Assignment 10

OPTI 570 Quantum Mechanics

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Problem I

a) The probabilities for x and y directions are:

$$P_x = |\langle \phi_x | \phi_\mu \rangle|^2 = \cos^2 \mu, \quad \text{and} \quad P_y = |\langle \phi_y | \phi_\mu \rangle|^2 = \sin^2 \mu.$$

For circular polarization we do the same:

$$P_{\sigma_+} = |\langle \sigma_+ | \phi_\mu \rangle|^2 = \left| -\frac{1}{\sqrt{2}} \left[(\langle \phi_x | - i \langle \phi_y |)(\cos \mu |\phi_x\rangle + \sin \mu |\phi_y\rangle) \right] \right|^2 = \frac{1}{2}(\sin^2 \mu + \cos^2 \mu) = \frac{1}{2}$$
$$P_{\sigma_-} = 1 - P_{\sigma_+} = \frac{1}{2}.$$

b) The state producted by the source is:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\phi_x^L\rangle|\phi_y^R\rangle - |\phi_x^L\rangle|\phi_y^R\rangle).$$

The joint measurement is performed through the following projector which is the tensorproduct of the measurement for each photon:

$$P_{jk} = P_j^L \otimes P_k^R = |\phi_j^L\rangle|\phi_k^R\rangle\langle\phi_j^L|\langle\phi_k^R|, \quad j,k \in \{x, y\}.$$

We are let j, k to vary in either direction so that we will have four types of measurements:

$$P_{xy} = \langle\psi|P_{xy}|\psi\rangle = |\langle\phi_x^L\phi_y^R|\psi\rangle|^2 = \frac{1}{2}$$
$$P_{yx} = \langle\psi|P_{yx}|\psi\rangle = |\langle\phi_y^L\phi_x^R|\psi\rangle|^2 = \frac{1}{2}$$
$$P_{xx} = \langle\psi|P_{xx}|\psi\rangle = |\langle\phi_x^L\phi_x^R|\psi\rangle|^2 = 0$$
$$P_{yy} = \langle\psi|P_{yy}|\psi\rangle = |\langle\phi_y^L\phi_y^R|\psi\rangle|^2 = 0$$

Only the non-zero probabilities are possible state the system may be left after the measurement, which are then:

$$|\psi\rangle \xrightarrow{P_{xy}} \frac{P_{xy}|\psi\rangle}{\sqrt{\langle\psi|P_{xy}|\psi\rangle}} = |\phi_x^L\phi_y^R\rangle, \quad \text{and} \quad |\psi\rangle \xrightarrow{P_{yx}} \frac{P_{yx}|\psi\rangle}{\sqrt{\langle\psi|P_{yx}|\psi\rangle}} = |\phi_y^L\phi_x^R\rangle.$$

On the other hand, measurement of a single photon is kind of a marginal measurement. The probabilities cn be constructed from the above probabilities:

$$\begin{aligned} P_x^L &= P_{xy} + P_{xx} = \frac{1}{2} + 0 = \frac{1}{2} \\ P_x^R &= P_{xx} + P_{yx} = 0 + \frac{1}{2} = \frac{1}{2} \\ P_y^L &= P_{yx} + P_{yy} = \frac{1}{2} + 0 = \frac{1}{2} \\ P_y^R &= P_{xy} + P_{yy} = \frac{1}{2} + 0 = \frac{1}{2} \end{aligned}$$

c) We first change of basis, by passing from $\phi_{x,y}$ to $\phi_{+,-}$. The transformation matrix is:

$$M = \frac{-1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \longrightarrow M^\dagger = \frac{-1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

. By extracting the equation from $M^\dagger$ ans subsitute them in $\phi_{x,y}$ we find the following equivalent state:

$$|\psi\rangle \propto \frac{1}{\sqrt{2}}(|\phi_+^L \phi_-^R\rangle - |\phi_-^L \phi_+^R\rangle).$$

The measurements are analogous to the previous part, but now in the circular polarizaed basis, with the extacly same measurement projector, but with j,k changed:

$$P_{jk} = P_j^L \otimes P_k^R = |\phi_j^L\rangle|\phi_k^R\rangle\langle\phi_j^L|\langle\phi_k^R|, \quad j,k \in \{-,+\}.$$

Therefore,

$$\begin{aligned} P_{++} &= |\langle\phi_+^L \phi_+^R|\psi\rangle|^2 = 0 \\ P_{+-} &= |\langle\phi_+^L \phi_-^R|\psi\rangle|^2 = \frac{1}{2} \\ P_{-+} &= |\langle\phi_-^L \phi_+^R|\psi\rangle|^2 = \frac{1}{2} \\ P_{--} &= |\langle\phi_-^L \phi_-^R|\psi\rangle|^2 = 0. \end{aligned}$$

We see the polarization for either basis is exactly the same, but the states after the measurement will be different.

