Nominally Due: Thurs, Oct. 16 (Main Campus); Tues, Oct. 21 (Online section). Late submission up until Exam 2 submission is also ok.

All questions below are modified from older take-home exam problems, and the first three involve working with the interaction picture (Field Guide, page 46). Make sure that you have worked and understood Problem I from Problem Set 5 before working the problems below. **Do as much on your own as you can, before working with others, as this problem set will double as your practice exam for the Harmonic Oscillator topic.**

Problem I. The 1D quantum harmonic oscillator Hamiltonian is defined as

$$\hat{H}_0 = \hbar\omega(\hat{N} + 1/2).$$

One of the main results of the harmonic oscillator problem is that if the state of the system is a coherent state at time $t = 0$, the system remains in a coherent state for any subsequent time as long as it evolves under a Hamiltonian of the form $\hat{H}_0$. In the problem below, we also investigate the dynamics of a quantum state that is *initially* a coherent state, but find that it does not remain a coherent state when it evolves due to a Hamiltonian that has an anharmonic term in it.

Here we will work with the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$, where $\hat{H}_1 = \hbar\Omega(\hat{N}^2 - 1/2)$ such that

$$\hat{H} = \hbar\omega(\hat{N} + 1/2) + \hbar\Omega(\hat{N}^2 - 1/2).$$

As can be seen by inspection, the eigenstates of $\hat{H}$ are the same as the eigenstates $\{|\varphi_n\rangle\}$ of the usual harmonic oscillator Hamiltonian $\hat{H}_0$, although the energy eigenvalues are different than those of $\hat{H}_0$.

Here we use a unitary time-dependent frame translation operator

$$\hat{\mathbb{F}}(t) = \hat{\mathbb{U}}_0^\dagger(t) = e^{i\hat{H}_0 t/\hbar},$$

which is the Hermitian conjugate of the time-evolution operator associated with $\hat{H}_0$ only (and *not* the time-evolution operator of the full Hamiltonian $\hat{H}$). The action of $\hat{\mathbb{F}}(t)$ on a state $|\psi(t)\rangle$ is to "undo" the effects of the evolution that are due only to $\hat{H}_0$, such that time evolution under the effective Hamiltonian $\hat{H}_E$ involves only $\hat{H}_1$. The reference frame thus obtained is called the **Interaction Picture** (Field Guide p.46), and the effective Hamiltonian is given by

$$\hat{H}_E(t) = \hat{\mathbb{F}}(t)\hat{H}_1\hat{\mathbb{F}}^\dagger(t) = \hat{\mathbb{U}}_0^\dagger(t)\hat{H}_1\hat{\mathbb{U}}_0(t).$$

Note that if $\hat{H}_1$ is zero ($\Omega = 0$), then the effective Hamiltonian $\hat{H}_E(t)$ would be zero, and quantum states would not evolve in time. Such a case would turn the interaction picture into the Heisenberg picture (see the text box at the bottom of the Field Guide, p. 45).

(a) It turns out that the effective Hamiltonian $\hat{H}_E(t)$ is just equal to $\hat{H}_1$ since $\hat{H}_1$ commutes with $\hat{\mathbb{U}}_0(t)$. So now that you know that $\hat{H}_E = \hat{H}_1 = \hbar\Omega(\hat{N}^2 - 1/2)$, write down the *effective* time evolution operator $\hat{\mathbb{U}}_E(t)$ that corresponds to evolution due to $\hat{H}_E$ only. This is the time evolution operator for use in the interaction picture.

In the questions below you will work with the operation $\hat{\mathbb{U}}_E(t)|\varphi_n\rangle$ for specific values of the time t. As a check, verify that $\hat{\mathbb{U}}_E(t = 2\pi/\Omega)|\varphi_n\rangle = -|\varphi_n\rangle$. If it does not, check your work now and fix any problems. However, there is an n-dependent phase factor imposed by $\hat{\mathbb{U}}_E(t)$ for evolution times other than integer multiples of $2\pi/\Omega$. In the following, keep all phase factors (even when you think they are global phase factors).

(b) Evaluate and simplify as much as possible $\hat{\mathbb{U}}_E(\tau)|\varphi_n\rangle$ for $\tau \equiv \frac{\pi}{2\Omega}$. In simplifying as much as possible, you will need to use the fact that $(-i)^{n^2} = 1$ for even n and $(-i)^{n^2} = -i$ for odd n, allowing you to write the phase factor in front of $|\varphi_n\rangle$ as a complex number of unit magnitude that has an n-dependent phase angle. Once you have obtained an expression for an n-dependent phase factor in exponential notation, write it as a complex number in the particular form $c_1 + ic_2$ for real numbers c_1 and c_2 (which you need to determine), where n-dependence will show up in this complex number.

