OPTI 570 LECTURE Tu Sep 2

Operators

Let $|\psi\rangle \in \mathcal{E}$

$$\hat{A}\,|\psi\rangle = c\,|\psi'\rangle$$

operator

complex scalar

Properties of operators

(1) linearity

$$\hat{A}\left(\lambda_1 |\psi_1\rangle + \lambda_2 |\psi_2\rangle\right) = \lambda_1 \hat{A}\,|\psi_1\rangle + \lambda_2 \hat{A}\,|\psi_2\rangle =$$
$$= \lambda_1 c\,|\psi_1'\rangle + \lambda_2 c\,|\psi_2'\rangle$$

(2) order of operations matters

$$\hat{A}\,(\hat{B}\,|\psi\rangle) \neq \hat{B}\,(\hat{A}\,|\psi\rangle)$$ in general

commutator: $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \neq 0$ in general

if $[\hat{A},\hat{B}] = 0$, $\hat{A}\hat{B} = \hat{B}\hat{A}$ – "$\hat{A}$ and $\hat{B}$ commute"

$[\hat{A},\hat{B}] \neq 0$ – —"— DO NOT COMMUTE

Bases and representations

Base: Let $|\psi\rangle \in \mathcal{E}$

Let $|\psi\rangle = \sum_{n=0} c_n |u_n\rangle$

$\{|u_n\rangle\}$ is a discrete basis for $\mathcal{E}$ ("spans" $\mathcal{E}$)

infinite number of bases in $\mathcal{E}$

sum over integers

$\Updownarrow$ if and only if

For every $|\psi\rangle \in \mathcal{E}$, there exists one set of $\{c_n\}$ that can define $|\psi\rangle$

* up to a global phase factor

$$|\psi\rangle = e^{i\phi} \sum_0^\infty c_n |u_n\rangle$$

real

- no physical effect on any measurement I can do.

ex: $|\psi\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle)$

$e^{i\pi}|\psi\rangle = -\frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle)$

global phase

global phase

rel. phase

$|\psi'\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle - i|u_2\rangle)$

Representations

$\{c_n\}$ represents $|\psi\rangle$

In our class, all bases are orthonormal.

$\{|u_n\rangle\}$ is an orthonormal basis IF AND ONLY IF

$\langle u_m | u_n\rangle = 1$ when $m = n$

0 when $m \neq n$

m m -

n n

$\langle u_m | u_n\rangle = \delta_{m,n}$ - Kronecker delta

all m, n

the dimension of a basis = # of elements in the basis
- equal to dimension of state space

ex: - 3 dimensional basis $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$

$$|\varphi\rangle = a|u_1\rangle + b|u_2\rangle$$

Q: what is $\langle\varphi|$?

$$\langle\varphi| = a^* \langle u_1| + b^* \langle u_2|$$

Evaluate $\langle u_1|\varphi\rangle = \langle u_1|\left(a|u_1\rangle + b|u_2\rangle\right) =$

$$= a \underbrace{\langle u_1|u_1\rangle}_{=1} + b \underbrace{\langle u_1|u_2\rangle}_{=0} =$$

$$= a$$

Q: $\langle u_3|\varphi\rangle = 0$

$\langle\varphi|u_1\rangle = a^*$

Q: $\underbrace{|u_2\rangle\langle u_1|}_{u(2,1)}$?

$$(|u_2\rangle\langle u_1|)|\varphi\rangle = |u_2\rangle\langle u_1|\varphi\rangle = |u_2\rangle \cdot a = a|u_2\rangle$$

- operator - "ket-bra" form

$|u_2\rangle\langle u_1|$ - operator

$\langle u_2|u_1\rangle$ - inner product - scalar

$$|\varphi\rangle\langle\varphi| : (a|u_1\rangle + b|u_2\rangle) \cdot (a^*\langle u_1| + b^*\langle u_2|) =$$

$$= |a|^2 |u_1\rangle\langle u_1| + |b|^2 \underbrace{|u_2\rangle\langle u_2|}_{\neq 0} + ab^* \underbrace{|u_1\rangle\langle u_2|}_{\neq 0} + ba^* |u_2\rangle\langle u_1|$$

$\langle u_2|u_1\rangle = 0$

$$(0\ \ 1\ \ 0)\begin{pmatrix}1\\0\\0\end{pmatrix} = 0$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix}(0\ \ 1\ \ 0) = \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix} \neq 0$$

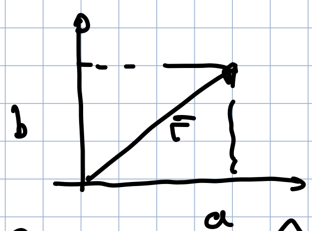
Projetors

For any $|\psi\rangle \in \mathcal{E}$, assumin $\langle\psi|\psi\rangle = 1$, "normalized"

