OPTI 570 LECTURE Tu Sep 30

Last time - QHO

dimensionless $\tilde{x}, \tilde{p}$ - scaled by ch. length $\sigma = \sqrt{\frac{\hbar}{m\omega}}$

Eigenvalue eqs

pos: $\frac{1}{2}\left(\tilde{x}^2 - \frac{\partial^2}{\partial \tilde{x}}\right) \varphi_n(\tilde{x}) = \tilde{E}_n \varphi_n(\tilde{x})$

mom: $\tilde{p}$, $\tilde{p}$, $\tilde{p}$, $\tilde{p}$

$\varphi_n(\tilde{x}) \xrightarrow{FT} \tilde{\varphi}_n(\tilde{p})$

functions: Hermite - Gaussian

QHO: Solve

- $\hat{H}$ given, find energy eigenvalues
- Find energy eigenstates

$\tilde{H}|\varphi_n\rangle = \tilde{E}_n|\varphi_n\rangle$, $\tilde{E}_n = \frac{E_n}{\hbar\omega}$, $\tilde{H} = \frac{H}{\hbar\omega}$

Define $\hat{a} = \frac{1}{\sqrt{2}}(\tilde{x} + i\tilde{p})$

$\hat{a}^+ = \frac{1}{\sqrt{2}}(\tilde{x} - i\tilde{p})$ $\Rightarrow$ $\tilde{x} = \frac{1}{\sqrt{2}}(\hat{a}^+ + \hat{a})$

$\tilde{p} = \frac{i}{\sqrt{2}}(\hat{a}^+ - \hat{a})$

$[\tilde{x}, \tilde{p}] = i \Rightarrow [a, a^+] = 1$

$\hat{H} = \frac{1}{2}\tilde{p}^2 + \frac{1}{2}\tilde{x}^2 \Rightarrow \tilde{H} = \underbrace{a^+ a}_{N} + \frac{1}{2}$

So: $\tilde{H}|\varphi_n\rangle = \tilde{E}_n|\varphi_n\rangle$

$\left(a^+ a + \frac{1}{2}\right)|\varphi_n\rangle = \tilde{E}_n|\varphi_n\rangle$

$a^+ a|\varphi_n\rangle = \underbrace{\left(\tilde{E}_n - \frac{1}{2}\right)}_{0,1,2,3,\dots}|\varphi_n\rangle$

$= n \geq 0$, integer

$$\boxed{\begin{aligned}\tilde{E}_n &= n + \frac{1}{2} \\ E_n &= \hbar\omega\left(n + \frac{1}{2}\right)\end{aligned}}\quad , \; n \geq 0, \text{ integer}$$

Lowest energy state - ground state

$$\underline{E_0 = \frac{1}{2}\hbar\omega,} \qquad \text{w/ ket } \underline{|\varphi_0\rangle}$$

pos. repr. $\langle x|\varphi_0\rangle = \varphi_0(x)$

$$\varphi_n(x) = \langle x|\varphi_n\rangle = \frac{1}{\sqrt{n!}}\langle x|(a^\dagger)^n|\varphi_0\rangle =$$

$$= \frac{1}{\sqrt{n!}}\underbrace{\left(\frac{1}{\sqrt{2}}\left(\frac{x}{\sigma} - \sigma\frac{d}{dx}\right)\right)^n}\ \underline{\varphi_0(x)}$$

generate Hermite - Gaussian functions

Representations

position $\{|x\rangle\}$

momentum $\{|p\rangle\}$

number $\{|n\rangle\}$-energy $|\varphi_n\rangle$

Closure relations

$$\mathbb{1} = \int_x |x\rangle\langle x|$$

$$\mathbb{1} = \int_p |p\rangle\langle p|$$

$$\mathbb{1} = \sum_{n=0}^{\infty} |\varphi_n\rangle\langle\varphi_n|$$

$$\{|\varphi_0\rangle, |\varphi_1\rangle, |\varphi_2\rangle \dots\}$$

$$\Rightarrow |\varphi_1\rangle_{\{\varphi\}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

Recap: $\hat{A}$ in the $\{|\varphi_n\rangle\}$ repr.

