OPTI 570 LECTURE Th 09/25

Known: $[\hat{a}, \hat{a}^+] = 1$

$\hat{N} = \hat{a}^+ \hat{a}$

Extra detail

- $\hat{N}\,|n\rangle = n\,|n\rangle$ $\quad \{n\}$ basis of $\hat{N}$
- $\{|n\rangle\}$ be non-degenerate
- $\{|n\rangle\}$ - discrete, orthonormal
- $\{|n\rangle\}$ spans $\mathcal{E}$, $|n\rangle \in \mathcal{E}$

Questions

- find the spectrum of $\hat{N}$
- calc. the action of $\hat{a}, \hat{a}^+$ on all $|n\rangle$
- do $\hat{a}, \hat{a}^+$ have eigenstates?
  ↳ what are they
- what are the units of $\hat{a}$? $\quad [a, a^+] = 1 = aa^+ - a^+a$
  - unitless
- — " — of $\hat{N}$?
  - unitless
- is $\hat{a}$ Hermitian? No,

$\{|n\rangle\} = ?$ - spectrum of N

$N\,|n\rangle = n\,|n\rangle = a^+ a\,|n\rangle$

Let $|\varphi\rangle = a\,|n\rangle$

may not be normalized $\qquad$ normalized

$|n\rangle \in \mathcal{E}$, are $|\varphi\rangle$ in $\mathcal{E}$?

$\langle\varphi|\varphi\rangle = \langle n|a^+ a|n\rangle = \langle n|N|n\rangle = n\underbrace{\langle n|n\rangle}_{1} =$

$$\boxed{\langle \varphi | \varphi \rangle = n}$$

Either : 1. $|\varphi\rangle = 0$

2. $n > 0$, $|\varphi\rangle$ has positive norm

$$\boxed{n \geq 0}$$

$|0\rangle \neq 0$

might be in spectrum

if $|0\rangle$ is an eigenket of N, then $|0\rangle \in \mathcal{E}$

then $\langle 0 | a^+ a | 0 \rangle = \langle 0 | N | 0 \rangle = 0$

$a|0\rangle = 0 \Rightarrow |\varphi\rangle = a|0\rangle \notin \mathcal{E}$

For $n > 0$

$|\varphi\rangle = a|n\rangle$

$N|\varphi\rangle = a^+ a |\varphi\rangle = a^+ a a |n\rangle$

$[a, a^+] = 1 \Rightarrow a^+ a = a a^+ - 1$

$N|\varphi\rangle = (a a^+ - 1) a |n\rangle =$

$= a \underbrace{a^+ a}_{N} |n\rangle - a|n\rangle =$

$= a N |n\rangle - a|n\rangle =$

$= a\, n|n\rangle - a|n\rangle =$

$= (n-1)(a|n\rangle) =$

$N|\varphi\rangle = (n-1)|\varphi\rangle \qquad |\varphi\rangle = a|n\rangle$

But : $N|n-1\rangle = (n-1)|n-1\rangle$

$\Rightarrow |\varphi\rangle \propto |n-1\rangle$

$a|n\rangle \propto |n-1\rangle$

If $|n\rangle$ is an eigenstate of $N \Rightarrow |n-1\rangle$ is also an eigenstate of N

If $|n\rangle$ is NOT an eigenstate of $N \Rightarrow |n-1\rangle$ is not an eigenstate of N

$a|n\rangle = c|n-1\rangle$

Normalize: $\langle n|a^\dagger a|n\rangle = |c|^2 \underbrace{\langle n-1|n-1\rangle}_{1} = |c|^2$

$\langle n|N|n\rangle = n\langle n|n\rangle = n = |c|^2$

$\Rightarrow c = \sqrt{n} \quad > 0$

for: $n>0, \quad a|n\rangle = \sqrt{n}|n-1\rangle$

$n=0, \quad a|0\rangle = 0$

consider $0<n<1$ ex: $n = 0.75$

$a\underbrace{|0.75\rangle}_{\text{does not exist.}} = \sqrt{0.75}\underbrace{|-0.25\rangle}_{\text{does not exist.}}$

$\Rightarrow |0.75\rangle \notin \{|n\rangle\}$

- by inference - no values $0<n<1$ are in the spectrum

$1<n<2$ ex: 1.5

$a\underbrace{|1.5\rangle}_{\text{not in spectrum}} = \sqrt{1.5}\underbrace{|0.5\rangle}_{\text{not in spectrum}}$

- infer that n integer ≥ 0, $n \in \mathbb{N}^0$

$a^\dagger|n\rangle = |\varphi\rangle$

$\langle\varphi|\varphi\rangle = \langle n|aa^\dagger|n\rangle =$

$= \langle n|(1+a^\dagger a)|n\rangle =$

$= \underline{\langle n|n\rangle} + \underline{\langle n|N|n\rangle} =$

$= 1 + n$

$N(a^\dagger|n\rangle) = a^\dagger a a^\dagger|n\rangle = a^\dagger(1+a^\dagger a)|n\rangle =$

$= a^\dagger(1+n)|n\rangle =$

$= (1+n)\underbrace{a^\dagger|n\rangle}_{\propto |n+1\rangle}$

Normalize: $a^\dagger|n\rangle = c\cdot|n+1\rangle$

$$|c|^2 \langle n+1 | n+1 \rangle = \langle n | a a^\dagger | n \rangle$$
$$= \langle n | 1 + a^\dagger a | n \rangle$$
$$|c|^2 = n+1 \qquad \underline{c = \sqrt{n+1}}$$

<u>Summary</u>: $a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$

$$a|n\rangle \begin{cases} \sqrt{n}\,|n-1\rangle & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases}$$

