Problem Set 7 Solutions

Problem I

a. $\hat{H}_E = \hat{H}_1 = \hbar\Omega(\hat{N}^2 - \frac{1}{2})$

$$\hat{U}_E = e^{-i\hat{H}_1 t/\hbar} = e^{-i\Omega t(\hat{N} - \frac{1}{2})}$$

check: $\hat{U}_E\left(t = \frac{2\pi}{\Omega}\right)|\varphi_m\rangle = e^{-i\cancel{\Omega}\cdot\frac{2\pi}{\cancel{\Omega}}(\hat{N}^2 - \frac{1}{2})}|\varphi_m\rangle =$

$$= \underbrace{e^{-i2\pi m^2}}_{1}\cdot\underbrace{e^{-i\pi}}_{-1}|\varphi_m\rangle =$$

$$\boxed{\hat{U}_E\left(t = \frac{2\pi}{\Omega}\right)|\varphi_m\rangle = -|\varphi_m\rangle}$$

b. $\hat{U}_E\left(\frac{\pi}{2\Omega}\right)|\varphi_m\rangle = e^{-i\cancel{\Omega}\frac{\pi}{2\cancel{\Omega}}(N^2 - \frac{1}{2})}|\varphi_m\rangle = \quad \tau = \frac{\pi}{2\Omega}$

$$= e^{\frac{i\pi}{4}}e^{-i\frac{\pi}{2}m^2}|\varphi_m\rangle =$$

$$= e^{\frac{i\pi}{4}}\left(e^{-i\frac{\pi}{2}}\right)^{m^2}|\varphi_m\rangle =$$

$$= e^{i\frac{\pi}{4}}(-i)^{m^2}|\varphi_m\rangle =$$

$$= e^{i\frac{\pi}{4}}|\varphi_m\rangle \cdot \begin{cases} 1 & \text{for } m \text{ even} \\ -i & \text{for } m \text{ odd} \end{cases}$$

$$= e^{i\frac{\pi}{4}(-1)^m}|\varphi_m\rangle =$$

$$= \left\{\cos\left[\frac{\pi}{4}(-1)^m\right] + i\sin\left[\frac{\pi}{4}(-1)^m\right]\right\}|\varphi_m\rangle =$$

$$= \left[\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}(-1)^m\right]|\varphi_m\rangle$$

c $|\Psi(t=0)\rangle = |\Psi_E(t=0)\rangle = |\alpha_0\rangle$ w/ $\hat{a}|\alpha_0\rangle = \alpha_0|\alpha_0\rangle$

$$|\Psi_E(t=\tau)\rangle = \hat{U}_E(\tau)|\Psi_E(0)\rangle =$$

$$= e^{-|\alpha_0|^2/2} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} \hat{U}_E(\tau) |\varphi_n\rangle =$$

$$= e^{-|\alpha_0|^2/2} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} \left[\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} (-1)^n \right] |\varphi_n\rangle =$$

$$= \frac{1}{\sqrt{2}} \left[e^{-|\alpha_0|^2/2} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} |\varphi_n\rangle + i e^{-|\alpha_0|^2/2} \sum_{n=0}^{\infty} \frac{(-\alpha_0)^n}{\sqrt{n!}} |\varphi_n\rangle \right] =$$

$$= \frac{1}{\sqrt{2}} \left[|\alpha_0\rangle + i |-\alpha_0\rangle \right]$$

d. $|\psi(\tau)\rangle = F^{\dagger}(\tau) |\psi_E(\tau)\rangle = \frac{1}{\sqrt{2}} e^{-i\hat{H}_0 \tau/\hbar} \left(|\alpha_0\rangle + i |-\alpha_0\rangle \right) =$

$\Leftarrow$ from class

$$= \frac{1}{\sqrt{2}} \left(|\alpha_0 e^{-i\omega\tau}\rangle + i |-\alpha_0 e^{-i\omega\tau}\rangle \right)$$

$$|\psi(\tau)\rangle = \frac{1}{\sqrt{2}} \left(|\alpha_0 e^{-i\omega\frac{\pi}{2\Omega}}\rangle + i |-\alpha_0 e^{-i\Omega\frac{\pi}{2\Omega}}\rangle \right)$$

