

Stationary Perturbation Theory

$q_n > 1$: degenerate energy e-state of H_0 w/ eols E_n^0

1. Determine $\hat{W}^{(n)}$, the part of \hat{W} that acts only within the q_n -dimensional subspace that is spanned by the degenerate state of H_0 .

$$\{ | \varphi_n^1 \rangle, \dots, | \varphi_n^{q_n} \rangle \}$$

2. Eigenvalues of $\hat{W}^{(n)}$ are the 1st order terms E_1 of $E_{n,j}$
 $E_{n,j} = E_n^0 + \lambda E_j$ for $j = 1, \dots, q_n$

3. Eigenstates of $\hat{W}^{(n)}$ are the corresponding eigenstates of H to 0th order

- find specific linear combination of $\{ | \varphi_n^i \rangle \}$ that are the e-states of $\hat{W}^{(n)}$

Example

Given: $H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} m \omega^2 (x^2 + y^2)$

$$W = \lambda m \omega^2 x y = \lambda \hat{W} \quad (\hat{W} = m \omega^2 x y)$$

I Estimate order of magnitude of terms

$$H_0: \sim \hbar \omega \quad E_n = \hbar \omega (n_x + n_y + 1) \quad E_0 = \hbar \omega \quad E_1 = 2\hbar \omega \quad E_2 = 3\hbar \omega \dots$$

$$\hat{W}: \quad x, y \sim \sigma = \sqrt{\frac{\hbar}{m\omega}} \quad W = m \omega^2 x y = m \omega^2 \sigma^2 \sim \hbar \omega$$

if $W \ll H_0 \Rightarrow \underline{\lambda \ll 1}$, ok to use perturbation theory

II. Choose a representation

Eigenstates of H_0 : $H_0 |n_x, n_y\rangle = E_{n_x + n_y}^0 |n_x, n_y\rangle$

$$\{ |00\rangle, |10\rangle, |01\rangle, |20\rangle, |11\rangle, |02\rangle, |30\rangle, \dots \}$$

or $\{ | \varphi_0 \rangle, | \varphi_1^1 \rangle, | \varphi_1^2 \rangle, | \varphi_2^1 \rangle, | \varphi_2^2 \rangle, | \varphi_2^3 \rangle, \dots \}$

H_0 as a matrix in this repr.

$$H_0 \rightarrow \hbar \omega_0 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 3 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 3 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

II. Construct \hat{W} matrix in the same representation

$$\begin{aligned} \hat{W} &= m\omega^2 xy = \\ &= \frac{1}{2} m\omega^2 \sigma^2 (a_x^\dagger + a_x)(a_y^\dagger + a_y) = \\ &= \frac{1}{2} m\omega^2 \sigma^2 (a_x^\dagger a_y^\dagger + a_x a_y^\dagger + a_x^\dagger a_y + a_x a_y) \end{aligned}$$

$$\hat{W} |0,0\rangle = \frac{1}{2} \hbar \omega |1,1\rangle$$

$$\hat{W} |1,0\rangle = \frac{1}{2} \hbar \omega (\sqrt{2} |2,1\rangle + |0,1\rangle)$$

$$\hat{W} |0,1\rangle = \frac{1}{2} \hbar \omega (\sqrt{2} |1,2\rangle + |1,0\rangle)$$

$$\hat{W} |2,0\rangle = \frac{1}{2} \hbar \omega (\sqrt{3} |3,1\rangle + \sqrt{2} |1,1\rangle)$$

$$\hat{W} |1,1\rangle = \frac{1}{2} \hbar \omega (2 |2,2\rangle + \sqrt{2} |0,2\rangle + \sqrt{2} |2,0\rangle + |0,0\rangle)$$

$$\hat{W} |0,2\rangle = \frac{1}{2} \hbar \omega (\sqrt{3} |1,3\rangle + \sqrt{2} |1,1\rangle)$$

$$\hat{W} \rightarrow \frac{\hbar \omega}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & \dots & \text{all zeros} \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \sqrt{2} 0 \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & \dots & \dots \\ 1 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} & \dots & \dots \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & \ddots & \dots \end{pmatrix}$$