(c) Define $|\Psi(t = 0)\rangle = |\Psi_E(t = 0)\rangle = |\alpha_0\rangle$, where $\hat{a}|\alpha_0\rangle = \alpha_0|\alpha_0\rangle$, $\hat{a}$ is the harmonic oscillator lowering operator, and α_0 is any complex number. Now evaluate and simplify as much as possible $|\Psi_E(t = \tau)\rangle = \hat{\mathbb{U}}_E(\tau)|\Psi_E(0)\rangle$. In simplifying, you should be able to finally express $|\Psi_E(t = \tau)\rangle$ as a superposition of two simply expressed quantum states (which are not energy eigenstates, and therefore each part of the superposition is itself a superposition of energy eigenstates).

(d) Evaluate $|\Psi(\tau)\rangle$, the quantum state in the original Schrödinger picture that is associated with your final calculation of $|\Psi_E(\tau)\rangle$ from part (c).

(e) Evaluate and simplify $|\Psi(\tau)\rangle$ for the case where $\Omega = \omega$. Now using two phase-space diagrams, one for time $t = 0$ and another for time $t = \tau = \frac{\pi}{2\omega} = \frac{\pi}{2\Omega}$, sketch a visual representation of the corresponding quantum states, making use of uncertainty "blobs." For the case associated with $t = \tau$, use different uncertainty blobs for each part of the superposition in your final answer to (c), rather than the usual approach of having a single blob centered at the position and momentum expectation values of the entire state.

Physical interpretation: The state $|\Psi(t = \tau)\rangle$ is a superposition of two different states that are each as close as possible to being like classical states of the harmonic oscillator. It is thus reminiscent of the concept behind Schrödinger's cat, and hence $|\Psi(t = \tau)\rangle$ is called a "cat state". As you can see, to generate a state like this, you would (1) prepare a coherent state in a harmonic oscillator, (2) suddenly turn on an extra Hamiltonian of the form of $\hat{H}_1$, (3) evolve for just the right amount of time and then turn off $\hat{H}_1$. Hamiltonians of this form can be generated in the lab, and cat states can be prepared, but they are extremely fragile: it is very challenging to maintain such quantum-state superpositions!

Problem II. As you saw in Problem Set 4, the position uncertainty $\Delta\hat{X}$ for a free particle wavepacket is time-dependent. Even if $\Delta\hat{X}$ is decreasing for some time period, it will eventually increase again, just like a laser beam diverges past its focus. An experimental technique that has been used in atom optics experiments to temporarily reverse such wavepacket spreading is to briefly apply a harmonic oscillator potential. Such a technique is the atom-optics equivalent of using a thin lens to refocus a diverging laser beam. This problem below briefly considers such a scenario, albeit in an unconventional way that may help make the following problem slightly easier.

Assume that for times $t < 0$, the state of a particle of mass m is evolving without the influence of any potential energy (ie, it is a free particle). The particle is constrained to move in the $\hat{x}$ direction only. For a brief period $0 \le t < \tau$, a 1D harmonic oscillator potential along $\hat{x}$ is turned on, then removed at $t = \tau$. The particle dynamics during the period $0 \le t < \tau$ could be evaluated using harmonic oscillator techniques in the Schrödinger Picture, but it is instructive to instead examine this problem in the Interaction Picture as follows. Let $\hat{H}_0$ and $\hat{\mathbb{U}}_0$ be the Hamiltonian and time-evolution operators (respectively) for the **free particle** problem. Let $\hat{W} = \frac{1}{2}m\omega^2\hat{X}^2$ be the additional harmonic oscillator potential that is briefly applied, so that for the time period $0 \le t < \tau$ the total system Hamiltonian is $\hat{H} = \hat{H}_0 + \hat{W} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega^2\hat{X}^2$.