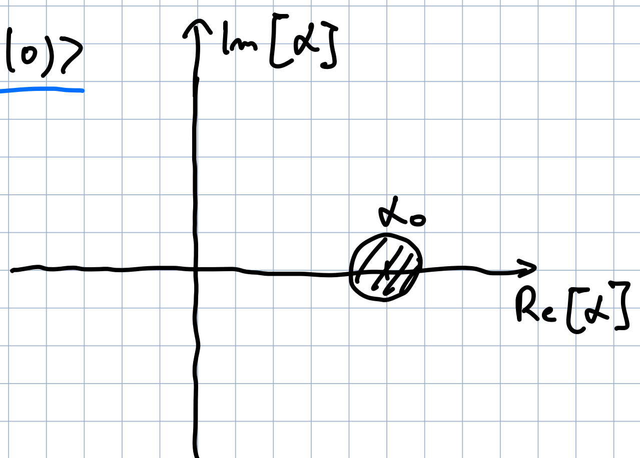
for $\omega = \Omega$

$$|\psi(\tau)\rangle = \frac{1}{\sqrt{2}} \left(|\alpha_0 e^{-i\frac{\pi}{2}}\rangle + i |-\alpha_0 e^{-i\frac{\pi}{2}}\rangle \right) =$$

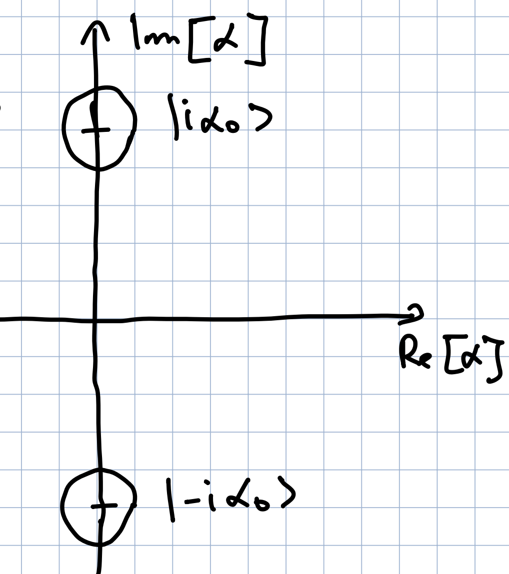
$$= \frac{1}{\sqrt{2}} \left(|-i\alpha_0\rangle + i |i\alpha_0\rangle \right)$$

Let α_0 be real

$|\psi(0)\rangle$



$|\psi(\tau)\rangle$



Problem II

$$\hat{H} = \hat{H}_0 + \hat{W} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega^2\hat{X}^2$$

p 46. : $\hat{H}_E = \hat{U}_0^+ \hat{W} \hat{U}_0 =$

$$= \frac{1}{2} m \omega^2 \; \hat{U}_0^+ \hat{X}^2 \hat{U}_0 =$$

$$= \frac{1}{2} m \omega^2 \left(\hat{U}_0^+ \hat{X} \hat{U}_0\right)^2$$

$$\hat{U}_0^+ \hat{X} \hat{U}_0 = e^{i\frac{\hat{P}^2 t}{2m\hbar}} \hat{X} e^{-i\frac{\hat{P}^2 t}{2m\hbar}}$$

In the momentum representation, for arbitrary state $|\Psi\rangle$:

$$\langle p|\hat{U}_0^+ \hat{x} \hat{U}_0|\Psi\rangle = \langle p|e^{i\frac{\hat{P}^2 t}{2m\hbar}} \cdot \hat{X} \cdot e^{-\frac{i\hat{P}^2 t}{2m\hbar}}|\Psi\rangle =$$

$$= e^{i\frac{p^2 t}{2m\hbar}} \cdot \left(i\hbar\frac{\partial}{\partial p}\right) e^{-\frac{ip^2 t}{2m\hbar}} \langle p|\Psi\rangle =$$

$$= i\hbar\, e^{\frac{ip^2 t}{2m\hbar}} \frac{\partial}{\partial p}\left[e^{-\frac{ip^2 t}{2m\hbar}} \bar{\Psi}(p)\right] =$$

$$= i\hbar\, \cancel{e^{\frac{ip^2 t}{2m\hbar}}} \cdot \left[-\frac{i\cancel{2}pt}{\cancel{2}m\hbar} \cancel{e^{-\frac{ip^2 t}{2m\hbar}}} \bar{\Psi}(p) + \cancel{e^{-\frac{ip^2 t}{2m\hbar}}} \frac{\partial}{\partial p}\bar{\Psi}(p)\right] =$$

