Assignment 9

OPTI 570 Quantum Mechanics

University of Arizona

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Total time: 7 hours

Problem I

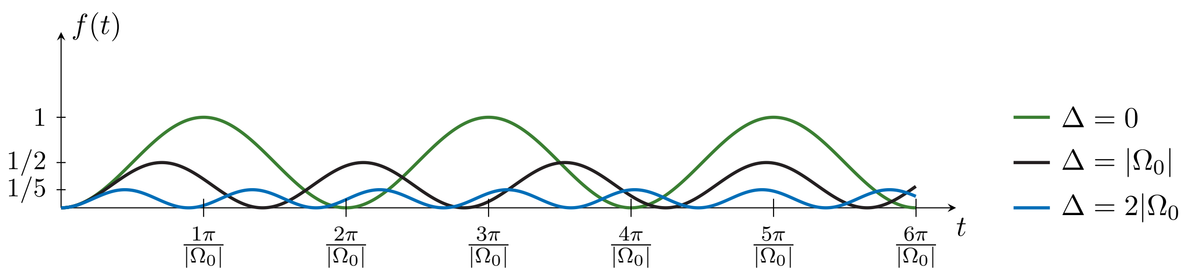
a) The general expression for the transition probability is:

$$P_{|+\rangle\to|-\rangle}(t) = \left|\frac{\Omega_0}{\Omega}\right|^2 \sin^2\frac{\Omega t}{2}, \quad \Omega = \sqrt{\Omega_0^2 + \Delta^2}.$$

By evaluating the different detuning given, we have:

$$\begin{aligned}
\Delta = 0: &\qquad \Omega = \Omega_0 \Longrightarrow P_{|+\rangle\to|-\rangle}(t) = \sin^2\frac{\Omega_0 t}{2} \\
\Delta = |\Omega_0|: &\qquad \Omega = \sqrt{2}\Omega_0 \Longrightarrow P_{|+\rangle\to|-\rangle}(t) = \frac{1}{2}\sin^2\frac{\sqrt{2}\Omega_0 t}{2} \\
\Delta = 2|\Omega_0|: &\qquad \Omega = \sqrt{5}\Omega_0 \Longrightarrow P_{|+\rangle\to|-\rangle}(t) = \frac{1}{5}\sin^2\frac{\sqrt{5}\Omega_0 t}{2}
\end{aligned}$$

For visualization, we will set $\Omega_0 = 1$.



b) We have the following.

c) The probability is the projection of the Bloch vector $\boldsymbol{r}(t)$ onto the measurement direction. For the cases given and that $\Delta = |\Omega_0|$ and $\beta = 0$, we have that:

$$P_{|x+\rangle}(t) = \frac{3 - \cos(\Omega t)}{4}, \quad \text{and} \quad P_{|y+\rangle}(t) = \frac{1 - \frac{1}{\sqrt{2}}\sin(\Omega t)}{2}.$$

The y are plotted in the following:

