OPTI 570 QM LECTURE Tu 10/7

What is the state of $|\varphi_0\rangle$ displaced from origin by x_0?

$$|\psi\rangle = \hat{S}(x_0)|\varphi_0\rangle = e^{-ix_0\hat{p}/\hbar}|\varphi_0\rangle =$$

$$= e^{-\frac{ix_0}{\cancel{\hbar}}\cdot\frac{i\cancel{\hbar}}{\sqrt{2}\sigma}(a^+ - a)}|\varphi_0\rangle =$$

$$= e^{\frac{x_0}{\sqrt{2}\sigma}(a^+ - a)}|\varphi_0\rangle =$$

BCH: $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$

$$A = \frac{x_0}{\sqrt{2}\sigma}a^+ \qquad B = -\frac{x_0}{\sqrt{2}\sigma}a$$

$$|\psi\rangle = e^{\frac{x_0}{\sqrt{2}\sigma}a^+} e^{-\frac{x_0}{\sqrt{2}\sigma}a} e^{-\frac{1}{2}\cdot\left[\frac{x_0}{\sqrt{2}\sigma}a^+, -\frac{x_0}{\sqrt{2}\sigma}a\right]}|\varphi_0\rangle =$$

$$= e^{\frac{x_0}{\sqrt{2}\sigma}a^+} e^{-\frac{x_0}{\sqrt{2}\sigma}a} e^{-\frac{1}{2}\left(\frac{x_0}{\sqrt{2}\sigma}\right)^2 \underbrace{[a,a^+]}_{1}}|\varphi_0\rangle =$$

$$= e^{\frac{x_0}{\sqrt{2}\sigma}a^+} e^{-\frac{x_0}{\sqrt{2}\sigma}a} \underline{e^{-\frac{x_0^2}{4\sigma^2}}}|\varphi_0\rangle =$$

$$= e^{-\frac{x_0^2}{4\sigma^2}} e^{\frac{x_0}{\sqrt{2}\sigma}a^+} \underline{e^{-\frac{x_0}{\sqrt{2}\sigma}a}|\varphi_0\rangle} =$$

$$= e^{-\frac{x_0^2}{4\sigma^2}} \underbrace{e^{\frac{x_0}{\sqrt{2}\sigma}a^+}}|\varphi_0\rangle =$$

$$= e^{-\frac{x_0^2}{4\sigma^2}} \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{x_0}{\sqrt{2}\sigma}\right)^n \underbrace{(a^+)^n|\varphi_0\rangle}_{\sqrt{n!}\,|\varphi_n\rangle} =$$

$$\underbrace{|\psi\rangle = e^{-\frac{x_0^2}{4\sigma^2}} \sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left(\frac{x_0}{\sqrt{2}\sigma}\right)^n|\varphi_n\rangle}$$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|\varphi_n\rangle \qquad \text{w/} \quad \alpha = \frac{x_0}{\sqrt{2}\sigma}$$

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

$$|\psi(t=0)\rangle = \left|\alpha = \frac{x_0}{\sqrt{2}\sigma}\right\rangle$$

Properties of coherent states

1. normalized but not orthogonal
2. Evolution in time?

$$\tilde{H} = \frac{H}{\hbar\omega}$$

$$\hat{U}(t) = e^{-i\hat{H}t/\hbar} = e^{-i\omega t\,(a^\dagger a + \frac{1}{2})}$$

$$|\psi(0)\rangle = |\alpha_0\rangle$$

$$|\psi(t)\rangle = e^{-i\omega t\,(\underline{a^\dagger a} + \frac{1}{2})}|\alpha_0\rangle =$$

$$= e^{-i\omega t/2}\, e^{-i\omega t\,\underline{a^\dagger a}}|\alpha_0\rangle =$$

$$= e^{-i\omega t/2}\,\underbrace{e^{-i\omega t N}}\,\underbrace{|\alpha_0\rangle} =$$

$$= e^{-\frac{i\omega t}{2}}\, e^{-i\omega t N}\, e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha_0^{\,n}}{\sqrt{n!}}|n\rangle =$$

