PRACTICE MIDTERM 3 SOLUTIONS

1 $\psi(r) = f(r) \cdot \left(\frac{z}{r}\right) \cdot \left(\frac{x}{r} + \frac{iy}{r}\right)$

$= f(r) \cos\theta \left(\sin\theta \cos\varphi + i \sin\theta \sin\varphi\right)$

$= f(r) \cos\theta \sin\theta \, e^{i\varphi} \propto Y_2^1 \quad l = 2, \; m_l = 1$

$\Rightarrow$ eigenvalue of L^2 is $l \cdot (l+1)\hbar^2 = 6\hbar^2$

eigenvalue of L_z is $m_l \hbar = \hbar$

2 $F \in \{5, 4, 3, 2\}$ $\quad \bar{F} = \bar{J} + \bar{I}$

$n^2 P_{3/2} \Rightarrow J = \frac{3}{2}$ $\quad \max(F) = J + I = 5 \Rightarrow I = \frac{7}{2}$

$\min(F) = |\bar{J} - \bar{I}| = \frac{7}{2} - \frac{3}{2} = 2$ ✓

3 $J_A = 2, \; J_B = \frac{3}{2}$

a. $|J_A = 2, J_B = \frac{3}{2}, m_A = 1, m_B = -\frac{1}{2}\rangle = \sqrt{\frac{12}{35}}\, |J = \frac{7}{2}, m_J = \frac{1}{2}\rangle +$

$+ \sqrt{\frac{5}{14}}\, |J = \frac{5}{2}, m_J = \frac{1}{2}\rangle - \sqrt{\frac{3}{10}}\, |J = \frac{1}{2}, m_J = \frac{1}{2}\rangle$

b. $|J_T = \frac{3}{2}, m_T = \frac{3}{2}\rangle = \sqrt{\frac{2}{5}}\, |J_A = 2, m_{J_A} = 2\rangle |J_B = \frac{3}{2}, m_{J_B} = -\frac{1}{2}\rangle -$

$- \sqrt{\frac{2}{5}}\, |J_A = 2, m_{J_A} = +1\rangle |J_B = \frac{3}{2}, m_{J_B} = \frac{1}{2}\rangle + \sqrt{\frac{1}{5}}\, |J_A = 2, m_{J_A} = 0\rangle |J_B = \frac{3}{2}, m_{J_B} = \frac{3}{2}\rangle$

4 $s = \frac{1}{2} \quad \gamma < 0, \; \bar{B}(t) = B_0 \exp(-t/\tau_0)\, \hat{x}, \; |\psi(0)\rangle = |+\rangle_z$

a. (i) $\hat{\mathbb{1}}$: no evolution

(ii) should precess fast first, then slow down as $|\bar{B}|$ decreases

(iii) it will depend on B_0 and τ_0

b. $H = -\vec{\mu} \cdot \vec{B} = -\gamma\, \bar{S} \cdot \bar{B} = -\gamma \cdot \frac{\hbar}{2} \cdot B_0\, e^{-t/\tau_0} \sigma_x = \frac{\hbar\omega}{2} e^{-t/\tau_0} \sigma_x$

w/ $\omega = -\gamma B_0 > 0$

$U(t) = e^{-\frac{i}{\hbar}\int_0^t H(t')\,dt'} = e^{-i\frac{\omega}{2}\sigma_x \int_0^t e^{-t'/\tau_0}\,dt'} =$

$$= e^{-\frac{i\omega\sigma_x}{2}(-\tau_0)e^{-t'/\tau_0}\Big|_0^t} =$$

$$= e^{\frac{i\omega\sigma_x\tau_0}{2}\left(e^{-t/\tau_0}-1\right)}$$

(i) $t=0 \qquad U(0) = \mathbb{1}$ ✓

(ii) fast first, then slower ✓

(iii) $U(t\to\infty) = e^{\frac{i\omega\sigma_x\tau_0}{2}}$ - depends on parameters ✓

c. $|+\rangle_z = \frac{1}{\sqrt{2}}\left(|+\rangle_x + |-\rangle_x\right)$

$$U(t)|+\rangle_z = \frac{1}{\sqrt{2}} e^{-\frac{i\omega\tau_0}{2}\left(e^{-t/\tau_0}-1\right)\sigma_x}\left(|+\rangle_x + |-\rangle_x\right) =$$

$$= \frac{1}{\sqrt{2}}\left[e^{\frac{i\omega\tau_0}{2}\left(e^{-t/\tau_0}-1\right)}|+\rangle_x + e^{-\frac{i\omega\tau_0}{2}\left(e^{-t/\tau_0}-1\right)}|-\rangle_x\right]$$

The probability of $|-\rangle_z$ is then:

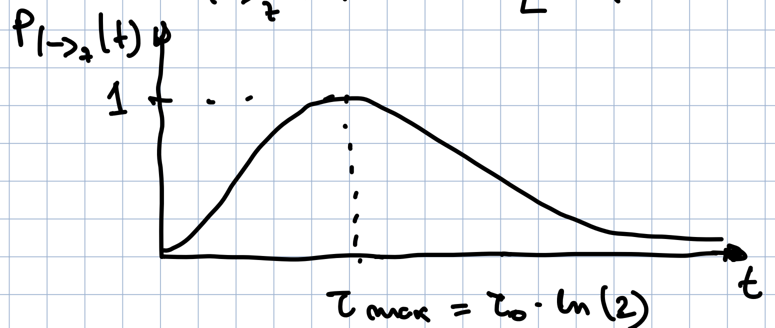
$$P_{|-\rangle_z} = \left|\langle -|_z U(t)|+\rangle_z\right|^2 = \left\{\frac{1}{2}\left[e^{\frac{i\omega\tau_0}{2}\left(e^{-t/\tau_0}-1\right)} + e^{-\frac{i\omega\tau_0}{2}\left(e^{-t/\tau_0}-1\right)}\right]\right\}^2 =$$

