OPTI 570 LECTURE Th Oct 2

3D QHO

$$V(x,y,z) = \frac{1}{2} m \left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)$$

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + V(x,y,z)$$

$$= \underbrace{\left(\frac{1}{2m}p_x^2 + \frac{1}{2}m\,\omega_x^2 x^2\right)}_{H_x \mathbb{1}_y \mathbb{1}_z} + \underbrace{\left(\frac{1}{2m}p_y^2 + \frac{1}{2}m\omega_y^2 y^2\right)}_{\mathbb{1}_x H_y \mathbb{1}_z} + \underbrace{\left(\frac{1}{2m}p_z^2 + \frac{1}{2}m\omega_z^2 z^2\right)}_{\mathbb{1}_x \mathbb{1}_y H_z}$$

$$\mathcal{E} = \mathcal{E}_x \otimes \mathcal{E}_y \otimes \mathcal{E}_z$$

basis: $\{|x\rangle\}\ \{|y\rangle\}\ \{|z\rangle\}$ $\Big\}$ $\{|x\rangle|y\rangle|z\rangle\}$

$\{|n_x\rangle\}\{|n_y\rangle\}\ \{|n_z\rangle\}$ $\Big\}$ $\{|n_x\rangle|n_y\rangle|n_z\rangle\} =$

$= \{|n_x, n_y, n_z\rangle\}$

tensor product basis

$$H|n_x, n_y, n_z\rangle = E_{n_x,n_y,n_z}|n_x, n_y, n_z\rangle$$

$$E_{n_x,n_y,n_z} = \underline{\hbar\omega_x\left(n_x+\tfrac{1}{2}\right) + \hbar\omega_y\left(n_y+\tfrac{1}{2}\right) + \hbar\omega_z\left(n_z+\tfrac{1}{2}\right)}$$

$$\mathbb{1} = \sum_{\substack{n_x=0\\ n_y=0\\ n_z=0}}^{\infty} |n_x, n_y, n_z\rangle\langle n_x, n_y, n_z|$$

$$= \sum |n_x\rangle|n_y\rangle\underbrace{|n_z\rangle\overset{0}{}\langle n_z|}\langle n_y|\langle n_x|$$

Define operators

$$a_j = \frac{1}{\sqrt{2}}\left(\frac{R_j}{\sigma_j} + \frac{i p_j \sigma_j}{\hbar}\right) \quad j = x, y, z$$

w/ $\{a_j, a_k^{+}\} = \delta_{jk}$

$$a_x^+ | n_x, n_y, n_z\rangle = \underline{a_x^+}\, \mathbb{1}_y \mathbb{1}_z\, | \underline{n_x}, n_y, \underline{n_z}\rangle =$$
$$= \sqrt{n+1}\, | \underline{n_x+1}, n_y, n_z\rangle$$

$$a_j |0,0,0\rangle = 0\, |0,0,0\rangle$$

$$\langle \vec{r} | 0,0,0\rangle = \langle z|\langle y| \underbrace{\underline{\langle x|}\,\overset{x}{\widetilde{\underline{|0\rangle}}}}\, \overset{y}{\widetilde{|\underline{0}\rangle}}\, \overset{z}{\widetilde{|\underline{0}\rangle}} =$$
$$= \langle z|\langle y|\ \varphi_0(x)\ |0\rangle\, |0\rangle =$$
$$= \varphi_0(x)\cdot\varphi_0(y)\cdot\varphi_0(z)$$

<u>ISOTROPIC 3D QHO</u> : $\omega_x = \omega_y = \omega_z = \omega$

$$\varphi_{0,0,0} = \left(\frac{1}{\pi\sigma^2}\right)^{3/2} e^{-\frac{1}{2\sigma^2}(x^2+y^2+z^2)}$$

$$\underline{E = \hbar\omega\left(\frac{3}{2} + n_x + n_y + n_z\right)} \qquad \underline{n = n_x + n_y + n_z}$$