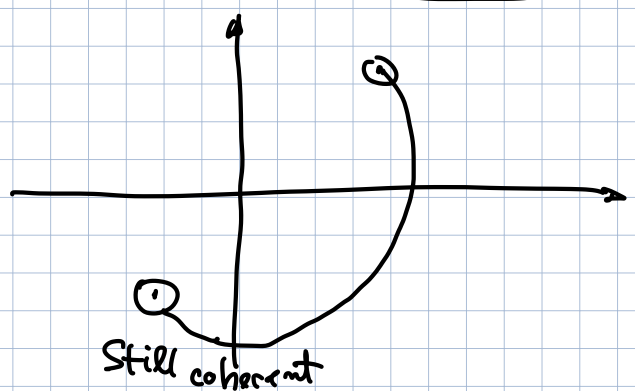
$$= e^{-\frac{i\omega t}{2}}\, e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha_0^{\,n}}{\sqrt{n!}}\, \underline{e^{-i\omega t N}|n\rangle} =$$

$$= e^{-\frac{i\omega t}{2}}\, e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}} \underbrace{(\alpha_0 e^{-i\omega t})}^{n}|n\rangle =$$

$$= e^{-\frac{i\omega t}{2}}|\alpha_0 e^{-i\omega t}\rangle =$$

$$= \underbrace{e^{-\frac{i\omega t}{2}}}|\alpha(t)\rangle$$

$$\underbrace{\alpha(t) = \alpha_0 \cdot e^{-i\omega t}}$$



3. Pos and momentum expectation values

$$a|\alpha(t)\rangle = \alpha(t)|\alpha(t)\rangle \qquad |\cdot \text{HC}$$

$$\langle\alpha(t)|a^\dagger = \alpha^*(t)\langle\alpha(t)|$$

$$\Rightarrow \langle x\rangle_t = \frac{\sigma}{\sqrt{2}}\langle\alpha(t)|a^\dagger + a|\alpha(t)\rangle =$$

$$= \frac{\sigma}{\sqrt{2}}\left[\langle\alpha(t)|a^\dagger|\alpha(t)\rangle + \langle\alpha(t)|a|\alpha(t)\rangle\right] =$$

$$= \frac{\sigma}{\sqrt{2}}\left[\alpha^*(t)\underbrace{\langle\alpha(t)|\alpha(t)\rangle}_{1} + \alpha(t)\underbrace{\langle\alpha(t)|\alpha(t)\rangle}_{1}\right]$$

$$= \frac{\sigma}{\sqrt{2}}\left[\alpha^*(t) + \alpha(t)\right] =$$

$$= \frac{\sigma}{\cancel{\sqrt{2}}}\cdot \cancel{2}^{\sqrt{2}}\, \mathrm{Re}\left[\alpha(t)\right]$$

$$\Rightarrow \boxed{\mathrm{Re}\left[\alpha(t)\right] = \frac{1}{\sqrt{2}}\frac{\langle x\rangle(t)}{\sigma}}$$

Similarly for momentum: $\boxed{\mathrm{Im}\left[\alpha(t)\right] = \frac{1}{\sqrt{2}}\left(\frac{\sigma\langle p\rangle(t)}{\hbar}\right)}$

$$\boxed{\alpha(t) = \frac{1}{\sqrt{2}}\left[\frac{\langle x\rangle(t)}{\sigma} + \frac{i\sigma\langle p\rangle(t)}{\hbar}\right]}$$

4. Position & momentum uncertainties

$$\langle x^2\rangle_\alpha = \frac{\sigma^2}{2}\langle a^{\dagger 2} + a^\dagger a + \underbrace{a a^\dagger}_{1+a^\dagger a} + a^2\rangle =$$

$$= \frac{\sigma^2}{2}\left[\langle\alpha|a^{\dagger 2}|\alpha\rangle + 2\cdot\langle\alpha|a^\dagger a|\alpha\rangle + \langle\alpha|1|\alpha\rangle + \langle\alpha|a^2|\alpha\rangle\right]$$

$$= \frac{\sigma^2}{2}\left[(\alpha^*)^2 + 2|\alpha|^2 + 1 + \alpha^2\right] =$$

$$= \frac{\sigma^2}{2} + \frac{\sigma^2}{2}\left[\alpha^*(t) + \alpha(t)\right]^2$$

