# Pset 2 solutions

$\boxed{\mathrm{I}}$ $\quad H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha\,\delta(x) \quad , \alpha > 0$

(a.1) $\quad H\,\varphi(x) = E\,\varphi(x)$

$$\int_{-\varepsilon}^{+\varepsilon}\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha\,\delta(x)\right]\varphi(x)\,dx = E\int_{-\varepsilon}^{\varepsilon}\varphi(x)\,dx$$

$$-\frac{\hbar^2}{2m}\left.\frac{d\,\varphi(x)}{dx}\right|_{-\varepsilon}^{+\varepsilon} - \alpha\,\varphi(0) = E\int_{-\varepsilon}^{\varepsilon}\varphi(x)\,dx$$

$$\left.\frac{d\,\varphi(x)}{dx}\right|_{-\varepsilon}^{+\varepsilon} = \frac{2m}{\hbar^2}\left[-\alpha\,\varphi(0) - E\int_{-\varepsilon}^{\varepsilon}\varphi(x)\,dx\right]$$

(a.2) $\quad \varepsilon \to 0 \Rightarrow \int_{-\varepsilon}^{\varepsilon}\varphi(x)\,dx = \int_{-\varepsilon}^{\varepsilon}\varphi(0)\,dx = \varphi(0)\cdot 2\varepsilon$

per hint, define discontinuity as

$$\Delta = \lim_{\varepsilon\to 0}\left(\left.\frac{d\,\varphi(x)}{dx}\right|_{\varepsilon} - \left.\frac{d\,\varphi(x)}{dx}\right|_{-\varepsilon}\right) =$$

$$= \lim_{\varepsilon\to 0}\left\{\frac{2m}{\hbar^2}\left[-\alpha\,\varphi(0) - 2\varepsilon E\,\varphi(0)\right]\right\} =$$

$$\boxed{\Delta = -\frac{2m\alpha}{\hbar^2}\,\varphi(0)}$$

(b.1) $\quad E < 0$ - bound state

$x < 0 \quad \varphi(x) = A_1 e^{\rho x} + A_1' e^{-\rho x}$

$x > 0 \quad \varphi(x) = A_2 e^{\rho x} + A_2' e^{-\rho x}$

$H\,\varphi(x) = E\,\varphi(x)$

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2} - \alpha\,\delta(x)\,\varphi(x) = E\,\varphi(x)$$

$$\left.\begin{aligned} x<0 \quad & \frac{d^2\varphi(x)}{dx^2} = A_1\,\rho^2 e^{\rho x} + A_1'\,\rho^2 e^{-\rho x} \\ x>0 \quad & \frac{d^2\varphi(x)}{dx^2} = A_2\,\rho^2 e^{\rho x} + A_2'\,\rho^2 e^{-\rho x} \end{aligned}\right\} \Rightarrow \frac{d^2\varphi(x)}{dx^2} = \rho^2\varphi(x)$$

for all $x \neq 0$

$$\left[-\frac{\hbar^2}{2m}\rho^2 - \alpha\,\delta(x)\right]\varphi(x) = E\,\varphi(x)$$

Evaluate at any $x$ :

$$-\frac{\hbar^2}{2m}\rho^2 = E \Rightarrow \boxed{\rho = \sqrt{\frac{2mE}{\hbar^2}}}$$

(b.2) Find $E$ :

- $\varphi(x)$ is square-integrable $\Rightarrow$ $\varphi(x) \to 0$ as $x \to \pm\infty$

$\Rightarrow A_1' = A_2 = 0$

- $\varphi(x)$ is continuous everywhere

$$\varphi(0)\Big|_{x<0} = \varphi(0)\Big|_{x>0} \Rightarrow A_1 = A_2' = A$$

- Normalize : $\int_{-\infty}^{+\infty} |\varphi(x)|^2 dx = 1$ $\quad \varphi(x) = A\cdot\rho^2 e^{-\rho x}$

for $x \neq 0$

$$|A|^2 \cdot \rho^2 \int_{-\infty}^{+\infty} e^{-2\cdot\rho\,|x|}\,dx = 1$$

$$|A|^2 \cdot \rho^2 \cdot 2\int_0^{\infty} e^{-2\rho x}\,dx = 1$$

$u = -2\rho x \qquad du = -2\rho\,dx$

$$|A|^2 \cdot \rho^{\cancel{x}} \cdot \cancel{2} \cdot \int_0^\infty e^{-u}\, du \;\frac{1}{-\cancel{2}\cancel{\rho}} = 1$$

$$-|A|^2 \cdot \rho \cdot \left[-e^{-u}\right|_0^\infty = 1$$

$$+|A|^2 \cdot \rho = 1$$

$A = \sqrt{\rho}$ choosing $A>0$ solution

So $\boxed{\varphi(x) = \sqrt{\rho}\, e^{-\rho|x|}}$

Now from part (a): $\lim_{\varepsilon \to 0} \left( \frac{d\varphi(x)}{dx} \right|_{-\varepsilon}^{\varepsilon} = -\frac{2m\alpha}{\hbar^2}\varphi(0)$

$$-\cancel{2}\cdot\rho\,\cancel{\sqrt{\rho}} \cdot 1 = -\frac{\cancel{2}m\alpha}{\hbar^2}\cancel{\sqrt{\rho}}$$

