Assignment 7

OPTI 570 Quantum Mechanics

University of Arizona

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Total time: 12 hours

Problem I

a) The evolution operator would be of the form

$$\hat{\mathbb{U}}_E(t) = e^{-i\hat{H}_1 t/\hbar} = e^{-i\Omega(\hat{N}^2 - 1/2)t}.$$

The checking is as follows:

$$\begin{aligned}
\hat{\mathbb{U}}_E(\frac{2\pi}{\Omega})|\varphi_n\rangle &= e^{-i\Omega(n^2-1/2)\frac{2\pi}{\Omega}}|\varphi_n\rangle \\
&= e^{-i2\pi(n^2-1/2)}|\varphi_n\rangle \\
&= (e^{-2\pi})^{n^2} e^{i\pi}|\varphi_n\rangle \\
\hat{\mathbb{U}}_E(\frac{2\pi}{\Omega})|\varphi_n\rangle &= -|\varphi_n\rangle.
\end{aligned}$$

b) For $\tau = \pi/2\Omega$, the evolution is

$$\begin{aligned}
\hat{\mathbb{U}}_E(\tau)|\varphi_n\rangle &= e^{-i\Omega(n^2-1/2)\frac{\pi}{2\Omega}}|\varphi_n\rangle \\
&= e^{-i\frac{\pi}{2}n^2} e^{i\frac{\pi}{4}}|\varphi_n\rangle \\
&= (e^{-i\frac{\pi}{2}})^{n^2} e^{i\frac{\pi}{4}}|\varphi_n\rangle \\
&= (-i)^{n^2} e^{i\frac{\pi}{4}}|\varphi_n\rangle \\
\hat{\mathbb{U}}_E(\tau)|\varphi_n\rangle &= \begin{cases} e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, & n \text{ even} \\ e^{-i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, & n \text{ odd} \end{cases} |\varphi_n\rangle.
\end{aligned}$$

c) We use the fact that in a coherent state, we can express it in terms of the energy eigenstates.

$$|\alpha_0\rangle = e^{-\frac{|\alpha_0|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}}|n\rangle.$$

We have found that

$$\hat{\mathbb{U}}_E(\tau) = \begin{cases} e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, & n \text{ even} \\ e^{-i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, & n \text{ odd} \end{cases}.$$

We then, must split the $|\alpha_0\rangle$ accordingly, in even and odd term so that the application of the evolution operator gives

$$|\psi_E(\tau)\rangle = e^{-\frac{|\alpha_0|^2}{2}} \left[e^{i\frac{\pi}{4}} S_{\text{even}} + e^{-i\frac{\pi}{4}} S_{\text{odd}} \right],$$

where

$$S_{\text{even}} = \sum_{n \text{ even}}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} |n\rangle, \quad \text{and} \quad S_{\text{odd}} = \sum_{n \text{ odd}}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} |n\rangle. \tag{1}$$

We then have that

$$\left. \begin{aligned} |\alpha_0\rangle &= e^{-\frac{|\alpha_0|^2}{2}} (S_{\text{even}} + S_{\text{odd}}) \\ |-\alpha_0\rangle &= e^{-\frac{|\alpha_0|^2}{2}} (S_{\text{even}} - S_{\text{odd}}) \end{aligned} \right\} \longrightarrow \begin{aligned} S_{\text{even}} &= \tfrac{1}{2} e^{\frac{|\alpha_0|^2}{2}} (|\alpha_0\rangle + |-\alpha_0\rangle) \\ S_{\text{odd}} &= \tfrac{1}{2} e^{\frac{|\alpha_0|^2}{2}} (|\alpha_0\rangle - |-\alpha_0\rangle) \end{aligned} .$$