degree of degeneracy $g_n = \frac{(n+1)(n+2)}{2}$

ex: meas. energy

#1: meas $\hat{H}_{tot}$ result is $\hbar\omega\left(1+\frac{3}{2}\right)$

$n_x + n_y + n_z = 1$

#2: meas $\hat{H}_x$ result is $n_x = 0$

$n_y + n_z = 1$

#3: meas $\hat{n}_z$ $n_z = 1$ ✓

$n_y = 0$ ✓

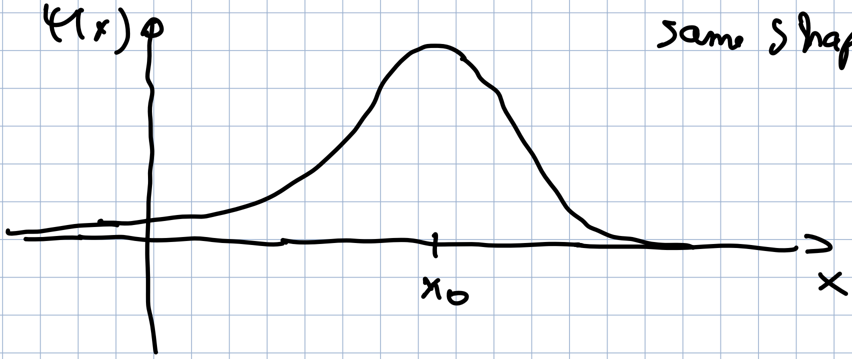
$$|0,0,1\rangle = |n_x=0, n_y=0, n_z=1\rangle$$

<u>CSCO</u> – in 3D isotropic osillator $\{H_x, H_y, H_z\}$, $\{H_{tot}, H_x, H_z\}$, $\{\underline{x}, \underline{p_y}, H_z\}$

non-isotropic $\frac{\omega_x}{\omega_z} = \sqrt{5}$ $\frac{\omega_y}{\omega_z} = \sqrt{7}$ → no degeneracies.

Coherent states

What is?



same shape as $\varphi_0(x)$ but displaced at x_0

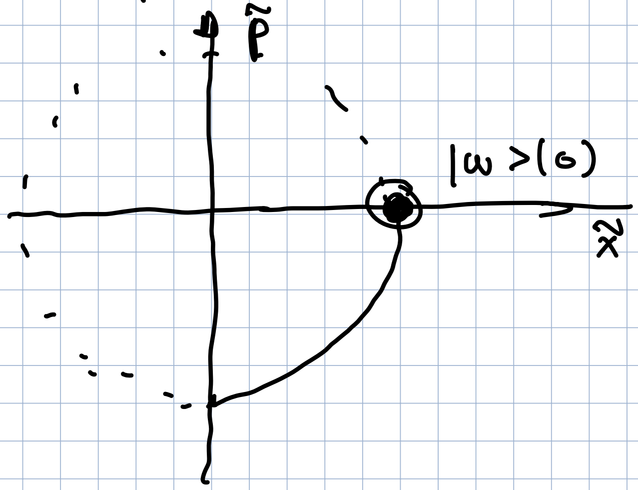
$$\Psi(x) = \langle x | \Psi \rangle = \langle x | \hat{S}(x_0) | \varphi_0 \rangle =$$
$$= \left(\frac{1}{\pi\sigma^2}\right)^{1/4} e^{-\frac{(x-x_0)^2}{2\sigma^2}}$$

$$\langle x \rangle (t) = ?$$

$$x_H(t) = x_S \cos(\omega t) + \frac{p_S}{m\omega} \sin(\omega t)$$
$$p_H(t) = p_S \cos(\omega t) + m\omega x_S \sin(\omega t)$$

$$\langle x \rangle (t) = \langle x \rangle (t=0) \cos(\omega t) + \frac{1}{m\omega} \langle p \rangle (t=0) \sin(\omega t)$$
$$\langle p \rangle (t) = \langle p \rangle (t=0) \cos(\omega t) - m\omega \langle x \rangle (t=0) \sin(\omega t)$$

$$\Rightarrow \langle x \rangle (t) = \langle x \rangle \cos(\omega t)$$
$$\langle p \rangle (t) = -m\omega \langle x_0 \rangle \sin(\omega t)$$



$$|\psi\rangle = \hat{S}(x_0)|\varphi_0\rangle = e^{-ix_0\hat{p}} \quad |\varphi_0\rangle$$

$$= e^{-\frac{ix_0}{\cancel{\hbar}}\cdot\frac{i\cancel{\hbar}}{\sqrt{2}\sigma}(a^\dagger - a)}|\varphi_0\rangle =$$

$$= e^{\frac{x_0}{\sqrt{2}\sigma}(a^\dagger - a)}|\varphi_0\rangle =$$

$$= e^{\frac{x_0 a^\dagger}{\sqrt{2}\sigma}} e^{-\frac{x_0 a}{\sqrt{2}\sigma}} e^{-\frac{1}{2}\underbrace{[a, a^\dagger]}_{1}\left(\frac{x_0}{\sqrt{2}\sigma}\right)^2}|\varphi_0\rangle =$$

