$\boxed{\text{Problem Set 5}}$

Problem 1 Part 1.

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle \qquad |\psi_E(t)\rangle = \hat{F}(t)|\psi(t)\rangle$$

$$\Rightarrow |\psi(t)\rangle = \hat{F}^{+}(t)\,|\psi_E(t)\rangle$$

LHS:

$$i\hbar \frac{\partial}{\partial t}\left(\hat{F}^{+}(t)|\psi_E(t)\rangle\right) = i\hbar\left(\frac{\partial}{\partial t}\hat{F}^{+}(t)\right)|\psi_E(t)\rangle + i\hbar \hat{F}^{+}(t)\frac{\partial}{\partial t}|\psi_E(t)\rangle$$

RHS:

$$\hat{H}(t)\,\hat{F}^{+}(t)|\psi_E(t)\rangle$$

$$i\hbar\,\hat{F}^{+}(t)\frac{\partial}{\partial t}|\psi_E(t)\rangle + i\hbar\left[\frac{\partial}{\partial t}\hat{F}^{+}(t)\right]|\psi_E(t)\rangle = \hat{H}(t)\hat{F}^{+}(t)|\psi_E(t)\rangle \Big| : \hat{F}^{+}(t)$$

$$i\hbar\frac{\partial}{\partial t}|\psi_E(t)\rangle = \left\{\underbrace{\left[\hat{F}^{+}(t)\right]^{-1}}_{\hat{F}(t)\text{ - unitary}}\hat{H}\,\hat{F}^{+}(t) - i\hbar\left[F^{+}(t)\right]^{-1}\frac{\partial}{\partial t}\hat{F}^{+}(t)\right\}|\psi_E(t)\rangle$$

$$\Rightarrow \underbrace{\hat{H}_E(t) = \hat{F}(t)\,\hat{H}\,\hat{F}^{+}(t) - i\hbar\,\hat{F}(t)\frac{\partial}{\partial t}\hat{F}^{+}(t)}\quad \checkmark$$

Part 2

$$|\psi_I(t)\rangle \equiv \hat{u}_0^{+}(t_1,t_0)|\psi_S(t)\rangle \qquad \hat{u}_0(t) = e^{-\frac{i}{\hbar}t\hat{H}_0}$$

$$\hat{H}_S(t) = \hat{H}_0 + \hat{W}(t)$$

Let $\hat{F}(t) = \hat{u}_0^{+}(t)$ above $\Rightarrow \frac{\partial}{\partial t}\hat{F}^{+}(t) = -\frac{i}{\hbar}\hat{H}_0\,\hat{F}^{+}(t)$

$$\Rightarrow \hat{F}^{+}(t) = \hat{u}_0(t)$$

$$\hat{H}_E(t) = \hat{F}(t)\left[\hat{H}_0 + \hat{W}(t)\right]\hat{F}^{+}(t) - \hat{F}(t)\,i\cancel{\hbar}\left(-\frac{i}{\cancel{\hbar}}\hat{H}_0\right)\hat{F}^{+}(t)$$

$$= \hat{u}_0^{+}\,\hat{W}\,\hat{u}_0 + \cancel{\hat{u}_0^{+}\hat{H}_0\hat{u}_0} - \cancel{\hat{u}^{+}\hat{H}\hat{u}_0} =$$

$$\boxed{\hat{H}_E = \hat{u}_0^{+}\,W\,\hat{u}_0}$$

Problem 2 CTU, #1

$$|\psi(0)\rangle = \sum_n c_n |\varphi_n\rangle \qquad H|\varphi_n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|\varphi_n\rangle$$

[a] $$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |\varphi_n\rangle$$

$$P(E>2\hbar\omega) = 1 - P(E=\hbar\omega) - P\left(E=\frac{3}{2}\hbar\omega\right) =$$
$$= 1 - \langle\psi|\hat{P}_0 + \hat{P}_1|\psi\rangle =$$
$$= 1 - \langle\psi|\left[|\varphi_0\rangle\langle\varphi_0| + |\varphi_1\rangle\langle\varphi_1|\right]|\psi\rangle =$$

$$\boxed{P(E>2\hbar\omega) = 1 - |c_0|^2 - |c_1|^2}$$

$P = 0 \Rightarrow |c_0|^2 + |c_1|^2 = 1 \Rightarrow c_2 = c_3 = \ldots = 0$, c_0, c_1 - non-zero

[b] $$|\psi(0)\rangle = c_0|\varphi_0\rangle + c_1|\varphi_1\rangle$$
$$\langle\psi(0)|\psi(0)\rangle = |c_0|^2 + |c_1|^2 = 1$$

$$\langle\hat{H}\rangle = \langle\psi(0)|\hat{H}|\psi(0)\rangle = |c_0|^2 \cdot \frac{1}{2}\hbar\omega + |c_1|^2 \frac{3}{2}\hbar\omega = \hbar\omega$$

$$\Rightarrow |c_0|^2 + 3|c_1|^2 = 2$$
$$|c_0|^2 = \frac{1}{2} \qquad |c_1|^2 = \frac{1}{2}$$

