# Assignment 2
## OPTI 570 Quantum Mechanics

University of Arizona

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## 1 Exercise 1

A particle under a delta function potential has the following hamiltonian:

$$H = -\frac{\hbar}{2m}\frac{d^2}{dx^2} - \alpha\delta(x), \tag{1}$$

where $\delta(x)$ is the potential to be analyzed which is centered at $x = 0$.

(a) (a.1) The integration of the eigenvalue equation

$$\begin{aligned} H\varphi(x) &= E\varphi(x) \\ -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi(x) - \alpha\delta(x)\varphi(x) &= E\varphi(x) \end{aligned}$$

in the range $x \in [-\epsilon, \epsilon]$ is

$$\int_{-\epsilon}^{\epsilon}\left[-\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2}(x) - \alpha\delta(x)\varphi(x)\right]\,dx = \int_{-\epsilon}^{\epsilon} E\varphi(x)\,dx,$$

where in this infinitesimal region around $x = 0$ different things will happen:

- By property of the delta function, we will have

$$\int_{-\infty}^{\infty}\alpha\delta(x)\varphi(x)\,dx = \alpha\varphi(0). \tag{2}$$

- The integration of the second derivative will become a substraction of two first derivatives:

$$\int_{-\epsilon}^{\epsilon}\left[-\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2}(x)\right]\,dx = -\frac{\hbar^2}{2m}\left[\frac{d\varphi}{dx}(\epsilon) - \frac{d\varphi}{dx}(-\epsilon)\right]. \tag{3}$$

- The right-hand term will be zero, as the function cannot change abruptly (to be physically realizable) in an infinitesimal range:

$$\int_{-\epsilon}^{\epsilon} E\varphi(x)\,dx = 0. \tag{4}$$

Plug in all in the eigenvalue equation yields

$$-\frac{\hbar^2}{2m}\left[\frac{d\varphi}{dx}(\epsilon)-\frac{d\varphi}{dx}(-\epsilon)\right]-\alpha\varphi(0)=0, \quad x\in[-\epsilon,\epsilon]. \tag{5}$$

(a.2) Recall that continuity of a function $f(x)$ at $x=x_0$ needs that both lateral limits are equal as they approach to the point, and that the value is equal to the function evaluated at that point: $f(x_0^-)=f(x_0^+)=f(x_0)$.
Taking the limit $\epsilon\to 0$ in a rearranged version of the equation (5) allow us to construct the difference of the derivative used to prove continuity in the first derivate of $\varphi(x)$:

$$\lim_{\epsilon\to 0}\left[\frac{d\varphi}{dx}(\epsilon)-\frac{d\varphi}{dx}(-\epsilon)\right]=-\frac{2m\alpha}{\hbar^2}\varphi(0). \tag{6}$$

We can see that lateral derivatives are not equal, but rather there is a finite jump of $-\frac{2m\alpha}{\hbar^2}\varphi(0)$ at $x=0$. Therefore, we conclude that $\varphi'(x)$ is not continue at that point.

(b) (b.1) Assuming that $E=\hbar\omega<0$, in order to obtain an expression for $\rho$ we replace the wavefunctions

$$x<0: \quad \varphi(x)=A_1e^{\rho x}+A_1'e^{-\rho x} \tag{7}$$
$$x>0: \quad \varphi(x)=A_2e^{\rho x}+A_2'e^{-\rho x}. \tag{8}$$

These solutions must be stable, therefore we evaluate them at $-\infty$ and $+\infty$ to see if any term blows up:

$$\varphi(-\infty)=A_1e^{\rho(-\infty)}+\cancel{A_1'e^{-\rho(-\infty)}}^{\text{blows up}},$$
$$\varphi(+\infty)=\cancel{A_2e^{\rho(\infty)}}^{\text{blows up}}+A_2'e^{-\rho(\infty)}.$$

Therefore, to keep with stable and normalizable functions we redefine the above to the following:

$$\varphi(x)=\begin{cases}A_1e^{\rho x}, & x<0\\ A_2'e^{-\rho x}, & x>0\end{cases}.$$

Now, we can proceed. In order to obtain an expression for $\rho$, we will study each domain separately.
For $x<0$, the function has not passed through the delta potential yet so the hamiltonian reduces to the derivative term only.

$$\begin{aligned}
H\varphi(x)&=E\varphi(x)\\
-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left[A_1e^{\rho x}\right]&=E\left[A_1e^{\rho x}\right]\\
-\frac{\hbar^2}{2m}\frac{d}{dx}\left[\rho A_1e^{\rho x}\right]&=EA_1e^{\rho x}\\
-\frac{\hbar^2}{2m}\rho^2\cancel{A_1e^{\rho x}}&=E\cancel{A_1e^{\rho x}}\\
\rho&=\sqrt{-\frac{2mE}{\hbar^2}}
\end{aligned}$$

For $x > 0$, despite that the delta function is not active, it has imposed initial conditions on the function in the first derivative.