$n = 0, 1, 2, 3, \ldots$

$|0\rangle$

$a^\dagger |0\rangle = |1\rangle$

$a^\dagger |1\rangle = \sqrt{2}\,|2\rangle$

$\vdots$

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle}$$

Question: Does a have eigenstates?

can I $|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$?

$$\underline{a|\alpha\rangle = \alpha |\alpha\rangle}$$

<u>LHS</u>: $a|\alpha\rangle = a \mathbb{1} |\alpha\rangle = a \sum_{n=0}^{\infty} |n\rangle \underbrace{\langle n|\alpha\rangle}_{c_n} =$

$$= \cancel{a|0\rangle}^{0} c_0 + \sum_{n=1}^{\infty} c_n a|\alpha\rangle =$$

$$= \sum_{n=1}^{\infty} \sqrt{n}\,|n-1\rangle c_n =$$

$$= \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1}\,|n\rangle$$

<u>RHS</u>: $\alpha|\alpha\rangle = \alpha \sum_{n=0}^{\infty} |n\rangle \underbrace{\langle n|\alpha\rangle}_{c_n} = \alpha \sum_{n=0}^{\infty} c_n |n\rangle$

LHS = RHS $\Rightarrow \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1}\,|n\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle$

• use bra $\langle m|$ on the left side:

$N|m\rangle = m|m\rangle$

$$\sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}\,\underbrace{\langle m|n\rangle} = \alpha \sum_{n=0}^{\infty} c_n \langle m|n\rangle$$

$$c_{m+1}\sqrt{m+1} = \alpha\, c_m$$

$$\Rightarrow c_1 = \alpha c_0$$

$$c_2 = \frac{\alpha}{\sqrt{2}} c_1 = \frac{\alpha^2}{\sqrt{2}} c_0$$

$$\vdots$$

$$c_m = \frac{\alpha^m}{\sqrt{m!}} c_0$$

$$\underline{|\alpha\rangle} = \sum_{n=0}^{\infty} c_n |n\rangle =$$

$$= \underline{c_0} \cdot \underline{\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$$

<u>Normalize</u>:

$$\langle\alpha|\alpha\rangle = 1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha^*)^m \alpha^n}{\sqrt{m!}\sqrt{n!}} c_0^* c_0 \underbrace{\langle m|n\rangle}_{\delta_{m,n}} =$$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!}}_{e^{|\alpha|^2}} \cdot |c_0|^2 =$$

$$= |c_0|^2 e^{|\alpha|^2} = 1 \Rightarrow \underline{c_0 = e^{-\frac{|\alpha|^2}{2}}}$$

$$\boxed{|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$$

$a|\alpha\rangle = \alpha|\alpha\rangle$, α can be any complex #

$|\alpha = 0\rangle = |n = 0\rangle$

$a|\alpha=0\rangle = 0|\alpha=0\rangle$ – eigenstate of $\hat{a}$ operator w/ eigenvalue 0

Language:

N – number operator

a, a^+ – ladder operators

a: annihilation, lowering operator

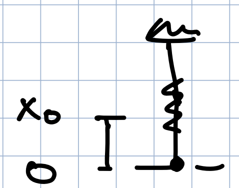
a^+: creation, raising operator

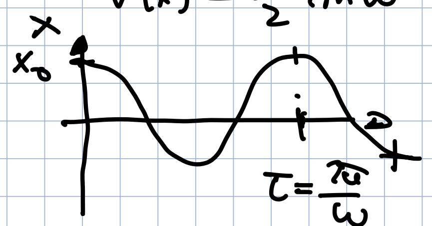
$$\hat{H} = \varepsilon_0 + \varepsilon_1 \hat{N}, \quad \hat{N} = a^+ a$$

Physical meaning

Classical HO

$$V(x) = \frac{1}{2} m\omega^2 x^2 \qquad E_{tot} = \frac{p^2(t)}{2m} + \frac{1}{2} m\omega^2 x(t)^2$$





QHO

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

Solve QHO: $\hat{H}|\varphi_n\rangle = E_n|\varphi_n\rangle$

$|\psi(t)\rangle$ – superpositions of $|\varphi_n\rangle$

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x \qquad \hat{a} = \ldots \hat{x} + \ldots \hat{p}$$

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right) \quad \text{where } N = a^+ a$$

$\Rightarrow E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ – energy eigenvalues

$|\varphi_n\rangle = ?$ $\quad |n\rangle$

Scaling: $\sigma = \sqrt{\frac{\hbar}{m\omega}}$ "Q. harm. os. length"

define: $\tilde{x} = \frac{\hat{x}}{\sigma}$

$\tilde{p} = \frac{\sigma \hat{p}}{\hbar}$

$\tilde{E} = E_n / \hbar\omega$

$\tilde{H} = \frac{H}{\hbar\omega}$ $\quad \tilde{H} = \frac{1}{2}\tilde{p}^2 + \frac{1}{2}\tilde{x}^2$ $\quad [\hat{x}, \tilde{p}] = 1$

$\tilde{H} | \varphi_n \rangle = \tilde{E}_n | \varphi_n \rangle$

pos representation

$\tilde{x} \to \bar{x}$

$\tilde{p} \to i \frac{\partial}{\partial x}$

mom repres:

$\tilde{p} \to p$

$\hat{x} \to i \frac{\partial}{\partial p}$

$$\frac{1}{2} \left(\hat{x}^2 - \frac{\partial^2}{\partial \tilde{x}^2} \right) \varphi_n(\tilde{x}) = E_n \varphi_n(\tilde{x})$$

$$\frac{1}{2} \left(\hat{p}^2 - \frac{\partial^2}{\partial \tilde{p}^2} \right) \varphi_n(\hat{p}) = E_n \varphi_n(\tilde{p})$$