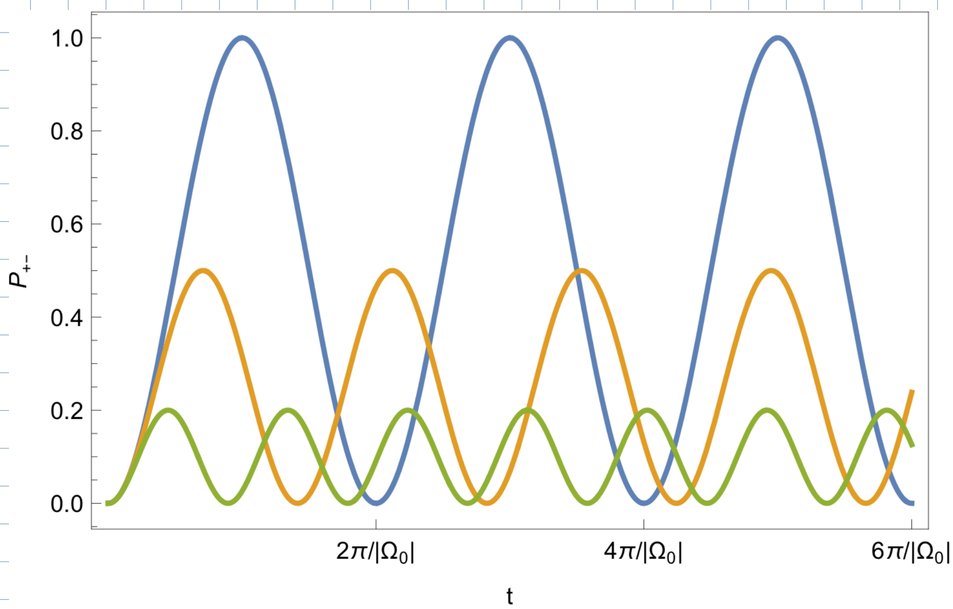
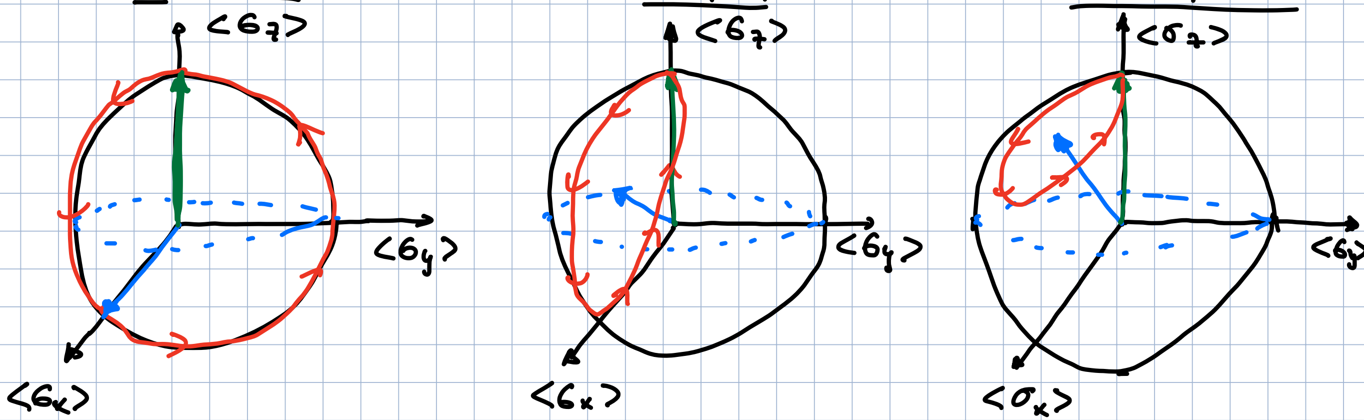
Problem Set 9 Solutions

$$|\psi(0)\rangle = |+\rangle_z$$

a.



b. $\Delta = 0$ $\qquad \Delta = |\Omega_0|$ $\qquad \Delta = 2|\Omega_0|$



c. $\Delta = |\Omega_0|$ $\quad \beta = 0$ $\qquad \sin\theta = \frac{|\Omega_0|}{\Omega} = \frac{1}{\sqrt{2}}$

