OPTI 570 MIDTERM 2 SOLUTIONS

Problem 1

a. $E_n = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right)$

$\frac{1}{3} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{3} = 1$ ✓ state is normalized

Outcomes:

$|000\rangle \rightarrow \frac{3}{2}\hbar\omega$ w/ prob $\frac{1}{3}$

$|100\rangle \rightarrow \frac{5}{2}\hbar\omega$ w/ prob $\frac{1}{9}$

$|010\rangle \rightarrow \frac{5}{2}\hbar\omega$ w/ prob. $\frac{1}{9}$ $\Big]\ \frac{2}{9}$

$|002\rangle \rightarrow \frac{7}{2}\hbar\omega$ w/ prob. $\frac{1}{9}$

$|101\rangle \rightarrow \frac{7}{2}\hbar\omega$ w/ prob $\frac{1}{3}$ $\Big]\ \frac{4}{9}$

these are stationary states so measurement results will be the same at later t

b. $\frac{5}{2}\hbar\omega \Rightarrow |\psi\rangle = C\cdot\left(\frac{1}{3}|100\rangle - \frac{i}{3}|010\rangle\right)$

$C^2\cdot\left(\frac{1}{9}+\frac{1}{9}\right) = 1 \Rightarrow C = \frac{3}{\sqrt{2}}$

$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|100\rangle - i\,|010\rangle\right)$

c. $|n(t)\rangle = e^{-iE_nt/\hbar}\,|n(0)\rangle$

$e^{-i\frac{3}{2}\hbar\omega\left(\frac{\pi}{\omega}\right)/\hbar} = i$

$e^{-i\frac{5}{2}\hbar\omega\left(\frac{\pi}{\omega}\right)/\hbar} = -i$

$e^{-i\frac{7}{2}\hbar\omega\left(\frac{\pi}{\omega}\right)/\hbar} = i$

$|\psi\left(\frac{\pi}{\omega}\right)\rangle = \frac{i}{\sqrt{3}}|000\rangle - \frac{i}{3}|100\rangle - \frac{1}{3}|010\rangle + \frac{i}{3}|002\rangle + \frac{1}{\sqrt{3}}|101\rangle$

OR usually first term is real and positive $\Rightarrow$ global phase multiply by $-i$

$|\psi\left(\frac{\pi}{\omega}\right)\rangle = \frac{1}{\sqrt{3}}|000\rangle - \frac{1}{3}|100\rangle + \frac{i}{3}|010\rangle + \frac{1}{3}|002\rangle - \frac{i}{\sqrt{3}}|101\rangle$

Problem 2

a. $H\{\phi\} = \frac{1}{2}\hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & i\sqrt{3} \\ 0 & -i\sqrt{3} & 5 \end{pmatrix}$ $\quad \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & i\sqrt{3} \\ 0 & -i\sqrt{3} & 5-\lambda \end{pmatrix} = 0$

$\Rightarrow \underline{\lambda=1} \Rightarrow (1-\lambda)(15-8\lambda-3+\lambda^2)=0$

$(1-\lambda)(6-\lambda)(2-\lambda)=0$

$\underline{\lambda=1}$ w/ eigenstate $\phi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\underline{\lambda=2}$ $\begin{pmatrix} 3 & i\sqrt{3} \\ -i\sqrt{3} & 5 \end{pmatrix}\begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2v_2 \\ 2v_2 \end{pmatrix} \Rightarrow 3v_2 + i\sqrt{3}v_3 = 2v_2 \Rightarrow v_2 = -i\sqrt{3}\,v_3$

$v_2 = 1$

$v_3 = \frac{i}{\sqrt{3}}$ $\quad \begin{pmatrix} 0 \\ 1 \\ i/\sqrt{3} \end{pmatrix} \xrightarrow{\text{norm}} \frac{1}{2}\begin{pmatrix} 0 \\ \sqrt{3} \\ i \end{pmatrix}$

$\underline{\lambda=6}$ $\begin{pmatrix} 3 & i\sqrt{3} \\ -i\sqrt{3} & 5 \end{pmatrix}\begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 6v_2 \\ 6v_3 \end{pmatrix}$ $\quad 3v_2 + i\sqrt{3}v_3 = 6v_2 \Rightarrow v_3 = -i\sqrt{3}\,v_2$

$\begin{pmatrix} 0 \\ 1 \\ -i\sqrt{3} \end{pmatrix} \xrightarrow{\text{norm}} \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -i\sqrt{3} \end{pmatrix}$

