Problem Set 3 Solutions

$\boxed{\text{CT II.2}}$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

a. $\sigma_y^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y$ so Yes! σ_y is Hermitian

eigenvalues: $\det(\sigma_y - \lambda I) = 0 \Rightarrow \begin{vmatrix} -\lambda & -i \\ -i & -\lambda \end{vmatrix} = 0$

$$\lambda^2 = 1 \Rightarrow \boxed{\lambda = \pm 1}$$

$\lambda_1 = 1$ with corresponding eigenvector $|v_1\rangle$

In the $\{|1\rangle, |2\rangle\}$ representation, write $|v_1\rangle$ as $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = 1 \cdot \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

$$-ib_1 = a_1 \qquad ia_1 = b_1$$

same equation

$\Rightarrow \begin{pmatrix} a_1 \\ ia_1 \end{pmatrix}$ is an eigenvector of σ_y for any a_1

Normalization: $\langle v_1 | v_1 \rangle = 1$

$$(a_1)^2 + (ia_1)^2 = 1$$

$$2a_1^2 = 1 \Rightarrow a_1 = \frac{1}{\sqrt{2}}$$

so $\boxed{|v_1\rangle \rightarrow \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}}$

$\lambda_2 = -1$ with corresponding eigenvector $|v_2\rangle$

$|v_2\rangle \to \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ in the $\{|1\rangle, |2\rangle\}$ representation

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = -1 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

$$-i\, b_2 = -a_2 \qquad i\, a_2 = -b_2$$

$\Rightarrow \begin{pmatrix} a_2 \\ -i\, a_2 \end{pmatrix}$ is an eigenvector

Normalize $|a_2|^2 + |-ia_2|^2 = 1 \Rightarrow a_2 = 1/\sqrt{2}$

so $\boxed{|v_2\rangle \to \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}}$

b $\hat{P}_{|v_1\rangle} = |v_1\rangle\langle v_1|$

in matrix representation:

$$\hat{P}_{|v_1\rangle} \to \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

complex conjugate for bra

$$\hat{P}_{|v_2\rangle} \to \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

Check orthogonality:

$$\hat{P}_{|v_1\rangle} \cdot \hat{P}_{|v_2\rangle} \to \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 + i^2 & i - i \\ i - i & i^2 + 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \checkmark$$

Check closure

$$\sum_{i=1}^{2} |v_i \rangle\langle v_i| = \hat{P}_1 + \hat{P}_2$$

$$\hat{P}_1 + \hat{P}_2 = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

$\boxed{\text{CT II. 3}}$

state space is $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$$

$$|\psi_1\rangle = \frac{1}{\sqrt{3}}|u_1\rangle + \frac{i}{\sqrt{3}}|u_3\rangle$$

$\boxed{a}$

$$\langle\psi_0|\psi_0\rangle = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 \Rightarrow \underline{|\psi_0\rangle \text{ is normalised}}$$

$$\langle\psi_1|\psi_1\rangle = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \Rightarrow \underline{|\psi_1\rangle \text{ is NOT normalized}}$$

$\boxed{b}$

$$\rho_0 = |\psi_0\rangle\langle\psi_0| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{2} \\ \frac{1}{2} \end{pmatrix} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{2} & \frac{1}{2} \end{pmatrix}}_{\text{Hermitian conjugate}} = \begin{pmatrix} 1/2 & -i/2\sqrt{2} & 1/2\sqrt{2} \\ i/2\sqrt{2} & +1/4 & i/4 \\ 1/2\sqrt{2} & -i/4 & 1/4 \end{pmatrix}$$

$\underline{\text{Hermitian}}$ ✓

For ρ_1, we need to normalize first:

Let $|\psi_1'\rangle = c|\psi_1\rangle$ s.t. $\langle\psi_1'|\psi_1'\rangle = 1$

$$\langle\psi_1|c^*c|\psi_1\rangle = 1 \Rightarrow c^2 = 3/2 \qquad c = \sqrt{\frac{3}{2}}$$

$$\underline{|\psi_1'\rangle = \sqrt{\frac{3}{2}}\,|\psi\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{\sqrt{2}}|u_2\rangle}$$

$$\hat{P}_1 = |\psi_1'\rangle\langle\psi_1'| \rightarrow \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}$$

is $\underline{\text{Hermitian}}$ ✓

$\boxed{\text{CT II 6}}$

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Prove : $e^{i\alpha\sigma_x} = I\cos\alpha + i\,\sigma_x \sin\alpha$

