

Assignment 6

OPTI 570 Quantum Mechanics

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Problem I

a) On the one hand, the action of $\tilde{a}(t)$ is

$$\begin{aligned}\tilde{a}(t)|\varphi_n\rangle &= U^\dagger(t, 0)aU(t, 0)|\varphi_n\rangle = U^\dagger(t, 0)ae^{-i(N+\frac{1}{2})\omega t}|\varphi_n\rangle = e^{-i(n+\frac{1}{2})\omega t}U^\dagger(t, 0)a|\varphi_n\rangle \\ &= \sqrt{n}e^{-i(n+\frac{1}{2})\omega t}U^\dagger(t, 0)|\varphi_{n-1}\rangle = \sqrt{n}e^{-i(n+\frac{1}{2})\omega t}e^{i(n-\frac{1}{2})\omega t}|\varphi_{n-1}\rangle = \sqrt{n}e^{-i\omega t}|\varphi_{n-1}\rangle.\end{aligned}$$

Therefore,

$$\tilde{a}(t)|\varphi_n\rangle = \sqrt{n}e^{-i\omega t}|\varphi_{n-1}\rangle = e^{-i\omega t}a|\varphi_n\rangle \longrightarrow \tilde{a}(t) = e^{-i\omega t}a.$$

On the other hand, the action of $\tilde{a}^\dagger(t)$ is

$$\begin{aligned}\tilde{a}^\dagger(t)|\varphi_n\rangle &= U^\dagger(t, 0)a^\dagger U(t, 0)|\varphi_n\rangle = e^{-i(n+\frac{1}{2})\omega t}U^\dagger(t, 0)a^\dagger|\varphi_n\rangle = \sqrt{n+1}e^{-i(n+\frac{1}{2})\omega t}U^\dagger(t, 0)|\varphi_{n+1}\rangle \\ &= \sqrt{n+1}e^{-i(n+\frac{1}{2})\omega t}e^{i(n+\frac{3}{2})\omega t}|\varphi_{n+1}\rangle.\end{aligned}$$

Consequently,

$$\tilde{a}^\dagger(t)|\varphi_n\rangle = \sqrt{n+1}e^{i\omega t}|\varphi_{n+1}\rangle = e^{i\omega t}a^\dagger|\varphi_n\rangle \longrightarrow \tilde{a}^\dagger(t) = e^{i\omega t}a^\dagger.$$

b) We can compute the operators if we stimulate them with a ket $|\varphi_n\rangle$. We make use of the a, a^\dagger expression for \tilde{X} and \tilde{P} .

$$\tilde{X}(t)|\varphi_n\rangle = \frac{\sigma}{\sqrt{2}} \left[U^\dagger a^\dagger U + U^\dagger a U \right] |\varphi_n\rangle.$$

But, we have already computed these operations in the previous part, so we will use it here:

$$\tilde{X}(t)|\varphi_n\rangle = \frac{\sigma}{\sqrt{2}} \left[e^{i\omega t}a^\dagger + e^{-i\omega t}a \right] |\varphi_n\rangle \implies \tilde{X}(t) = \frac{\sigma}{\sqrt{2}} \left[e^{i\omega t}a^\dagger + e^{-i\omega t}a \right].$$

We can further simplified if we develop the complex exponential, obtaining:

$$\tilde{X}(t) = X \cos \omega t + \frac{\sigma^2 P}{\hbar} \sin \omega t.$$

In the same manner, we have for \tilde{P} :

$$\tilde{P}(t)|\varphi_n\rangle = \frac{i\hbar}{\sqrt{2}\sigma} \left[U^\dagger a^\dagger U - U^\dagger a U \right] |\varphi_n\rangle = \frac{i\hbar}{\sqrt{2}\sigma} \left[e^{i\omega t} a^\dagger - e^{-i\omega t} a \right] |\varphi_n\rangle$$

Therefore,

$$\tilde{P}(t) = \frac{i\hbar}{\sqrt{2}\sigma} \left[e^{i\omega t} a^\dagger - e^{-i\omega t} a \right].$$

And,

$$\tilde{P}(t) = P \cos \omega t + i \frac{\sqrt{2}}{\sigma} X \sin \omega t.$$

They are like trigonometric functions cosine and sine, with the different that the operator a, a^\dagger is in the middle.

