OPTI 570 FINAL EXAM SOLUTIONS

Problem 1

a. Need to keep at least 3×3, since we are going to x^4:

$$\hat{X}^2 = \frac{\sigma^2}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 & \ddots \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 & \ddots \end{pmatrix} = \frac{\sigma^2}{2}\begin{pmatrix} 1 & 0 & \sqrt{2} \\ 0 & 3 & 0 \\ \sqrt{2} & 0 & 2 & \ddots \end{pmatrix}$$

$$\hat{X}^4 = \frac{\sigma^4}{4}\begin{pmatrix} 1 & 0 & \sqrt{2} \\ 0 & 3 & 0 \\ \sqrt{2} & 0 & 2 & \ddots \end{pmatrix}\begin{pmatrix} 1 & 0 & \sqrt{2} \\ 0 & 3 & 0 \\ \sqrt{2} & 0 & 2 & \ddots \end{pmatrix} = \frac{\sigma^4}{4}\begin{pmatrix} 3 & 0 & 3\sqrt{2} \\ 0 & 9 & 0 \\ 3\sqrt{2} & 0 & 6 & \ddots \end{pmatrix}$$

notice these already need more states

Alternative:

$$\hat{X} = \frac{\sigma}{\sqrt{2}}\left(a^+ + a\right) \qquad \hat{X}^2 = \frac{\sigma^2}{2}\left(a^{+2} + a^2 + a^+a + aa^+\right)$$

keep only terms that bring $|0\rangle$ back to $|0\rangle$

$$\hat{X}^4 = \frac{\sigma^4}{4}\left(a^2a^{+2} + a^+aaa^+ + \dots\right)$$

$$\langle 0|\hat{X}^4|0\rangle = \langle 0|\frac{\sigma^4}{4}\cdot\left(a^2a^{+2} + a^+aaa^+\right)|0\rangle = \frac{\sigma^4}{4}\left(\sqrt{1}\cdot\sqrt{2}\cdot\sqrt{2}\cdot\sqrt{1} + \sqrt{1}\cdot\sqrt{1}\cdot\sqrt{1}\cdot\sqrt{1}\right)$$

$$= \frac{\sigma^4}{4}\cdot 3$$

b. $$E_0^{(1)} = \langle 0,0|W|0,0\rangle = \lambda\hbar\omega\cdot\cancel{\frac{4}{\sigma^4}}\cdot\cancel{\frac{\sigma^4}{4}}\cdot 3 = 3\lambda\hbar\omega$$

$$E_0^{(2)} = \lambda^2\sum_{p_x\neq 0}\frac{\left|\langle p_x,0|\,\hbar\omega\left(\frac{\sqrt{2}x}{\sigma}\right)^4|0,0\rangle\right|^2}{\hbar\omega - \hbar\omega(p_x+1)}$$

Summing only non-zero matrix elements

$$E_0^{(2)} = \lambda^2\hbar^2\omega^2\cdot\left(\frac{72}{\underbrace{-2\hbar\omega}_{p_x=2}} + \frac{24}{\underbrace{-4\hbar\omega}_{p_x=4}}\right) = \lambda^2\hbar\omega(-36-6) =$$

$$= -42\lambda^2\hbar\omega$$

$$\boxed{E_0 = \hbar\omega + 3\lambda\hbar\omega - 42\lambda^2\hbar\omega + \ldots}$$

c. $$|\psi_0\rangle \simeq |0,0\rangle + \lambda \sum_{p_x \neq 0} \frac{\langle p_x,0|\,\cancel{\hbar\omega}\left(\frac{\sqrt{2}\,x}{\sigma}\right)^4|0,0\rangle}{\cancel{\hbar\omega}(-p_x)}\,|p_x,0\rangle$$

$$\simeq |0,0\rangle + \lambda\cdot\left(\underbrace{\frac{6\sqrt{2}}{-2}\,|2,0\rangle}_{p_x=2} + \underbrace{\frac{2\sqrt{6}}{-4}\,|4,0\rangle}_{p_x=4}\right)$$

$$\boxed{|\psi_0\rangle \simeq |0,0\rangle - 3\sqrt{2}\,\lambda\,|2,0\rangle - \frac{\sqrt{6}}{2}\lambda\,|4,0\rangle}$$

d. degenerate states $|1,0\rangle$ and $|0,1\rangle$ with energy $2\hbar\omega$

subspace matrix is: $$\begin{pmatrix} \langle 01|W|01\rangle & \langle 01|W|10\rangle \\ \langle 10|W|01\rangle & \langle 10|W|10\rangle \end{pmatrix} = \begin{pmatrix} \langle 0|W|0\rangle\langle 1|1\rangle & \langle 0|W|1\rangle\langle 1|0\rangle \\ \langle 1|W|0\rangle\langle 0|1\rangle & \langle 1|W|1\rangle\langle 0|0\rangle \end{pmatrix} =$$

