

Stationary Perturbation Theory

$q_m > 1$: degenerate energy e-state of H_0 with eigenvalue E_n^0

1. Determine $\hat{W}^{(m)}$, the part of \hat{W} that acts only within the q_m -dimensional subspace that is spanned by the degenerate states of H_0

$$\{ | \varphi_n^1 \rangle, \dots, | \varphi_n^{q_m} \rangle \}$$

2. Eigenvalues of $\hat{W}^{(m)}$ are the 1st order terms ϵ_i of E_{nij}
 $E_{nji} = E_n^0 + \lambda \epsilon_j \quad \text{for } j = 1, \dots, q_m$

3. Eigenstates of $\hat{W}^{(m)}$ are the corresponding eigenstates of H to 0th order
• find specific linear combination of $\{ | \varphi_n^i \rangle \}$ that are the e-states of $\hat{W}^{(m)}$

Example

Given: $H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} m \omega^2 (x^2 + y^2)$

$$W = \lambda m \omega^2 xy = \lambda \hat{W} \quad (\hat{W} = m \omega^2 xy)$$

I Estimate order of magnitude of terms

$$H_0: \propto \hbar \omega \quad E_m = \hbar \omega (m_x + m_y + 1) \quad E_0 = \hbar \omega \quad E_1 = \frac{2 \hbar \omega}{\dots} \quad E_2 = \frac{3 \hbar \omega}{\dots}$$

$$\hat{W}: \propto x, y \propto \sigma = \sqrt{\frac{\hbar}{m \omega}} \quad W = m \omega^2 xy = m \omega^2 \zeta^2 \propto \hbar \omega$$

if $W \ll H_0 \Rightarrow \lambda \ll 1$, ok to use perturbation theory

II Choose a representation

Eigenstates of H_0 : $H_0(m_x, m_y) = E_n^0 (m_x, m_y)$

$$\{|00\rangle, |10\rangle, |01\rangle, |20\rangle, |11\rangle, |02\rangle, |30\rangle, \dots\}$$

$$\text{or } \{ | \varphi_0 \rangle, | \varphi_1^1 \rangle, | \varphi_1^2 \rangle, | \varphi_2^1 \rangle, | \varphi_2^2 \rangle, | \varphi_2^3 \rangle, \dots \}$$

H_0 as a matrix in this repr.

$$H_0 \rightarrow \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

III. Construct \hat{W} matrix in the same representation

$$\begin{aligned} \hat{W} &= m\omega^2 xy = \\ &= \frac{1}{2} m\omega^2 \sigma^2 (c_x^\dagger + c_x)(a_y^\dagger + a_y) = \\ &= \frac{1}{2} m\omega^2 \sigma^2 (c_x^\dagger c_y^\dagger + c_x a_y^\dagger + c_x^\dagger a_y + c_x a_y) \end{aligned}$$

$$\hat{W}|0,0\rangle = \frac{1}{2}\hbar\omega|1,1\rangle$$

$$\hat{W}|1,0\rangle = \frac{1}{2}\hbar\omega(\sqrt{2}|2,1\rangle + |0,1\rangle)$$

$$\hat{W}|0,1\rangle = \frac{1}{2}\hbar\omega(\sqrt{2}|1,2\rangle + |1,0\rangle)$$

$$\hat{W}|2,0\rangle = \frac{1}{2}\hbar\omega(\sqrt{3}|3,1\rangle + \sqrt{2}|1,1\rangle)$$

$$\hat{W}|1,1\rangle = \frac{1}{2}\hbar\omega(2|2,2\rangle + \sqrt{2}|0,2\rangle + \sqrt{2}|2,0\rangle + |0,0\rangle)$$

$$\hat{W}|0,2\rangle = \frac{1}{2}\hbar\omega(\sqrt{3}|1,3\rangle + \sqrt{2}|1,1\rangle)$$

$$\hat{W} \rightarrow \frac{\hbar\omega}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{all zeros}$$

$$\hat{W}^{(0)} \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{W}^{(1)} \quad \begin{pmatrix} 1 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

not zero

$\hat{W}^{(2)}$

$$\hat{w}|10\rangle = \frac{1}{2}\hbar\omega (\underbrace{a_x^+ a_y^+ + a_x a_y^\dagger + a_x^\dagger a_y + a_x a_y}_0)|10\rangle =$$

$$= \frac{1}{2}\hbar\omega (\sqrt{2} \cdot \sqrt{1} \cdot |21\rangle + \sqrt{1} \cdot \sqrt{1} \cdot |01\rangle + 0 + 0) =$$

