OPTI 570, Fall 2025 **Midterm Exam 1**

2-hour written exam.

On-campus students: exam time and location: 5 - 7pm (Tucson time), Sep 24 in Meinel 305.

Distance students: exam will be available on the D2L **Assignments** page, starting 7pm (Tucson time), Sep 24, after submitting a **Quiz** acknowledging that you have read these instructions. You must complete and return by 9pm (Tucson time), Sep 29, unless you have made other arrangements with me. Distance students have 2 hours to work on the exam questions, plus *up to* 30 minutes for downloading and printing the exam, changing locations, scanning answers, and uploading answers to D2L as a single PDF (or email to me if there's a problem uploading). These extra 30 minutes are *not* to be used to work on the exam.

Instructions

- **The 2 hours that you have available for the exam begin *after* you finish reading this instructions page and as soon as you start reading the problems of the exam.**
- You may consult the following items during the exam: PDF or physical/printed copies of the course notes and the notes from lectures, recap sessions, and recitation sections; QM Field Guide; OPTI 570 problem sets and solutions (yours and mine); and any of your own notes or anything you have personally written or typed. You do not need a calculator, and you must not use one. Computers may be used only to access allowed material that is stored on your computer, or allowed internet resources. Allowed internet resources are only the OPTI 570 D2L site for distance students to access and return the exam, and email to communicate with me if needed. You must **not** consult other people, or accept or provide help to anyone else in the class. You are on your honor to adhere to these rules; violation of these rules will result in a failing grade.
- There are 6 problems on the following 3 pages. **120 points are available**, although the exam is graded out of 100 points, so there are **20 extra points** available. It is possible that the exam scores may also be further scaled up. The highest final grade that will be recorded will be 100, even for those who earn more than 100 points.
- Use your own paper to solve all problems. Show enough work that I can follow your reasoning and give you partial credit for problems that are not fully correct.
- It is up to you to convince me that you know how to solve the problems, and to write legibly enough that I do not need to struggle to interpret your work. However, I expect you to work quickly, and that the neatness of your solutions might consequently suffer. That's OK as long as I can interpret your solutions. Draw a box around final answers if your final results are not obvious. If you have a mess of equations all over the page, direct my attention to your line of thought if it is not otherwise obvious. If you have obtained an answer that you know is not correct and you do not have enough time to fix the error, please tell me that you know the answer is wrong, why you know that it is wrong, and guess an appropriate answer – this may help you earn significant partial credit.
- If you are convinced that there is a significant mistake in a problem that may affect the answer or interpretation, please ask about it. Or if a mistake is obvious to you, you may indicate what you think is wrong, what should be changed to make the problem solvable in the manner that you think I intended, then solve the problem. Make sure that I can understand how you have modified the problem to make it solvable. Part of the challenge of learning a new subject is to try to identify mistakes and speculate about the original intention of given problems!

For problems 1-3, let $\{\,|u_1\rangle,\,|u_2\rangle,\,|u_3\rangle\}$ be an orthonormal basis for a state space $\mathcal{E}$.

1. [15 pts.] Give the scalar, column vector, row vector, or matrix that corresponds to each one of the following items in the representation defined by the above basis. You are not required to show any work if you know the answer immediately.

(a) $|u_3\rangle$

(b) $|w\rangle = \frac{1}{\sqrt{2}}|u_1\rangle - \frac{e^{i\theta}}{2}|u_2\rangle - \frac{i}{2}|u_3\rangle$

(c) $\langle w|$, the Hermitian conjugate of the above

(d) $|u_3\rangle\langle u_2|$

(e) $|u_1\rangle\langle u_1| + |u_2\rangle\langle u_2| + |u_3\rangle\langle u_3|$

(f) $\displaystyle\sum_{m=1}^{m=3}\sum_{n=1}^{n=3} |u_m\rangle\langle u_n|$

(g) The projector onto $|u_3\rangle$

(h) $\langle w|w\rangle$, with $|w\rangle$ defined in (b)

(i) $\hat{P}_w = |w\rangle\langle w|$

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2. [25 pts.] The operator $\hat{B}$ is defined by the following relationships:

$\hat{B}\,|u_1\rangle = i\,|u_2\rangle$
$\hat{B}\,|u_2\rangle = i\,|u_3\rangle$
$\hat{B}\,|u_3\rangle = i\,|u_1\rangle$

(a) Construct the matrix associated with $\hat{B}$ using the $\{\,|u_1\rangle,\,|u_2\rangle,\,|u_3\rangle\}$ representation.

