OPTI 570, Fall 2025 **Problem Set 2** Prof. Cris D. Panda

**Due: Thurs, Sept 4 (Main Campus); Tues, Sept 9 (Online section)**

**I. Modification of problem 2 from CT Chapter I, complement $K_I$, reproduced below.** This problem involves a few challenging parts, but does not make use of Dirac notation. It is a good problem to jog your memory regarding integration, solving energy eigenvalue problems, Fourier transforms, estimating physical quantities, characteristics of physically realizable wavefunctions, and just being clever regarding thinking about what is needed when solving problems rather than just marching through a sequence of math steps. Some parts of this problem may seem tricky. Feel free to work with someone else, especially if this is your first time in a quantum mechanics course or your math skills are rusty.

**Bound state of a particle in a "delta function potential."** Consider a particle of mass $m$ whose Hamiltonian $H$ [operator defined by formula (D-10) of CT Chapter 1] is:

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha\delta(x)$$

where $\alpha$ is a positive constant whose dimensions are to be found and $x$ is a position variable.

**(a.1)** Integrate the eigenvalue equation of $H$ over the range of positions $-\epsilon$ to $+\epsilon$. Hint: first write down the eigenvalue equation

$$H\varphi(x) = E\varphi(x)$$

where $\varphi(x)$ is a function that solves this equation and is associated with the eigenvalue $E$. You don't know what $\varphi(x)$ is yet, but you will determine it soon. Then, integrate both sides of the equation over the limits given.

**(a.2)** Letting $\epsilon$ approach 0, show that the **derivative** of the eigenfunction $\varphi(x)$ presents a discontinuity at $x = 0$ and determine it in terms of $\alpha$, $m$, and $\varphi(0)$. Hint: a discontinuity in the derivative of a function $f(x)$ at $x_0$ is the value of the slope of $f(x)$ on one side of $x_0$ at a position $x_0 - \epsilon$, minus the value of the slope of $f(x)$ at $x_0 + \epsilon$, determined in the limit that $\epsilon$ approaches 0. Translate the words of this hint into a mathematical expression for this problem.

**(b.1)** Assume that the energy $E$ of the particle is negative (which means that the particle is in a bound state of the delta function potential, since an energy of 0 is associated with no potential energy and the particle being free). $\varphi(x)$ can then be written:

$$x < 0 \qquad \varphi(x) = A_1 e^{\rho x} + A_1' e^{-\rho x} \tag{1}$$
$$x > 0 \qquad \varphi(x) = A_2 e^{\rho x} + A_2' e^{-\rho x}. \tag{2}$$

Express the constant $\rho$ in terms of $E$ and $m$ (and constants). For this, you will need to use the energy eigenvalue equation,

$$H\varphi(x) = E\varphi(x).$$

**(b.2)** For this part, you will need to use the characteristics of a physically realizable wavefunctions, see pages 32-35 of the *Field Guide*. In particular, the condition that $\varphi(x)$ must be continuous everywhere will help you, as will the condition that $\varphi(x)$ must be square-integrable over all space.

With these conditions, find the possible values of the energy eigenvalue $E$. As a check of your work, you should find that there is only one possible value of $E$ for a given value of $\alpha$, and $E$ should be expressed in terms of $\alpha$, $m$, and $\hbar$. Finally, put everything you have learned together to calculate and write out the normalized wavefunction $\varphi(x)$ that corresponds to the energy $E$. Write $\varphi(x)$ in terms of $\rho$.

**(c)** Sketch $\varphi(x)$ vs $x$. Give an order of magnitude estimate of the width $\Delta x$.

*Hint: Don't be overly concerned with how you define "widths" $\Delta x$ and $\Delta p$. Sketch the functions first and then just pick something reasonable. Feel free to choose something like full-width at half-maximum, or half-width at half-maximum.*

**(d)** The Fourier transform of $\varphi(x)$ is a function of momentum $p$, and we designate it here as $\bar{\varphi}(p)$. Write out an expression for $\bar{\varphi}(p)$. You can look up the Fourier transform of $\varphi(x)$, but make sure you get the constants right and make sure you use the correct definition of Fourier transform in this case. Or, you can calculate the Fourier transform yourself. It takes less than 10 lines or so of math, and it is not very difficult and is good practice. Once you have found an answer, sketch $\bar{\varphi}(p)$ vs $p$. Then, estimate a nominal order-of-magnitude width $\Delta p$ for $\bar{\varphi}(p)$. Finally, give an order of magnitude for the product $\Delta x \Delta p$.

**II. Complete problem 1 from CT Ch. II.** This is available in the Ch II PDF on D2L (the problem is given in Complement $\mathrm{H}_{II}$). You will need to have read Chapter II Section B first. For (d)-(e) you should have also read through CT Ch. II, Section D-2, and you will likely need to refer to complement $\mathrm{B}_{II}$ (you'll find a description of the Trace of an operator here). If this is your first time through this kind of math, give it your best shot, and use this problem as a way of playing around with the mechanics of Dirac notation. Also, when you see the word "prove," just read this as "show that." We don't need do be doing formal math proofs in this class.