$$\langle x^2\rangle_\alpha = \frac{\sigma^2}{2} + \langle x\rangle^2(t)$$

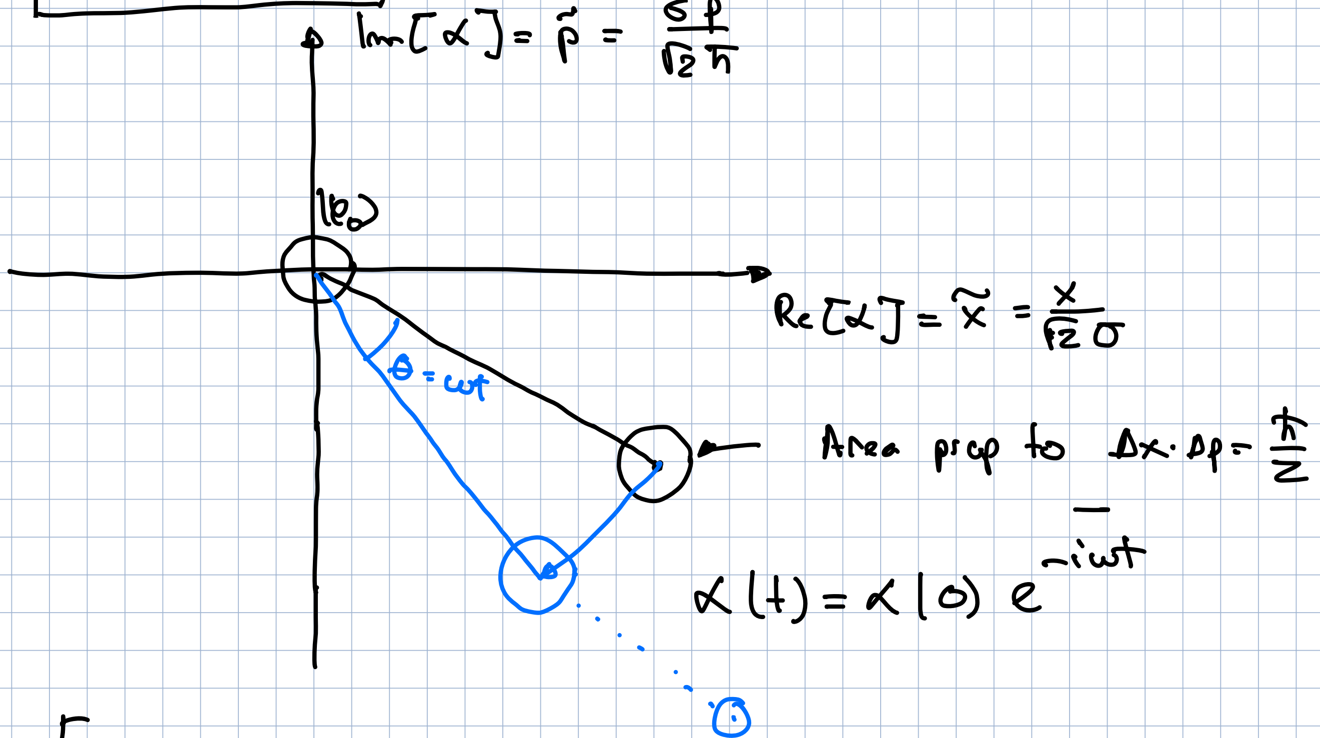
$$\Rightarrow \Delta x = \sqrt{\langle x^2\rangle_\alpha - \langle x\rangle^2_\alpha} =$$

$$\underbrace{= \frac{\sigma}{\sqrt{2}}}$$

- same as $|\varphi_0\rangle$
- time independent

$$\underbrace{\Delta p = \frac{\hbar}{\sqrt{2}\sigma}}$$

$$\boxed{\Delta x \cdot \Delta p = \frac{\hbar}{2}}$$



$$\alpha(t) = \alpha(0)\, e^{-i\omega t}$$

5. Energy

$$\langle N\rangle_\alpha = \langle \alpha | a^\dagger a | \alpha\rangle = \underline{|\alpha|^2}$$

$$\underbrace{\langle H\rangle_\alpha = \hbar\omega\left(|\alpha|^2 + \frac{1}{2}\right)}$$

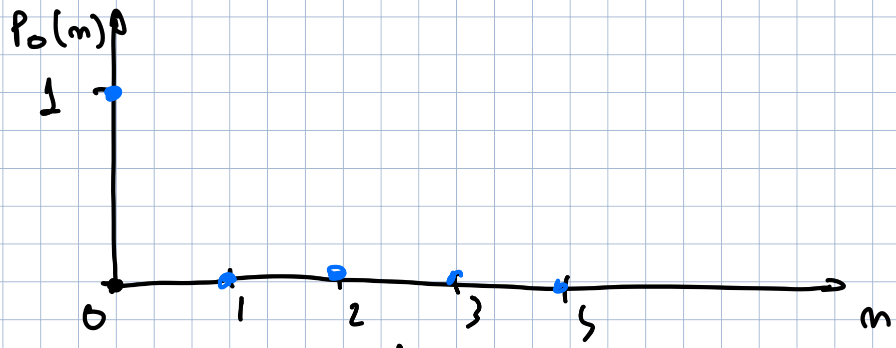
$$\langle H^2\rangle_\alpha = \hbar^2\omega^2\left(|\alpha|^4 + 2|\alpha|^2 + \frac{1}{4}\right)$$