(b) Calculate the commutator $[\hat{B},\,\hat{B}^\dagger]$.

(c) The operator $\hat{C}$ is defined as $\hat{C} \equiv \hat{B} + \hat{B}^\dagger$. Is $e^{i\hat{C}}$ hermitian? Is $e^{i\hat{C}}$ unitary? Justify your answers with math or reasoning.

(d) Determine if $e^{i\hat{C}}$ is equal to $e^{i\hat{B}}e^{i\hat{B}^\dagger}$, and justify your answer with math or reasoning.

(e) Find the eigenvalues of $\hat{C}$.

(f) Evaluate $e^{i\hat{C}}\,|\varphi\rangle$, where $|\varphi\rangle \equiv \frac{1}{\sqrt{3}}\{|u_1\rangle + |u_2\rangle + |u_3\rangle\}$ is an eigenstate of $\hat{C}$.

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3. [15 pts.] A new base $\{|w_1\rangle, |w_2\rangle, |w_3\rangle\}$ lives in the same state space $\mathcal{E}$. The expansions of $|w_1\rangle$ and $|w_2\rangle$ in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis are as follows:

$$|w_1\rangle = N_1\left(|u_1\rangle + 2|u_2\rangle - i|u_3\rangle\right)$$

$$|w_2\rangle = N_2\left(4|u_1\rangle - |u_2\rangle + 2i|u_3\rangle\right),$$

where N_1 and N_2 are normalization coefficients that you will not need to find. Determine the normalized expansion of $|w_3\rangle$ in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis such that $|w_3\rangle$ is orthogonal to both $|w_1\rangle$ and $|w_2\rangle$, and choose a global phase factor for this expansion such that the first non-zero superposition coefficient is real and positive.

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4. [15 pts.] Suppose that a Hamiltonian $\hat{H}$ has the eigenvalue equation

$$\hat{H}|u_n^j\rangle = E_n|u_n^j\rangle$$

where $n \in \{1, 2, 3, 4\}$, and the index j accounts for degeneracies in the spectrum of $\hat{H}$. In the representation defined by its eigenkets (arranged from lowest to highest associated eigenvalues), $\hat{H}$ is expressed as the matrix

$$\begin{pmatrix} 2\epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 5\epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 5\epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 8\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 10\epsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 10\epsilon \end{pmatrix}$$

where ϵ is a constant. $\hat{H}$ can act on elements of some state space $\mathcal{E}$. Consider one element of $\mathcal{E}$:

$$|\psi\rangle = \frac{1}{3}|u_1\rangle + \frac{i}{\sqrt{3}}|u_2^1\rangle - \frac{1}{3}|u_2^2\rangle + \frac{\sqrt{2}}{3}|u_3\rangle - \frac{\sqrt{2}}{3}|u_4^2\rangle.$$

(a) An energy measurement is to be made of the system for which $\hat{H}$ is the Hamiltonian. Specify all of the possible outcomes of the measurement, regardless of the state the system is in. Then for the state $|\psi\rangle$, specify the probabilities of each of these outcomes being obtained.

(b) If the measurement of part **(a)** results in the value 5ϵ, what is the **normalized** state of the system immediately after the measurement? Write your answer in terms of the eigenkets of $\hat{H}$.

(c) Calculate $\langle H\rangle$ for state $|\psi\rangle$.

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5. [25 pts.] The state of a particle is described in the position representation by wavefunction $\psi(x,t)$. Let $\mathcal{J}(x,t) = \frac{\hbar}{(2mi)}(\psi^*(x,t)\frac{\partial\psi(x,t)}{\partial x} - \psi(x,t)\frac{\partial\psi^*(x,t)}{\partial x})$. Note: parts **(b)** and **(c)** can be solved independently.

