OPTI 570, Fall 2025 **Problem Set 4** Prof. C. D. Panda

Due: Thurs, Sept. 18 (Main Campus); Tues, Sept 23 (Online section)

There are some optional questions at the end of the problem set. **You do not need to complete or even attempt the optional problems!** They involve important results and concepts, and are provided to give you additional challenging practice with mathematical formalism only if you would like to try tackling such problems.

Problem I.

(a) Using the *sixth postulate* in CT chapter III, show that $\frac{d}{dt}\langle\psi(t)|\phi(t)\rangle = 0$, where $|\psi(t)\rangle$ and $|\phi(t)\rangle$ are arbitrary states in the same state space. Based on this result show that Schrödinger evolution over some time interval $[t_0, t]$ is a *unitary transformation*. (See CT III, eqs. D-2 and D-3, and complement F_{III} if you get stuck).

(b) Let $|\psi(t_0)\rangle = |\phi_n^i\rangle$, where $|\phi_n^i\rangle$ is an eigenvector of the Hamiltonian: $H|\phi_n^i\rangle = E_n|\phi_n^i\rangle$. Assuming that H does not explicitly depend on time, show that

$$|\psi(t)\rangle = e^{-i(E_n/\hbar)(t-t_0)}|\psi(t_0)\rangle$$

is a solution to the Schrödinger Equation. Since the global phase factor $e^{-i(E_n/\hbar)t}$ has no physical significance, the physical properties of a system described by a quantum state $|\psi(t)\rangle$ of the above form do not change over time, and the eigenstates of H are therefore called *stationary states.*

(c) More generally, show that if $|\psi(t_0)\rangle = \sum_{n,i} c_n^i|\phi_n^i\rangle$, then $|\psi(t)\rangle = \sum_{n,i} c_n^i e^{-i(E_n/\hbar)(t-t_0)}|\phi_n^i\rangle$ is a solution to the Schrödinger Equation. Show that this state is not a stationary state for arbitrary coefficients c_n^i.

Problem II.

CT III, Complement L-III, problem 1.

Note: check that the dimensional units of all answers make sense, and that probabilities and expressions of physical quantities are real.

Problem III.

CT III, Complement L-III, problem 12, **ONLY parts (a) and (d)**. Assume $|\psi(0)\rangle$ is properly normalized. This should be very quick to complete.

Problem IV.

CT III, Complement L-III, problem 14.

Problem V.

CT III, Complement L-III, problem 4, **ONLY parts (a) and (b)**. Do not use the position or momentum representations in this problem.

REMAINING PROBLEMS ARE OPTIONAL

Problem VI: OPTIONAL: you do *not* need to attempt or complete this problem.

Deriving the Generalized Uncertainty Principle

PART 1.

This is a straightforward math and graphing/visualization problem. It is really rather simple, so don't over-analyze it! The point of the problem is to help you build a visual interpretation of mathematical expressions, especially as related to CT Chapter III, complement C_{III}. The relevance of this problem should become clear later in this problem set.

Consider the function $F(\lambda) = a\lambda^2 + c$, where both a and c are real and positive scalars, and λ is a real variable. Consider a second function $G(\lambda) = b\lambda$, where b is a real scalar (positive or negative).

(a) *Qualitatively* plot both $F(\lambda)$ and $G(\lambda)$ vs λ on each of 3 separate sketches so that your plots correspond to and graphically show the following 3 different cases:
(i) $F(\lambda) = G(\lambda)$ at exactly 2 values of λ,
(ii) $F(\lambda) = G(\lambda)$ at exactly one value of λ, and
(iii) $F(\lambda) > G(\lambda)$ for all values of λ.
Your relative magnitudes of the constants do not matter here as long as they meet the conditions listed above. The sign of b is up to you to choose.

(b) For condition (i) above, write a familiar equation for the two values of λ that solve $F(\lambda) = G(\lambda)$. In this equation you should have the expression $b^2 - 4ac$. We will label this quantity D, that is: $D = b^2 - 4ac$. What condition on D (ie, what range of values of D) corresponds to there being two different solutions to $F(\lambda) = G(\lambda)$ where λ is any real number?

(c) Write an inequality for D that would correspond to the plots (ii) and (iii) above, that is: $F(\lambda) \geq G(\lambda)$ for all real λ. This condition will ensure that $a\lambda^2 - b\lambda + c \geq 0$ for all real values of λ. (Note: $D = b^2 - 4ac$ is called the discriminant of the polynomial $a\lambda^2 - b\lambda + c$.)

