OPTI 570, Fall 2025 **Problem Set 6** Prof. C. D. Panda

Due: Thurs, Oct. 9 (Main Campus); Tues, Oct. 14 (Online section)

Problem I. CT V, complement $\mathbf{M}_V$, problem 8.

Solve parts (a)-(c) without making use of the position or momentum representations. In parts (d)-(f), you *will* need to work with the position representation.

This problem gets you to derive some important properties of the quantum harmonic oscillator in a clever (although somewhat tricky) way. **But... the problem is long** — please don't wait until the last minute to do this problem, but also don't let this problem stand in the way of completing the other problems on this assignment. This is one of the most challenging homework problems of the semester. **Hint:** As is often the case with physics problems, if you are asked to calculate something in one part of a problem, there is a good chance that the result will be needed in a later part of the problem. This is the case here. Start early enough on this problem that you have time to give it your best shot and try various approaches prior to asking for help, but don't stress if you're having trouble figuring it all out. Just giving the problem a serious attempt is valuable.

Also, there is a mistake in the problem (at least in early CT editions): part (f-*ii*) should have $\psi(x,0) = e^{-\rho|x|}$ in order for the state to be normalizable over all x.

Problem II. Use phase-space diagrams and visualization methods to solve parts of this problem.

Consider a particle of mass m in a 1-D quantum harmonic oscillator of oscillator frequency ω. The particle is in the ground state $|\phi_0\rangle$ at times $t < 0$. At $t = 0$, the particle is given a momentum "kick" of magnitude p_0 in the positive-x direction. The resulting state is $|\psi(0)\rangle = \hat{T}(p_0)|\phi_0\rangle$, where $\hat{T}(p_0) = \exp\{ip_0\hat{X}/\hbar\}$.

(a) We know that $|\psi(0)\rangle$ is a coherent state, which is an eigenstate of the lowering operator $\hat{a}$. This state can be labeled as $|\alpha_0\rangle = |\psi(0)\rangle$, where α_0 is the associated eigenvalue of $\hat{a}$ for this state. Specify the relationship between $\langle\psi(0)|\hat{P}|\psi(0)\rangle$ and α_0, and the relationship between $\langle\psi(0)|\hat{X}|\psi(0)\rangle$ and α_0. Here, $\hat{P}$ and $\hat{X}$ are the (dimensional) momentum and position operators.

(b) Evaluate $\langle\psi(0)|\hat{H}|\psi(0)\rangle$, where $\hat{H}$ is the 1D harmonic oscillator Hamiltonian. Give your answer in terms of α_0, and also in terms of p_0 [use the relationship you stated in part (a)].

(c) The state now evolves from time $t = 0$ to time $t = t_1$ in the harmonic potential, so that $|\psi(t_1)\rangle = U(t_1, 0)|\psi(0)\rangle$, with U the time evolution operator for the harmonic potential. The state $|\psi(t_1)\rangle$ is also an eigenstate of $\hat{a}$. What is its associated eigenvalue? (You can just write down the answer if you know it, or look it up. A derivation is not needed.)

(d) Evaluate $[\hat{a}, \hat{X}]$ and $[\hat{a}, \hat{T}(p_0)]$. Your answer to this second commutator will have a $\hat{T}(p_0)$ term, but $\hat{a}$ should not appear.

(e) At time $t = t_1$, the particle is given a second momentum kick identical to the first, mathematically described again by $\hat{T}(p_0)$. The value of p_0 is identical to that of the first kick. Is the particle in a coherent state after this second kick? Find the answer to this question by determining whether or not $\hat{T}(p_0)|\psi(t_1)\rangle$ is an eigenstate of $\hat{a}$. If it is, give its eigenvalue. Hint: use the answer for part (d).

(f) Evaluate $\langle\hat{H}\rangle$ immediately after the second kick.

(g) Give a time evolution duration t_1 less than $2\pi/\omega$ for which $\langle\hat{H}\rangle$ is left *unchanged* by the action of the second momentum kick.

