OPTI 570 LECTURE Th OCT 23

Last time

Spin 1/2 , $s = \frac{1}{2}$, $m_u = \pm \frac{1}{2}$

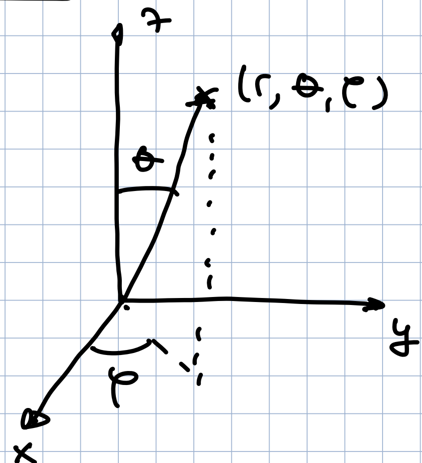
$$S^2 |\pm\rangle_u = \frac{3}{4}\hbar^2 |\pm\rangle_u$$
$$S_u |\pm\rangle_u = \pm \frac{\hbar}{2} |\pm\rangle_u$$

$$\bar{S} = (S_x, S_y, S_z)$$
$$\bar{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

$$\bar{S} = \frac{\hbar}{2}\bar{\sigma}$$
$$\bar{S}_u = \frac{\hbar}{2}\bar{\sigma}_u$$

$$\hat{u} = \begin{pmatrix} \frac{x}{r} \\ \frac{y}{r} \\ \frac{z}{r} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

$$S_u^{(z)} = \frac{\hbar}{2}\underbrace{\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}}_{\sigma_u}$$



Ex: $\hat{u} = \hat{z}$, $\theta = 0$

$$S_z^{(z)} = \frac{\hbar}{2}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_z}$$

$\hat{u} = \hat{y}$ $\quad \theta = \frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$

$$S_y^{(z)} = \frac{\hbar}{2}\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_y}$$

$\hat{u} = \hat{x}$ $\quad \theta = \frac{\pi}{2}$ $\quad \varphi = 0$ $\quad \Rightarrow S_x^{(z)} = \frac{\hbar}{2}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_x}$

For any $\hat{u}$, what are the eigenvalues of $\hat{S}_u$?: $\pm \frac{\hbar}{2}$

$+\frac{\hbar}{2}$ w/ eigenvector $\begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\varphi/2} \\ \sin\frac{\theta}{2}\, e^{i\varphi/2} \end{pmatrix}$

$-\frac{\hbar}{2}$ w/ eigenvector $\begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\varphi/2} \\ \cos\frac{\theta}{2}\, e^{i\varphi/2} \end{pmatrix}$

$|+\rangle_z^{(z)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\uparrow\rangle$

$|-\rangle_z^{(z)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\downarrow\rangle$

$|+\rangle_u = \cos\frac{\theta}{2}\, e^{-i\frac{\varphi}{2}}\, |+\rangle_z + \sin\frac{\theta}{2}\, e^{i\frac{\varphi}{2}}\, |-\rangle_z$

$|-\rangle_u = -\sin\frac{\theta}{2}\, e^{-i\frac{\varphi}{2}}\, |+\rangle_z + \cos\frac{\theta}{2}\, e^{i\frac{\varphi}{2}}\, |-\rangle_z$

OR

$|+\rangle_u = \underbrace{\cos\frac{\theta}{2}}_{a}\, |+\rangle_z + \underbrace{\sin\frac{\theta}{2}\, e^{i\varphi}}_{b}\, |-\rangle_z$ FG p.68

$|-\rangle_u = \sin\frac{\theta}{2}\, |+\rangle_z - \cos\frac{\theta}{2}\, e^{i\varphi}\, |-\rangle_z$

$|a|^2 + |b|^2 = 1$

any $|\psi\rangle \in \mathcal{E}_{1/2}$

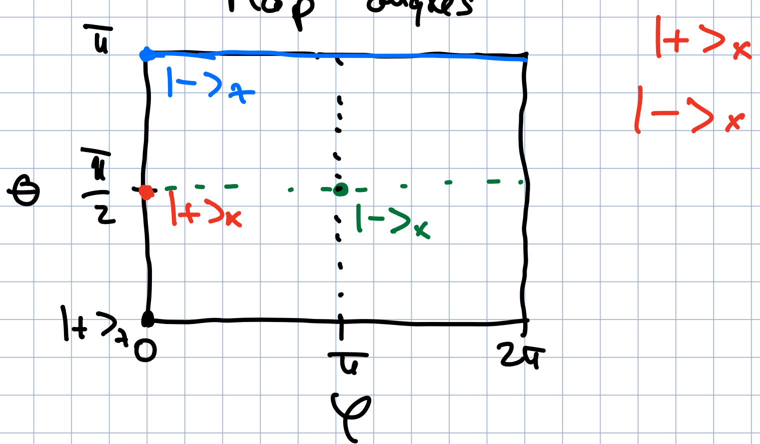
$|\psi\rangle = a\,|+\rangle_z + b\,|-\rangle_z$