Labels on the left: $\hat{W}^{(0)}$, $\hat{W}^{(1)}$, $\hat{W}^{(2)}$

Label at the bottom: not zero

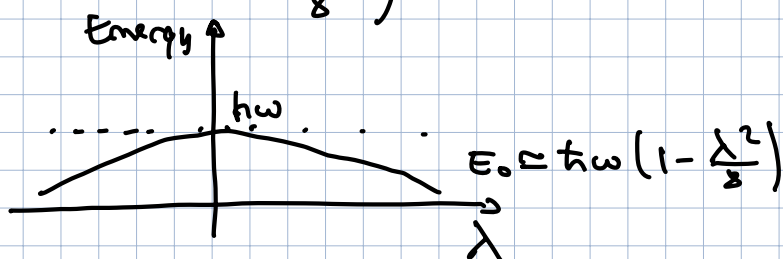
$$\begin{aligned}
 \hat{W}|10\rangle &= \frac{1}{2}\hbar\omega (\underline{a_x^\dagger a_y^\dagger} + a_x a_y^\dagger + a_x^\dagger a_y + a_x a_y)|10\rangle = \\
 &= \frac{1}{2}\hbar\omega (\sqrt{2}\cdot\sqrt{1}\cdot|21\rangle + \sqrt{1}\cdot\sqrt{1}\cdot|01\rangle + 0 + 0) = \\
 &= \frac{1}{2}\hbar\omega \cdot (\sqrt{2}|21\rangle + |01\rangle)
 \end{aligned}$$

Q: What is the ground state energy, eigenvalue and eigenstate to 2nd order in λ ?

$n=0, \quad \rho_{n=0}=1 \Rightarrow$ use non-degenerate stationary pert. theory.

$$\begin{aligned}
 E_0 &= E_0^{(0)} + \lambda \langle 00|\hat{W}|00\rangle + \lambda^2 \sum_{\substack{k_x, k_y \\ \neq (0,0)}} \frac{|\langle k_x k_y|\hat{W}|00\rangle|^2}{\hbar\omega - (k_x + k_y + 1)\hbar\omega} > 0 \\
 &= \hbar\omega + 0 + \frac{\left(\frac{\lambda}{2}\hbar\omega\right)^2}{-2\hbar\omega} < 0
 \end{aligned}$$

$$= \hbar\omega \left(1 - \frac{\lambda^2}{8}\right)$$



• What does ground state look like?

$$|\psi\rangle = |00\rangle + \sum_{\substack{k_x, k_y \\ \neq (0,0)}} \frac{\langle k_x k_y|\hat{W}|00\rangle}{\hbar\omega - (k_x + k_y + 1)\hbar\omega} |k_x k_y\rangle$$

$$= |00\rangle + \frac{\lambda \left(\frac{\hbar\omega}{2}\right)}{-2\hbar\omega} |11\rangle =$$

$$|\psi\rangle = |00\rangle - \frac{\lambda}{4} |11\rangle$$

meas energy \Rightarrow What is prob of meas $3\hbar\omega$?

Need to normalize:

$$|\psi\rangle_{\text{norm}} = \frac{1}{\sqrt{1 + \frac{\lambda^2}{16}}} \left(|00\rangle - \frac{\lambda}{4} |11\rangle \right)$$

$$P = \frac{\lambda^2}{\sqrt{16 + \lambda^2}}$$

$m=1$, use deg. stationary perturbation theory

IV. Find solutions

$$W^{(m=1)} |\psi_{1,j=1,2}\rangle = \lambda \hat{W}^{(m=0)} |\psi_{1,j=1,2}\rangle = E_{1,j=1,2} |\psi_{1,j=1,2}\rangle$$