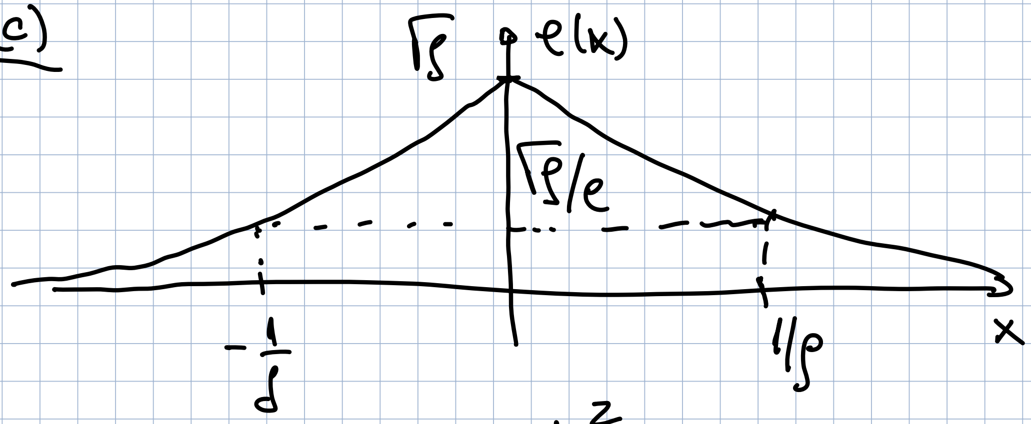
$$\rho = \frac{m\alpha}{\hbar^2}$$

$$\rho = \sqrt{\frac{2mE}{\hbar}} \Rightarrow -\frac{\cancel{m^2}\alpha^2}{\hbar\,\cancel{\hbar^3}} = \frac{2\cancel{m}E}{\cancel{\hbar}}$$

$\boxed{E = -\frac{m\alpha^2}{2\hbar^3}}$ $<0$ solution

$$\boxed{\varphi(x) = \sqrt{\rho}\, e^{-\rho|x|} \quad \text{where} \quad \rho = \frac{m\alpha}{\hbar^2}}$$

(c)




$$\Delta x = \frac{2}{\rho} = \frac{2\hbar^2}{m\alpha}$$

(d) $\varphi(x) = \sqrt{\rho}\, e^{-\rho|x|}$

$$\bar{\varphi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-\frac{ipx}{\hbar}} \sqrt{\rho}\, e^{-\rho|x|}\, dx =$$

$$= \frac{\sqrt{\rho}}{\sqrt{2\pi\hbar}} \left( \int_{-\infty}^{0} e^{-\frac{ipx}{\hbar}} e^{+\rho x}\, dx + \int_{0}^{\infty} e^{-\frac{ipx}{\hbar}} e^{-\rho x}\, dx \right) =$$

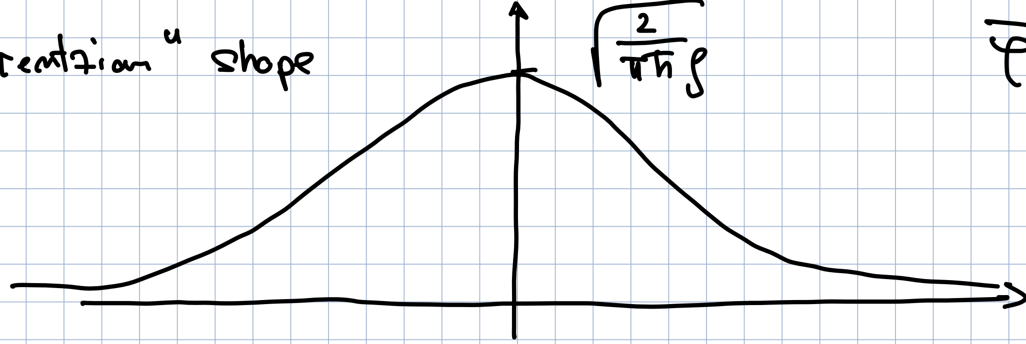
$$= \sqrt{\frac{\rho}{2\pi\hbar}} \left( \frac{1}{\rho - \frac{ip}{\hbar}} e^{\left(\rho - \frac{ip}{\hbar}\right)x} \Big|_{-\infty}^{0} - \frac{1}{\rho + \frac{ip}{\hbar}} e^{-\left(\rho + \frac{ip}{\hbar}\right)x} \Big|_{0}^{\infty} \right) =$$

