Assignment 7

OPTI 570 Quantum Mechanics

University of Arizona

Nicolás Hernández Alegría

October 15, 2025
Total time: 15 hours

Problem I

a) The evolution operator would be of the form

$$\hat{\mathbb{U}}_E(t) = e^{-i\hat{H}_1 t/\hbar} = e^{-i\Omega(\hat{N}^2-1/2)t}.$$

The checking is as follows:

$$\begin{aligned}
\hat{\mathbb{U}}_E(\frac{2\pi}{\Omega})|\varphi_n\rangle &= e^{-i\Omega(n^2-1/2)\frac{2\pi}{\Omega}}|\varphi_n\rangle \\
&= e^{-i2\pi(n^2-1/2)}|\varphi_n\rangle \\
&= (e^{-2\pi})^{n^2} e^{i\pi}|\varphi_n\rangle \\
\hat{\mathbb{U}}_E(\frac{2\pi}{\Omega})|\varphi_n\rangle &= -|\varphi_n\rangle.
\end{aligned}$$

b) For $\tau = \pi/2\Omega$, the evolution is

$$\begin{aligned}
\hat{\mathbb{U}}_E(\tau)|\varphi_n\rangle &= e^{-i\Omega(n^2-1/2)\frac{\pi}{2\Omega}}|\varphi_n\rangle \\
&= e^{-i\frac{\pi}{2}n^2} e^{i\frac{\pi}{4}}|\varphi_n\rangle \\
&= (e^{-i\frac{\pi}{2}})^{n^2} e^{i\frac{\pi}{4}}|\varphi_n\rangle \\
&= (-i)^{n^2} e^{i\frac{\pi}{4}}|\varphi_n\rangle \\
\hat{\mathbb{U}}_E(\tau)|\varphi_n\rangle &= \begin{cases} e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, & n \text{ even} \\ e^{-i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, & n \text{ odd} \end{cases} |\varphi_n\rangle.
\end{aligned}$$

c) We use the fact that in a coherent state, we can express it in terms of the energy eigenstates.

$$|\alpha_0\rangle = e^{-\frac{|\alpha_0|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}}|n\rangle.$$

We have found that

$$\hat{\mathbb{U}}_E(\tau) = \begin{cases} e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, & n \text{ even} \\ e^{-i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, & n \text{ odd} \end{cases}.$$

We then, must split the $|\alpha_0\rangle$ accordingly, in even and odd term so that the application of the evolution operator gives

$$|\psi_E(\tau)\rangle = e^{-\frac{|\alpha_0|^2}{2}}\left[e^{i\frac{\pi}{4}}S_{\text{even}} + e^{-i\frac{\pi}{4}}S_{\text{odd}}\right],$$

where

$$S_{\text{even}} = \sum_{n\text{ even}}^{\infty}\frac{\alpha_0^n}{\sqrt{n!}}|n\rangle, \quad \text{and} \quad S_{\text{odd}} = \sum_{n\text{ odd}}^{\infty}\frac{\alpha_0^n}{\sqrt{n!}}|n\rangle. \tag{1}$$

We then have that

$$\left.\begin{aligned}|\alpha_0\rangle &= e^{-\frac{|\alpha_0|^2}{2}}(S_{\text{even}} + S_{\text{odd}})\\ |-\alpha_0\rangle &= e^{-\frac{|\alpha_0|^2}{2}}(S_{\text{even}} - S_{\text{odd}})\end{aligned}\right\} \longrightarrow \begin{aligned}S_{\text{even}} &= \tfrac{1}{2}e^{\frac{|\alpha_0|^2}{2}}(|\alpha_0\rangle + |-\alpha_0\rangle)\\ S_{\text{odd}} &= \tfrac{1}{2}e^{\frac{|\alpha_0|^2}{2}}(|\alpha_0\rangle - |-\alpha_0\rangle)\end{aligned}.$$

Substituting those in the evolution equation and rearranging:

$$|\psi_E(\tau)\rangle = \frac{1}{2}\left[(e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}})|\alpha_0\rangle + (e^{i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}})|-\alpha_0\rangle\right] = \frac{1}{\sqrt{2}}[\alpha_0 + i|-\alpha_0\rangle],$$

where

$$|\pm\alpha_0\rangle = e^{-\frac{|\alpha_0|^2}{2}}\sum_{n=0}^{\infty}\frac{(\pm\alpha_0)^n}{\sqrt{n!}}|n\rangle.$$

d) The transfrmation from the Interaction picture to the Schrodinger picture is

$$\begin{aligned}|\psi(\tau)\rangle &= \hat{\mathbb{U}}_0(\tau)|\psi_E(\tau)\rangle, \qquad \hat{\mathbb{U}}_0(\tau) = e^{-iH_0\tau/\hbar}\\ &= e^{-i\omega\tau(\hat{N}+1/2)}|\psi_E(\tau)\rangle\\ &= \frac{1}{\sqrt{2}}e^{-i\omega\tau(\hat{N}+1/2)}[\alpha_0 + i|-\alpha_0\rangle]\\ &= \frac{1}{\sqrt{2}}e^{-\frac{|\alpha_0|^2}{2}}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left[\alpha_0^n e^{-i\omega\tau(n+1/2)} + i(-\alpha_0)^n e^{-i\omega\tau(n+1/2)}\right]|n\rangle\\ &= \frac{1}{\sqrt{2}}e^{-i\frac{\omega}{2}\tau}e^{-\frac{|\alpha_0|^2}{2}}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left[\left(\alpha_0 e^{-i\omega\tau}\right)^n|n\rangle + i\left(-\alpha_0 e^{-i\omega\tau}\right)|n\rangle\right]\\ |\psi(\tau)\rangle &= \frac{1}{\sqrt{2}}e^{-i\frac{\omega}{2}\tau}\left[|\alpha_0 e^{-i\omega\tau}\rangle + i|-\alpha_0 e^{-i\omega\tau}\rangle\right].\end{aligned}$$