$m=1$

$$W^{(1)} \rightarrow \lambda \underbrace{\frac{\hbar\omega}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\hat{W}^{(1)}}$$

repres basis $\{|10\rangle, |01\rangle\}$

Eigenvalues and eigenvectors

evals: $\pm \lambda \frac{\hbar\omega}{2}$

evecs: $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

to 1st order
in λ

$$\begin{cases} E_{1,1} = 2\hbar\omega - \lambda \frac{\hbar\omega}{2} = 2\hbar\omega \left(1 - \frac{\lambda}{4}\right) \\ |\psi_{1,1}\rangle = \frac{1}{\sqrt{2}} (|10\rangle - |01\rangle) \end{cases}$$

$$E_{1,2} = 2\hbar\omega + \lambda \frac{\hbar\omega}{2} = 2\hbar\omega \left(1 + \frac{\lambda}{4}\right)$$

$$|\psi_{1,2}\rangle = \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle)$$

if $\lambda > 0$, then $E_{1,1} < E_{1,2}$

$\lambda < 0$, then $E_{1,2} < E_{1,1}$

$$\frac{n=2}{W^{(2)}} \rightarrow \lambda \hbar \omega \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \lambda \hbar \omega J_x^{(1)}$$

eigvals
eigenvectors

$$\lambda \hbar \omega$$

$$0$$

$$-\lambda \hbar \omega$$

$$\begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}$$

Basis: $\{|20\rangle, |11\rangle, |02\rangle\}$

eigvals: $E_{2,1} = 3\hbar\omega - \lambda\hbar\omega = 3\hbar\omega \left(1 - \frac{\lambda}{3}\right)$

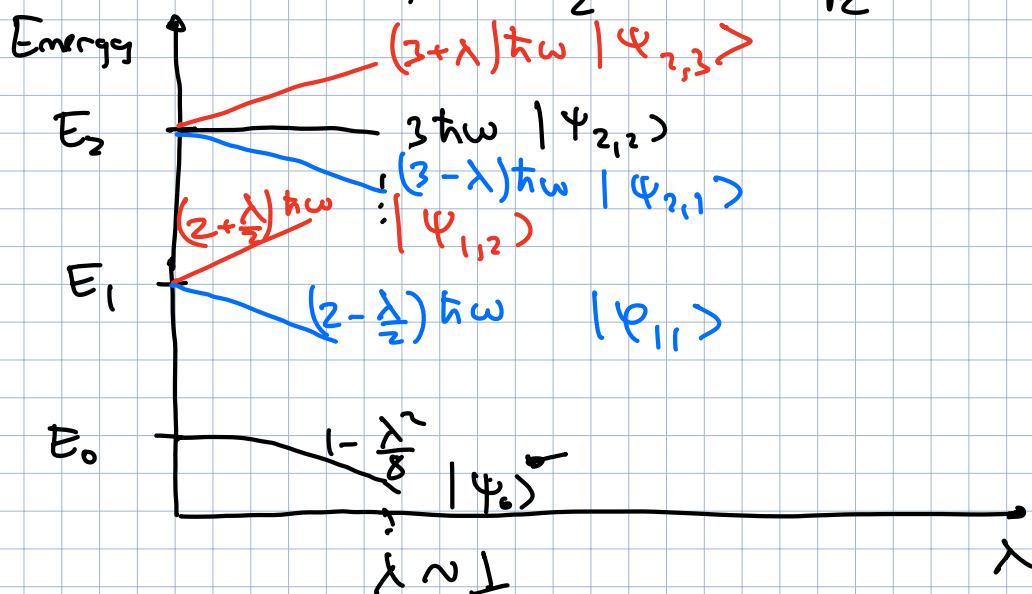
$$|\Psi_{2,1}\rangle = \frac{1}{2}|20\rangle - \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{2}|02\rangle$$

$$E_{2,2} = 3\hbar\omega$$

$$|\Psi_{2,2}\rangle = \frac{1}{\sqrt{2}}|20\rangle + \frac{1}{\sqrt{2}}|02\rangle$$

$$E_{2,3} = 3\hbar\omega + \lambda\hbar\omega = 3\hbar\omega \left(1 + \frac{\lambda}{3}\right)$$

$$|\Psi_{2,3}\rangle = \frac{1}{2}|20\rangle + \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{2}|02\rangle$$



Example energy level diagrams

Perturbation energy diagram or NOT?