Problem II

In the deuterium atom we have that $I = 1$ and $S = 1/2$.

a) In the state $1s$, we have $n = 1$ and $l = 0$. The quantum number J is:

$$J \in \{|L-S|, |L-S|+1, \cdots, L+S-1, L+S\} = \left\{\frac{1}{2}\right\}.$$

For this unique value, we have

$$J = \frac{1}{2}: \quad F \in \{|J-I|, |J-I|+1, \cdots, J+I-1, J+I\} = \left\{\frac{1}{2}, \frac{3}{2}\right\}.$$

b) For the state $2p$, we have $n = 2$ and $l = 1$. The quantum number J is:

$$J \in \left\{ |1 - \frac{1}{2}|, 1 + \frac{1}{2} \right\} = \left\{ \frac{1}{2}, \frac{3}{2} \right\}.$$

For each J, we have F:

$$\begin{aligned} J = \frac{1}{2}: \quad & F \in \{ |\frac{1}{2} - 1|, \frac{1}{2} + 1 \} = \left\{ \frac{1}{2}, \frac{3}{2} \right\} \\ J = \frac{3}{2}: \quad & F \in \{ |\frac{3}{2} - 1|, |\frac{3}{2} - 1| + 1, \frac{3}{2} + 1 \} = \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \right\}. \end{aligned}$$

Problem III

We are asked to compute the matrix element of A, but the initial and final states are in the TAM basis, which is not suitable due to the action of A only on $|L, M_L\rangle$; in the TAM basis those are encoded. We need therefore to convert from TAM basis to TP basis each $|\phi\rangle$ and $|\phi'\rangle$. We need to do the following for eacth state:

$$|F, M_F\rangle \xrightarrow{CG} |J, M_J\rangle |I, M_I\rangle \xrightarrow{CG} |L, M_L\rangle |S, M_S\rangle |I, M_I\rangle.$$

We do that for each state then,

a) We first have that

$$5^2 S_{1/2} \Longrightarrow n = 5,\ S = \frac{1}{2},\ L = 0,\ J = 1/2.$$

We also ommit those quantities as are the same during all the computations. We then need to express $|F = 1, M_F = 1\rangle$, and we select $J_1 = I = 3/2,\ J_2 = J = 1/2$ for convention. To decouple this ket in J and I, we use the coefficient from the table to say that:

$$3/2 \times 1/2 \rfloor \quad |1, 1\rangle_F = \sqrt{\frac{3}{4}} |1/2, -1/2\rangle_J |3/2, 3/2\rangle_I - \sqrt{\frac{1}{4}} |1/2, 1/2\rangle_J |3/2, 1/2\rangle_I.$$

If we look at the following decomposition in the table, with $J_1 = S = 1/2, J_2 = L = 0$, we wont get anything. However, we can use the conservation of angular momentum $M_J = M_S + M_L$ to select the states that corresponds. For $L = 0$ we only have $M_L = 0$. For $S = 1/2$, we have $M_S = \pm 1/2$ and only the negative one satisfies the conservation of AM. Therefore,

$$M_J = M_S + M_L \rfloor \quad |1/2, -1/2\rangle_J = |0, 0\rangle_L |1/2, -1/2\rangle_S.$$

We do the same for the remaining ket:

$$M_J = M_S + M_L \rfloor \quad |1/2, 1/2\rangle_J = |0, 0\rangle_L |1/2, 1/2\rangle_S.$$

Thus, putting all together:

$$|\phi\rangle = |1, 1\rangle_F = \sqrt{\frac{3}{4}} |0, 0\rangle_L |1/2, -1/2\rangle_S |3/2, 3/2\rangle_I - \sqrt{\frac{1}{4}} |0, 0\rangle_L |1/2, 1/2\rangle_S |3/2, 1/2\rangle_I.$$

We do this process for $|\phi'\rangle$, which is completely analogous to this one.

$$3/2 \times 1/2 \rfloor \quad |1, 1\rangle_{F'} = \sqrt{\frac{3}{4}} |1/2, -1/2\rangle_{J'} |3/2, 3/2\rangle_{I'} - \sqrt{\frac{1}{4}} |1/2, 1/2\rangle_{J'} |3/2, 1/2\rangle_{I'}$$