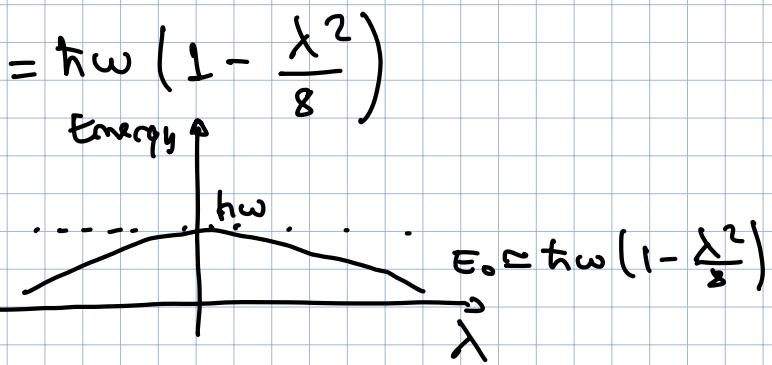
$$= \frac{1}{2}\hbar\omega \cdot (\sqrt{2}|21\rangle + |01\rangle)$$

Q: What is the ground state energy, eigenvalue and eigenstate to 2nd order in λ ?

$M=0, \beta_{m=0}=1 \Rightarrow$ use non-degenerate stationary pert. theory.

$$E_0 = E_0^0 + \lambda \langle 00 | \hat{w} | 00 \rangle + \lambda^2 \sum_{\substack{k_x, k_y \\ \neq (0,0)}} \frac{|\langle k_x k_y | \hat{w} | 00 \rangle|}{\hbar\omega - (k_x + k_y + 1)\hbar\omega}$$

$$= \hbar\omega + 0 + \frac{\left(\frac{\lambda}{2}\hbar\omega\right)^2}{-2\hbar\omega}$$



- What does ground state look like?

$$|\Psi\rangle = |00\rangle + \sum_{\substack{k_x, k_y \\ \neq (0,0)}} \frac{\langle k_x k_y | \hat{w} | 00 \rangle}{\hbar\omega - (k_x + k_y + 1)\hbar\omega} |k_x k_y\rangle$$

$$= |00\rangle + \frac{\lambda \left(\frac{\hbar\omega}{2}\right)}{-2\hbar\omega} |11\rangle =$$

$$|\Psi\rangle = |00\rangle - \frac{\lambda}{4} |11\rangle,$$

meas energy \Rightarrow What is prob of meas $\geq \hbar\omega$?

Need to normalize:

$$|\Psi\rangle_{\text{norm}} = \frac{1}{\sqrt{1 + \frac{\lambda^2}{T_0}}} (|00\rangle - \frac{\lambda}{\sqrt{2}} |11\rangle)$$

$$P = \frac{\lambda^2}{\sqrt{16 + \lambda^2}}$$

$m=1$, use deg. stationary perturbation theory

IV. Find solutions

$$W^{(m=1)} |\Psi_{1,j=1,2}\rangle = \lambda \hat{W}^{(m=0)} |\Psi_{1,j=1,2}\rangle = E_{1,j=1,2} |\Psi_{1,j=1,2}\rangle$$

$m=1$

$$W^{(1)} \rightarrow \lambda \frac{\hbar\omega}{2} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\hat{W}^{(1)}} \quad \text{repres basis } \{|10\rangle, |01\rangle\}$$

