Problem Set 10 Solutions

a. $|\varphi_u\rangle = \cos(\mu)|\varphi_x\rangle + \sin(\mu)|\varphi_y\rangle$

$\boxed{P(x) = \cos^2(\mu)}$

$\boxed{P(y) = \sin^2(\mu)}$

$|\varphi_+\rangle = -\frac{1}{\sqrt{2}}\left(|\varphi_x\rangle + i|\varphi_y\rangle\right)$

$|\varphi_-\rangle = -\frac{1}{\sqrt{2}}\left(|\varphi_x\rangle - i|\varphi_y\rangle\right)$

$\Big|-\quad\Big|+$

$$|\varphi_y\rangle = \frac{i}{\sqrt{2}}\left(|\varphi_+\rangle - |\varphi_-\rangle\right)$$

$$|\varphi_x\rangle = -\frac{1}{\sqrt{2}}\left(|\varphi_+\rangle + |\varphi_-\rangle\right)$$

$$|\varphi_u\rangle = -\frac{1}{\sqrt{2}}\cos(\mu)\left(|\varphi_+\rangle + |\varphi_-\rangle\right) + \frac{i}{\sqrt{2}}\sin(\mu)\left(|\varphi_+\rangle - |\varphi_-\rangle\right) =$$

$$= \left[-\frac{1}{\sqrt{2}}\cos(\mu) + \frac{i}{\sqrt{2}}\sin(\mu)\right]|\varphi_+\rangle + \left[-\frac{1}{\sqrt{2}}\cos(\mu) - \frac{i}{\sqrt{2}}\sin(\mu)\right]|\varphi_-\rangle$$

$\boxed{P(\sigma_+) =}\ \frac{1}{2}\cdot(\cos^2\mu + \sin^2\mu) \boxed{= \frac{1}{2}}$

$\boxed{P(\sigma_-) =}\ \frac{1}{2}\cdot(\cos^2\mu + \sin^2\mu) \boxed{= \frac{1}{2}}$

b. $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\varphi_x^L\rangle|\varphi_y^R\rangle - |\varphi_y^L\rangle|\varphi_x^R\rangle\right)$

Product states:

meas L is x, R is x : projector $P = |\varphi_x^L\rangle|\varphi_x^R\rangle\langle\varphi_x^R|\langle\varphi_x^L|$

$$\mathcal{P}(L=x, R=x) = \langle\psi|P|\psi\rangle = \frac{1}{2}\left(\langle\varphi_y^R|\langle\varphi_x^L| - \langle\varphi_x^R|\langle\varphi_y^L|\right)P\left(|\varphi_x^L\rangle|\varphi_y^R\rangle - |\varphi_y^L\rangle|\varphi_x^R\rangle\right) =$$

$$= \frac{1}{2}\cdot(0 + 0 + 0 + 0) \underline{= 0}$$

$$\mathcal{P}(L=y, R=y) = \langle\psi|\overbrace{\varphi_y^L\rangle|\varphi_y^R\rangle\langle\varphi_y^R|\langle\varphi_y^L|}^{\text{Projector}}\psi\rangle =$$

$$= \frac{1}{2}(0 + 0 + 0 + 0) \underline{= 0}$$

$$\mathcal{P}(L=y, R=x) = \langle\varphi| \overbrace{\varphi_y^L\rangle|\varphi_x^R\rangle\langle\varphi_x^R|\langle\varphi_y^L|}^{\text{Projector}}\varphi\rangle =$$

$$= \frac{1}{2}\cdot\left(+ 1\right) =$$

$$\underbrace{= \frac{1}{2}}$$

$$\mathcal{P}(L=x, R=y) = \langle\varphi|\varphi_x^L\rangle|\varphi_y^R\rangle\langle\varphi_y^R|\langle\varphi_x^L|\varphi\rangle =$$

$$\underbrace{= \frac{1}{2}}$$

<u>Single direction polarizations</u>

$$\mathcal{P}(L=x) = \langle\varphi|\overbrace{\varphi_x^L\rangle\langle\varphi_x^L|}^{P}\varphi\rangle = \frac{1}{2}\cdot(1) = \frac{1}{2}$$

$$\mathcal{P}(L=y) = \langle\varphi|\varphi_y^L\rangle\langle\varphi_y^L|\varphi\rangle = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

$$\mathcal{P}(R=x) = \langle\varphi|\varphi_x^R\rangle\langle\varphi_x^R|\varphi\rangle = \frac{1}{2}$$

$$\mathcal{P}(R=y) = \langle\varphi|\varphi_y^R\rangle\langle\varphi_y^R|\varphi\rangle = \frac{1}{2}$$