In this case we have $J_1 = L = 1$ and $J_2 = S = 1/2$, so that we use the $1 \times 1/2$ table.

$$1 \times 1/2\rfloor \quad |1/2, -1/2\rangle_{J'} = \sqrt{\frac{1}{3}}|1,0\rangle_{L'}|1/2,-1/2\rangle_{S'} - \sqrt{\frac{2}{3}}|1,-1\rangle_{L'}|1/2,1/2\rangle_{S'}$$
$$|1/2, 1/2\rangle_{J'} = \sqrt{\frac{2}{3}}|1,1\rangle_{L'}|1/2,-1/2\rangle_{S'} - \sqrt{\frac{1}{3}}|1,0\rangle_{L'}|1/2,1/2\rangle_{S'}$$

Putting all together:

$$\begin{aligned}|\phi'\rangle = |1,1\rangle_{F'} =& \sqrt{\frac{3}{4}}\left[\sqrt{\frac{1}{3}}|1,0\rangle_{L'}|1/2,-1/2\rangle_{S'} - \sqrt{\frac{2}{3}}|1,-1\rangle_{L'}|1/2,1/2\rangle_{S'}\right]|3/2,3/2\rangle_{I'}\\ &- \sqrt{\frac{1}{4}}\left[\sqrt{\frac{2}{3}}|1,1\rangle_{L'}|1/2,-1/2\rangle_{S'} - \sqrt{\frac{1}{3}}|1,0\rangle_{L'}|1/2,1/2\rangle_{S'}\right]|3/2,1/2\rangle_{I'}\\ =& \sqrt{\frac{1}{4}}|1,0\rangle_{L'}|1/2,-1/2\rangle_{S'}|3/2,3/2\rangle_{I'} - \sqrt{\frac{1}{2}}|1,-1\rangle_{L'}|1/2,1/2\rangle_{S'}|3/2,3/2\rangle_{I'}\\ &- \sqrt{\frac{1}{6}}|1,1\rangle_{L'}|1/2,-1/2\rangle_{S'}|3/2,1/2\rangle_{I'} + \sqrt{\frac{1}{12}}|1,0\rangle_{L'}|1/2,1/2\rangle_{S'}|3/2,1/2\rangle_{I'}.\end{aligned}$$

Finally, we compute the matrix element considering that only $L' - L = \pm 1$ and $M_L' - M_L = 0$ for z polarization. Also, recall that A only acts on the L-basis:

$$T = |\langle\phi'|A|\phi\rangle|^2 = \left|\left[\sqrt{\frac{1}{4}}\sqrt{\frac{3}{4}} - \sqrt{\frac{1}{4}}\sqrt{\frac{1}{12}}\right]A_{1000}\right|^2 = \frac{1}{12}|A_{1000}|^2.$$

b) For $|\phi\rangle$, $n = 5,\ S = 1/2,\ L = 0,\ J = 1/2$:

$$3/2 \times 1/2\rfloor \quad |1,0\rangle_F = \sqrt{\frac{1}{2}}|1/2,-1/2\rangle_J|3/2,1/2\rangle_I - \sqrt{\frac{1}{2}}|1/2,1/2\rangle_J|3/2,-1/2\rangle_I.$$

$$M_J = M_L + M_S\rfloor \quad |1/2,-1/2\rangle_J = |0,0\rangle_L|1/2,-1/2\rangle_S$$
$$M_J = M_L + M_S\rfloor \quad |1/2,1/2\rangle_J = |0,0\rangle_L|1/2,1/2\rangle_S$$

$$|\phi\rangle = |1,0\rangle_F = \sqrt{\frac{1}{2}}|0,0\rangle_L|1/2,-1/2\rangle_S|3/2,1/2\rangle_I - \sqrt{\frac{1}{2}}|0,0\rangle_L|1/2,1/2\rangle_S|3/2,-1/2\rangle_I.$$