$\cos\theta = \frac{1}{\sqrt{2}}$

$\varphi = 0$

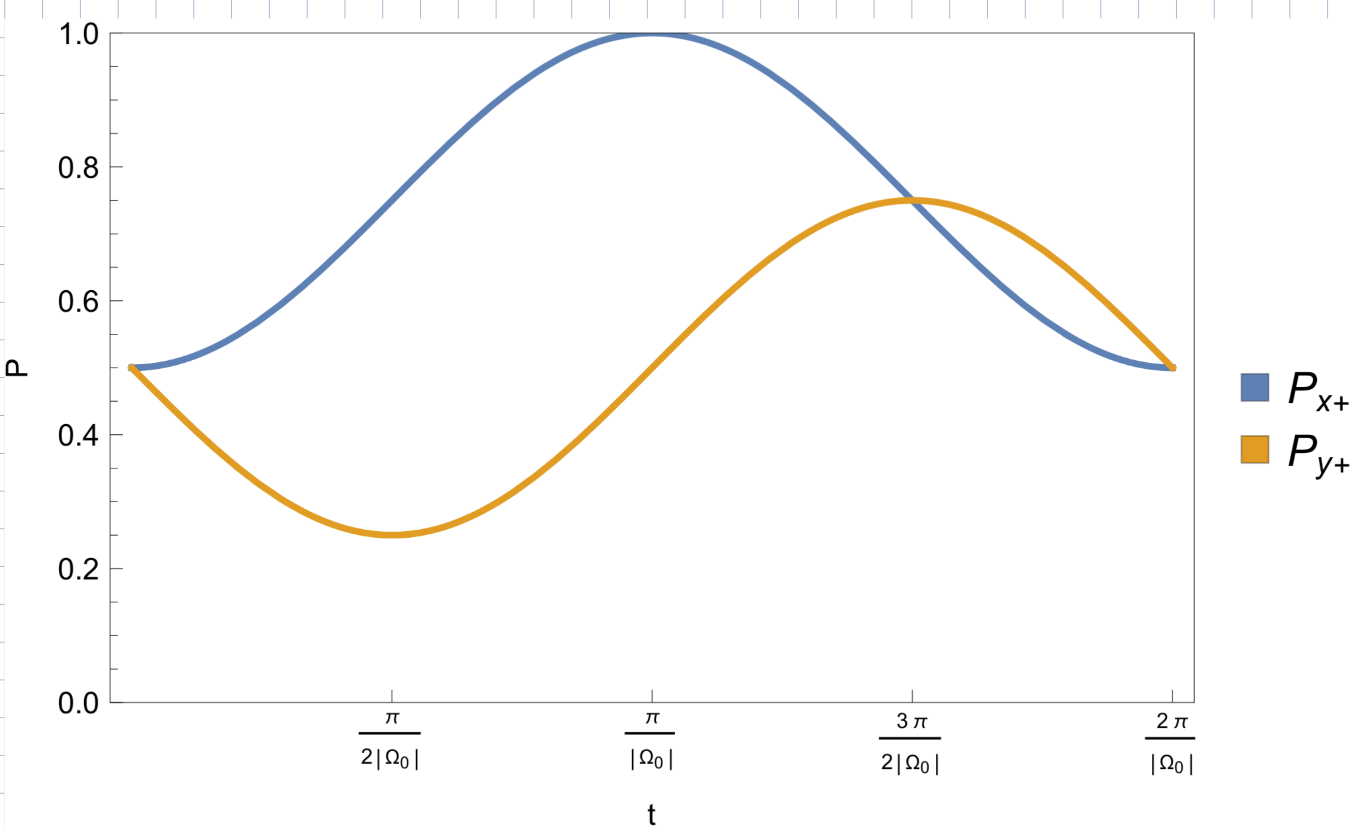
$$\langle\sigma_x\rangle = \frac{1}{2}\left[1 - \cos(\Omega t)\right] = \sin^2\left(\frac{\Omega t}{2}\right)$$

$$\langle\sigma_y\rangle = -\frac{1}{2}\sin(\Omega t)$$

$$\langle\sigma_x\rangle = P_{|x+\rangle} - P_{|x-\rangle} \qquad P_{|x+\rangle} + P_{|x-\rangle} = 1$$

$$P_{|x+\rangle} = \frac{1}{2}\left(1 + \langle\sigma_x\rangle\right) = \frac{1}{2}\left[1 + \sin^2\left(\frac{\Omega t}{2}\right)\right]$$

$$P_{|y+\rangle} = \frac{1}{2}\left(1 + \langle\sigma_y\rangle\right) = \frac{1}{2}\left[1 - \frac{1}{2}\sin\left(\Omega t\right)\right]$$



Problem II

a.