- c) To show that a ket is an eigenvector of an operator, it will have to satisfy its eigenequation with a constant representing the eigenvalue: $P|\psi\rangle = \lambda|\psi\rangle$.
- d) asfa
- e) asfafasf
- f) asfsa

Problem II

- a) We know that $\alpha_0 = |\psi(0)\rangle$ is a coherent state, so $a|\alpha\rangle = \alpha|\alpha\rangle$. Then, replcaing the definitions of X and P in terms of a, a^\dagger and doing algebra we find:

$$\langle\alpha_0|P|\alpha_0\rangle = \frac{i\hbar}{\sqrt{2}\sigma} \langle\alpha_0|(a^\dagger - a)|\alpha_0\rangle = \frac{i\hbar}{\sqrt{2}\sigma} \left[\langle\alpha_0|a^\dagger|\alpha_0\rangle - \langle\alpha_0|a|\alpha_0\rangle \right] = \frac{i\hbar}{\sqrt{2}\sigma} [\alpha_0^* - \alpha_0] = -\frac{i2\hbar}{\sqrt{2}\sigma} \text{Im}(\alpha_0).$$

In the same manner,

$$\langle\alpha_0|X|\alpha_0\rangle = \frac{\sigma}{\sqrt{2}} \langle\alpha_0|(a^\dagger + a)|\alpha_0\rangle = \frac{\sigma}{\sqrt{2}} \left[\langle\alpha_0|a^\dagger|\alpha_0\rangle + \langle\alpha_0|a|\alpha_0\rangle \right] = \frac{\sigma}{\sqrt{2}} [\alpha_0^* + \alpha_0] = \frac{2\sigma}{\sqrt{2}} \text{Re}(\alpha_0).$$

Then,

$$\alpha_0 = \frac{1}{\sqrt{2}} \left[\frac{\langle X \rangle(0)}{\sigma} + i \frac{\sigma \langle P \rangle(0)}{\hbar} \right].$$

These mean values are the ones we have computed.

- b) Due to the momentum kick, we know that

$$\alpha_0 = \frac{ip_o\sigma}{\sqrt{2}\hbar}.$$

Then, doing the same as in the previous part, we have:

$$\langle\alpha_0|H|\alpha_0\rangle = \hbar\omega \langle\alpha_0|(a^\dagger a + 1/2)|\alpha_0\rangle = \hbar\omega \left[|\alpha_0|^2 + \frac{1}{2} \right] = \frac{p_0^2 \sigma^2 \omega}{2\hbar} + \frac{\hbar\omega}{2}.$$

But, the oscillator length is given by:

$$\sigma = \sqrt{\frac{\hbar}{m\omega}},$$

so

$$\langle \alpha_0 | H | \alpha_0 \rangle = \frac{p_0^2}{2m} + \frac{\hbar\omega}{2}.$$

c) The eigenvalue is $\alpha(t)$:

$$\alpha(t) = \frac{ip_0\sigma}{\sqrt{2}\hbar} e^{-i\omega t}.$$

This is because the state vector remains a coherent state over time, and under that condition the eigenequation of a gives you α_0 when acting on such a vector.

d) sgag

$$[a, X] = \left[\frac{1}{\sqrt{2}} \left\{ \frac{X}{\sigma} + i \frac{\sigma}{\hbar} P \right\}, X \right] = i \frac{\sigma}{\sqrt{2}\hbar} [P, X] = i \frac{\sigma}{\sqrt{2}\hbar} (-i\hbar) = \frac{\sigma}{\sqrt{2}}.$$