For $|\phi'\rangle$, $n = 5,\ S = 1/2,\ L = 1,\ J = 1/2$:

$$3/2 \times 1/2\rfloor \quad |1,0\rangle_{F'} = \sqrt{\frac{1}{2}}|1/2,-1/2\rangle_{J'}|3/2,1/2\rangle_{I'} - \sqrt{\frac{1}{2}}|1/2,1/2\rangle_{J'}|3/2,-1/2\rangle_{I'}.$$

$$1 \times 1/2\rfloor \quad |1/2,-1/2\rangle_{J'} = \sqrt{\frac{1}{3}}|1,0\rangle_{L'}|1/2,-1/2\rangle_{S'} - \sqrt{\frac{2}{3}}|1,-1\rangle_{L'}|1/2,1/2\rangle_{S'}$$
$$1 \times 1/2\rfloor \quad |1/2,1/2\rangle_{J'} = \sqrt{\frac{2}{3}}|1,1\rangle_{L'}|1/2,-1/2\rangle_{S'} - \sqrt{\frac{1}{3}}|1,0\rangle_{L'}|1/2,1/2\rangle_{S'}$$

$$\begin{aligned}|\phi'\rangle = |1,0\rangle_{F'} =&\sqrt{\frac{1}{2}}\left[\sqrt{\frac{1}{3}}|1,0\rangle_{L'}|1/2,-1/2\rangle_{S'} - \sqrt{\frac{2}{3}}|1,-1\rangle_{L'}|1/2,1/2\rangle_{S'}\right]|3/2,1/2\rangle_{I'}\\ &-\sqrt{\frac{1}{2}}\left[\sqrt{\frac{2}{3}}|1,1\rangle_{L'}|1/2,-1/2\rangle_{S'} - \sqrt{\frac{1}{3}}|1,0\rangle_{L'}|1/2,1/2\rangle_{S'}\right]|3/2,-1/2\rangle_{I'}\\ &= \sqrt{\frac{1}{6}}|1,0\rangle_{L'}|1/2,-1/2\rangle_{S'}|3/2,1/2\rangle_{I'} - \sqrt{\frac{1}{3}}|1,-1\rangle_{L'}|1/2,1/2\rangle_{S'}|3/2,1/2\rangle_{I'}\\ &-\sqrt{\frac{1}{3}}|1,1\rangle_{L'}|1/2,-1/2\rangle_{S'}|3/2,-1/2\rangle_{I'} + \sqrt{\frac{1}{6}}|1,0\rangle_{L'}|1/2,1/2\rangle_{S'}|3/2,-1/2\rangle_{I'}.\end{aligned}$$

$$T = |\langle\phi'|A|\phi\rangle|^2 = \left|\left[\sqrt{\frac{1}{6}}\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{6}}\sqrt{\frac{1}{2}}\right]A_{1000}\right|^2 = 0.$$

c) For $|\phi\rangle$, $n=5,\ S=1/2,\ L=0,\ J=1/2$:

$$|\phi\rangle = |1,1\rangle_F = \sqrt{\frac{3}{4}}|0,0\rangle_L|1/2,-1/2\rangle_S|3/2,3/2\rangle_I - \sqrt{\frac{1}{4}}|0,0\rangle_L|1/2,1/2\rangle_S|3/2,1/2\rangle_I.$$

For $|\phi'\rangle$, $n=5,\ S=1/2,\ L=1,\ J=1/2$:

$$3/2\times 1/2\rfloor\quad |2,1\rangle_{F'} = \sqrt{\frac{1}{4}}|1/2,-1/2\rangle_{J'}|3/2,3/2\rangle_{I'} + \sqrt{\frac{3}{4}}|1/2,1/2\rangle_{J'}|3/2,1/2\rangle_{I'}.$$

$$\begin{aligned}1\times 1/2\rfloor\quad |1/2,-1/2\rangle_{J'} &= \sqrt{\frac{1}{3}}|1,0\rangle_{L'}|1/2,-1/2\rangle_{S'} - \sqrt{\frac{2}{3}}|1,-1\rangle_{L'}|1/2,1/2\rangle_{S'}\\ 1\times 1/2\rfloor\quad |1/2,1/2\rangle_{J'} &= \sqrt{\frac{2}{3}}|1,1\rangle_{L'}|1/2,-1/2\rangle_{S'} - \sqrt{\frac{1}{3}}|1,0\rangle_{L'}|1/2,1/2\rangle_{S'}.\end{aligned}$$