$$= \left\{-i\sin\left[\frac{1}{2}\omega\tau_0\left(1-e^{-t/\tau_0}\right)\right]\right\}^2 =$$

$$= \sin^2\left[\frac{1}{2}\omega\tau_0\left(1-e^{-t/\tau_0}\right)\right]$$

d. $\tau_0 = -2\pi/(\gamma B_0) = \frac{2\pi}{\omega}$

$$P_{|-\rangle_z}(t) = \sin^2\left[\pi\left(1-e^{-t/\tau_0}\right)\right]$$



e. $P_{|-\rangle_z}(t_{max}) = 1 \Rightarrow \pi\left(1-e^{-t_{max}/\tau_0}\right) = \frac{\pi}{2} \qquad e^{-t_{max}/\tau_0} = \frac{1}{2}$

$$\underline{t_{max} = \tau_0 \cdot \ln(2)}$$

$\boxed{5}$ $R_n^{(j)}(\Phi) = e^{-i\Phi \hat{J}_n/\hbar}$

a. start w/ $|+\rangle_x$ and precess by $\frac{\pi}{2}$ around $\hat{z}$ using right-hand rule. Look at Bloch sphere, I end up at $\underline{|+\rangle_y}$

b. $|\psi\rangle = R_z^{(1/2)}(\pi/2)|+\rangle_x = e^{-i\frac{\pi}{4}\sigma_z}|+\rangle_x =$

$$= e^{-i\frac{\pi}{4}\sigma_z}\frac{1}{\sqrt{2}}\left(|+\rangle_z + |-\rangle_z\right) =$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\frac{\pi}{4}}|+\rangle_z + e^{i\frac{\pi}{4}}|-\rangle_z\right) =$$

$$= e^{-i\frac{\pi}{4}}\frac{1}{\sqrt{2}}\left(|+\rangle_z + i|-\rangle_z\right) =$$

$= |+\rangle_y$ matches intuition above ✓

c. $|+\rangle_y$

d. $|-\rangle_z$

e. $\frac{1}{\sqrt{2}}\left(|+\rangle_y + |+\rangle_z\right)$

f. $|-\rangle_z$

g. $|+\rangle_x$

h. $|\psi\rangle = R_z^{(1)}\left(\frac{\pi}{2}\right)|j=1, m_x=1\rangle = e^{-i\frac{\pi}{2}J_z/\hbar}|j=1, m_x=1\rangle$

in z basis: $|j=1, m_x=1\rangle = \frac{1}{2}|j=1, m_z=1\rangle + \frac{1}{\sqrt{2}}|j=1, m_z=0\rangle + \frac{1}{2}|j=1, m_z=-1\rangle$

$$|\psi\rangle = \frac{1}{2}e^{-i\frac{\pi}{2}}|m_z=1\rangle + \frac{1}{\sqrt{2}}e^{0}|m_z=0\rangle + \frac{1}{2}e^{i\frac{\pi}{2}}|m_z=-1\rangle =$$

$$= -i\cdot\left(\frac{1}{2}|m_z=1\rangle + \frac{i}{\sqrt{2}}|m_z=0\rangle - \frac{1}{2}|m_z=-1\rangle\right) =$$

$= -i\cdot|+\rangle_y$ matches expectation from $\underline{b}$

i. $|m_y=2\rangle = R_z^{(2)}\left(\frac{\pi}{2}\right)|m_x=2\rangle = e^{-i\frac{\pi}{2}J_z/\hbar}|m_x=2\rangle =$

$$= \frac{1}{4}\cdot e^{-i\frac{\pi}{2}\cdot 2}|m_z=2\rangle + \frac{1}{2}e^{-i\frac{\pi}{2}}|m_z=1\rangle + \sqrt{\frac{3}{8}}e^{0}|m_z=0\rangle + \frac{1}{2}e^{i\frac{\pi}{2}}|m_z=-1\rangle +$$

$$+ \frac{1}{4}e^{i\frac{\pi}{2}\cdot 2}|m_z=-2\rangle =$$

$$= -\frac{1}{4}\lvert m_z=2\rangle - \frac{i}{2}\lvert m_z=1\rangle + \sqrt{\frac{3}{8}}\lvert m_z=0\rangle + \frac{i}{2}\lvert m_z=-1\rangle - \frac{1}{4}\lvert m_z=-2\rangle$$

$$\lvert m_x=-2\rangle = R_z^{(2)}(\pi)\lvert m_x=2\rangle = e^{-i\pi J_z/\hbar}\lvert m_x=2\rangle =$$

$$= \frac{1}{4}e^{-i\pi\cdot 2}\lvert m_z=2\rangle + \frac{1}{2}e^{-i\pi}\lvert m_z=1\rangle + \sqrt{\frac{3}{8}}e^{0}\lvert m_z=0\rangle + \frac{1}{2}e^{i\pi}\lvert m_z=-1\rangle + \frac{1}{4}e^{i\pi\cdot 2}\lvert m_z=-2\rangle$$

$$= \frac{1}{4}\lvert m_z=2\rangle - \frac{1}{2}\lvert m_z=1\rangle + \sqrt{\frac{3}{8}}\lvert m_z=0\rangle - \frac{1}{2}\lvert m_z=-1\rangle + \frac{1}{4}\lvert m_z=-2\rangle$$