$\hat P_\psi = |\psi\rangle\langle\psi|$ is the "projector onto $|\psi\rangle$"

$$\underset{\substack{\uparrow\\ \text{projection}\\ \text{onto }\psi}}{\hat P_\psi}\,|\varphi\rangle = |\psi\rangle\underbrace{\langle\psi|\varphi\rangle}_{\text{scalar}} = \underbrace{\langle\psi|\varphi\rangle}_{\substack{\text{overlap bw}\\ \psi\text{ and }\varphi}}\,\underset{\text{scaled }|\psi\rangle}{\underline{|\psi\rangle}}$$

ex: $\bar r = a\hat x + b\hat y$



proj onto $\hat x \Rightarrow a\hat x$

$$\begin{aligned}
\hat P_\psi^2|\varphi\rangle &= \hat P_\psi \hat P_\psi |\varphi\rangle = \\
&= \hat P_\psi \langle\psi|\varphi\rangle|\psi\rangle = \\
&= \langle\psi|\varphi\rangle\, \hat P_\psi|\psi\rangle = \\
&= \langle\psi|\varphi\rangle\, |\psi\rangle \underbrace{\langle\psi|\psi\rangle}_{=1} = \\
&= \langle\psi|\varphi\rangle|\psi\rangle = \\
&= \hat P_\psi|\varphi\rangle
\end{aligned}$$

$$\boxed{\begin{array}{l}\hat P_\psi = |\psi\rangle\langle\psi| \text{ a projetor IF AND ONLY IF} \\ \hat P_\psi^2 = \hat P_\psi \quad \text{and} \quad |\psi\rangle \text{ is normalized}\end{array}}$$

ex: $\{ |u_1\rangle, |u_2\rangle, |u_3\rangle \} \in \mathcal{E}_{1,2,3}$

Define a ket $|\varphi\rangle = c_1 |u_1\rangle + c_2 |u_2\rangle + c_3 |u_3\rangle$

$$\hat{P}_m = |u_m\rangle\langle u_m|$$

$$\hat{P}_2 = |u_2\rangle\langle u_2|$$

$$\hat{P}_1 = |u_1\rangle\langle u_1|$$

$$\hat{P}_{1,2} = |u_1\rangle\langle u_1| + |u_2\rangle\langle u_2| = \hat{P}_1 + \hat{P}_2$$

- "subspace projector"
- projects from $\mathcal{E}_{1,2,3}$ to $\mathcal{E}_{1,2}$

$$\hat{P}_{1,2}^2 = \hat{P}_{1,2}$$

$\hat{P}_1 \cdot \hat{P}_2 = \hat{P}_2 \hat{P}_1 = 0 \Rightarrow$ projectors are <u>orthogonal</u>

$$\hat{P}_{1,2,3} = \hat{P}_1 + \hat{P}_2 + \hat{P}_3 =$$
$$= |u_1\rangle\langle u_1| + |u_2\rangle\langle u_2| + |u_3\rangle\langle u_3|$$

$$\hat{P}_{1,2,3} |\psi\rangle = |\psi\rangle$$

$\hat{P}_{1,2,3} = \mathbb{1}$ - identity operator

More generally, $\{|u_m\rangle\}$ is a basic for $\mathcal{E}$

then: $\underline{\sum_{\text{all } m} |u_m\rangle\langle u_m| = \hat{\mathbb{1}}}$ - identity operator

"closure relation" $\Updownarrow$ IF and ONLY IF.

defines $\{|u_m\rangle\}$ as a basis for $\mathcal{E}$

Hermitian conjugation

rules:
$$|\psi\rangle \to \langle\psi|$$
$$\langle\psi| \to |\psi\rangle$$
Scalar $c \to c^*$
$$\hat{A} \to \hat{A}^+$$ – Hermitian conjugate, adjoint
$$(\hat{A}^+)^+ \to \hat{A}$$ –

ex: $\hat{A}|\psi\rangle = |\psi'\rangle \;\Big|^+ \quad \in \mathcal{E}$
$$\langle\psi|\hat{A}^+ = \langle\psi'|$$

$$\Big(\underbrace{\lambda}_{\text{scalar}} \underbrace{\langle u|A|v\rangle}_{\text{scalar}} \underline{|\varphi\rangle\langle\psi|}\Big)^+ = \underline{|\psi\rangle\langle\varphi|\langle v|A^+|u\rangle\lambda^*}$$

IF $\hat{A} = \hat{A}^+$, then $\hat{A}$ is "Hermitian"