$$\boxed{\Delta H = \hbar\omega|\alpha|}$$

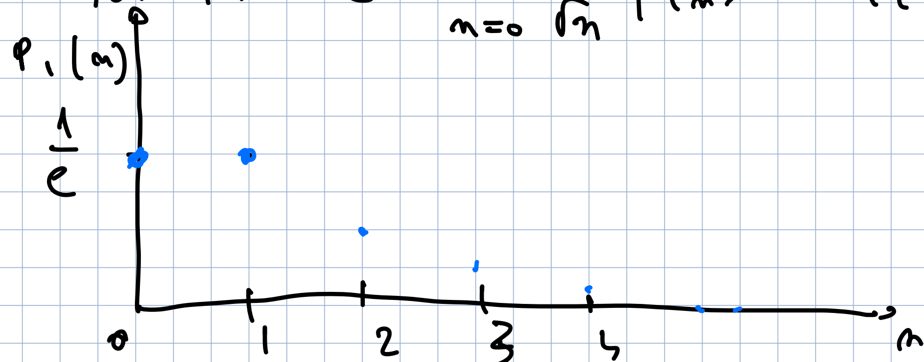
$$P_\alpha(E_n) = \langle \alpha | \hat{P}_n | \alpha\rangle = \langle \alpha | \varphi_n\rangle\langle \varphi_n | \alpha\rangle =$$

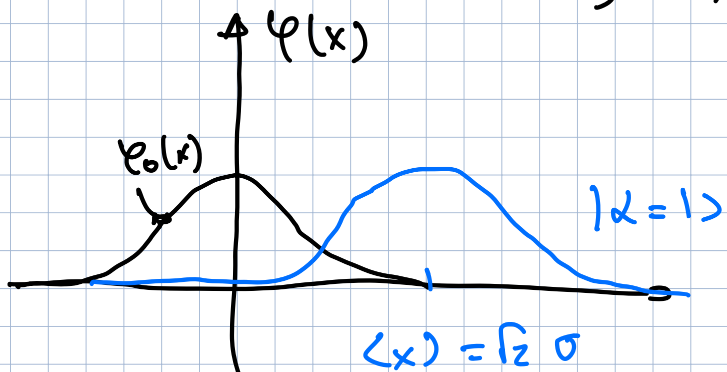
$$\underbrace{P_\alpha(E_n) = |\langle \alpha | \varphi_n\rangle|^2 = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}}$$

ex: $|\alpha = 0\rangle = |\varphi_0\rangle$ $\qquad P_0(n) = \frac{0^{2n}}{n!} e^{-0} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$



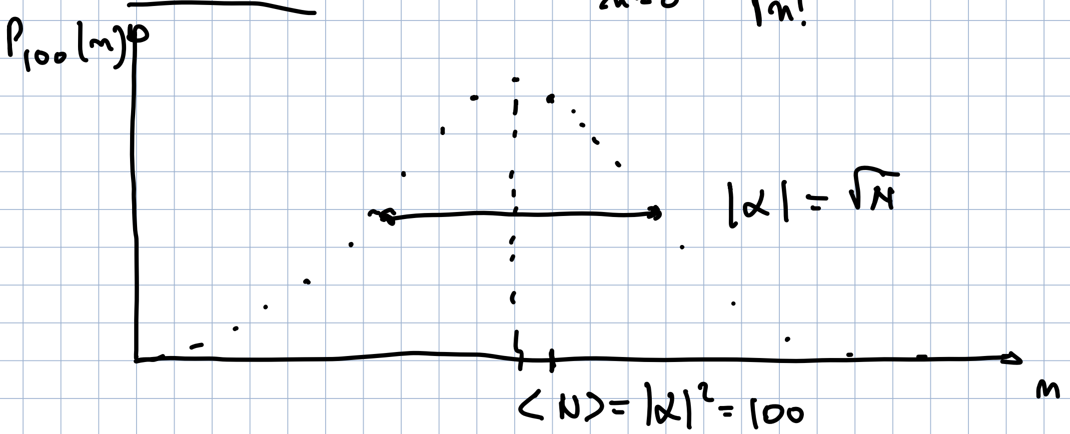
$$|\alpha = 1\rangle = e^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}} |\varphi_n\rangle \qquad P_1(n) = \frac{1}{n!} e^{-1}$$





$$P(E) = E_0 = \frac{1}{e}$$

$$|\alpha = 10i\rangle = e^{-50} \sum_{n=0}^{\infty} \frac{(10i)^n}{\sqrt{n!}} |\varphi_n\rangle \qquad P_{100}(n) = \frac{100^n}{n!} e^{-100}$$



