OPTI 570, Fall 2025 **Problem Set 1** Prof. Cristian (Cris) Panda

Due in D2L-Brightspace: Thurs, Aug 28 (Main Campus); Mon, Sept 1 (Online section)

Indicate the total time that you spent on this problem set at the top of your first page.

This problem set is to be solved entirely on your own without any help from other people or artificial intelligence (AI) tools, but you can look up definitions and formula as needed.

Solve the following problems.

1. Assuming that a and C are real and $a \geq 0$, show that

$$\int_{-\infty}^{\infty} dx\, C\, \exp\{-\frac{x^2}{2a^2} + b\, x\} = \sqrt{2\pi}\, a\, C\, \exp\{a^2 b^2/2\}.$$

Use the technique of "completing the squares" in the exponent on the left (look this up if you do not know or remember what this technique involves). You will use this very basic technique throughout the course, so start to get re-familiar with it now and keep it in mind when you are working with expressions of the form $Ax^2 + Bx$. You may use the information given on page 122 of the *Field Guide to Quantum Mechanics* **after** you have done the "completing the squares" step (ie, don't just write down the answer above because it exists in the *Field Guide*).

2. In problem 1 above, assume that x is a one-dimensional spatial coordinate with dimensional units of length. What are the dimensional units of a? What are the dimensional units of b? What are the dimensional units of C if the entire integral is to have no dimensional units?

3. Solve the following integral **by inspection** by considering the parity (symmetry properties) of the integrand. Assume a is real and positive. **Don't use a computer to help you, and don't write down anything (even on scratch paper that I will never see)** except the answer and a justification for your answer.

$$\int_{-\infty}^{\infty} dx\; x^{11}\, \exp\{-\frac{x^2}{2a^2}\}.$$

4. The definition of a one-dimensional Fourier transform that we will use in this class is the following: given a wave function $\psi(x)$ that is a function of position x, its Fourier transform $\widetilde{\psi}(p)$ over momentum p is

$$\widetilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx\, e^{-ipx/\hbar}\, \psi(x),$$

where $i = \sqrt{-1}$. (For all of you engineers, we use i in this class, not j. I will expect you to do the same in all of your work.) Assume x is a spatial coordinate with dimensional units of length and p has dimensional units of momentum.

(a) Without looking up the answer, and **only** examining the argument of the exponential term in the integral, determine and state the dimensional units of $\hbar$ in the SI system of units.

(b) If $\psi(x)$ is a properly normalized 1D wave function, i.e. $\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = 1$, what are the dimensional units of $\psi(x)$? This question requires no knowledge of quantum mechanics. (If you are about to say "A wave function has no dimensional units," please don't; instead, state the correct answer from an analysis of the information given, despite what your intuition might tell you.)

(c) What are the dimensional units of $\widetilde{\psi}(p)$?

5. Suppose $\psi(x) = A \exp\{-\frac{x^2}{2a^2}\}$, where A is a scalar. Using both the relation given in problem 1, and the definition of Fourier transform given in problem 4, determine $\tilde{\psi}(p)$, the Fourier transform of $\psi(x)$, with as little work as possible.

6. Using Euler's formula (look it up if you need to), prove the following two trig identities:

$$\sin(2\theta) \;=\; 2\sin(\theta)\cos(\theta) \qquad \text{and} \qquad \cos^2(\theta) = \frac{1}{2}[1+\cos(2\theta)].$$

There's more than one approach, and don't get hung up on the word "prove." All I want to do here is to make sure that you're comfortable using complex number notations to work with trigonometric functions.

7. Let $\frac{\partial}{\partial x} y(x) = m\, y(x)$. **This is an eigenvalue equation!** Give a function $y(x)$ (other than $y(x) = 0$) that solves this equation for any unspecified scalar m. Just one correct example is fine, you don't need to write down the most general expression possible.

8. Let $\frac{\partial^2}{\partial x^2} y(x) = -m^2\, y(x)$. **Another eigenvalue equation!** Give a non-zero function $y(x)$ that solves this equation for a scalar m. Just one correct example is fine, you don't need to write down the most general expression possible.

9. Suppose $\mathbf{M} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. List the eigenvalues of $\mathbf{M}$. (Feel free to look up *eigenvalues* or *eigenvalue equations* if you need to.)

10. Suppose $\mathbf{M} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Give the eigenvalues and associated eigenvectors of $\mathbf{M}$. Extra admiration given for writing eigenvectors normalized to 1, where the first element is real and positive.

11. The **norm** of a vector whose components are *real* scalars is the (positive) square-root of the sum of the squares of the components (look up *norm* if you need to). However, for a vector with complex scalars as components, the norm is *not* obtained by first just squaring the components. If you do not know/remember how to find the norm of a vector with complex elements, you

should look that up somewhere now. Now suppose $\mathbf{v} = \begin{pmatrix} -3 \\ 4i \end{pmatrix}$. Evaluate $||\mathbf{v}||$, the norm of $\mathbf{v}$.

12. Suppose $\mathbf{M} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 1 \\ 3 & 2 & 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$. Evaluate $\mathbf{M} \cdot \mathbf{v}$.

13. Let $y(x) = e^{-x^2/a^2}$ for a positive real scalar a. Sketch this function, $y(x)$ vs x, including labeled axes. Indicate the points $x = a$ and $x = -a$ on your x axis. What is the common name given to the function that you have sketched? (Hint: *exponential* is not the correct answer. This is one of the most important functional forms that you will encounter for the *rest of your scientific life!* It's good to know now how to talk about it.)

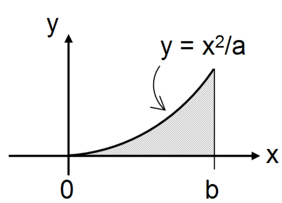
14. Let $f(t) = \sin^2(\Omega t/2)$. Sketch $f(t)$ vs. t for $t = 0$ to $t = 8\pi/\Omega$. Label the axes. Indicate important points on the axes (i.e., limiting values of $f(t)$, values of t where $f(t) = 0$).

15. Given that $\int_{-\infty}^{\infty} du\, e^{-u^4} = 1.81 \pm 0.01$ for a real scalar u. Also given:

$$1 = A^2 \int_{-\infty}^{\infty} dx\, e^{-2x^4/a^4},$$

where $a = 10^{-6}$ m. Solve for A, assuming that A is real and positive. Don't forget to check whether A has units (or does it?). **The most common integration mistake in OPTI 570 involves substitution of variables** You *must* get comfortable with this technique to do well in this course.

16. A section of the curve $y(x) = x^2/a$ is plotted below. If x has dimensional units of meters [m], and $a = 4$ m and $b = 2$ m, give a value (with dimensional units) for the area of the shaded region.



17. Give 3 *different* solutions for the cube-root of 1, allowing complex numbers as solutions.

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