OPTI 570 Practice Exam 1

Problem 1

a. $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ b. $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$ c. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$

d. $\begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}$ e. $-1/\sqrt{2} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}$ f. $\mathbb{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ closure rel.

g. $\begin{pmatrix} -\hbar\omega & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hbar\omega \end{pmatrix}$ h. $\begin{pmatrix} e^{i\omega t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\omega t} \end{pmatrix}$

i. $\begin{pmatrix} e^{-i\omega t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\omega t} \end{pmatrix} \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} e^{-i\omega t} \\ -1/\sqrt{2} \\ \frac{1}{2} e^{i\omega t} \end{pmatrix}$

j. $\hat{P}_S = |a_3\rangle\langle a_3|$ $\hat{P}_H = \underbrace{\hat{U}^+ |a_3\rangle}_{i} \underbrace{\langle a_3| U}_{\text{conj of } i} = \begin{pmatrix} \frac{1}{2} e^{-i\omega t} \\ -1/\sqrt{2} \\ \frac{1}{2} e^{i\omega t} \end{pmatrix} \left(\frac{1}{2} e^{i\omega t} \; -\frac{1}{\sqrt{2}} \; \frac{1}{2} e^{-i\omega t} \right)$

$$= \begin{pmatrix} \frac{1}{4} & -\frac{1}{2\sqrt{2}} e^{-i\omega t} & \frac{1}{4} e^{-2i\omega t} \\ -\frac{1}{2\sqrt{2}} e^{+i\omega t} & \frac{1}{2} & -\frac{1}{2\sqrt{2}} e^{-i\omega t} \\ \frac{1}{4} e^{2i\omega t} & -\frac{1}{2\sqrt{2}} e^{i\omega t} & \frac{1}{4} \end{pmatrix}$$

Hermitian ✓

Trace $= 1$ ✓

Problem 2

$$FT[\psi(x - x_0)] = \langle p | \hat{S}(x_0) | \psi \rangle =$$
$$= e^{-i x_0 p/\hbar} \langle p | \psi \rangle =$$
$$= e^{-i x_0 p \hbar} \tilde{\psi}(p)$$

Problem 3

a.

$e^{+ip_0\hat{x}/\hbar}$ – translates in momentum space by p_0

$e^{-ip_0\hat{x}/\hbar}$ – " by $-p_0$

$e^{-ix_0\hat{p}/\hbar}$ – " in position space by x_0

$e^{ix_0\hat{p}/\hbar}$ – " by $-x_0$

Overall, the operator brings state back to initial position

$\hat{A} = 1^n = \mathbb{1}$

b. Projector onto the $x_0 - \sigma$ to $x_0 + \sigma$ part of the position axis

c.
$$\hat{C} = \int_{-\infty}^{+\infty} dx' \, |x'\rangle F(x') \langle x'|$$

in pos:
$$\langle x|\hat{C}| = \int_{-\infty}^{+\infty} dx' \, \underbrace{\langle x|x'\rangle}_{\delta(x-x')} F(x') \langle x'| =$$

$$= F(x)\langle x|$$

in pos. representation, $\hat{C}$ multiplies state by $F(x)$

To be projector, $\hat{C}^2 = \hat{C}$

$$\hat{C}^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx' dx'' \, |x'\rangle\langle x'|F\rangle \underbrace{\langle x'|x''\rangle}_{\delta(x'-x'')} \langle x''|F\rangle \langle x''| =$$

$$= \int_{-\infty}^{+\infty} dx' \, |x'\rangle\langle x'|F\rangle \langle x'|F\rangle\langle x'| =$$

$$= \int_{-\infty}^{+\infty} |\langle x'|F\rangle|^2 \, |x'\rangle\langle x'| \, dx'$$

$$\hat{C} = \int_{-\infty}^{\infty} (\langle x'|F\rangle)^2 \, |x'\rangle\langle x'| dx'$$

$\hat{C}^2 = \hat{C}$ only if $F(x)^2 = F(x)$ for all x

so $F(x) = \begin{cases} 0 \\ 1 \end{cases}$ or piecewise

Problem 4 $\hat{B} = \sum_{m=-2}^{2} |m| \, |-m\rangle\langle m|$

[a] $\langle \psi | \hat{N} | \psi \rangle = ?$

$$\hat{N}|\psi\rangle = \begin{pmatrix} 2 \cdot \frac{1}{2} \\ 1 \cdot \frac{1}{\sqrt{2}} \\ 0 \cdot \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/\sqrt{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\langle \psi | \hat{N} | \psi \rangle = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1/\sqrt{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} + \frac{1}{2} \boxed{= 1}$$