$$= e^{-\frac{x_0^2}{4\sigma^2}} e^{-\frac{x_0 a^\dagger}{2\sigma}} \underline{e^{-\frac{x_0 a}{\sqrt{2}\sigma}}|\varphi_0\rangle} = \quad e^0|\varphi_0\rangle$$

$$= e^{-\frac{x_0^2}{4\sigma^2}} e^{\frac{x_0 a^\dagger}{\sqrt{2}\sigma}}|\varphi_0\rangle =$$

$$= \underline{e^{-\frac{x_0^2}{4\sigma^2}}} \underline{\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left(\frac{x_0}{\sqrt{2}\sigma}\right)^n (a^\dagger)^n \quad |\varphi_0\rangle} =$$

Last time: $a|\alpha\rangle = \alpha|\alpha\rangle$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|\varphi_n\rangle$$

$$\alpha = \frac{x_0}{\sqrt{2}\sigma} \qquad |\psi(t=0)\rangle = |\alpha = \frac{x_0}{\sqrt{2}\sigma}\rangle$$

coherent state

Clasical H.O. $\tilde{x}(t) = \frac{x}{\sigma}$ $\hat{p}(t) = p\cdot\sigma$

$t=0$ $\tilde{x}_0, \tilde{p}_0$

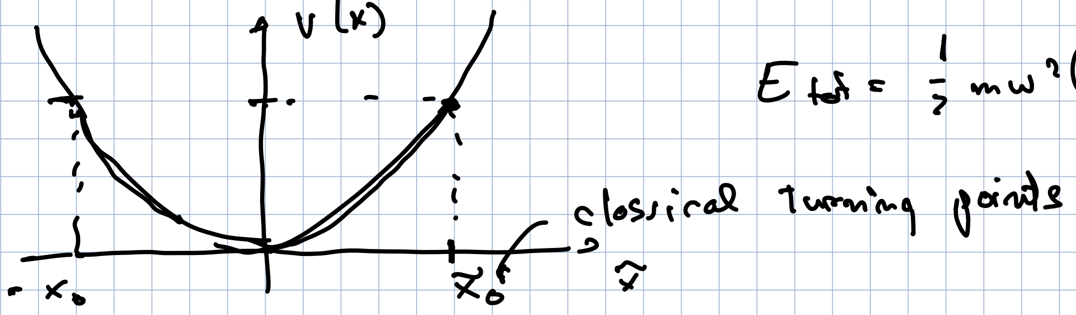
$$\hat{x}(t) = \tilde{x}\cos(\omega t) + \hat{p}_0\sin(\omega t)$$

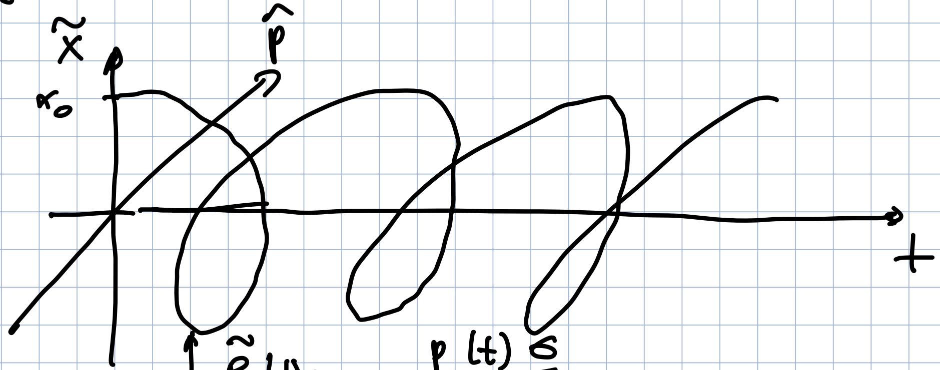
$$\tilde{p}(t) = -\hat{x}_0\sin(\omega t) + \tilde{p}_0\cos(\omega t)$$

