Exam 3 Solutions

$\boxed{1.}$

$$\frac{dE_k}{d\xi} = \langle \psi_k | \frac{d\hat{H}}{d\xi} | \psi_k \rangle$$

$$\hat{H} = \frac{p^2}{2m} - \frac{e^2}{\hat{R}} \qquad \langle \frac{1}{\hat{R}} \rangle = \langle \psi_k | \frac{1}{\hat{R}} | \psi_k \rangle = ?$$

Use theorem by diff w.r.t. e :

$$\langle \psi_k | \frac{d\hat{H}}{de} | \psi_k \rangle = \frac{d\left(-\frac{me^4}{2\hbar^2 n^2}\right)}{de}$$

$$-\cancel{2}\cancel{e} \cdot \langle \psi_k | \frac{1}{\hat{R}} | \psi_k \rangle = -\frac{m \cdot 4 \cdot \cancel{e^3}^{\,2}}{\cancel{2}\hbar^2 n^2}$$

$$\underbrace{\langle \frac{1}{\hat{R}} \rangle = \langle \psi_k | \frac{1}{\hat{R}} | \psi_k \rangle = \frac{me^2}{\hbar^2 n^2} = \frac{1}{a_0 n^2}}$$

$\boxed{2}$ $n^2 P_{3/2}$ $\quad S = \frac{1}{2}, L = 1, J = \frac{3}{2}, I = ?$

$F = 4, 3, 2, 1$ $\quad \vec{F} = \vec{J} + \vec{I}$

a.

$\max(F) = J + I \Rightarrow I = \cancel{4}^{\,8/2} - \frac{3}{2} = \frac{5}{2}$

$\min(F) = |J - I| = \left| \frac{3}{2} - \frac{5}{2} \right| = 1$ ✓

b. $n^2 S_{1/2}$ $\quad S = \frac{1}{2}, L = 0, J = \frac{1}{2}, I = \frac{5}{2}$

$\vec{F} = \vec{J} + \vec{I}$

$\max(F) = J + I = 3$ $\qquad F \in \{2, 3\}$

$\min(F) = |J - I| = \left|\frac{1}{2} - \frac{5}{2}\right| = 2$

$n^2 P_{1/2}$ $\quad S = \frac{1}{2}, L = 1, J = \frac{1}{2}, I = \frac{5}{2}$

$\vec{F} = \vec{J} + \vec{I}$

$\max(F) = 3$

$\min(F) = 2$ $\qquad \Rightarrow F \in \{2, 3\}$

$\boxed{3}$ $|\phi\rangle = |n=5, L=2, m_L=-1\rangle |S=\tfrac{1}{2}, m_S=\tfrac{1}{2}\rangle |I=\tfrac{3}{2}, m_I=\tfrac{1}{2}\rangle$

Prob $F^2 = 0$ same as $F=0$?

Need to add AM

First $\bar{J} = \bar{L} + \bar{S}$

$$|L=2, m_L=-1\rangle |S=\tfrac{1}{2}, m_S=\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{5}}\,|J=\tfrac{5}{2}, m_J=-\tfrac{1}{2}\rangle - \sqrt{\tfrac{3}{5}}\,|J=\tfrac{3}{2}, m_J=-\tfrac{1}{2}\rangle$$

Then we need $\bar{F} = \bar{J} + \bar{I}$

$$|\phi\rangle = \underbrace{\sqrt{\tfrac{2}{5}}\,|J=\tfrac{5}{2}, m_J=-\tfrac{1}{2}\rangle |I=\tfrac{3}{2}, m_I=\tfrac{1}{2}\rangle}_{F\in\{1,2,3,4\} \text{ so } F=0 \text{ not possible}} - \sqrt{\tfrac{3}{5}}\,|J=\tfrac{3}{2}, m_J=-\tfrac{1}{2}\rangle |I=\tfrac{3}{2}, m_I=\tfrac{1}{2}\rangle$$

next go to F TAM base for the second term:

$$|J=\tfrac{3}{2}, m_J=-\tfrac{1}{2}\rangle |I=\tfrac{3}{2}, m_I=\tfrac{1}{2}\rangle = \sqrt{\tfrac{3}{20}}\,|F=3, m_F=0\rangle - \sqrt{\tfrac{1}{4}}\,|F=2, m_F=0\rangle -$$

$$- \sqrt{\tfrac{1}{20}}\,|F=1, m_F=0\rangle + \sqrt{\tfrac{1}{4}}\,|F=0, m_F=0$$

$$\Rightarrow P_{F=0} = \tfrac{1}{4}\cdot\tfrac{3}{5} \boxed{= \tfrac{3}{20}}$$

$\boxed{4}$ $H_1 = \hbar\omega_0(\sigma_z + \sigma_x)$ for τ

$H_2 = \hbar\omega_0(-\sigma_z + \sigma_x)$ for τ

$t=0$ $\quad |+\rangle_z$

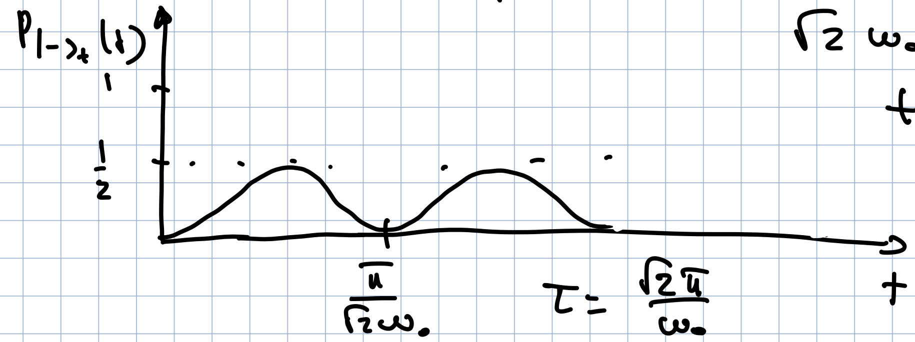
a. $\tau = \dfrac{\sqrt{2}\,\pi}{\omega_0}$

$$H_1 = \hbar\omega_0 \cdot \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \hbar\omega_0 \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H_{\text{book}} = \frac{\hbar}{2}\begin{bmatrix} \Delta & \Omega_0^* \\ \Omega_0 & -\Delta \end{bmatrix} \Rightarrow \Delta = 2\omega_0 \qquad \Omega_0 = 2\omega_0$$

$$P_{|-\rangle_z}(t) = \frac{|\Omega_0|^2}{\Omega^2} \sin^2\left(\frac{\Omega t}{2}\right) = \qquad \Omega = \sqrt{\Delta^2 + |\Omega_0|^2} = 2\sqrt{2}\,\omega_0$$

$$= \frac{1}{2} \sin^2\left(\sqrt{2}\,\omega_0 t\right)$$



$$\sqrt{2}\,\omega_0 t_\pi = \pi$$

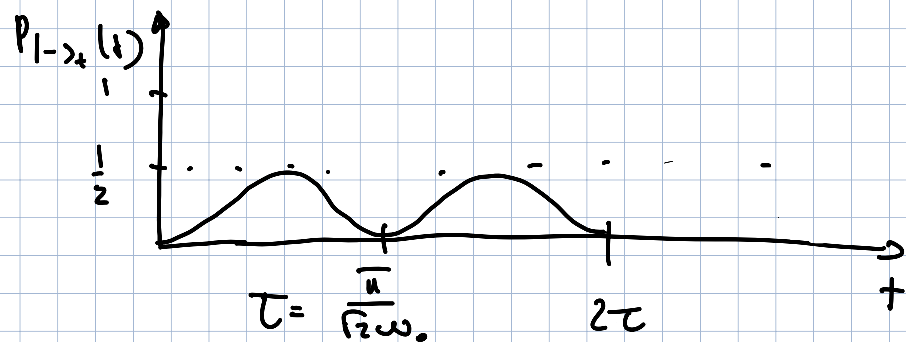
$$t_\pi = \frac{\pi}{\sqrt{2}\,\omega_0}$$

b. $\tau = \frac{1}{2} \cdot \frac{\sqrt{2}\pi}{\omega_0} \Rightarrow$ system in $|+\rangle_z$ at $t = \tau$

$$H_2 = \hbar\omega_0 \cdot \left[-\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \hbar\omega_0 \begin{pmatrix} -1 & 1 \\ 1 & +1 \end{pmatrix}$$

$$\Delta = -2\omega_0 \qquad \Omega_0 = 2\omega_0$$

negative sign $\Rightarrow$ doesn't matter for prob plot.



