Due: Thurs, Nov. 6. (Main Campus); Tues, Nov. 11 (Online section)

Problem I.

Consider a two-level system, and quantum states of the system that can be expressed in a basis denoted $\{|+\rangle, |-\rangle\}$. In the representation defined by this basis, the system has a Hamiltonian that is given by the matrix

$$H = \frac{\hbar}{2}\begin{pmatrix} \Delta & \Omega_0^* \\ \Omega_0 & -\Delta \end{pmatrix} = \frac{\hbar|\Omega_0|}{2}\begin{pmatrix} \Delta/|\Omega_0| & e^{-i\beta} \\ e^{i\beta} & -\Delta/|\Omega_0| \end{pmatrix},$$

where $\Omega_0 = |\Omega_0|e^{i\beta}$ is a complex number, and all quantities are time-independent. Let the initial state of the system be $|\psi(0)\rangle = |+\rangle$, which we will define to correspond to the Bloch vector $\langle\vec{\sigma}\rangle = (0, 0, 1)$.

(a) Plot the time-dependent transition probability $\mathcal{P}_{|+\rangle\to|-\rangle}(t)$ for the cases (i) $\Delta = 0$, (ii) $\Delta = |\Omega_0|$, and (iii) $\Delta = 2|\Omega_0|$. Have your plot cover the times $t = 0$ to $t = \frac{6\pi}{|\Omega_0|}$. Preferably, use a computational package (Matlab, Mathematica, Python, etc) to create the plots, and place all three plots on one graph so that it is easy to compare the three sets of Rabi oscillations. If you draw your plots by hand, please make the oscillation frequencies and amplitudes as accurate as possible.

(b) Sketch by hand OR plot by computer three separate illustrations showing the motion of the Bloch vector corresponding to the three cases above, for a range of times $t = 0$ to $t = \frac{2\pi}{\Omega}$, where Ω is the generalized Rabi frequency for each case. Let $\beta = 0$ for all three cases.

(c) For the case $\Delta = |\Omega_0|$ and $\beta = 0$, and for a range of times $t = 0$ to $t = \frac{2\pi}{\Omega}$, plot the time-dependent probabilities of the system being found in state $|x+\rangle$ and in state $|y+\rangle$, where

$$|x+\rangle \equiv \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

$$|y+\rangle \equiv \frac{1}{\sqrt{2}}|+\rangle + \frac{i}{\sqrt{2}}|-\rangle.$$

Problem II.

(a) The unit vector $\hat{r}$ can be expressed in the $(\hat{x}, \hat{y}, \hat{z})$ basis as $\hat{r} = x\hat{x} + y\hat{y} + z\hat{z}$. Re-write this expression for $\hat{r}$, again in the $(\hat{x}, \hat{y}, \hat{z})$ basis, but use spherical harmonics to write these expansion coefficients instead of x, y, and z. The spherical harmonics are defined on page 100 of the Field Guide, with spherical coordinates defined on page 115 of the Field Guide.

(b) Write the function

$$F(x, y, z) = \frac{x + y + z}{\sqrt{x^2 + y^2 + z^2}} = \frac{x + y + z}{r}$$

in terms of spherical harmonics instead of x, y, and z.

Problem III. It is possible, and intended, to solve this problem using conceptual insight rather than complete quantum-mechanical calculations from scratch. Consider a spin-$\frac{1}{2}$ particle with gyromagnetic ratio $\gamma < 0$. At time $t = 0$, the particle is in the state $|+\rangle_z$ (spin-up along $\hat{z}$). The experimenter sequentially pulses on two orthogonal magnetic fields for two different temporal periods, where the angles in this expression are arbitrary and are the same as those we used in class for spherical coordinates. The fields are turned on and off as follows:

- For times $0 < t \le \tau_y$, a magnetic field of amplitude $B_0 > 0$ is applied and points in the $\hat{y}$ direction.
- For times $\tau_y < t \le \tau_y + \tau_z$, a magnetic field of the same amplitude B_0 is applied but points in the $\hat{z}$ direction.

For all other times, there is no magnetic field.

Specify the values of τ_y and τ_z that produce a state that is spin-up along $\hat{u} = (\,\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta\,)$ by writing answers in terms of quantities given above.

