Assignment 3

OPTI 570 Quantum Mechanics

University of Arizona

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Part I

Problem 2

a. The operator σ_y is Hermitian as

$$\sigma_y^\dagger = \left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^*\right)^T = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \sigma_y.$$

Its eigenvalues are then, the roots of the characteristic polynomial

$$P(\lambda) = \det(\sigma_y - \lambda I) = \lambda^2 - 1 = 0,$$

from which we have

$$\lambda \in \{-1, 1\} \in \mathbb{R}. \tag{1}$$

The eigenvalues are obtained evaluating each eigenvalue in the eigenvalue problem $(\sigma_y - \lambda)\boldsymbol{v} = \mathbf{0}$. We only list the final results as they were calculated in the assignment 1:

$$\boldsymbol{v} \in \{\boldsymbol{v}_1, \boldsymbol{v}_2\} = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \right\} = \left\{ \frac{1}{\sqrt{2}}(|1\rangle - i|2\rangle), \frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle) \right\}. \tag{2}$$

b. The projectors is

$$P = \sum_{i=1}^{2} |\boldsymbol{v}_i\rangle\langle\boldsymbol{v}_i| = |\boldsymbol{v}_1\rangle\langle\boldsymbol{v}_1| + |\boldsymbol{v}_2\rangle\langle\boldsymbol{v}_2|.$$

It consists of the sum of two outer products, which are

$$|\boldsymbol{v}_1\rangle\langle\boldsymbol{v}_1| = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

$$|\boldsymbol{v}_2\rangle\langle\boldsymbol{v}_2| = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$$

The orthonormality relation states that that multiplication of the two outerproducts will produces a zero matrix:

$$|\boldsymbol{v}_1\rangle\langle\boldsymbol{v}_1|\boldsymbol{v}_2\rangle\langle\boldsymbol{v}_2| = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ -i & 1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\ i & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1-1 & -i+i\\ -i+i & -1+1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = \mathbf{0},$$

which could also be verified by simply computing

$$\langle\boldsymbol{v}_1|\boldsymbol{v}_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1-1\end{bmatrix} = 0.$$

On the other hand, the closure relation states that the sum of each outer product must results in the identity matrix:

$$|\boldsymbol{v}_1\rangle\langle\boldsymbol{v}_1| + |\boldsymbol{v}_2\rangle\langle\boldsymbol{v}_2| = \frac{1}{2}\begin{bmatrix}1 & i\\ -i & 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1 & -i\\ i & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \mathbb{1}.$$

c. Omitted.

Problem 3

a. To verify wether they are normalized, we must compute the norm in each ket and see if it is one. First, we represent them in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis:

$$|\psi_0\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{2}\\ \frac{1}{2}\end{bmatrix}, \quad \text{and} \quad |\psi_1\rangle = \begin{bmatrix}\frac{1}{\sqrt{3}}\\ 0\\ \frac{i}{\sqrt{3}}\end{bmatrix}.$$

Then, the norm is:

$$\langle\psi_0|\psi_0\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{-i}{2} & \frac{1}{2}\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{2}\\ \frac{1}{2}\end{bmatrix} = \frac{1}{2}+\frac{1}{4}+\frac{1}{4} = 1,$$

$$\langle\psi_1|\psi_1\rangle = \begin{bmatrix}\frac{1}{\sqrt{3}} & 0 & \frac{-i}{\sqrt{3}}\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{\sqrt{3}}\\ 0\\ \frac{i}{\sqrt{3}}\end{bmatrix} = \frac{1}{3}+0+\frac{1}{3} = \frac{2}{3} \neq 1.$$

