Recitation

OPTI 570 Quantum Mechanics

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Exercise HII-4

1. sfaf

$$K = |\varphi\rangle\langle\psi| \longrightarrow K^\dagger = (|\varphi\rangle\langle\psi|)^\dagger = |\psi\rangle\langle\varphi|.$$

   This to be Hermitian we must have that $|\varphi\rangle = |\psi\rangle$, the other is implicity stated as well.

2.

$$K^2 = |\varphi\rangle\langle\psi|\varphi\rangle\langle\psi| = \langle\psi|\varphi\rangle K,$$

   To be a projector, we need that $K^2 = K$, therefore $\langle\psi|\varphi\rangle = 1$.

3. sgasg

$$K = \lambda P_1 P_2, \quad K = |\varphi\rangle\langle\psi|,\ P_1 = |\varphi\rangle\langle\varphi|,\ P_2 = |\psi\rangle\langle\psi|.$$

$$P_1 P_2 = |\varphi\rangle\langle\varphi|\psi\rangle\langle\psi| = \langle\varphi|\psi\rangle|\varphi\rangle\langle\psi|,$$

   so

$$K = \frac{1}{\langle\varphi|\psi\rangle} P_1 P_2.$$

   But this is not going to work because $\langle\varphi|\psi\rangle = 0$ when the vectors are orthonogal. This incise is not possible. We may need to use absurb rule: proove the negation (counterexample).

HII-5

What is necessary to be a projector is the idempotency: $(P_1P_2)^2 = P_1P_2$:

$$(P_1P_2)^2 = P_1P_2P_1P_2 \overset{(a)}{=} P_1P_1P_2P_2 = P_1^2P_2^2 = (P_1P_2)^2$$

In (a) we have asummed that $[P_1, P_2] = 0$.

The subspace they project onto is the intersection of them: $\mathscr{E}_1 \cap \mathscr{E}_2$. The only thing that survivies is the same element components of them, all other projections that are orthogonal will die .

HII-8

a. The hint is the following commutator:

$$\begin{aligned}
[X,H] &= [X, \frac{P^2}{2m} + V(X)] = [X, \frac{P^2}{2m}] + [X, V(X)] \\
&= [X, \frac{P^2}{2m}] \\
&= \frac{1}{2m} i\hbar(2P) \\
[X,H] &= \frac{i\hbar P}{m}.
\end{aligned}$$

We insert the above expression:

$$\begin{aligned}
\langle\varphi_n|P|\varphi_{n'}\rangle &= \frac{m}{i\hbar}\langle\varphi_n|[X,H]|\varphi_{n'}\rangle \\
&= \frac{m}{i\hbar}[\langle\varphi_n|XH|\varphi_{n'}\rangle - \langle\varphi_n|HX|\varphi_{n'}\rangle] \\
&= \frac{m}{i\hbar}[E_{n'}\langle\varphi_n|X|\varphi_{n'}\rangle - E_n\langle\varphi_n|X|\varphi_{n'}\rangle] \\
\langle\varphi_n|P|\varphi_{n'}\rangle &= \frac{m}{i\hbar}(E_{n'} - E_n)\langle\varphi_n|X|\varphi_{n'}\rangle.
\end{aligned}$$

b. gasgag

$$\begin{aligned}
\sum_{n'}(E_n - E_{n'})^2|\langle\varphi_n|X|\varphi_{n'}\rangle|^2 &= \frac{\hbar}{m^2}\langle\varphi_n|P^2|\varphi_{n'}\rangle \\
&= \frac{\hbar^2}{m^2}\langle\varphi_n|P\left(\sum_k |\varphi_k\rangle\langle\varphi_k|\right)P|\varphi'_n\rangle \\
&= \frac{\hbar^2}{m^2}\sum_k \langle\varphi_n|P|\varphi_k\rangle\langle\varphi_k|P|\varphi_{n'}\rangle \\
&= \frac{\hbar^2}{m^2}\left[\frac{m}{i\hbar}(E_k - E_n)\langle\varphi_n|X|\varphi_k\rangle\right]\left[\frac{m}{i\hbar}(E_{n'} - E_k)\langle\varphi_k|X|\varphi_{n'}\rangle\right] \\
\sum_{n'}(E_n - E_{n'})^2|\langle\varphi_n|X|\varphi_{n'}\rangle|^2 &= \sum_{n'}(E_n - E_{n'})^2|\langle\varphi_n|X|\varphi_{n'}\rangle|^2.
\end{aligned}$$

Correct this

HII-10

Inserting closure relation:

$$
\begin{aligned}
\langle x|XP|\varphi\rangle &= \int dx'\ \langle x|X|x'\rangle\langle x'|P|\varphi\rangle \\
&= \int dx'\ x'\langle x|x'\rangle\langle x'|P|\varphi\rangle \\
&= \int dx'\ x'\delta(x-x')\langle x'|P|\varphi\rangle \\
\langle x|XP|\varphi\rangle &= x\langle x|P|\varphi\rangle \\
&= x\int dp\ \langle x|p\rangle\langle p|P|\varphi\rangle \\
&= x(2\pi\hbar)^{-1/2}\int dp\ e^{ixp/\hbar}\langle p|P|\varphi\rangle \\
&= x(2\pi\hbar)^{-1/2}\int dp\ e^{ixp/\hbar}\langle p|P|\varphi\rangle \\
&= x(2\pi\hbar)^{-1/2}\int dp\ e^{ixp/\hbar}[p\langle p|\varphi\rangle] \\
&= x(2\pi\hbar)^{-1/2}\int dp\ e^{ixp/\hbar}[p\tilde{\psi}(p)] \\
&= x\frac{\hbar}{i}(2\pi\hbar)^{-1/2}\int dp\ e^{ixp/\hbar}\left[\frac{ip}{\hbar}\tilde{\psi}(p)\right] \qquad \left(\mathscr{F}[\psi^{(n)}(x)] = \left(\frac{ip}{\hbar}\right)^n\tilde{\psi}(p)\right) \\
\langle x|XP|\varphi\rangle &= -i\hbar x\partial_x\psi(x).
\end{aligned}
$$