Now, using the methods described on p.46 of the *Field Guide to Quantum Mechanics*, determine the effective Hamiltonian $\hat{H}_E$ for this case, valid for times $0 \le t < \tau$. Express your answer in terms of the Schrödinger Picture position and momentum operators $\hat{X}$ and $\hat{P}$, and simplify as much as possible. You must not have the operator $\hat{\mathbb{U}}_0$ appearing in your final answer. Your result should look *somewhat* like a harmonic oscillator Hamiltonian, along with extra terms that can be associated with the wavepacket focusing action. *Hints:* (1) the calculations here could take very little work if you carefully interpret your expressions involving time evolution operators before doing much math; and (2) For any operator $\hat{A}$ and an operator $\hat{B}$ such that $\hat{B}^\dagger = -\hat{B}$ (which makes $e^{\hat{B}}$ unitary),

$$e^{\hat{B}}\,\hat{A}^2\,e^{-\hat{B}} = e^{\hat{B}}\,\hat{A}\,e^{-\hat{B}}\,e^{\hat{B}}\,\hat{A}e^{-\hat{B}} = (e^{\hat{B}}\,\hat{A}e^{-\hat{B}})^2.$$

Problem III. *(Work Problem II first.)* The goal here is for you to develop some understanding of current topics in quantum mechanics by working through this problem.

To begin, we make some definitions: First, define $\hat{H}_0 \equiv \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$. Next, define

$$\hat{W}(t) \equiv i\frac{\hbar\Omega}{2}(\hat{a}^2 e^{2i\omega t} - (\hat{a}^\dagger)^2 e^{-2i\omega t}),$$

so that $\hat{H}(t) = \hat{H}_0 + \hat{W}(t)$. In the context of a quantized electromagnetic field, for example, $\hat{W}(t)$ can show up as a Hamiltonian term associated with two-photon absorption and emission processes, but we will not further develop or need that physical context here.

In this problem, we work in the Interaction Picture, so we define the evolution operator $\hat{\mathbb{U}}_0(t)$ associated with $\hat{H}_0$ *only*, and t is a time duration referenced to an initial time of $t = 0$. As you have seen before, the Interaction Picture is the result of a frame transformation based on $\hat{\mathbb{U}}_0(t)$, so that if we have a quantum state $|\psi(t)\rangle$ in the Schrödinger Picture, there is a corresponding effective state $|\psi_E(t)\rangle \equiv \hat{\mathbb{U}}_0^\dagger(t)|\psi(t)\rangle$ in the Interaction Picture (this is the same as $|\psi_I(t)\rangle$ in the Field Guide, page 46). Here and in what follows, the subscript E indicates either the *effective* state or the *effective* Hamiltonian in the Interaction Picture.

We assume that for times $t < 0$, the system Hamiltonian is just $\hat{H}_0$, and that the state of the system is the ground state $|0\rangle$ of $\hat{H}_0$. The extra Hamiltonian term $\hat{W}(t)$ is then turned on at time $t = 0$, and remains on until $t = \tau \equiv 4\pi/\omega$. We are interested in the state of the system $|\psi(\tau)\rangle$ (or $|\psi_E(\tau)\rangle$) at the end of this period, and will not be concerned with how the system's state evolves for times $0 \le t < \tau$.

(a) Once $|\psi_E(\tau)\rangle$ is known, we may wish to determine the expression for the state in the Schrödinger Picture using: $|\psi(\tau)\rangle = \hat{\mathbb{U}}_0(\tau)|\psi_E(\tau)\rangle$. Given what you know about harmonic oscillator dynamics, specify the precise relationship between $|\psi(\tau)\rangle$ and $|\psi_E(\tau)\rangle$ for $\tau = 4\pi/\omega$.

(b) Calculate the effective Hamiltonian for the Interaction Picture, $\hat{H}_E$, valid for $0 \le t < \tau$. Simplify your result as much as possible while leaving it in terms of the Schrödinger-Picture harmonic oscillator ladder operators $\hat{a}$ and $\hat{a}^\dagger$. Check to make sure that your answer has the correct dimensional units, that it is Hermitian, and is independent of time. Hint: you have done most of the necessary calculations in a previous problem set. See also the hint given in the previous problem.

(c) Express $\hat{H}_E$ in terms of the Schrödinger-Picture position and momentum operators $\hat{X}$ and $\hat{P}$. You will not use this particular way of writing $\hat{H}_E$ in the remainder of this problem, but just examining this form of $\hat{H}_E$ in comparison to other work you have done may give you some insight into a quantum state's evolution as determined by the Hamiltonian $\hat{H}(t)$ or $\hat{H}_E$.

(d) Calculate and simplify the Interaction-Picture time-evolution operator $\hat{\mathbb{U}}_E(\tau)$ in terms of the ladder operators $\hat{a}$ and $\hat{a}^\dagger$, associated with a duration of time $\tau = 4\pi/\omega$. Since $\hat{H}_E$ is time-independent, calculating $\hat{\mathbb{U}}_E(\tau)$ is straightforward: $\hat{\mathbb{U}}_E(\tau) = \exp\{-i\tau\hat{H}_E/\hbar\}$. To save some writing, define the dimensionless quantity $b \equiv \Omega\tau$ and write your answer in terms of b.