Substituting those in the evolution equation and rearranging:

$$|\psi_E(\tau)\rangle = \frac{1}{2} \left[(e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}})|\alpha_0\rangle + (e^{i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}})|-\alpha_0\rangle \right] = \frac{1}{\sqrt{2}} [|\alpha_0\rangle + i|-\alpha_0\rangle],$$

where

$$|\pm\alpha_0\rangle = e^{-\frac{|\alpha_0|^2}{2}} \sum_{n=0}^{\infty} \frac{(\pm\alpha_0)^n}{\sqrt{n!}} |n\rangle.$$

d) The transfrmation from the Interaction picture to the Schrodinger picture is

$$\begin{aligned} |\psi(\tau)\rangle &= \hat{\mathbb{U}}_0(\tau)|\psi_E(\tau)\rangle, \qquad \hat{\mathbb{U}}_0(\tau) = e^{-iH_0\tau/\hbar} \\ &= e^{-i\omega\tau(\hat{N}+1/2)} |\psi_E(\tau)\rangle \\ &= \frac{1}{\sqrt{2}} e^{-i\omega\tau(\hat{N}+1/2)} [\alpha_0 + i|-\alpha_0\rangle]|n\rangle \\ &= \frac{1}{\sqrt{2}} e^{-\frac{|\alpha_0|^2}{2}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left[\alpha_0^n e^{-i\omega\tau(n+1/2)} + i(-\alpha_0)^n e^{-i\omega\tau(n+1/2)} \right] |n\rangle \\ &= \frac{1}{\sqrt{2}} e^{-i\frac{\omega}{2}\tau} e^{-\frac{|\alpha_0|^2}{2}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left[\left(\alpha_0 e^{-i\omega\tau}\right)^n |n\rangle + i\left(-\alpha_0 e^{-i\omega\tau}\right)|n\rangle \right] \\ |\psi(\tau)\rangle &= \frac{1}{\sqrt{2}} e^{-i\frac{\omega}{2}\tau} \left[|\alpha_0 e^{-i\omega\tau}\rangle + i|-\alpha_0 e^{-i\omega\tau}\rangle \right]. \end{aligned}$$

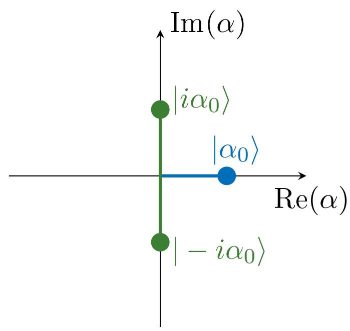
e) Evaluating with $\tau = \pi/2\omega$,

$$\begin{aligned} |\psi(\frac{\pi}{2\omega})\rangle &= \frac{1}{\sqrt{2}} e^{-i\frac{\omega}{2}\frac{\pi}{2\omega}} \left[|\alpha_0 e^{-i\omega\frac{\pi}{2\omega}}\rangle + i|-\alpha_0 e^{-i\omega\frac{\pi}{2\omega}}\rangle \right] \\ &= \frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}} \left[|\alpha_0 e^{-i\frac{\pi}{2}}\rangle + i|-\alpha_0 e^{-i\frac{\pi}{2}}\rangle \right] \\ |\psi(\frac{\pi}{2\omega})\rangle &= \frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}} \left[|-i\alpha_0\rangle + i|i\alpha_0\rangle \right]. \end{aligned}$$

In addition, at $t = 0$ we have

$$|\psi_E(0)\rangle = |\alpha_0\rangle.$$

Then,



Problem II

The Hamiltonian in the whole range is:

$$\hat{H} = \hat{H}_0 + \hat{W} = \begin{cases} \frac{\hat{P}^2}{2m}, & t < 0 \\ \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2, & 0 \leq t < \tau \ , \\ \frac{\hat{P}^2}{2m}, & t \geq 0 \end{cases} \qquad \hat{W} = \frac{1}{2}m\omega^2\hat{X}^2.$$