By looking the results, we can say that $|\psi_0\rangle$ is normalized but $|\psi_1\rangle$ does not. If we want to normalize it we must divide the ket by the root of the value we have obtained:

$$|\psi_1'\rangle = \sqrt{\frac{3}{2}}|\psi_1\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\\ \frac{i}{\sqrt{2}}\end{bmatrix}.$$

b. The projections operators onto each state $|\psi_0\rangle$ and $||\psi_1\rangle$ are:

$$\rho_0 = |\psi_0\rangle\langle\psi_0| = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{2} \\ \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{i}{4} \\ \frac{1}{2\sqrt{2}} & \frac{-i}{4} & \frac{1}{4} \end{bmatrix},$$

$$\rho_1' = |\psi_1'\rangle\langle\psi_1'| = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{i}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{-i}{2} \\ 0 & 0 & 0 \\ \frac{i}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Notice we have used $|\psi_1'\rangle$ instead of $|\psi_1\rangle$. At first glance, both projectors look Hermitian. We can confirm it mathematically:

$$\rho_0^\dagger = \left(\begin{bmatrix} \frac{1}{2} & \frac{-i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{i}{4} \\ \frac{1}{2\sqrt{2}} & \frac{-i}{4} & \frac{1}{4} \end{bmatrix}^*\right)^T = \left(\begin{bmatrix} \frac{1}{2} & \frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{-i}{2\sqrt{2}} & \frac{1}{4} & \frac{-i}{4} \\ \frac{1}{2\sqrt{2}} & \frac{i}{4} & \frac{1}{4} \end{bmatrix}\right)^T = \begin{bmatrix} \frac{1}{2} & \frac{-i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{i}{4} \\ \frac{1}{2\sqrt{2}} & \frac{-i}{4} & \frac{1}{4} \end{bmatrix} = \rho_0,$$

$$\rho_1'^\dagger = \left(\begin{bmatrix} \frac{1}{2} & 0 & \frac{-i}{2} \\ 0 & 0 & 0 \\ \frac{i}{2} & 0 & \frac{1}{2} \end{bmatrix}^*\right)^T = \left(\begin{bmatrix} \frac{1}{2} & 0 & \frac{i}{2} \\ 0 & 0 & 0 \\ \frac{-i}{2} & 0 & \frac{1}{2} \end{bmatrix}\right)^T = \begin{bmatrix} \frac{1}{2} & 0 & \frac{-i}{2} \\ 0 & 0 & 0 \\ \frac{i}{2} & 0 & \frac{1}{2} \end{bmatrix} = \rho_1'.$$

Problem 6

We can use Taylor expansion to bring down the matrix:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \mathbb{1} + A + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \cdots \tag{3}$$

Then, the matrix σ_x is elevated to an increasing power. It is then important to know how it behaves:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_x^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times2}, \quad \sigma_x^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_x, \quad \sigma_x^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \sigma_x^2 = I_{2\times2}, \quad \cdots$$

We conclude the following:

$$\sigma_x^n = \begin{cases} \sigma_x, & n \text{ odd} \\ I_{2\times2}, & n \text{ even} \end{cases}.$$

The other part of the term, $i\alpha$, is merely a constant we can take out of the matrix. Now, using the Taylor expression with the results above:

$$\begin{aligned}
e^{i\alpha\sigma_x} &= I_{2\times 2} + (i\alpha)\sigma_x + \frac{(i\alpha)^2}{2!}\sigma_x^2 + \frac{(i\alpha)^3}{3!}\sigma_x^3 + \frac{(i\alpha)^4}{4!}\sigma_x^4 + \cdots \\
&= I_{2\times 2} + i\alpha\sigma_x - \frac{\alpha^2}{2!}I_{2\times 2} - i\frac{\alpha^3}{3!}\sigma_x + \frac{\alpha^4}{4!}I_{2\times 2} + \cdots \\
&= I_{2\times 2}\left[1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \cdots\right] + i\sigma_x\left[\alpha - \frac{\alpha^3}{3!} + \cdots\right] \\
e^{i\alpha\sigma_x} &\overset{(a)}{=} I_{2\times 2}\cos\alpha + i\sigma_x\sin\alpha. \qquad (4)
\end{aligned}$$

In (a) we have used the very well-known series expansion of $\cos\alpha$ and $\sin\alpha$.