[b] $$\hat{B} \to \begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \boxed{\begin{array}{l} B \text{ is Hermitian} \\ B \text{ in not unitary} \end{array}}$$

[c] $\mathrm{Tr}\, \hat{B} = 0$

[d]

$|\varphi\rangle \equiv c \hat{B} |\psi\rangle$ - normalized

$$|\varphi\rangle \to c \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \\ 0 \\ 0 \end{pmatrix} = c \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/\sqrt{2} \\ 1 \end{pmatrix}$$

$$\langle \varphi | \varphi \rangle = 1 \Rightarrow c^2 \cdot \left(\frac{1}{2} + 1 \right) = 1 \Rightarrow \boxed{c = \sqrt{\frac{2}{3}}}$$

$$|\varphi\rangle \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{1/3} \\ \sqrt{2/3} \end{pmatrix} \qquad \hat{N}|\varphi\rangle \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\sqrt{1/3} \\ -2\sqrt{\frac{2}{3}} \end{pmatrix}$$

$$\boxed{\langle \varphi | \hat{N} | \varphi \rangle = -1/3 - \frac{4}{3} = -\frac{5}{3}}$$

e) $|\beta_1\rangle \equiv \mu\,|m=-1\rangle + \nu\,|m=1\rangle \qquad \hat{B}\,|\beta_1\rangle = |\beta_1\rangle$

$$\hat{B}\,|\beta_1\rangle = \mu\,|-1\rangle + \nu\,|1\rangle \Rightarrow \boxed{|\beta_1\rangle = \frac{1}{\sqrt{2}}\left(|-1\rangle + |1\rangle\right)}$$

f) $\underline{\lambda = 2} \Rightarrow |\beta_2\rangle = \frac{1}{\sqrt{2}}\left(|2\rangle + |-2\rangle\right)$

$\underline{\lambda = -1} \Rightarrow |\beta_{-1}\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle - |-1\rangle\right)$

$\underline{\lambda = -2} \Rightarrow |\beta_{-2}\rangle = \frac{1}{\sqrt{2}}\left(|2\rangle - |-2\rangle\right)$

$\underline{\lambda = 0} \Rightarrow |\beta_0\rangle = |0\rangle$

g) Frob above: $\quad |2\rangle = \frac{1}{\sqrt{2}}\left(|\beta_2\rangle + |\beta_{-2}\rangle\right)$

$\quad |1\rangle = \frac{1}{\sqrt{2}}\left(|\beta_1\rangle + |\beta_{-1}\rangle\right)$

$$\Rightarrow \boxed{|\psi\rangle = \frac{1}{2\sqrt{2}}\,|\beta_2\rangle + \frac{1}{2\sqrt{2}}\,|\beta_{-2}\rangle + \frac{1}{2}\,|\beta_1\rangle + \frac{1}{2}\,|\beta_{-1}\rangle + \frac{1}{2}\,|\beta_0\rangle}$$

h) $\langle 1|\, e^{i\theta\hat{B}}\, |1\rangle \qquad e^{i\theta\hat{B}} \cdot \frac{1}{\sqrt{2}}\left(|\beta_1\rangle + |\beta_{-1}\rangle\right) \qquad e^{i\theta\hat{B}}_{\{\beta\}} = \begin{pmatrix} e^{2i\theta} & & & & 0 \\ & e^{i\theta} & & & \\ & & 0 & & \\ & & & e^{-i\theta} & \\ 0 & & & & e^{-2i\theta} \end{pmatrix}$

$$\Rightarrow e^{i\theta\hat{B}}\,|1\rangle = \frac{1}{\sqrt{2}}\left(e^{i\theta}\,|\beta_1\rangle + e^{-i\theta}\,|\beta_{-1}\rangle\right)$$

$$\boxed{\langle 1|\, e^{i\theta\hat{B}}\, |1\rangle = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \cos\theta}$$

Problem 5

$\boxed{a}$ $\hat{H} \to \frac{\hbar\omega}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -i\sqrt{3} \\ 0 & +i\sqrt{3} & 5 \end{pmatrix} \Rightarrow \underline{\frac{\hbar\omega}{2}}$ and $\underline{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

For other two $\begin{vmatrix} 3-\lambda & -i\sqrt{3} \\ i\sqrt{3} & 5-\lambda \end{vmatrix} = 0$ $\quad (3-\lambda)(5-\lambda) - 3 = 0$

