# Assignment 7

## OPTI 570 Quantum Mechanics


University of Arizona

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Total time: 15 hours

## Problem I

a) The evolution operator would be of the form

$$\hat{\mathbb{U}}_E(t) = e^{-i\hat{H}_1 t/\hbar} = e^{-i\Omega(\hat{N}^2-1/2)t}.$$

The checking is as follows:

$$\begin{aligned}
\hat{\mathbb{U}}_E(\frac{2\pi}{\Omega})|\varphi_n\rangle &= e^{-i\Omega(n^2-1/2)\frac{2\pi}{\Omega}}|\varphi_n\rangle \\
&= e^{-i2\pi(n^2-1/2)}|\varphi_n\rangle \\
&= (e^{-2\pi})^{n^2} e^{i\pi}|\varphi_n\rangle \\
\hat{\mathbb{U}}_E(\frac{2\pi}{\Omega})|\varphi_n\rangle &= -|\varphi_n\rangle.
\end{aligned}$$

b) For $\tau = \pi/2\Omega$, the evolution is

$$\begin{aligned}
\hat{\mathbb{U}}_E(\tau)|\varphi_n\rangle &= e^{-i\Omega(n^2-1/2)\frac{\pi}{2\Omega}}|\varphi_n\rangle \\
&= e^{-i\frac{\pi}{2}n^2} e^{i\frac{\pi}{4}}|\varphi_n\rangle \\
&= (e^{-i\frac{\pi}{2}})^{n^2} e^{i\frac{\pi}{4}}|\varphi_n\rangle \\
&= (-i)^{n^2} e^{i\frac{\pi}{4}}|\varphi_n\rangle \\
\hat{\mathbb{U}}_E(\tau)|\varphi_n\rangle &= \begin{cases} e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, & n \text{ even} \\ e^{-i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, & n \text{ odd} \end{cases} |\varphi_n\rangle.
\end{aligned}$$

c) We use the fact that in a coherent state, we can express it in terms of the energy eigenstates.

$$|\alpha_0\rangle = e^{-\frac{|\alpha_0|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}}|n\rangle.$$

We have found that

$$\hat{\mathbb{U}}_E(\tau) = \begin{cases} e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, & n \text{ even} \\ e^{-i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, & n \text{ odd} \end{cases}.$$

We then, must split the $|\alpha_0\rangle$ accordingly, in even and odd term so that the application of the evolution operator gives

$$|\psi_E(\tau)\rangle = e^{-\frac{|\alpha_0|^2}{2}}\left[e^{i\frac{\pi}{4}}S_{\text{even}} + e^{-i\frac{\pi}{4}}S_{\text{odd}}\right],$$

where

$$S_{\text{even}} = \sum_{n \text{ even}}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}}|n\rangle, \quad \text{and} \quad S_{\text{odd}} = \sum_{n \text{ odd}}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}}|n\rangle. \tag{1}$$

We then have that

$$\left.\begin{array}{l} |\alpha_0\rangle = e^{-\frac{|\alpha_0|^2}{2}}(S_{\text{even}} + S_{\text{odd}}) \\ |-\alpha_0\rangle = e^{-\frac{|\alpha_0|^2}{2}}(S_{\text{even}} - S_{\text{odd}}) \end{array}\right\} \longrightarrow \begin{array}{l} S_{\text{even}} = \frac{1}{2}e^{\frac{|\alpha_0|^2}{2}}(|\alpha_0\rangle + |-\alpha_0\rangle) \\ S_{\text{odd}} = \frac{1}{2}e^{\frac{|\alpha_0|^2}{2}}(|\alpha_0\rangle - |-\alpha_0\rangle) \end{array}.$$