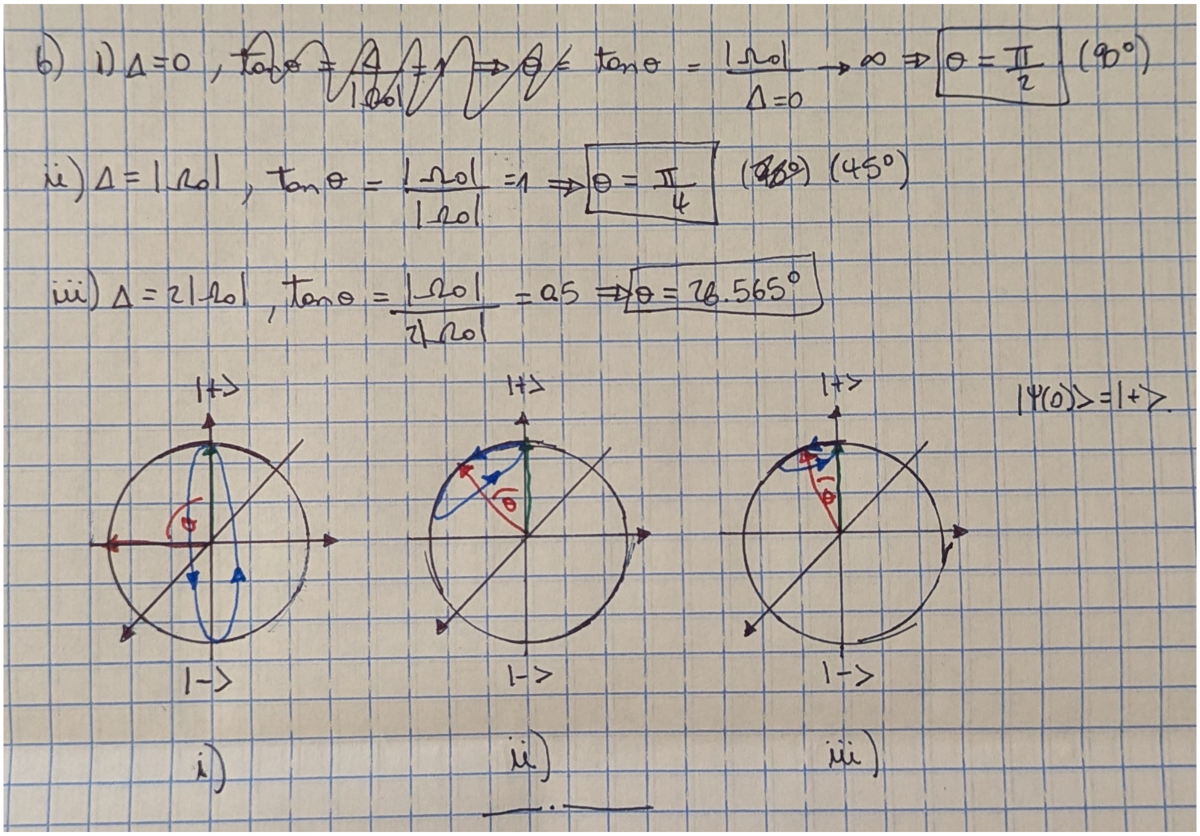
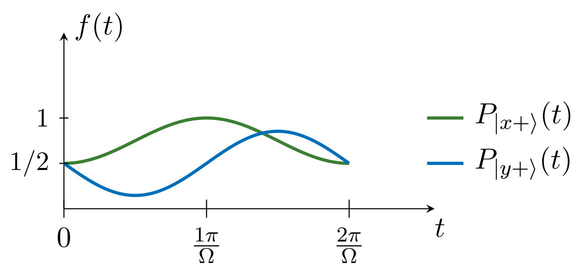


Figure 1: Problem I, part b)



Problem II

a) The unit-vector in cartesian coordinates expressed in terms of the spherical quantities is:

$$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\hat{\boldsymbol{x}} + \sin\theta\sin\phi\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{z}}.$$

We now try to use the table given in the Field guide to substitute these coefficients and express $\hat{\boldsymbol{r}}$ in terms of the Spherical harmonics. The z-direction is the easiest as it only has one quantitiy involved. We now that Y_1^0 has cosine of that angle, so we can use it to say that:

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \longrightarrow \cos\theta = \sqrt{\frac{4\pi}{3}}Y_1^0.$$

For the x-direction, we have the term $\sin\theta\cos\phi$ meaning we need to combine some spherical to have this product form. Using $Y_1^{\pm 1}$ we can create a cosine by considering both sign and collect the expoential:

$$Y_1^{-1} - Y_1^1 = \sqrt{\frac{3}{8\pi}}\sin\theta\left[e^{-i\phi} + e^{i\phi}\right] = \sqrt{\frac{3}{2\pi}}\sin\theta\cos\phi \longrightarrow \sin\theta\cos\phi = \sqrt{\frac{2\pi}{3}}(Y_1^{-1} - Y_1^1).$$

Similarly, we play with $Y_1^{\pm 1}$ to get the $\sin\phi$ term:

$$Y_1^{-1} + Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\sin\theta\left[-e^{-i\phi} + e^{i\phi}\right] = 2i\sqrt{\frac{3}{8\pi}}\sin\theta\sin\phi \longrightarrow \sin\theta\sin\phi = -i\sqrt{\frac{2\pi}{3}}(Y_1^{-1} + Y_1^1).$$

Therefore, we finally have

$$\hat{\boldsymbol{r}} = \sqrt{\frac{2\pi}{3}}(Y_1^{-1} - Y_1^1)\hat{\boldsymbol{x}} - i\sqrt{\frac{2\pi}{3}}(Y_1^{-1} + Y_1^1)\hat{\boldsymbol{y}} + \sqrt{\frac{4\pi}{3}}Y_1^0\hat{\boldsymbol{z}}.$$

b) We have already substituted x for $\sin\theta\cos\phi$ and in terms of the spherical harmonics. The only quantity we need to compute is the r, which is obtained by squaring the compoents.

$$\begin{aligned} r^2 = x^2 + y^2 + z^2 &= \left[\sqrt{\frac{2\pi}{3}}(Y_1^{-1} - Y_1^1)\right]^2 + \left[-i\sqrt{\frac{2\pi}{3}}(Y_1^{-1} + Y_1^1)\right]^2 + \left[\sqrt{\frac{4\pi}{3}}Y_1^0\right]^2 \\ &= \frac{2\pi}{3}\left[(Y_1^{-1})^2 - 2Y_1^{-1}Y_1^1 + (Y_1^1)^2 + (Y_1^{-1})^2 + 2Y_1^{-1}Y_1^1 + (Y_1^1)^2\right] + \frac{4\pi}{3}(Y_1^0)^2 \\ r^2 &= \frac{4\pi}{3}\left[(Y_1^{-1})^2 + (Y_1^1)^2 + (Y_1^0)^2\right]. \end{aligned}$$