Problem 7

The matrix to use is:

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Taking the first four powers of σ_y:

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_y^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}, \quad \sigma_y^3 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \sigma_y, \quad \sigma_y^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}.$$

Therefore,

$$\sigma_y^n = \begin{cases} \sigma_y, & n \text{ odd} \\ I_{2\times 2}, & n \text{ even} \end{cases}.$$

Performing the same expansion as before:

$$\begin{aligned}
e^{i\alpha\sigma_y} &= I_{2\times 2} + (i\alpha)\sigma_y + \frac{(i\alpha)^2}{2!}\sigma_y^2 + \frac{(i\alpha)^3}{3!}\sigma_y^3 + \frac{(i\alpha)^4}{4!}\sigma_y^4 \\
&= I_{2\times 2} + i\alpha\sigma_y - \frac{\alpha^2}{2!}I_{2\times 2} - i\frac{\alpha^3}{3!}\sigma_y + \frac{\alpha^4}{4!}I_{2\times 2} \\
&= I_{2\times 2}\left[1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \cdots\right] + i\sigma_y\left[\alpha - \frac{\alpha^3}{3!} + \cdots\right] \\
e^{i\alpha\sigma_y} &= I_{2\times 2}\cos\alpha + i\sigma_y\sin\alpha. \qquad (5)
\end{aligned}$$

Now, we consider the general case where $\sigma_u = \lambda\sigma_x + \mu\sigma_y$:

$$e^{i\alpha\sigma_u} = e^{i\alpha(\lambda\sigma_x + \mu\sigma_y)} = \sum_{n=1}^{\infty} \frac{(i\alpha)^n(\lambda\sigma_x + i\alpha\mu\sigma_y)^n}{n!}.$$

We will verify if σ_x and σ_y commute in order to simplify the above expresion:

$$[\sigma_x, \sigma_y] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

They dont commute, so we cannot simplfy further the series expansion. However, we now know that $\sigma_x\sigma_y + \sigma_y\sigma_x = 0$. We have to develop the $(\lambda\sigma_x + \mu\sigma_y)^n$ to derive something. First, we compute the first four terms:

$$\begin{aligned}
\sigma_u^1 = (\lambda\sigma_x + \mu\sigma_y) &= \sigma_u \\
\sigma_u^2 = (\overbrace{\lambda^2 + \mu^2}^{1})I_{2\times2} + \lambda\mu(\overbrace{\sigma_x\sigma_y + \sigma_y\sigma_x}^{0}) &= I_{2\times2} \\
\sigma_u^3 = \sigma_u^2\sigma_u = I_{2\times2}\sigma_u &= \sigma_u \\
\sigma_u^4 = \sigma_u^2\sigma_u^2 &= I_{2\times2} \\
&\vdots
\end{aligned}$$

We have then,

$$\sigma_u^n = \begin{cases} \sigma_u, & n \text{ odd} \\ I_{2\times2}, & n \text{ even} \end{cases}.$$

Therefore,

$$\begin{aligned}
e^{i\alpha\sigma_u} &= I_{2\times2} + (i\alpha)\sigma_u + \frac{(i\alpha)^2}{2!}\sigma_u^2 + \frac{(i\alpha)^3}{3!}\sigma_u^3 + \frac{(i\alpha)^4}{4!}\sigma_u^4 + \cdots \\
&= I_{2\times2} + i\alpha\sigma_u - \frac{\alpha^2}{2!}I_{2\times2} - i\frac{\alpha^3}{3!}\sigma_u + \frac{\alpha^4}{4!}I_{2\times2} + \cdots \\
&= I_{2\times2}\left[1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \cdots\right] + i\sigma_u\left[\alpha - \frac{\alpha^3}{3!} + \cdots\right] \\
e^{i\alpha\sigma_u} &= I_{2\times2}\cos\alpha + i\sigma_u\sin\alpha.
\end{aligned}$$