Substituting those in the evolution equation and rearranging:

$$|\psi_E(\tau)\rangle = \frac{1}{2}\left[(e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}})|\alpha_0\rangle + (e^{i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}})|-\alpha_0\rangle\right] = \frac{1}{\sqrt{2}}[|\alpha_0\rangle + i|-\alpha_0\rangle],$$

where

$$|\pm\alpha_0\rangle = e^{-\frac{|\alpha_0|^2}{2}}\sum_{n=0}^{\infty}\frac{(\pm\alpha_0)^n}{\sqrt{n!}}|n\rangle.$$

d) The transfrmation from the Interaction picture to the Schrodinger picture is

$$\begin{aligned}
|\psi(\tau)\rangle &= \hat{\mathbb{U}}_0(\tau)|\psi_E(\tau)\rangle, \qquad \hat{\mathbb{U}}_0(\tau) = e^{-iH_0\tau/\hbar} \\
&= e^{-i\omega\tau(\hat{N}+1/2)}|\psi_E(\tau)\rangle \\
&= \frac{1}{\sqrt{2}}e^{-i\omega\tau(\hat{N}+1/2)}[\alpha_0 + i|-\alpha_0\rangle]|n\rangle \\
&= \frac{1}{\sqrt{2}}e^{-\frac{|\alpha_0|^2}{2}}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left[\alpha_0^n e^{-i\omega\tau(n+1/2)} + i(-\alpha_0)^n e^{-i\omega\tau(n+1/2)}\right]|n\rangle \\
&= \frac{1}{\sqrt{2}}e^{-i\frac{\omega}{2}\tau}e^{-\frac{|\alpha_0|^2}{2}}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left[\left(\alpha_0 e^{-i\omega\tau}\right)^n|n\rangle + i\left(-\alpha_0 e^{-i\omega\tau}\right)|n\rangle\right] \\
|\psi(\tau)\rangle &= \frac{1}{\sqrt{2}}e^{-i\frac{\omega}{2}\tau}\left[|\alpha_0 e^{-i\omega\tau}\rangle + i|-\alpha_0 e^{-i\omega\tau}\rangle\right].
\end{aligned}$$

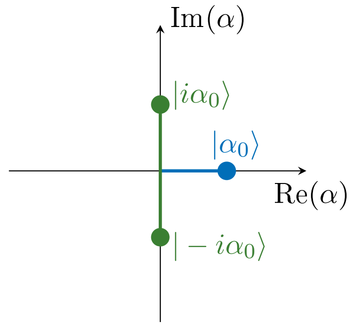
e) Evaluating with $\tau = \pi/2\omega$,

$$\begin{aligned}
|\psi(\frac{\pi}{2\omega})\rangle &= \frac{1}{\sqrt{2}}e^{-i\frac{\omega}{2}\frac{\pi}{2\omega}}\left[|\alpha_0 e^{-i\omega\frac{\pi}{2\omega}}\rangle + i|-\alpha_0 e^{-i\omega\frac{\pi}{2\omega}}\rangle\right] \\
&= \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}\left[|\alpha_0 e^{-i\frac{\pi}{2}}\rangle + i|-\alpha_0 e^{-i\frac{\pi}{2}}\rangle\right] \\
|\psi(\frac{\pi}{2\omega})\rangle &= \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}\left[|-i\alpha_0\rangle + i|i\alpha_0\rangle\right].
\end{aligned}$$

In addition, at $t = 0$ we have

$$|\psi_E(0)\rangle = |\alpha_0\rangle.$$

Then,




## Problem II

The Hamiltonian in the whole range is:

$$\hat{H} = \hat{H}_0 + \hat{W} = \begin{cases} \frac{\hat{P}^2}{2m}, & t < 0 \\ \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2, & 0 \leq t < \tau \ , \\ \frac{\hat{P}^2}{2m}, & t \geq 0 \end{cases} \qquad \hat{W} = \frac{1}{2}m\omega^2\hat{X}^2.$$