Then, we have

$$F(x,y,z) = \frac{(Y_1^{-1} - Y_1^1 - iY_1^{-1} - iY_1^1 + \sqrt{2}Y_1^0)}{\sqrt{2}[(Y_1^{-1})^2 + (Y_1^1)^2 + (Y_1^0)^2]}.$$

Problem III

The action of a B-field gives a precession about it with angular frequency of

$$\omega = -\gamma B_0.$$

The angular displacement in the duration of the pulse is:

$$\alpha = \omega_1\tau.$$

These two quantities must be considerd for each B-field.

First, we start at $|+\rangle_z$. The first pulse is along $+y$ for a duration of τ_y, which produces a rotation of the polar angle of

$$\theta = -\gamma B_0\tau_y.$$

The vector still lives in the $x - z$ plane. Nesxt, the second pulse is along $+z$ for a time of τ_z. This is a rotation about the $+z$ axis, or a rotatin in the azimuth angle of the spherical coordinates ϕ. The rotation is

$$\phi = -\gamma B_0\tau_z.$$

Solving for each equation for the pulse duration yields:

$$\tau_y = -\frac{\theta}{\gamma B_0}, \quad \text{and} \quad \tau_z = -\frac{\phi}{\gamma B_0}.$$

Problem IV

a) Writing the Hailtonian yields:

$$H = pc\mathbb{1} + \frac{c^4}{2pc}\begin{bmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{bmatrix}.$$

b) We can write it in that form if we compare the elements:

$$\begin{aligned} H_1 &= pc + \frac{c^4}{2pc}m_1^2 E_m + \epsilon \\ H_2 &= pc + \frac{c^4}{2pc}m_2^2 = E_m - \epsilon. \end{aligned}$$

Then,

$$\begin{aligned} H_1 + H_2 = 2E_m &= [2pc + \frac{c^4}{2pc}(m_1^2 + m_1^2)] \longrightarrow E_m = pc + \frac{c^4}{4pc}(m_1^2 + m_2^2) \\ H_1 - H_2 = 2\epsilon &= \frac{c^4}{2pc}(m_1^2 - m_2^2) \longrightarrow \epsilon = \frac{c^4}{4pc}(m_1^2 - m_2^2) = -\frac{c^4}{4pc}\delta_m^2. \end{aligned}$$

Thus, our Hamiltonian takes the following form:

$$H = \left[pc + \frac{c^4}{4pc}(m_1^2 + m_2^2)\right]\mathbb{1} - \frac{c^4}{4pc}\delta_m^2\sigma_z.$$

c) The transformation matrix M is:

$$M = \begin{bmatrix} \sin\beta & \cos\beta \\ -\cos\beta & \sin\beta \end{bmatrix}.$$

Then, to transform the hamiltonian we do:

$$\begin{aligned} H_{\{|v_e\rangle,|v_\mu\rangle\}} &= MH_{\{|v_1\rangle,|v_2\rangle\}}M^\dagger \\ &= \epsilon\begin{bmatrix} \sin\beta & \cos\beta \\ -\cos\beta & \sin\beta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \sin\beta & -\cos\beta \\ \cos\beta & \sin\beta \end{bmatrix} \\ &= \epsilon\begin{bmatrix} \sin\beta & \cos\beta \\ -\cos\beta & \sin\beta \end{bmatrix}\begin{bmatrix} \sin\beta & -\cos\beta \\ -\cos\beta & -\sin\beta \end{bmatrix} \\ H_{\{|v_e\rangle,|v_\mu\rangle\}} &= -\frac{c^4}{4pc}\delta_m^2\begin{bmatrix} \cos 2\beta & -2\sin 2\beta \\ -2\sin 2\beta & -\cos 2\beta \end{bmatrix}. \end{aligned}$$

d) By comparing the above result with the form required, we can conclude by looking the element 11 and 21 that:

$$\begin{aligned} \frac{\hbar}{2}\Delta &= -\frac{c^4}{4pc}\delta_m^2\cos 2\beta \longrightarrow \Delta = -\frac{c^4}{2\hbar pc}\delta_m^2\cos 2\beta \\ \frac{\hbar}{2}\Omega_0 &= \frac{c^4}{4pc}\delta_m^2\sin 2\beta \longrightarrow \Omega_0 = \frac{c^4}{2\hbar pc}\delta_m^2\sin 2\beta. \end{aligned}$$

