Assignment 5

OPTI 570 Quantum Mechanics

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Problem I

Part 1.

We define the effective state in the second frame $|\psi_E(t)\rangle = \mathbb{F}(t)|\psi(t)\rangle$, where $\mathbb{F}(t)$ is some unitary time-dependend operator. Substituting $|\psi(t)\rangle = \mathbb{F}^\dagger(t)|\psi_E(t)\rangle$ into the Schrodinger equation yields:

$$\begin{aligned}
i\hbar\partial_t \left[\mathbb{F}^\dagger(t)|\psi_E(t)\rangle\right] &= H(t)\left[\mathbb{F}^\dagger(t)|\psi_E(t)\rangle\right] \\
i\hbar\left[\partial_t\mathbb{F}^\dagger(t)|\psi_E(t)\rangle + \mathbb{F}^\dagger(t)\partial_t|\psi_E(t)\rangle\right] &= H(t)\mathbb{F}^\dagger(t)|\psi_E(t)\rangle \\
i\hbar\mathbb{F}^\dagger(t)\partial_t|\psi_E(t)\rangle &= \left[H(t)\mathbb{F}^\dagger(t) - i\hbar\partial_t\mathbb{F}^\dagger(t)\right]|\psi_E(t)\rangle \Big/ \mathbb{F}(t) \\
i\hbar\partial_t|\psi_E(t)\rangle &= \left[\mathbb{F}(t)H(t)\mathbb{F}^\dagger(t) - i\hbar\mathbb{F}(t)\partial_t\mathbb{F}^\dagger(t)\right]|\psi_E(t)\rangle \\
i\hbar\partial_t|\psi_E(t)\rangle &= H_E(t)|\psi_E(t)\rangle,
\end{aligned}$$

where $H_E(t)$ is the effective Hamiltonian:

$$H_E(t) = \mathbb{F}(t)H(t)\mathbb{F}^\dagger(t) - i\hbar\mathbb{F}(t)\partial_t\mathbb{F}^\dagger(t).$$

Part 2.

We know that

$$|\psi_I(t)\rangle = \mathbb{U}_0^\dagger(t,t_0)|\psi_S(t)\rangle, \quad \text{with} \quad \mathbb{U}_0(t,t_0) = e^{-i(t-t_0)H_0/\hbar}.$$

Then,

$$\begin{aligned}
i\hbar\partial_t\left[\mathbb{U}_0^\dagger|\psi_S(t)\rangle\right] &= i\hbar\partial_t\mathbb{U}^\dagger|\psi_S(t)\rangle + i\hbar\mathbb{U}_0^\dagger\partial_t|\psi_S(t)\rangle \\
&= i\hbar\partial_t\mathbb{U}_0^\dagger|\psi_S(t)\rangle + \mathbb{U}_0^\dagger H_S(t)|\psi_S\rangle \\
&= \left[i\hbar(\partial_t\mathbb{U}_0^\dagger)\mathbb{U}_0 + \mathbb{U}_0^\dagger H_S(t)\mathbb{U}_0\right]|\psi_I(t)\rangle \\
&= \left[-\mathbb{U}_0^\dagger H_0\mathbb{U}_0 + \mathbb{U}_0^\dagger(H_0 + W(t))\mathbb{U}_0\right]|\psi_I(t)\rangle \quad (i\hbar\partial_t\mathbb{U}_0^\dagger = -\mathbb{U}_0^\dagger H_0) \\
i\hbar\partial_t|\psi_I(t)\rangle &= \left[\mathbb{U}_0(t,t_0)^\dagger W(t)\mathbb{U}_0(t,t_0)\right]|\psi_I(t)\rangle \\
i\hbar\partial_t|\psi_I(t)\rangle &= H_E(t)|\psi_I(t)\rangle,
\end{aligned}$$

where $H_E(t)$ is the effective Hamiltonian:

$$H_E(t) = \mathbb{U}_0(t,t_0)^\dagger W(t) \mathbb{U}_0(t,t_0).$$

Problem II

1. The probability for energies greater than $2\hbar\omega$ is then

$$P(E > 2\hbar\omega) = \sum_{n\geq 2} |c_n|^2, \quad c_n = \langle n|\psi(t)\rangle.$$

If $P = 0$, then all $c_n = 0$, $n \geq 2$. Only c_0 and c_1 may be non-zero.

