}{2}|1\rangle + \sin\frac{\theta}{2} e^{i\varphi} \underbrace{e^{-i(E_2-E_1)t/\hbar}}_{\frac{E_2-E_1}{\hbar}\ \text{"Bohr frequency"}}|2\rangle\right]$$

Example: $|\psi(0)\rangle = |1\rangle \Rightarrow \theta = 0$

$|\psi(t)\rangle = |1\rangle$

Case 2: Generic time independent

$$H = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2|$$

$$W = W_{21}|2\rangle\langle 1| + W_{21}^{*} \underbrace{|1\rangle\langle 2|}_{\text{Connectors}}$$

$$W = \frac{1}{2}\hbar\Omega_0 |2\rangle\langle 1| + \frac{1}{2}\hbar\Omega_0^* |1\rangle\langle 2|$$

$\Omega_0 = |\Omega_0| e^{i\varphi}$ $\qquad |\Omega_0|$ - strength of W- coupling

$$H_{\{1,2\}} = \begin{pmatrix} E_1 & \frac{1}{2}\hbar|\Omega_0|e^{-i\varphi} \\ \frac{1}{2}\hbar|\Omega_0|e^{i\varphi} & E_2 \end{pmatrix}$$

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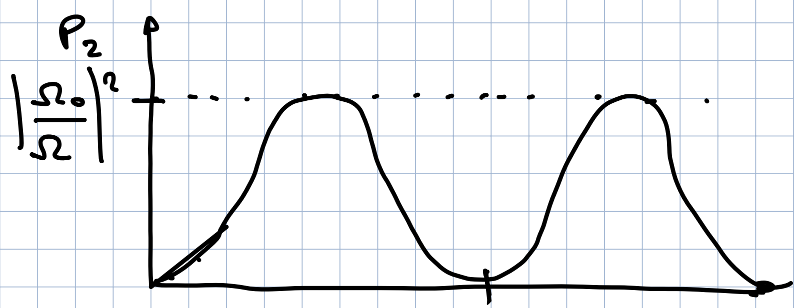

"Rabi oscillation.