OPTI 570, Fall 2025 **Practice Final Exam**

Instructions for all students, including distance students.

3-hour written exam.

- You are on your honor to adhere to the exam rules below; violation of these rules will result in a failing grade.
- You may consult the following items during the exam: Any printed or previously saved notes or reference materials, including course notes, practice exams and problem sets and their provided solutions, QM Field Guide **(please have this handy!)** and any textbook including Cohen-Tannoudji, and any of your own notes. **You will need a calculator for a few basic calculations. You may instead use a computer for these, but not for any programming.** During the exam, you may otherwise use a computer only to access allowed materials that are stored on your computer or on the OPTI 570 D2L site (including videos if you think you have time to watch any), to email me, or to access and return the exam. You must **not** consult other people or AI/chatbots, or accept or provide help to anyone else for the entire duration of time that this exam is available for the class to take.
- The first 3 problems on this exam are worth 110 total points, and are the main problems on the exam. **Problems 4-6 are extra credit problems worth 25 total points, but should only be attempted after the indicated main problems have been completed.**
- Use your own paper or tablet to solve all problems. Show enough work that I can follow your reasoning and give you partial credit for problems that are not fully correct.
- It is up to you to convince me that you know how to solve the problems, and to write legibly enough that I do not need to struggle to interpret your work. However, I expect you to work quickly, and that the neatness of your solutions might consequently suffer. That's OK as long as I can interpret your solutions. Draw a box around final answers if your final results are not obvious. If you have a mess of equations all over the page, direct my attention to your line of thought if it is not otherwise obvious. If you have obtained an answer that you know is not correct and you do not have enough time to fix the error, please tell me that you know the answer is wrong, why you know that it is wrong, and guess an appropriate answer – this may help you earn significant partial credit.
- If you are convinced that there is a significant mistake in a problem that may affect the answer or interpretation, you may try to ask or email me about it, but I may not be able to respond before your exam time is up. Alternatively, if I have made an error that is obvious to you, clearly indicate what you think is wrong, what should be changed to make the problem solvable in the manner that you think I intended, then solve the problem. Make sure that I can understand how you have modified the problem to make it solvable.

Useful information for Problem 3. (No exam questions on this page.)

The two-dimensional isotropic harmonic oscillator has energy eigenstates denoted in the usual tensor-product basis by $|n_x, n_y\rangle = |n_x\rangle|n_y\rangle$, where $E_{n_x,n_y} = \hbar\omega(n_x + n_y + 1)$ are the energy eigenvalues in the usual notation.

The **upper-left** 3×3 portions of the matrices representing the position operators X and Y in the tensor-product representation $\{|0,0\rangle, |1,0\rangle, |0,1\rangle, ...\}$ are as follows, where $\sigma = \sqrt{\frac{\hbar}{m\omega}}$:

$$X = \frac{\sigma}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & \\ 0 & 0 & 0 & \\ \vdots & & & \end{pmatrix},$$

$$Y = \frac{\sigma}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \\ 1 & 0 & 0 & \\ \vdots & & & \end{pmatrix}.$$

Due to energy degeneracies, other energy-eigenstate bases can be constructed. The first 3 elements of one alternative energy-eigenstate basis are $\{|0,0\rangle, |\psi_+\rangle, |\psi_-\rangle, ...\}$, where

$$|\psi_+\rangle \equiv \frac{1}{\sqrt{2}}|1,0\rangle + \frac{i}{\sqrt{2}}|0,1\rangle,$$

$$|\psi_-\rangle \equiv \frac{1}{\sqrt{2}}|1,0\rangle - \frac{i}{\sqrt{2}}|0,1\rangle.$$

In the $\{|0,0\rangle, |\psi_+\rangle, |\psi_-\rangle, ...\}$ representation, the **upper-left** 3×3 portions of the matrices representing the position operators X and Y are

$$X = \frac{\sigma}{2} \begin{pmatrix} 0 & 1 & 1 & \cdots \\ 1 & 0 & 0 & \\ 1 & 0 & 0 & \\ \vdots & & & \end{pmatrix},$$

$$Y = \frac{\sigma}{2} \begin{pmatrix} 0 & i & -i & \cdots \\ -i & 0 & 0 & \\ i & 0 & 0 & \\ \vdots & & & \end{pmatrix}.$$