The evolution operator is $U_0(t) = e^{-iH_0 t/\hbar} = e^{-i\hat{P}^2 t/2m\hbar}$. The effective Hamiltonian in terms of the Schrodinger picture position and momentum operators is:

$$H_E = U_0^\dagger(t,0)H_1U_0(t,0) = e^{i\frac{\hat{P}^2 t}{2m\hbar}}\left[\frac{1}{2}m\omega^2\hat{X}^2\right]e^{-i\hat{P}^2 t/2m\hbar} = \frac{1}{2}m\omega^2 e^{i\hat{P}^2 t/2m\hbar}\hat{X}^2 e^{-i\hat{P}^2 t/2m\hbar}$$
$$\stackrel{(a)}{=} \frac{1}{2}m\omega^2\left[e^{i\hat{P}^2 t/2m\hbar}\hat{X}e^{-i\hat{P}^2 t/2m\hbar}\right]^2.$$

In $(a)$, we used the property. We can see the term iinside the brackets as the product $ABC$ of operators, where we would like to switch the position of $\hat{X}$ with the right exponential, that why we use

$$ABC = A[B,C] + ACB.$$

The commutator $[B,C]$ is

$$[B,C] = [\hat{X}, e^{-i\frac{\hat{P}^2 t}{2m\hbar}}] = i\hbar\partial_{\hat{P}} e^{-i\frac{\hat{P}^2 t}{2m\hbar}} = i\hbar\frac{-i2\hat{P}t}{2m\hbar}e^{-i\frac{\hat{P}^2 t}{2m\hbar}} = \frac{\hat{P}t}{m}e^{-i\frac{\hat{P}^2 t}{2m\hbar}}.$$

Then, substituting this commutator in the above relation

$$e^{i\frac{\hat{P}^2 t}{2m\hbar}}\hat{X}e^{-i\frac{\hat{P}^2 t}{2m\hbar}} = e^{i\frac{\hat{P}^2 t}{2m\hbar}}\frac{\hat{P}t}{m}e^{-i\frac{\hat{P}^2 t}{2m\hbar}} + e^{i\frac{\hat{P}^2 t}{2m\hbar}}e^{-i\frac{\hat{P}^2 t}{2m\hbar}}\hat{X} = \hat{X} + \frac{\hat{P}t}{m}.$$

Finally,

$$H_E = \frac{1}{2}m\omega^2\left[\hat{X} + \frac{\hat{P}t}{m}\right]^2.$$

# Problem III

The Hamiltonian is

$$H = \begin{cases} H_0, & t < 0 \\ H_0 + W(t), & 0 \leq t < \tau = \frac{4\pi}{\omega} \\ H_0, & t > \tau \end{cases}.$$

a) sa

$$|\psi_I(t)\rangle = U_0^\dagger(\tau, 0)|\psi_S(t)\rangle = e^{-i4\pi(\hat{N}+1/2)}|\psi_S(t)\rangle = e^{-i4\pi n}e^{-i2\pi}|\psi_S(t)\rangle = |\psi_S(t)\rangle.$$

b) The effective Hamiltonian is:

$$\begin{aligned} H_E &= U_0^\dagger W(t) U_0 \\ &= \frac{i\hbar\Omega}{2}\left[e^{i\omega(\hat{N}+1/2)t}(\hat{a}^2 e^{i2\omega t} - (\hat{a}^\dagger)^2 e^{-i2\omega t})e^{-i\omega(\hat{N}+1/2)t}\right] \\ &= \frac{i\hbar\Omega}{2}\left\{e^{i2\omega t}[e^{i\omega(\hat{N}+1/2)t}\hat{a}e^{-i\omega(\hat{N}+1/2)t}]^2 - e^{-i2\omega t}[e^{i\omega(\hat{N}+1/2)t}\hat{a}^\dagger e^{-i\omega(\hat{N}+1/2)t}]^2\right\} \\ H_E &= \frac{i\hbar\Omega}{2}\left\{\hat{a}^2 - (\hat{a}^\dagger)^2\right\}. \end{aligned}$$

c) We use the expression of the $\hat{a}$ operators in terms of $\hat{X}$ and $\hat{P}$:

$$\begin{aligned} \hat{H}_E &= \frac{i\hbar\Omega}{2}\left\{\frac{1}{2}\left(\frac{\hat{X}}{\sigma} + i\frac{\hat{P}\sigma}{\hbar}\right)^2 - \frac{1}{2}\left(\frac{\hat{X}}{\sigma} - i\frac{\hat{P}\sigma}{\hbar}\right)^2\right\} \\ &= \frac{i\hbar\Omega}{4}\left\{\frac{\hat{X}^2}{\sigma^2} + \frac{i}{\hbar}(\hat{X}\hat{P} + \hat{P}\hat{X}) - \frac{\hat{P}^2\sigma^2}{\hbar^2} - \left[\frac{\hat{X}^2}{\sigma^2} - \frac{i}{\hbar}(\hat{X}\hat{P} + \hat{P}\hat{X}) - \frac{\hat{P}^2\sigma^2}{\hbar^2}\right]\right\} \\ &= -\frac{\Omega}{2}(\hat{X}\hat{P} + \hat{P}\hat{X}) \\ \hat{H}_E &= -\frac{\Omega}{2}\{\hat{X}, \hat{P}\}, \quad \{\cdot\} = \text{anti-commutator}. \end{aligned}$$

d) gasgas

e) asgagasga

f) asgasg

g) asgag

h) asgasg

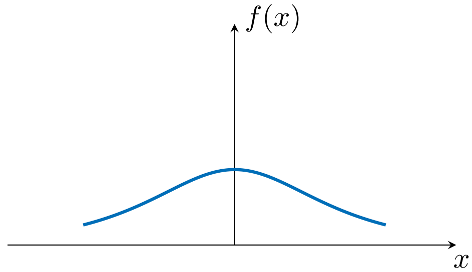
i) asgasgasgasg

j) asgasgasgasg

k) asgag

l) asgasg

m) asfas




## Problem IV

a) We plot the function $\text{sech}(x)$ to verify its parity. We can see that it is **even**.

This fact will facilitate us when computing $\Delta X$, as we must integrate over $|\phi(x)|^2$ which therefore, is also even. We then have,

$$\langle X \rangle = \int_{-\infty}^{\infty} x|\phi(x)|^2 \, dx = \frac{1}{2\beta} \int_{-\infty}^{\infty} x \, \text{sech}(x/\beta) \, dx = 0$$
$$\langle X^2 \rangle = \int_{-\infty}^{\infty} x^2|\phi(x)|^2 \, dx = \frac{1}{2\beta} \int_{-\infty}^{\infty} x^2 \text{sech}(x/\beta) \, dx = \frac{\beta^2}{2} \int_{-\infty}^{\infty} u^2 \text{sech}^2(u) \, du = \frac{\pi^2\beta^2}{12}.$$

The $X$ uncertainty is

$$\Delta X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \frac{\pi\beta}{2\sqrt{3}}.$$

Similarly, for the Fourier transform we have:

$$\langle P \rangle = \int_{-\infty}^{\infty} p|\hat{\phi}(p)|^2 \, dp = \frac{\pi\beta}{4\hbar} \int_{-\infty}^{\infty} p \, \text{sech}^2(\frac{\pi\beta p}{2\hbar}) \, dp = 0$$
$$\langle P^2 \rangle = \int_{-\infty}^{\infty} p^2|\hat{\phi}(p)|^2 \, dp = \frac{\pi\beta}{4\hbar} \int_{-\infty}^{\infty} p^2 \text{sech}^2(\frac{\pi\beta p}{2\hbar}) \, dp = \frac{2\hbar^2}{\pi^2\beta^2} \int_{-\infty}^{\infty} u^2 \text{sech}^2(u) \, du = \frac{\hbar^2}{\beta^2 3}.$$

Thus

$$\Delta P = \sqrt{\langle P^2 \rangle - \langle P \rangle^2} = \frac{\hbar}{\beta\sqrt{3}}.$$

