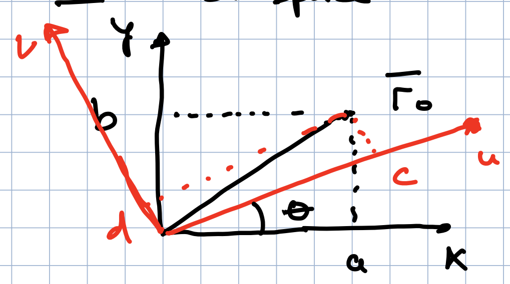
OPTI 570 LECTURE Th Sep 4

Representations CT II C

ex: 2D space



$\bar{r}_0 = a\hat{x} + b\hat{y} \Rightarrow (a, b)$ represents $\bar{r}_0$ in the $\{x, y\}$ basis

$\bar{r}_0 = c\hat{u} + d\hat{v} \Rightarrow (c, d)$ represents $\bar{r}_0$ in the $\{u, v\}$ basis

$$(a, b) \neq (c, d) \neq \vec{r}_0$$

Notation: $\bar{r}_{0,\{\hat{x},\hat{y}\}} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\bar{r}_0 \rightarrow \begin{pmatrix} a \\ b \end{pmatrix}$ in the $\{\hat{x}, \hat{y}\}$ basis

$$\begin{pmatrix} c \\ d \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{\text{transformation representation matrix}} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

Representations in Quantum Mechanics

Let $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ a basis for $\mathcal{E}_{1,2,3}$

$$|\psi\rangle = a|u_1\rangle + b|u_2\rangle + c|u_3\rangle$$

$$|\psi\rangle \rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = |\psi\rangle_{\{u\}}$$

"matrix" represents $|\psi\rangle$ in the u basis "representation"

In general: $\{|u_n\rangle\}$, n - integer

$$|\psi\rangle = \sum_{\text{all } n} c_n |u_n\rangle$$

$$|\psi\rangle \rightarrow \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$$

$$\langle\psi| \rightarrow \begin{pmatrix} c_1^* & c_2^* & c_3^* & \dots \end{pmatrix}$$

Express $|\psi\rangle$ in the $\{|u_n\rangle\}$ representation

1. Write closure relation for $\{|u_n\rangle\}$ basis

$$\hat{\mathbb{1}} = \sum_n |u_n\rangle\langle u_n|$$

2. Apply closure relation to $|\psi\rangle$

$$|\psi\rangle = \hat{\mathbb{1}}|\psi\rangle = \sum_n |u_n\rangle\underline{\langle u_n|\psi\rangle} = \sum_n \langle u_n|\psi\rangle \underline{|u_n\rangle}$$

3.

$$|\psi\rangle \rightarrow \begin{pmatrix} \langle u_1|\psi\rangle \\ \langle u_2|\psi\rangle \\ \vdots \end{pmatrix} \leftarrow \text{coefficients of } |\psi\rangle \text{ and basis elements}$$

Ex: $\{|\uparrow\rangle, |0\rangle, |\downarrow\rangle\}$ - "arrow basis"

$$\mathbb{1} = |\uparrow\rangle\langle\uparrow| + |0\rangle\langle 0| + |\downarrow\rangle\langle\downarrow|$$

$$|\uparrow\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |0\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |\downarrow\rangle \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{in "arrow" basis}$$

Transforming between representations

<u>example</u>: • 3D state space $\mathcal{E}$
• $|\psi\rangle \in \mathcal{E}$

how to represent in two different bases $\{u\}, \{v\}$

$$\hat{\mathbb{1}} = \sum_{j=1}^{3} |u_j\rangle\langle u_j| \quad \text{and} \quad \hat{\mathbb{1}} = \sum_{j=1}^{3} |v_j\rangle\langle v_j|$$

<u>Q</u>: $\langle u_m | u_n \rangle = \delta_{m,n}$

<u>Q</u>: $\langle v_m | v_n \rangle = \delta_{m,n}$

<u>Q</u>: $\langle u_m | v_n \rangle =$ scalar $\in \mathbb{C}$
$\hookrightarrow 0 <$ norm < 1

1. Express $|\psi\rangle$ in the $\{u\}$ basis representation:

$$|\psi\rangle = \mathbb{1}|\psi\rangle =$$
$$= \sum_{j=1}^{3} |u_j\rangle \underline{\langle u_j | \psi\rangle} =$$
$$= \sum_{j=1}^{3} \langle u_j|\psi\rangle |u_j\rangle$$
$$\Rightarrow |\psi\rangle_{\{u\}} = \begin{pmatrix} \langle u_1|\psi\rangle \\ \langle u_2|\psi\rangle \\ \langle u_3|\psi\rangle \end{pmatrix}$$

Corresponding $\langle\psi|_{\{u\}} = \left(\langle\psi|u_1\rangle \;\; \langle\psi|u_2\rangle \;\; \langle\psi|u_3\rangle\right)$
$= (\langle u_1|\psi\rangle)^*$

Same ... same in $\{v\}$ basis

$$|\psi\rangle_{\{v\}} = \begin{pmatrix} \langle v_1|\psi\rangle \\ \langle v_2|\psi\rangle \\ \langle v_3|\psi\rangle \end{pmatrix}$$

If we know $|\psi\rangle$ in the $\{u\}$ repres, Q: what is $|\psi\rangle$ in the $\{v\}$ repres?

