Notes of Quantum Mechanics

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Preface

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Chapter 1

One-dimensional harmonic oscillator

1.1 Introduction

1.1.1 Importance of the harmonic oscillator in physics

The simplest example is a particle of mass m moving in a potential which depends only on x and has the form

$$V(x) = \frac{1}{2}kx^2, \quad k > 0.$$

The particle is attracted towards the $x = 0$ by a restoring force:

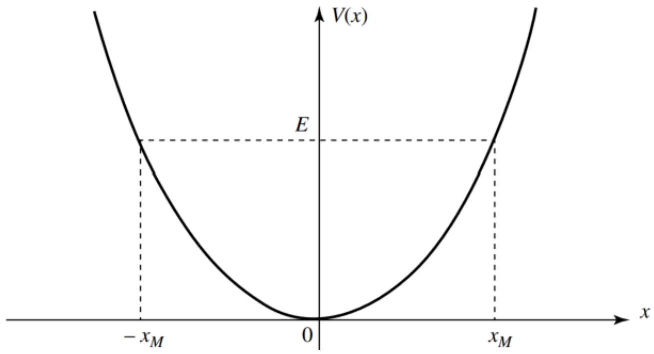

Figure 1.1 Potential energy $V(x)$ of a 1D harmonic oscillator.

$$F_x = \frac{dV}{dx} = -kx.$$

In classical mechanics, the motion of the particle is a sinusoidal oscillation about $x = 0$ with angular frequency $\omega = \sqrt{k/m}$.

> **Various systems are governed by the harmonic oscillator equations**
>
> Whenever one studies the behavior of a system in the neighborhood of a stable equilibrium position, one arrives at equations which, in the limit of small oscillations, are those of a harmonic oscillator.

1.1.2 The harmonic oscillator in classical mechanics

The motion of the particle is governed by the dynamics equation

$$m\frac{d^2x}{dt^2} = -\frac{dV}{dx} = -kx \longrightarrow x = x_M \cos(\omega t - \varphi). \tag{1.1}$$

The kinetic energy of the particle is

$$T = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{p^2}{2m}, \tag{1.2}$$

where $p = mv$ is the momentum of the paticle. The total energy is

$$E = T + V = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 = \frac{1}{2}m\omega^2x_M^2.$$

- The potential can be expanded in Taylor's series around x_0:

$$V(x) = \underbrace{V(x_0)}_{a} + V'(x_0)(x-x_0) + \underbrace{\frac{1}{2!}V^{(2)}(x_0)}_{b}(x-x_0)^2 + \underbrace{\frac{1}{3!}V^{(3)}(x_0)}_{c}(x-x_0)^3 + \cdots$$

The force derived from the potential in the neighborhood of x_0 is

$$F_x = -\frac{dV}{dx} = -2b(x-x_0) - 3c(x-x_0)^2 + \cdots \tag{1.3}$$

The point $x = x_0$ is a stable equilibrium for the particle: $F_x(x_0) = 0$. In adittion, if the amplitude of the motion of the particle about x_0 is sufficiently small, we can keep with the linear term only and we have a harmonic oscillator since the dynamics equation can be approximated by

$$m\frac{d^2x}{dt^2} \approx -2b(x-x_0).$$

For higher energies E, the particle will be in period but not sinusoidal motion (as signal in Fourier series) between the limits x_1 and x_2. We then say that we are dealing with an **anharmonic oscillator**.

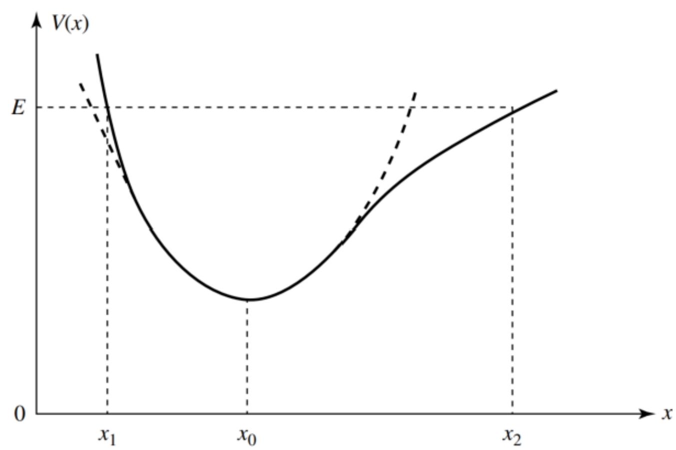

Figure 1.2 Any potential can be approximated by a parabolic potential. In $V(x)$, a classical particle of energy E oscillates between x_1 and x_2.