$$= \lambda\hbar\omega\begin{pmatrix} 3 & 0 \\ 0 & 15 \end{pmatrix}.$$ No mixing, so already diagonal $\Rightarrow$

$$E_+ = 2\hbar\omega + 15\lambda\hbar\omega$$

$$E_- = 2\hbar\omega + 3\lambda\hbar\omega$$

Problem 2

$$W(t) = i\frac{\hbar\Omega}{2}\left[a^2 e^{2i\omega t} - (a^\dagger)^2 e^{-2i\omega t}\right]$$

a. we need matrix elements that are non-zero

$\langle 2|a^{\dagger 2}|0\rangle = \sqrt{2}$ is the only non-zero

$$\Rightarrow \lambda b_2^{(1)}(t) = \frac{1}{i\hbar}\int_0^t dt'\, e^{i\omega_{20}t'}\, W_{20}(t') =$$

$$= \frac{1}{\cancel{i\hbar}}\cdot\frac{\cancel{i\hbar}\Omega}{2}\int_0^t dt'\, e^{i\cdot 2\omega t'}\cdot(-\sqrt{2})\cdot e^{-2i\omega t'}$$

$$= -\frac{\Omega}{\sqrt{2}}\,t \Rightarrow P_2^{(1)}(t) = \frac{\Omega^2 t^2}{2}$$

Second order, we can couple to 2 or 4

$$\langle 4 | a^{+2} | 2 \rangle = \sqrt{3} \cdot \sqrt{4}$$

$$\lambda^2 b_4^{(2)}(t) = \frac{1}{i\hbar} \int_0^t dt' \, e^{i\omega_{42} t'} W_{42}(t) \cdot \lambda b_2^{(1)}(t) =$$

$$= \frac{1}{\cancel{i\hbar}} \int_0^t dt' \, e^{i2\omega t'} \cdot \cancel{2}\sqrt{3} \, \frac{\cancel{i\hbar}\Omega}{\cancel{2}} \left(-e^{-i2\omega t'}\right) \left(-\frac{\Omega}{\sqrt{2}} \cdot t\right) =$$

$$= \Omega^2 \cdot \sqrt{\frac{3}{2}} \cdot \frac{t^2}{2} =$$

$$= \frac{\sqrt{6}}{4} (\Omega t)^2 \qquad P_4^{(2)}(t) = \frac{3}{8} (\Omega t)^4$$

b. $\max \left[P_{0\to 2}(\tau) \right] = 0.005 = \frac{\Omega^2 \tau_{max}^2}{2}$

$$\tau_{max} = \frac{\Omega}{10}$$

$$\gamma = \sigma e^{-\frac{1}{10}} = \sigma \left(1 - \frac{1}{10}\right) \sim \frac{9}{10} \sigma. \quad \text{A little bit of squeezing.}$$

Problem 3

$$W(t) = \frac{\lambda}{2} m\omega_0^2 x^2 \exp\left(-|t|/\tau\right)$$

a $$\lambda b_1^{(1)}(0) = \frac{1}{i\hbar} \int_{-\infty}^0 dt' \, e^{i\omega_0 t'} e^{-|t'|/\tau} \underbrace{\langle 1 | \frac{\lambda}{2} m\omega_0^2 x^2 | 0 \rangle}_{=0}$$

$$= 0$$

$\Rightarrow$ $P_1 = 0$ to all orders since W only couples states 2 away from eachother

b. $$\lambda b_2^{(1)}(0) = \frac{1}{i\hbar} \int_{-\infty}^0 dt' e^{i2\omega_0 t'} e^{t'/\tau} \langle 2 | \frac{\lambda}{2} m\omega_0^2 x^2 | 0 \rangle =$$

$$= \frac{\lambda m\omega_0^2}{i\hbar} \int_{-\infty}^0 e^{i2\omega_0 t'} e^{t'/\tau} \cdot \frac{\sigma^2}{\sqrt{2}} \, dt' =$$

$$x^2 = \frac{\sigma^2}{2} \left(a^{+2} + a^2\right) \Rightarrow \langle 2 | x^2 | 0 \rangle = \frac{\sigma^2}{2} \sqrt{2}$$

$$= \frac{\lambda m \omega_0^2 \sigma^2}{2\sqrt{2}\, i\hbar} \int_{-\infty}^{0} e^{\left(2i\omega_0 + \frac{1}{\tau}\right)t'} dt'$$