The evolution operator is $U_0(t) = e^{-iH_0t/\hbar} = e^{-i\hat{P}^2t/2m\hbar}$. The effective Hamiltonian in terms of the Schrodinger picture position and momentum operators is:

$$H_E = U_0^\dagger(t,0)H_1U_0(t,0) = e^{i\frac{\hat{P}^2t}{2m\hbar}}\left[\frac{1}{2}m\omega^2\hat{X}^2\right]e^{-i\hat{P}^2t/2m\hbar} = \frac{1}{2}m\omega^2 e^{i\hat{P}^2t/2m\hbar}\hat{X}^2e^{-i\hat{P}^2t/2m\hbar}$$
$$\stackrel{(a)}{=} \frac{1}{2}m\omega^2\left[e^{i\hat{P}^2t/2m\hbar}\hat{X}e^{-i\hat{P}^2t/2m\hbar}\right]^2.$$

In (a), we used the property. We can see the term iinside the brackets as the product ABC of operators, where we would like to switch the position of $\hat{X}$ with the right exponential, that why we use

$$ABC = A[B,C] + ACB.$$

The commutator $[B,C]$ is

$$[B,C] = [\hat{X}, e^{-i\frac{\hat{P}^2t}{2m\hbar}}] = i\hbar\partial_{\hat{P}}e^{-i\frac{\hat{P}^2t}{2m\hbar}} = i\hbar\frac{-i2\hat{P}t}{2m\hbar}e^{-i\frac{\hat{P}^2t}{2m\hbar}} = \frac{\hat{P}t}{m}e^{-i\frac{\hat{P}^2t}{2m\hbar}}.$$

Then, substituting this commutator in the above relation

$$e^{i\frac{\hat{P}^2t}{2m\hbar}}\hat{X}e^{-i\frac{\hat{P}^2t}{2m\hbar}} = e^{i\frac{\hat{P}^2t}{2m\hbar}}\frac{\hat{P}t}{m}e^{-i\frac{\hat{P}^2t}{2m\hbar}} + e^{i\frac{\hat{P}^2t}{2m\hbar}}e^{-i\frac{\hat{P}^2t}{2m\hbar}}\hat{X} = \hat{X} + \frac{\hat{P}t}{m}.$$

Finally,

$$H_E = \frac{1}{2}m\omega^2\left[\hat{X} + \frac{\hat{P}t}{m}\right]^2.$$

Problem III

The Hamiltonian is

$$H = \begin{cases} H_0, & t < 0 \\ H_0 + W(t), & 0 \le t < \tau = \frac{4\pi}{\omega} \\ H_0, & t > \tau \end{cases}.$$

a)

$$|\psi_I(t)\rangle = U_0^\dagger(\tau,0)|\psi_S(t)\rangle = e^{-i4\pi(\hat{N}+1/2)}|\psi_S(t)\rangle = e^{-i4\pi n}e^{-i2\pi}|\psi_S(t)\rangle = |\psi_S(t)\rangle.$$

b) The effective Hamiltonian is:

$$\begin{aligned} H_E &= U_0^\dagger W(t) U_0 \\ &= \frac{i\hbar\Omega}{2}\left[e^{i\omega(\hat{N}+1/2)t}(\hat{a}^2 e^{i2\omega t} - (\hat{a}^\dagger)^2 e^{-i2\omega t})e^{-i\omega(\hat{N}+1/2)t}\right] \\ &= \frac{i\hbar\Omega}{2}\left\{e^{i2\omega t}[e^{i\omega(\hat{N}+1/2)t}\hat{a}e^{-i\omega(\hat{N}+1/2)t}]^2 - e^{-i2\omega t}[e^{i\omega(\hat{N}+1/2)t}\hat{a}^\dagger e^{-i\omega(\hat{N}+1/2)t}]^2\right\} \\ H_E &= \frac{i\hbar\Omega}{2}\left\{\hat{a}^2 - (\hat{a}^\dagger)^2\right\}. \end{aligned}$$

c) We use the expression of the $\hat{a}$ operators in terms of $\hat{X}$ and $\hat{P}$:

$$\begin{aligned} \hat{H}_E &= \frac{i\hbar\Omega}{2}\left\{\frac{1}{2}\left(\frac{\hat{X}}{\sigma} + i\frac{\hat{P}\sigma}{\hbar}\right)^2 - \frac{1}{2}\left(\frac{\hat{X}}{\sigma} - i\frac{\hat{P}\sigma}{\hbar}\right)^2\right\} \\ &= \frac{i\hbar\Omega}{4}\left\{\frac{\hat{X}^2}{\sigma^2} + \frac{i}{\hbar}(\hat{X}\hat{P} + \hat{P}\hat{X}) - \frac{\hat{P}^2\sigma^2}{\hbar^2} - \left[\frac{\hat{X}^2}{\sigma^2} - \frac{i}{\hbar}(\hat{X}\hat{P} + \hat{P}\hat{X}) - \frac{\hat{P}^2\sigma^2}{\hbar^2}\right]\right\} \\ &= -\frac{\Omega}{2}(\hat{X}\hat{P} + \hat{P}\hat{X}) \\ \hat{H}_E &= -\frac{\Omega}{2}\{\hat{X},\hat{P}\}, \quad \{\cdot\} = \text{anti-commutator}. \end{aligned}$$

d) We use the effective Hamiltonian of part b):

$$\hat{\mathbb{U}}_E(\tau) = e^{-\frac{i\tau}{\hbar}\frac{i\hbar\Omega}{2}(\hat{a}^2 - (\hat{a}^\dagger)^2)} = e^{\frac{\Omega\tau}{2}(\hat{a}^2 - (\hat{a}^\dagger)^2)} = e^{\frac{b}{2}(\hat{a}^2 - (\hat{a}^\dagger)^2)}.$$

e) In this case, we compare the definition the above operator with the new definition, and we look at the exponent of each function:

$$e^{-\hat{B}} = \hat{\mathbb{U}}_E(\tau) = e^{\frac{b}{2}(\hat{a}^2 - (\hat{a}^\dagger)^2)} \Longrightarrow \hat{B} = -\frac{b}{2}(\hat{a}^2 - (\hat{a}^\dagger)^2).$$

As a verification, the adjoint is:

$$\hat{B}^\dagger = -\frac{b}{2}((\hat{a}^\dagger)^2 - \hat{a}^2) = \frac{b}{2}(\hat{a}^2 - (\hat{a}^\dagger)^2) = -\hat{B}.$$

So $\hat{B}$ is anti-Hermitian.

f)

$$[\hat{B},\hat{a}] = -\frac{b}{2}\{[\hat{a}^2,\hat{a}] - [(\hat{a}^\dagger)^2,\hat{a}]\} = \frac{b}{2}[(\hat{a}^\dagger)^2,\hat{a}] = \frac{b}{2}\{\hat{a}^\dagger[\hat{a}^\dagger,\hat{a}] + [\hat{a}^\dagger,\hat{a}]\hat{a}^\dagger\} = -b\hat{a}^\dagger$$
$$[\hat{B},\hat{a}^\dagger] = -\frac{b}{2}\{[\hat{a}^2,\hat{a}^\dagger] - [(\hat{a}^\dagger)^2,\hat{a}^\dagger]\} = -\frac{b}{2}[\hat{a}^2,\hat{a}^\dagger] = -\frac{b}{2}\{\hat{a}[\hat{a},\hat{a}^\dagger] + [\hat{a},\hat{a}^\dagger]\hat{a}\} = -b\hat{a}.$$

g) We now from the previous part the commutators, and also

$$\hat{\mathbb{Q}}(b) = e^{-\hat{B}} = e^{\frac{b}{2}(\hat{a}^2-(\hat{a}^\dagger)^2)}, \quad \hat{\mathbb{Q}}^\dagger(b) = e^{\hat{B}} = e^{-\frac{b}{2}(\hat{a}^2-(\hat{a}^\dagger)^2)}, \quad \hat{\mathbb{Q}}(b)\hat{\mathbb{Q}}^\dagger(b) = \mathbb{1}.$$

To the application of the formula provided, we compute the following commutators:

$$[\hat{B},\hat{a}] = -b\hat{a}^\dagger, \quad [\hat{B},[\hat{B},\hat{a}]] = [\hat{B},-b\hat{a}^\dagger] = b^2\hat{a}, \quad [\hat{B},[\hat{B},[\hat{B},\hat{a}]]] = -b^3\hat{a}^\dagger, \quad \cdots$$