Obtaining a similar relation as before:

$$e^{i\alpha\sigma_u} = e^{i\alpha(\lambda\sigma_x + \mu\sigma_y)} = I_{2\times2}\cos\alpha + i\sigma_u\sin\alpha, \quad \sigma_u = \lambda\sigma_x + \mu\sigma_y,\ \lambda^2 + \mu^2 = 1. \tag{6}$$

- The others exponential required can used the formula we have just obtained. For $e^{i2\sigma_x}$ $(\alpha = 2)$ and $(e^{i\sigma_x})^2$ $(\alpha = 1)$ we have:

$$e^{i2\sigma_x} = I_{2\times2}\cos 2 + i\sigma_x\sin 2 \quad \text{versus} \quad \begin{aligned}
(e^{i\sigma_x})^2 &= (I_{2\times2}\cos 1 + i\sigma_x\sin 1)(I_{2\times2}\cos 1 + i\sigma_x\sin 1) \\
&= [\cos^2(1)I_{2\times2} - \sin^2(1)\sigma_x^2] + i[2\cos(1)\sin(1)\sigma_x] \\
&= [\cos^2(1) - \sin^2(1)]I_{2\times2} + i[2\cos(1)\sin(1)]\sigma_x \\
(e^{i\sigma_x})^2 &\overset{(a)}{=} [\cos 2]I_{2\times2} + i[\sin 2]\sigma_x,
\end{aligned}$$

where in (a) we have used the following trigonometric identities:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \text{and} \quad \sin 2\theta = 2\cos\theta\sin\theta.$$

We conclude that:

$$e^{2i\sigma_x} = (e^{i\sigma_x})^2 = I_{2\times2}\cos 2 + i\sigma_x\sin 2.$$

This is expected, as the same operator can commute with itself.

- On the other hand, the next test involve both σ_x and σ_y and because we know they dont commute, the terms $e^{i(\sigma_x+\sigma_y)}$ and $e^{i\sigma_x}e^{i\sigma_y}$ will be different. First, recall that $\lambda^2+\mu^2=1$ and in this case $\lambda=\mu=1$. We need to normalize σ_u:

$$\sigma_u' = \frac{\sigma_u}{\sqrt{2}} = \frac{1}{\sqrt{2}}\sigma_x + \frac{1}{\sqrt{2}}\sigma_y \Longrightarrow \lambda = \mu = \frac{1}{\sqrt{2}}.$$

Then, the first exponential is ($\alpha = 1, \lambda = \mu = 1\sqrt{2}$):

$$e^{i(\frac{1}{\sqrt{2}}\sigma_x + \frac{1}{\sqrt{2}}\sigma_y)} = I_{2\times 2}\cos 1 + i\frac{\sigma_x+\sigma_y}{\sqrt{2}}\sin 1$$

versus ($\alpha = 1/\sqrt{2}$):

$$\begin{aligned} e^{i\frac{1}{\sqrt{2}}\sigma_x}e^{i\frac{1}{\sqrt{2}}\sigma_y} &= [I_{2\times 2}\cos(1/\sqrt{2}) + i\sigma_x\sin(1/\sqrt{2})][I_{2\times 2}\cos(1/\sqrt{2}) + i\sigma_y\sin(1/\sqrt{2})] \\ &= \left[\cos^2(1/\sqrt{2})I_{2\times 2} - \sin^2(1/\sqrt{2})\sigma_x\sigma_y\right] + i\left[\cos(1/\sqrt{2})\sin(1/\sqrt{2})(\sigma_x+\sigma_y)\right]. \end{aligned}$$

We see that both are different.

