OPTI 570 LECTURE 7 Tu 09/16

$\boxed{\text{IV}}$ Probability postulate

$$\mathcal{A} \; (\Leftrightarrow \hat{A}) \;, \qquad |\psi\rangle \in \mathcal{E}$$

$$P(a_n) = \langle\psi|\hat{P}_{a_n}|\psi\rangle$$

Cases: (i) discrete, non-degenerate

$$\hat{P}_{a_n} = |a_n\rangle\langle a_n|$$

$$P(a_n) = \langle\psi|a_n\rangle\langle a_n|\psi\rangle =$$

$$= \underbrace{|\langle a_n|\psi\rangle|^2}$$

expansion coefficients of state $|\psi\rangle$ in the eigenket basis.

$$|\psi\rangle = \sum_n P_{a_n}|\psi\rangle = \sum_n |a_n\rangle\underbrace{\langle a_n|\psi\rangle} =$$

$$= \sum_n \underbrace{\langle a_n|\psi\rangle}|a_n\rangle$$

expansion coefficients. $\Rightarrow$ probability amplitudes

(ii) discrete, degenerate: $\hat{A}|a_n^i\rangle = \underline{a_n}|a_n^i\rangle$

$$\underline{\hat{P}}(a_n) = \sum_{i=1}^{g_n} |a_n^i\rangle\langle a_n^i|$$

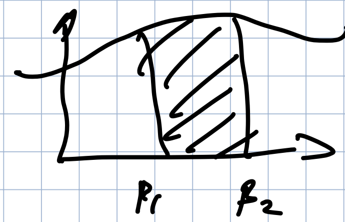
$$\Rightarrow \quad \underline{P}(a_n) = \sum_{i=1}^{g_n} |\langle a_n^i|\psi\rangle|^2$$

(iii) Continuous spectrum $\quad \hat{B}|\beta\rangle = \beta|\beta\rangle$

$$\hat{P}_{\beta_1 \leq \beta < \beta_2} = \int_{\beta=\beta_1}^{\beta_2} |\beta\rangle\langle\beta| \, d\beta$$

$$P(\beta_1 \leq \beta < \beta_2) = \langle\psi|\hat{P}|\psi\rangle =$$

$$= \int_{\beta_1}^{\beta_2} \langle\psi|\beta\rangle\langle\beta|\psi\rangle =$$



$$\boxed{P(\beta_1 \leq \beta < \beta_2) = \int_{\beta_1}^{\beta_2} \underbrace{|\langle\beta|\psi\rangle|^2}_{\text{probability density}} d\beta}$$

Normalization: $\int_{\text{all } \beta} dP = 1$

$dP = |\langle\beta|\psi\rangle|^2 \, d\beta \qquad dP$ - prob of meas. between β and $\beta + d\beta$

Q: $\quad \psi(x) \quad$ meas x_0 ?

prob of meas particle between x_0 and $x_0 + dx$ for small dx ?

$$|\psi(x_0)|^2 dx$$

(iv) continuous, but degenerate spectrum.

ex: 3D space coordinates $\quad$ in $\mathcal{E}$ basis ?

$|x, y, z\rangle \quad , \{|x\rangle, |y\rangle, |z\rangle\}$

Formally: - tensor product: $\mathcal{E} = \mathcal{E}_x \otimes \mathcal{E}_y \otimes \mathcal{E}_z$

$$\{x \in \mathcal{E}_x, \; y \in \mathcal{E}_y, \; z \in \mathcal{E}_z\} \Rightarrow \{|x\rangle\} \; \{|y\rangle\} \; \{|z\rangle\}$$

infinity.

Closure relation:

$$\mathbb{1} = \int_{\text{all } x,y,z} |x,y,z\rangle\langle x,y,z|\, dx\, dy\, dz.$$

$$\underbrace{\hat{P}_{x_1 \le x < x_2} = \int_{x_1}^{x_2} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, \underbrace{|x,y,z\rangle\langle x,y,z|}_{f(x)\, f(y)\, f(z)}}$$

Collapse:

$\hat{A}\,|a_n\rangle = a_n|a_n\rangle$ state $|\psi\rangle \in \mathcal{E}$

$\boxed{\text{I}}$ $P(a_n) = \hat{P}_{a_n}|\psi\rangle$

$\boxed{\text{II}}$ re-normalize

Consequences of postulates

1. Constant global phase factors have no physical significance.

ex: $$|\psi\rangle = c_1 e^{i\theta_1}|u_1\rangle + c_2 e^{i\theta_2}|u_2\rangle =$$
$$= \underbrace{e^{i\theta_1}}_{\substack{\text{global}\\ \text{phase}}}\left[c_1|u_1\rangle + c_2 \underbrace{e^{i(\theta_2-\theta_1)}}_{\substack{\text{relative phase}\\ \text{factor.}}}|u_2\rangle\right]$$

no physical.