$\boxed{5.}$ $\hat{J}^2 = \hat{J} \cdot \hat{J} = (\hat{J}_1 + \hat{J}_2)(\hat{J}_1 + \hat{J}_2) = \hat{J}_1^2 + \hat{J}_2^2 + 2\hat{J}_1 \cdot \hat{J}_2$

$$\Rightarrow \hat{J}_1 \cdot \hat{J}_2 = \frac{1}{2}\left(\hat{J}^2 - \hat{J}_1^2 - \hat{J}_2^2\right)$$

$$\hat{J}_1 \cdot \hat{J}_2 |j_1, j_2, J, m\rangle = \frac{1}{2}\left(\hat{J}^2 - \hat{J}_1^2 - \hat{J}_2^2\right)|j_1, j_2, J, m\rangle =$$

$$= \underbrace{\frac{1}{2}\left[j(j+1) - j_1(j_1+1) - j_2(j_2+1)\right]\hbar^2}\ |j_1, j_2, J, m\rangle$$

$$\hat{J} \cdot \hat{J}_1 = \hat{J}_1^2 + \hat{J}_1 \cdot \hat{J}_2$$

$$\hat{J}\cdot\hat{J}_1\,|j_1,j_2,J,m\rangle = (\hat{J}_1^{\,2}+\hat{J}_1\cdot\hat{J}_2)\,|j_1,j_2,J,m\rangle =$$

$$=\underbrace{\frac{1}{2}\left[j(j+1)+j_1(j_1+1)-j_2(j_2+1)\right]\hbar^2\,|j_1,j_2,J,m\rangle}\,,$$

similarly:

$$\hat{J}\cdot\hat{J}_2\,|j_1,j_2,J,m\rangle = \frac{1}{2}\left[j(j+1)+j_2(j_2+1)-j_1(j_1+1)\right]\hbar^2\,|j_1,j_2,J,m\rangle$$

$\boxed{6}$ $|\psi\rangle = a|2,m_z=2\rangle + b|2,m_z=1\rangle + c|2,m_z=0\rangle +$
$+ b|2,m_z=-1\rangle + a|2,m_z=-2\rangle$

a. $s=2$

$$S_+|\psi\rangle = 0 + b\cdot\hbar\sqrt{2\cdot3-1\cdot2}\,|2,2\rangle + c\cdot\hbar\sqrt{2\cdot3-0\cdot1}\,|2,1\rangle +$$
$$+ b\cdot\hbar\cdot\sqrt{2\cdot3-(-1)\cdot0}\,|2,0\rangle + a\cdot\hbar\cdot\sqrt{2\cdot3-(-2)\cdot(-1)}\,|2,-1\rangle =$$
$$=2\cdot b\cdot\hbar\,|2,2\rangle + c\hbar\sqrt{6}\,|2,1\rangle + b\hbar\sqrt{6}\,|2,0\rangle + a\hbar\,2\,|2,-1\rangle$$

$$S_-|\psi\rangle = a\cdot\hbar\cdot\sqrt{2\cdot3-2\cdot1}\,|2,1\rangle + b\cdot\hbar\cdot\sqrt{2\cdot3-1\cdot0}\,|2,0\rangle +$$
$$+ c\cdot\hbar\sqrt{2\cdot3-0\cdot(-1)}\,|2,-1\rangle + b\cdot\hbar\cdot\sqrt{2\cdot3-(-1)\cdot(-2)}\,|2,-2\rangle =$$
$$= a\hbar\,2\,|2,1\rangle + b\cdot\hbar\sqrt{6}\,|2,0\rangle + c\hbar\sqrt{6}\,|2,-1\rangle + b\hbar\cdot2\,|2,-2\rangle$$