Then,

$$\begin{aligned}
\hat{\mathbb{Q}}^\dagger(b)\hat{a}\hat{\mathbb{Q}}(b) &= \hat{a} + (-b\hat{a}^\dagger) + \frac{1}{2!}[\hat{B},[\hat{B},\hat{a}]] + \frac{1}{3!}[\hat{B},[\hat{B},[\hat{B},\hat{a}]]] + \cdots \\
&= \hat{a} - b\hat{a}^\dagger + \frac{b^2}{2!}\hat{a} - \frac{b^3}{3!}\hat{a}^\dagger + \cdots \\
&= \left(1 + \frac{b^2}{2!} + \cdots\right)\hat{a} - \left(b - \frac{b^3}{3!}\right)\hat{a}^\dagger \\
\hat{\mathbb{Q}}^\dagger(b)\hat{a}\hat{\mathbb{Q}}(b) &= \cosh(b)\hat{a} - \sinh(b)\hat{a}^\dagger,
\end{aligned}$$

and,

$$[\hat{\mathbb{Q}}^\dagger(b)\hat{a}\hat{\mathbb{Q}}(b)]^\dagger = \hat{\mathbb{Q}}^\dagger(b)\hat{a}^\dagger\hat{\mathbb{Q}}(b) = \cosh(b)\hat{a}^\dagger - \sinh(b)\hat{a}.$$

h) We use both result from above

$$\begin{aligned}
\hat{\mathbb{Q}}^\dagger(b)\hat{X}\hat{\mathbb{Q}}(b) &= \frac{\sigma}{\sqrt{2}}\hat{\mathbb{Q}}^\dagger(b)(\hat{a}^\dagger + \hat{a})\hat{\mathbb{Q}}(b) = \frac{\sigma}{\sqrt{2}}\left[\hat{\mathbb{Q}}^\dagger(b)\hat{a}^\dagger\hat{\mathbb{Q}}(b) + \hat{\mathbb{Q}}^\dagger(b)\hat{a}\hat{\mathbb{Q}}(b)\right] \\
&= \frac{\sigma}{\sqrt{2}}\left[\cosh(b)\hat{a}^\dagger - \sinh(b)\hat{a} + \cosh(b)\hat{a} - \sinh(b)\hat{a}^\dagger\right] \\
&= \frac{\sigma}{\sqrt{2}}[(\hat{a}^\dagger + \hat{a})(\cosh(b) - \sinh(b))] \\
\hat{\mathbb{Q}}^\dagger(b)\hat{X}\hat{\mathbb{Q}}(b) &= e^{-b}\hat{X}.
\end{aligned}$$

For the momentum operator, we have similarly

$$\begin{aligned}
\hat{\mathbb{Q}}^\dagger(b)\hat{P}\hat{\mathbb{Q}}(b) &= \frac{i\hbar}{\sqrt{2}\sigma}\hat{\mathbb{Q}}^\dagger(b)(\hat{a}^\dagger - \hat{a})\hat{\mathbb{Q}}(b) = \frac{i\hbar}{\sqrt{2}\sigma}\left[\hat{\mathbb{Q}}^\dagger(b)\hat{a}^\dagger\hat{\mathbb{Q}}(b) - \hat{\mathbb{Q}}^\dagger(b)\hat{a}\hat{\mathbb{Q}}(b)\right] \\
&= \frac{i\hbar}{\sqrt{2}\sigma}\left[\cosh(b)\hat{a}^\dagger - \sinh(b)\hat{a} - \cosh(b)\hat{a} + \sinh(b)\hat{a}^\dagger\right] \\
&= \frac{i\hbar}{\sqrt{2}\sigma}[(\hat{a}^\dagger + \hat{a})(\cosh(b) + \sinh(b))] \\
\hat{\mathbb{Q}}^\dagger(b)\hat{P}\hat{\mathbb{Q}}(b) &= e^{b}\hat{P}.
\end{aligned}$$

