Problem Set 12 Solutions

$\boxed{I}$ 1D QHO

$$W(t) = \begin{cases} -q\mathcal{E}x & \text{for } 0 \le t \le \tau \\ 0 & t<0,\ t>\tau \end{cases} \qquad |\psi(t=0)\rangle = |\varphi_0\rangle$$

a. $x = \frac{\sigma}{\sqrt{2}}(a^\dagger + a)$

to 0th order: $b_0^{(0)} = 1$, $b_1^{(0)} = 0$, $b_2^{(0)} = 0 \ldots$

to 1st order: $\lambda b_1^{(1)}(t) = \frac{\lambda}{i\hbar}\int_0^\tau \sum_k dt'\, e^{i\omega_{1k}t'}\, W_{1k}(t')\, b_k^{(0)}(t')$

$$= \frac{\lambda}{i\hbar}\int_0^\tau dt'\, e^{i\omega_0 t'} \cdot \langle\varphi_1| -q\cdot\mathcal{E}\cdot\frac{\sigma}{\sqrt{2}}(a^\dagger + a)|\varphi_0\rangle \cdot 1 =$$

$$= \frac{iq\epsilon\lambda\sigma}{\sqrt{2}\hbar}\int_0^\tau dt'\, e^{i\omega_0 t'}\langle\varphi_1|\varphi_1\rangle\cdot 1 =$$

$$= \frac{\cancel{i}q\epsilon\lambda\cdot\sigma}{\sqrt{2}\hbar}\left.\frac{e^{i\omega_0 t'}}{\cancel{i}\omega_0}\right|_0^\tau =$$

$$= \underbrace{\frac{q\epsilon\sigma}{\sqrt{2}\hbar\omega_0}\lambda\cdot\left(e^{i\omega_0 t'} - 1\right)}$$

$$P_{01}(t>\tau) = \left(\frac{q\epsilon\sigma}{\sqrt{2}\hbar\omega_0}\right)^2 \left(e^{i\omega_0\tau} - 1\right)\left(e^{-i\omega_0\tau} - 1\right) =$$

$$= \left(\frac{q\epsilon\sigma}{\sqrt{2}\hbar\omega_0}\right)^2 \cdot \left[2 - \left(e^{i\omega_0\tau} + e^{-i\omega_0\tau}\right)\right] =$$

$$= \left(\frac{q\epsilon\sigma}{\sqrt{2}\hbar\omega_0}\right)^2 \cdot \left[2 - 2\cos(\omega_0\tau)\right] =$$

$$= 4\left(\frac{q\epsilon\sigma}{\sqrt{2}\hbar\omega_0}\right)^2 \cdot \sin^2\left(\frac{\omega_0\tau}{2}\right)$$

$\boxed{b}$ $\lambda b_2^{(1)}(t) = \frac{\lambda}{i\hbar} \int_0^\tau \sum_k dt' \, e^{i\omega_{1k}t'} W_{2k}(t') \, b_k^{(0)}(t')$

Since the perturbation only couples adjacent energy eigenstates and only $b_0^{(0)} \neq 0$, that means $b_2^{(1)} = b_3^{(1)} = b_4^{(1)} = \ldots = 0$.

to second order:

$$\lambda^2 b_2^{(2)}(t) = \frac{\lambda^2}{i\hbar} \int_0^\tau \sum_k dt' \, e^{i\omega_{2k}t'} W_{2k}(t') \, b_k^{(1)}(t')$$

based on above, we only have $k=1$ being non-zero

$$\lambda^2 b_2^{(2)}(t) = \frac{\lambda^2}{i\hbar} \int_0^\tau dt' \, e^{i\omega_0 t'} W_{21}(t') \, b_1^{(1)}(t')$$

$$W_{21} = -q\epsilon \langle \varphi_2 | \frac{\sigma}{\sqrt{2}} (a^+ + a) | \varphi_1 \rangle = -q\epsilon\sigma$$

$$\lambda^2 b_2^{(2)}(t) = \frac{\lambda^2}{i\hbar} (-q\epsilon\sigma) \cdot \int_0^\tau dt' \, e^{i\omega_0 t'} \cdot \frac{q\epsilon\sigma}{\sqrt{2}\hbar\omega_0} \left(e^{i\omega_0 t'} - 1\right) =$$

