

Assignment 8
OPTI 570 Quantum Mechanics
University of Arizona

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Problem I

Given that $j = 1$, we have the magnetic quantum number $m \in \{1, 0, -1\}$ and $\{|z_+\rangle, |z_0\rangle, |z_-\rangle\} = \{|1, 1\rangle, |1, 0\rangle, |1, -1\rangle\}$. The state vector can then be spanned as:

$$|\psi\rangle = a|z_+\rangle + b|z_0\rangle + c|z_-\rangle \longrightarrow |\psi\rangle = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

- a) i) In the state vector method, we want to compute $\langle\psi|J_x|\psi\rangle$. Using the eigenequation for the ladder operators:

$$J_{\pm}|j, m\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}|j, m \pm 1\rangle,$$

we have for $j = 1$

$$j_+|1, 0\rangle = \hbar\sqrt{2}|1, 1\rangle, \quad J_+|1, -1\rangle = \hbar\sqrt{2}|1, 0\rangle, \quad J_-|1, 1\rangle = \hbar\sqrt{2}|1, 0\rangle, \quad J_-|1, 0\rangle = \hbar\sqrt{2}|1, -1\rangle.$$

In matrix form, we have

$$J_+ = \hbar\sqrt{2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_- = \hbar\sqrt{2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow J_x = \frac{1}{2}(J_+ + J_-) = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Its action on the state is a matrix-multiplication:

$$\langle\psi|J_x|\psi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} a^* & b^* & c^* \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{\hbar}{\sqrt{2}}(a^*b + b^*a + b^*c + c^*b) \equiv \frac{2\hbar}{\sqrt{2}}[\text{Re}(a^*b) + \text{Re}(b^*c)].$$

- ii) In the density operator method, the expectation we want to compute is

$$\langle J_x \rangle = \text{Tr}(\rho J_x).$$

In a pure state, we have that

$$|\psi\rangle = a|z_+\rangle + b|z_0\rangle + c|z_-\rangle, \quad \rho = |\psi\rangle\langle\psi|.$$

In matrix form, the density operator is

$$\rho = \begin{bmatrix} |a|^2 & ab^* & ac^* \\ ba^* & |b|^2 & bc^* \\ ca^* & cb^* & |c|^2 \end{bmatrix}.$$

The trace then mean to sum over the diagonal of this matrix multiplied by J_x , that is, over the following elements:

$$\begin{aligned} (\rho J_x)_{11} &= \rho_{12} J_{21} + \rho_{13} J_{31} = ab^* \frac{\hbar}{\sqrt{2}} \\ (\rho J_x)_{22} &= \rho_{21} J_{12} + \rho_{23} J_{32} = ba^* \frac{\hbar}{\sqrt{2}} + bc^* \frac{\hbar}{\sqrt{2}} \\ (\rho J_x)_{33} &= \rho_{31} J_{13} + \rho_{32} J_{23} = cb^* \frac{\hbar}{\sqrt{2}} \end{aligned}$$

Then, the trace, and therefore the meanvalue is,

$$\langle J_x \rangle = (\rho J_x)_{11} + (\rho J_x)_{22} + (\rho J_x)_{33} = \frac{\hbar}{\sqrt{2}} [ab^* + ba^* + bc^* + cb^*] = \frac{2\hbar}{\sqrt{2}} [\text{Re}(a^*b) + \text{Re}(b^*c)].$$

b) For $j = 1$,

$$J^2 = J_x^2 + J_y^2 + J_z^2 = j(j+1)\hbar^2 \mathbf{1} = 2\hbar^2 \mathbf{1}.$$

Then, for any state we have that

$$\langle J^2 \rangle = \langle \psi | 2\hbar^2 \mathbf{1} | \psi \rangle = 2\hbar^2 (|a|^2 + |b|^2 + |c|^2) = 2\hbar^2.$$

c) We know that J operator matrices for the spin-1 are:

$$J_x = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_y = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad J_z = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