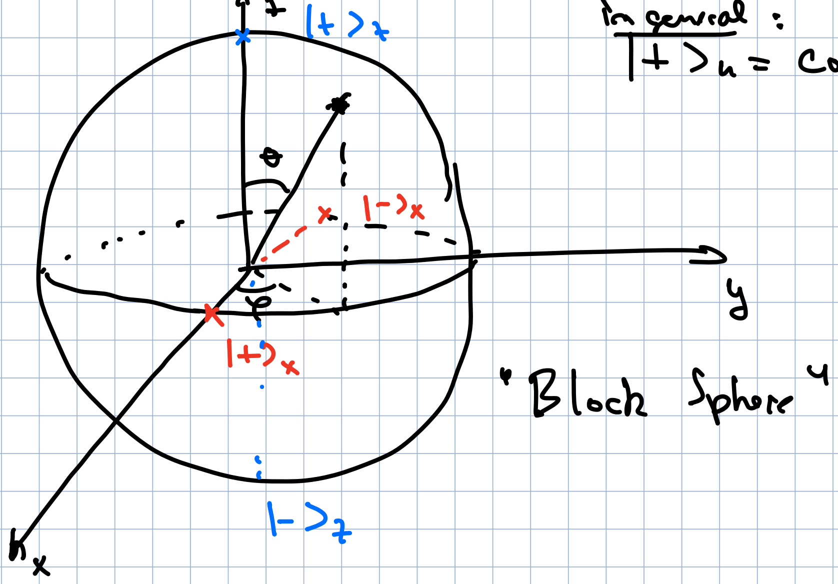
$= \cos\frac{\theta}{2}\, |+\rangle_z + \sin\frac{\theta}{2}\, e^{i\varphi}\, |-\rangle_z =$

$= |+\rangle_u$

Graphical Representations

Map angles





in general:

$$|+\rangle_u = \cos\frac{\theta}{2}\,|+\rangle_z + \sin\frac{\theta}{2}\,e^{i\varphi}|-\rangle_z$$

"Bloch Sphere"

Examples: $\{|+\rangle_z, |-\rangle_z\}$ basis for $\mathcal{E}_{\frac{1}{2}}$

$|+\rangle_x : \theta = \frac{\pi}{2}, \varphi = 0$ $\qquad |+\rangle_x = \frac{1}{\sqrt{2}}|+\rangle_z + \frac{1}{\sqrt{2}}|-\rangle_z$

$$|-\rangle_x = \frac{1}{\sqrt{2}}|+\rangle_z - \frac{1}{\sqrt{2}}|-\rangle_z$$

$$|+\rangle_y = \frac{1}{\sqrt{2}}|+\rangle_z + \frac{i}{\sqrt{2}}|-\rangle_z$$

$$|-\rangle_y = \frac{1}{\sqrt{2}}|+\rangle_z - \frac{i}{\sqrt{2}}|-\rangle_z$$

Particle w/ angular momentum in a magnetic field

classically: $\vec{\mu} \propto \vec{\mathcal{L}}$ ← ang. mom.

mag. moment.

$$\vec{\mu} = \gamma_e \vec{\mathcal{L}}$$

"gyromagnetic ratio"

quantum: $\vec{\mu} = \gamma \vec{J}$ ← gen. AM vector operator

operator

ex: electron in a state w/ orb. AM about z axis w/ $\hbar$

$$\vec{\mu}_L = -\underbrace{\left(\frac{\mu_B}{\hbar}\right)}_{\gamma\text{ - gyromagnetic ratio}}\vec{L}$$

$$\vec{\mu}_L = -\underbrace{\left(\frac{e\hbar}{2m_e}\right)}\underbrace{\frac{\vec{L}}{\hbar}}_{\text{unitless}}$$

μ_B – Bohr magneton $= 9.3 \cdot 10^{-24} \frac{J}{T}$

In general : $\bar{\mu} = \gamma \bar{S}$

γ depends on the system or particle.

