OPTI 570 GRAD QM LECTURE Tu Oct 21

Ang Momentum recap

$$J^2 |j\, m_z\rangle = j(j+1)\hbar^2 |j\, m_z\rangle$$
$$J_z |j\, m_z\rangle = m_z \hbar |j\, m_z\rangle$$

j: any non-negative integer or $\frac{1}{2}$ integer

$m_z \in \{j, j-1, \ldots -j\}$

Examples: $\mathcal{E}_{j=1}$ state space of AM with $j=1$

one basis $\{|j=1, m_z=1\rangle, |j=1, m_z=0\rangle, |j=1, m_z=-1\rangle\}$

$\{|z+\rangle, |z0\rangle, |z-\rangle\}$

$\sqrt{2}\,\hbar \to$ "length of A.M vector"

another $\{|j=1, m_x=1\rangle, |j=1, m_x=0\rangle, |j=1, m_x=-1\rangle$

$\{|x_+\rangle, |x_0\rangle, |x_-\rangle\}$

standard representation - z basis $\{|z_+\rangle, |z_0\rangle, |z_-\rangle\}$

- z representation

$$J^2 \to 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\{|j\ m_z\rangle\}$

$$J_z \to \hbar \begin{array}{c} \begin{array}{ccc} |z_+\rangle & |z_0\rangle & |z_-\rangle \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{array} = J_z^{(z)}$$

$$J_\pm |j\, m_z\rangle = \hbar \sqrt{j(j+1) - m_z(m_z \pm 1)}\ |j\, m_z = \pm 1\rangle$$

(unles 0 for m_z at limit)

$$J_+ = \frac{1}{2}(J_x + i J_y)$$

$$J_- = \frac{1}{2}(J_x - i J_y)$$

$$J_+ |z_+\rangle = 0$$

$$J_+ |z_0\rangle = \hbar \sqrt{2}\, |z_+\rangle$$

$$J_+ |z_-\rangle = \hbar \sqrt{2}\, |z_0\rangle$$

$$J_+ \to \sqrt{2}\hbar \begin{array}{c} \begin{array}{ccc} |z_+\rangle & |z_0\rangle & |z_-\rangle \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array} = J_+^{(z)}$$

$$J_- \to \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = J_-^{(z)}$$

$$J_x = \frac{1}{2}(J_+ + J_-) \qquad\qquad J_y = -\frac{i}{2}(J_+ - J_-)$$

$$J_x^{(z)} = \frac{\hbar}{\sqrt{2}} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{\text{eigenvalues } -\hbar,\, 0,\, \hbar} \qquad\qquad J_y^{(z)} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

Q: Express J_x in x representation

$$J_x^{(x)} = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

check: obtain $J_x^{(z)}$ from $J_x^{(x)}$ using M matrices.

$$J_x^{(z)} = M^\dagger_{x\to z}\, J_x^{(x)}\, M_{x\to z}$$

$$M_{x\to z} = \begin{pmatrix} \langle x_+|z_+\rangle & \langle x_+|z_0\rangle & \langle x_+|z_-\rangle \\ \langle x_0|z_+\rangle & \dots & - \\ \vdots & & \end{pmatrix}$$ FG p 70

$$M^{\dagger}_{x\to z} = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} = M_{z\to x}$$

Orbital AM

Classical: $\bar{L} = \underbrace{\vec{R}}_{\text{pos}} \times \underbrace{\vec{P}}_{\text{momentum}}$

QM: $\bar{L} = \bar{R} \times \bar{P}$

$J^2, J_z \rightarrow L^2, L_z$

$j \Rightarrow l$

$\vec{R} \rightarrow \bar{r}$ $\qquad\qquad$ $\vec{P} \rightarrow i\hbar \bar{\nabla}$