If we square them, we have:

$$J_x^2 = \frac{\hbar^2}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad J_y^2 = \frac{\hbar^2}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad J_z^2 = \hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The expectation $\langle \psi | J_i^2 | \psi \rangle$ are then:

$$\begin{aligned} \langle J_x^2 \rangle &= \frac{\hbar^2}{2} (|a|^2 + 2|b|^2 + |c|^2 + 2\text{Re}(a^*c)) \\ \langle J_y^2 \rangle &= \frac{\hbar^2}{2} (|a|^2 + 2|b|^2 + |c|^2 - 2\text{Re}(a^*c)) \\ \langle J_z^2 \rangle &= \hbar^2 (|a|^2 + |b|^2) \\ \langle J^2 \rangle &= \langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle = 2\hbar^2 (|a|^2 + |b|^2 + |c|^2) = \langle J^2 \rangle \end{aligned}$$

d) We have from our previous parts both expectation required, so it becomes an algebra problem

$$\begin{aligned}
 (\Delta J_x)^2 &= \langle J_x^2 \rangle - \langle J_x \rangle^2 \\
 &= \frac{\hbar^2}{2} [|a|^2 + 2|b|^2 + |c|^2 + 2\text{Re}(a^*c) - |a^*b + b^*a + b^*c + c^*b|^2] \\
 (\Delta J_x)^2 &= \frac{\hbar^2}{2} [1 + |b|^2 + 2\text{Re}(a^*c) - 4(\text{Re}(a^*b) + \text{Re}(b^*c))^2] \quad (|a|^2 + |b|^2 + |c|^2 = 1)
 \end{aligned}$$

e) If $b = 0$, then the normalization condition becomes $|a|^2 + |c|^2 = 1$, and

$$\langle J_x \rangle = 0, \quad \langle J_x^2 \rangle = \frac{\hbar^2}{2} [|a|^2 + |c|^2 + 2\text{Re}(a^*c)] = \frac{\hbar^2}{2} [1 + 2\text{Re}(a^*c)].$$

Thus, we need to minimize the last result, which means that $\text{Re}(a^*c)$ must take its minimum value. If $a = |a|e^{i\alpha}$ and $c = |c|e^{i\gamma}$, then we can express the real part as $|a||c|\cos(\alpha - \gamma)$. The minimum value of the cosine is when $\alpha - \gamma = \pi$. Assuming that both magnitudes are equal (simplest scenario), we therefore have $c = -a$ and :

$$a = \frac{e^{i\phi}}{\sqrt{2}}, \quad b = 0, \quad c = -\frac{e^{i\phi}}{\sqrt{2}}.$$

Which then gives

$$\langle J_x^2 \rangle = \frac{\hbar^2}{2} (1 - 1) = 0 \implies (\Delta J_x)^2 = 0.$$

f) We can just pick an eigenstate of J_z ,

$$a = 0, \quad b = e^{i\phi}, \quad c = 0.$$

Thus,

$$|\psi\rangle = |z_0\rangle \quad (\Delta J_z)^2 = 0.$$

We could also pick a non-zero value for a or c .

g) We use the generalized Heisenberg uncertainty relation to compute the required uncertainty product:

$$\Delta J_y \Delta J_z \geq \frac{1}{2} |\langle [J_y, J_z] \rangle| = \frac{1}{2} |\langle i\hbar J_x \rangle| = \frac{\hbar^2}{\sqrt{2}} |\langle \text{Re}(a^*b) + \text{Re}(b^*c) \rangle|.$$

We see that the uncertainty relation of two perpendicular angular momentum components J_y and J_z depends on how much angular momentum you have along the third component x . This relation is telling us that the more precisely the J_x is known on average, the more uncertain are the perpendicular components.

Problem II

a) The expansion of $|x_+\rangle$ for $\lambda = +\hbar$ is

$$|x_+\rangle = \frac{1}{2}|z_+\rangle + \frac{1}{\sqrt{2}}|z_0\rangle + \frac{1}{2}|z_-\rangle.$$

If $|\psi(0)\rangle = |z_0\rangle$, then

$$\langle x_+|\psi(0)\rangle = \langle x_+|z_0\rangle = \frac{1}{\sqrt{2}}.$$

The probability to measure $J_x\hbar$ is therefore

$$P_x(\hbar) = |\langle x_+|z_0\rangle|^2 = \frac{1}{2}.$$

b) Now for $\lambda = 0$, the expansion is

$$|x_0\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle - |z_-\rangle).$$

The inner product $\langle x_0|\psi(0)\rangle$ is now zero, so the probability of measuring $J_x = 0$ is zero.

c) For $\lambda = -\hbar$,

$$|x_-\rangle = \frac{1}{2}|z_+\rangle - \frac{1}{\sqrt{2}}|z_0\rangle + \frac{1}{2}|z_-\rangle.$$

The projection of $|\psi(0)\rangle$ onto $|x_-\rangle$ is $1/\sqrt{2}$, so that

$$P_x(-\hbar) = |\langle x_-|z_0\rangle|^2 = \frac{1}{2}.$$

d) Since $|\psi(0)\rangle = |z_0\rangle$, it is evident that the probability of getting $|z_+\rangle$ is not possible. So,

$$P_z(\hbar) = |\langle z_+|z_0\rangle|^2 = 0.$$

e) After the measurement, the state will evolve and live between the eigenstate of J_x with certain probabilities:

$$\rho = \frac{1}{2}|x_+\rangle\langle x_+| + \frac{1}{2}|x_-\rangle\langle x_-| \quad (P_x(0) = 0).$$