PROBLEM 1. [55 pts.] In this problem we return to the one-dimensional delta-function potential well that you examined at the beginning of the semester. Here we combine the delta-function potential with a one-dimensional harmonic oscillator to create a potential well for a particle of mass m written in two different ways as

$$V(x) \;=\; \frac{1}{2}m\omega^2x^2 - \frac{\hbar^2}{m\,d}\delta(x) \;=\; \frac{1}{2}\hbar\omega(x/\sigma)^2 - \left(\frac{\sigma}{d}\right)\hbar\omega\sigma\cdot\delta(x),$$

where $\sigma \equiv \sqrt{\hbar/(m\omega)}$. In the problems below, you will examine the combined potential in two different limits of the ratio σ/d in parts of the problem labeled A and B, although Part A begins with a potential well that has a different form altogether. Parts A and B are fully independent and can be solved independently.

Part A. A Hamiltonian H_0 expressed in the position representation is defined as

$$H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{\hbar^2}{m\,d}\,\delta(x)$$

where $\delta(x)$ is the Dirac delta function and d is a length scale. This potential well has the ground-state energy eigenfunction

$$\varphi_g(x) = \sqrt{\frac{1}{d}}\,e^{-|x|/d}$$

that is associated with the energy $E_g^0 = -\hbar^2/(2\,m\,d^2)$. (The superscript on E_g^0 indicates that this is an energy eigenvalue of H_0.) In this part, we examine small perturbations W to the above potential so that the full Hamiltonian is given by $H = H_0 + W$. In both cases below, W is given in the position representation.

A(i). Let $W = -\lambda\,E_g^0\,x/d$, where λ is a dimensionless real number, and $|\lambda| \ll 1$. Calculate the ground-state energy of H to first order in λ.

A(ii). Now instead of the perturbation given in part A(i), let $W = \frac{1}{2}m\omega^2x^2 = \frac{\lambda}{2}\hbar\omega(x/d)^2$. Again, λ is a dimensionless real number, and $|\lambda| \ll 1$, but here we have defined $\lambda \equiv (d/\sigma)^2$. Calculate the ground-state energy of H to first order in λ, and define this energy as E_A. Express your answer in terms of σ and d, without the variable λ explicitly appearing in your answer.

A(iii). By using your expression for E_A above, determine the value of d/σ that will shift the ground state energy of the system by 1% of the unperturbed value. In other words, what value of d/σ leads to a first-order energy *correction* (the first-order term in the energy expansion) that has a magnitude that is 1% of the magnitude of E_g^0?

Part B. This is a new scenario, not a continuation of Part A. In this part, a harmonic oscillator Hamiltonian H_0 is expressed in the position representation as

$$H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}\hbar\omega\,(x/\sigma)^2.$$

In the questions below, we examine just one small perturbation W to the above potential so that the full Hamiltonian is given by $H = H_0 + W$.

B(i). Let $W = -\frac{\hbar^2}{m\,d}\delta(x) = -\lambda\,\hbar\omega\,\sigma\cdot\delta(x)$ where λ is a dimensionless real positive number defined as $\lambda \equiv \sigma/d$, and we assume for this part $|\lambda| \ll 1$. Calculate the ground-state energy of H to first order in λ, and define this energy as E_B. Express your answer in terms of σ and d, without λ appearing in your answer.

B(ii). By using the expression for E_B found in B(i), determine the value of σ/d that will shift the ground-state energy of the system by 1% of the unperturbed value.

B(iii). Now calculate the approximate ground-state energy of H to second order in λ, and express your result in terms of σ/d rather than λ. To do this calculation, you will need to make use of the following result, where $H_k(0)$ is the k^{th} order Hermite polynomial evaluated at zero:

$$\sum_{k=1}^{k=\infty} \frac{1}{k}\,\frac{1}{2^k k!}\,|H_k(0)|^2 = \ln 2.$$

PROBLEM 2. [40 pts.] Consider a particle of mass m confined in a one-dimensional harmonic oscillator potential well of frequency ω. The particle's potential energy is given by

$$V(x) = \frac{1}{2}m\omega^2[x - d\cdot f(t)]^2,$$

where d is a positive length constant much smaller than $\sigma \equiv \sqrt{\frac{\hbar}{m\omega}}$, and $f(t)$ is a function of time t that has no dimensional units. At any given time, the center of the potential well is located at position $d\cdot f(t)$. In the questions below, we only consider cases such that $f(t)$ has a maximum magnitude of 1 (although $f(t)$ can be positive or negative), and $f(t) = 0$ for $t = \pm\infty$. Note that $V(x)$ can be re-written as

$$V(x) = \frac{1}{2}m\omega^2x^2 - m\omega^2x\,d\,f(t) + \frac{1}{2}m\omega^2d^2f(t)^2.$$

The last term in the expression above can be neglected in all that follows since it just involves an energy offset that has no effect on system dynamics or coupling between states. We can then consider $W(t) = -m\omega^2x\,d\,f(t) = -\lambda\hbar\omega\cdot(x/\sigma)\cdot f(t)$ as a small time-dependent "pulse" perturbation added to a harmonic oscillator potential, with $\lambda \equiv d/\sigma \ll 1$.

