OPTI 570 Lecture 9

Approach II

free - particle wave packet $\langle \hat{p} \rangle = p_0$, $\langle \hat{x} \rangle = x_0$

Gaussian wave packet s.t. $|\tilde{\psi}(p)|^2$ - half-width (1/e) of a

$$\underbrace{\langle p | \psi(t_0) \rangle = \left(\frac{1}{\pi a^2}\right)^{1/4} \underbrace{e^{-i x_0 p/\hbar}}_{\text{-pos transl.}} e^{-(p-p_0)^2/2a^2}}_{\text{in p - representation}}$$

translation at 0, width

- in position representation.

$$\langle x | \psi(t_0) \rangle = \left(\frac{1}{\pi w^2}\right)^{1/4} \underbrace{e^{-i x_0 p_0/\hbar}}_{\text{global phase factor}} e^{i p_0 x/\hbar} \; e^{-(x-x_0)^2/2w^2} \qquad w = \frac{\hbar}{a}$$

Goal: $\langle \hat{x} \rangle (t) = \langle x(t) | \hat{x} | \psi(t) \rangle$

$$|\psi(t)\rangle = \underline{\hat{u}(t, t_0)} \, |\psi(t_0)\rangle$$

$$\hat{u}(t, t_0) = e^{-i \hat{H}(t-t_0)/\hbar}$$

free particle: $\hat{u}(t, t_0) = e^{-i \frac{\hat{p}^2}{2m} (t-t_0)/\hbar}$

$$\langle p | \hat{u} | \psi(t_0) \rangle = \underbrace{\left(\frac{1}{\pi a^2}\right)^{1/4}}_{\text{normalisation}} \underbrace{e^{-\frac{i p^2 (t-t_0)}{2m\hbar}}}_{\text{- time dependent part}} \underbrace{e^{-\frac{i x_0 p}{\hbar}} e^{-\frac{(p-p_0)^2}{2a^2}}}_{\text{in. state}}$$

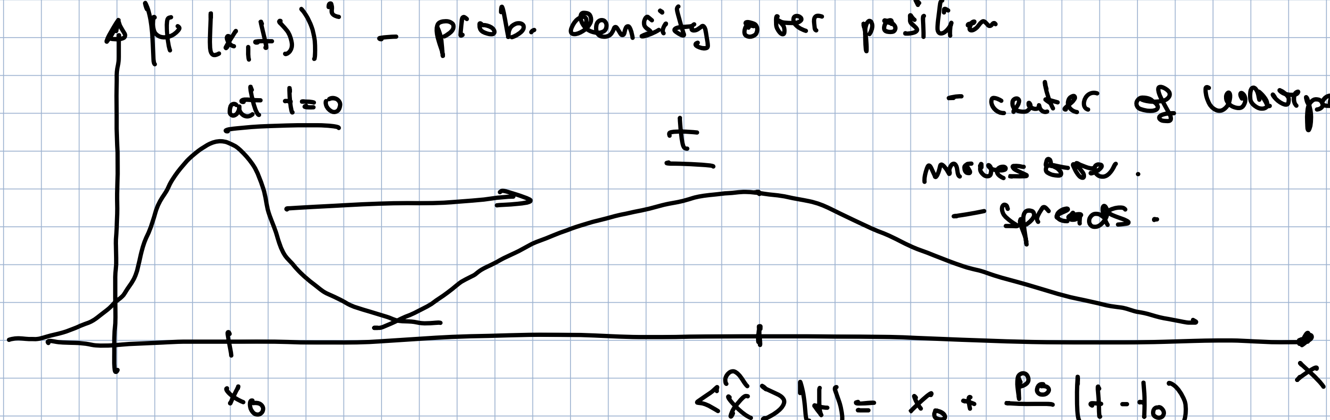
$|\psi(p,t)|^2$ - does not change in time $\Rightarrow$ $|\psi(x,t)|^2$ - will change

Calculate: $\psi(x, t)$?