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

$$\hat{r} = \sin\theta \cos\phi \,\hat{x} + \sin\theta \sin\phi \,\hat{y} + \cos\theta \,\hat{z}$$

$$Y_1^1 + Y_1^{-1} = \sqrt{\frac{3}{8\pi}} \sin\theta \left(e^{-i\phi} - e^{i\phi}\right) =$$

$$= -\sqrt{\frac{3}{2\pi}}\, i \sin\theta \sin\phi$$

$$Y_1^1 - Y_1^{-1} = -\sqrt{\frac{3}{8\pi}} \sin\theta \left(e^{-i\phi} + e^{i\phi}\right) =$$

$$= -\sqrt{\frac{3}{2\pi}} \sin\theta \cos\phi$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$\hat{r} = -\sqrt{\frac{2\pi}{3}} \left(Y_1^1 - Y_1^{-1}\right) \hat{x} + \sqrt{\frac{2\pi}{3}}\, i \left(Y_1^1 + Y_1^{-1}\right) \hat{y} + \sqrt{\frac{4\pi}{3}}\, Y_1^0 \,\hat{z}$$

$$\boxed{\hat{r} \to \left\{ -\sqrt{\frac{2\pi}{3}} \left(Y_1^1 - Y_1^{-1}\right), \sqrt{\frac{2\pi}{3}}\, i \left(Y_1^1 + Y_1^{-1}\right), \sqrt{\frac{4\pi}{3}}\, Y_1^0 \right\}}$$

b. $F(x,y,z) = \frac{x+y+z}{r} = \frac{x}{r} + \frac{y}{r} + \frac{z}{r} =$

$$= -\sqrt{\frac{2\pi}{3}}\left(Y_1^1 - Y_1^{-1}\right) + \sqrt{\frac{2\pi}{3}}\, i\left(Y_1^1 + Y_1^{-1}\right) + \sqrt{\frac{4\pi}{3}}\, Y_1^0 =$$

$$\underline{= \sqrt{\frac{2\pi}{3}}(i-1)\, Y_1^1 + \sqrt{\frac{2\pi}{3}}(i+1)\, Y_1^{-1} + \sqrt{\frac{4\pi}{3}}\, Y_1^0}$$

Problem VI

$\gamma < 0$, $|\Psi(t=0)\rangle = |+\rangle_z$

$$\bar{B} = \begin{cases} B_0 \hat{y} & \text{for } 0 < t \le \tau_y \\ B_0 \hat{z} & \text{for } \tau_y < t \le \tau_y + \tau_z \end{cases}$$

$|\Psi(\tau_y + \tau_z)\rangle = |+\rangle_u$ w/ $\hat{u} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

Conceptual solution:

$0 < t \le \tau_y$: spin vector precesses around $\hat{y}$ w/ angle $\theta = \omega\tau_y$

the vector stays in the x-y plane, $\varphi = 0$

$\omega = -\gamma B_0 \Rightarrow \boxed{\tau_y = -\frac{\theta}{\gamma B_0}}$

$\tau_y < t \le \tau_y + \tau_z$: spin vector precesses around $\hat{z}$ w/ angle $\varphi = \omega\tau_z$

θ will stay constant

$$\boxed{\tau_z = \frac{\varphi}{\omega} = -\frac{\varphi}{\gamma B_0}}$$

These two steps give $|\Psi(\tau_y + \tau_z)\rangle = |+\rangle_u$

Problem IV

a. $$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} = pc \cdot \mathbb{1} + \frac{c^4}{2pc} \cdot \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}$$

b. $$E_m = \frac{E_1 + E_2}{2} = pc + \frac{c^4}{4pc} \cdot (m_1^2 + m_2^2) \qquad m_1^2 = \frac{pc}{2c^4} \cdot (E_m + \mathcal{E}) - pc$$

$$\mathcal{E} = \frac{E_1 - E_2}{2} = \frac{c^4}{4pc} \cdot (m_1^2 - m_2^2) = -\frac{c^4}{4pc}\,\delta m^2 \qquad m_2^2 = \frac{pc}{2c^4}(E_m - \mathcal{E}) - pc$$

$$H = E_m \cdot \mathbb{1} + \mathcal{E} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \checkmark$$

c. use $\{|\nu_1\rangle, |\nu_2\rangle\}$ representation. $\quad |\psi\rangle_{\nu_{1,2}} = M^\dagger \, |\psi\rangle_{\nu_{e,N}}$

Transformation matrix $\quad M = \begin{pmatrix} \sin\beta & -\cos\beta \\ \cos\beta & \sin\beta \end{pmatrix}$

$$H_{\{e,N\}} = M^\dagger H M = \begin{pmatrix} \sin\beta & \cos\beta \\ -\cos\beta & \sin\beta \end{pmatrix} \mathcal{E} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sin\beta & -\cos\beta \\ \cos\beta & \sin\beta \end{pmatrix} =$$

$$= \mathcal{E} \begin{pmatrix} \sin\beta & \cos\beta \\ -\cos\beta & \sin\beta \end{pmatrix} \begin{pmatrix} \sin\beta & -\cos\beta \\ -\cos\beta & -\sin\beta \end{pmatrix} =$$