For each of the following cases, the particle is in the ground state of the harmonic oscillator before the perturbation has been applied. Using first-order time-dependent perturbation theory, determine the probability of finding the particle in the first excited state of the harmonic oscillator after the perturbation pulse has finished (ie, at $t = \infty$ or perhaps earlier, depending on the pulse).

(a.) For this first case, the potential well does not move at all until time $t = -\tau$, at which time it moves at a constant speed d/τ until the center is at position $x = d$ at time $t = 0$. The potential well then reverses direction and moves at the same speed until it has returned to being centered at $x = 0$ for all times $t \geq \tau$. For this case, $f(t) = \Lambda(t/\tau)$, where $\Lambda(t/\tau) \equiv 1 - |t|/\tau$ for times $|t| \leq \tau$, and $\Lambda(t/\tau) = 0$ otherwise. Calculate the first-order probability for the system to be found in the first excited state at any time $t \geq \tau$. HINT: The Fourier transform of $\Lambda(t/\tau)$ is $\tau\,\mathrm{sinc}^2(\omega\tau/2)$. **However, the *Field Guide* lists an incorrect Fourier transform of $\Lambda(t/\tau)$ that I just discovered when writing this question! Please correct your copy!**

(b.) A smoothed version of the function $f(t)$ given in part (a) is instead the function

$$f(t) = [1 + 9t^2/\tau^2]^{-1}$$

where the value of τ is identical to that of part (a). Using this new $f(t)$, calculate the first-order probability of finding the system in the first excited state of the harmonic oscillator at $t = \infty$.

(c.) Calculate the ratio of the result of part (a) to the result of part (b) for $\tau = \pi/\omega$. Which of these two pulses is more likely to excite the system into the first excited state, and by what factor?

(d.) Making careful note of the minus sign below, let $f(t)$ be defined by

$$f(t) = \begin{cases} -\sin(2\omega t) & \text{for } 0 \le t \le \pi/\omega \\ 0 & \text{otherwise.} \end{cases}$$

Before calculating the transition probability, first calculate the first-order expansion coefficient term $\lambda b_1^{(1)}(t)$ evaluated at $t = \pi/\omega$. Then calculate the transition probability. The notation is the same as in class, with the subscript 1 corresponding to the first excited state of the oscillator. **To do this, see the information at the end of this problem instead of doing this problem using Fourier transform tables!**

(e.) Do the same as part (d) but for

$$f(t) = \begin{cases} \sin^2(\omega t) & \text{for } 0 \le t \le \pi/\omega \\ 0 & \text{otherwise.} \end{cases}$$

Again, see the information below instead of doing this problem using Fourier transform tables!

Extra information:

$$\int_0^\pi du\, e^{iu} \sin(2u) = \frac{4}{3}.$$

$$\int_0^\pi du\, e^{iu} \sin^2(u) = \frac{4i}{3}.$$

PROBLEM 3. [15 pts.] Complete Problem 2 first! In this problem, a particle of mass m is confined to a time-dependent two-dimensional potential well defined by the potential energy

$$V(x) = \frac{1}{2} m\omega^2 \{[x - d\cdot f_x(t)]^2 + [y - d\cdot f_y(t)]^2\}.$$

As in Problem 2, d is a positive length much smaller than $\sigma \equiv \sqrt{\frac{\hbar}{m\omega}}$, and here $f_x(t)$ and $f_y(t)$ are functions of time t defined by

$$f_x(t) = \begin{cases} -\sin(2\omega t) & \text{for } 0 \le t \le \pi/\omega \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_y(t) = \begin{cases} \sin^2(\omega t) & \text{for } 0 \leq t \leq \pi/\omega \\ 0 & \text{otherwise.} \end{cases}$$