(h) We now return to the harmonic oscillator ground state $|\phi_0\rangle$. Describe a sequence of momentum kicks given by identical $\hat{T}(p_0)$ operators ($p_0 \neq 0$) and identical time evolution operators $U_0 = U(t+t_1, t)$ [each with the value of t_1 you found in part (g)] that will act on the particle and return it to the ground state after a few operations.

In other words, find $\hat{Q}$ such that $|\phi_0\rangle$ is an eigenstate of $\hat{Q}$, where $\hat{Q}$ consists of a sequence of $\hat{T}(p_0)$ and U_0 operators that you are to define. For example, one possible sequence is $Q = \hat{T}(p_0)U(\pi/\omega, 0)\hat{T}(p_0)$. You should construct something similar with a different value of the time duration, and a different sequence of operations.

Problem III. A ^{87}Rb atom (mass $m = 1.5 \times 10^{-25}$ kg) is trapped in an isotropic 3D harmonic oscillator potential well created by electro-magnetic fields (the details of which are irrelevant to this problem). The oscillator frequency is $\omega = 100$ radians/s. The atom-trapping apparatus is bolted down in the cargo area of a truck, which is being driven by a physics professor at a constant speed $v_0 = 20$ m/s in the $+\hat{x}$ direction. There is also a grad student in the cargo area maintaining and observing the atom-trapping apparatus. In the grad student's frame of reference, for times $t < 0$, the atom is in the ground state of the 3D potential well. Let the $+\hat{z}$ direction be "up," pointing away from the surface of the earth. The $\hat{y}$ direction is orthogonal to $\hat{x}$ and $\hat{z}$ following the usual Cartesian coordinate conventions.

Precisely at time $t = 0$, two things happen: (1) First, the height of the road suddenly drops by 1 mm, so the truck and the atom-trapping apparatus (but not the atom itself) instantaneously experience a -1 mm shift in their $\hat{z}$ positions at $t = 0$; (2) Second, the professor, who has instantaneously fast reflexes and is attuned to every small bump in the road, is momentarily surprised by the sudden change in the height of the road. Because of this surprise, precisely at $t = 0$ she instantaneously applies the brakes and then nearly immediately releases the brakes, with the net effect of causing a decrease in the speed of the truck from v_0 to $v_0 - \delta v$ at $t = 0$ to an excellent approximation. Because of these two actions, in the grad student's frame of reference, it appears as if at $t = 0$ the atom was given an instantaneous position translation in one direction and a momentum kick in an orthogonal direction.

(a) For $t > 0$, the atom is bouncing around in the trap. Relative to the trap center at coordinate $(x, y, z) = (0, 0, 0)$ in the student's frame of reference, what are the maximum spatial displacements of the center of the atom's wavepacket along each of the $\hat{x}$ and $\hat{z}$ directions that are reached by the atom as it bounces around? In other words, what are the amplitudes of wavepacket oscillations along $\hat{x}$ and $\hat{z}$, as defined by position expectation values? The maximum displacements will not be simultaneously reached for the two different directions. Give numbers with correct units where possible, or analytical expressions in term of the variables given and other known quantities, and make it clear which spatial directions your results refer to.

(b) For one particular value of δv, the center of the atom's wavepacket will undergo circular orbits in space (not phase space) about the center of the trap for $t > 0$. What is this value of δv? Give a number.

(c) For the value of δv determined in part (b), if an energy measurement is to be performed on the atom, what energy measurement result has the highest likelihood of being obtained compared with all other possibilities, and what would be the resulting state of the system after such a measurement result? Give an expression in Dirac notation for the state of the atom, and make sure that whatever symbols and labels you use are clearly identified so that there is no ambiguity in what you mean with your notation.

(d) If the atom's total energy is to be measured at any time $t > 0$, what are the $\hat{x}$-direction position and momentum expectation values (in the grad student's frame of reference) of the atom immediately **after** the energy measurement, regardless of the outcome? Justify your answer.