Problem 9

a. For a Hamiltonian operator H with $H|\psi_n\rangle = E_n|\psi_n\rangle$ and an arbitrary operator A, we have:

$$\begin{aligned} \langle\varphi_n|[A,H]|\varphi_n\rangle &= \langle\varphi_n|(AH-HA)|\varphi_n\rangle \\ &= \langle\varphi_n|AH|\varphi_n\rangle - \langle\varphi_n|HA|\varphi_n\rangle \\ &= E_n\langle\varphi_n|A|\varphi_n\rangle - E_n^*\langle\varphi_n|A|\varphi_n\rangle \\ &= E_n\langle\varphi_n|A|\varphi_n\rangle - E_n\langle\varphi_n|A|\varphi_n\rangle \quad (E_n = E_n^*) \\ &= E_n\left[\langle\varphi_n|A|\varphi_n\rangle - \langle\varphi_n|A|\varphi_n\rangle\right] \\ \langle\varphi_n|[A,H]|\varphi_n\rangle &= 0, \quad \forall A. \end{aligned} \tag{7}$$

b. α. Given that H is

$$H = \frac{1}{2m}P^2 + V(X),$$

where $V(X)$ can be thought of as a function of an operator and P, by comparison with the standard Hamiltonian, is $P = -i\hbar\partial/\partial x$.
We will use algebra of commutator, function of an operator and assume that X and P are the position and momentum operators, whose commutation is $[X,P] = i\hbar$.

- The commutator between the Hamiltonian and momentum is:

$$\begin{aligned} [H,P] &= \left[\frac{P^2}{2m} + V(X), P\right] \\ &= \left[\frac{P^2}{2m}, P\right] + [V(X),P] \\ &= \frac{P^3}{2m} - \frac{P^3}{2m} + V(X)P - PV(X) \\ &= V(X)P - PV(X) \\ [H,P] &= [V(X),P]. \end{aligned}$$

The last equation can be further developed if we note that

$$[X, P] = i\hbar, \quad [X, i\hbar] = 0, \quad [P, i\hbar] = 0.$$

Then, we apply the property $[F(A), B] = [A, B]F'(A)$:

$$[H, P] = i\hbar\frac{dV(X)}{dX}. \tag{8}$$

- The commutator of the Hamiltonian and the position is

$$\begin{aligned}[H, X] &= \left[\frac{P^2}{2m} + V(X), X\right] \\ &= \left[\frac{P^2}{2m}, X\right] + [V(X), X] \\ [H, X] &\overset{(a)}{=} \left[\frac{P^2}{2m}, X\right] + 0.\end{aligned}$$

In (a), any function of X commute with X. We can demonstrate it by expressing $V(X)$ as a Taylor expansion and computing the commutator with X:

$$V(X) = \sum_{n=0}^{\infty} a_n X^n \Longrightarrow \left[\sum_{n=0}^{\infty} a_n X^n, X\right] = \sum_{n=0}^{\infty} a_n [X^n, X],$$

but $[X^n, X]$ is:

$$[X^n, X] = X^n X - X X^n = X^{n+1} - X^{n+1} = 0 \Longrightarrow [V(X), X] = 0.$$

Continuing the problem,

$$\begin{aligned}[H, X] &= \frac{1}{2m}[P^2 X - X P^2] \\ &= \frac{1}{2m}[P(PX) - (XP)P] \\ &\overset{(b)}{=} \frac{1}{2m}[P(XP - i\hbar) - (i\hbar + PX)P] \qquad ([X, P] = i\hbar) \\ &= \frac{1}{2m}[PXP - i\hbar P - i\hbar P - PXP] \\ &= \frac{1}{2m}[-i2\hbar P] \\ [H, X] &= -\frac{i\hbar}{m}P.\end{aligned} \tag{9}$$

Where in (b) we have used the canonical commutator relation between position and momentum operators.

- The commutator of the Haimltonian and the product of the position and momentum is

$$\begin{aligned}[H, XP] &= [H, X]P + X[H, P] \\ &= \left(-\frac{i\hbar}{m}P\right)P + X\left(i\hbar\frac{dV(X)}{dx}\right) \\ [H, XP] &= -\frac{i\hbar P^2}{m} + i\hbar X\frac{dV(X)}{dX}.\end{aligned} \tag{10}$$

We have used the result of previous commutators in the process.