Taken together, $f_x(t)$ and $f_y(t)$ correspond to motion of the harmonic oscillator's center in one small clockwise circle of radius $d/2$, returning it back to its initial origin at time π/ω, at which point the potential well remains stationary for all later times. Now, using the notation that $|n_x, n_y\rangle$ specifies the energy eigenstates of the harmonic oscillator in the usual tensor-product basis, we define the following two states:

$$|\psi_+\rangle \equiv \frac{1}{\sqrt{2}}|1,0\rangle + \frac{i}{\sqrt{2}}|0,1\rangle,$$

$$|\psi_-\rangle \equiv \frac{1}{\sqrt{2}}|1,0\rangle - \frac{i}{\sqrt{2}}|0,1\rangle.$$

(a.) You are given that the particle is in the ground state of the harmonic oscillator for all times up until $t = 0$. Using first-order time-dependent perturbation theory, determine the probabilities $\mathcal{P}_+^{(1)}$ and $\mathcal{P}_-^{(1)}$, both evaluated for $t = \pi/\omega$, of finding the particle in state $|\psi_+\rangle$ or $|\psi_-\rangle$, respectively. To solve this, you could either (i) use the approach discussed in recap and mentioned in class regarding transitions to superposition states, or (ii) use the position operators X and Y expressed in the $\{|0,0\rangle, |\psi_+\rangle, |\psi_-\rangle, ...\}$ representation rather than the more familiar $\{|0,0\rangle, |1,0\rangle, |0,1\rangle, ...\}$ representation. If you wish to use approach (ii), **see the information given on page 2 of this exam.** I suspect that approach (ii) will make the most sense for most people.

(b.) The probabilities $\mathcal{P}_+^{(1)}$ and $\mathcal{P}_-^{(1)}$ should be substantially different from each other. Briefly give a physical justification for why there is such a significant difference. A couple of sentences should be sufficient.

END OF REQUIRED PROBLEMS

EXTRA CREDIT PROBLEMS

PROBLEM 4. Extra Credit: 10 pts. You must complete Problem 2(a) first. (This is probably the most challenging of the 3 extra credit problems, but quick to complete once you understand what to do.) The transition probability of Problem 2(a) can actually be obtained exactly, without using perturbation theory. Calculate this exact transition probability. There is one quick way to obtain the answer with a lot of conceptual insight, and another way that takes just a little bit longer and also some insight. Either way, show or explain all steps.

PROBLEM 5. Extra Credit: 5 pts. You must complete Problems 2(a)-2(c) first. Give a convincing physical physical justification for the conclusion obtained in Problem 2(c).

(problem 6 on next page)

PROBLEM 6. Extra Credit: 10 pts. You must complete Problem 3 first.

Like with Problem 1 on your **Exam 2**, Problem 3 above involves a quantum analog of the motion of a marble in a swirling bowl. Both approaches to exciting the system involve spatial translations of a harmonic oscillator, although the excitations occur over different total times ($2\pi/\omega$ from Exam 2, and π/ω in Problem 3 above). In both cases, the displacements are put in terms of a length constant d. So to reduce confusion, let the parameter d from Exam 2 be called d' instead. The maximum position shifts of the harmonic oscillator away from its initial origin for the two cases are then identical for $d = \sqrt{2}d'$.

For the final coherent state that is a result of the instantaneous shifts explored in Exam 2 Problem 1(a), determine the probabilities of finding the system in state $|\psi_+\rangle$ and in $|\psi_-\rangle$, as defined in Problem 3 of this exam. These calculations do not involve perturbation theory. Then for $d/\sigma \ll 1$ and $d = \sqrt{2}d'$, compare these results with the probabilities $\mathcal{P}_+^{(1)}$ and $\mathcal{P}_-^{(1)}$ of Problem 3. Which method of exciting the system leads to a larger transition probability, and by what factor is this probability larger than the other? You will need to make an approximation valid for $d/\sigma \ll 1$ to compare the results and obtain a numerical value free of variables.

Since the two different approaches to swirling the potential well occur over different time scales, it is helpful to determine an overall *transition rate* T for each case. This is just a probability that you have already calculated divided by the total time of the perturbation ($2\pi/\omega$ for the Exam 2 problem, and π/ω in Problem 3 above). For the two different types of swirling of the harmonic oscillator, notice whether or not one approach is clearly more efficient than the other in driving the system into state $|\psi_+\rangle$ or $|\psi_-\rangle$, or whether one transition probability of one method may be comparable but still somewhat larger than that of the other method of swirling. Propose a physical explanation that justifies your result about the relative transition rates of the different approaches to exciting the ground state into either $|\psi_+\rangle$ or $|\psi_-\rangle$.

End of exam.