$$= \frac{pt}{m}\bar{\Psi}(p) + i\hbar\frac{\partial}{\partial p}\bar{\Psi}(p)$$

$$\underbrace{\hat{U}_0^+ \hat{x} \hat{U}_0 = \frac{\hat{P}_S t}{m} + \hat{X}_S}$$ - makes sense as this is the Heisenberg picture operator, $\hat{P}_S, \hat{X}_S$ are Sch. picture operators

$$\Rightarrow H_E = \frac{1}{2} m \omega^2 \left(\frac{\hat{P}t}{m} + \hat{X}\right)^2 =$$

$$= \frac{1}{2} m \omega^2 \left(\frac{\hat{P}^2 t^2}{m^2} + \hat{X}^2 + \frac{\hat{P}\hat{X}t}{m} + \frac{\hat{X}\hat{P}t}{m}\right) =$$

$$= \frac{P^2}{2m}(\omega t)^2 + \frac{1}{2}m\omega^2 x^2 + \underbrace{\frac{1}{2}\omega^2 t\,(xP + Px)}_{\text{focusing terms}}$$

Problem III

a $4\pi/\omega$ is two oscillation periods. We know the state returns to the initial state each period, which means: $\hat{U}_0\left(\frac{4\pi}{\omega}\right) = \mathbb{1}$

$$|\psi\left(\tfrac{4\pi}{\omega}\right)\rangle = |\psi_E\left(\tfrac{4\pi}{\omega}\right)\rangle$$

b $$H_E = U_0^+ W U_0 = i\frac{\hbar\Omega}{2}\left(U_0^+ \hat{a}^2 U_0 e^{2i\omega t} - U^+ a^{+2} U_0 e^{-2i\omega t}\right)$$

$$= i\frac{\hbar\Omega}{2}\left[\left(U_0^+ \hat{a} U_0\right)^2 e^{i2\omega t} - \left(U^+ a^+ U_0\right)^2 e^{-2i\omega t}\right] =$$

in PSET 6 Problem 1

$$= \frac{i\hbar\Omega}{2}\left[\left(a e^{-i\omega t}\right)^2 e^{i2\omega t} - \left(a^+ e^{i\omega t}\right)^2 e^{-2i\omega t}\right] =$$

$$= \frac{i\hbar\Omega}{2}\left(a^2 - a^{+2}\right)$$ $\underline{\hbar\omega \text{ units}}$ ✓

$$H_E^+ = \frac{-i\hbar\Omega}{2}\left[(aa)^+ - (a^+a^+)^+\right] = \frac{-i\hbar\Omega}{2}\left[a^+a^+ - aa\right] =$$

$$= \frac{i\hbar\Omega}{2}\left[a^2 - a^{+2}\right]$$ ✓

c $$a = \frac{1}{\sqrt{2}}\left(\frac{x}{\sigma} + \frac{i\sigma p}{\hbar}\right)$$

$$a^2 = \frac{1}{2}\left(\frac{x^2}{\sigma^2} - \frac{\sigma^2 p^2}{\hbar^2}\right) + \frac{i}{2\hbar}(xp + px)$$

$$a^{+2} = \frac{1}{2}\left(\frac{x^2}{\sigma^2} - \frac{\sigma^2 p^2}{\hbar^2}\right) - \frac{i}{2\hbar}(xp + px)$$

$$H_E = \frac{i\cancel{\hbar}\Omega}{2}\cdot\frac{\cancel{2}i}{\cancel{2}\cancel{\hbar}}(xp + px) =$$

$$= -\frac{\Omega}{2}(xp + px)$$ ⇒ same form as Problem II. Expect focusing action

d. $$U_E(\tau) = e^{-i\tau H_E/\hbar} = e^{\frac{\Omega\tau}{2}\left(a^2 - a^{+2}\right)} =$$

$$= e^{\frac{b}{2}\left(a^2 - a^{+2}\right)} \quad \text{w/} \quad b = \Omega\tau$$

e. $$Q_b(b) = e^{\frac{b}{2}\left(a^2 - a^{+2}\right)} = e^{-\hat{B}} \Rightarrow \hat{B} = \frac{b}{2}\left(a^{+2} - a^2\right)$$

