Pset 1 Solutions OPTI 570

1 $\displaystyle\int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2a^2} + bx\right) =$

$$= \int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2a^2} + bx - \frac{a^2b^2}{2}\right) \cdot \exp\left(\frac{a^2b^2}{2}\right) dx =$$

$$= \int_{-\infty}^{+\infty} \exp\left[-\left(\frac{x}{\sqrt{2}a} - \frac{ab}{\sqrt{2}}\right)^2\right] \cdot \exp\left(\frac{a^2b^2}{2}\right) dx =$$

$$= \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2}\left(\frac{x}{a} - ab\right)^2\right] \exp\left(\frac{a^2b^2}{2}\right) dx =$$

$$= \exp\left(\frac{a^2b^2}{2}\right) \cdot a\sqrt{2} \cdot \int_{-\infty}^{+\infty} \exp[-u^2]\, du$$

$$u^2 = \frac{1}{2}\left(\frac{x}{a} - ab\right)^2$$

$$du = \frac{1}{\sqrt{2}} \cdot \frac{1}{a}\, dx$$

$$dx = a\sqrt{2} \cdot du$$

$$\boxed{= \sqrt{2\pi}\; a \; \exp\left(\frac{a^2b^2}{2}\right)}$$

polar integrals

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-r^2} r\, dr\, d\theta} =$$

$$= \sqrt{2\pi \cdot \left(-\frac{1}{2} e^{-r^2}\right|_0^{\infty}} =$$

$$= \sqrt{\pi \cdot 1} =$$

$$= \sqrt{\pi}$$

$$\frac{d\, e^{-r^2}}{dr} = -e^{-r^2} \cdot 2r$$

$\boxed{2}$ argument $-\frac{x^2}{2a^2} + bx$ must be dimensionless

$\Rightarrow$ a has units of length [meter]
b has units of inverse length [1/meter]

$\boxed{3}$ $\int_{-\infty}^{\infty} dx \, x^{n} \exp\left(-\frac{x^2}{2a^2}\right) = 0$

bounds symmetric around 0 — odd — even

$\boxed{4}$ $\bar{\varphi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx \, e^{-ipx/\hbar} \varphi(x)$

a $\frac{px}{\hbar}$ must be unitless

$\Rightarrow$ $\hbar$ has dimensions of [momentum · length]
or $\left[kg \cdot \frac{m}{s} \cdot m\right] = kg \frac{m^2}{s}$

b $\int_{-\infty}^{+\infty} |\varphi(x)|^2 \underbrace{dx}_{\text{length}} = 1$ $\Rightarrow$ $\varphi(x)$ units of $1/\sqrt{\text{length}}$
or $m^{-\frac{1}{2}}$

c $\bar{\varphi}(p) = \underbrace{\frac{1}{\sqrt{2\pi\hbar}}}_{\left[\frac{1}{\sqrt{kg \frac{m^2}{s}}}\right]} \int_{-\infty}^{+\infty} \underbrace{e^{-ipx/\hbar}}_{\text{unitless}} \underbrace{\varphi(x)}_{[m^{-1/2}]} \underbrace{dx}_{[m]}$

$\Rightarrow$ $\bar{\varphi}(p)$ has units of $\frac{1}{\sqrt{kg \; m/s}}$

$\boxed{5}$

$$\tilde{\varphi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-ipx/\hbar} A \cdot e^{-\frac{x^2}{2a^2}} dx$$

$$b = -\frac{ip}{\hbar}$$

$$\bar{\varphi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-ipx/\hbar} \cdot A \cdot e^{-\frac{x^2}{2a^2}} dx =$$

$$= \frac{A}{\sqrt{\cancel{2\pi}\hbar}} \cdot \sqrt{\cancel{2\pi}} \cdot a \cdot \exp\left[\frac{a^2 \cdot \left(-\frac{ip}{\hbar}\right)^2}{2}\right] =$$

$$= \frac{A \cdot a}{\sqrt{\hbar}} \; e^{-\frac{a^2 p^2}{2\hbar^2}}$$

$\boxed{6}$

$$e^{i\theta} = \cos\theta + i\sin\theta \,)^2$$

$$e^{2i\theta} = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$$

also $$e^{2i\theta} = \cos 2\theta + i\sin 2\theta$$

$$\Rightarrow \cos 2\theta = \cos^2\theta - \sin^2\theta + 2 \cdot i \sin\theta \cos\theta - i\sin 2\theta \quad (1)$$

$$\sin 2\theta = \frac{1}{2i}\left(e^{2i\theta} - e^{-2i\theta}\right)^2 =$$

$$= \frac{1}{2i}\left(e^{i\theta} + e^{-i\theta}\right)\left(e^{i\theta} - e^{-i\theta}\right) =$$

$$= 2 \cdot \sin\theta \cdot \cos\theta$$

From (1): $$\cos 2\theta = \cos^2\theta - \sin^2\theta + \cancel{2i\sin\theta\cos\theta - 2i\sin\theta\cos\theta}$$

$$\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta}$$

$$\boxed{\frac{1}{2}\left[1 + \cos 2\theta\right] =} \; \frac{1}{2}\left[1 + \frac{1}{2}\left(e^{2i\theta} + e^{-2i\theta}\right)\right] =$$

$$= \frac{1}{4} \cdot \left[e^{2i\theta} + 2 + e^{-2i\theta}\right] = \frac{1}{4} \cdot \left(e^{i\theta} + e^{-i\theta}\right)^2 \boxed{= \cos^2\theta}$$

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$y(x) = A \cdot e^{mx}$

$\frac{d}{dx} y(x) = A \cdot m \cdot e^{mx} = m \cdot y(x)$ ✓

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$y(x) = A e^{imx} + B e^{-imx}$