Background: This problem is inspired by a true story! In 1993, Harvard University physics professor G. Gabrielse and his graduate student C. Tseng built an electron-trapping apparatus in the back of a truck and attempted to drive across the US from California to Boston with the same electrons that were loaded into the apparatus in California. The problem above may give insight into how difficult it would be to keep a trapped atom in the ground state of a potential well that is subjected to position and momentum fluctuations, like if it were in a trap on the International Space Station (this is also being done!) and subject to vibrations etc.

Problem IV.

Part 1. Suppose a hole is drilled through the earth (mass $M_E \approx 6 \times 10^{24}\,\mathrm{kg}$, radius $R_E \approx 6.4 \times 10^6\,\mathrm{m}$), from one point on the surface, through the center (assumed solid), to a point on the opposite side. In this problem, we consider the gravitational potential energy of a particle of mass m inside this hole, and neglect all other forces (such as air resistance, friction with the sides of the hole, rotation) and heating. The center of the earth defines the origin of our coordinate system. For points **outside** the hole, the gravitational potential energy is given by $V(r) = -G\,M_E\,m/r$, where r is the distance from the origin, and $G \approx 6.7 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$. The zero of the energy scale is at points infinitely far away from the earth, where there is no gravitational attraction, hence our potential energies are negative.

For $r < R_E$, as r gets smaller, the gravitational potential energy decreases less steeply than $1/r$, since only the portion of the earth's mass contained within a sphere of radius r affects a particle at distance r from the origin. By treating the earth as a sphere of constant mass density, it is straightforward to calculate that inside the hole $V(r) = V(0) + \frac{G\,M_E\,m\,r^2}{2R_E^3}$, where the potential energy $V(0) = -\frac{3}{2}\frac{G\,M_E\,m}{R_E}$ at $r = 0$ is negative due to the choice of $V(r = \infty) = 0$.

(a) Now consider a particle or mass m dropped into the hole. The particle will reach a speed v_c when it passes through the center of the earth, reach the other side of the earth, then turn around, returning to its original drop point after duration T. Calculate T and v_c. You can compare these with the speed and orbital period of the International Space Station, 28,000 km/hr and 92.7 minutes, respectively.

Part 2. We now adapt this problem to one of relevance to atomic physics, and consider the *Lorentz oscillator model*, a simple model of a hydrogen atom (a proton and electron bound by their attractive Coulomb interaction). In this model, the proton is assumed to be a point particle of electric charge e (where $e = 1.6 \times 10^{-19}$ C) and free to move, and the electron of charge $-e$ is assumed to be a stationary cloud of **uniform** charge density centered at the origin of a three-dimensional coordinate system. The electron cloud has a radius R much larger than the proton. The coordinate r is the distance between the proton and the center of the electron cloud; for $r > R$, the proton feels an electrostatic potential energy

$$V_C = -k_e \frac{e^2}{r},$$

where $k_e = (4\pi\epsilon_0)^{-1} \approx 9 \times 10^9\ \mathrm{N\,m^2\,C^{-2}}$. This has the same mathematical form as the gravitational problem above, including the zero of the energy scale at $r = \infty$. In analogy with the gravitational problem, for $r < R$, it is only the total charge within a sphere of radius r that contributes to the Coulomb interaction, leading to an r^2 dependence of potential energy.

(b) By relating this problem to the gravitational problem, calculate the angular frequency ω of the oscillations of the proton within the electron cloud. You will need to use the mass of the proton, $m_p \approx 1.7 \times 10^{-27}$ kg, and assume that the radius of the electron cloud is $R = 2.5 \times 10^{-11}$ m.

(c) Now for quantum mechanics: if the proton is in the ground state of this electrostatic potential well, what is the approximate length scale over which it is localized? (For this model to be reasonable, this number should be less than the electron cloud radius R and larger than the proton radius $\sim 10^{-15}$ m.)

(d) What wavelength of laser light would be needed to excite the system into one of the first excited energy states (assuming the validity of this model)?

(e) In this model, what is the speed of the proton as it passes through the origin, if it is released at rest at $r = R$? Compare this to the speed calculated in part (a) for an object falling in Earth's gravitational potential.