[c] $c_1 = |c_1|e^{i\theta_1}$ $\quad |\psi(0)\rangle = c_0|\varphi_0\rangle + c_1|\varphi_1\rangle \quad \langle\hat{x}\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}} = \frac{1}{2}\sigma$

$$\langle\hat{x}\rangle = \langle\psi(0)|\hat{x}|\psi(0)\rangle \Rightarrow \qquad \hat{x} = \frac{1}{\sqrt{2}}(a^\dagger + a)\,\sigma$$

$$\frac{1}{2} = \frac{1}{\sqrt{2}}\langle\psi(0)|a^\dagger + a|\psi(0)\rangle =$$
$$= \frac{1}{\sqrt{2}}\left(\langle\psi(0)|a^\dagger|\psi(0)\rangle + \langle\psi(0)|a|\psi(0)\rangle\right) =$$
$$= \frac{1}{\sqrt{2}}\left(c_1^* c_0 \langle\varphi_1|a^\dagger|\varphi_0\rangle + c_0^* c_1\langle\varphi_0|a|\varphi_1\rangle + \underbrace{\ldots}_{\text{all others are } 0}\right) =$$
$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}e^{-i\theta_1}\langle\varphi_1|a^\dagger|\varphi_0\rangle + \frac{1}{\sqrt{2}}e^{i\theta_1}\langle\varphi_0|a|\varphi_1\rangle\right) =$$

$$= \frac{1}{2}\left(e^{-i\theta_1} \langle \varphi_1 | \sqrt{1} | \varphi_1 \rangle + e^{i\theta_1} \langle \varphi_0 | 1 | \varphi_0 \rangle \right) =$$

$$= \frac{1}{2}\left(e^{-i\theta_1} + e^{i\theta_1} \right) = \cos(\theta_1)$$

$$\Rightarrow \boxed{\theta_1 = \frac{\pi}{4}}$$

$\boxed{d}$ $$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\frac{1}{2}\omega t} |\varphi_0\rangle + e^{i\pi/4} e^{-i\frac{3}{2}\omega t} |\varphi_1\rangle \right]$$

$$\boxed{= \frac{1}{\sqrt{2}}\left[|\varphi_0\rangle + e^{-i(\omega t - \pi/4)} |\varphi_1\rangle \right]}$$

$$\boxed{\theta_1(t) = \frac{\pi}{4} - \omega t}$$

$$\langle x \rangle = \frac{\sigma}{\sqrt{2}} \langle \psi(t) | a^+ + a | \psi(t) \rangle =$$

$$= \frac{\sigma}{\sqrt{2}} \cdot \frac{1}{2} \left[\langle \varphi_0 | + e^{i(\omega t - \pi/4)} \langle \varphi_1 | \right] \Big| a^+ + a \Big| \left[|\varphi_0\rangle + e^{-i(\omega t - \frac{\pi}{4})} |\varphi_1\rangle \right] =$$

$$= \frac{\sigma}{2\sqrt{2}} \cdot \left[e^{i(\omega t - \pi/4)} \underbrace{\langle \varphi_1 | a^+ | \varphi_0 \rangle}_{1} + e^{-i(\omega t - \pi/4)} \underbrace{\langle \varphi_0 | a | \varphi_1 \rangle}_{1} \right] =$$

$$= \frac{\sigma}{2\sqrt{2}} \left[e^{i\theta_1} + e^{-i\theta_1} \right] =$$

$$\boxed{\langle x \rangle = \frac{\sigma}{\sqrt{2}} \cdot \cos(\omega t - \pi/4)}$$

Problem III   CT III, #17

$|x_\ell\rangle$ eigenvectors of ρ, thus

$$\boxed{\rho = \sum_\ell \pi_\ell |x_\ell\rangle\langle x_\ell|}$$

$$\rho^2 = \sum_k \sum_\ell \pi_k \pi_\ell |x_\ell\rangle\underbrace{\langle x_\ell|x_k\rangle}_{\delta_{k\ell}}\langle x_k| =$$

$$\boxed{\rho^2 = \sum_\ell \pi^2 |x_\ell\rangle\langle x_\ell|}$$

Pure state: $Tr\{\rho\} = \sum_m \langle x_m|\left(\sum_\ell \pi_\ell |x_\ell\rangle\langle x_\ell|\right)|x_m\rangle =$

$$= \sum_m \sum_\ell \langle x_m|\pi_\ell |x_\ell\rangle\underbrace{\langle x_\ell|x_m\rangle}_{\delta_{\ell m}} =$$

$$= \sum_\ell \pi_\ell \langle x_\ell|x_\ell\rangle =$$

$$= \sum_\ell \pi_\ell$$

$$Tr\{\rho^2\} = \sum_m \langle x_m|\left(\sum_\ell \pi_\ell^2 |x_\ell\rangle\langle x_\ell|\right)|x_m\rangle =$$

$$= \sum_\ell \pi_\ell^2$$

$Tr\{\rho^2\} = Tr\{\rho\} = 1$ for pure state $\Rightarrow \sum_\ell \pi_\ell = \sum_\ell \pi_\ell^2 = 1$