e^- spin, $\gamma_e = \frac{\mu_B}{\hbar}\underbrace{(-2.002\ldots)}_{} < 0$

n^0 spin, $\gamma_N = \frac{\mu_N}{\hbar}(-3.8\ldots)$ $\underset{<0}{\text{imaginary}}$

p^+ spin, $\gamma_p = \frac{\mu_N}{\hbar}(5,59\ldots) > 0$ $\quad \mu_N = \frac{e\hbar}{2m_p}$ mass of proton

Spin $\frac{1}{2}$ particle in magnetic field

$$H = -\bar{\mu}\cdot\bar{B}$$

$$\bar{\mu} = \gamma\bar{S} = \frac{\hbar}{2}\gamma\bar{\sigma}$$

$$H = -\frac{\hbar\gamma}{2}\bar{\sigma}\cdot\bar{B}$$

case : $\bar{B} = B_0\hat{u}$, $B_0 > 0$ constant in space and time

$$H = -\frac{\hbar\gamma B_0}{2}\hat{\sigma}_u \qquad \omega_0 = -\gamma B_0$$

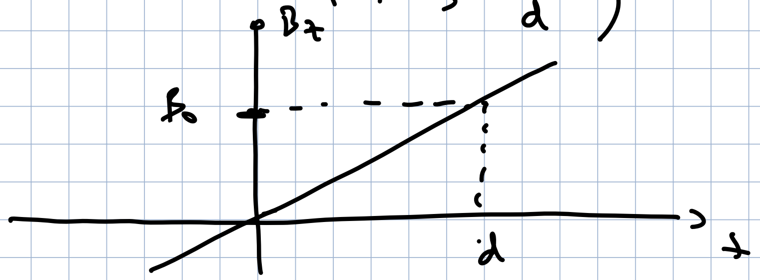
$$H = \frac{1}{2}\hbar\omega_0\hat{\sigma}_u$$

H eigenstates $|+\rangle_u$ $\quad |-\rangle_u$

eigenvalues $E_+ = \frac{1}{2}\hbar\omega_0$ $\quad E_- = -\frac{1}{2}\hbar\omega_0$

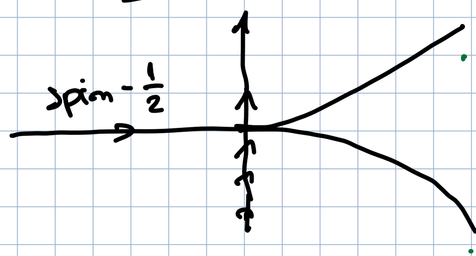
case 2 $\bar{B}$ constant in time, but a linear gradient in space

$$\bar{B} = \left(0, 0, \frac{B_0 \cdot z}{d}\right)$$



$$H = -\frac{\bar{\gamma}\hbar}{2} \cdot \underbrace{\frac{\bar{B}_0}{d}\, z}\ \hat{\sigma}_z$$ Mol operator

Stern-Gerlach



	$\gamma > 0$	$\gamma < 0$
up	$\|+\rangle_z$	$\|-\rangle_z$
down	$\|-\rangle_z$	$\|+\rangle_z$

Spin 1/2, constant uniform magnetic field

$$\bar{B} = B_0 \hat{z}$$

$$H = \frac{1}{2}\hbar\omega_0 \sigma_z \qquad \omega_0 = -\gamma B_0$$

$$E_\pm = \pm\frac{\hbar\omega_0}{2}$$

$$|\psi(0)\rangle = |+\rangle_u = \cos\frac{\theta}{2}|+\rangle_z + \sin\frac{\theta}{2} e^{i\varphi}|-\rangle_z$$

$$|\psi(t)\rangle = ?$$

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle = e^{-\frac{iHt}{\hbar}}|+\rangle_u = e^{-\frac{i}{2}\omega_0 t\hat{\sigma}_z}|+\rangle_u$$

$$= e^{-i\omega_0 t/2}\cos\frac{\theta}{2}|+\rangle_z + e^{i\omega_0 t/2}\sin\frac{\theta}{2}e^{i\varphi}|-\rangle_z =$$

$$= e^{-i\omega_0 t/2}\left[\cos\frac{\theta}{2}|+\rangle_z + \sin\frac{\theta}{2}\underline{e^{i\varphi(t)}}|-\rangle_z\right]$$

where $\varphi(t) = \varphi + \omega_0 t$