Ex: $|\psi\rangle_{\{u\}} = \begin{pmatrix} 1/\sqrt{3} \\ i\sqrt{2/3} \\ 0 \end{pmatrix}$ $\Rightarrow$ $|\psi\rangle_{\{v\}} = ?$

Let: $|v_1\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + 0|u_2\rangle + \frac{i}{\sqrt{2}}|u_3\rangle$

$|v_2\rangle = 0\cdot|u_1\rangle + |u_2\rangle + 0|u_3\rangle$

$|v_3\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + 0|u_2\rangle - \frac{i}{\sqrt{2}}|u_3\rangle$

Representation:

$|v_1\rangle_{\{u\}} = \begin{pmatrix} \langle u_1|v_1\rangle \\ \langle u_2|v_1\rangle \\ \langle u_3|v_1\rangle \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ i/\sqrt{2} \end{pmatrix}$

$|v_2\rangle_{\{u\}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\qquad |v_3\rangle_{\{u\}} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -i/\sqrt{2} \end{pmatrix}$

Q: $|\psi\rangle_{\{v\}} = ?$

Need $\underbrace{\langle v_k|\psi\rangle} = \langle v_k|\hat{\mathbb{1}}_u|\psi\rangle =$

$= \langle v_k\Big(\sum_{j=1}^{3}|u_j\rangle\langle u_j|\Big)\psi\rangle =$

$= \sum_{j=1}^{3}\underbrace{\langle v_k|u_j\rangle}\underline{\langle u_j|\psi\rangle}$

$|\psi\rangle = \sum_{k=1}^{3}|v_k\rangle\langle v_k|\psi\rangle = \sum_{k=1}^{3}\underline{\langle v_k|\psi\rangle}\,|v_k\rangle$

$= \sum_{k=1}^{3}\Big(\sum_{j=1}^{3}\underline{\langle v_k|u_j\rangle}\,\underline{\langle u_j|\psi\rangle}\Big)|v_k\rangle$

So:

$$|\psi\rangle_{\{v\}} = \begin{pmatrix} \langle v_1|\psi\rangle \\ \langle v_2|\psi\rangle \\ \langle v_3|\psi\rangle \end{pmatrix} = \begin{pmatrix} \langle v_1|u_1\rangle\langle u_1|\psi\rangle + \langle v_1|u_2\rangle\langle u_2|\psi\rangle + \langle v_1|u_3\rangle\langle u_3|\psi\rangle \\ \langle v_2|u_1\rangle\langle u_1|\psi\rangle + \langle v_2|u_2\rangle\langle u_2|\psi\rangle + \dots \\ \dots \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} \langle v_1|u_1\rangle & \langle v_1|u_2\rangle & \langle v_1|u_3\rangle \\ \langle v_2|u_1\rangle & \langle v_2|u_2\rangle & \langle v_2|u_3\rangle \\ \langle v_3|u_1\rangle & \langle v_3|u_2\rangle & \langle v_3|u_3\rangle \end{pmatrix}}_{M^\dagger\text{, the Herm conj of } M} \begin{pmatrix} \langle u_1|\psi\rangle \\ \langle u_2|\psi\rangle \\ \langle u_3|\psi\rangle \end{pmatrix}$$

$$|\psi\rangle_{\{v\}} = M^\dagger\, |\psi\rangle_{\{u\}}$$

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ i & 0 & -i \end{pmatrix} \xrightarrow{H.C} M^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -i \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & i \end{pmatrix}$$

$$|\psi\rangle_{\{v\}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -i \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & i \end{pmatrix}\begin{pmatrix} 1/\sqrt{3} \\ i\sqrt{2/3} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1/\sqrt{3} \\ 2i/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

<u>check</u> – "norm" is still 1?

$$\frac{1}{6} + \frac{4}{6} + \frac{1}{6} = 1 \checkmark$$

<u>Recap</u>: $|\psi\rangle_{\{v\}} = M^\dagger\, |\psi\rangle_{\{u\}}$

<u>Field Guide page 18</u>

Operator representation

$$\hat{A} = \hat{\mathbb{1}}\,\hat{A}\,\hat{\mathbb{1}} =$$

$$= \sum_{j=1}^{3} |u_j\rangle\langle u_j|\ \hat{A} \sum_{m=1}^{3} |u_m\rangle\langle u_m| =$$

$$= \sum_j \sum_m |u_j\rangle \underline{\langle u_j|\hat{A}|u_m\rangle} \langle u_m| =$$

$$= \sum_j \sum_m \underbrace{\langle u_j|\hat{A}|u_m\rangle}\ \underline{|u_j\rangle\langle u_m|}$$

Q: What is $|u_j\rangle\langle u_m|$ in the u rep?

$$\Rightarrow \hat{A} \rightarrow \begin{pmatrix} \langle u_1|\hat{A}|u_1\rangle & \langle u_1|\hat{A}|u_2\rangle & \cdots \\ \langle u_2|\hat{A}|u_1\rangle & & \\ & & \ddots \end{pmatrix} = A_{\{u\}}$$

$$\boxed{\begin{array}{l} A \text{ in the } \{v\} \text{ representation} \\ A_{\{v\}} = M^{\dagger} A_{\{u\}} M \quad \text{w/ } M \text{ as defined previously} \end{array}}$$

CT II-D - Observables

$\hat{A}\,|\lambda_n^i\rangle = \lambda_n |\lambda_n^i\rangle$ ← i index of degeneracy

For any λ_n, there are $i = 1, 2, \ldots g_n$ - degree of degeneracy