$\bar{L} = -i\hbar\, \bar{r} \times \bar{\nabla}$

Cartesian:
$$L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$
$$L_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$
$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Sperical:
$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

$$L_z = -i\hbar\frac{\partial}{\partial\varphi}$$

$$L_z F(\varphi) = -i\hbar \frac{\partial}{\partial \varphi} F(\varphi)$$

$$-i\cancel{\hbar}\frac{\partial}{\partial \varphi} F(\varphi) = m_z \cancel{\hbar} F(\varphi)$$

$$L_z F(\varphi) = m_z \hbar \, F(\varphi)$$

$$\frac{\partial}{\partial \varphi} F(\varphi) = i m_z F(\varphi)$$

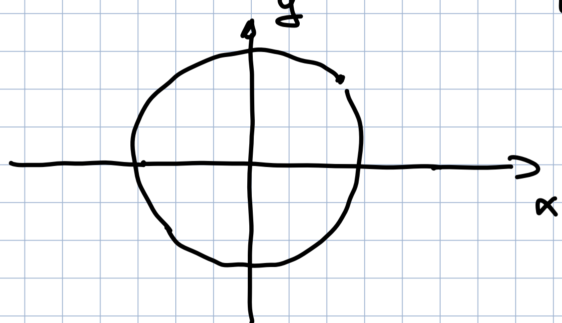
$$F(\varphi) \propto e^{i m_z \varphi}$$

Full solution:

$$L^2 Y_\ell^m(\theta,\varphi) = \hbar\, \ell(\ell+1) Y_\ell^m(\theta,\varphi)$$

$$L_z Y_\ell^m(\theta,\varphi) = \hbar m \underbrace{Y_\ell^m(\theta,\varphi)}_{\text{spherical harmonics}}$$

$$Y_\ell^m(\theta,\varphi) = N e^{im\varphi} \underbrace{P_\ell^m(\cos\theta)}_{\text{Legendre polynomials}}$$



$e^{im0} = e^{im2\pi} \Rightarrow$ m- must be integer

spherical harmonics:

$$\delta_{m',m}\,\delta_{\ell',\ell} = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi \, Y_{\ell'}^{m'}(\theta,\varphi)\, Y_\ell^m(\theta,\varphi)$$

$$1 = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi \, |Y_\ell^m(\theta,\varphi)|^2$$

$$\mathbb{1}_{\theta,\varphi} = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi \, |\theta,\varphi\rangle\langle\theta,\varphi|$$

Examples:

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{\frac{1}{2}}$$

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \cos\theta$$

$$Y_1^{\pm 1} = \mp \left(\frac{3}{8\pi}\right)^{\frac{1}{2}} \sin\theta \, e^{\pm i\varphi}$$

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$$\cos\theta = \frac{z}{r} \Rightarrow$$

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \frac{z}{\sqrt{x^2+y^2+z^2}}$$

For problems w/ spherical symmetry

central potential : Coulomb, gravity, 3D QHO

solutions to energy eigenvalues :

$$\psi_{n,l,m}(r,\theta,\varphi) = \underbrace{F_{n,l}(r)}_{\text{radial part.}} \underbrace{Y_l^m(\theta,\varphi)}_{\text{sph harmonics}}$$

Spin AM

Spin $\frac{1}{2}$ system

- operator matrices : 2×2

Notation : $\bar{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$

$s = \frac{1}{2}$

$$S^2 |s=\tfrac{1}{2}, m_u = \pm\tfrac{1}{2}\rangle = \tfrac{3}{4}\hbar^2 |s=\tfrac{1}{2}, m_u = \pm\tfrac{1}{2}\rangle$$

$$S_u |s=\tfrac{1}{2}, m_u = \pm\tfrac{1}{2}\rangle = m_u\hbar |s=\tfrac{1}{2}, m_u = \pm\tfrac{1}{2}\rangle$$

$\hat{u}$: any direction in 3D

choose $\hat{u} = \hat{z}$

$$\{|\tfrac{1}{2}, m_z = \tfrac{1}{2}\rangle, |\tfrac{1}{2}, m = -\tfrac{1}{2}\rangle\}$$

$$\{|z_+\rangle, |z_-\rangle\}, \{|+\rangle_z, |-\rangle_z\}$$

$$S^2 \rightarrow \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad S_x^{(x)} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_x^{(z)} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x \qquad S_y^{(z)} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_y$$

$$S_z^{(z)} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\sigma_z$$

$\sigma_x, \sigma_y, \sigma_z$ – Pauli matrices

Define: $\bar{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z) \Rightarrow \bar{S} = \frac{\hbar}{2}\bar{\sigma}$

$$\hat{S}_u = \bar{S}\cdot\hat{u} = \hat{S}_x \sin\theta\cos\varphi + \hat{S}_y \sin\theta\sin\varphi + \hat{S}_z\cos\theta$$

$$= \frac{\hbar}{2}\underbrace{\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}}_{\sigma_u}$$