<u>c.</u> $$|\Psi\rangle = -\frac{1}{\sqrt{2}}\cdot\frac{i}{2}\left(|\varphi_+^L\rangle + |\varphi_-^L\rangle\right)\left(|\varphi_+^R\rangle - |\varphi_-^R\rangle\right) + \frac{1}{\sqrt{2}}\frac{i}{2}\left(|\varphi_+^R\rangle - |\varphi_-^R\rangle\right)\left(|\varphi_+^L\rangle + |\varphi_-^L\rangle\right) =$$

$$= \frac{i}{2\sqrt{2}}\cdot\Big(\cancel{-|\varphi_+^L\rangle|\varphi_+^R\rangle} + |\varphi_+^L\rangle|\varphi_-^R\rangle - |\varphi_-^L\rangle|\varphi_+^R\rangle + \cancel{|\varphi_-^L\rangle|\varphi_-^R\rangle} +$$

$$+ \cancel{|\varphi_+^L\rangle|\varphi_+^R\rangle} - |\varphi_-^L\rangle|\varphi_+^R\rangle + |\varphi_+^L\rangle|\varphi_-^R\rangle - \cancel{|\varphi_-^L\rangle|\varphi_-^R\rangle}\Big) =$$

$$= \frac{i}{\sqrt{2}}\left(|\varphi_+^L\rangle|\varphi_-^R\rangle - |\varphi_-^L\rangle|\varphi_+^R\rangle\right)$$

$$\mathcal{P}(L=+, R=+) = \mathcal{P}(L=-, R=-) = 0$$

$$\mathcal{P}(L=+, R=-) = \mathcal{P}(L=-, R=+) = \frac{1}{2}$$

$$\mathcal{P}(L=+) = \mathcal{P}(R=+) = \mathcal{P}(L=-) = \mathcal{P}(R=-) = \frac{1}{2}$$

The state $|\Psi\rangle$ is not a product state of the individual photon polarizabilities, so the two photons must be considered together as a single quantum system.

We see this above. The state of each photon is fully determined only once the state of the other photon has been measured.

Problem 2 $\quad I = 1 \quad S = \frac{1}{2} \quad \bar{J} = \bar{L} + \bar{S}, \quad \bar{F} = \bar{J} + \bar{I}$

a. 1S state: $L = 0$

$J \in \{\frac{1}{2}\} \qquad F \in \{\frac{3}{2}, \frac{1}{2}\}$

b. 2p state $\Rightarrow L = 1$

- $J = \frac{1}{2} \Rightarrow F \in \{\frac{3}{2}, \frac{1}{2}\}$
- $J = \frac{3}{2} \Rightarrow F \in \{\frac{5}{2}, \frac{3}{2}, \frac{1}{2}\}$

Problem 3

a. $|5^2S_{1/2}, F=1, M_F=1\rangle \Rightarrow |5^2P_{1/2}, F'=1, M_F'=1\rangle$

We need the TAM states decomposed in the TP basis

$5^2S_{1/2}$ states: $L=0, S=\frac{1}{2}, J=\frac{1}{2}, I=\frac{3}{2} \quad \vec{F} = \vec{J} + \vec{I}$

$$|\phi\rangle = |F=1, M_F=1\rangle = \sqrt{\frac{3}{4}}\,|I=\frac{3}{2}, m_I=\frac{3}{2}\rangle|J=\frac{1}{2}, m_J=-\frac{1}{2}\rangle -$$
$$-\sqrt{\frac{1}{4}}\,|I=\frac{3}{2}, m_I=\frac{1}{2}\rangle|J=\frac{1}{2}, m_J=\frac{1}{2}\rangle$$

Need $\vec{J} = \vec{L} + \vec{S}$

$$\Rightarrow \left.\begin{aligned} |J=\tfrac{1}{2}, m_J=\tfrac{1}{2}\rangle &= |L=0, m_L=0\rangle|S=\tfrac{1}{2}, m_S=\tfrac{1}{2}\rangle \\ |J=\tfrac{1}{2}, m_J=-\tfrac{1}{2}\rangle &= |L=0, m_L=0\rangle|S=\tfrac{1}{2}, m_S=-\tfrac{1}{2}\rangle \end{aligned}\right\} \text{Eqs (1)}$$