Summary:

$\frac{1}{2}\hbar\omega$ w/ eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |\varphi_1\rangle = |u_1\rangle$

$\hbar\omega$ w/ $\frac{1}{2}\begin{pmatrix} 0 \\ \sqrt{3} \\ i \end{pmatrix} = \frac{\sqrt{3}}{2}|\varphi_2\rangle + \frac{i}{2}|\varphi_3\rangle = |u_2\rangle$

$3\hbar\omega$ w/ $\frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -i\sqrt{3} \end{pmatrix} = \frac{1}{2}|\varphi_2\rangle - i\frac{\sqrt{3}}{2}|\varphi_3\rangle = |u_3\rangle$

b. $|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$, $|\psi(0)\rangle = |\varphi_2\rangle$

$\frac{3}{2}|\varphi_2\rangle + \frac{i\sqrt{3}}{2}|\varphi_3\rangle = \sqrt{3}\,|u_2\rangle$

$\frac{1}{2}|\varphi_2\rangle + \frac{i\sqrt{3}}{2}|\varphi_3\rangle = |u_3\rangle$

+ then div. by 2

$|\varphi_2\rangle = \frac{\sqrt{3}}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$

$\Rightarrow |\psi(t)\rangle = \frac{\sqrt{3}}{2}e^{-i\omega t}|u_2\rangle + \frac{1}{2}e^{-i3\omega t}|u_3\rangle$

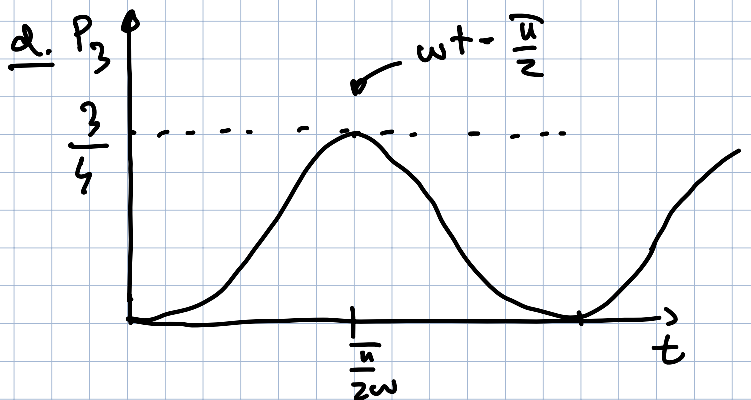
c. $|\psi(t)\rangle = \frac{\sqrt{3}}{2} e^{-i\omega t}\left(\frac{\sqrt{3}}{2}|\varphi_2\rangle + \frac{i}{2}|\varphi_3\rangle\right) + \frac{1}{2} e^{-3i\omega t}\left(\frac{1}{2}|\varphi_2\rangle - i\frac{\sqrt{3}}{2}|\varphi_3\rangle\right)$

$$= \left(\frac{3}{4} e^{-i\omega t} + \frac{1}{4} e^{-3i\omega t}\right)|\varphi_2\rangle + \left(\frac{i\sqrt{3}}{4} e^{-i\omega t} - \frac{i\sqrt{3}}{4} e^{-3i\omega t}\right)|\varphi_3\rangle$$

$$P_3(t) = \frac{3}{16}\left|e^{-i\omega t} - e^{-3i\omega t}\right|^2 = \frac{3}{16}(2\sin\omega t)^2 =$$

$$\boxed{P_3(t) = \frac{3}{4}\sin^2\omega t}$$

d.

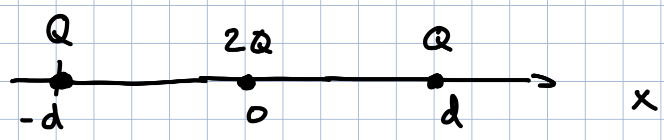


Interpretation: adding W makes it such that states $|\varphi_2\rangle$ and $|\varphi_3\rangle$ are not stationary. There is now a probability of transitioning from $|\varphi_2\rangle$ to $|\varphi_3\rangle$ that varies sinusoidally in time.