1.1.3 General properties of the quantum mechanical Hamiltonian

In QM, the classical quantities x and p are replaced respectively by the observables X and P, which satisfy

$$[X, P] = i\hbar.$$

It is then easy to obtain the Hamiltonian operator of the system from the total energy

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2.$$

Since H is time-independent (conservative system), the quantum mechanical study of the harmonic oscillator reduces to the solution of the eigenequation:

$$H|\varphi\rangle = E|\varphi\rangle$$

which is written, in the $\{|x\rangle\}$ representation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega^2x^2\right]\varphi(x)=E\varphi(x).$$

Let us indicate some properties of the potential function:

- **The eigenvalues of the Hamiltonian are positive.** If $V(x)$ has a lower bound, the eigenvalues E of H are greater than the minimum of $V(x)$:

  $$V(x)\leq V_m \quad \text{requires} \quad E>V_m.$$

  We have chosen for the harmonic oscillator that $V_m=0$.

- **The eigenfunctions of H have a definite parity** due to that $V(-x)=V(x)$ is an even function. We shall see that the eigenvalues of H are not degenerate; the wave functions associated with the stationary tates are necessarily either even or odd.

- **The energy spectrum is discrete**.

1.2 Eigenvalues of the Hamiltonian

1.2.1 Notation

It is easy to see that the observables $\hat{X}$ and $\hat{P}$

$$\text{dimensionless observables} \qquad \boxed{\hat{X}=\sqrt{\frac{m\omega}{\hbar}}X, \quad \hat{P}=\frac{1}{\sqrt{m\hbar\omega}}P}$$

are dimensionless. With these new operators, the canonical commutation is

$$\text{Canonical commutation} \qquad \boxed{[\hat{X},\hat{P}]=i} \tag{1.4}$$

and the Hamiltonian can be put in the form

$$H=\hbar\omega\hat{H}, \quad \text{with} \quad \hat{H}=\frac{1}{2}(\hat{X}+\hat{P}). \tag{1.5}$$

In consequence, we seek the solutions of the following eigenequation

$$\hat{H}|\varphi_\nu^i\rangle=\epsilon_\nu|\varphi_\nu^i\rangle,$$

where the operator $\hat{H}$ and the eigenvalues ϵ_ν are **dimensionless.**

If $\hat{X}$ and $\hat{P}$ were numbers and not operators, we could write the sum $\hat{X}^2+\hat{P}^2$ appearing in the definition of $\hat{H}$ in the form of a product $(\hat{X}-i\hat{P})(\hat{X}+i\hat{P})$. However, the introuction of operators proportional to $\hat{H}\pm i\hat{P}$ enables us to simplify considerably out search for eigenvalues and eigenvectors of $\hat{H}$. We therefore set

$$\begin{aligned} a&=\frac{1}{\sqrt{2}}(\hat{X}+i\hat{P}) \\ a^\dagger&=\frac{1}{\sqrt{2}}(\hat{X}-i\hat{P}) \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \hat{X}&=\frac{1}{\sqrt{2}}(a^\dagger+a) \\ \hat{P}&=\frac{i}{\sqrt{2}}(a^\dagger-a) \end{aligned}. \tag{1.6}$$

The commutator of a and $a^\dagger$ is

$$[a, a^\dagger] = \frac{1}{2}[\hat{X} + i\hat{P}, \hat{X} - i\hat{P}] = \frac{i}{2}[\hat{P}, \hat{X}] - \frac{i}{2}[\hat{X}, \hat{P}] = 1 \longrightarrow \boxed{[a, a^\dagger] = 1}\,. \tag{1.7}$$