(a) What are the units of $\mathcal{J}(x,t)$?

(b) Let P_{ab} be the probability of finding the particle in the range $a < x < b$. Show that $\frac{dP_{ab}}{dt} = \mathcal{J}(a,t) - \mathcal{J}(b,t)$ for any $\psi(x,t)$. *Hint:* Integration by parts might become helpful in the second part of the problem: $\int_a^b f\frac{\partial g}{\partial x}dx = fg|_a^b - \int_a^b \frac{\partial f}{\partial x} g dx$ for functions f and g.

(c) In general, we can write any wavefunction as the product of amplitude and phase factors $\psi(x,t) = |\psi(x,t)|e^{i\phi(x,t)}$. Assume that the phase varies linearly with position as $\phi(x,t) = xp'/\hbar$, where p' is a scalar. What are the units of p'? Show that, in this case, $\mathcal{J}(x,t)$ is proportional to $|\psi(x,t)|^2$. Interpret the resulting equation and proportionality factor.

6. **[25 pts.]** Consider the state space $\mathcal{E}$ associated with a particle constrained to move in a 1-dimensional coordinate system over position x. In this problem, you will use the momentum basis $\{|p\rangle\}$, where $\hat{P}\,|p\rangle = p\,|p\rangle$ and $\hat{P}$ is the 1-dimensional momentum operator. We define a new operator $\hat{\Pi}$ by the following expression (assuming integration over all momenta in this and all integrals below):

$$\hat{\Pi} = \int dp'\,|-p'\rangle\langle p'|.$$

Note that this is *not* a closure relation! With this definition, we can determine that the action of $\hat{\Pi}$ on the basis element $|p_0\rangle$ is simply $\hat{\Pi}\,|p_0\rangle = |-p_0\rangle$, where $\hat{P}\,|-p_0\rangle = -p_0|-p_0\rangle$. In detail:

$$\hat{\Pi}\,|p_0\rangle = \int dp'\,|-p'\rangle\langle p'|p_0\rangle = \int dp'\,|-p'\rangle\,\delta(p'-p_0) = |-p_0\rangle.$$

(a) Evaluate $\hat{\Pi}^2\,|p_0\rangle$.

(b) Evaluate $\langle p|\hat{\Pi}\,|p_0\rangle$.

(c) For an arbitrary state-space element $|\psi\rangle$, where the wavefunction $\tilde{\psi}(p)$ is defined by $\tilde{\psi}(p) = \langle p|\psi\rangle$, evaluate $\langle p|\hat{\Pi}\,|\psi\rangle$. Hint: use the closure relation $\hat{\mathbb{1}} = \int dp'\,|p'\rangle\langle p'|$.

(d) Now define a new operator $\hat{F}_+$ by the relation

$$\hat{F}_+ = \frac{1}{2}(\hat{\mathbb{1}} + \hat{\Pi}).$$

Evaluate $\langle p|\hat{F}_+\,|\psi\rangle$ for arbitrary $|\psi\rangle$.

(e) Under the condition that $\tilde{\psi}(p) = \langle p|\psi\rangle$ is a function with even parity, meaning that $\tilde{\psi}(-p) = \tilde{\psi}(p)$, evaluate and simplify your answer to part **(d)**.

(f) Under the condition that $\tilde{\psi}(p) = \langle p|\psi\rangle$ is a function with odd parity, meaning that $\tilde{\psi}(-p) = -\tilde{\psi}(p)$, evaluate and simplify your answer to part **(d)**.

(g) Show that $|\varphi\rangle = |p\rangle + |-p\rangle$ is an eigenstate of $\hat{\Pi}$. What is its associated eigenvalue?

(h) Sketch $|\varphi(x)|^2$ vs x for some reasonable range of positions, where $\varphi(x)$ is the **position** representation of $|\varphi\rangle$.

(i) Determine whether or not $\hat{\Pi}$ is hermitian.

END OF EXAM