Problem 3

configuration:

$$V_c = k_e \frac{q_1 q_2}{r}$$



$$V(x) = 2k_e Q^2 \left(\frac{1}{x+d} - \frac{1}{x-d} \right)$$

a. Harmonic approximation $\frac{1}{x+d} - \frac{1}{x-d} = \frac{x-d-(x+d)}{(x^2-d^2)} = -\frac{2d}{x^2-d^2} = \frac{2}{d} \cdot \frac{1}{1-\frac{x^2}{d^2}}$

Taylor expand: $\frac{1}{1-\frac{x^2}{d^2}} = 1 + \frac{x^2}{d^2} + \ldots$

$$V(x) = 2k_e Q^2 \cdot \frac{2}{d} \cdot \left(1 + \frac{x^2}{d^2} + \ldots\right)$$

quad. term $V_{QHO}(x) = \frac{4 k_e Q^2}{d^3} x^2 = \frac{1}{2} m \omega^2 x^2$

$$\boxed{\omega = \sqrt{\frac{8 k_e Q^2}{m d^3}}}$$

b. $d = 10^{-10}\ m$ $m = 2.7 \cdot 10^{-26}\ kg$

$$\omega = \sqrt{\frac{8 \cdot 9 \cdot 10^9\ Nm^2C^{-2} \cdot (1.6 \cdot 10^{-19} C)^2}{2.7 \cdot 10^{-26} kg \cdot (10^{-10}\ m)^3}} = \sqrt{\frac{8 \cdot 9 \cdot 1.6^2}{2.7}} \cdot 10^{(9-38+26+30)/2} \frac{rad}{s} =$$

$$\boxed{\omega = 2.6 \cdot 10^{14} \frac{rad}{s}}$$

c. $k_B T = M v^2$ $v = 240 \frac{m}{s}$

$p_0 = Mv = \sqrt{k_B T M} = 1.74 \cdot 10^{-23}\ kg \frac{m}{s}$

Coherent state: $\alpha_0 = \frac{i p_0 \sigma}{\sqrt{2}\hbar}$ $\sigma = \sqrt{\frac{\hbar}{m\omega}} = 3.87 \cdot 10^{-12}\ m$

$\alpha_0 = i \cdot 0.45$

$\langle x_{max} \rangle = \sqrt{2}\sigma |\alpha_0| \boxed{= 2.5 \cdot 10^{-12}\ m = 0.025 \cdot d}$

so the oscillation is a small fraction of mol. spacing

$P_{n=0} = e^{-|\alpha_0|^2} = 0.8$

Problem 4

$\hat{B}(p_0) = \hat{\mathbb{1}} + \hat{T}(p_0)$

a. $|\psi(0)\rangle = C\,\hat{B}(p_0)\,|\varphi_0\rangle = C\left[\,|\varphi_0\rangle + |\alpha_0 = \frac{i p_0 \sigma}{\sqrt{2}\,\hbar}\rangle\right]$

$$\langle\psi(0)|\psi(0)\rangle = 1 = |C|^2\left[\langle\varphi_0|\varphi_0\rangle + \langle\varphi_0|\alpha_0\rangle + \langle\alpha_0|\varphi_0\rangle + \langle\alpha_0|\alpha_0\rangle\right]$$

$$C^2\cdot\left(1 + e^{-|\alpha_0|^2/2} + e^{-|\alpha_0|^2/2} + 2\right) = 1$$

$$\underline{C = 1/\sqrt{2 + 2\cdot e^{-|\alpha_0|^2/2}}}$$

b. $$\psi(x,t=0) = C\left[\left(\frac{1}{\pi\sigma^2}\right)^{\frac14} e^{-\frac{x^2}{2\sigma^2}} + \left(\frac{1}{\pi\sigma^2}\right)^{\frac14} e^{-\frac{x^2}{2\sigma^2}} e^{\frac{i p_0 x}{\hbar}}\right] =$$