$$= i \frac{\lambda^2}{\sqrt{2}} \left(\frac{q\epsilon\sigma}{\hbar}\right)^2 \cdot \frac{1}{\omega_0} \cdot \int_0^\tau e^{2i\omega_0 t'} - e^{i\omega_0 t'} \, dt' =$$

$$= \cancel{i} \frac{\lambda^2}{\sqrt{2}\omega_0} \left(\frac{q\epsilon\sigma}{\hbar}\right)^2 \cdot \left(\frac{e^{2i\omega_0 t'}}{2\cancel{i}\omega_0} - \frac{e^{i\omega_0 t'}}{\cancel{i}\omega_0} \right|_0^\tau =$$

$$= \frac{\lambda^2}{\sqrt{2}\omega_0^2} \left(\frac{q\epsilon\sigma}{\hbar}\right)^2 \cdot \left(\frac{e^{2i\omega_0\tau}}{2} - e^{i\omega_0\tau} - \frac{1}{2} + 1 \right) =$$

$$= \lambda^2 \frac{1}{\sqrt{2}\omega_0^2} \left(\frac{q\epsilon\sigma}{\hbar}\right)^2 \cdot e^{i\omega_0\tau} \left[\frac{1}{2}\left(e^{i\omega_0\tau} + e^{-i\omega_0\tau}\right) - 1 \right] =$$

$$= \lambda^2 \frac{1}{\sqrt{2}\omega_0^2} \left(\frac{q\epsilon\sigma}{\hbar}\right)^2 \cdot e^{i\omega_0\tau} \cdot \left[\cos(\omega_0\tau) - 1 \right] =$$

$$= -\frac{2}{\sqrt{2}} \lambda^2 \cdot \left(\frac{q\epsilon\sigma}{\hbar\omega_0}\right)^2 \cdot e^{i\omega_0\tau} \sin^2\left(\frac{\omega_0\tau}{2}\right)$$

So: $\boxed{P_{02}^{(2)} = 2 \left(\frac{q\epsilon\sigma}{\hbar\omega_0}\right)^4 \sin^4\left(\frac{\omega_0\tau}{2}\right)}$

$\boxed{\text{II}}$ $|\psi(0)\rangle = |\varphi_1\rangle$

$$W(x,t) = \lambda\hbar\omega \exp\left[-\frac{(x-vt)^2}{2a^2}\right]$$

a. $b_0^{(0)} = 1$, $b_1^{(0)} = b_2^{(0)} = \dots = 0$

$$\lambda b_1^{(1)}(t) = \frac{\lambda}{i\hbar}\int_0^t \sum_k dt' e^{i\omega t'} W_{1k}(t')\, b_k^{(0)}(t') .$$

$$= \frac{\lambda}{i\cancel{\hbar}}\int_{-\infty}^{+\infty} dt' e^{i\omega t'} \cancel{\hbar}\omega \langle 1| e^{-\frac{(x-vt)^2}{2a^2}} |0\rangle$$

$u = x - vt \Rightarrow du = v\,dt \qquad t = \frac{x-u}{v}$

$$= -\frac{i\lambda\omega}{v}\langle 1|\int_{-\infty}^{+\infty} du\, e^{i\omega\left(\frac{x-u}{v}\right)} e^{-\frac{u^2}{2a^2}}|0\rangle =$$

$$= -\frac{i\lambda\omega}{v}\langle 1|\underbrace{e^{i\omega\frac{x}{v}}}_{\text{momentum translation}}|0\rangle \int_{-\infty}^{+\infty} du\, e^{-\frac{u^2}{2a^2}} e^{-\frac{i\omega u}{v}} \; b$$

$$= -\frac{i\lambda\omega}{v}\langle 1|T\left(\frac{\hbar\omega}{v}\right)|0\rangle \cdot \sqrt{2\pi}\cdot a \cdot \exp\left(-\frac{\omega^2 a^2}{2v^2}\right)$$

$$= -\frac{i\lambda\omega}{v}\sqrt{2\pi}\cdot a\, e^{-\frac{\omega^2a^2}{2v^2}} e^{-\frac{|\alpha|^2}{2}}\alpha \quad \text{w/} \quad \alpha = \frac{i}{\sqrt{2}}\frac{\langle p\rangle\sigma}{\hbar} = \frac{i\omega\sigma}{\sqrt{2}v}$$