$\frac{d^2}{dx^2} y(x) = \frac{d}{dx} im\, y(x) = (im)^2 y(x) =$

$= -m^2 y(x)$ ✓

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$M = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow$ eigenvalues are 3, 4, 2, values along diagonal

or

$|M - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$ $\quad (3-\lambda)(4-\lambda)(2-\lambda) = 0$

solutions are 3, 4, 2

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$|M - \lambda I| = 0$

$\begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = 0$

$\lambda^2 - (-i) i = 0$

$\lambda^2 - 1 = 0$

$\boxed{\lambda = \pm 1}$

$\lambda_1 = 1$

$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$

$-i v_{12} = v_{11}, \quad i v_{11} = v_{12}$

$\Rightarrow \boxed{\bar{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}}$

$\lambda_2 = -1$

$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} -v_{11} \\ -v_{12} \end{pmatrix}$

$-i v_{12} = -v_{11}, \quad i v_{11} = -v_{12}$

$\Rightarrow \boxed{\bar{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}}$

11 $\|\bar{v}\| = \sqrt{\bar{v}^{+} \cdot \bar{v}} =$

$$= \sqrt{\begin{pmatrix} -3 & -4i \end{pmatrix} \begin{pmatrix} -3 \\ 4i \end{pmatrix}} =$$

$$= \sqrt{9 - 16 i^2} =$$

$$= \sqrt{9 + 16} =$$

$$\boxed{\|\bar{v}\| = 5}$$

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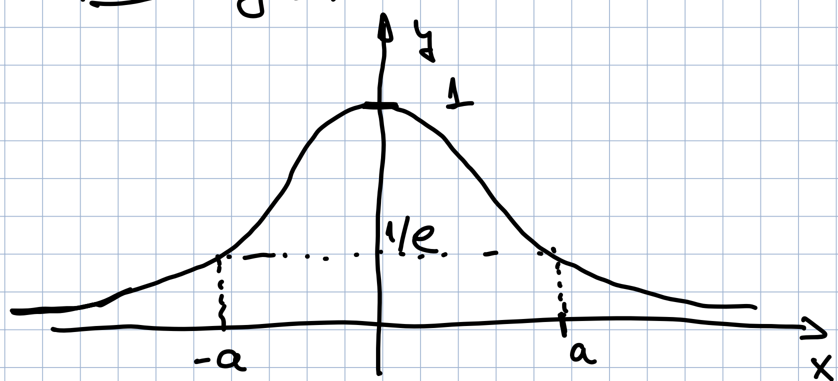
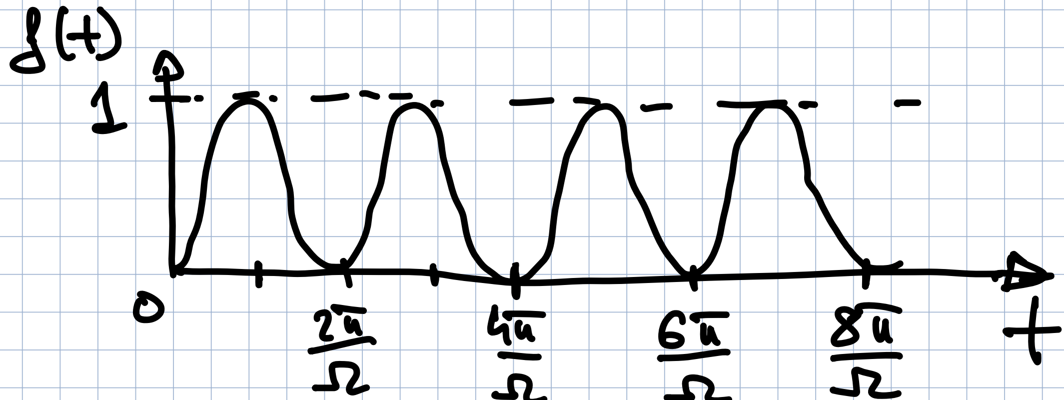
$$\bar{M} \cdot \bar{v} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 1 \\ 3 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ 12 \end{pmatrix}$$

13 $y(x) = e^{-x^2/a^2}$ - Gaussian function



14 $f(t) = \sin^2\left(\frac{\Omega t}{2}\right)$



$\boxed{15}$ $\int_{-\infty}^{+\infty} du\, e^{-u^4} = 1.81 \pm 1$

Let $u = 2^{1/4} \frac{x}{a}$ $\quad du = 2^{1/4} \frac{dx}{a}$

$$1 = A^2 \int_{-\infty}^{+\infty} a / 2^{1/4} \cdot du \cdot e^{-u^4}$$

$$1 = a \cdot A^2 \cdot 2^{-1/4} \cdot \int_{-\infty}^{+\infty} du\, e^{-u^4}$$

$$A = \sqrt{\frac{1}{10^{-6} m \cdot 2^{-1/4} \cdot 1.81}} = 810 / \sqrt{m}$$

$\boxed{16}$

$$A = \int_0^b \frac{x^2}{a} dx =$$

$$= \frac{x^3}{3a} \Big|_0^b = \frac{b^3}{3a} = \frac{(2m)^3}{3 \cdot (4m)} = \frac{2}{3} m^2$$

$\boxed{17}$ $x^3 - 1 = 0$

$$(x-1)(x^2 + x + 1) = 0$$

$\underline{x_1 = 1}$ $\quad \underline{x_{2,3}} = \frac{-2 \pm \sqrt{1-4}}{2} = \underline{-1 \pm i\sqrt{\frac{3}{2}}}$

Also $\underline{x_{2,3}} = e^{i 2\pi/3}, e^{i 4\pi/3}$