$$= \frac{\lambda m \omega_0^2 \sigma^2}{2\sqrt{2}\, i\hbar} \left. \frac{e^{2i\omega_0 + \frac{1}{\tau} t'}}{2i\omega_0 + \frac{1}{\tau}} \right|_{-\infty}^{0} =$$

$$\sigma = \sqrt{\frac{\hbar}{m\omega}}$$

$$= \frac{\lambda m \omega_0^2 \sigma^2}{2\sqrt{2}\, i\hbar} \cdot \frac{\tau}{1 + 2i\omega_0\tau} = \frac{\lambda \cancel{m} \omega_0^{\cancel{2}} \frac{\cancel{\hbar}}{\cancel{m\omega_0}}}{2\sqrt{2}\, i\cancel{\hbar}} \frac{\tau}{1 + 2i\omega_0\tau}$$

$$\boxed{P_2^{(1)}(0) = \frac{\lambda^2 \omega_0^2 \tau^2}{8 \cdot (1 + 4\omega_0^2\tau^2)}}$$

c. $\lambda b_2^{(1)}(\infty) = \frac{1}{i\hbar} \int_{-\infty}^{+\infty} dt' \, e^{i2\omega_0 t'} e^{-|t'|/\tau} \frac{\lambda}{2} m\omega_0^2 \cdot \frac{\sigma^2}{\sqrt{2}} =$

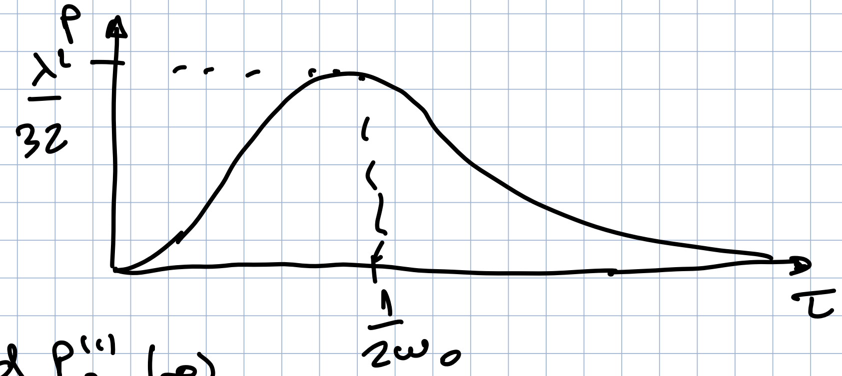
$$= -i \frac{\lambda m \omega_0^2 \sigma^2}{2\sqrt{2}\,\hbar} \int_{-\infty}^{+\infty} e^{-|t'|/\tau} e^{2i\omega_0 t'} dt =$$

$$= -i \frac{\lambda \cancel{m} \omega_0^{\cancel{2}} \frac{\cancel{\hbar}}{\cancel{m\omega_0}}}{2\sqrt{2} \cdot \cancel{\hbar}} \left. \frac{2\tau}{1 + \Omega^2\tau^2} \right|_{\Omega = 2\omega_0}$$

$$= -i \frac{\lambda \omega_0 \tau}{2\sqrt{2}} \frac{2\tau}{1 + 4\omega_0^2\tau^2} = -i \frac{\lambda \omega_0 \tau}{\sqrt{2}\,(1 + 4\omega_0^2\tau^2)}$$

$$\boxed{P_2^{(1)}(\infty) = \frac{\lambda^2 \omega_0^2 \tau^2}{2\,(1 + 4\omega_0^2\tau^2)^2}}$$

d.



$$\frac{d P_2^{(1)}(\infty)}{d\tau} = 0 \Rightarrow$$

$$\frac{2\tau}{(1+4\omega_0^2\tau^2)^2} - \frac{2\tau^2}{(1+4\omega_0^2\tau^2)^3} \cdot 4\omega_0^2 \cdot 2 \cdot \tau = 0$$

$$\cancel{2\tau}(1+4\omega_0^2\tau^2) = \cancel{16}^{\,8}\,\omega_0^2\,\tau^{\cancel{3}\,2}$$

$$4\omega_0^2\tau^2 = 1 \qquad \omega_0^2\tau_{max}^2 = \frac{1}{4} \qquad \tau_{max} = \frac{1}{2\omega_0}$$

$$P_{max} = \frac{\lambda^2 \cdot \frac{1}{4}}{2\left(1+4\cdot\frac{1}{4}\right)^2} = \frac{\lambda^2 \cdot \frac{1}{4}}{2\cdot 4} = \frac{\lambda^2}{32}$$