1. $$[\hat{B}, \hat{a}] = \frac{b}{2}\left(a^{+2} - a^2\right) a - \frac{b}{2} a \left(a^{+2} - a^2\right) =$$

$$= \frac{b}{2}\left[a^{+2} a - \cancel{a^3} - a a^{+2} + \cancel{a^3} \right] =$$

$$[a, a^+] = 1$$
$$a a^+ - a^+ a = 1$$

$$= \frac{b}{2}\left[a^+ a^+ a - \left(a^+ a + 1\right) a^+ \right] =$$

$$= \frac{b}{2}\left[a^+ \underbrace{[a^+, a]}_{-1} - a^+ \right] = \underline{-b a^+}$$

$$[\hat{B}, \hat{a}^+] = \frac{b}{2} \underbrace{[a^{+2}, a^+]}_{0} - [a^2, a^+] = -\frac{b}{2}\left(a a a^+ - a^+ a a \right) =$$

$$= -\frac{b}{2}\left[a(1 + a^+ a) - (a a^+ - 1) a \right] = -\frac{b}{2} \cdot 2a$$

$$\underline{= -b a}$$

g. $$Q^+ a Q = e^{B} a e^{-B} = a + [B, a] + \frac{1}{2!}[B,[B,a]] + \frac{1}{3!}[B,[B,[B,a]]] + \dots$$

$$= a - b a^+ + \frac{1}{2!}[B, -b a^+] + \frac{1}{3!}[B,[B, -b a^+]] + \dots =$$

$$= a - b a^+ + \frac{1}{2!}(-b)^2 a + \frac{1}{3!}[B, -b^2 a] + \dots =$$

$$= a - b a^+ + \frac{b^2}{2!} a - \frac{b^3}{3!} a^+ + \dots =$$

$$= a \cdot \left(1 + \frac{b^2}{2!} + \frac{b^4}{4!} + \dots\right) - a^+ \left(b + \frac{b^3}{3!} + \frac{b^5}{5!} + \dots\right) =$$

$$= a \cosh b - a^+ \sinh b$$

$$Q^+ a^+ Q = \left[Q a Q^+ \right]^+ = a^+ \cosh b - a \sinh b$$

h $$Q^+ x Q = \frac{\sigma}{\sqrt{2}}\left(Q^+ a^+ Q + Q^+ a Q \right) =$$

$$= \frac{\sigma}{\sqrt{2}}\left(a^+ \cosh b - a \sinh b + a \cosh b - a^+ \sinh b \right) =$$

$$= \frac{\sigma}{\sqrt{2}}\left(a^+ + a \right)\left(\cosh b - \sinh b \right) = x \left(\frac{e^{b} + e^{-b}}{2} - \frac{e^{b} - e^{-b}}{2} \right) =$$

$$\underbrace{Q^+ x Q = x \cdot e^{-b}}$$

$$Q^{+} p\, Q = \frac{i\hbar}{\sqrt{2}\,\sigma}\left(Q^{+} a^{+} Q - Q^{+} a\, Q\right) =$$

$$= \frac{i\hbar}{\sqrt{2}\,\sigma}\left(a^{+} - a\right)(\cosh b + \sinh b) =$$

$$\underline{Q^{+} p\, Q = \hat{p}\, e^{b}}$$

$$Q^{+} x^{2} Q = \left(Q^{+} x\, Q\right)^{2} = x^{2} e^{-2b}$$

$$Q^{+} p^{2} Q = \left(Q^{+} p\, Q\right)^{2} = p^{2} e^{2b}$$

i.

$$\langle \hat{x} \rangle = \langle 0 |\, Q^{+} x\, Q\, | 0 \rangle = e^{-b} \langle 0 | x | 0 \rangle = 0$$

$$\langle \hat{p} \rangle = e^{b} \langle 0 | p | 0 \rangle = 0$$

$$\langle \hat{x}^{2} \rangle = \langle 0 |\, Q^{+} x^{2} Q\, | 0 \rangle = e^{-2b} \langle 0 | x^{2} | 0 \rangle = e^{-2b}\, \frac{\sigma^{2}}{2}$$

$$\langle \hat{p}^{2} \rangle = e^{2b} \langle 0 | p^{2} | 0 \rangle = e^{2b}\, \frac{\hbar^{2}}{2\sigma^{2}}$$

j.

$$\Delta x = \sqrt{\langle x^{2} \rangle - \langle x \rangle^{2}} = e^{-b}\, \frac{\sigma}{\sqrt{2}}$$

$$\Delta p = e^{b}\, \frac{\hbar}{\sigma\sqrt{2}}$$

$$\Delta x \cdot \Delta p = \frac{\hbar}{2}$$ ← minimum uncertainty state

k.