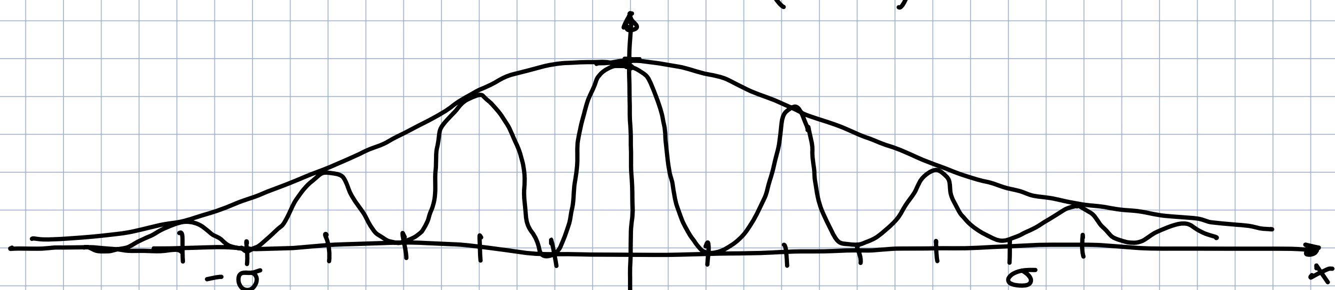
$$= C\left(\frac{1}{\pi\sigma^2}\right)^{\frac14} e^{-\frac{x^2}{2\sigma^2}}\left[1 + e^{\frac{i p_0 x}{\hbar}}\right] =$$

$$= C\left(\frac{1}{\pi\sigma^2}\right)^{\frac14} e^{-\frac{x^2}{2\sigma^2}} e^{\frac{i p_0 x}{2\hbar}}\left[e^{-\frac{i p_0 x}{2\hbar}} + e^{\frac{i p_0 x}{2\hbar}}\right] =$$

$$= 2C\left(\frac{1}{\pi\sigma^2}\right)^{\frac14} e^{-\frac{x^2}{2\sigma^2}} e^{\frac{i p_0 x}{2\hbar}} \cos\left(\frac{p_0 x}{2\hbar}\right)$$

c. $|\psi|^2 = 4C^2 \frac{1}{\sqrt{\pi}\,\sigma} e^{-\frac{x^2}{\sigma^2}} \cos^2\left(\frac{p_0 x}{2\hbar}\right)$ $\qquad p_0 = \frac{5\pi\hbar}{\sigma}$

$|\psi|^2 = 4C^2\,|\varphi_0(x)|^2 \cos^2\left(\frac{5\pi x}{\sigma}\right) = 0 \Rightarrow x_0 = \frac{\sigma}{5}, \frac{\sigma}{3}, 6$

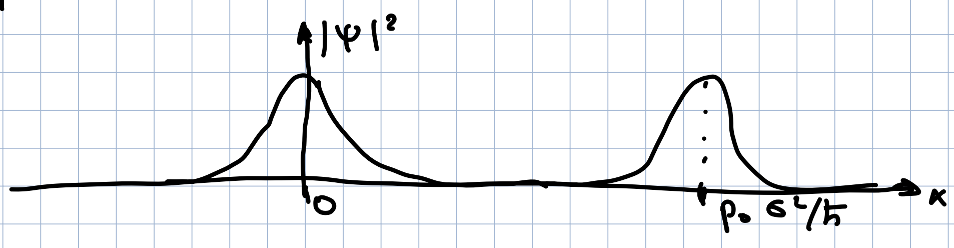


d. Quarter-period time evolution:

$$\psi\left(x, t = \frac{\pi}{2\omega}\right) = C\cdot\left(\langle x|\varphi_0\rangle + \langle x|-i\alpha_0\rangle\right)$$

$$= C\cdot\left(\frac{1}{\pi\sigma^2}\right)^{\frac14}\left[e^{-\frac{x^2}{2\sigma^2}} + e^{-\frac{(x-x_0)^2}{2\sigma^2}}\right] \qquad \text{w/ } x_0 = \frac{p_0\sigma^2}{\hbar}$$

$$\left|\psi\left(x, \frac{\pi}{2\omega}\right)\right|^2 = \frac{C^2}{\sigma\sqrt{\pi}}\left[e^{-\frac{x^2}{\sigma^2}} + e^{-\frac{(x-x_0)^2}{\sigma^2}} + \underbrace{2e^{-\frac{x^2}{2\sigma^2}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}}_{\text{small since minimal overlap}}\right]$$



Problem 5

a. $|\psi(0)\rangle$ is an eigenstate of the lowering operator $\Rightarrow$ coherent state
<u>NOT</u> energy eigenstate

$$\langle H\rangle = \hbar\omega\left(|\alpha_x|^2 + |\alpha_y|^2 + 1\right) = 51\hbar\omega$$

b. $$\Delta X \cdot \Delta P_x = \Delta Y \cdot \Delta P_y = \frac{\hbar}{2}$$

c. $|\alpha_x = 5\rangle \qquad |\alpha_y = 5i\rangle$

$\langle x\rangle = 5\sqrt{2}\,\sigma \qquad \langle x\rangle = 0$

$\langle p_x\rangle = 0 \qquad \langle p_y\rangle = 5\sqrt{2}\,\hbar/\sigma$

$$\psi(x,y,0) = \frac{1}{\sqrt{\pi}\,\sigma}\, e^{(x-5\sqrt{2}\sigma)^2/(2\sigma^2)}\, e^{-y^2/(2\sigma^2)}\, e^{i5\sqrt{2}\,y/\sigma}$$