6. Displacement Operator

$$\hat{D}(\alpha_0)|\varphi_0\rangle = |\alpha_0\rangle$$

$$\hat{D}(\alpha_0) = \hat{T}(\langle p\rangle)\,\hat{S}(\langle x\rangle)\,e^{-i\langle x\rangle\langle p\rangle/\hbar} =$$

$$= \hat{S}(\langle x\rangle)\,\hat{T}(\langle p\rangle)\,e^{i\langle x\rangle\langle p\rangle/\hbar} =$$

$$= e^{\frac{i}{\hbar}(\hat{x}\langle p\rangle + \hat{p}\langle\hat{x}\rangle)}$$

$$a = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{\sigma} + \frac{i\sigma\hat{p}}{\hbar}\right) \qquad \alpha_0 = \frac{1}{\sqrt{2}}\left(\frac{\langle x\rangle}{\sigma} + \frac{i\sigma\langle p\rangle}{\hbar}\right)$$

$$a^+ = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{\sigma} - \frac{i\sigma\hat{p}}{\hbar}\right) \qquad \alpha_0^* = \frac{1}{\sqrt{2}}\left(\frac{\langle x\rangle}{\sigma} - \frac{i\sigma\langle p\rangle}{\hbar}\right)$$

$$\underline{\hat{D}(\alpha_0) = e^{\alpha_0\hat{a}^+ - \alpha_0^*\hat{a}}}$$

$D(\alpha_0)$ - pos translation by $\langle x\rangle = \sqrt{2}\,\sigma\,\mathrm{Re}[\alpha_0]$

mom trans. by $\langle p\rangle = \frac{\sqrt{2}\,\hbar}{\sigma}\,\mathrm{Im}[\alpha_0]$

$D(\alpha_0)|\varphi_1\rangle$ -

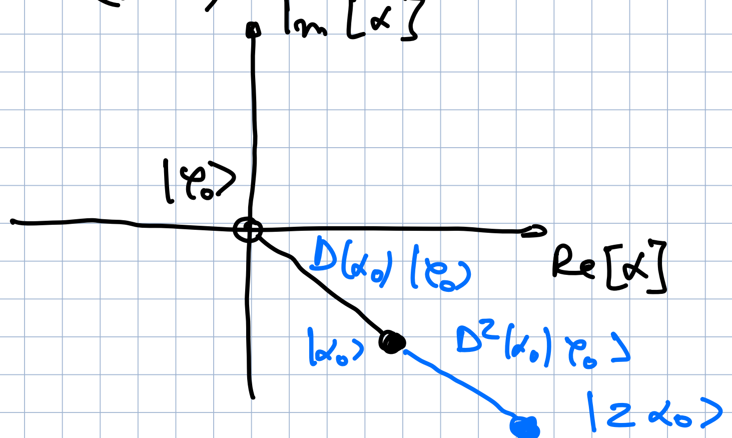
ex: $D(\alpha_0)|\varphi_0\rangle \qquad \hat{D} = e^{\underbrace{\frac{1}{\sqrt{2}}(5+10i)}_{\alpha_0}a^+ - \underbrace{\frac{1}{\sqrt{2}}(5-10i)}_{\alpha_0^*}a}$

$\alpha_0 = 5 + 10i \Rightarrow \langle x\rangle = 5\sigma \qquad \langle p\rangle = 10\frac{\hbar}{\sigma}$

$$\langle x|\varphi_0\rangle = \left(\frac{1}{\pi\sigma^2}\right)^{1/4} e^{-\frac{x^2}{2\sigma^2}}$$

$$\langle x|\hat{D}|\varphi_0\rangle = \left(\frac{1}{\pi\sigma^2}\right)^{1/4} e^{\frac{ix}{\hbar}\left(10\frac{\hbar}{\sigma}\right)}\; e^{-\frac{(x-5\sigma)^2}{2\sigma^2}}$$

Phase space



$$\hat{D}(\alpha_0)\,\hat{D}(\alpha_0)\,|\varphi_0\rangle = |2\alpha_0\rangle$$

$$\hat{D}(\alpha_1)\,\hat{D}(\alpha_0)\,|\varphi_0\rangle = |\alpha_0 + \alpha_1\rangle$$

<u>Q:</u> $\langle x \rangle$ for $\hat{D}(-\alpha_0)\,\hat{D}(\alpha_0)|\varphi_0\rangle = 0$

<u>$\langle p \rangle$</u> $\hat{D}(\alpha_0^*)\,\hat{D}(\alpha_0)\,|\varphi_0\rangle = 0$