If we do $aa^\dagger$ we obtain

$$a^\dagger a = \frac{1}{2}(\hat{X} - i\hat{P})(\hat{X} + i\hat{P}) = \frac{1}{2}(\hat{X}^2 + \hat{P}^2 + i\hat{X}\hat{P} - i\hat{P}\hat{X}) = \frac{1}{2}(\hat{X}^2 + \hat{P}^2 - 1).$$

Comparing with $\hat{H}$ we see that

$$\boxed{\hat{H} = a^\dagger a + \frac{1}{2} = aa^\dagger - \frac{1}{2}}\,.$$

We see that we canot put $\hat{H}$ in a product of linear terms, due to the non-commutatitivty of $\hat{X}$ and $\hat{P}$ ($1/2$ term).

We introduce another operator:

$$\text{Operator } N \qquad \boxed{N = a^\dagger a}\,. \tag{1.8}$$

This operator is Hermitian

$$N^\dagger = a^\dagger (a^\dagger)^\dagger = a^\dagger a = N. \tag{1.9}$$

And its relation with $\hat{H}$ is

$$\hat{H} = N + \frac{1}{2} \tag{1.10}$$

so that the eigenvectors of $\hat{H}$ are eigenvectors of N, and viceversa. The commutators with a and $a^\dagger$ are:

$$[N, a] = [a^\dagger a, a] = a^\dagger [a, a] + [a^\dagger, a]a = -a \longrightarrow \boxed{[N, a] = -a} \tag{1.11}$$

$$[N, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger [a, a^\dagger] + [a^\dagger, a^\dagger]a = a^\dagger \longrightarrow \boxed{[N, a^\dagger] = a^\dagger}\,. \tag{1.12}$$

The study of the harmonic oscilator is based on these operatores a, $a^\dagger$, and N. The eigenequation for N is

$$\text{Eigenequation of } N \qquad \boxed{N|\varphi_\nu^i\rangle = \nu|\varphi_\nu^i\rangle}\,. \tag{1.13}$$

When this is solved, we know that the eigenvector $|\varphi_\nu^i\rangle$ of N is also an eigenvector of H with the eigenvalue $E_\nu = (\nu + 1/2)\hbar\omega$:

$$H|\varphi_\nu^i\rangle = (\nu + 1/2)\hbar\omega|\varphi_n u^i\rangle. \tag{1.14}$$

The solution of the eigenequation of N will be based on the commutation relation $[a, a^\dagger] = 1$.

1.2.2 Determination of the spectrum

Lemmas

- **Properties of the eigenvalues of N** The eigenvalues ν of the operator N are positive or zero. We can see this by looking the qaure of the norm of the vector $a|\varphi_n u^i\rangle$

$$\|a|\varphi_\nu^i\rangle\|^2 = \langle\varphi_\nu^i|a^\dagger a|\varphi_\nu^i\rangle = \langle\varphi_\nu^i|N|\varphi_\nu^i\rangle = \nu\langle\varphi_\nu^i|\varphi_\nu^i\rangle \geq 0 \Longrightarrow \boxed{\nu \geq 0}.$$

- **Properties of the vector $a|\varphi_\nu^i\rangle$**
  i) $\nu = 0 \Longrightarrow a|\varphi_{\nu=0}^i\rangle = 0$. If $\nu = 0$ is an eigenvalue of N, all eigenvectors $|\varphi_0^i\rangle$ associated with this eigenvalue satisfy the relation

$$a|\varphi_0^i\rangle = 0. \tag{1.15}$$

  Anyn vector which satisfy this relation is therefore an eigenvector of N with the eigenvalue $\nu = 0$.
  ii) $\nu > 0 \Longrightarrow a|\varphi_\nu^i\rangle$ is a non-zero eigenvector of N with eigenvalue $\nu - 1$.

$$\begin{aligned} [N,a]|\varphi_\nu^i\rangle &= -a|\varphi_\nu^i\rangle \\ Na|\varphi_\nu^i\rangle &= aN|\varphi_\nu^i\rangle - a|\varphi_\nu^i\rangle \\ N[a|\varphi_\nu^i\rangle] &= a\nu|\varphi_\nu^i\rangle - a|\varphi_\nu^i\rangle \end{aligned} \Longrightarrow \boxed{N[a|\varphi_\nu^i\rangle] = (\nu-1)[a|\varphi_\nu^i\rangle]}.$$