(e) Since τ is a fixed quantity here, but b is variable (due to the flexibility in choosing Ω), let's give $\hat{\mathbb{U}}_E(\tau)$ a new name to explicitly indicate that it depends on b: we define the unitary operator $\hat{\mathbb{Q}}(b) \equiv \hat{\mathbb{U}}_E(\tau)$ and also define an operator $\hat{B}$ such that $\hat{\mathbb{Q}}(b) = e^{-\hat{B}}$. Write down what $\hat{B}$ is equal to. As a check that you're on the right track, you should find that $\hat{B}$ is dimensionless, and that $\hat{B}^\dagger = -\hat{B}$ (so that it is "anti-Hermitian"), making $\hat{\mathbb{Q}}(b)$ unitary.

(f) Calculate and simplify as much as possible the two commutators $[\hat{B}, \hat{a}]$ and $[\hat{B}, \hat{a}^\dagger]$.

(g) A useful relation in operator algebra is the following: For operators $\hat{A}$ and $\hat{B}$,

$$e^{\hat{B}}\hat{A}e^{-\hat{B}} = \hat{A} + [\hat{B}, \hat{A}] + \frac{1}{2!}[\hat{B}, [\hat{B}, \hat{A}]] + \frac{1}{3!}[\hat{B}, [\hat{B}, [\hat{B}, \hat{A}]]] + \frac{1}{4!}[\hat{B}, [\hat{B}, [\hat{B}, [\hat{B}, \hat{A}]]]] + ...$$

Using this relation, calculate the quantities

$$\hat{\mathbb{Q}}^\dagger(b)\,\hat{a}\,\hat{\mathbb{Q}}(b) \qquad \text{and} \qquad \hat{\mathbb{Q}}^\dagger(b)\,\hat{a}^\dagger\,\hat{\mathbb{Q}}(b).$$

Simplify your answers as much as possible by expressing them in terms of the sinh and cosh functions below (hyperbolic sine and cosine). This will take a little bit of math, but your answers shouldn't look too complicated.

$$\cosh(u) = \cos(iu) = \frac{e^u + e^{-u}}{2} = 1 + \frac{1}{2!}u^2 + \frac{1}{4!}u^4 + \frac{1}{6!}u^6 + ...$$
$$\sinh(u) = -i\sin(iu) = \frac{e^u - e^{-u}}{2} = u + \frac{1}{3!}u^3 + \frac{1}{5!}u^5 + \frac{1}{7!}u^7 + ...$$
$$\cosh^2(u) - \sinh^2(u) = 1$$

(h) Using your results from above, calculate and simplify as much as possible the following four quantities:

$$\hat{\mathbb{Q}}^\dagger(b)\hat{X}\hat{\mathbb{Q}}(b) \qquad \hat{\mathbb{Q}}^\dagger(b)\hat{P}\hat{\mathbb{Q}}(b) \qquad \hat{\mathbb{Q}}^\dagger(b)\hat{X}^2\hat{\mathbb{Q}}(b) \qquad \hat{\mathbb{Q}}^\dagger(b)\hat{P}^2\hat{\mathbb{Q}}(b).$$

Your final expressions for each should look very simple, and only contain exponenential functions of b (no cosh or sinh functions remaining).

(i) We're finally ready to put all of these calculations to use! We are interested in how the system evolves under the influence of $\hat{H}_E$ in the Interaction Picture, or equivalently what operators of the form $\hat{\mathbb{Q}}(b)$ do to quantum states of the harmonic oscillator state space. To make this as simple as possible, we'll define the ket $|\varphi\rangle \equiv \hat{\mathbb{Q}}(b)|0\rangle$. For the state $|\varphi\rangle$, we wish to calculate quantities such as $\langle\hat{X}\rangle = \langle\varphi|\hat{X}|\varphi\rangle = \langle 0|\hat{\mathbb{Q}}^\dagger(b)\hat{X}\hat{\mathbb{Q}}(b)|0\rangle$. Now, for the state $|\varphi\rangle$, calculate the following four quantities:

$$\langle\hat{X}\rangle \qquad \langle\hat{P}\rangle \qquad \langle\hat{X}^2\rangle \qquad \langle\hat{P}^2\rangle.$$

(j) Give the standard deviations $\Delta\hat{X}$ and $\Delta\hat{P}$ for $|\varphi\rangle$, and calculate the uncertainty product $(\Delta\hat{X})(\Delta\hat{P})$, simplifying your answer as much as possible.