$$\hat{A} = \hat{\mathbb{1}}\hat{A}\hat{\mathbb{1}} = \sum_{\substack{n=0 \\ m=0}}^{\infty} \underbrace{\langle\varphi_m|\hat{A}|\varphi_n\rangle}_{\text{matrix elements}}\underbrace{|\varphi_m\rangle\langle\varphi_n|}_{\text{connectors}}$$

ex: $N|\varphi_n\rangle = n|\varphi_n\rangle$

$$\langle\varphi_m|N|\varphi_n\rangle = n\langle\varphi_m|\varphi_n\rangle = n\cdot\delta_{mn}$$

$$N \to \begin{pmatrix} 0 & & & & \\ & 1 & & 0 & \\ & & 2 & & \\ & & & 3 & \\ & 0 & & & 4 \\ & & & & & \ddots \end{pmatrix} \qquad N = \sum_{n=0}^{\infty} n |\varphi_n\rangle\langle\varphi_n|$$

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

$$H \to \hbar\omega \begin{pmatrix} \frac{1}{2} & & & \\ & \frac{3}{2} & & 0 \\ & & \frac{5}{2} & \\ & 0 & & \ddots \end{pmatrix}$$

$\underline{\hat{a}}$

$$a_{mn} = \langle\varphi_m|a|\varphi_n\rangle$$

$$a^\dagger|\varphi_m\rangle = \sqrt{m+1}\,|\varphi_{m+1}\rangle \overset{H.C.}{\Rightarrow} \langle\varphi_m|a = \sqrt{m+1}\,\langle\varphi_{m+1}|$$

$$\Rightarrow a_{mn} = \sqrt{m+1}\,\langle\varphi_{m+1}|\varphi_n\rangle = \underline{\sqrt{m+1}}\,\delta_{m+1,n}$$

$$a \to \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & & \\ 0 & 0 & \sqrt{2} & 0 & & 0 \\ 0 & 0 & 0 & \sqrt{3} & & \\ & & & 0 & \sqrt{4} & \\ & 0 & & & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}$$

$$a^\dagger \to \begin{pmatrix} 0 & 0 & 0 & & & \\ \sqrt{1} & 0 & 0 & \cdots & & \\ 0 & \sqrt{2} & 0 & & & \\ 0 & 0 & \sqrt{3} & & & \\ & \vdots & & \ddots & & \\ & \vdots & & & & \end{pmatrix}$$

$a, a^+ \Rightarrow \tilde{x} = \frac{1}{\sqrt{2}}(a + a^+)$ -

$$\tilde{x} \to \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & & \\ \sqrt{1} & 0 & \sqrt{2} & 0 & & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & & \\ 0 & 0 & \sqrt{3} & 0 & \ddots & \\ & 0 & & \ddots & \ddots & \ddots \end{pmatrix} \qquad \tilde{p} \to \frac{i\hbar}{\sqrt{2}\sigma} \begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 & \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & \\ 0 & 0 & \sqrt{3} & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

choose $\{x\}$, what is $\varphi(x) = \langle x | \varphi \rangle$?

<u>know:</u> $a|\varphi_0\rangle = 0\,|\varphi_0\rangle$

$$\langle x | a | \varphi_0 \rangle = \langle x | \frac{1}{\sqrt{2}} \left(\frac{\tilde{x}}{\sigma} + \frac{i\tilde{p}\sigma}{\hbar} \right) | \varphi_0 \rangle = 0$$

$$\frac{1}{\sqrt{2}} \left(\frac{x}{\sigma} + \sigma \frac{\partial}{\partial x} \right) \varphi_0(x) = 0$$