- **Properties of the vector $a^\dagger|\varphi_\nu^i\rangle$**
  i) $a^\dagger|\varphi_\nu^i\rangle$ is always non-zero. We study it with the square of the norm:

$$\|a^\dagger|\varphi_\nu^i\rangle\|^2 = \langle\varphi_\nu^i|aa^\dagger|\varphi_\nu^i\rangle = \langle\varphi_\nu^i|(N+1)|\varphi_\nu^i\rangle = (\nu+1)\langle\varphi_\nu^i|\varphi_\nu^i\rangle.$$

  As $\nu \geq 0$ by lemma 1, the ket $a^\dagger|\varphi_\nu^i\rangle$ always has non-zero norm and, consequently, is never zero.
  ii) $a^\dagger|\varphi_\nu^i\rangle$ is an eigenvector of N with eigenvalue $N+1$. We do it analoguisly to lemma IIb):

$$\begin{aligned} [N,a^\dagger]|\varphi_\nu^i\rangle &= a^\dagger|\varphi_\nu^i\rangle \\ Na^\dagger|\varphi_\nu^i\rangle &= a^\dagger N|\varphi_\nu^i\rangle + a^\dagger|\varphi_\nu^i\rangle \\ N[a^\dagger|\varphi_\nu^i\rangle] &= \nu a^\dagger|\varphi_\nu^i\rangle + a^\dagger|\varphi_\nu^i\rangle \end{aligned} \Longrightarrow \boxed{N[a^\dagger|\varphi_\nu^i\rangle] = (\nu+1)[a^\dagger|\varphi_\nu^i\rangle]}.$$

The spectrum of N is composed of non-negative integers

If ν is non-integral, we can therefore construct a non-zero eigenvector of N with a strictly negative eigenvalue. Since this is impossible by lemma 1, the hypothesis of non-integral ν must be rejected.

> ν can only be a non-negative integer.

We conclude that the eigenvalues of H are of the form

$$\text{Eigenvalue of } H \qquad \boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \quad n \in \mathbb{N}_0^+}. \tag{1.16}$$

In QM, the energy of the harmonic oscillator is **quantized**. The smallest value (ground state) is $\hbar\omega/2$.

Interpretation of the a and $a^\dagger$ operators

We have seen that, given $|\varphi_n^i\rangle$ with eigenvalue E_n, application of a gives an eigenvector associated with E_{n-1} while application of $a^\dagger$ yields the energy E_{n+1}.

Thats why $a^\dagger$ is said to be a **creation operator** and a an **annihilation operator**; their action on an eigenvector of N makes an energy quantum $\hbar\omega$ appear or dissapear.

1.2.3 Degeneracy of the eigenvalues

The grounds state is non-degenerate

The eigenstates of H associated with $E_0 = \hbar\omega/2$ (or eigenvector of N associated with $n = 0$), according to lemma II, must all satisfy the equation

$$a|\varphi_0^i\rangle = 0.$$

To find the degeneracy of the E_0 level, all we must do is see how many li kets satisfy the above. We can write the above equation using the definition of $\hat{X}, \hat{P}$ and a in terms of them, in the form

$$\frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}X + \frac{i}{\sqrt{m\hbar\omega}}P\right]|\varphi_0^i\rangle = 0.$$

In the $\{|x\rangle\}$ representation, this relation becomes

$$\left(\frac{m\omega}{\hbar}x + \frac{d}{dx}\right)\varphi_0^i(x) = 0, \quad \text{where} \quad \varphi_0^i(x) = \langle x|\varphi_0^i\rangle.$$

Therefore we msut solve a first-order differential equation, which solution is

$$\varphi_0^i(x) = ce^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2} \tag{1.17}$$

The various solutions of the ODE are all proportional to each other. Consequently, there exists only one ket $|\varphi_0\rangle$ that satisfies the initial equation: the ground sate $E_0 = \hbar\omega/2$ is not degenerate.

All the states are non-degenerate

We use recurrence to show that all other states are also non-degenerate. We need to prove that if E_n is non degenerate, the level E_{n+1} is not either.