From CT, since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{i\alpha\sigma_x} = \sum_{m=0}^{\infty} \frac{(i\alpha\sigma_x)^m}{m!} = \underbrace{\sum_{m=2k}^{\infty}}_{\text{even}} \frac{(i\alpha\sigma_x)^{2k}}{(2k)!} + \underbrace{\sum_{m=2k+1}^{\infty}}_{\text{odd}} \frac{(i\alpha\sigma_x)^{2k+1}}{(2k+1)!} =$$

Check: $\sigma_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}$

$\sigma_x^3 = \sigma_x$

$\vdots$

even : $\sigma_x^{2k} = \mathbb{I}$

odd : $\sigma_x^{2k+1} = \sigma_x$

So : $$e^{i\alpha\sigma_x} = \mathbb{I}\sum_{k=0}^{\infty} \underbrace{\frac{(-1)^k \alpha^{2k}}{(2k)!}}_{\underbrace{1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!}}_{\cos\alpha}} + \sigma_x i \underbrace{\sum_{k=0} \frac{(-1)^k \alpha^{2k+1}}{(2k+1)!}}_{\underbrace{\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \dots}_{\sin\alpha}}$$

$$\boxed{e^{i\alpha\sigma_x} = \mathbb{I}\cos(\alpha) + i\,\sigma_x \sin\alpha}$$

$\boxed{CT\ \text{V} 7}$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_y \cdot \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}$$

Same as for σ_x

$$\Rightarrow \boxed{e^{i\beta\sigma_y} = \sum_{n=0}^{\infty} \frac{1}{n!}(i\beta\sigma_y)^n = \sum_{n\ \text{even}} \frac{1}{n!}(i\beta)^n \sigma_y^n + \sum_{n\ \text{odd}} \frac{1}{n!}(i\beta)^n \sigma_y^n = \mathbb{I}\cos(\beta) + i\sigma_y \sin(\beta)}$$

<u>In general</u>: $\sigma_u = \lambda\sigma_x + \mu\sigma_y$ w/ $\lambda^2 + \mu^2 = \mathbb{I}$

$$\sigma_u^2 = (\lambda\sigma_x + \mu\sigma_y)^2 = \lambda^2\sigma_x^2 + \lambda\mu\cdot(\sigma_x\sigma_y + \sigma_y\sigma_x) + \mu^2\sigma_y^2 =$$

$$= \mathbb{I}\underbrace{(\lambda^2 + \mu^2)}_{1} + \lambda\mu\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] =$$

$$= \mathbb{I} + \lambda\mu\cdot\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right] =$$

$$= \mathbb{I} + \lambda\mu\cdot\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} =$$

$\boxed{\sigma_u^2 = \mathbb{I}}$ So the previous derivation applies

$$\Rightarrow \boxed{e^{i\gamma\sigma_u} = \mathbb{I}\cos(\gamma) + i\sigma_u\sin(\gamma)}$$

$$e^{2i\sigma_x} = \mathbb{I}\cos(2) + i\sigma_x \sin(2)$$

$$(e^{i\sigma_x})^2 = [\mathbb{I}\cos(1) + i\sigma_x\sin(1)]^2$$

$$= \mathbb{I}\cos^2(1) - \mathbb{I}\sin^2(1) + 2i\sigma_x\sin(1)\cos(1) =$$

$$= \mathbb{I}\cdot[\cos^2(1) - \sin^2(1)] + i\sigma_x\cdot[2\sin(1)\cos(1)] =$$

$$= \mathbb{I}\cos(2) + i\sigma_x \sin(2)$$

$$\boxed{\text{So Yes, } e^{2i\sigma_x} = \left(e^{i\sigma_x}\right)^2}$$

$$e^{i(\sigma_x+\sigma_y)} = \left[\mathbb{I}\cos(1) + i\sigma_x\sin(1)\right]\left[\mathbb{I}\cos(1) + i\sigma_y\sin(1)\right] =$$

$$= \mathbb{I}\cos^2(1) - \sigma_x\sigma_y\sin^2(1) + i\sin(1)\cos(1)\cdot\left[\sigma_x + \sigma_y\right]$$

$$\boxed{\text{so No, } e^{i(\sigma_x+\sigma_y)} \neq e^{i\sigma_x}e^{i\sigma_y}}$$