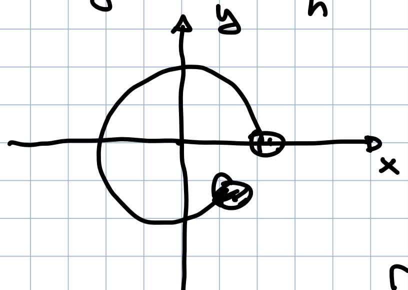
d. Yes, it will still be an eigenstate of $\hat{a}_x$ and $\hat{a}_y$ since it is a coherent state.

eigenvalue of a_x: $5\cdot(-i) = -5i$

—— " —— $a_y = 5i\cdot(-i) = 5$

e. $\langle x\rangle(t) = \langle x\rangle(0)\cos(\omega t) = 5\sqrt{2}\,\sigma\cos(\omega t)$

$$\langle y\rangle(t) = \frac{\sigma^2}{\hbar}\langle p_y\rangle(0)\sin(\omega t) = \frac{\sigma^{\cancel{2}}}{\cancel{\hbar}}\cdot\frac{5\sqrt{2}\cancel{\hbar}}{\cancel{\sigma}}\sin(\omega t) = 5\sqrt{2}\,\sigma\sin(\omega t)$$



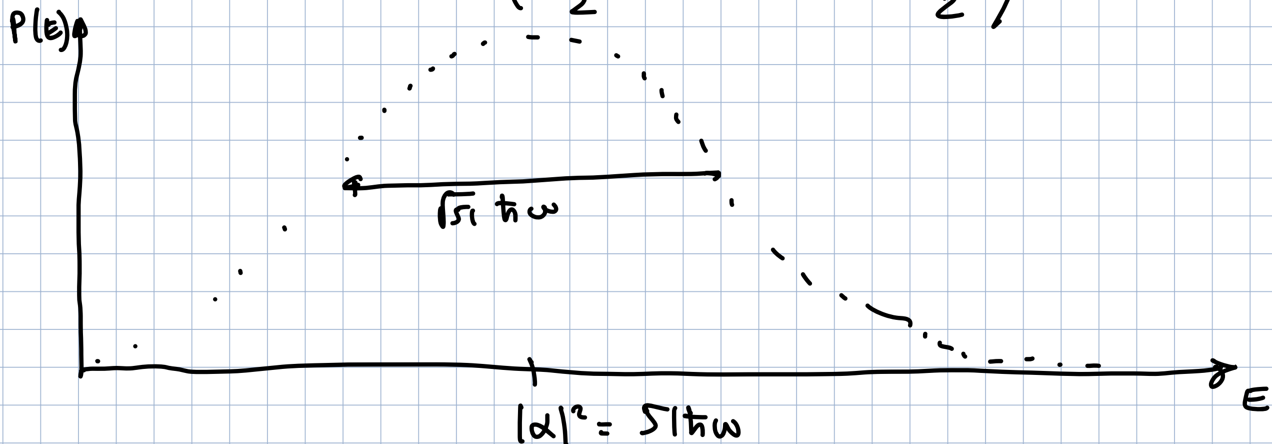
circle w/ radius $5\sqrt{2}\,\sigma$

coherent state $\Rightarrow$ Gaussian wavepacket w/ radius of σ that is constant in time

f. $$P(E=\hbar\omega) = \left|e^{-\frac{5^2}{2}}e^{-\frac{5^2}{2}}\right|^2 = e^{-25\cdot 2} = e^{-50}$$

$$P(E=2\hbar\omega) = e^{-50}\left(|5|^2 + |5i|^2\right) = 50\cdot e^{-50}$$

$$P(E=3\hbar\omega) = e^{-50}\left(\frac{25^2}{2} + 25\cdot 25 + \frac{25^2}{2}\right) = 1250\cdot e^{-50}$$



Most likely result is <u>$51\hbar\omega$</u>