$\Delta x = e^{-b}\, \frac{\sigma}{\sqrt{2}}$ was $\frac{\sigma}{\sqrt{2}}$ for $|0\rangle$

So we have $|0\rangle$ but $\sigma' \to e^{-b}\sigma$ $\quad \gamma = \sigma e^{-b}$

$$\varphi(x) = \left(\frac{1}{\pi\, \sigma'^{2}}\right)^{\frac{1}{4}} e^{-\frac{x^{2}}{2\sigma'^{2}}} = \left(\frac{1}{\pi\, \sigma^{2} e^{-2b}}\right)^{\frac{1}{4}} e^{-\frac{x^{2}}{2\sigma^{2} e^{-2b}}} = \left(\frac{1}{\pi \gamma^{2}}\right)^{\frac{1}{4}} e^{-\frac{x^{2}}{2\gamma^{2}}}$$

e.

$$\langle \hat{H}_0 \rangle = \langle \varphi |\, \hat{H}_0\, | \varphi \rangle =$$

$$= \hbar\omega\left(\langle 0 |\, Q^{+} a^{+} a\, Q\, | 0 \rangle + \frac{1}{2}\right) =$$

$$= \hbar\omega\left(\langle 0 |\, Q^{+} a^{+} Q\, Q^{+} a\, Q\, | 0 \rangle + \frac{1}{2}\right) =$$

$$= \hbar\omega\left[\langle 0 | \left(a^{+} \cosh b - a \sinh b\right)\left(a \cosh b - a^{+} \sinh b\right) | 0 \rangle + \frac{1}{2}\right] =$$

$$= \hbar\omega\Big[\langle 0|\, a^\dagger a \cosh^2 b - \underbrace{a a^\dagger}_{1+a^\dagger a} \sinh^2 b - \sinh b \cosh b \underbrace{(a^2 + a^{\dagger 2})}_{\substack{0 \text{ because} \\ a^2, a^{\dagger 2} \text{ don't connect} \\ |0\rangle \text{ and } |0\rangle}} |0\rangle + \frac{1}{2}\Big] =$$

$$= \hbar\omega\Big[\langle 0|\sinh^2 b|0\rangle + \frac{1}{2}\Big] =$$

$$\boxed{\langle \hat{H}_0\rangle = \hbar\omega\left(\sinh^2 b + \frac{1}{2}\right)}$$

For $b = 0$, we get $\langle \hat{H}_0\rangle = \frac{\hbar\omega}{2}$ ✓

m. $Q(b)$ focuses a state of the QHO in position or momentum, while maintaining its initial uncertainty product.

Problem IV

a) Need $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}} = \operatorname{sech}(-x) \Rightarrow \langle x\rangle = \langle p\rangle = 0$$

$$\langle x^2\rangle = \int_{-\infty}^{+\infty} \frac{1}{2\beta} x^2 \operatorname{sech}^2\left(\frac{x}{\beta}\right) dx = \quad \text{let } u = \frac{x}{\beta} \quad du = \frac{dx}{\beta}$$

$$= \int_{-\infty}^{\infty} \frac{\beta^2}{2} u^2 \operatorname{sech}^2 u = \frac{\pi^2\beta^2}{12}$$

$$\langle p^2\rangle = \int_{-\infty}^{+\infty} \frac{\pi\beta}{4\hbar} p^2 \operatorname{sech}^2\left(\frac{\pi\beta p}{2\hbar}\right) dp = \quad \text{let } u = \frac{\pi\beta p}{2\hbar} \quad du = \frac{\pi\beta}{2\hbar}\, dp$$

$$= \frac{\cancel{\pi}\cancel{\beta}}{\cancel{4}\cancel{\hbar}} \cdot \frac{\cancel{4}\hbar^{\cancel{2}}}{\pi^{\cancel{2}}\beta^2} \frac{2\hbar}{\pi\cancel{\beta}} \int_{-\infty}^{+\infty} u^2 \operatorname{sech}^2(u)\, du =$$