$\boxed{\text{CT II g}}$ $\quad H|\varphi_n\rangle = E_n|\varphi_n\rangle$

a) $$\langle\varphi_n|[A,H]|\varphi_n\rangle = \langle\varphi_n|\hat{A}\hat{H} - \hat{H}\hat{A}|\varphi_n\rangle =$$

$$= \langle\varphi_n|\hat{A}\,E_n|\varphi_n\rangle - \langle\varphi_n|E_n\hat{A}|\varphi_n\rangle =$$

$$= E_n\cdot\left(\langle\varphi_n|\hat{A}|\varphi_n\rangle - \langle\varphi_n|\hat{A}|\varphi_n\rangle\right) =$$

$$= 0$$

b) $$\hat{H} = \frac{1}{2m}\hat{P}^2 + V(\hat{x})$$

α) $$[\hat{H},\hat{P}] = \hat{H}\hat{P} - \hat{P}\hat{H} = \cancel{\frac{1}{2m}\hat{P}^3} + V(\hat{x})\hat{P} - \cancel{\frac{1}{2m}\hat{P}^3} - \hat{P}\hat{V}(x) =$$

$$\boxed{[\hat{H},\hat{P}] = \hat{V}(x)\hat{P} - \hat{P}\hat{V}(x)}$$

$$[\hat{H},\hat{x}] = \frac{1}{2m}[\hat{P}^2,\hat{x}] + \underbrace{[V(\hat{x}),\hat{x}]}_{=0} =$$

$$= \frac{1}{2m}\left(\hat{P}^2\hat{x} - \hat{x}\hat{P}^2\right) =$$

$$= \frac{1}{2m}\left(\hat{P}\hat{P}\hat{x} - \hat{x}\hat{P}\hat{P}\right) =$$

$$= \frac{1}{2m}\left[\hat{P}\left(\hat{x}\hat{P} - i\hbar\right) - \left(i\hbar + \hat{P}\hat{x}\right)\hat{P}\right] = \quad \left\{\text{use } [\hat{x},\hat{P}] = i\hbar\right\}$$

$$= \frac{1}{2m}\left[\cancel{\hat{P}\hat{x}\hat{P}} - i\hbar\hat{P} - i\hbar\hat{P} - \cancel{\hat{P}\hat{x}\hat{P}}\right] =$$

$$\boxed{[\hat{H},\hat{x}] = -\frac{i\hbar}{m}\hat{P}}$$

$$[\hat H, \hat x \hat p] = \frac{1}{2m}[\hat p^2, \hat x \hat p] + [V(\hat x), \hat x \hat p] =$$

$$= \frac{1}{2m}(\hat p^2 \hat x \hat p - \hat x \hat p^3) + [V(\hat x)\hat x \hat p - \hat x \hat p V(\hat x)] =$$

$$= \frac{1}{2m}(\hat p^2 \hat x - \hat x \hat p^2)\hat p + [\hat x V(\hat x)\hat p - \hat x \hat p V(\hat x)] =$$

$$= \frac{1}{\cancel{2}m}\Big(\underbrace{-\cancel{2}i\hbar \hat p}_{\text{See above}}\Big)\hat p + \hat x [V(\hat x), \hat p]$$

$$\boxed{[\hat H, \hat x \hat p] = -\frac{i\hbar}{m}\hat p^2 + \hat x [V(\hat x), \hat p]}$$

$\boxed{\beta}$

$$\langle \varphi_n | \hat p | \varphi_n \rangle = \langle \varphi_n | [\hat H, \hat x]\cdot\left(-\frac{m}{i\hbar}\right) | \varphi_n \rangle =$$

$$= \frac{m}{i\hbar}\langle \varphi_n | [\hat x, \hat H] | \varphi_n \rangle =$$

$$= 0 \quad (\text{from } \boxed{a})$$

$\boxed{\gamma}$ $E_k = \langle \varphi_n | \frac{\hat p^2}{2m} | \varphi_n \rangle$

