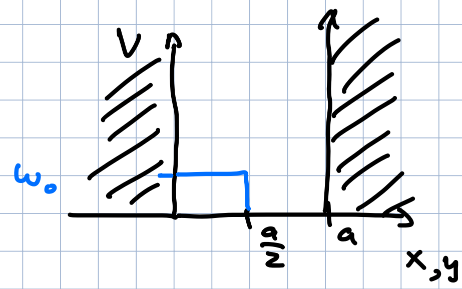
Problem Set 11 Solutions

Problem 1 CT H_{XI} E2



2D Inf. square well

a $\varphi_{n_x,n_y}(x,y) = \frac{2}{a}\sin\left(\frac{n_x \pi x}{a}\right)\sin\left(\frac{n_y \pi y}{a}\right)$

$E_{n_x+n_y} = \left(n_x^2 + n_y^2\right)\frac{\hbar^2\pi^2}{2ma^2}$

ground state is non-degenerate : $E_2^0 = \frac{\hbar^2\pi^2}{ma^2}$

To first order in w_0:

$E_2 = E_2^0 + w_0 \langle \varphi_{1,1} | \hat{W} | \varphi_{1,1}\rangle$

$\langle \varphi_{1,1} | \hat{W} | \varphi_{1,1}\rangle = \int_{x=0}^{\frac{a}{2}}\int_{y=0}^{\frac{a}{2}} dx\,dy \; \frac{4}{a^2}\sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{\pi y}{a}\right) =$

$= \frac{4}{a^2}\cdot\left[\int_{x=0}^{\frac{a}{2}} dx \, \sin^2\left(\frac{\pi x}{a}\right)\right]^2 =$ $\quad u = \frac{\pi x}{a}$, $dx = du \cdot \frac{a}{\pi}$

$= \frac{4}{\cancel{a^2}}\cdot\frac{\cancel{a^2}}{\pi^2}\int_{u=0}^{\frac{\pi}{2}} \sin^2(u)\,du =$

$= \frac{4}{\pi^2}\cdot\left(\frac{\pi}{4}\right)^2 = \frac{1}{4}$

$E_2 = \frac{\hbar^2\pi^2}{ma^2} + \frac{w_0}{4}$

b. $E_3^0 = (1+4)\frac{\hbar^2\pi^2}{2ma^2} = \frac{5\hbar^2\pi^2}{2ma^2}$

two degeneracies : $\varphi_{2,1}$ and $\varphi_{1,2}$

$\Rightarrow$ we need to build subspace matrix $W^{(3)}$, with elements

$$W^{(3)} = \begin{pmatrix} \langle \varphi_{12}|W|\varphi_{12}\rangle & \langle \varphi_{12}|W|\varphi_{21}\rangle \\ \langle \varphi_{21}|W|\varphi_{12}\rangle & \langle \varphi_{21}|W|\varphi_{21}\rangle \end{pmatrix}$$

$$\langle \varphi_{12} | W | \varphi_{12} \rangle = W_0 \int_{x=0}^{\frac{a}{2}} \int_{y=0}^{\frac{a}{2}} dx\, dy\ \frac{4}{a^2} \sin^2\left(\frac{\pi x}{a}\right) \sin^2\left(\frac{2\pi y}{a}\right) =$$

$$= W_0 \cdot \frac{4}{a^2} \cdot \int_{x=0}^{a/2} \sin^2\left(\frac{\pi x}{a}\right) dx \int_{y=0}^{a/2} \sin^2\left(\frac{2\pi y}{a}\right) dy$$

$$u = \frac{2\pi y}{a} \qquad du = \frac{2\pi}{a} dy$$

$$= W_0 \cdot \frac{\cancel{4}}{\cancel{a^2}} \cdot \frac{\cancel{a}}{\cancel{4}} \cdot \int_{u=0}^{\pi} \sin^2(u)\, du \cdot \frac{\cancel{a}}{2\pi} =$$