$$= \mathcal{E} \begin{pmatrix} \sin^2\beta - \cos^2\beta & -2\sin\beta\cos\beta \\ -2\sin\beta\cos\beta & \cos^2\beta - \sin^2\beta \end{pmatrix} =$$

$$= -\mathcal{E} \cdot \begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix}$$

d. $$-\mathcal{E} \cdot \sin 2\beta = \frac{\hbar}{2}\Omega_0 \Rightarrow \Omega_0 = +\frac{\cancel{2}\sin 2\beta}{\hbar} \cdot \frac{c^4}{{}_2\cancel{4}pc}\,\delta m^2$$

$$\underbrace{\Omega_0 = +\frac{\sin 2\beta}{2\hbar pc}\, c^4 \cdot \delta m^2}$$

$$-\mathcal{E}\cos 2\varphi = \frac{\hbar}{2}\Delta \Rightarrow \Delta = +\frac{\cancel{2}\cos 2\beta}{\hbar} \cdot \frac{c^4}{{}_2\cancel{4}pc} \cdot \delta m^2$$

$$\underbrace{\Delta = \frac{\cos 2\beta}{2\hbar pc}\, c^4 \cdot \delta m^2}$$

e. $\Omega = \sqrt{\Omega_0^2 + \Delta^2} = \frac{c^4}{2\hbar p c}\delta m^2 \cdot \sqrt{\cos^2 2\beta + \sin^2 2\beta} =$

$$= \frac{c^4}{2\hbar p c}\delta m^2 = \frac{2.5\cdot 10^{-3}\,\cancel{eV^2}}{2\cdot 6.6\cdot 10^{-16}\,\cancel{eV}\cdot s\cdot 10^{10}\,\cancel{eV}} =$$

$$\underbrace{= 190\ \frac{rad}{s}}$$

f. $|\nu(t=0)\rangle = |\nu_\mu\rangle$

$$P_{|\nu_e\rangle}(t) = \left|\frac{\Omega_0}{\Omega}\right|^2 \cdot \sin^2\left(\frac{\Omega}{2}t\right) =$$

$$= \left(\frac{\cancel{\frac{c^4\delta m^2}{2\hbar pc}}\sin 2\beta}{\cancel{\frac{c^4\delta m^2}{2\hbar pc}}}\right)^2 \sin^2\left(\frac{c^4\delta m^2}{4\hbar pc}t\right) =$$

$$\boxed{P_{|\nu_e\rangle}(t) = \sin^2(2\beta)\ \sin^2\left(\frac{c^4\delta m^2}{4\hbar pc}t\right)}$$

g. $d = c\cdot t \Rightarrow t = d/c$

$$P_{|\nu_e\rangle}(d) = \sin^2(2\beta)\cdot\sin^2\left(\frac{c^4\delta m^2}{4\hbar pc}\cdot\frac{d}{c}\right)$$

h.

$$\frac{\Omega_0}{2}\cdot\frac{d_0}{c} = \frac{\pi}{2} \Rightarrow d_0 = \frac{\pi c}{\Omega_0} = 5\cdot 10^6\,m = 5000\,km$$

i. $P_{|\nu_e\rangle}(300\,km) = \sin^2(2\cdot 80^\circ)\cdot\sin^2\left(\frac{190\,\frac{rad}{s}}{2}\cdot\frac{300\,km}{3\cdot 10^8\,\frac{m}{s}}\right) =$

$$= 0.001 = 0.1\ \%$$

j. $P_{|\nu_e\rangle} = \frac{1}{4}$ $\quad P_{|\nu_\mu\rangle} = \frac{3}{4}$

$$\rho_{\{\nu_e,\nu_\mu\}} = \begin{pmatrix}\frac{1}{4} & 0\\ 0 & \frac{3}{4}\end{pmatrix} \qquad \rho_{\{\nu_1,\nu_2\}} = M\rho_{\{\nu_e,\nu_\mu\}}M^{+} =$$

$$= \begin{pmatrix}\sin\beta & -\cos\beta\\ \cos\beta & \sin\beta\end{pmatrix}\begin{pmatrix}\frac{1}{4} & 0\\ 0 & \frac{3}{4}\end{pmatrix}\begin{pmatrix}\sin\beta & \cos\beta\\ -\cos\beta & \sin\beta\end{pmatrix} = \frac{1}{4}\begin{pmatrix}\sin\beta & -\cos\beta\\ \cos\beta & \sin\beta\end{pmatrix}\begin{pmatrix}\sin\beta & \cos\beta\\ -3\cos\beta & 3\sin\beta\end{pmatrix}$$

$$= \frac{1}{4}\cdot\begin{pmatrix}\sin^2\beta + 3\cos^2\beta & -2\sin\beta\cos\beta\\ -2\sin\beta\cos\beta & \cos^2\beta + 3\sin^2\beta\end{pmatrix}$$