(k) Express the position representation of $|\varphi\rangle$ as previously defined; that is, write the function $\varphi(x) = \langle x|\varphi\rangle$. To do this you will need to make use of information that you have calculated, as well as some extra work relating uncertainties to the terms that should appear in the wavefunction. You should have an answer that depends on b. *Hint:* The position representation of a 1D minimum uncertainty state *must* be of the form $C\exp\{-\frac{(x-x_0)^2}{2\gamma^2}\}\exp\{ip_0x/\hbar\}$, where C is a normalization coefficient, x_0 and p_0 are any real-valued position and momentum, and γ is any real-valued scalar with dimensional units of length (but not necessarily the harmonic oscillator length σ).

(l) Calculate $\langle\hat{H}_0\rangle$ for the state $\hat{\mathbb{Q}}(b)|0\rangle$, simplifying as much as possible, and ensure that your answer reduces to the known result for the specific case $b = 0$.

(m) In a few sentences, interpret the action of $\hat{\mathbb{Q}}(b)$ on a quantum state of the harmonic oscillator. Getting to this interpretation is a primary motivation for working all the way through this problem, so please try to be clear but concise and reasonably complete with your answer.

Problem IV. As you already know, the Fourier transform of a Hermite-Gaussian function of order n is also a Hermite-Gaussian of order n. For example, for any real length constant b, the Fourier transform of the normalized Gaussian wavefunction over position (x)

$$\psi(x) = \left(\frac{1}{\pi b^2}\right)^{1/4} e^{-\frac{x^2}{2b^2}}$$

is the normalized wavefunction over momentum (p)

$$\tilde{\psi}(p) = \left(\frac{b^2}{\pi\hbar^2}\right)^{1/4} e^{-\frac{p^2b^2}{2\hbar^2}}.$$

The hyperbolic secant function, sech(u), is another such function that is "its own Fourier transform." Look up or plot the sech(u) function so that you have a sense of what the function looks like. The inverse of the sech(u) function is the hyperbolic cosine:

$$\text{sech}^{-1}(u) \equiv \cosh(u) = \cos(iu) \equiv \frac{1}{2}(e^u + e^{-u}).$$

The Fourier transform of the normalized wavefunction

$$\phi(x) = \frac{1}{\sqrt{2\beta}} \operatorname{sech}(x/\beta)$$

is the normalized wavefunction

$$\tilde{\phi}(p) = -\sqrt{\frac{\pi\beta}{4\hbar}} \operatorname{sech}\left(\frac{\pi\beta p}{2\hbar}\right).$$

(a) Calculate the uncertainty product $(\Delta X)(\Delta P)$ for the wavefunction $\phi(x)$ above. You will likely want to use this result: $\int_{-\infty}^{\infty} du\, u^2 \operatorname{sech}^2(u) = \pi^2/6$.

(b) Consider a particle of mass m in a 1D harmonic oscillator potential well (in x), with oscillator angular frequency ω. At time $t = 0$, the particle is in a quantum state represented by the wavefunction

$$\Phi(x, 0) = \frac{1}{\sqrt{2\beta}} \operatorname{sech}[(x - x_0)/\beta]$$

such that the probability density distribution $|\Phi(x, 0)|^2$ is peaked at the position x_0. Now for a subsequent evolution time of $\frac{\pi}{2\omega}$, give an expression for the wavefunction $\Phi(x, t = \frac{\pi}{2\omega})$.

(c) Determine the value of β in terms of $\sigma \equiv \sqrt{\frac{\hbar}{m\omega}}$ that allows $|\Phi(x, \frac{\pi}{2\omega})|^2$ to maintain the same width as at $t = 0$. Conclusion: while it is not the wavefunction for a coherent state, the $\operatorname{sech}(u)$ function is in some respects similar to one, and it also plays an important role in nonlinear optics and in designing and fabricating optical waveguides.

Problem V. Consider the one-dimensional potential well $V(x)$ defined as

$$V(x) = \begin{cases} -V_0\sqrt{1 - (x/b)^2} & |x| \le b \\ 0 & |x| > b \end{cases}$$

where b is a length constant ($b > 0$) and V_0 is a positive energy constant. Assume that for a particle of mass m, the difference in energy between the first excited state and the ground state is very much less than V_0, thereby allowing the regions of $V(x)$ near $x = 0$ to be approximated reasonably well as a harmonic oscillator potential. In terms of the constants given above, find the ground and first excited state energy eigenvalues when making the harmonic oscillator approximation.