Lets assume there exists only one vector $|\varphi_n\rangle$ such that

$$N|\varphi_n\rangle = n|\varphi_n\rangle.$$

Then consider an eigenvector $|\varphi_{n+1}^i\rangle$ corresponding to the eigenvalue $n+1$

$$N|\varphi_{n+1}^i\rangle = (n+1)|\varphi_{n+1}^i\rangle.$$

We know that the ket $a|\varphi_{n+1}^i\rangle$ is not zero and that it is an eigenvector of N with eigenvalue n. Since this ket is not degenerae by hypothesis, there exists a number c^i such that

$$a|\varphi_{n+1}^i\rangle = c^i|\varphi_n\rangle / a^\dagger \longrightarrow a^\dagger a|\varphi_{n+1}^i\rangle = N|\varphi_{n+1}^i\rangle = (n+1)|\varphi_{n+1}^i\rangle = c^i a^\dagger|\varphi_n\rangle.$$

We have,

$$|\varphi_{n+1}^i\rangle = \frac{c^i}{n+1}a^\dagger|\varphi_n\rangle.$$

We see that all kets $|\varphi^i_{n+1}\rangle$ associated with the eigenvalue $n+1$ are proportional to $a^\dagger|\varphi_n\rangle$. They are proportional to each other: the eigenvalue $n+1$ is not degenerate.

Since the eigenvalue $n=0$ is not degenerate, the eigenvalue $n=1$ is not either, nor is $n=2$, etc.: all the eigenvalues of N and, consequently, all those of H, are non-degenerate. Now, we can just write $|\varphi_n\rangle$ for the eigenvector of H associated with E_n.

1.3 Eigenstates of the Hamiltonian

1.3.1 The $\{\varphi_n\}$ representation

Since none of the eigenvalues of N (H) is degenerate, N (H) alon constitutes a CSCO in $\mathscr{E}_c$.

The basis vectors in terms of $|\psi_0\rangle$

We assume that the vector $|\varphi_0\rangle$ which satsfies $a|\varphi_0\rangle = 0$, is normalized. According to lemma III, the vector $|\varphi_1\rangle$ is proportional to $a^\dagger|\varphi_0\rangle$ in the form

$$|\varphi_1\rangle = c_1 a^\dagger|\varphi_0\rangle.$$

We shall determine c_1 by requiring $|\varphi_1\rangle$ to e normalized and choosing the phase of $|\varphi_1\rangle$ such that c_1 is real and positive. The square of the norm of $|\varphi_1\rangle$ is

$$\langle\varphi_1|\varphi_1\rangle = |c_1|^2\langle\varphi_0|aa^\dagger|\varphi_0\rangle = |c_1|^2\langle\varphi_0|(a^\dagger a+1)|\varphi_0\rangle = |c_1|^2[\underbrace{\langle\varphi_0|N|\varphi_0\rangle}_{0\langle\varphi_0|\varphi_0\rangle} + \langle\varphi_0|\varphi_0\rangle] = |C_1|^2.$$

We find that $c_1 = 1$:

$$\langle\varphi_1|\varphi_1\rangle = |c_1|^2 = 1 \Longrightarrow |\varphi_1\rangle = a^\dagger|\varphi_0\rangle. \tag{1.18}$$

We can do the same to constrcut $|\varphi_2\rangle$ from $|\varphi_1\rangle$ and get c_2 and so on. In general, if we know $|\varphi_{n-1}\rangle$ (normalized), then the normalized vector $|\varphi_n\rangle$ is written

$$|\varphi_n\rangle = c_n a^\dagger|\varphi_{n-1}\rangle, \quad \text{so that} \quad c_n = \frac{1}{\sqrt{n}}.$$

In fact, we can express all $|\varphi_n\rangle$ in terms of $|\varphi_0\rangle$ by recursion:

$$\boxed{|\varphi_n\rangle = \frac{1}{\sqrt{n}}(a^\dagger)^n|\varphi_0\rangle}\,. \tag{1.19}$$