$$= W_0 \ \frac{1}{2\cancel{\pi}} \cdot \frac{\cancel{\pi}}{2} = \frac{W_0}{4}$$

$\langle \varphi_{21} | W | \varphi_{21} \rangle = \frac{W_0}{4}$ since we can swap variables x and y

$$\langle \varphi_{12} | W | \varphi_{21} \rangle = W_0 \int_{x=0}^{\frac{a}{2}} \int_{y=0}^{\frac{a}{2}} \frac{4}{a^2} \cdot \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) \sin\left(\frac{2\pi y}{a}\right) dx\, dy$$

$$= W_0 \cdot \frac{4}{a^2} \cdot \left[\int_{x=0}^{\frac{a}{2}} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) dx \right]^2 =$$

$$= \frac{4W_0}{a^2} \left\{ \int_{x=0}^{a/2} \frac{1}{2} \cdot \left[\cos\left(\frac{\pi x}{a}\right) - \cos\left(\frac{3\pi x}{a}\right) \right] \right\}^2 =$$

$$= \frac{W_0}{a^2} \left\{ \left[\frac{a}{\pi} \sin\left(\frac{\pi x}{a}\right) - \frac{a}{3\pi} \sin\left(\frac{3\pi x}{a}\right) \right] \Big|_0^{\frac{a}{2}} \right\}^2 =$$

$$= \frac{W_0}{\cancel{a^2}} \cdot \frac{\cancel{a^2}}{\pi^2} \cdot \left[1 \cdot 1 - \frac{1}{3} \cdot (-1) \right]^2 =$$

$$= \frac{W_0}{\pi^2} \cdot \frac{16}{9} = \frac{W_0\, 16}{9\pi^2}$$

$$\Rightarrow W^{(3)} = \frac{W_0}{4} \cdot \begin{pmatrix} 1 & \frac{64}{9\pi^2} \\ \frac{64}{9\pi^2} & 1 \end{pmatrix}$$

eigenvalues are $\det(W^{(3)} - \lambda I) = 0$

$$\det \begin{pmatrix} 1-\lambda & \frac{64}{9\pi^2} \\ \frac{64}{9\pi^2} & 1-\lambda \end{pmatrix} = 0 \qquad (1-\lambda)^2 - \left(\frac{64}{9\pi^2}\right)^2 = 0$$

$$\left(1 - \lambda - \frac{64}{9\pi^2}\right)\left(1 - \lambda + \frac{64}{9\pi^2}\right) = 0$$

$$\lambda_1 = \frac{w_0}{4}\left(1 - \frac{64}{9\pi^2}\right) \quad \text{w/ eigenvector } \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\lambda_2 = \frac{w_0}{4}\left(1 + \frac{64}{9\pi^2}\right) \quad \text{w/ eigenvector } \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

<u>to 1st order:</u> $E_{3,1} = \dfrac{5\hbar^2\pi^2}{2ma^2} + \dfrac{w_0}{4}\left(1 - \dfrac{64}{9\pi^2}\right)$

w/ state $\psi_{3,1}(x,y) = \dfrac{1}{\sqrt{2}}\left[\varphi_{12}(x,y) - \varphi_{21}(x,y)\right] =$

$$= \frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right) - \sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)\right]$$

$$E_{3,2} = \frac{5\hbar^2\pi^2}{2ma^2} + \frac{w_0}{4}\left(1 + \frac{64}{9\pi^2}\right)$$

w/ state $\psi_{3,2}(x,y) = \dfrac{\sqrt{2}}{a}\left[\sin\left(\dfrac{\pi x}{a}\right)\sin\left(\dfrac{2\pi y}{a}\right) + \sin\left(\dfrac{2\pi x}{a}\right)\sin\left(\dfrac{\pi y}{a}\right)\right]$

<u>Problem II</u> CT x1 5

$H_0 = aJ_z + \dfrac{b}{\hbar}J_z^2 \qquad a,b > 0$ $\qquad\qquad J_z|\psi\rangle = m_z\hbar|\psi\rangle$