β. To prove that $\langle\varphi_n|P|\varphi_n\rangle = 0$, we use equation (7) and (9):

$$\langle\varphi_n|[A,H]|\varphi_n\rangle = \langle\varphi_n|[H,A]|\varphi_n\rangle = 0, \quad \text{and} \quad [H,X] = -\frac{i\hbar}{m}P.$$

If we solve for P in the second equation and perform $\langle\varphi_n|\cdot|\varphi_n\rangle$ over P we have:

$$\begin{aligned} P &= -\frac{m}{i\hbar}[H,X] \;/\langle\varphi_n|\cdot|\varphi_n\rangle \\ \langle\varphi_n|P|\varphi_n\rangle &= -\frac{m}{i\hbar}\langle\varphi_n|[H,X]|\varphi_n\rangle \\ \langle\varphi_n|P|\varphi_n\rangle &= \frac{m}{i\hbar}\langle\varphi_n|[X,H]|\varphi_n\rangle. \end{aligned}$$

Since we have derivated (7) for an arbitrary operator, we can do $A = X$ and set directly zero:

$$\langle\varphi_n|P|\varphi_n\rangle = 0. \tag{11}$$

γ. To establish a relation between $\langle\varphi_n|\frac{P^2}{2m}|\varphi_n\rangle$ and $\langle\varphi_n|X\frac{dV}{dX}|\varphi_n\rangle$, we will use equation (7), and (10). We employ equation (7)

$$\langle\varphi_n|[H,A]|\varphi_n\rangle = 0, \quad \forall A,$$

to define $A = XP$ and use equation (10), which we know is

$$[H,XP] = -\frac{i\hbar P^2}{m} + i\hbar X\frac{dV(X)}{dX}.$$

Then,

$$\begin{aligned} \langle\varphi_n|[H,XP]|\varphi_n\rangle &= 0 \\ \langle\varphi_n| -\frac{i\hbar P^2}{m} + i\hbar X\frac{dV(X)}{dX}|\varphi_n\rangle &= \\ -i2\hbar\langle\varphi_n|\frac{P^2}{2m}|\varphi_n\rangle + i\hbar\langle\varphi_n|X\frac{dV(X)}{dX}|\varphi_n\rangle &= 0 \\ \langle\varphi_n|X\frac{dV(X)}{dX}|\varphi_n\rangle &= 2\langle\varphi_n|\frac{P^2}{2m}|\varphi_n\rangle = 2E_n. \end{aligned}$$

Consequently, both quantities are related by a constant term of 2.
To now relate $\langle\varphi_n|V(X) = V_oX^s|\varphi_n\rangle$ with $\langle\varphi_n|\frac{P^2}{2m}|\varphi_n\rangle$, we use the above result and evaluate $V(X) = V_oX^s$ which after differentiation becomes $V'(X) = V_osX^{s-1}$. Then,

$$\begin{aligned} 2\langle\varphi_n|\frac{P^2}{2m}|\varphi_n\rangle &= \langle\varphi_n|X\frac{d(V_oX^s)}{dX}|\varphi_n\rangle \\ &= \langle\varphi_n|V_osXX^{s-1}|\varphi_n\rangle \\ &= s\langle\varphi_n|V_oX^s|\varphi_n\rangle \\ 2\langle\varphi_n|\frac{P^2}{2m}|\varphi_n\rangle &= s\langle\varphi_n|V(X)|\varphi_n\rangle. \end{aligned}$$

Thus, the relation between both quantities is:

$$\langle\varphi_n|V(X)|\varphi_n\rangle = 2s\langle\varphi_n|\frac{P^2}{2m}|\varphi_n\rangle = 2sE_n, \quad s = 2,4,6,\cdots, \; V_0 > 0. \tag{12}$$