First, we know that the function must be continuous throught the whole domain. Matematically, that means that

$$\varphi(0^-) = \varphi(0^+) = \varphi(0) \Longrightarrow A_1 = A_2' = A.$$

Then, again

$$\varphi(x) = Ae^{-\rho|x|} = \begin{cases} Ae^{\rho x}, & x < 0 \\ Ae^{-\rho x}, & x > 0 \end{cases}.$$

Now, if we look at the equation of discontinuity we have derived in (a.2) and replace the function above we can solve for $\rho$.

$$\begin{aligned} \varphi'(0^+) - \varphi(0^+) &= -\frac{2m\alpha}{\hbar^2}\varphi(0) \\ -\rho Ae^{-\rho(0)} - \rho Ae^{\rho(0)} &= -\frac{2m\alpha}{\hbar^2}\left[Ae^{-\rho(0)}\right] \\ -2\rho &= -\frac{2m\alpha}{\hbar^2} \\ \rho &= \frac{m\alpha}{\hbar^2}. \end{aligned}$$

Because the variable $\rho$ must be equal in either region, we conclude that

$$\rho = \sqrt{-\frac{2mE}{\hbar^2}} = \frac{m\alpha}{\hbar^2}. \tag{9}$$

The dimension of $\alpha$ can now be found easily. We know that $\rho$ must have dimensions of $[L]^{-1}$ in order to have a dimensionless quantity in the argument of the exponential of the wave $\varphi(x)$. Knowing also that the mass have dimension of $[M]$ and $\hbar^2$ of $[M^2L^4T^{-2}]$, doing dimensional analysis with the last equation above yields

$$[L^{-1}] = [M][M^2L^4T^{-2}]^{-1}[\alpha] \longrightarrow [\alpha] = [L]^{-1}[M]^{-1}[M^2L^4T^{-2}] = [ML^3T^{-2}].$$

(b.2) Above we have already limited the general equation to only terms that dont blow up when $x$ goes to $\pm\infty$, meaning that they are square-interable over the space. We have also implicitly set continuity in $\varphi(x)$.

The energy possible upon these constraints is obtained solving for $E$ in the previous equation derived.

$$\sqrt{-\frac{2mE}{\hbar^2}} = \frac{m\alpha}{\hbar^2} \longrightarrow E = -\frac{m\alpha^2}{2\hbar^2}.$$

The wavefunction must be normalized in order to satisfy the normalization condition and be interpreted as a probability density function:

$$\int_{-\infty}^{\infty} |\varphi(x)|^2 \, dx = 1. \tag{10}$$

For that, we will separate the domain in two regions, one for $x < 0$ and other for $x > 0$ and in each subdomain we will use the respective wavefunction.

$$\begin{aligned}
\int_{-\infty}^{0} |Ae^{\rho x}|^2 \, dx + \int_{0}^{\infty} |Ae^{-\rho x}|^2 &= 1 \\
\int_{-\infty}^{0} A^2 e^{2\rho x} \, dx + \int_{0}^{\infty} A^2 e^{-2\rho x} \, dx &= \\
\frac{A^2}{2\rho}\left[e^{2\rho x}\right]\Big|_{-\infty}^{0} - \frac{A^2}{2\rho}\left[e^{-2\rho x}\right]\Big|_{0}^{\infty} &= \\
\frac{A^2}{2\rho}[1-0] - \frac{A^2}{2\rho}[0-1] &= \\
\frac{A^2}{\rho} &= 1 \\
A &= \sqrt{\rho} \quad [L]^{-1/2}.
\end{aligned}$$

Therefore, the normalized wavefunctions are:

$$\varphi(x) = \sqrt{\rho} e^{-\rho|x|} = \begin{cases} \sqrt{\rho} e^{\rho x}, & x < 0 \\ \sqrt{\rho} e^{-\rho x}, & x > 0 \end{cases}. \tag{11}$$

(c) The plot of the wavefunction is illustrated below with the respective defined $\Delta x$ we will derive shortly.

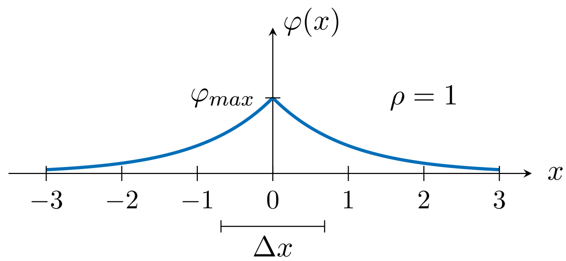


Figure 1: Normalized wavefunction $\varphi(x)$ with its respective $\Delta x$.