$\lambda^2 - 8\lambda + 12 = 0 \quad (\lambda-6)(\lambda-2) = 0$

$\underline{\lambda = 2}$ $\begin{pmatrix} 3 & -i\sqrt{3} \\ i\sqrt{3} & 5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2v_1 \\ 2v_2 \end{pmatrix} \Rightarrow -i\sqrt{3}\, v_2 = v_1$

$\Rightarrow \begin{pmatrix} 1 \\ -\frac{i}{\sqrt{3}} \end{pmatrix} \Rightarrow \frac{1}{\sqrt{1+\frac{1}{3}}}\begin{pmatrix} 1 \\ -\frac{i}{\sqrt{3}} \end{pmatrix} = \underline{\begin{pmatrix} \sqrt{3}/2 \\ -i/2 \end{pmatrix}}$

$\underline{\lambda = 6}$ $\begin{pmatrix} 3 & -i\sqrt{3} \\ i\sqrt{3} & 5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 6v_1 \\ 6v_2 \end{pmatrix} \Rightarrow -i\sqrt{3}\, v_2 = 3v_1$

$\Rightarrow \begin{pmatrix} 1 \\ i\sqrt{3} \end{pmatrix} \Rightarrow \frac{1}{\sqrt{1+3}}\begin{pmatrix} 1 \\ i\sqrt{3} \end{pmatrix} = \underline{\begin{pmatrix} 1/2 \\ i\sqrt{3}/2 \end{pmatrix}}$

$\underline{\hbar\omega/2}$	$\underline{\hbar\omega}$	$\underline{3\hbar\omega}$
$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ \sqrt{3}/2 \\ -i/2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1/2 \\ i\sqrt{3}/2 \end{pmatrix}$

$\boxed{b}$ $\hat{u}(t,0) = e^{-i\hat{H}t/\hbar}$

$\hat{u}_{\{u\}} = \begin{pmatrix} e^{-i\omega t/2} & 0 & 0 \\ 0 & e^{-i\omega t} & 0 \\ 0 & 0 & e^{-i3\omega t} \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -i/2 & i\sqrt{3}/2 \end{pmatrix}$

$\hat{u}_{\{u\}} = M^{+}\hat{u}_{\{\phi\}}M \Rightarrow \hat{u}_{\{\phi\}} = M\,\hat{u}_{\{u\}}M^{+} =$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -i/2 & i\sqrt{3}/2 \end{pmatrix}\begin{pmatrix} e^{-i\omega t/2} & 0 & 0 \\ 0 & e^{-i\omega t} & 0 \\ 0 & 0 & e^{-i3\omega t} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & i/2 \\ 0 & 1/2 & -i\sqrt{3}/2 \end{pmatrix} =$

$$= \begin{pmatrix} e^{-i\omega t/2} & 0 & 0 \\ 0 & \frac{3}{4}e^{-i\omega t} + \frac{1}{4}e^{-i3\omega t} & i\frac{\sqrt{3}}{4}e^{-i\omega t} - i\frac{\sqrt{3}}{4}e^{-i3\omega t} \\ 0 & -\frac{i\sqrt{3}}{4}e^{-i\omega t} + \frac{i\sqrt{3}}{4}e^{-i3\omega t} & \frac{1}{4}e^{-i\omega t} + \frac{3}{4}e^{-i3\omega t} \end{pmatrix}$$

$\boxed{c}$ $|\Psi(0)\rangle = |\phi_2\rangle$

$$\Rightarrow |\Psi(t)\rangle = \hat{U}_{4\phi 4} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3/4\, e^{i\omega t} + \frac{1}{4}e^{-i3\omega t} \\ i\frac{\sqrt{3}}{4}e^{-i\omega t} - i\frac{\sqrt{3}}{4}e^{-i3\omega t} \end{pmatrix}$$

$\boxed{d}$ $P_3(t) = |\langle \phi_3 | \Psi(t)\rangle|^2 =$

$$= \frac{3}{16}\left(e^{-i\omega t} - e^{-i3\omega t}\right)^2 = \frac{3}{16}\underbrace{\left(e^{-i2\omega t}\right)^2}_{1} \cdot \underbrace{\left(e^{i\omega t} - e^{-i\omega t}\right)^2}_{2\cdot \sin\omega t}$$

$$= \frac{3}{16} \cdot 4 \cdot \sin^2\omega t \boxed{= \frac{3}{4}\sin^2\omega t}$$

$\boxed{e}$