Part II

II-1

The ket is already defined, we need to find the constant c to make it orthonormal: $\langle\psi|\psi\rangle = 1$. First, we interpret the coefficients as the projections of ψ onto the $\{|u_n\rangle\}$ basis and use it to construct a column matrix:

$$|\psi\rangle_{\{|u_n\rangle\}} = \begin{bmatrix} \langle u_1|\psi\rangle \\ \langle u_2|\psi\rangle \\ \langle u_3|\psi\rangle \\ \langle u_4|\psi\rangle \end{bmatrix} = c \begin{bmatrix} 2 \\ -i\sqrt{3} \\ -3e^{i\theta} \\ 3 \end{bmatrix}.$$

We now compute the scalar product as a matrix multiplication:

$$\langle\psi|\psi\rangle = c^2 \begin{bmatrix} 2 & i\sqrt{3} & -3e^{-i\theta} & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -i\sqrt{3} \\ -3e^{i\theta} \\ 3 \end{bmatrix} = c^2(4+3+9+9) = 25c^2 = 1 \longrightarrow c = 1/5.$$

Therefore, the ket in matrix form is:

$$|\psi\rangle_{\{|u_n\rangle\}} = \frac{1}{5} \begin{bmatrix} 2 \\ -i\sqrt{3} \\ -3e^{i\theta} \\ 3 \end{bmatrix}, \quad \forall\theta \in \mathbb{R}. \tag{13}$$

II-2

(a) For $|u_2\rangle$, we arrange the representation as a column vector:

$$|u_2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(b) For $\langle u_3|$, we arrange the representation os a row vector:

$$\langle u_3| = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

(c) The term $|u_2\rangle\langle u_3|$ is an operator that project the input vector $|\psi\rangle$ onto $|u_3\rangle$ and then assign it to $|u_2\rangle$. The representation will be the outer product (matrix multiplication) of both elements, which will yield a matrix:

$$|u_2\rangle\langle u_3| = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This means that it will only gives you non-zero vectors $|u_2\rangle$ for collinear vectors of $|u_3\rangle$.

(d) The projector onto $|u_2\rangle$ can be described as $P_{u_2} = |u_2\rangle\langle u_2|$. The matrix representation is then the product of its column vector times the adjoint of the column vector (row vector of complex conjugates elements):

$$P_{u_2} = |u_2\rangle\langle u_2| = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(e) This expression projects the vector $|\psi\rangle$ onto $|u_n\rangle$ and assign it to $|u_m\rangle$. the sum on i project the input to the nth-element of the basis $\{|u_1\rangle, |u_2\rangle, |u_3\rangle, |u_4\rangle\}$. Then, the other sum assign it to the mth-element of the same basis. Rearranging the summations:

$$\sum_{m=1}^{m=4} \sum_{n=1}^{m=4} |u_m\rangle\langle u_n| = \left[\sum_{m=1}^{m=4} |u_m\rangle\right] \cdot \left[\sum_{n=1}^{m=4} \langle u_n|\right].$$

Each state $|u_m\rangle$ provides a one non-zero element at the mth-position of the column vector. However, summing them all produces a vector full of ones. The same applies to the bra $\langle u_n|$. Consequently, the matrix representation is:

$$\left[\sum_{m=1}^{m=4} |u_m\rangle\right] \cdot \left[\sum_{n=1}^{m=4} \langle u_n|\right] = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

II-3

We know the action of the operator Q on each ket of the basis. The result is a different element within the same basis. The basis is orthonormal, meaning that they are all linearly independent each other. Because each element $|u_n\rangle$ has one single non-zero element in its matrix representation, every column of Q will have a single non-zero element as well.

Given $Q|u_i\rangle = \alpha|u_j\rangle$, $\alpha \in \mathbb{C}$, the problem then reduces to find the constant of proportionality α_{ji} in the matrix to provide the equality. Doing this with the four equations and constructing the matrix representation results in the following operator:

$$Q = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix},$$

which is a Hermitian operator: $Q = Q^\dagger$.