<u>a.</u> $H_0|\psi\rangle = \left(aJ_z + \dfrac{b}{\hbar}J_z^2\right)|\psi\rangle = \left(a m_z\hbar + \dfrac{b}{\cancel{\hbar}} m_z^2\hbar^{\cancel{2}}\right)|\psi\rangle =$

$$= \left(a m_z + b m_z^2\right)\hbar|\psi\rangle$$

energy levels are eigenvalues of $H_0 \Rightarrow E_+ = (a+b)\hbar$

$E_0 = 0$

$E_- = (-a+b)\hbar$

<u>$E_- = E_0$ when $b = a$</u>

<u>b.</u> $\bar{M} = \gamma\bar{J} \qquad \gamma < 0$

$W = w_0 J_u \qquad J_u = J_z\cos\theta + J_x\sin\theta\cos\varphi + J_y\sin\theta\sin\varphi$

In the z - representation:

$$J_z \to \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix} \qquad J_x \to \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \qquad J_y \to \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}$$

$$W \to \omega_0 \cdot J_u = \hbar\omega_0 \begin{pmatrix} \cos\theta & \frac{1}{\sqrt{2}}\sin\theta(1-i\sin\varphi) & 0 \\ \frac{1}{\sqrt{2}}\sin\theta(1+i\sin\varphi) & 0 & \frac{1}{\sqrt{2}}\sin\theta(1-i\sin\varphi) \\ 0 & \frac{1}{\sqrt{2}}\sin\theta(1+i\sin\varphi) & -\cos\theta \end{pmatrix}$$

c. $b=a$, $\hat{u}=(1,0,0)$, $\omega_0 \ll a$

$\theta=\frac{\pi}{2}$ $\quad \varphi=0$

$$H_0 \to \hbar \cdot \begin{pmatrix} 2a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad W \to \hbar\omega_0 \cdot \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$\underline{E_+ = 2a\hbar}$ non-degenerate

$$\Rightarrow E'_+ = E_+ + \langle + | W | + \rangle =$$
$$= 2a\hbar + 0 = 2a\hbar$$

$\underline{E_0 = E_- = 0}$ degenerate

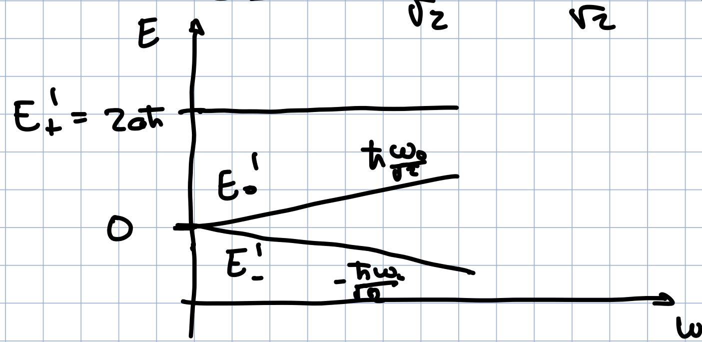
$$W' \to \frac{\hbar\omega_0}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

eigenvalues and states: $+\frac{\hbar\omega_0}{\sqrt{2}}$ w/ $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$-\frac{\hbar\omega_0}{\sqrt{2}}$ w/ $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

so: $E'_0 = 0 + \frac{\hbar\omega_0}{\sqrt{2}} = \frac{\hbar\omega_0}{\sqrt{2}}$ w/ state $\frac{1}{\sqrt{2}}(|0\rangle + |-\rangle)$

$E'_- = 0 - \frac{\hbar\omega_0}{\sqrt{2}} = -\frac{\hbar\omega_0}{\sqrt{2}}$ w/ state $\frac{1}{\sqrt{2}}(|0\rangle - |-\rangle)$



d. $b = 2a$

$$H_0 \to \hbar \begin{pmatrix} 3a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}$$

ground state is $|0\rangle$ - non-degenerate perturbation theory

$$|\psi_0\rangle = |0\rangle + \frac{\langle + | W | 0 \rangle}{E_0^0 - E_+^0} |+\rangle + \frac{\langle - | W | 0 \rangle}{E_0^0 - E_-^0} |-\rangle$$