The uncertainty product is

$$\Delta X \Delta P = \frac{\pi\beta}{2\sqrt{3}} \frac{\hbar}{\beta\sqrt{3}} = \frac{\hbar\pi}{6}.$$

b) The evolution in $\pi/2\omega$ gives a well-known quantity, a scaled Fourier transform of the wavefunction.

$$\Phi(x, \frac{\pi}{2\omega}) = U(\frac{\pi}{2\omega}, 0)\Phi(x, 0) = e^{-i\pi/4}\sqrt{\frac{\hbar}{\sigma^2}}\mathcal{F}\{\Phi(x, 0)\}\big|_{p=\hbar x/\sigma^2}$$

We can see that the function to be computed its Fourier transform is spatially shifted by $x_0$ so we could directly use the respective property of Fourier transform of a shifter function:

$$\mathcal{F}\{\Phi(x, 0)\} = \hat{\Phi}(p, 0) \Longrightarrow \mathcal{F}\{\Phi(x - x_0, 0)\} = e^{-ipx_0/\hbar}\hat{\Phi}(p, 0).$$

So,

$$\Phi(x, \frac{\pi}{2\omega}) = -e^{-i\pi/4}\sqrt{\frac{\hbar}{\sigma^2}} \left[e^{-ipx_0/\hbar}\hat{\Phi}(p, 0)\right]\Big|_{p=\hbar x/\sigma^2} = -\sqrt{\frac{\pi\beta}{4\sigma^2}} e^{-i\pi/4} e^{-i\frac{xx_0}{\sigma^2}} \, \text{sech}(\frac{\pi\beta x}{2\sigma^2}).$$

c) To maintain the width $\Delta X = \frac{\pi\beta}{2\sqrt{3}}$, we compute $\Delta X$ for $\Phi(0, \pi/2\omega)$ and equate it to the uncertainty at $t = 0$:

$$\left.\begin{aligned} &\langle X \rangle = 0 \\ &\langle X^2 \rangle = \frac{\pi\beta}{4\sigma^2}\int_{-\infty}^{\infty} x^2 \text{sech}^2(\frac{\pi\beta x}{2\sigma^2})\, dx = \frac{\sigma^4}{3\beta^2}. \end{aligned}\right\} \Delta X = \sqrt{\langle X^2 \rangle} = \frac{\sigma^2}{\sqrt{3}\beta}.$$

Equating it with the uncertainty of the wavefunction at $t = 0$:

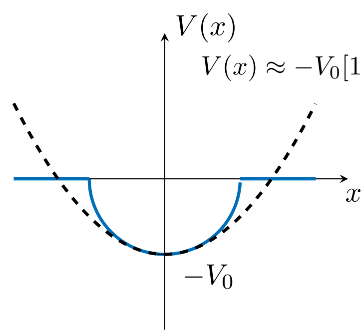
$$\frac{\pi\beta}{2\sqrt{3}} = \frac{\sigma^2}{\sqrt{3}\beta} \longrightarrow \beta = \sqrt{\frac{2\sigma^2}{\pi}}.$$

## Problem V

We handle the problem by approximating the potential given with its second-order Taylor expansion:

$$V(x) = -V_0 - \frac{V_0}{2b^2}x^2 + O(x^3) + \cdots = \frac{1}{2}m\omega^2 x^2.$$

The figure below represents the behavior of this approximation versus the real potential.




Comparing the quadratic term of the expansion with the QHO yields the following frequency:

$$\omega = \sqrt{\frac{V_0}{mb^2}}.$$

The energy levels in the QHO is:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \cdots.$$

So, in this case they will be shifted

$$E_n' = -V_0 + E_n = -V_0 + \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \cdots.$$

The ground and first excited state energy eigenvalues are:

$$E_0 = -V_0 + \frac{1}{2}\hbar\omega, \quad E_1 = -V_0 + \frac{3}{2}\hbar\omega.$$