Use order $\underline{L, S, I} \Rightarrow$

$$|\phi\rangle = |F=1, M_F=1\rangle = \sqrt{\frac{3}{4}}\,|L=0, m_L=0\rangle|S=\frac{1}{2}, m_S=-\frac{1}{2}\rangle|I=\frac{3}{2}, m_I=\frac{3}{2}\rangle -$$
$$-\sqrt{\frac{1}{4}}\,|L=0, m_L=0\rangle|S=\frac{1}{2}, m_S=\frac{1}{2}\rangle|I=\frac{3}{2}, m_I=\frac{1}{2}\rangle =$$
$$= \underline{\sqrt{\frac{3}{4}}\,|0,0\rangle|\frac{1}{2}, -\frac{1}{2}\rangle|\frac{3}{2}, \frac{3}{2}\rangle} - \underline{\sqrt{\frac{1}{4}}\,|0,0\rangle|\frac{1}{2}, \frac{1}{2}\rangle|\frac{3}{2}, \frac{1}{2}\rangle}$$

$5^2P_{1/2}$ states: $L=1$, $S=\frac{1}{2}$, $J=\frac{1}{2}$, $I=\frac{3}{2}$ $\quad \vec{F}=\vec{J}+\vec{I}$

$$|\phi'\rangle = |F=1, M_F'=1\rangle = \sqrt{\frac{3}{4}}\,\underbrace{|\tfrac{3}{2},\tfrac{3}{2}\rangle}_{I}\underbrace{|\tfrac{1}{2},-\tfrac{1}{2}\rangle}_{J} - \sqrt{\frac{1}{4}}\,\underbrace{|\tfrac{3}{2},\tfrac{1}{2}\rangle}_{I}\underbrace{|\tfrac{1}{2},\tfrac{1}{2}\rangle}_{J}$$

Need $\vec{J}=\vec{L}+\vec{S}$ $\quad L=1,\ S=\frac{1}{2}$

$$\left.\begin{aligned}
|J=\tfrac{1}{2}, m_J=-\tfrac{1}{2}\rangle &= \sqrt{\tfrac{1}{3}}\,|L=1, m_L=0\rangle|S=\tfrac{1}{2}, m_S=-\tfrac{1}{2}\rangle \\
&- \sqrt{\tfrac{2}{3}}\,|L=1, m_L=-1\rangle|S=\tfrac{1}{2}, m_L=\tfrac{1}{2}\rangle \\
|J=\tfrac{1}{2}, m_J=\tfrac{1}{2}\rangle &= \sqrt{\tfrac{2}{3}}\,|L=1, m_L=1\rangle|S=\tfrac{1}{2}, m_S=-\tfrac{1}{2}\rangle \\
&- \sqrt{\tfrac{1}{3}}\,|L=1, m_L=0\rangle|S=\tfrac{1}{2}, m_S=\tfrac{1}{2}\rangle
\end{aligned}\right\}\ \text{Eqs (2)}$$

$$\Rightarrow |\phi'\rangle = \sqrt{\tfrac{1}{4}}\,|1,0\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{3}{2},\tfrac{3}{2}\rangle - \sqrt{\tfrac{1}{2}}\,|1,-1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{3}{2},\tfrac{3}{2}\rangle -$$
$$- \sqrt{\tfrac{1}{6}}\,|1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{3}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{12}}\,|1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{3}{2},\tfrac{1}{2}\rangle$$

Finally, the transition strength is $T=|\langle\phi|A|\phi'\rangle|^2$

$$T=\left(\sqrt{\tfrac{3}{4}}\cdot\sqrt{\tfrac{1}{4}}-\sqrt{\tfrac{1}{4}}\cdot\sqrt{\tfrac{1}{12}}\right)^2\cdot|A_{1000}|^2 =$$
$$=\left(\sqrt{\tfrac{3}{16}}-\sqrt{\tfrac{1}{48}}\right)^2|A_{1000}|^2 =$$
$$=\left(\frac{3-1}{\sqrt{48}}\right)^2|A_{1000}|^2 \boxed{=\frac{1}{12}|A_{1000}|^2}$$

b. $|\phi\rangle = |5^2S_{1/2}, F=1, M_F=0\rangle$

$5^2S_{1/2}$ states: $L=0$, $S=\frac{1}{2}$, $J=\frac{1}{2}$, $I=\frac{3}{2}$ $\quad F=\vec{J}+\vec{I}$

$$|\phi\rangle = |F=1, M_F=0\rangle = \sqrt{\tfrac{1}{2}}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{2}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

For $\vec{J}=\vec{L}+\vec{S}$, use Eqs (1)

$$= \sqrt{\tfrac{1}{2}}\,|0,0\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{3}{2},\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{2}}\,|0,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{3}{2},-\tfrac{1}{2}\rangle$$