So: $P_1 = \lambda^2\frac{\omega^2}{v^2}\cdot 2\pi a^2 \cdot e^{-\frac{\omega^2a^2}{v^2}} e^{-|\alpha|^2}|\alpha|^2 =$

$$= \cancel{2}\pi\lambda^2\frac{\omega^2}{v^2}a^2 e^{-\frac{\omega^2a^2}{v^2}}\cdot e^{-\frac{\omega^2\sigma^2}{2v}}\cdot\frac{\omega^2\sigma^2}{\cancel{2}v^2} =$$

$$= \frac{\lambda^2\pi a^2\sigma^2\omega^4}{v^4}\exp\left[-\frac{\omega^2}{v^2}\left(a^2+\frac{\sigma^2}{2}\right)\right] \quad \text{w/} \quad \sigma = \sqrt{\frac{\hbar}{m\omega}}$$

$\boxed{b}$ $\frac{dP_1}{dv} = 0 \Rightarrow \frac{d}{dv}\cdot\left[\frac{1}{v^4} e^{-\frac{\omega^2}{v^2}\left(a^2+\frac{\sigma^2}{2}\right)}\right] = 0$

$$\frac{2e^{-\frac{\omega^2a^2}{v^2}}\left[\omega^2\left(a^2+\frac{\sigma^2}{2}\right)-v^2\right]}{v^5} = 0 \Rightarrow v_{max} = \omega\sqrt{a^2+\frac{\sigma^2}{2}}$$

$$P_1(v_{max}) = \lambda^2 \frac{\pi a^2 \cancel{\omega^4} \sigma^2}{\cancel{\omega^4}\left(a^2+\frac{\sigma^2}{2}\right)^2} \quad e^{-1} =$$

$$= \lambda^2 \frac{\pi a^2 \sigma^2}{\left(a^2+\frac{\sigma^2}{2}\right)^2} e^{-1}$$

$\boxed{c}$ $$P_1(a) = \lambda^2 \frac{\pi a^2 \sigma^2}{\left(a^2+\frac{\sigma^2}{2}\right)^2} e^{-1}$$

$$\frac{dP_1}{da} = 0 \Rightarrow \quad 2a\left(a^2 - \frac{\sigma^2}{2}\right) = 0$$

$$\underline{a_{max} = \frac{\sigma}{\sqrt{2}}}$$

$$P_1(a_{max}) = \lambda^2 \frac{\pi \cancel{\frac{\sigma^2}{2}} \cancel{\sigma^2}}{\cancel{\sigma^4}} e^{-1} \boxed{= \frac{\pi}{2}\lambda^2 e^{-1}}$$

$\boxed{\text{III}}$ $$W(t) = \lambda \hbar \omega_0 \exp\left(-\frac{x^2}{d^2}\right) \exp\left(-\frac{t^2}{2\tau^2}\right) \qquad |\psi(-\infty)\rangle = |\varphi_0\rangle$$

a. since W only couples even parity states,

$P_{0\to1}^{(1)}(\infty) = 0$, exact to all orders.

b. $$W_{20} = \langle \varphi_2 | W | \varphi_0 \rangle =$$

$$= \lambda \hbar \omega \langle \varphi_2 | e^{-x^2/d^2} | \varphi_0 \rangle =$$

$$= \lambda \hbar \omega \frac{\cancel{2}}{\cancel{2}\sqrt{2}} \frac{1}{\sqrt{\pi}\,\sigma} \int_{-\infty}^{+\infty} dx\, e^{-\frac{x^2}{d^2}} e^{-x^2/\sigma^2}\left(2\frac{x^2}{\sigma^2} - 1\right) =$$