Orthonormalization and closure relations

Action of the various operators

1.3.2 Wave functions associated with the stationary states

1.4 Discussion

1.4.1 Mean values and rms eviations of X and P in a state $|\varphi_n\rangle$

1.4.2 Properties of the ground state

1.4.3 Time evolution of the mean values

Formula sheet

1.4.4 Useful formulas

Closure relation (discrete)	$\sum_k \sum_{i=1}^{g_k} \lvert v_k^i\rangle\langle v_k^i\rvert = \mathbb{1}$	Closure relation (continuous)	$\int_\beta d\beta \, \lvert\omega_\beta\rangle\langle\omega_\beta\rvert = \mathbb{1}$
Glauber Formula	$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$	Generalized uncertainty relation	$\Delta A \Delta B \geq \frac{1}{2}\lvert\langle[A,B]\rangle\rvert$
Function of an operator	$F(A) = \sum_{n=0}^{\infty} f_n (A-a)^n$		$\Delta Q = \sqrt{\langle Q^2\rangle - \langle Q\rangle^2}$
Eigenequation of $F(A)$	$F(A)\lvert\psi\rangle = F(\lambda)\lvert\psi\rangle$		
Transformation $\{u\} \to \{v\}$	$\mathbb{M}_{jk} = \langle u_j \vert v_k\rangle$	$\lvert\psi\rangle_{\{u\}} = \mathbb{M}\lvert\psi\rangle_{\{v\}}$ $A_{\{u\}} = \mathbb{M} A_{\{v\}} \mathbb{M}^\dagger$	$\lvert\psi\rangle_{\{v\}} = \mathbb{M}^\dagger\lvert\psi\rangle_{\{u\}}$ $A_{\{v\}} = \mathbb{M}^\dagger A_{\{u\}} \mathbb{M}$

1.4.5 Basis

Quantity	Discrete basis (sum over j, k)	Continuous basis (integrate over β, β')
$\mathbb{1}$	$= \sum \lvert v_k\rangle\langle v_k\rvert$	$= \int d\beta \, \lvert\omega_\beta\rangle\langle\omega_\beta\rvert$
$\lvert\psi\rangle = \mathbb{1}\lvert\psi\rangle$	$= \sum \lvert v_k\rangle\langle v_k\vert\psi\rangle$	$= \int d\beta \, \lvert\omega_\beta\rangle\langle\omega_\beta\vert\psi\rangle$
$\langle\varphi\rvert = \langle\varphi\rvert\mathbb{1}$	$= \sum \langle\varphi\vert v_k\rangle\langle v_k\rvert$	$= \int d\beta \, \langle\varphi\vert\omega_\beta\rangle\langle\omega_\beta\rvert$
$A = \mathbb{1}A\mathbb{1}$	$= \sum\sum \lvert v_j\rangle\langle v_j\vert A\vert v_k\rangle\langle v_k\rvert$	$= \iint d\beta \, d\beta' \, \lvert\omega_\beta\rangle\langle\omega_\beta\vert A\vert\omega_{\beta'}\rangle\langle\omega_{\beta'}\rvert$

Quantity	X representation	P_x representation
X	x	$i\hbar \, \partial/\partial p$
P_x	$-i\hbar \, \partial/\partial x$	p
$\lvert x'\rangle$	$\langle x\vert x'\rangle = \delta(x-x')$	$\langle p\vert x'\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp(-ix'p/\hbar)$
$\lvert p'\rangle$	$\langle x\vert p'\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp(ixp'/\hbar)$	$\langle p\vert p'\rangle = \delta(p-p')$
$\lvert\psi\rangle$	$\langle x\vert\psi\rangle = \psi(x)$	$\langle p\vert\psi\rangle = \tilde{\psi}(p)$