We are going to use the Full Width at Half Maximum to define $\Delta x$. It is defined as the diameter (full width) at which half of the peak is reached. We will use the $x > 0$ function and the value of $x$ obtained will be multiplied bw two because of symmetry.

$$\begin{aligned}
\varphi(x) &= \frac{\varphi_{max}}{2} \\
\sqrt{\rho} e^{-\rho x} &= \frac{\sqrt{\rho}}{2} \Big/ \ln(\cdot) \\
-\rho x &= -\ln 2 \\
x &= \frac{\ln 2}{\rho} \longrightarrow \Delta x = \frac{2 \ln 2}{\rho}.
\end{aligned}$$

The order of magnitude of $\Delta x$ would be something proprotional to the inverse of $\rho$ as the other is constant:

$$\Delta x \sim \frac{1}{\rho}.$$

(d) The Fourier transform is applied to ouy wavefunction is:

$$
\begin{aligned}
\tilde{\varphi}(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \varphi(x) \exp\left\{-i\frac{p}{\hbar}x\right\}\, dx \\
&= \frac{\sqrt{\rho}}{\sqrt{2\pi\hbar}} \left[\int_{-\infty}^{0} \exp\left\{\rho x - i\frac{p}{\hbar}x\right\}\, dx + \int_{0}^{\infty} \exp\left\{-\rho x - i\frac{p}{\hbar}x\right\}\, dx\right] \\
&= \sqrt{\frac{\rho}{2\pi\hbar}} \left[\int_{-\infty}^{0} \exp\left\{\left(\frac{\rho\hbar - ip}{\hbar}\right)x\right\}\, dx + \int_{0}^{\infty} \exp\left\{\left(-\frac{\rho\hbar + ip}{\hbar}\right)x\right\}\, dx\right] \\
&= \sqrt{\frac{\rho}{2\pi\hbar}} \left[\frac{\hbar}{\rho\hbar - ip} \exp\{u\}\Big|_{-\infty}^{0} - \frac{\hbar}{\rho\hbar + ip} \exp\{u\}\Big|_{0}^{-\infty}\right] \\
&= \sqrt{\frac{\rho}{2\pi\hbar}} \left[\frac{\hbar}{\rho\hbar - ip} + \frac{\hbar}{\rho\hbar + ip}\right] \\
\tilde{\varphi}(p) &= \sqrt{\frac{\rho}{2\pi\hbar}} \frac{2\hbar^2\rho}{\rho^2\hbar^2 + p^2}.
\end{aligned}
$$

We then define the FWHM as we did for $\varphi(x)$.

$$
\begin{aligned}
\tilde{\varphi}(p) &= \frac{\tilde{\varphi}_{max}}{2} \\
\sqrt{\frac{\rho}{2\pi\hbar}} \frac{2\hbar^2\rho}{\rho^2\hbar^2 + p^2} &= \sqrt{\frac{\rho}{2\pi\hbar}} \frac{1}{\rho} \\
\frac{2\hbar^2\rho}{\rho^2\hbar^2 + p^2} &= \frac{1}{\rho} \\
2\hbar^2\rho &= \rho\hbar^2 + \frac{p^2}{\rho} \\
p &= \hbar\rho \longrightarrow \Delta p = 2\hbar\rho.
\end{aligned}
$$

The figure of $\tilde{\varphi}(p)$ as well as $\Delta p$ is illustrated.

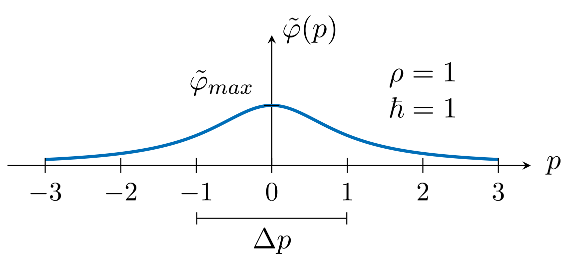


Figure 2: Fourier transform $\tilde{\varphi}(p)$ with a normalized $\hbar = 1$ for visualization.

The prduct of the widths is then:

$$
\Delta x \Delta p = \frac{2\ln 2}{\rho} 2\hbar\rho = 4\ln 2\ \hbar. \tag{12}
$$

The order of magnitude of $\Delta p$ and $\Delta x\Delta p$ would be:

$$
\Delta p \sim \hbar\rho, \quad \text{and} \Delta x \Delta p \sim \hbar.
$$

## 2 Exercise 2