For the other commutator, we can use the commutator of a function property:

$$[a, T(p_0)] = [a, e^{ip_0 X/\hbar}] = [a, X] \frac{d(e^{ip_0 X/\hbar})}{dX} = \frac{\sigma}{\sqrt{2}} \frac{ip_0}{\hbar} e^{ip_0 X/\hbar} = \alpha_0 T(p_0).$$

Using the definition of the commutator and the result above:

$$aT(p_0) - T(p_0)a = \alpha_0 T(p_0) \longrightarrow aT(p_0) = (a + \alpha_0)T(p_0).$$

e) Yes, it remains a coherent state. The only adding is a global phase factor. The eigenvalue at $t = t_1$ is $\alpha(t_1) = \alpha_0 e^{-i\omega t_1}$.

$$a(T(p_0)|\psi(t_1)\rangle) = T(p_0)(a + \alpha_0)|\psi(t_1)\rangle = (\alpha(t_1) + \alpha_0)T(p_0)|\psi(t_1)\rangle.$$

so that

$$\alpha_2 = \alpha(t_1) + \alpha_0 = \alpha_0(e^{-i\omega t_1} + 1).$$

f) We do the same as before, but with the new α .

$$\begin{aligned} \langle \alpha_2 | H | \alpha_2 \rangle &= \hbar\omega \left[|\alpha_0(1 + e^{-i\omega t_1})|^2 + 1/2 \right] = \hbar\omega |\alpha_0|^2 (2 + 2\cos\omega t_1) + \frac{\hbar\omega}{2} = \hbar\omega \frac{2p_0^2\sigma^2}{2\hbar^2} (1 + \cos\omega t_1) + \frac{\hbar\omega}{2} \\ \langle \alpha_2 | H | \alpha_2 \rangle &= \frac{p_0^2}{m} (1 + \cos\omega t_1) + \frac{\hbar\omega}{2}. \end{aligned}$$

g) This means that the mean value $\langle H \rangle$ before and after the second kick must be equal:

$$\langle H \rangle_{\text{before}} = \frac{p_0^2}{2m} + \frac{\hbar\omega}{2}, \quad \text{and} \quad \langle H \rangle_{\text{after}} = \frac{p_0^2}{m} (1 + \cos\omega t_1) + \frac{\hbar\omega}{2}.$$

We set both to be equal and solve for t_1 :

$$\begin{aligned} \frac{p_0^2}{m} (1 + \cos\omega t_1) + \frac{\hbar\omega}{2} &= \frac{p_0^2}{2m} + \frac{\hbar\omega}{2} \\ \cos\omega t_1 &= -\frac{1}{2} \\ t_1 &\in \left\{ \frac{2\pi}{3\omega}, \frac{4\pi}{3\omega} \right\}. \end{aligned}$$

- h) Each time evolution given by $t_1 = 2\pi/3\omega$ rotates a third of the full cycle, so with three we complete a full period. We know that a kick adds α , while a time evolution rotates along the phase-space. The sequence is therefore:

$$Q = T(p_0)U(t_1)T(p_0)U(t_1)T(p_0).$$

Problem III

We consider the system as a tensor product of each dimension.

- a) The changes in position can be represented in the following way:

$$|\alpha_x\rangle = T(m\delta_v)|\varphi_0\rangle, \quad |\alpha_y\rangle = |\varphi_0\rangle, \quad |\alpha_z\rangle = S(1 \text{ mm})|\varphi_0\rangle.$$

At $t = 0^+$, we have

$$\alpha_x(0) = \frac{1}{\sqrt{2}} \frac{i\sigma m\delta_v}{\hbar}, \quad \alpha_y(0) = 0, \quad \alpha_z(0) = \frac{1}{\sqrt{2}} \frac{1 \text{ mm}}{\sigma}.$$

The maximum displacements in z is $A_z = 1 \text{ mm}$, whereas for x we know that $p_x(0) = m\omega\delta_v$ so:

$$A_x(0) = \frac{\delta_v}{m\omega} = \frac{\delta_v}{100}.$$

- b) In this case, $A_x = A_z$ and

$$\frac{\delta_v}{\omega} = A_z = 1 \text{ mm} \longrightarrow \delta_v = (1 \text{ mm})\omega = 100 \text{ mm/s} = 0.1 \text{ m/s}.$$