$$= |0\rangle - \frac{\frac{\cancel{\hbar}\omega_0}{\sqrt{2}} \sin\theta (1 - i\sin\varphi)}{3a\cancel{\hbar}} |+\rangle - \frac{\frac{\cancel{\hbar}\omega_0}{\sqrt{2}} \sin\theta (1 + i\sin\varphi)}{a\cancel{\hbar}} |-\rangle$$

$$= |0\rangle - \frac{\omega_0}{3\sqrt{2}a} \sin\theta (1 - i\sin\varphi) |+\rangle - \frac{\omega_0}{\sqrt{2}a} \sin\theta (1 + i\sin\varphi) |-\rangle$$

Problem III

$$V_S(r) = \begin{cases} -\frac{q^2}{r} & \text{for } r > b \\ -\frac{q^2}{b} & \text{for } r < b \end{cases}$$

$$W = \begin{cases} 0 & r > b \\ 2E_I a_0 \left(\frac{1}{r} - \frac{1}{b}\right) & r \le b \end{cases}$$

$b/a_0 = 10^{-5}$

a. We need matrix elements, W only depends on r

$$\langle n'\ell'm' | W | n\ell m \rangle = \delta_{\ell'\ell} \delta_{m'm} \int_0^\infty r^2 R^*_{n'l'}(r) R_{nl}(r) W(r)\, dr$$

$$= \delta_{\ell'\ell} \delta_{m'm} \cdot \int_0^b 2E_I a_0 \left(r - \frac{r^2}{b}\right) \cdot R^*_{n'\ell'}(r) R_{n\ell}(r)$$

$$\langle 100 | W | 100 \rangle = \int_0^b 2E_I \cancel{a_0} \left(r - \frac{r^2}{b}\right) \cdot \frac{4}{a_0^{\cancel{3}2}} e^{-\frac{2r}{a_0}} dr$$

$$= \frac{8E_I}{a_0^2} \int_0^b \left(r - \frac{r^2}{b}\right) e^{-\frac{2r}{a_0}} dr$$

we can expand $e^{-\frac{2r}{a_0}} \simeq 1 - \underbrace{\frac{2r}{a_0}}_{\substack{\text{small} \\ \simeq 0}} + \underbrace{\frac{1}{2} \cdot \left(\frac{2r}{a_0}\right)^2}_{\substack{\text{smaller} \\ \simeq 0}} \dots$$

$$= \frac{8E_I}{a_0^2} \int_0^b \left(r - \frac{r^2}{b}\right) dr =$$

$$= \frac{8E_I}{a_0^2} \cdot \left(\frac{r^2}{2} - \frac{r^3}{3b}\right|_0^b =$$

$$= \frac{8E_I}{a_0^2} \left(\frac{b^2}{2} - \frac{b^2}{3}\right) \boxed{= \frac{4E_I}{3}\left(\frac{b}{a}\right)^2}$$

$$\langle 200|W|200\rangle = \int_0^b \cancel{2} E_I a_0 \left(r - \frac{r^2}{b}\right) \frac{1}{\cancel{2} a_0^3} \left(1 - \frac{r}{2a_0}\right)^2 \underbrace{e^{-\frac{r}{a_0}}}_{1} dr$$

$$= \frac{E_I}{a_0^2} \int_0^b \left(r - \frac{r^2}{a_0} + \frac{r^3}{4a_0^2} - \frac{r^2}{b} + \frac{r^3}{ba_0} - \frac{r^4}{4a^2 b}\right) dr$$

$$= \frac{E_I}{a_0^2} \cdot \left| \frac{r^2}{2} - \frac{r^3}{3a_0} - \frac{r^3}{3b} + \dots \right|_0^b =$$

$$= E_I \cdot \left(\frac{b^2}{2a_0^2} - \underbrace{\frac{b^3}{3a_0^3}}_{\text{small}} - \frac{b^{\cancel{3}\,2}}{3\cancel{b}a_0^2}\right) =$$

$$\boxed{= E_I \cdot \frac{1}{6} \cdot \left(\frac{b}{a_0}\right)^2}$$

b. equal to 0 because the angular wavefunctions are orthogonal.