1. $\langle x | \hat{u} | \psi(t_0) \rangle = ?$
2. $\psi(x,t) = FT^{-1}\{ \tilde{\psi}(p,t) \}$

3. Reason its properties

3. $\langle \hat{x} \rangle (t) = \int_{-\infty}^{\infty} \psi^*(t)\,\psi(t)\, x \, dx$ -

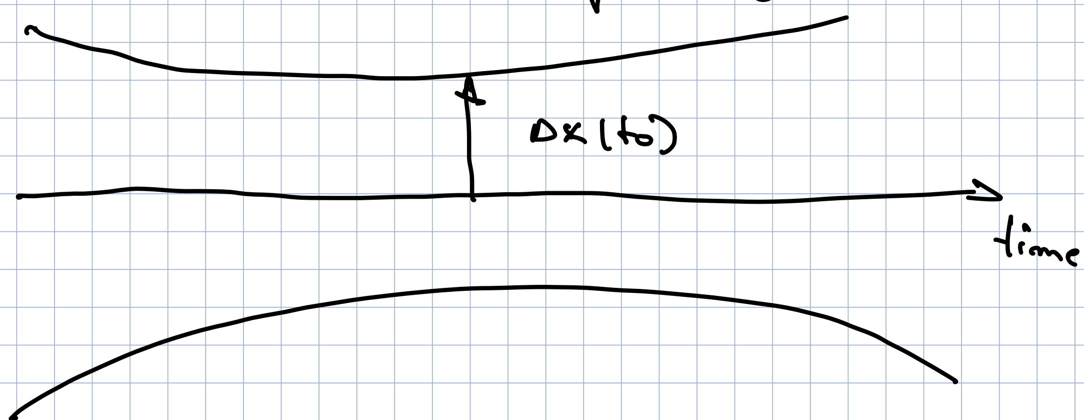


- center of wavepacket moves over.
- spreads.

$\langle \hat{x} \rangle (t) = x_0 + \frac{p_0}{m}(t - t_0)$

- Gen unc. principle - Opt. PSET 4, P V
- $\Delta p(t)$ - const. in time, $\langle p \rangle$, $\langle p^2 \rangle$ - const in time
- $\Delta x(t)$ - changes in time

$$(\Delta x)(t') = \Delta x(t_0)\sqrt{1 + \frac{t^2}{\tau^2}} \qquad \tau = \frac{m(\Delta x)(t=0)}{\Delta p(t=0)}$$



Approch 3

$$\langle \hat{x}(t)\rangle = \langle \psi(t)|\hat{x}|\psi(t)\rangle =$$

$$= \langle \psi(t_0)|\underbrace{\hat{U}^\dagger(t,t_0)\,\hat{x}\,\hat{U}(t,t_0)}_{\hat{X}_H(t)}|\psi(t_0)\rangle$$

$$\hat{X}_H(t) = e^{\frac{i\hat{p}^2(t-t_0)}{2m\hbar}}\,\hat{x}\,e^{\frac{-i\hat{p}^2(t-t_0)}{2m\hbar}}$$

$\hat{x} \to i\hbar\frac{\partial}{\partial p}$

$|G\rangle \in \mathcal{E}$

$\langle p|G\rangle = \bar{G}(p)$

$$\langle p|\underbrace{\hat{X}_H(t)|G\rangle}\ = e^{\frac{ip^2(t-t_0)}{2m\hbar}}\left\{i\hbar\frac{\partial}{\partial p}\left[e^{-\frac{ip^2(t-t_0)}{2m\hbar}}\bar{G}(p)\right]\right\}$$

$$= \ldots = \left[\frac{p}{m}(t-t_0) + i\hbar\frac{\partial}{\partial p}\right]\bar{G}(p)$$

$$\boxed{\hat{X}_H(t) = \frac{\hat{p}(t-t_0)}{m} + \hat{x}}$$

$$\langle \hat{x}_H(t)\rangle = \langle \hat{p}\rangle_{t_0}\frac{(t-t_0)}{m} + \langle \hat{x}\rangle t_0$$

$$\langle \hat{x}_H(t)\rangle = \underbrace{p_0\frac{(t-t_0)}{m}}\ + x_0$$

time-dependence

Heisenberg picture: time-evolution attached to operators

Schrödinger picture: — // ——— states

In general:

$$\hat{A}_H(t) = \hat{U}^\dagger(t)\,\hat{A}_S\,\hat{U}(t)$$

$$\langle \hat{A}\rangle(t) = \langle \psi(t_0)|\hat{A}_H(t)|\psi(t_0)\rangle$$

Equation of motion: - CT Ch3 G

$$\boxed{i\hbar \frac{d}{dt}\hat{A}_H(t) = [\hat{A}_H, \hat{H}_H] + i\hbar\, \hat{U}^+ \left[\frac{d}{dt}\hat{A}(t)\right]\hat{U}}$$

Free particle

$$\hat{H}_H = \hat{U}^+ \hat{H}_S \hat{U} =$$
$$= \underbrace{\hat{U}^+ \hat{U}}\, \hat{H}_S =$$
$$= \hat{H}_S \quad \text{- time independent}$$

$$i\hbar \frac{d}{dt}\hat{P}_H(t) = [\hat{P}_H, \hat{H}_H] + i\hbar\, U^+ \cancel{\left(\frac{d}{dt}\hat{P}_S\right)}^{0} U =$$

$$\boxed{\begin{aligned}\hat{P}_H &= \hat{U}^+ \hat{P}_S \hat{U} = \\ &= \hat{P}_S\end{aligned}}$$

$$\frac{d}{dt}\hat{X}_H(t) = \frac{1}{i\hbar}[\hat{x}_H(t), \hat{H}_H(t)] =$$

$\hat{H}_S$

$$\frac{d}{dt}\hat{x}_H(t) = \frac{1}{i\hbar}(i\hbar\, \hat{P}_H/m)$$

$$\boxed{\frac{d\hat{x}_H(t)}{dt} = \frac{\hat{P}_H}{m}}$$

$$\boxed{\frac{d}{dt}\hat{P}_H(t) = 0}$$

$$P_H(t_0) = P_S = P_0$$
$$x_H(t) = x_S$$

$$\Rightarrow \boxed{\hat{X}_H(t) = \frac{1}{m}\hat{P}_S(t-t_0) + \hat{x}_S}$$

$$x(t) = \frac{P}{m}(t-t_0) + x_0$$

classical   QM

eq. of motion:

$$x(t) \longrightarrow \hat{X}_H(t)$$
$$p(t) \longrightarrow \hat{P}_H(t)$$

Example: classical 1D Harmonic osillator

$$E = \frac{p(t)^2}{2m} + \frac{1}{2} m\omega^2 x^2(t)$$

Eq. of motion:

$$x(t) = x_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t)$$

$$p(t) = p_0 \cos(\omega t) - m\omega x_0 \sin(\omega t)$$

In QM:

$$\hat{X}_H(t) = \hat{X}_S \cos(\omega t) + \frac{\hat{P}_S}{m\omega} \sin(\omega t)$$

$$\underbrace{\hat{P}_H(t) = \hat{P}_S \cos(\omega t) - m\omega \hat{X}_S \sin(\omega t)}$$

exp values:

$$\underbrace{\langle x \rangle(t) = \langle x \rangle(t=0) \cos(\omega t) + \frac{\langle p \rangle(t=0)}{m\omega} \sin(\omega t)}$$

One last thing:

Commut.: $[\hat{x}, \hat{p}] = i\hbar$ $\Rightarrow$ $\langle x | \psi \rangle = \psi(x) \in \mathcal{F}_x$

$\overset{\Sigma}{}$ F.T. " $\langle p | \psi \rangle = \tilde{\psi}(p) \in \mathcal{F}_p$

– define operators through comm. relations

On Thursday: QHO

def $\hat{a}$ s.t. $[\hat{a}, \hat{a}^+] = 1$

$\hat{N}$ s.t. $\hat{N} = \hat{a}^+ a$

$\hat{N} | n \rangle = n | n \rangle$

$\hat{N}$ – Hermitian?

$\hat{a} | \psi \rangle = \ldots | \psi \rangle$

$\hat{a}^+ | \psi \rangle = \ldots | \psi \rangle$