- c) The probability follows a Poisson distribution. First, we have that

$$\alpha_x = \frac{i}{\sqrt{2}} \sqrt{\frac{m}{\hbar\omega}} \delta_v, \quad \alpha_z = \frac{1 \text{ mm}\sigma}{\sqrt{2}}.$$

The square of them corresponds to the Poisson mean and we will take the floor of that value as the most likelihood:

$$|\alpha_x|^2 = \frac{1}{2} \frac{m\delta_v^2}{\hbar\omega} = \frac{1}{2} \frac{(1.5 \times 10^{-25})(0.1)^2}{(1.054571817 \times 10^{-34})(100)} = 71118.91.$$

Therefore,

$$|n_x = 71118\rangle, \quad |n_z = 71118\rangle, \quad |n_y = 0\rangle.$$

- d) They both are zero:

$$\langle X \rangle_{\text{after}} = \langle P \rangle_{\text{after}} = 0.$$

Problem IV

Part 1.

- a) We can treat the problem as a harmonic oscillator. The potential makes a force of the form

$$F_r = -\partial_r V(r) = -\frac{GM_E m}{R_E^3} r = -m\omega^2 r, \quad \omega = \sqrt{\frac{GM_E}{R_E^3}}.$$

Using the information provided, we can obtain the period as:

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{R_E^3}{GM_E}} = 5085.24 \text{ s} \equiv 84.75 \text{ min.}$$

For the case of the velocity, we know that because we are ignoring other forces the system must conserve the energy and it will be distributed in potential and kinetic energy. So,

$$-\frac{3}{2} \frac{GM_E m}{R_E} = \frac{1}{2} m v_c^2 - \frac{3}{2} \frac{GM_E m}{R_E} \rightarrow v_c = \sqrt{\frac{GM_E}{R_E}} = 7907.67 \text{ m/s} \equiv 28.467 \text{ km/h.}$$

Part 2.

- b) For the potential $r < R$, we can take Gauss law to give a structure and then force through the whole space to be something continuous. However, I will compare it with the form of the gravitational force to deduce that:

$$V_C(r) = -\frac{3}{2} \frac{k_e e^2}{R} + \frac{k_e e^2}{2R^3} r^2$$

Now, the force can be derived similarly and equated to the classical harmonic oscillator:

$$F = -\partial_r V_C(r) = -\frac{k_e e^2}{R^3} r = -m_p \omega^2 r \rightarrow \omega = \sqrt{\frac{k_e e^2}{m_p R^3}} = 2.945 \times 10^{15} \text{ rad/s.}$$

- c) The oscillator length σ is:

$$\sigma = \sqrt{\frac{\hbar}{m_p \omega}} = 4.588 \times 10^{-12} \text{ m.}$$

- d) The step of the energy is $\Delta E = \hbar \omega$, and this value is also hc/λ so equating them and solving for λ :

$$\lambda = \frac{hc}{\hbar \omega} = \frac{2\pi c}{\omega} = 640 \text{ nm.}$$

- e) Using the general potential we have derived evaluated at $r = R$ yields

$$V_C(R) = -\frac{k_e e^2}{R}.$$

At that position, there is no kinetic energy as it is at rest. Equating it with the total energy of an arbitrary position provides:

$$\frac{1}{2} m_p v^2 + V_C(r) = -\frac{k_e e^2}{R} \rightarrow v = \sqrt{\frac{k_e e^2}{m_p R^3} (R^2 - r^2)}.$$

At the origin, we have $r = 0$ so

$$v = \sqrt{\frac{k_e e^2}{m_p R}} = 7.4 \times 10^4 \text{ m/s} \equiv 2.66 \times 10^5 \text{ km/h}.$$

It is about 10 times faster than under gravitational potential.