$\Rightarrow \sum_\ell \pi_\ell \cdot (\pi_\ell - 1) = 0 \Rightarrow$ all $\pi_\ell = 0$ and only one is 1

$$\rho = \rho^2 \rightarrow \begin{pmatrix} 0 & & & & & \\ & 0 & & & & 0 \\ & & \ddots & & & \\ & & & 0 & & \\ & & & & 1 & \\ & 0 & & & & 0 \\ & & & & & & \ddots \\ & & & & & & & 0 \end{pmatrix}$$ for pure state

Mixed state $\rho^2 \neq \rho$

$$\rho \rightarrow \begin{pmatrix} \pi_1 & & & 0 \\ & \pi_2 & & \\ & & \pi_3 & \\ 0 & & & \ddots \end{pmatrix} \qquad \rho^2 \rightarrow \begin{pmatrix} \pi_1^2 & & & 0 \\ & \pi_2^2 & & \\ & & \pi_3^2 & \\ 0 & & & \ddots \end{pmatrix} \qquad 0 \leq \pi_\ell < 1$$

Problem IV $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ $\mathcal{E}$

$\hat{H}|u_m\rangle = E_m|u_m\rangle$

[a] $$\mathrm{Tr}(\hat{P}_\psi) = \sum_{j=1}^{3} \langle u_j | \psi(t)\rangle\langle \psi(t)|u_j\rangle =$$

$$= \sum_{j=1}^{3} |\langle \psi(t)|u_j\rangle|^2 = \sum_{j=1}^{3} |\langle \psi(t_0)|u_j\rangle|^2$$ since unitary time evolution

$$\boxed{\mathrm{Tr}(\hat{P}_\psi) = 1}$$ since $\langle \psi(t_0)\rangle$ is normalized.

[b] $\hat{P}_{\psi(t)}$ at $t=0$

$$\hat{P}_{\psi(0)} = |\psi(0)\rangle\langle\psi(0)|$$

$$\Rightarrow \hat{P}_{\psi(0)\{u\}} = \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} |c_1|^2 & c_1c_2^* & c_1c_3^* \\ c_2c_1^* & |c_2|^2 & c_2c_3^* \\ c_3c_1^* & c_3c_2^* & |c_3|^2 \end{pmatrix}$$

$\mathrm{Tr}\{\hat{P}_{\psi(0)\{u\}}) = |c_1|^2 + |c_2|^2 + |c_3|^2 = 1$ because normalization

[c] new basis $\{v\} = \{|\psi(0)\rangle, |v_2\rangle, |v_3\rangle\}$

$$\hat{P}_{\psi(0)\{v\}} = |\psi(0)\rangle\langle\psi(0)| =$$

$$= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\mathrm{Tr}[\hat{P}_{\psi(0)\{v\}}] = 1}$$ ✓

[d] $$P(\lambda_m) = \langle \psi(t)| \hat{P}_m |\psi(t)\rangle = \qquad \hat{P}_m = |w_m\rangle\langle w_m|$$

$$= \langle \psi(t)| \hat{P}_m \sum_m |u_m\rangle\langle u_m|\psi(t)\rangle =$$

$$= \sum_m \langle \psi(t)|\hat{P}_m|u_m\rangle\langle u_m|\psi(t)\rangle =$$

$$= \sum_m \langle u_m | \psi(t) \rangle\langle \psi(t) | \hat{P}_m | u_m \rangle =$$

$$= \mathrm{Tr}\left\{ \underbrace{|\psi(t)\rangle\langle\psi(t)|} \hat{P}_m \right\} =$$

$$= \mathrm{Tr}\left\{ \hat{P}_{\psi(t)} \hat{P}_m \right\}$$

$\boxed{e}$ $$\frac{d}{dt}\hat{P}_{\psi(t)} = \frac{d}{dt}\left(|\psi(t)\rangle\langle\psi(t)|\right) =$$ $\underline{SE}$: $i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$

$$= \left(\frac{d}{dt}|\psi(t)\rangle\right)\langle\psi(t)| + |\psi(t)\rangle\left(\frac{d}{dt}\langle\psi(t)|\right) =$$

$$= \frac{1}{i\hbar}\hat{H}\underbrace{|\psi(t)\rangle\langle\psi(t)|}_{\hat{P}_{\psi(t)}} - \frac{1}{i\hbar}\underbrace{|\psi(t)\rangle\langle\psi(t)|}\hat{H} =$$

$$= \frac{1}{i\hbar}\left(\hat{H}\hat{P}_{\psi(t)} - \hat{P}_{\psi(t)}\hat{H}\right) =$$

$$\boxed{\frac{d}{dt}\hat{P}_{\psi}(t) = \frac{1}{i\hbar}\left[\hat{H}, \hat{P}_{\psi(t)}\right]}$$