2. The normalization condition means that

$$\sum_{n<2} |c_n|^2 = 1 \Longrightarrow |c_0|^2 + |c_1|^2 = 1.$$

The mean value of the energy is

$$\langle H\rangle = \langle\psi|H|\psi\rangle = |c_0|^2 E_0 + |c_1|^2 E_1 = \frac{1}{2}\hbar\omega|c_0|^2 + \frac{3}{2}\hbar\omega|c_1|^2.$$

If $\langle H\rangle = \hbar\omega$, we have a system of equation composed of the normalization and mean value expression:

$$\left.\begin{array}{l} \frac{1}{2}\hbar\omega|c_0|^2 + \frac{3}{2}\hbar\omega|c_1|^2 = \hbar\omega \\ |c_0|^2 + |c_1|^2 = 1 \end{array}\right] \longrightarrow |c_0|^2 = |c_1|^2 = \frac{1}{2}.$$

3. First, we develop the mean value of X:

$$\langle X\rangle = \langle\psi|X|\psi\rangle = \frac{1}{2}(\langle 0| + e^{-i\theta_1}\langle 1|)X(|0\rangle + e^{i\theta_1}|1\rangle) = \frac{1}{2}\left[\langle 0|X|0\rangle + e^{i\theta_1}\langle 0|X|1\rangle + e^{-i\theta_1}\langle 1|X|0\rangle + \langle 1|X|1\rangle\right].$$

The last result is due to the result we have obtained in the previous incise. Now, we use the matrix element of X of the harmonic oscillator:

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \quad \text{where} \quad a|n\rangle = \sqrt{n}|n-1\rangle, \;\; a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

We compute the terms separately,

$$\langle 0|X|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 0|(a + a^\dagger)|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\langle 0|a|0\rangle + \langle 0|a^\dagger|0\rangle\right] = \sqrt{\frac{\hbar}{2m\omega}}\left[\langle 0|0\rangle + \langle 0|1\rangle\right] = 0,$$

$$\langle 1|X|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\langle 1|a|1\rangle + \langle 1|a^\dagger|1\rangle\right] = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{1}\langle 1|0\rangle + \sqrt{2}\langle 1|2\rangle\right] = 0,$$

$$\langle 0|X|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\langle 0|a|1\rangle + \langle 0|a^\dagger|1\rangle\right] = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{1}\langle 0|0\rangle + \sqrt{2}\langle 0|2\rangle\right] = \sqrt{\frac{\hbar}{2m\omega}},$$

$$\langle 1|X|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\langle 1|a|0\rangle + \langle 1|a^\dagger|0\rangle\right] = \sqrt{\frac{\hbar}{2m\omega}}\left[\langle 1|0\rangle + \sqrt{1}\langle 1|1\rangle\right] = \sqrt{\frac{\hbar}{2m\omega}}.$$

We put these results in $\langle X\rangle$:

$$\langle X\rangle = \sqrt{\frac{\hbar}{2m\omega}}\frac{e^{i\theta_1} + e^{i\theta_1}}{2} = \sqrt{\frac{\hbar}{2m\omega}}\cos\theta_1 = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}.$$

The last relation means that

$$\cos\theta_1 = \frac{\sqrt{2}}{2} \longrightarrow \theta_1 = \pm\frac{\pi}{4} \quad \text{(inside one period)}.$$

4. The time evolution is:

$$|\psi(t)\rangle = \sum_{n=0}^{1} c_n e^{-iE_n t/\hbar}|n\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega t/2}|0\rangle + e^{i\theta_1}e^{-i3\omega t/2}|1\rangle\right)$$

We can factor out the common phase that translates to global phase factor so that we have

$$|\psi(t)\rangle \propto \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i(\theta_1-\omega t)|1\rangle}\right) \longrightarrow \theta_1(t) = \theta_1 - \omega t.$$

We use out previous result of $\langle X\rangle$ and replace θ_1 by $\theta_1(t)$:

$$\langle X\rangle(t) = \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t - \theta_1).$$

The argument of the cosine is reversedas the one in part c) due to the restriction of $\cos\theta_1 = 1/\sqrt{2}$.

Problem III

Problem IV