Problem Set 8 Solutions

Problem 1

$$|\psi\rangle = a|z+\rangle + b|z0\rangle + c|z-\rangle$$

a. state vector

$$J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\langle J_x\rangle = \langle\psi|J_x|\psi\rangle = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} a^* & b^* & c^* \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} =$$

$$= \frac{\hbar}{\sqrt{2}}\begin{pmatrix} a^* & b^* & c^* \end{pmatrix}\begin{pmatrix} b \\ a+c \\ b \end{pmatrix} =$$

$$= \underbrace{\frac{\hbar}{\sqrt{2}}\left(a^* b + b^* a + b^* c + c^* b\right)}$$

density matrix $\rho = |\psi\rangle\langle\psi|$ - pure state

$$\rho = \begin{pmatrix} a \\ b \\ c \end{pmatrix}\begin{pmatrix} a^* & b^* & c^* \end{pmatrix} = \begin{pmatrix} aa^* & ab^* & ac^* \\ ba^* & bb^* & bc^* \\ ca^* & cb^* & cc^* \end{pmatrix}$$

$$\langle J_x\rangle = T_r(\rho J_x) =$$

$$= \frac{\hbar}{\sqrt{2}} T_r\left[\begin{pmatrix} aa^* & ab^* & ac^* \\ ba^* & bb^* & bc^* \\ ca^* & cb^* & cc^* \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right] =$$

$$= \frac{\hbar}{\sqrt{2}} T_r\begin{pmatrix} ab^* & aa^* + ac^* & ab^* \\ bb^* & ba^* + bc^* & bb^* \\ cb^* & ca^* + cc^* & cb^* \end{pmatrix} =$$

$$= \underbrace{\frac{\hbar}{\sqrt{2}}\left(ab^* + ba^* + bc^* + cb^*\right)}$$

b. $J^2 |j\, m_u\rangle = j\cdot(j+1)\hbar^2 |j\, m_u\rangle$ w/ $j=1$

$$\langle J^2\rangle = 1\cdot 2\cdot \hbar^2 = \underline{2\hbar^2}$$

c.

$$J_x^2 = \frac{\hbar^2}{2}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} = \frac{\hbar^2}{2}\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}$$

$$\langle J_x^2\rangle = \frac{\hbar^2}{2}(a^* \; b^* \; c^*)\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} =$$

$$= \frac{\hbar^2}{2}(a^* \; b^* \; c^*)\begin{pmatrix}a+c\\2b\\a+c\end{pmatrix} =$$

$$= \frac{\hbar^2}{2}\left(a^*a + a^*c + 2b^*b + c^*a + c^*c\right)$$

$$J_y^2 = \frac{\hbar^2}{2}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix} = \frac{\hbar^2}{2}\begin{pmatrix}1&0&-1\\0&2&0\\-1&0&1\end{pmatrix}$$

$$\langle J_y^2\rangle = \frac{\hbar^2}{2}(a^* \; b^* \; c^*)\begin{pmatrix}1&0&-1\\0&2&0\\-1&0&1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} =$$

$$= \frac{\hbar^2}{2}(a^* \; b^* \; c^*)\begin{pmatrix}a-c\\2b\\-a+c\end{pmatrix} =$$

$$= \frac{\hbar^2}{2}\left(a^*a - a^*c + 2b^*b - ac^* + cc^*\right)$$

$$J_z^2 = \frac{\hbar^2}{2}\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}$$

$$\langle J_z^2\rangle = \frac{\hbar^2}{2}(a^* \; b^* \; c^*)\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} =$$

$$= \frac{\hbar^2}{2}\cdot\left(a^*a + c^*c\right)$$

d. $$\langle J_x\rangle^2 = \frac{\hbar^2}{2}\cdot\left[b\cdot(a^*+c^*) + b^*(a+c)\right]^2 =$$

$$= \frac{\hbar^2}{2}\left[b^2(a^*+c^*)^2 + b^{*2}(a+c)^2 + 2bb^*(a+c)(a^*+c^*)\right] =$$

$$(\Delta J_x)^2 = \langle J_x^2\rangle - \langle J_x\rangle^2 =$$

$$= \frac{\hbar^2}{2}\left[a^*a + a^*c + 2b^*b + c^*a + c^*c - b^2(a^*+c^*)^2 - b^{*2}(a+c)^2 - 2bb^*(a+c)(a^*+c^*)\right]$$

e. $(\Delta J_x)^2$ is smallest if J_x is known, i.e. an eigenstate of J_x

$b = 0$

$$(\Delta J_x)^2 = \frac{\hbar^2}{2}\left[a^*a + a^*c + c^*a + c^*c\right] = 0$$

$$(a^* + c^*)(a+c) = 0 \Rightarrow a = -c$$

for example $a = \frac{1}{\sqrt{2}}$, $b = 0$, $c = -\frac{1}{\sqrt{2}}$ $\quad |\psi\rangle_{\{z\}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

f. for example $|+\rangle_z = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

g. $$(\Delta J_y)^2 (\Delta J_z)^2 \geq \frac{1}{4} |\langle \underbrace{[J_y, J_z]}_{i\hbar J_x} \rangle|^2$$

$$(\Delta J_y)^2 (\Delta J_z)^2 \geq \frac{\hbar^2}{4} \langle J_x \rangle^2$$

$$(\Delta J_y)^2 (\Delta J_z)^2 \geq \frac{\hbar^4}{8} (ab^* + ba^* + bc^* + cb^*)^2$$

in part e: $(\Delta J_y)^2 (\Delta J_z)^2 \geq 0$

this means the system could be in an eigenstate of J_y or J_z while also having $\langle J_x \rangle = 0$.