c. basis we use is $\{|100\rangle, |200\rangle, |211\rangle, |210\rangle, |21\text{-}1\rangle\}$
in this basis representation:

$$W \to E_I \cdot \frac{1}{3}\left(\frac{b}{a}\right)^2 \cdot \begin{pmatrix} 4 & 3b & 0 & 0 & 0 \\ 3b & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 0 & \epsilon \end{pmatrix}$$

d. $E_{100} \simeq E^0_{100} + \langle 100|W|100\rangle = -E_I + \frac{4}{3}E_I\left(\frac{b}{a_0}\right)^2$

$$= -E_I\left[1 - \frac{4}{3}\left(\frac{b}{a_0}\right)^2\right]$$

$$E_{200} \simeq E^{0}_{200} + \langle 200|W|200\rangle = -\frac{E_I}{4} + \frac{1}{6}E_I\left(\frac{b}{a_0}\right)^2$$

$$= -\frac{E_I}{4}\left[1 - \frac{2}{3}\left(\frac{b}{a_0}\right)^2\right]$$

$$E_{210} = E_{211} = E_{21-1} = -\frac{E_I}{4}$$

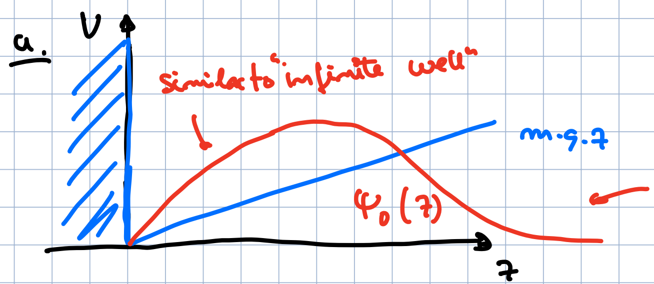
e. $$\left(E_{210} - E_{200}\right)/_{2\pi\hbar} = \frac{E_I}{4\cdot(2\pi\hbar)}\cdot\frac{2}{3}\left(\frac{b}{a_0}\right)^2 = \frac{\cancel{E_I/2\pi}\cdot 3.3\cdot 10^{15}\,\mathrm{Hz}}{4\cdot\cancel{(2\pi\hbar)}}\cdot\frac{2}{3}\left(10^{-5}\right)^2 =$$

$= 55$ kHz much smaller than Lamb shift at 1 GHz

Problem IV

$$E_0 \leq \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \qquad V(z) = \begin{cases}\infty & z\leq 0\\ mgz & z>0\end{cases}$$

a.

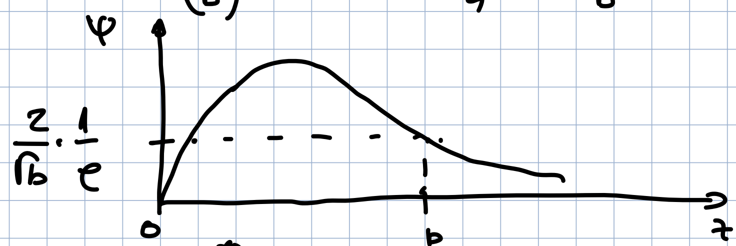


b. $$H(z) = \frac{p^2}{2m} + V(z)$$

c. guess $\psi(z) = A\left(\frac{z}{b}\right)e^{-\frac{z}{b}}$

Normalize: $$\int|\psi(z)|^2\,dz = 1 \Rightarrow \frac{A^2}{b^2}\cdot\int_{z=0}^{\infty} z^2 e^{-\frac{2z}{b}}\,dz = 1$$

$$\frac{A^2}{b^2}\cdot\frac{2}{\left(\frac{2}{b}\right)^3} = 1 \qquad \frac{A^2}{4} = \frac{1}{b} \qquad A = \frac{2}{\sqrt{b}}$$



d. $$\langle z\rangle = \int_0^{\infty} z|\psi(z)|^2\,dz = \frac{4}{b}\cdot\frac{1}{b^2}\int_{z=0}^{\infty} z^3 e^{-\left(\frac{2}{b}\right)z}\,dz = \frac{4}{b^3}\cdot\frac{2\cdot 3}{\left(\frac{2}{b}\right)^4} = \boxed{\frac{3}{2}b}$$

