Assignment 7

OPTI 570 Quantum Mechanics

University of Arizona

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Total time: 15 hours

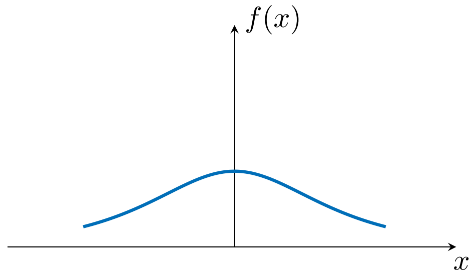
Problem I

a) sagasg
b) asgasg
c) asgag
d) gasgas
e) asgagasga

Problem II

Problem III

a) sagasg
b) asgasg
c) asgag
d) gasgas
e) asgagasga
f) asgasg
g) asgag
h) asgasg
i) asgasgasgasg
j) asgasgasgasg
k) asgag
l) asgasg
m) asfas



Problem IV

a) We plot the function $\text{sech}(x)$ to verify its parity. We can see that it is **even**.

This fact will facilitate us when computing ΔX, as we must integrate over $|\phi(x)|^2$ which therefore, is also even. We then have,

$$\begin{aligned}\langle X\rangle &= \int_{-\infty}^{\infty} x|\phi(x)|^2\,dx = \frac{1}{2\beta}\int_{-\infty}^{\infty} x\,\text{sech}(x/\beta)\,dx = 0\\ \langle X^2\rangle &= \int_{-\infty}^{\infty} x^2|\phi(x)|^2\,dx = \frac{1}{2\beta}\int_{-\infty}^{\infty} x^2\text{sech}(x/\beta)\,dx = \frac{\beta^2}{2}\int_{-\infty}^{\infty} u^2\text{sech}^2(u)\,du = \frac{\pi^2\beta^2}{12}.\end{aligned}$$

The X uncertainty is

$$\Delta X = \sqrt{\langle X^2\rangle - \langle X\rangle^2} = \frac{\pi\beta}{2\sqrt{3}}.$$

Similarly, for the Fourier transform we have:

$$\begin{aligned}\langle P\rangle &= \int_{-\infty}^{\infty} p|\bar{\phi}(p)|^2\,dp = \frac{\pi\beta}{4\hbar}\int_{-\infty}^{\infty} p\,\text{sech}^2(\frac{\pi\beta p}{2\hbar})\,dp = 0\\ \langle P^2\rangle &= \int_{-\infty}^{\infty} p^2|\bar{\phi}(p)|^2\,dp = \frac{\pi\beta}{4\hbar}\int_{-\infty}^{\infty} p^2\text{sech}^2(\frac{\pi\beta p}{2\hbar})\,dp = \frac{2\hbar^2}{\pi^2\beta^2}\int_{-\infty}^{\infty} u^2\text{sech}^2(u)\,du = \frac{\hbar^2}{\beta^2 3}.\end{aligned}$$

Thus

$$\Delta P = \sqrt{\langle P^2\rangle - \langle P\rangle^2} = \frac{\hbar}{\beta\sqrt{3}}.$$

The uncertainty product is

$$\Delta X\Delta P = \frac{\pi\beta}{2\sqrt{3}}\frac{\hbar}{\beta\sqrt{3}} = \frac{\hbar\pi}{6}.$$

b) The evolution in $\pi/2\omega$ gives a well-known quantity, a scaled Fourier transform of the wavefunction.

$$\Phi(x,\frac{\pi}{2\omega}) = U(\frac{\pi}{2\omega},0)\Phi(x,0) = e^{-i\pi/4}\sqrt{\frac{\hbar}{\sigma^2}}\mathcal{F}\{\Phi(x,0)\}\big|_{p=\hbar x/\sigma^2}$$

We can see that the function to be computed its Fourier transform is spatially shifted by x_0 so we could directly use the respective property of Fourier transform of a shifter function:

$$\mathcal{F}\{\Phi(x,0)\} = \bar{\Phi}(p,0) \Longrightarrow \mathcal{F}\{\Phi(x-x_0,0)\} = e^{-ipx_0/\hbar}\bar{\Phi}(p,0).$$

So,

$$\Phi(x,\frac{\pi}{2\omega}) = -e^{-i\pi/4}\sqrt{\frac{\hbar}{\sigma^2}}\left[e^{-ipx_0/\hbar}\bar{\Phi}(p,0)\right]\Big|_{p=\hbar x/\sigma^2} = -\sqrt{\frac{\pi\beta}{4\sigma^2}}e^{-i\pi/4}e^{-i\frac{xx_0}{\sigma^2}}\ \text{sech}(\frac{\pi\beta x}{2\sigma^2}).$$

c) To maintain the width $\Delta X = \frac{\pi\beta}{2\sqrt{3}}$, we compute ΔX for $\Phi(0, \pi/2\omega)$ and equate it to the uncertainty at $t = 0$:

$$\left.\begin{array}{l} \langle X \rangle = 0 \\ \langle X^2 \rangle = \dfrac{\pi\beta}{4\sigma^2} \displaystyle\int_{-\infty}^{\infty} x^2 \mathrm{sech}^2(\frac{\pi\beta x}{2\sigma^2})\, dx = \dfrac{\sigma^4}{3\beta^2}. \end{array}\right\} \Delta X = \sqrt{\langle X^2 \rangle} = \frac{\sigma^2}{\sqrt{3}\beta}.$$

Equating it with the uncertainty of the wavefunction at $t = 0$:

$$\frac{\pi\beta}{2\sqrt{3}} = \frac{\sigma^2}{\sqrt{3}\beta} \longrightarrow \beta = \sqrt{\frac{2\sigma^2}{\pi}}.$$

Problem V