This is a mixture of the eigenstate with the coefficient being the probability. If now we take a measurement with J_z , we will have, for a given angular momentum m

$$P_z(m\hbar) = \text{Tr}(|z_m\rangle\langle z_m|\rho) = \frac{1}{2} [|\langle z_m|x_+\rangle|^2 + |\langle z_m|x_-\rangle|^2].$$

In $|x_+\rangle$ and $|x_-\rangle$ we substitute their expression in terms of the eigenstate of J_z and evaluate for $m \in \{\hbar, 0, -\hbar\}$:

$$P_z(\hbar) = \frac{1}{2} \left(\frac{1}{4} + \frac{1}{4} \right) = \frac{1}{4}, \quad P_z(0) = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2}, \quad P_z(-\hbar) = \frac{1}{2} \left(\frac{1}{4} + \frac{1}{4} \right) = \frac{1}{4}.$$

f) If the measurement is uncertain, then we can express it as a mixed state. However, if we know the result then it collapses into a single eigenstate of J_x , that is, a pure state. We can see for the statistical mixture that the trace of the density operator squared is less than 1, reinforcing this fact: $\text{Tr}(\rho^2) = 1/2 < 1$.

g) This is the same we have computed previously, its matrix form is

$$\rho^{(x)} = \frac{1}{2}|x_+\rangle\langle x_+| + \frac{1}{2}|x_-\rangle\langle x_-| = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

h) The transformation matrix is needed to pass from x to z , so we use the above definition in terms of x to express in terms of z and the formulation given in the field guide.

$$\begin{aligned} |x_+\rangle &= \frac{1}{2}|z_+\rangle + \frac{1}{\sqrt{2}}|z_0\rangle + \frac{1}{2}|z_-\rangle \\ |x_0\rangle &= \frac{1}{\sqrt{2}}|z_+\rangle - \frac{1}{\sqrt{2}}|z_-\rangle \\ |x_-\rangle &= \frac{1}{2}|z_+\rangle - \frac{1}{\sqrt{2}}|z_0\rangle + \frac{1}{2}|z_-\rangle \end{aligned}$$

We concatenate them to construct the transformation matrix:

$$M = \begin{bmatrix} | & | & | \\ |x_+\rangle & |x_0\rangle & |x_-\rangle \\ | & | & | \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}.$$

We use it to transform the state vector and the density operator as:

$$|\psi\rangle_z = M|\psi\rangle_x, \quad \text{and} \quad \rho^{(z)} = M\rho^{(x)}M^\dagger.$$

This is exactly the same process shown in the field guide, because these elements on M correspond to the inner product $M_{ij} = \langle z_i|x_j\rangle$.

i) The specific matrix $\rho^{(z)}$ is

$$\rho^{(z)} = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix}$$

Using it in the expectation values yields

$$\begin{aligned} \langle J_z \rangle &= \text{Tr}(\rho^{(z)} J_z) = \hbar \text{Tr} \left(\begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right) = \hbar \left(\frac{1}{4} - \frac{1}{4} \right) = 0 \\ \langle J_x \rangle &= \text{Tr}(\rho^{(z)} J_x) = \frac{\hbar}{\sqrt{2}} \text{Tr} \left(\begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right) = 0. \end{aligned}$$

j) They should match the answer from part e as the transformation matrix shouldn't change the meaning

of them.

$$\begin{aligned}
 P_z(\hbar) &= \text{Tr}(\rho^{(z)} P_+) = \text{Tr} \left(\begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \text{Tr} \left(\begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 0 & 0 \\ 1/4 & 0 & 0 \end{bmatrix} \right) = \frac{1}{4} \\
 P_z(0) &= \text{Tr}(\rho^{(z)} P_0) = \text{Tr} \left(\begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \text{Tr} \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \frac{1}{2} \\
 P_z(-\hbar) &= \text{Tr}(\rho^{(z)} P_-) = \text{Tr} \left(\begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \text{Tr} \left(\begin{bmatrix} 0 & 0 & 1/4 \\ 0 & 0 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \right) = \frac{1}{4}
 \end{aligned}$$

We see that these probabilities coincide with part e).

Problem III

- a) sgasg
- b) asgsag
- c) sagasg
- d) asgsagasg
- e) asgsag
- f) asgsag
- g) asgsag

Problem IV