Problem IV. A neutron (n^0) can interact with a charge-neutral *electron neutrino* (ν_r), producing a proton (p^+) and an electron (e^-). This process is characterized by the expression

$$\nu_e \;+\; n^0 \to p^+ \;+\; e^-.$$

Similarly, if a *muon neutrino* ν_μ interacts with a neutron, a proton and a *muon* (μ^-) can be produced:

$$\nu_\mu \;+\; n^0 \to p^+ \;+\; \mu^-.$$

Since $\nu_e \;+\; n^0 \to p^+ \;+\; \mu^-$ and $\nu_\mu \;+\; n^0 \to p^+ \;+\; e^-$ do not occur, the electron or muon neutrino types ("flavors") can be determined by examining the particles produced in neutrino interactions with neutrons. (A third neutrino flavor exists, which we neglect in this problem.) Neutrinos are also *produced* as one of these flavors (ν_e or ν_μ) in different processes. Although neutrinos have non-zero mass, the ν_e and ν_μ particles strangely *do not have precise masses*! This topic was the subject of the 2015 Physics Nobel Prize. The problem below gives an example of how neutrino mass differences can be measured using a quantum oscillation effect, and challenges our intuition on the meaning of mass. The underlying theory is based on the idea that the ν_e and ν_μ particles are actually *two different quantum superpositions* of neutrinos of different but definite masses.

In an accelerator, neutrinos are produced with well-defined momentum p (from here on, p refers to momentum, rather than symbolizing a proton, and is a scalar quantity rather than an operator). Since a neutrino's mass m is so small, and since neutrinos travel very nearly at the speed of light c, neutrino energy $E = \sqrt{p^2c^2 + m^2c^4}$ is well approximated by $E \simeq pc + \frac{m^2c^4}{2pc}$.

Let $\hat{H}$ be the Hamiltonian of a free neutrino of momentum p. We label the eigenstates of $\hat{H}$ as $|\nu_1\rangle$ and $|\nu_2\rangle$. Suppose that

$$\hat{H}|\nu_1\rangle = E_1|\nu_1\rangle, \quad \hat{H}|\nu_2\rangle = E_2|\nu_2\rangle$$

where

$$E_1 = pc + \frac{m_1^2c^4}{2pc}, \quad E_2 = pc + \frac{m_2^2c^4}{2pc}.$$

Here, m_1 and m_2 are the masses of the two energy eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$, and we assume $m_1 < m_2$. We can say that $|\nu_1\rangle$ and $|\nu_2\rangle$ are not only energy eigenstates, but also *mass eigenstates* (to make sense of this, we might assume that a mass operator could be constructed). However, these statements do not mean that a particle must necessarily exist at any arbitrary point in time in one of the Hamiltonian's eigenstates, i.e., in a state with definite mass. A neutrino might instead exist in a superposition of mass eigenstates, as follows.

Let $|\nu_e\rangle$ and $|\nu_\mu\rangle$ indicate the electron and muon neutrino quantum states, respectively. Based on the processes defined at the beginning of this problem, these are the states that are produced or detected in an experiment, rather than a state of well-defined mass ($|\nu_1\rangle$ or $|\nu_2\rangle$). In other words, we are supposing that neutrino mass is not directly measurable. So we will also suppose the following:

$$|\nu_e\rangle = |\nu_1\rangle \sin\beta + |\nu_2\rangle \cos\beta,$$

$$|\nu_\mu\rangle = -|\nu_1\rangle \cos\beta + |\nu_2\rangle \sin\beta,$$

where β is some constant real scalar related to what is called the mixing angle (which has nothing to do with mixed states). β just defines a transformation from one basis to another, and is not an angle in space.

(a) Write the Hamiltonian for this problem as a matrix using the $\{|\nu_1\rangle, |\nu_2\rangle\}$ representation.