Fourier transforms for 3D wavefunctions

$\tilde{\psi}(\boldsymbol{p}) = \mathscr{F}[\psi(\boldsymbol{r})] = \left(\frac{1}{2\pi\hbar}\right)^{3/2} \int_{-\infty}^{\infty} d^3\boldsymbol{r} \, e^{-i\boldsymbol{r}\cdot\boldsymbol{p}/\hbar}\psi(\boldsymbol{r})$	$\psi(\boldsymbol{r}) = \mathscr{F}^{-1}\left[\tilde{\psi}(\boldsymbol{p})\right] = \left(\frac{1}{2\pi\hbar}\right)^{3/2} \int_{-\infty}^{\infty} d^3\boldsymbol{p} \, e^{i\boldsymbol{r}\cdot\boldsymbol{p}/\hbar}\tilde{\psi}(\boldsymbol{p})$
$\mathscr{F}\left[\psi^{(n)}(x)\right] = \left(\frac{ip}{\hbar}\right)^n \tilde{\psi}(p)$	$\tilde{\psi}^{(n)}(p) = \mathscr{F}\left[\left(-\frac{ix}{\hbar}\right)^n \psi(x)\right]$
$\tilde{\psi}(p-p_0) = \mathscr{F}\left[e^{ip_0x/\hbar}\psi(x)\right]$	$e^{-ipx_0/\hbar}\tilde{\psi}(p) = \mathscr{F}[\psi(x-x_0)]$
$\mathscr{F}[\psi(cx)] = \tilde{\psi}(p/c)/\lvert c\rvert$	$\int_{-\infty}^{\infty} dx \, \varphi^*(x)\psi(x) = \int_{-\infty}^{\infty} dp \, \tilde{\varphi}^*(p)\tilde{\psi}(p)$
$\psi(x)$ real: $[\tilde{\psi}(p)]^* = \tilde{\psi}(-p)$	$\psi(x)$ imaginary: $[\tilde{\psi}(p)]^* = -\tilde{\psi}(-p)$
$\Delta x \Delta p \geq \hbar$	

Commutators

Key points

- When a matrix has a block form, we can compute the eigenvalues in each block submatrix.
- The eigenpairs allows you to diagonalize $A = V\Lambda V^{-1}$ in the eigenbasis, where $V = [\boldsymbol{u}_1\vert\boldsymbol{u}_2\vert\cdots]$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots)$, and $A\lvert\boldsymbol{u}_i\rangle = \lambda_i\lvert\boldsymbol{u}_i\rangle$. In the eigenbasis we can do $F(A) = VF(\Lambda)V^{-1}$.
- When A is Hermitian, V is unitary: $V^{-1} = V^\dagger$.

$[A,B] = -[B,A]$
$[A,B]^\dagger = [B^\dagger, A^\dagger]$
$[AB,CD] = A[B,C]D + AC[B,D] + [A,C]DB + C[A,D]B$
$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$
$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$ $([A,[A,B]] = [B,[A,B]] = 0)$
$[X,P] = i\hbar$
$[H,P] = i\hbar \frac{dV(X)}{dX}$

$[A+B, C+D] = [A,C] + [A,D] + [B,C] + [B,D]$
$[F(A), A] = 0$
$[A,B] = 0 \Longrightarrow [F(A),B] = [A,F(B)] = [F(A),F(B)] = 0$
$[A,[A,B]] = [B,[A,B]] = 0 \Longrightarrow [A,F(B)] = [A,B]\frac{dF(B)}{dB}$
$[A,[A,B]] = [B,[A,B]] = 0 \Longrightarrow [F(A),B] = [A,B]\frac{dF(A)}{dA}$
$[H,X] = -\frac{i\hbar}{m}P$
$\langle \varphi_n | [A,H] | \varphi_n \rangle = 0, \ \forall A$

- If the matrix is diagonal, the exponential acts directly onto the elements.
- The evolution operator is $U = e^{-iHt/\hbar}$ and it evolves the state by matrix multiplication $U|\psi\rangle$.
- The eigenequation show you the relation of the eigenvectors that must be considered to construct the eigenvectors of the eigenbasis: $A|u_i\rangle = \lambda|u_j\rangle$. Its matrix representation is λ in the ji position.
- You can reduce the dimension of an operator to its eigensubspace when only acting inside it.
- To know the action of an operator you can stimulate it by applying $|\psi\rangle$ or $\langle\psi|$.
- In the operation $|u_i\rangle\langle u_j|$, the element will be located at ij in the matrix.
- Conservative=H time-independent, Stationary state=$|\psi\rangle$ projects in a single eigenstate of H.
- Constant of motion=A time-independent and $[A,H] = 0$.

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