$\underline{e.}$ $\langle H \rangle = \frac{1}{2m}\langle p^2 \rangle + mg\langle z \rangle =$ $\qquad p = i\hbar \frac{\partial}{\partial z}$

$$= -\frac{\hbar^2}{2m}\left\langle \frac{\partial^2}{\partial z^2} \right\rangle + mg\langle z \rangle$$

$$\left\langle \frac{\partial^2}{\partial z^2} \right\rangle = \int_0^\infty \frac{\partial^2}{\partial z^2} |\psi(z)|^2 dz = \frac{4}{b^3} \cdot \int_{z=0}^{\infty} z e^{-\frac{2}{b}z} \frac{\partial^2}{\partial z^2}\left(z e^{-\frac{2}{b}z}\right) dz =$$

$$= \frac{4}{b^3} \int_{z=0}^{\infty} z e^{-\frac{2}{b}z} \frac{\partial}{\partial z}\left[e^{-\frac{2}{b}z} + z\left(-\frac{2}{b}\right)e^{-\frac{2}{b}z}\right] dz =$$

$$= \frac{4}{b^3} \int_{z=0}^{\infty} z e^{-\frac{2}{b}z}\left[-\left(\frac{2}{b}\right)e^{-\frac{2}{b}z} + \left(-\frac{2}{b}\right)e^{-\frac{2}{b}z} + \left(-\frac{2}{b}\right)\cdot z \cdot \left(-\frac{2}{b}\right)e^{-\frac{2}{b}z}\right] dz$$

$$= \frac{4}{b^3} \cdot \left(-\frac{2}{b}\right) \int_{z=0}^{\infty} 2z e^{-\left(\frac{4}{b}\right)z} - \frac{2}{b} z^2 e^{-\left(\frac{4}{b}\right)z} dz =$$

$$= -\frac{8}{b^4} \cdot \left[2 \frac{1}{\left(\frac{4}{b}\right)^2} - \frac{2}{b} \cdot \frac{2}{\left(\frac{4}{b}\right)^3}\right] = -\frac{8}{b^4} \cdot \left(\frac{1}{8} \cdot b^2 - \frac{1}{16} b^2\right) =$$

$$= -\frac{1}{b^2}$$

$$\Rightarrow \boxed{\langle H \rangle = \frac{\hbar^2}{2m} \cdot \frac{1}{b^2} + \frac{3}{2} mgb}$$

$\underline{f.}$ $\frac{\partial \langle H \rangle}{\partial b} = 0 \qquad \frac{\hbar^2}{\cancel{2}m} \cdot (-\cancel{2} \cdot b^{-3}) + \frac{3}{2} mg = 0 \Rightarrow \frac{\hbar^2}{m b^3} = \frac{3}{2} mg$

$$b^3 = \frac{\hbar^2}{m^2 g} \cdot \frac{2}{3} \Rightarrow b = \left(\frac{2\hbar^2}{3m^2 g}\right)^{\frac{1}{3}}$$

$\underline{g.}$ $m = 1.5 \cdot 10^{-25} kg \to b = 3.3 \cdot 10^{-7} m = 0.33\ \mu m$

$\underline{h.}$ $\langle H \rangle = \frac{\hbar^2}{2m} \cdot \frac{1}{b^2} + \frac{3}{2} mgb \approx 1 \cdot 10^{-30} J$

$\underline{i.}$ $\frac{\langle H \rangle}{k_B} = 7.7 \cdot 10^{-8} K = \underline{77\ nK}$

$\underline{j.}$ $b = 4.5 \cdot 10^{-25} m \qquad \langle H \rangle \sim 10 \cdot 10^{-22} J \qquad \frac{\langle H \rangle}{k_B} \sim 67\ K$