(b) Now re-write the matrix from part (a) in the form

$$H = E_m \cdot \mathbb{I} + \epsilon \cdot \sigma_z$$

where $\mathbb{I}$ is the 2×2 identity matrix, σ_z is a Pauli spin matrix following the usual notation, and E_m and ϵ are scalars that you are to determine in terms of the quantities and constants given above. You should find that ϵ is proportional to $m_2^2 - m_1^2$, so write ϵ in terms of a new parameter $\delta m^2 \equiv m_2^2 - m_1^2$. From here on, you are to neglect E_m from all parts of the problem since it is a constant term that won't affect the dynamics of neutrino quantum states. In other words, just to keep things as simple as possible, we are redefining the Hamiltonian such that it takes the form $\epsilon \cdot \sigma_z$ when expressed as a matrix in the $\{|\nu_1\rangle, |\nu_2\rangle\}$ representation. *HINT:* the main step here involves writing the elements of the Hamiltonian matrix in terms of a mean energy $E_m = (E_1 + E_2)/2$ and $\epsilon = (E_1 - E_2)/2$.

(c) Now write the Hamiltonian from part (b) (neglecting the E_m term) as a matrix in the $\{|\nu_e\rangle, |\nu_\mu\rangle\}$ representation instead of the $\{|\nu_1\rangle, |\nu_2\rangle\}$ representation. Be very careful with your math in this step, maybe even doing this part twice. A mistake here will throw off many answers below.

(d) Write your answer from part (c) in the form

$$H = \frac{\hbar}{2} \begin{pmatrix} \Delta & \Omega_0^* \\ \Omega_0 & -\Delta \end{pmatrix}.$$

In solving this part, all you are doing is defining expressions for Δ and Ω_0 in terms of δm^2 and the other scalar quantities and constants of the problem. Simplify your answers as much as possible.

(e) In 1998, convincing evidence for neutrino oscillations was reported using atmospheric neutrino data resulting from a 535-day exposure of the Super-Kamiokande detector [Fukuda et al., PRL 81, 1562 (1998)]. Other recent experiments with reactors, accelerators, and high-precision detectors have also provided evidence for neutrino oscillations. Based on the experimental results, we can realistically suppose that that $\delta m^2 \cdot c^4 \approx 2.5 \times 10^{-3}$ (eV)^2 and $\beta = 80°$. Using these numbers, along with $pc = 10$ GeV $= 10^{10}$ eV, and $\hbar = 6.6 \times 10^{-16}$ eV·s, **calculate a numerical value for the *generalized* Rabi frequency Ω of this problem.**

(f) Suppose that a muon neutrino of momentum p is produced in a particle accelerator at time $t = 0$, so that we write the neutrino's initial quantum state as $|\nu(t = 0)\rangle = |\nu_\mu\rangle$. **Write an expression for the probability of detecting this neutrino in the state $|\nu_e\rangle$ at a later time** t. Express the result in terms of symbols used above, not their numerical values.

(g) Suppose that after the neutrino described in part (f) is produced, its state is to be detected in a target located a distance d from the production point. **Express the probability of part (f) in terms of d instead of t.** Use the approximation that the neutrino travels at the speed of light, c.

(h) You should see that there will be many distances d where the probability of detecting the neutrino in state $|\nu_e\rangle$ is maximized, given that it was initially produced in state $|\nu_\mu\rangle$. **Calculate a numerical value for the shortest distance d_0 where this probability is maximized, using the numerical values that were given in part (e).**

(i) Based on your results above, **what is the probability of a neutrino being found in the $|\nu_e\rangle$ state given that it was produced in the $|\nu_\mu\rangle$ state 300 km away from the detector?** Give a numerical value. (This is the distance that a new accelerator in Japan will be located away from a new neutrino detector, with neutrinos travelling through the earth.) Your answer should be small but not negligible, and give some insight into why it is difficult but not impossible to detect neutrino oscillations in experiments.

(j) Neutrinos can be produced when cosmic ray protons enter the earth's atmosphere, and it turns out there are significantly more ν_μ particles as ν_e detected. So suppose that a neutrino traveling through the atmosphere reaches the earth's surface in a statistical mixture of states with probabilities of 1/4 to be found in the $|\nu_e\rangle$ state, and 3/4 to be found in the $|\nu_\mu\rangle$ state. **Write the density matrix for this neutrino's quantum state in both the $\{|\nu_e\rangle, |\nu_\mu\rangle\}$ representation, and in the $\{|\nu_1\rangle, |\nu_2\rangle\}$ representation in terms of β (use the symbol β, not the numerical value of β).** Label your answers appropriately.