b. $S_x|\psi\rangle = \frac{1}{2}(S_+ + S_-)|\psi\rangle$

$$S_x|\psi\rangle = b\cdot\hbar\,|2,2\rangle + \hbar\cdot\frac{1}{2}(c\sqrt{6}+2a)\,|2,1\rangle + \hbar\cdot b\sqrt{6}\,|2,0\rangle +$$
$$+ \frac{\hbar}{2}(2a+c\sqrt{6})\,|2,-1\rangle + b\hbar\,|2,-2\rangle$$

c. $S_x|\psi\rangle = 2\hbar\,|\psi\rangle =$
$$= 2\hbar\left(a|2,2\rangle + b|2,1\rangle + c|2,0\rangle + b|2,-1\rangle + a|2,-2\rangle\right)$$

$\Rightarrow$ $2a = b$

$2b = \frac{1}{2}(c\sqrt{6}+2a)$

$2c = b\sqrt{6}$ $\quad c = \frac{2a\sqrt{6}}{2} = a\cdot\sqrt{6}$

check: $4a = \frac{1}{2}(6a+2a)$

$4a = \frac{1}{2}\cdot 8a$ ✓

Normalization: $\langle \psi | \psi \rangle = 1 \Rightarrow (2\hbar)^2 \cdot (2a^2 + 2b^2 + c^2) = 1$

$$\left(2a^2 + 2 \cdot 4a^2 + a^2 \cdot 6\right) = 1$$

$$a^2 \cdot (\underbrace{2 + 8 + 6}) = 1$$

$$a = \frac{1}{4}$$

$$b = \frac{1}{2}$$

$$c = \frac{\sqrt{6}}{4}$$

$\underline{s = \frac{1}{2}}$ $\quad 2s = 1 \quad \sqrt{\frac{1}{2}}\,|\frac{1}{2}, \frac{1}{2}\rangle + \sqrt{\frac{1}{2}}\,|\frac{1}{2}, -\frac{1}{2}\rangle$

$\underline{s = 1}$ $\quad 2 \quad \sqrt{\frac{1}{4}}\,|1, 1\rangle + \sqrt{\frac{2}{4}}\,|1, 0\rangle + \sqrt{\frac{1}{4}}\,|1, -1\rangle$

$\underline{s = \frac{3}{2}}$ $\quad 3 \quad \sqrt{\frac{1}{8}}\,|\frac{3}{2}, \frac{3}{2}\rangle + \sqrt{\frac{3}{8}}\,|\frac{3}{2}, \frac{1}{2}\rangle + \sqrt{\frac{3}{8}}\,|\frac{3}{2}, -\frac{1}{2}\rangle + \sqrt{\frac{1}{8}}\,|\frac{3}{2}, -\frac{3}{2}\rangle$

$\underline{s = 2}$ $\quad 4 \quad \sqrt{\frac{1}{16}}\,|2, 2\rangle + \sqrt{\frac{4}{16}}\,|2, 1\rangle + \sqrt{\frac{6}{16}}\,|2, 0\rangle + \sqrt{\frac{4}{16}}\,|2, -1\rangle + \sqrt{\frac{1}{16}}\,|2, -2\rangle$

Pattern is that 2^{2s} in numerator. At the denominator, coeff. are:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

This gives:

$$|s = \tfrac{5}{2}, m_z\rangle = \sqrt{\frac{1}{32}}\,|\tfrac{5}{2}, \tfrac{5}{2}\rangle + \sqrt{\frac{5}{32}}\,|\tfrac{5}{2}, \tfrac{3}{2}\rangle + \sqrt{\frac{10}{32}}\,|\tfrac{5}{2}, \tfrac{1}{2}\rangle + \sqrt{\frac{10}{32}}\,|\tfrac{5}{2}, -\tfrac{1}{2}\rangle +$$

$$+ \sqrt{\frac{5}{32}}\,|\tfrac{5}{2}, -\tfrac{3}{2}\rangle + \sqrt{\frac{1}{32}}\,|\tfrac{5}{2}, -\tfrac{5}{2}\rangle$$