The remaining ones are computed easily with the formular used in past exercise:

$$\hat{\mathbb{Q}}^\dagger(b)\hat{X}^2\hat{\mathbb{Q}}(b) = [\hat{\mathbb{Q}}^\dagger(b)\hat{X}\hat{\mathbb{Q}}(b)]^2 = e^{-2b}\hat{X}^2$$
$$\hat{\mathbb{Q}}^\dagger(b)\hat{P}^2\hat{\mathbb{Q}}(b) = [\hat{\mathbb{Q}}^\dagger(b)\hat{P}\hat{\mathbb{Q}}(b)]^2 = e^{2b}\hat{P}^2.$$

i) The ground state of QHO is:

$$\langle 0|\hat{X}|0\rangle = 0, \quad \langle 0|\hat{P}|0\rangle = 0, \quad \langle 0|\hat{X}^2|0\rangle = \frac{\sigma^2}{2}, \quad \langle 0|\hat{P}|0\rangle = \frac{\hbar^2}{2\sigma^2}.$$

Using the above results, and the definition of the state $|\varphi\rangle = \hat{\mathbb{Q}}(b)|0\rangle$

$$\begin{aligned}
\langle\varphi|\hat{X}|\varphi\rangle &= e^{-b}\langle 0|\hat{X}|0\rangle = 0\\
\langle\varphi|\hat{P}|\varphi\rangle &= e^{b}\langle 0|\hat{P}|0\rangle = 0\\
\langle\varphi|\hat{X}^2|\varphi\rangle &= e^{-2b}\langle 0|\hat{X}^2|0\rangle = e^{-2b}\frac{\sigma^2}{2}\\
\langle\varphi|\hat{P}^2|\varphi\rangle &= e^{2b}\langle 0|\hat{P}^2|0\rangle = e^{2b}\frac{\hbar^2}{2\sigma^2}.
\end{aligned}$$

j) The uncertainty is therefore

$$\left.\begin{aligned}
\Delta\hat{X} &= \sqrt{\langle\hat{X}^2\rangle} = e^{-b}\tfrac{\sigma}{\sqrt{2}}\\
\Delta\hat{P} &= \sqrt{\langle\hat{P}^2\rangle} = e^{b}\tfrac{\hbar}{\sqrt{2}\sigma}
\end{aligned}\right\}\Delta\hat{X}\Delta\hat{P} = \frac{\hbar}{2}.$$

k) The wavefunction in the $\{|x\rangle\}$ representation is $\varphi(x) = \langle x|\varphi\rangle$. We need to know the action of t

$$\hat{X}(\hat{\mathbb{Q}}(b)|x\rangle) = \hat{\mathbb{Q}}(b)\hat{\mathbb{Q}}^\dagger(b)\hat{X}\hat{\mathbb{Q}}(b)|x\rangle = \hat{\mathbb{Q}}(b)e^{-b}\hat{X}|x\rangle = xe^{-b}(\hat{\mathbb{Q}}(b)|x\rangle).$$

Therefore,

$$\hat{X}(\hat{\mathbb{Q}}(b)|x\rangle) = xe^{-b}(\hat{\mathbb{Q}}(b)|x\rangle) \Longrightarrow \hat{\mathbb{Q}}(b)|x\rangle = C|e^{-b}x\rangle.$$

The eigenstate $|e^{-b}x\rangle$ is proportional by the factor C. We find the coeficient c

$$\langle x'|x\rangle = \langle x|\hat{\mathbb{Q}}^\dagger(b)\hat{\mathbb{Q}}(b)|x\rangle = |c|^2\langle e^{-b}x'|e^{-b}x\rangle = |c|^2\delta[e^{-b}(x'-x)] = |c|^2e^{b}\delta(x'-x) = 1.$$