ex: $$|\psi\rangle = \frac{e^{i\pi/5}}{\sqrt{2}}|u_1\rangle + \frac{e^{-i\pi\frac{4}{5}}}{\sqrt{2}}|u_2\rangle =$$
$$= \cancel{e^{i\pi/5}}\cdot\left(\frac{1}{\sqrt{2}}|u_1\rangle - \frac{1}{\sqrt{2}}|u_2\rangle\right)$$

$$\psi'(x) = e^{i\theta}\psi(x)$$

$$|\psi'(x)|^2 = \psi'^*(x)\,\psi'(x) = e^{-i\theta}\psi(x)\,e^{i\theta}\psi(x) =$$

$$\underbrace{|\psi'(x)|^2 = |\psi(x)|^2}$$

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle \Leftrightarrow i\hbar\frac{\partial}{\partial t}\left(e^{i\theta}|\psi(t)\rangle\right) = \hat{H}(t)\left|e^{i\theta}|\psi(t)\rangle\right)$$

<u>2.</u> Measurement of a large # of measurements:

<u>weighted sum</u> of the possible <u>outcomes</u>:

eigenvalues of $\hat{A}$: a_n

$$\underbrace{\langle\hat{A}\rangle_\psi}_{\substack{\text{expectation}\\ \text{value}}} = \sum_{\text{all } n} a_n P(a_n) =$$

$\langle\hat{A}\rangle$

$$= \sum_n a_n \langle\psi|\hat{P}_{a_n}|\psi\rangle =$$

$$= \sum a_n \langle\psi|a_n\rangle\langle a_n|\psi\rangle =$$

$$= \sum \langle\psi|\underbrace{a_n|a_n}\rangle\langle a_n|\psi\rangle =$$

$$= \sum\langle\psi|\hat{A}|a_n\rangle\langle a_n|\psi\rangle =$$

$$= \langle\psi|\hat{A}|\underbrace{\sum_n|a_n\rangle\langle a_n|}_{\mathbb{1}}\psi\rangle =$$

$$\boxed{\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle}$$

expectation value $\neq$ expected value

3. Expected spread of measurement results

$$\Delta\hat{A} = \sqrt{\langle \hat{A}^2 \rangle - \langle A \rangle^2} =$$

$$= \sqrt{\langle \psi | \hat{A}^2 | \psi \rangle - (\langle \psi | A | \psi \rangle)^2}$$

Ex: $(\Delta\hat{A})(\Delta\hat{B}) \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle|$

ex: $\hat{x}, \hat{p}$ $\quad \underbrace{\Delta\hat{x} \; \Delta\hat{p} \geq \frac{\hbar}{2}}$

Q: $\hat{H} | u_n \rangle = \underline{E_n} | u_n \rangle$

ΔH for $|u_n\rangle = ?$ $\quad \Delta H = 0$ because E_n

$\Delta\hat{A} = 0$ for any $|a_n\rangle$ always.

Language:

- <u>Compatible observable</u> $\Leftrightarrow$ CSCO

$\hat{A}$ & $\hat{B}$ $\Leftrightarrow$ $[\hat{A}, \hat{B}] = 0$ $\Leftrightarrow$ share set of eigenvectors

ex: meas x and then p $\quad [\hat{x}, \hat{p}] = i\hbar$ $\Rightarrow$ loss information

Cons. measurement w/ compatible observables - no loss of information.

- <u>Bohr frequencies</u> : differences b/w eigenvalues of $\hat{H}$

$\nu_{nm} = (E_n - E_m)/h$ $\quad$ <u>cycle freq.</u> $\quad$ Hz.

$\omega_{nm} = (E_n - E_m)/\hbar$ $\quad$ <u>ang. freq.</u> $\quad \frac{rad}{s}$

<u>AHO</u>: $\quad \omega = 2\pi \cdot \_\_ \quad \frac{rad}{s}$

Time - evolution operator

$$|\psi(t_0)\rangle \longrightarrow |\psi(t)\rangle$$

$$|\psi(t)\rangle = \underbrace{\hat{U}(t,t_0)}\,|\psi(t_0)\rangle$$

$\hat{U}(t,t_0) =$ (i) $\hat{H}$ is time-independent.

$$\hat{U}(t,t_0) = e^{-i\hat{H}(t-t_0)/\hbar}$$

(ii) $\hat{H}$ is time-dependent but $\underbrace{[\hat{H}(t), \hat{H}(t')]}_{=0}$

then

$$\hat{U}(t,t_0) = e^{-i\int_{t_0}^{t} dt'\,\hat{H}(t')/\hbar}$$