We can also compute Ω:

$$\Omega = \sqrt{\Omega_0^2 + \Delta^2} = \frac{c^4}{2\hbar pc}\delta_m^2.$$

e) Using the numerical values of the variables, by evaluating we have that:

$$\Omega_0 = \frac{2.5\times10^{-3}(eV)^2}{2(6.6\times10^{-16}\ eV\cdot s)(10^{10}\ eV)}\sin 160^\circ = 64.776\ s^{-1}$$
$$\Delta = -\frac{2.5\times10^{-3}\ (eV)^2}{2(6.6\times10^{-16}\ eV\cdot s)(10^{10}\ eV)}\cos 160^\circ = 177.972\ s^{-1}.$$

Then,

$$\Omega = \sqrt{\Omega_0^2+\Delta^2} = 189.394\ s^{-1}.$$

f) The probability asked is the transition probability, given we are initially in $|\nu_\mu\rangle$. This probability is:

$$P_{\mu\to e}(t) = |\langle\nu_e|e^{-iHt\hbar}|\nu_\mu\rangle|^2 = \frac{|\Omega_0|^2}{\Omega^2}\sin^2\frac{\Omega t}{2} = \sin^2(2\beta)\sin^2\left(\frac{c^4\delta_m^2}{4\hbar pc}t\right).$$

g) We need to do the change of variable $t = d/c$ for the above probability:

$$P_{\mu\to e}(d) = \sin^2(2\beta)\sin^2\left(\frac{c^3\delta_m^2}{4\hbar pc}d\right).$$

Note that now we have c^3.

h) To maximize the probability, we need the sin squared function of distance to be 1. It have the same period as $\sin(\cdot)$ which is ieasier to manipulate, so:

$$\sin\frac{c^3\delta_m^2}{4\hbar pc}d = 1/\sin^{-1}(\cdot)$$
$$\frac{c^3\delta_m^2}{4\hbar pc}d = \frac{\pi(2n-1)}{2},\quad n\in\mathbb{Z}.$$

The shortest distance is obtained by picking the first zero, $n = 0$,

$$\frac{c^3\delta_m^2}{4\hbar pc}d = \frac{\pi}{2}$$
$$d = \frac{2\pi\hbar pc}{c^3\delta_m^2} = \frac{2\pi(6.6\times10^{-16}\ eV)(10^{10}\ eV)(3\times10^8\ m/s)}{2.5\times10^{-3}\ (eV)^2} = 4.976\times10^3\ km.$$

i) Evaluating the probability with $d = 300\ km$ yields:

$$P_{\mu\to e}(300\times10^3) = \sin^2(160^\circ)\sin^2\left(\frac{2.5\times10^{-3}\ (eV)^2}{4(6.6\times10^{-16}\ eV)(10^{10}\ eV)(3\times10^8\ m/s)}300\times10^3\right) = 0.0011.$$

j) The density operator in the $\{|\nu_e\rangle,|\nu_\mu\rangle\}$ is:

$$\rho_{\{|\nu_e\rangle,|\nu_\mu\rangle\}} = \frac{1}{4}|\nu_e\rangle\langle\nu_e| + \frac{3}{4}|\nu_\mu\rangle\langle\nu_\mu| = \begin{bmatrix}\frac{1}{4} & 0\\ 0 & \frac{3}{4}\end{bmatrix}.$$

In the $\{|\nu_1\rangle, |\nu_2\rangle\}$, we use the transformation matrix used in prevous part to convert this operator:

$$
\begin{aligned}
\rho_{\{|\nu_1\rangle,|\nu_2\rangle\}} &= M^\dagger \rho_{\{|\nu_e\rangle,|\nu_\mu\rangle\}} M \\
&= \begin{bmatrix} \sin\beta & -\cos\beta \\ \cos\beta & \sin\beta \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{bmatrix} \begin{bmatrix} \sin\beta & \cos\beta \\ -\cos\beta & \sin\beta \end{bmatrix} \\
&= \begin{bmatrix} \sin\beta & -\cos\beta \\ \cos\beta & \sin\beta \end{bmatrix} \begin{bmatrix} \frac{1}{4}\sin\beta & \frac{1}{4}\cos\beta \\ -\frac{3}{4}\cos\beta & \frac{3}{4}\sin\beta \end{bmatrix} \\
\rho_{\{|\nu_1\rangle,|\nu_2\rangle\}} &= \begin{bmatrix} \frac{1}{2} + \frac{1}{4}\cos 2\beta & -\frac{1}{4}\sin 2\beta \\ -\frac{1}{4}\sin 2\beta & \frac{1}{2} - \frac{1}{4}\cos 2\beta \end{bmatrix}.
\end{aligned}
$$

We see that in both representation, the trace is unitary.