The coeffient is:

$$|c|^2e^b = 1 \longrightarrow c = e^{-b/2}.$$

Because the expression for the ground state is a gaussian of the form:

$$\psi_0(x) = \left(\frac{1}{\pi\sigma^2}\right)^{1/4} e^{-\frac{x^2}{2\sigma^2}},$$

we construct our function as:

$$\varphi(x) = \langle x|\hat{\mathbb{Q}}(b)|0\rangle = e^{-b/2}\langle e^bx|0\rangle = C\psi_0(e^bx) = e^{-b/2}\left(\frac{1}{\pi\sigma^2}\right)^{1/4} e^{b/2}e^{-\frac{e^{2b}x^2}{2\sigma^2}} = \left(\frac{1}{\pi\gamma^2}\right)^{1/4} e^{-\frac{x^2}{2\gamma^2}},$$

with $\gamma = \sigma e^{-b}$.

l) Knowing that

$$\hat{H}_0 = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2, \quad \langle\hat{X}^2\rangle_\varphi = e^{-2b}\frac{\sigma^2}{2}, \quad \langle\hat{P}^2\rangle_\varphi = e^{2b}\frac{\hbar^2}{2\sigma^2}.$$

The mean value of the Hamiltonian can be expressed in terms of the mean values of the position and momentum operators:

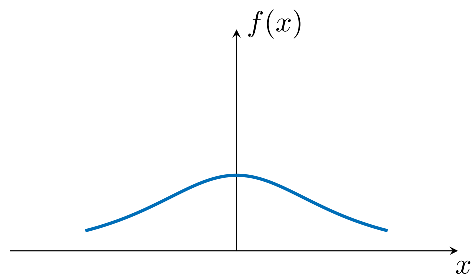
$$\langle H_0\rangle_\varphi = \frac{1}{2m}e^{2b}\frac{\hbar^2}{2\sigma^2} + \frac{1}{2}m\omega^2 e^{-2b}\frac{\sigma^2}{2} = \frac{\hbar\omega}{2}\cosh(2b),$$

where we substituted $\sigma = \sqrt{\hbar/m\omega}$ to simplify further the expression. For $b = 0$, we see that $\langle H_0\rangle_\varphi = \hbar\omega/2$ the ground state energy level.

m) The operator $\hat{\mathbb{Q}}(p)$ changes the uncertainty of the quadratures increasing one and reducing the other respectively so that the uncertainty product is maintained.

Problem IV

a) We plot the function $\text{sech}(x)$ to verify its parity. We can see that it is **even**.



This fact will facilitate us when computing ΔX, as we must integrate over $|\phi(x)|^2$ which therefore, is also even. We then have,

$$\langle X\rangle = \int_{-\infty}^{\infty} x|\phi(x)|^2\,dx = \frac{1}{2\beta}\int_{-\infty}^{\infty} x\,\text{sech}(x/\beta)\,dx = 0$$

$$\langle X^2\rangle = \int_{-\infty}^{\infty} x^2|\phi(x)|^2\,dx = \frac{1}{2\beta}\int_{-\infty}^{\infty} x^2\text{sech}(x/\beta)\,dx = \frac{\beta^2}{2}\int_{-\infty}^{\infty} u^2\text{sech}^2(u)\,du = \frac{\pi^2\beta^2}{12}.$$

The X uncertainty is

$$\Delta X = \sqrt{\langle X^2\rangle - \langle X\rangle^2} = \frac{\pi\beta}{2\sqrt{3}}.$$

Similarly, for the Fourier transform we have:

$$\langle P\rangle = \int_{-\infty}^{\infty} p|\hat{\phi}(p)|^2\,dp = \frac{\pi\beta}{4\hbar}\int_{-\infty}^{\infty} p\,\text{sech}^2(\frac{\pi\beta p}{2\hbar})\,dp = 0$$

$$\langle P^2\rangle = \int_{-\infty}^{\infty} p^2|\hat{\phi}(p)|^2\,dp = \frac{\pi\beta}{4\hbar}\int_{-\infty}^{\infty} p^2\text{sech}^2(\frac{\pi\beta p}{2\hbar})\,dp = \frac{2\hbar^2}{\pi^2\beta^2}\int_{-\infty}^{\infty} u^2\text{sech}^2(u)\,du = \frac{\hbar^2}{\beta^2 3}.$$

Thus

$$\Delta P = \sqrt{\langle P^2\rangle - \langle P\rangle^2} = \frac{\hbar}{\beta\sqrt{3}}.$$