$$= \sqrt{\frac{\rho}{2\pi\hbar}} \left( \frac{1}{\rho - \frac{ip}{\hbar}} + \frac{1}{\rho + \frac{ip}{\hbar}} \right) =$$

$$= \sqrt{\frac{\rho}{2\pi\hbar}}\; \frac{\rho + \cancel{\frac{ip}{\hbar}} + \rho - \cancel{\frac{ip}{\hbar}}}{\rho^2 + \frac{p^2}{\hbar^2}} =$$

$$\boxed{\bar{\varphi}(p) = \sqrt{\frac{2\rho}{\pi\hbar}} \cdot \frac{\rho}{\rho^2 + \frac{p^2}{\hbar^2}}}$$

"Lorentzian" shape




$\tilde{\varphi}(0) = \sqrt{\frac{2g}{\pi\hbar}} \cdot \frac{1}{g} = \sqrt{\frac{2}{\pi\hbar g}}$

At half maximum: $g^2 = \frac{p_{1/2}^2}{\hbar^2} \Rightarrow \boxed{p_{1/2} = \hbar g}$

Full width at half maximum: $\Delta p = 2\hbar g$

$\Delta x \cdot \Delta p = \frac{2}{\cancel{g}} \cdot 2\hbar \cancel{g} = 4\hbar \sim$ few $\hbar$ depending on how we define width

$\boxed{\text{II}}$ $U(m,n) = |\varphi_m\rangle\langle\varphi_n|$ $\qquad H|\varphi_n\rangle = E_n|\varphi_n\rangle$

a. $U^{+}(m,n) = \left(|\varphi_m\rangle\langle\varphi_n|\right)^{+} =$

$= |\varphi_n\rangle\langle\varphi_m|$

b. $[H, U(m,n)] = H\,U(m,n) - U(m,n)\,H =$

$= H|\varphi_m\rangle\langle\varphi_n| - |\varphi_m\rangle\langle\varphi_n|H$

$H|\varphi_n\rangle = E_n|\varphi_n\rangle \;\Big|^{+}$

$\langle\varphi_n|H^{+} = E_n^{*}\langle\varphi_n|$

H is Hermitian $\Rightarrow H^{+} = H \qquad E_n^{*} = E$

$\langle\varphi_n|H = E_n\langle\varphi_n|$

$[H, U(m,n)] = E_m|\varphi_m\rangle\langle\varphi_n| - E_n|\varphi_m\rangle\langle\varphi_n\rangle =$

$= (E_m - E_n)\,U(m,n)$

c. $U(m,n)\,U^{+}(p,q) = |\varphi_m\rangle\langle\varphi_n|\left(|\varphi_p\rangle\langle\varphi_q|\right)^{+} =$

$= |\varphi_m\rangle\underbrace{\langle\varphi_n|\varphi_q\rangle}_{\delta_{n,q}}\langle\varphi_p| =$

$= \delta_{n,q}|\varphi_m\rangle\langle\varphi_p| = \delta_{n,q}\,U(m,p)$

d. $\mathrm{Tr}\{U(m,n)\} \equiv \sum_p \langle\varphi_p|\varphi_m\rangle\langle\varphi_n|\varphi_p\rangle =$

$= \sum_p \delta_{p,m}\,\delta_{n,p} =$

$= \delta_{m,n}$

e. $A_{mn} = \langle \varphi_m | A | \varphi_n \rangle$

$$\sum_{m,n} A_{mn}\, u(m,n) = \sum_{m,n} \underbrace{\langle \varphi_m | A | \varphi_n \rangle}_{\text{scalar so can move around}} | \varphi_m \rangle \langle \varphi_n |$$

$$= \sum_{m,n} | \varphi_m \rangle \langle \varphi_m | A | \varphi_n \rangle \langle \varphi_n | =$$

use $\sum_m | \varphi_m \rangle \langle \varphi_m | = \mathbb{I}$ closure relation

$$= \mathbb{I} \cdot A \cdot \mathbb{I} =$$

$$= A$$

f. $$\mathrm{Tr}\{A\, u^\dagger(p,q)\} = \sum_m \langle \varphi_m | A | \varphi_q \rangle \langle \varphi_p | \varphi_m \rangle =$$

$$= \sum_m A_{mq} \cdot \delta_{p,m} =$$

$$= A_{pq}$$