e) asgagasga

Problem II

Problem III

a) sagasg

b) asgasg

c) asgag

d) gasgas

e) asgagasga

f) asgasg

g) asgag

h) asgasg

i) asgasgasgasg

j) asgasgasgasg

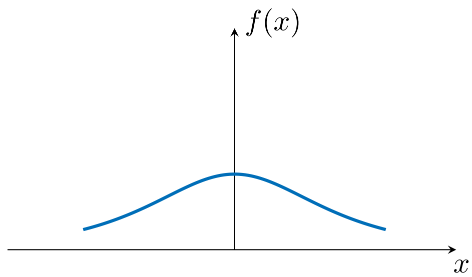
k) asgag

l) asgasg

m) asfas

Problem IV

a) We plot the function $\text{sech}(x)$ to verify its parity. We can see that it is **even**.



This fact will facilitate us when computing ΔX, as we must integrate over $|\phi(x)|^2$ which therefore, is also even. We then have,

$$\langle X \rangle = \int_{-\infty}^{\infty} x|\phi(x)|^2 \, dx = \frac{1}{2\beta}\int_{-\infty}^{\infty} x \, \text{sech}(x/\beta) \, dx = 0$$
$$\langle X^2 \rangle = \int_{-\infty}^{\infty} x^2|\phi(x)|^2 \, dx = \frac{1}{2\beta}\int_{-\infty}^{\infty} x^2 \text{sech}(x/\beta) \, dx = \frac{\beta^2}{2}\int_{-\infty}^{\infty} u^2 \text{sech}^2(u) \, du = \frac{\pi^2\beta^2}{12}.$$

The X uncertainty is

$$\Delta X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \frac{\pi\beta}{2\sqrt{3}}.$$

Similarly, for the Fourier transform we have:

$$\langle P \rangle = \int_{-\infty}^{\infty} p|\bar{\phi}(p)|^2 \, dp = \frac{\pi\beta}{4\hbar}\int_{-\infty}^{\infty} p \, \text{sech}^2(\frac{\pi\beta p}{2\hbar}) \, dp = 0$$
$$\langle P^2 \rangle = \int_{-\infty}^{\infty} p^2|\bar{\phi}(p)|^2 \, dp = \frac{\pi\beta}{4\hbar}\int_{-\infty}^{\infty} p^2 \text{sech}^2(\frac{\pi\beta p}{2\hbar}) \, dp = \frac{2\hbar^2}{\pi^2\beta^2}\int_{-\infty}^{\infty} u^2 \text{sech}^2(u) \, du = \frac{\hbar^2}{\beta^2 3}.$$

Thus

$$\Delta P = \sqrt{\langle P^2 \rangle - \langle P \rangle^2} = \frac{\hbar}{\beta\sqrt{3}}.$$

The uncertainty product is

$$\Delta X \Delta P = \frac{\pi\beta}{2\sqrt{3}} \frac{\hbar}{\beta\sqrt{3}} = \frac{\hbar\pi}{6}.$$

b) The evolution in $\pi/2\omega$ gives a well-known quantity, a scaled Fourier transform of the wavefunction.

$$\Phi(x, \frac{\pi}{2\omega}) = U(\frac{\pi}{2\omega}, 0)\Phi(x, 0) = e^{-i\pi/4}\sqrt{\frac{\hbar}{\sigma^2}}\mathcal{F}\{\Phi(x, 0)\}\big|_{p=\hbar x/\sigma^2}$$

We can see that the function to be computed its Fourier transform is spatially shifted by x_0 so we could directly use the respective property of Fourier transform of a shifter function:

$$\mathcal{F}\{\Phi(x, 0)\} = \bar{\Phi}(p, 0) \Longrightarrow \mathcal{F}\{\Phi(x - x_0, 0)\} = e^{-ipx_0/\hbar}\bar{\Phi}(p, 0).$$

So,

$$\Phi(x, \frac{\pi}{2\omega}) = -e^{-i\pi/4}\sqrt{\frac{\hbar}{\sigma^2}}\left[e^{-ipx_0/\hbar}\bar{\Phi}(p, 0)\right]\Bigg|_{p=\hbar x/\sigma^2} = -\sqrt{\frac{\pi\beta}{4\sigma^2}}e^{-i\pi/4}e^{-i\frac{xx_0}{\sigma^2}}\ \mathrm{sech}(\frac{\pi\beta x}{2\sigma^2}).$$

c) To maintain the width $\Delta X = \frac{\pi\beta}{2\sqrt{3}}$, we compute ΔX for $\Phi(0, \pi/2\omega)$ and equate it to the uncertainty at $t = 0$:

$$\left.\begin{aligned} &\langle X \rangle = 0 \\ &\langle X^2 \rangle = \frac{\pi\beta}{4\sigma^2}\int_{-\infty}^{\infty} x^2 \mathrm{sech}^2(\frac{\pi\beta x}{2\sigma^2})\, dx = \frac{\sigma^4}{3\beta^2}. \end{aligned}\right\} \Delta X = \sqrt{\langle X^2 \rangle} = \frac{\sigma^2}{\sqrt{3}\beta}.$$

Equating it with the uncertainty of the wavefunction at $t = 0$:

$$\frac{\pi\beta}{2\sqrt{3}} = \frac{\sigma^2}{\sqrt{3}\beta} \longrightarrow \beta = \sqrt{\frac{2\sigma^2}{\pi}}.$$

Problem V