$$\tilde{x}_0 > 0, \quad \tilde{p}_0 = 0$$

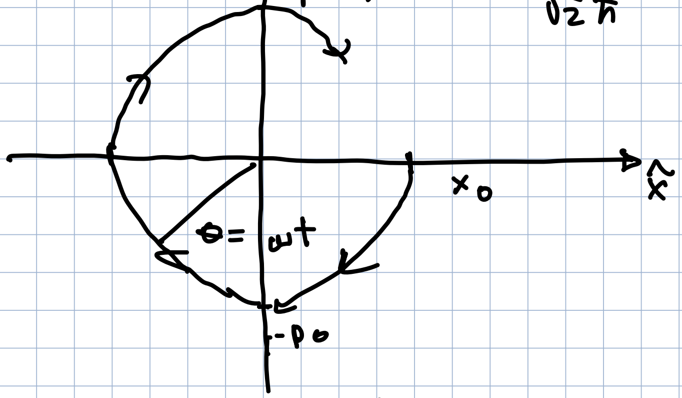
$$\tilde{x}(t) = \tilde{x}_0\cos(\omega t) \qquad \tilde{p}(t) = -\hat{x}_0\sin(\omega t)$$

$$E_{tot} = \frac{1}{2}m\omega^2(\sigma\tilde{x}_0)^2$$





$$\tilde{p}(t) = \frac{p(t)\,\sigma}{\sqrt{2}\,\hbar}$$



$$\hat{x} = \frac{1}{\sqrt{2}}\,\frac{x(t)}{\sigma}$$

$$\alpha(t) \equiv \frac{1}{\sqrt{2}}\left[\underline{\hat{x}(t)} + i\,\tilde{p}(t)\right]$$

$$\tilde{x}(t) = \sqrt{2}\,\mathrm{Re}\left[\alpha(t)\right]$$

$$\hat{p}(t) = \sqrt{2}\,\mathrm{Im}\left[\alpha(t)\right.$$

$$\alpha(t) = \alpha(0)\,e^{-i\omega t}$$

Recap QHO:

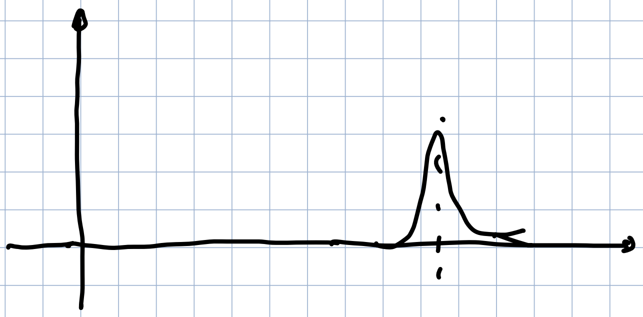
classical	quantum
$E = \frac{p^2(t)}{2m} + \frac{1}{2} m\omega^2 x^2(t)$	$\hat{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$
$\tilde{x}(t) = \frac{x(t)}{\sigma}$	$\tilde{x} = \frac{x}{\sigma}$
$\tilde{p}(t) = \frac{\sigma p(t)}{\hbar}$	$\tilde{p} = \frac{\sigma p}{\hbar}$
$\alpha(t) = \frac{1}{\sqrt{2}}[\tilde{x}(t) + i\tilde{p}(t)]$	$\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\tilde{p})$
$E = \frac{1}{2}\hbar\omega[\tilde{p}^2(t) + \tilde{x}^2(t)] =$ $= \hbar\omega[\alpha(t)]^2 =$ $= \hbar\omega[\underline{\alpha}^* \cdot \alpha]$	$H = \hbar\omega(\underline{a}^{+}\underline{a} + \frac{1}{2})$

QHO:

eigenstate of $\tilde{x}$: $\Delta\hat{x} = 0$ $\Delta\tilde{p} = \infty$

— " — $\hat{p}$: $\Delta\tilde{x} = \infty$ $\Delta\hat{p} = 0$

$a|\alpha\rangle = \alpha|\alpha\rangle$ $\Delta x \cdot \Delta p = \frac{\hbar}{2}$



Properties of coherent states

1. **Orthogonal ?** $\hat{a}|\alpha_1\rangle = \alpha_1|\alpha_1\rangle$ $\hat{a}|\alpha_2\rangle = \alpha_2|\alpha_2\rangle$

$$\langle \alpha_1 | \alpha_2 \rangle = e^{-\frac{1}{2}|\alpha_1|^2} e^{-\frac{1}{2}|\alpha_2|^2} \sum_{n=0}^{\infty} \frac{(\alpha_2^* \alpha_1)^n}{n!}$$

$$|\langle \alpha_1 | \alpha_2 \rangle|^2 = \underbrace{e^{-|\alpha_2-\alpha_1|^2}}_{} \neq 0$$ not ortogonal.

$$|\langle \alpha_1 | \alpha_1 \rangle|^2 = 1 \checkmark$$