$$= \frac{\cancel{2}\hbar^2}{\cancel{\pi^2}\beta^2} \frac{\cancel{\pi^2}}{\cancel{6}_3} = \frac{\hbar^2}{3\beta^2}$$

$$\Delta x \cdot \Delta p = \sqrt{\frac{\pi^2\cancel{\beta^2}}{12} \cdot \frac{\hbar^2}{3\cancel{\beta^2}}} = \sqrt{\frac{\pi^2\hbar^2}{36}} = \frac{\hbar\pi}{6} > \frac{\hbar}{2}$$ ✓

b. $\Phi(x,0) = \frac{1}{\sqrt{2\beta}} \operatorname{sech}[(x-x_0)/\beta]$

$\Phi\left(x, \frac{\pi}{2\omega}\right)$ will be the Fourier transform shifted

$$\Phi\left(x, \frac{\pi}{2\omega}\right) = \underbrace{e^{-i\frac{\pi}{4}}}_{\text{global phase}} \cdot \sqrt{\frac{\hbar}{\sigma^2}}\; FT\left[\langle x| e^{-i x_0 \hat{p}/\hbar} |\Phi\rangle\right] \Big|_{p \to \frac{\hbar x}{\sigma^2}}$$

$$= e^{-i\frac{\pi}{4}} \cdot (-1) \sqrt{\frac{\cancel{\hbar}}{\sigma^2}} \cdot e^{-i x_0 p/\hbar} \sqrt{\frac{\pi\beta}{4\cancel{\hbar}}} \operatorname{sech}\left(\frac{\pi\beta p}{2\hbar}\right) \Big|_{p \to \frac{\hbar x}{\sigma^2}}$$

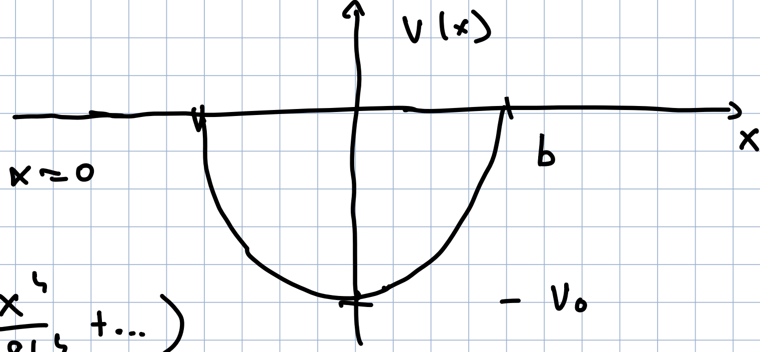
$$= -e^{-i\frac{\pi}{4}} \sqrt{\frac{\pi\beta}{4\sigma^2}}\; e^{-i x_0 \frac{\cancel{\hbar} x}{\sigma^2} \cdot \frac{1}{\cancel{\hbar}}} \cdot \operatorname{sech}\left(\frac{\pi\beta}{2\cancel{\hbar}} \frac{\cancel{\hbar} x}{\sigma^2}\right) =$$

$$= -e^{-i\frac{\pi}{4}} \sqrt{\frac{\pi\beta}{4\sigma^2}}\; e^{-i x_0 x/\sigma^2} \operatorname{sech}\left(\frac{\pi\beta x}{2\sigma^2}\right)$$

c. $$\frac{\pi\beta \cancel{x}}{2\sigma^2} = \frac{\cancel{x}}{\beta} \Rightarrow \beta = \sqrt{\frac{2\sigma^2}{\pi}}$$

Problem V

Need to Taylor expand at $x=0$



$$V(x) = -V_0\left(1 - \frac{x^2}{2b^2} - \frac{x^4}{8b^4} + \dots\right)$$

QHO approx: $V_0 \frac{x^2}{\cancel{2}b^2} = \frac{1}{\cancel{2}} m\omega^2 x^2 \Rightarrow \omega = \sqrt{\frac{V_0}{mb^2}}$

Energies are: $E_n = \hbar\omega\left(n + \frac{1}{2}\right) - V_0$

$$\underbrace{E_0 = \frac{1}{2}\hbar\sqrt{\frac{V_0}{mb^2}} - V_0}$$

$$\underbrace{E_1 = \frac{3}{2}\hbar\sqrt{\frac{V_0}{mb^2}} - V_0}$$