$\underline{5\,^2P_{1/2} \text{ states}}$: $L=1$, $S=\frac{1}{2}$, $J=\frac{1}{2}$, $I=\frac{3}{2}$ $\quad \vec{F}=\vec{J}+\vec{I}$

$$|\phi'\rangle = |F=1, M_F=0\rangle = \sqrt{\frac{1}{2}}\,|I=\tfrac{3}{2}, m_I=\tfrac{1}{2}\rangle|J=\tfrac{1}{2}, m_J=-\tfrac{1}{2}\rangle -$$

$$-\sqrt{\frac{1}{2}}\,|I=\tfrac{3}{2}, m_I=-\tfrac{1}{2}\rangle|J=\tfrac{1}{2}, m_J=\tfrac{1}{2}\rangle$$

Need $\vec{J}=\vec{L}+\vec{S}$, use Eqs (2)

$$|\phi'\rangle = \sqrt{\frac{1}{6}}\,|1,0\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{3}{2},\tfrac{1}{2}\rangle - \sqrt{\frac{1}{3}}\,|1,-1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{3}{2},\tfrac{1}{2}\rangle -$$

$$-\sqrt{\frac{1}{3}}\,|1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{3}{2},-\tfrac{1}{2}\rangle + \sqrt{\frac{1}{6}}\,|1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{3}{2},-\tfrac{1}{2}\rangle$$

$$T = \left|\sqrt{\frac{1}{12}}\,A_{1000} - \sqrt{\frac{1}{12}}\,A_{1000}\right|^2 =$$

$$=0$$

$\underline{c.}$ Already have $|\phi\rangle$ from $\underline{a.}$

$$|\phi\rangle = \underline{\sqrt{\frac{3}{4}}\,|0,0\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{3}{2},\tfrac{3}{2}\rangle} - \underline{\sqrt{\frac{1}{4}}\,|0,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{3}{2},\tfrac{1}{2}\rangle}$$

$$|\phi'\rangle = |5\,^2P_{1/2}, F'=2, M_F'=1\rangle$$

$\underline{5\,^2P_{1/2} \text{ states}}$: $L=1$, $S=\frac{1}{2}$, $J=\frac{1}{2}$, $I=\frac{3}{2}$ $\quad \vec{F}=\vec{J}+\vec{I}$

$$|\phi'\rangle = |F'=2, M_F'=1\rangle = \sqrt{\frac{1}{4}}\,|\tfrac{3}{2},\tfrac{3}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle + \sqrt{\frac{3}{4}}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

Use Eqs (2):

$$|\phi'\rangle = \underline{\sqrt{\frac{1}{12}}\,|1,0\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{3}{2},\tfrac{3}{2}\rangle} - \sqrt{\frac{1}{6}}\,|1,-1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{3}{2},\tfrac{3}{2}\rangle$$

$$+\sqrt{\frac{1}{2}}\,|1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{3}{2},\tfrac{1}{2}\rangle - \underline{\sqrt{\frac{1}{4}}\,|1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{3}{2},\tfrac{1}{2}\rangle}$$

$$T = \left|\sqrt{\frac{\cancel{3}}{4}}\cdot\sqrt{\frac{1}{\cancel{12}_4}}\,A_{1000} + \sqrt{\frac{1}{4}}\cdot\sqrt{\frac{1}{4}}\,A_{1000}\right|^2 =$$

$$=\frac{1}{4}\,|A_{1000}|^2$$

d. We have $|\phi\rangle$ from b:

$$|\phi\rangle = \sqrt{\tfrac{1}{2}}\,|0,0\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{1}{2}}\,|0,0\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle$$

$$|\phi'\rangle = |5\,{}^2P_{1/2},\ F'=2,\ M_F'=0\rangle$$

$5\,{}^2P_{1/2}$ states: $L=1,\ S=\frac{1}{2},\ J=\frac{1}{2},\ I=\frac{3}{2}$ $\quad \vec{F} = \vec{J} + \vec{I}$

$$|\phi'\rangle = |F'=2, M_F'=0\rangle = \sqrt{\tfrac{1}{2}}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{2}}\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

Use Eqs (2)

$$|\phi'\rangle = \sqrt{\tfrac{1}{6}}\,|1,0\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{1}{3}}\,|1,-1\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle +$$
$$+ \sqrt{\tfrac{1}{3}}\,|1,1\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{1}{6}}\,|1,0\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle$$

$$T = \left(\sqrt{\tfrac{1}{12}} + \sqrt{\tfrac{1}{12}}\right)^2 |A_{1000}|^2 =$$
$$= \tfrac{1}{3}\,|A_{1000}|^2$$