The hint is to use what we had in $\boxed{a}$, try the comutator $[\hat H, \hat x \hat p]$

$$\frac{\hat p^2}{2m} = \frac{1}{2i\hbar}\hat x [V(\hat x), \hat p] - \frac{1}{(2i\hbar)}[\hat H, \hat x \hat p]$$

$$E_k = \frac{1}{2i\hbar}\Big\{\langle \varphi_n | \hat x [V(\hat x), \hat p] | \varphi_n \rangle - \underbrace{\langle \varphi_n | [\hat H, \hat x \hat p] | \varphi_n \rangle}_{0 \text{ from } \boxed{a}}\Big\}$$

$$= \frac{1}{2i\hbar}\langle \varphi_n | \hat x \cdot (-[\hat p, V(\hat x)]) | \varphi_n \rangle =$$

$$= \frac{1}{2i\hbar}\langle \varphi_n | \hat x \left(-[\hat p, \hat x]\frac{dV(\hat x)}{dx}\right) | \varphi_n \rangle = \quad \Big\{ \text{used Eq. 51 in Comp B-II}$$

$$= \frac{1}{2\cancel{i\hbar}}\langle \varphi_n | \hat x\, \cancel{i\hbar}\frac{d V(\hat x)}{d\hat x} | \varphi_n \rangle =$$

$$\boxed{E_k = \frac{1}{2} \langle \varphi_n | \hat{x} \frac{d V(\hat{x})}{d\hat{x}} | \varphi_n \rangle}$$

When $V(\hat{x}) = V_0 \hat{x}^{\lambda}$ ω $\lambda = 2, 4, 6 \ldots$, $V_0 > 0$

$$\frac{d V(\hat{x})}{d\hat{x}} = \lambda V_0 \hat{x}^{\lambda - 1}$$

$$\hat{x} \frac{d V(\hat{x})}{d\hat{x}} = \lambda V_0 \hat{x}^{\lambda} = \lambda V(\hat{x})$$

$$\Rightarrow \quad E_k = \frac{1}{2} \langle \varphi_n | \lambda V(\hat{x}) | \varphi_n \rangle =$$

$$\boxed{E_k = \frac{\lambda}{2} \langle \varphi_n | V(\hat{x}) | \varphi_n \rangle \quad \text{for } \lambda = 0, 2, 4, 6 \ldots}$$

$\boxed{\text{Part II}}$

$\mathcal{E}_{1,2,3,4}$ spanned by $\{|u_1\rangle, |u_2\rangle, |u_3\rangle, |u_4\rangle\}$

$\boxed{\text{II}-1}$

$$|\psi\rangle = c\left(2|u_1\rangle - i\sqrt{3}|u_2\rangle - 3e^{i\theta}|u_3\rangle + 3|u_4\rangle\right)$$

$$\langle\psi|\psi\rangle = |c|^2 \begin{pmatrix} 2 & i\sqrt{3} & -3e^{-i\theta} & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -i\sqrt{3} \\ -3e^{i\theta} \\ 3 \end{pmatrix} =$$

$$= |c|^2\left[4 + 3 + 9 + 9\right] =$$

$$= 25\ c^2 = 1 \Rightarrow \boxed{c = 1/5}$$

$\boxed{\text{II}-2}$

a. $|u_2\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

b. $\langle u_3| \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$

c. $|u_2\rangle\langle u_3| \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

d. $P_{|u_2\rangle} = |u_2\rangle\langle u_2| \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

e. $\sum_{m=1}^{4} \sum_{n=1}^{4} |u_m\rangle\langle u_n| = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

$\boxed{\text{II-3}}$ $\hat{Q}|u_1\rangle = i|u_4\rangle$ $\quad \hat{Q}|u_2\rangle = 2|u_3\rangle$

$\hat{Q}|u_3\rangle = 2|u_2\rangle$ $\quad \hat{Q}|u_4\rangle = -i|u_1\rangle$

$$\hat{Q} \to \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix}$$

$\hat{Q}|u_1\rangle = i\,|u_4\rangle$

$$\hat{Q}|u_1\rangle \to \begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \\ c_{41} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} \qquad \begin{pmatrix} c_{12} \\ c_{22} \\ c_{32} \\ c_{42} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} c_{13} \\ c_{23} \\ c_{33} \\ c_{43} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_{14} \\ c_{24} \\ c_{34} \\ c_{44} \end{pmatrix} = \begin{pmatrix} -i \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

<u>Overall</u>: $$Q \to \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$