Part 2.

An operator A is termed "anti-Hermitian" if $A = -A^\dagger$.

(a) Show that if an operator F is Hermitian, then $G = iF$ is an anti-hermitian operator.

(b) Show that if an operator G is anti-Hermitian, its eigenvalues are purely imaginary and the expectation value $\langle G \rangle = \langle \psi | G | \psi \rangle$ for any quantum state $|\psi\rangle$ is also purely imaginary.

(c) Show that if A and B are observables, $[A, B]$ is anti-Hermitian, and therefore that $\langle [A, B] \rangle$ is a purely imaginary number.

Part 3.

Suppose A and B are observables, and let $|\psi\rangle$ indicate a properly normalized quantum state. Define the Hermitian operator C by the expression $[A, B] = iC$. (Although this is not strictly necessary for the following questions, it can be convenient to help with notation.) Also define a new — and not necessarily normalized — quantum state $|\phi\rangle$ as

$$|\phi\rangle = (A + i\lambda B)|\psi\rangle$$

where λ is any real number.

(a) Calculate the scalar product $\langle\phi|\phi\rangle$, showing the result as a second order polynomial in λ. What condition must the discriminant satisfy? (Remember that this polynomial is equal to a scalar product!) Use your answer to show that

$$\left\langle A^2 \right\rangle \left\langle B^2 \right\rangle \geq \frac{1}{4} \left|\langle [A, B] \rangle\right|^2 .$$

(b) Define two new operators,

$$A' = A - \langle A\rangle \quad \text{and} \quad B' = B - \langle B\rangle.$$

Show that $[A', B'] = [A, B]$ and therefore that

$$\left\langle (A')^2 \right\rangle \left\langle (B')^2 \right\rangle \geq \frac{1}{4} \left|\langle [A, B] \rangle\right|^2 .$$

(c) Show that

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} \left|\langle [A, B] \rangle\right|^2 .$$

where ΔA is the r.m.s deviation or standard deviation of A (and similarly for B). This result is the most general form of Heisenberg's uncertainty principle. As you can see, it was derived for **any** two observables A and B.

(d) Using the results from above, write Heisenberg's uncertainty principle for the specific case of any two observables Q and P that have the commutation relation $[Q, P] = i\hbar$.

Problem VII: <u>OPTIONAL</u>: you do not need to attempt or complete this problem.

This problem considers mathematical relationships relevant to upcoming topics.

We consider a Hamiltonian $H(t)$ that explicitly depends on time through at least one of its terms. This can happen if a particle with a non-zero magnetic dipole interacts with a time-varying magnetic field, for example. Although it may seem counter-intuitive, it is not necessarily the case that the Hamiltonian commutes with itself at different times: $[H(t),\, H(t')]$ is not necessarily equal to zero if $t \neq t'$.

(a) If $[H(t),\, H(t')] \neq 0$ for arbitrary times t and t', show that $[H(t),\, \int_{t_0}^{t} H(t')dt'] \neq 0$.

(b) Define the operator $F(t) = -\frac{i}{\hbar}\int_{t_0}^{t} H(t')dt'$. Derive the following relationships:

$$\left[F(t),\, \frac{d}{dt}F(t)\right] = -(1/\hbar^2)\left[\int_{t_0}^{t} H(t')dt',\, H(t)\right] = (1/\hbar^2)\left[H(t),\, \int_{t_0}^{t} H(t')dt'\right].$$

(c) With $F(t)$ defined as above, show that if $[F(t),\, \frac{d}{dt}F(t)] \neq 0$, then $\frac{d}{dt}(e^{F(t)}) \neq \frac{dF(t)}{dt}\, e^{F(t)}$. Also show that if $[F(t),\, \frac{dF(t)}{dt}] = 0$, then $\frac{d}{dt}(e^{F(t)}) = \frac{dF(t)}{dt}\, e^{F(t)}$.

(d) The Schrödinger evolution operator $U(t, t_0)$ is defined such that $|\psi(t)\rangle = U(t, t_0)\,|\psi(t_0)\rangle$. After substitution of $|\psi(t)\rangle$ into the Schrödinger Equation, this leads to the equation for $U(t, t_0)$:

$$i\hbar\frac{d}{dt}U(t, t_0) = H(t)\; U(t, t_0).$$

Show that the evolution operator is given by the expression

$$U(t, t_0) = e^{-\frac{i}{\hbar}\int_{t_0}^{t} H(t')dt'}$$

IF and ONLY IF $[H(t),\, H(t')] = 0$.