In general, that is possible, for example $|+\rangle_y$

in part f: $(\Delta J_y)^2 (\Delta J_z)^2 \geq 0$

makes sense since $\Delta J_z = 0$ for $|+\rangle_z$, which is an eigenstate of the system.

Problem II $\quad |\psi(t=0)\rangle = |z0\rangle$

a. We need $|x+\rangle$, $|x-\rangle$, $|x0\rangle$ in z-rep.

$$J_x^{(z)} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{w/ eigenvalues } \pm\hbar, 0$$

$\lambda = +1$ $\quad J_x|x+\rangle = |x+\rangle$

$$\frac{\cancel{\hbar}}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \cancel{\hbar}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$\frac{1}{\sqrt{2}} b = a$

$\frac{1}{\sqrt{2}}(a+c) = b$

$\frac{1}{\sqrt{2}} b = c$

$a = c$

$b = \sqrt{2}a$

$$\Rightarrow |x+\rangle_{\{z\}} = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$\lambda = -1$ $\quad \Rightarrow |x-\rangle_{\{z\}} = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\hbar\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$\frac{1}{\sqrt{2}} b = -a$

$\frac{1}{\sqrt{2}}(a+c) = -b$

$\frac{1}{\sqrt{2}} b = -c$

$a = c$

$b = -\sqrt{2}a$

$\lambda = 0$

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$b = 0$

$a + c = 0$

$$|x0\rangle_{\{z\}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Then:

$$P(J_x = \hbar) = \langle z0|\hat{P}_{|x+\rangle}|z_0\rangle = \qquad \hat{P}_x = |x+\rangle\langle x+|$$

$$= \langle z0|x+\rangle\langle x+|z0| =$$

$$= |\langle z_0|x+\rangle|^2 = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$$

b. $P(J_x = 0) = \langle z0|\, P_{|x0\rangle}|z_0\rangle =$

$$= |\langle z_0|x0\rangle|^2 = 0$$

c. $P(J_x = -\hbar) = |\langle z_0|x-\rangle|^2 = \frac{1}{2}$

d. $P(J_z = \hbar) = |\langle z_0|z+\rangle|^2 = 0$

e. $J_x = \begin{cases} \hbar & \text{w/ } \frac{1}{2} \\ -\hbar & \text{w/ } \frac{1}{2} \end{cases}$ state after is

$$|x+\rangle = \frac{1}{2}|z+\rangle + \frac{1}{\sqrt{2}}|z0\rangle + \frac{1}{2}|z-\rangle$$

$$|x-\rangle = \frac{1}{2}|z+\rangle - \frac{1}{\sqrt{2}}|z0\rangle + \frac{1}{2}|z-\rangle$$

When meas J_z, prob of $\hbar$ is $\frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{4}$

$-\hbar$ is $\frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{4}$

0 is $\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$

f. Since the result of the measurement is unknown, we only have the probabilities for the different states, so the state after the meas. is a <u>mixed state.</u>

g. $J_x = \begin{cases} \hbar & w/\ \text{prob.}\ \frac{1}{2} \\ -\hbar & w/\ \text{prob.}\ \frac{1}{2} \end{cases}$

$$\Rightarrow \rho^{(x)} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad Tr\left[\rho^{(x)2}\right] = Tr\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} = \frac{1}{2} \neq 1$$

mixed state

h. $M^{+} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$

$$\rho^{(z)} = M^{+}\rho^{(x)}M = \frac{1}{4}\cdot\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} =$$

$$= \frac{1}{4}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ 0 & 0 & 0 \\ 1/2 & -1/\sqrt{2} & 1/2 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix} =$$

$$\rho^{(z)} = \begin{pmatrix} 1/4 & 0 & 1/4 \\ 0 & 1/2 & 0 \\ 1/4 & 0 & 1/4 \end{pmatrix}$$

(i) $$\langle J_z\rangle = Tr\{\rho J_z\} = \hbar\cdot\frac{1}{8}\cdot Tr\left\{\begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right\}$$

$$= \frac{\hbar}{8} Tr\begin{pmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{pmatrix} = 0$$

$$\langle J_x\rangle = Tr\{\rho J_x\} = \frac{\hbar}{\sqrt{2}}\cdot\frac{1}{8}\, Tr\left\{\begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right\} =$$

$$= \frac{\hbar}{8\sqrt{2}}\, Tr\begin{pmatrix} 0 & 4 & 0 \\ 1 & 0 & 1 \\ 0 & 4 & 0 \end{pmatrix} = 0$$

i. $P(J_z = \hbar) = \mathrm{Tr}\{\rho\, |z+\rangle\langle z+|\}$

$= \mathrm{Tr}\{\langle z+|\rho|z+\rangle\} =$

$= \langle z+|\rho|z+\rangle =$

$= \rho_{11}^{(z)} = \frac{1}{4}$

$P(J_z = 0) = \rho_{22}^{(z)} = \frac{1}{2}$

$P(J_z = -\hbar) = \rho_{33}^{(z)} = \frac{1}{4}$

makes sense - same answers as above