$$\begin{aligned}|\phi'\rangle = |2,1\rangle_{F'} =&\sqrt{\frac{1}{4}}\left[\sqrt{\frac{1}{3}}|1,0\rangle_{L'}|1/2,-1/2\rangle_{S'} - \sqrt{\frac{2}{3}}|1,-1\rangle_{L'}|1/2,1/2\rangle_{S'}\right]|3/2,3/2\rangle_{I'}\\ &+\sqrt{\frac{3}{4}}\left[\sqrt{\frac{2}{3}}|1,1\rangle_{L'}|1/2,-1/2\rangle_{S'} - \sqrt{\frac{1}{3}}|1,0\rangle_{L'}|1/2,1/2\rangle_{S'}\right]|3/2,1/2\rangle_{I'}\\ =&\sqrt{\frac{1}{12}}|1,0\rangle_{L'}|1/2,-1/2\rangle_{S'}|3/2,3/2\rangle_{I'} - \sqrt{\frac{1}{6}}|1,-1\rangle_{L'}|1/2,1/2\rangle_{S'}|3/2,3/2\rangle_{I'}\\ &+\sqrt{\frac{1}{2}}|1,1\rangle_{L'}|1/2,-1/2\rangle_{S'}|3/2,1/2\rangle_{I'} - \sqrt{\frac{1}{4}}|1,0\rangle_{L'}|1/2,1/2\rangle_{S'}|3/2,1/2\rangle_{I'}.\end{aligned}$$

$$T = |\langle\phi'|A|\phi\rangle|^2 = \left|\left[\sqrt{\frac{1}{12}}\sqrt{\frac{3}{4}} - \sqrt{\frac{1}{4}}\sqrt{\frac{1}{4}}\right]A_{1000}\right|^2 = \frac{1}{4}|A_{1000}|^2.$$

d) For $|\phi\rangle$, $n=5,\ S=1/2,\ L=0,\ J=1/2$:

$$|1,0\rangle_F = \sqrt{\frac{1}{2}}|0,0\rangle_L|1/2,-1/2\rangle_S|3/2,1/2\rangle_I - \sqrt{\frac{1}{2}}|0,0\rangle_L|1/2,1/2\rangle_S|3/2,-1/2\rangle_I.$$

For $|\phi'\rangle$, $n=5,\ S=1/2,\ L=1,\ J=1/2$:

$$3/2\times 1/2\rfloor \quad |2,0\rangle_{F'}=\sqrt{\frac{1}{2}}|1/2,-1/2\rangle_{J'}|3/2,1/2\rangle_{I'}+\sqrt{\frac{1}{2}}|1/2,1/2\rangle_{J'}|3/2,-1/2\rangle_{I'}.$$

$$1\times 1/2\rfloor \quad |1/2,-1/2\rangle_{J'}=\sqrt{\frac{1}{3}}|1,0\rangle_{L'}|1/2,-1/2\rangle_{S'}-\sqrt{\frac{2}{3}}|1,-1\rangle_{L'}|1/2,1/2\rangle_{S'}$$
$$1\times 1/2\rfloor \quad |1/2,1/2\rangle_{J'}=\sqrt{\frac{2}{3}}|1,1\rangle_{L'}|1/2,-1/2\rangle_{S'}-\sqrt{\frac{1}{3}}|1,0\rangle_{L'}|1/2,1/2\rangle_{S'}.$$

$$\begin{aligned}|\phi'\rangle=|2,0\rangle_{F'}=&\sqrt{\frac{1}{2}}\left[\sqrt{\frac{1}{3}}|1,0\rangle_{L'}|1/2,-1/2\rangle_{S'}-\sqrt{\frac{2}{3}}|1,-1\rangle_{L'}|1/2,1/2\rangle_{S'}\right]|3/2,1/2\rangle_{I'}\\&+\sqrt{\frac{1}{2}}\left[\sqrt{\frac{2}{3}}|1,1\rangle_{L'}|1/2,-1/2\rangle_{S'}-\sqrt{\frac{1}{3}}|1,0\rangle_{L'}|1/2,1/2\rangle_{S'}\right]|3/2,-1/2\rangle_{I'}\\=&\sqrt{\frac{1}{6}}|1,0\rangle_{L'}|1/2,-1/2\rangle_{S'}|3/2,1/2\rangle_{I'}-\sqrt{\frac{1}{3}}|1,-1\rangle_{L'}|1/2,1/2\rangle_{S'}|3/2,1/2\rangle_{I'}\\&+\sqrt{\frac{1}{3}}|1,1\rangle_{L'}|1/2,-1/2\rangle_{S'}|3/2,-1/2\rangle_{I'}-\sqrt{\frac{1}{6}}|1,0\rangle_{L'}|1/2,1/2\rangle_{S'}|3/2,-1/2\rangle_{I'}.\end{aligned}$$

$$T=|\langle\phi'|A|\phi\rangle|^2=\left|\left[\sqrt{\frac{1}{2}}\sqrt{\frac{1}{6}}+\sqrt{\frac{1}{2}}\sqrt{\frac{1}{6}}\right]A_{1000}\right|^2=\frac{1}{3}|A_{1000}|^2.$$