The uncertainty product is

$$\Delta X \Delta P = \frac{\pi\beta}{2\sqrt{3}}\frac{\hbar}{\beta\sqrt{3}} = \frac{\hbar\pi}{6}.$$

b) The evolution in $\pi/2\omega$ gives a well-known quantity, a scaled Fourier transform of the wavefunction.

$$\Phi(x, \frac{\pi}{2\omega}) = U(\frac{\pi}{2\omega}, 0)\Phi(x, 0) = e^{-i\pi/4}\sqrt{\frac{\hbar}{\sigma^2}}\mathcal{F}\{\Phi(x, 0)\}\big|_{p=\hbar x/\sigma^2}$$

We can see that the function to be computed its Fourier transform is spatially shifted by x_0 so we could directly use the respective property of Fourier transform of a shifter function:

$$\mathcal{F}\{\Phi(x, 0)\} = \hat{\Phi}(p, 0) \Longrightarrow \mathcal{F}\{\Phi(x - x_0, 0)\} = e^{-ipx_0/\hbar}\hat{\Phi}(p, 0).$$

So,

$$\Phi(x, \frac{\pi}{2\omega}) = -e^{-i\pi/4}\sqrt{\frac{\hbar}{\sigma^2}}\left[e^{-ipx_0/\hbar}\hat{\Phi}(p, 0)\right]\Bigg|_{p=\hbar x/\sigma^2} = -\sqrt{\frac{\pi\beta}{4\sigma^2}}e^{-i\pi/4}e^{-i\frac{xx_0}{\sigma^2}}\,\mathrm{sech}(\frac{\pi\beta x}{2\sigma^2}).$$

c) To maintain the width $\Delta X = \frac{\pi\beta}{2\sqrt{3}}$, we compute ΔX for $\Phi(0, \pi/2\omega)$ and equate it to the uncertainty at $t = 0$:

$$\left.\begin{aligned} &\langle X\rangle = 0 \\ &\langle X^2\rangle = \frac{\pi\beta}{4\sigma^2}\int_{-\infty}^{\infty} x^2\mathrm{sech}^2(\frac{\pi\beta x}{2\sigma^2})\,dx = \frac{\sigma^4}{3\beta^2}. \end{aligned}\right\}\Delta X = \sqrt{\langle X^2\rangle} = \frac{\sigma^2}{\sqrt{3}\beta}.$$

Equating it with the uncertainty of the wavefunction at $t = 0$:

$$\frac{\pi\beta}{2\sqrt{3}} = \frac{\sigma^2}{\sqrt{3}\beta} \longrightarrow \beta = \sqrt{\frac{2\sigma^2}{\pi}}.$$

Problem V

We handle the problem by approximating the potential given with its second-order Taylor expansion:

$$V(x) = -V_0 - \frac{V_0}{2b^2}x^2 + O(x^3) + \cdots = \frac{1}{2}m\omega^2x^2.$$

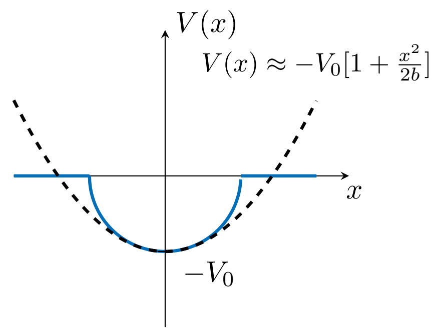
The figure below represents the behavior of this approximation versus the real potential.

Comparing the quadratic term of the expansion with the QHO yields the following frequency:

$$\omega = \sqrt{\frac{V_0}{mb^2}}.$$

The energy levels in the QHO is:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \cdots.$$



So, in this case they will be shifted

$$E_n' = -V_0 + E_n = -V_0 + \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \cdots .$$

The ground and first excited state energy eigenvalues are:

$$E_0 = -V_0 + \frac{1}{2}\hbar\omega, \quad E_1 = -V_0 + \frac{3}{2}\hbar\omega.$$