$$= \frac{\lambda \hbar \omega}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{dx}{\sigma}\left(2\left(\frac{x}{\sigma}\right)^2 - 1\right) e^{-\frac{x^2}{\sigma^2}\left(1+\frac{\sigma^2}{a^2}\right)} \qquad u = \frac{x}{\sigma}$$

$$= \frac{\lambda \hbar \omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du\,(2u^2 - 1)\, e^{-u\left(1+\frac{\sigma^2}{a^2}\right)} =$$

$$= \frac{\lambda \hbar \omega}{\sqrt{2\pi}} \left[2 \int_{-\infty}^{+\infty} du\, u^2 e^{-u\left(1+\frac{\sigma^2}{d^2}\right)} - \int_{-\infty}^{+\infty} du\, e^{-u\left(1+\frac{\sigma^2}{d^2}\right)} \right] =$$

$$= \frac{\lambda \hbar \omega}{\sqrt{2\pi}} \left[\frac{\sqrt{\pi}}{\left(1+\frac{\sigma^2}{d^2}\right)^{3/2}} - \frac{\sqrt{\pi}}{\sqrt{1+\frac{\sigma^2}{d^2}}} \right] =$$

$$= \frac{\lambda \hbar \omega}{\sqrt{2}} \frac{-\frac{\sigma^2}{d^2}}{\left(1+\sigma^2/d^2\right)^{3/2}} =$$

$$= -\frac{\lambda \hbar \omega}{\sqrt{2}} \frac{\sigma^2}{d^2} \left(1+\frac{\sigma^2}{d^2}\right)^{-3/2}$$

$$\Rightarrow P^{(1)}_{0\to 2}(\infty) = \lambda^2 \frac{\cancel{\hbar^2} \omega^2}{2 \cancel{\hbar^2}} \cdot \left(\frac{\sigma^2}{d^2}\right)^2 \frac{1}{\left(1+\frac{\sigma^2}{d^2}\right)^3} \left| \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2\tau^2}} \cdot e^{i\omega t}\, dt \right|^2 =$$

$$= \frac{\lambda^2 \omega^2}{\cancel{2}} \left(\frac{\sigma^2}{d^2}\right)^2 \frac{1}{\left(1+\frac{\sigma^2}{d^2}\right)^3} \cdot e^{-4\omega^2\tau^2} \cdot \tau^2 \cdot \cancel{2}\pi =$$

$$\underbrace{= \lambda^2 \pi (\omega\tau)^2 \cdot \frac{(\sigma^2/d^2)^2}{(1+\sigma^2/d^2)^3} e^{-4\omega^2\tau^2}}$$

c. Let $c = \frac{\sigma^2}{d^2}$

$\Rightarrow P_{0\to 2} \propto \frac{c^2}{(1+c)^3}$ max $P_{0\to 2}$ w.r.t. c

$$\Rightarrow \frac{\partial P_{0\to 2}}{\partial c} = \frac{2c}{(1+c)^3} - \frac{3c^2}{(1+c)^4} = 0$$

$$2c \cdot (1+c) - 3c^2 = 0$$

$$-c^2 + 2c = 0$$

$$c \cdot (2-c) = 0 \qquad c = 2$$

$$\Rightarrow \boxed{d_{max} = \frac{\sigma}{\sqrt{2}}}$$

d. $P_{0\to 2} = \lambda^2 \pi (\omega \tau)^2 \cdot \frac{1/2}{\left(1+\frac{1}{2}\right)^3} e^{-4\omega^2\tau^2}$

$$\frac{\partial P_{0\to 2}}{\partial \tau} = 0 \Rightarrow 2\tau\, e^{-4\omega^2\tau^2} + \tau^2 \cdot e^{-4\omega^2\tau^2} \cdot (-4\omega^2 2\tau) = 0$$

$$4\omega^2\tau^2 = 1 \qquad \boxed{\tau = \frac{1}{2\omega}}$$

e. $P_{0\to 2}^{max} = \lambda^2 \pi \, \frac{1}{\cancel{4}} \cdot \frac{\cancel{1/2}}{27/\cancel{8}} e^{-1} =$

$$\boxed{= \frac{\lambda^2 \pi e^{-1}}{27}}$$