Eigenvalues and eigenvectors

$$\text{evals: } \pm \lambda \frac{\hbar\omega}{2} \quad \text{evecs: } \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{aligned} \text{to 1st order } & \left\{ \begin{array}{l} E_{1,1} = 2\hbar\omega - \lambda \frac{\hbar\omega}{2} = 2\hbar\omega \left(1 - \frac{\lambda}{4}\right) \\ \text{in } \lambda \end{array} \right. \\ & |\Psi_{1,1}\rangle = \frac{1}{\sqrt{2}} (|10\rangle - |01\rangle) \end{aligned}$$

$$E_{1,2} = 2\hbar\omega + \lambda \frac{\hbar\omega}{2} = 2\hbar\omega \left(1 + \frac{\lambda}{4}\right)$$

$$|\Psi_{1,2}\rangle = \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle)$$

if $\lambda > 0$, then $E_{1,1} < E_{1,2}$

$\lambda < 0$, then $E_{1,2} < E_{1,1}$

$$\overline{m=2} \quad \omega^{(2)} \rightarrow \lambda \hbar \omega \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & \\ 0 & \frac{1}{\sqrt{2}} & 0 & \end{pmatrix} = \lambda \hbar \omega \underbrace{\mathbf{J}_x^{(1)}}_{\text{J}_x^{(1)}}$$

levels

vectors

$$\lambda \hbar \omega$$

$$0$$

$$-\lambda \hbar \omega$$

$$\begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}$$

Basis: $\{|20\rangle, |11\rangle, |02\rangle\}$

$$\text{evols: } E_{2,1} = 3\hbar\omega - \lambda \hbar \omega = 3\hbar\omega \left(1 - \frac{\lambda}{3}\right)$$

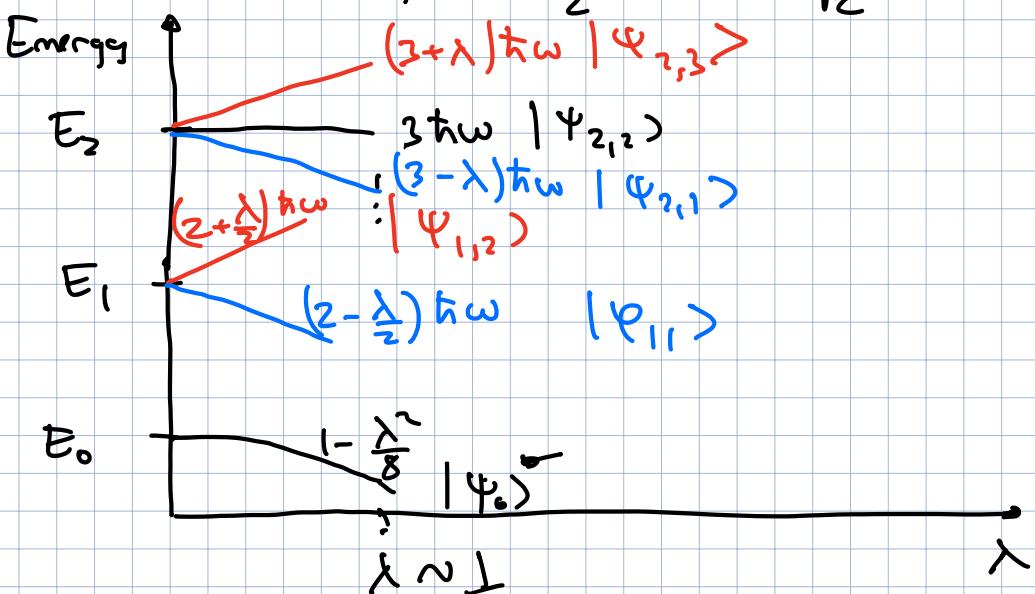
$$|\Psi_{2,1}\rangle = \frac{1}{2} |20\rangle - \frac{1}{\sqrt{2}} |11\rangle + \frac{1}{2} |02\rangle$$

$$E_{2,2} = 3\hbar\omega$$

$$|\Psi_{2,2}\rangle = \frac{1}{\sqrt{2}} |20\rangle + \frac{1}{\sqrt{2}} |02\rangle$$

$$E_{2,3} = 3\hbar\omega + \lambda \hbar \omega = 3\hbar\omega \left(1 + \frac{\lambda}{3}\right)$$

$$|\Psi_{2,3}\rangle = \frac{1}{2} |20\rangle + \frac{1}{\sqrt{2}} |11\rangle + \frac{1}{2} |02\rangle$$



Example energy level diagrams

Perturbation energy diagram OR NOT?