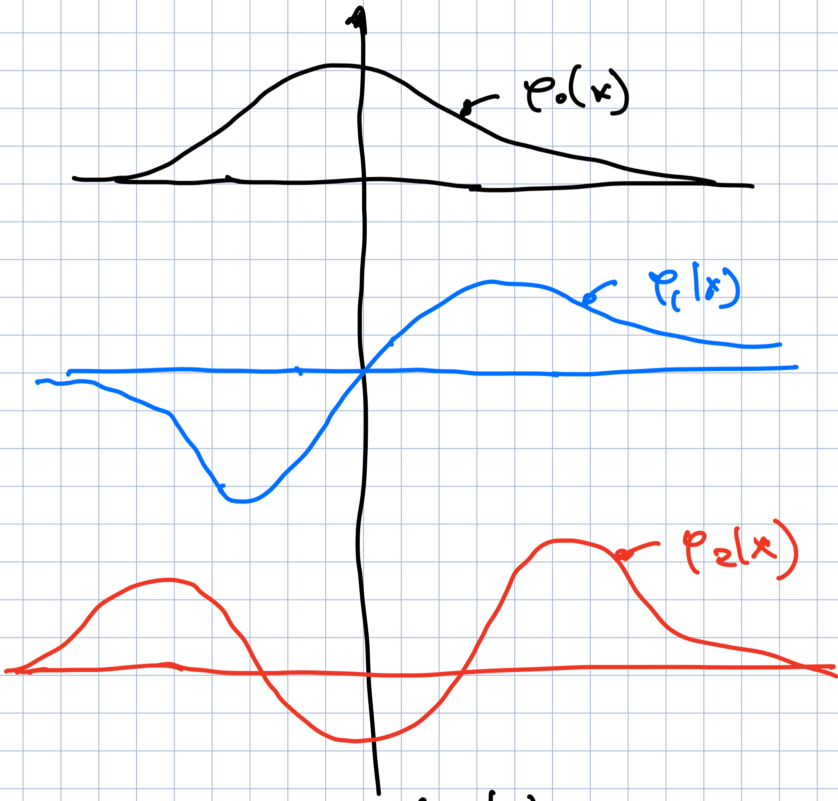
$$\frac{\partial \varphi_0(x)}{\partial x} = -\frac{x}{\sigma^2} \varphi_0(x)$$

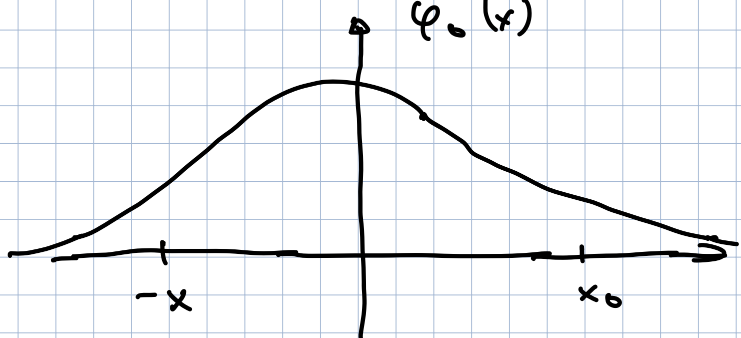
<u>Solutions:</u> $\varphi_0(x) = \underline{c} \cdot e^{-\frac{x^2}{2\sigma^2}}$

<u>Normalize:</u> $\varphi_0(x) = \left(\frac{1}{\pi \sigma^2} \right)^{1/4} e^{-\frac{x^2}{2\sigma^2}}, \quad \sigma = \sqrt{\frac{\hbar}{m\omega}}$

$$\varphi_n(x) = \frac{1}{\sqrt{2^n n!}} \underbrace{H_n\left(\frac{x}{\sigma}\right)}_{\text{Hermite polynomials}} \varphi_0(x)$$

$$\varphi_{n+1}(x) = \frac{x}{\sigma} \sqrt{\frac{2}{n+1}}\, \varphi_n(x) - \sqrt{\frac{n}{n+1}}\, \varphi_{n-1}(x) \qquad (n \geq 1)$$





Energy eigenstates

$|\varphi_n\rangle \qquad \langle H\rangle = \langle \varphi_n | H | \varphi_n\rangle = E_n$

$\langle H^2\rangle = \langle \varphi | H^2 | \varphi_n\rangle = E_n^2$

$\Delta H = \sqrt{\langle H\rangle^2 - \langle H^2\rangle} = 0$ ✓

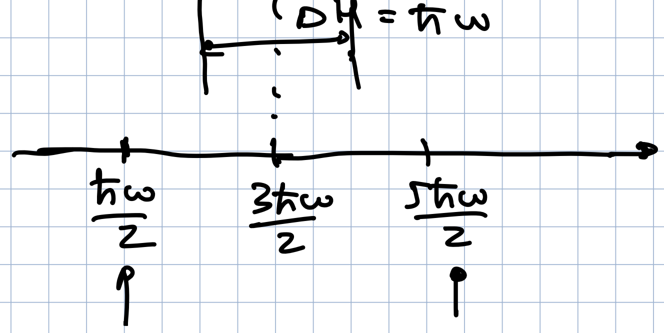
Super: $\{ |\varphi_n\rangle \}$

$$|\psi\rangle = \frac{1}{\sqrt{2}} |\varphi_0\rangle + \frac{1}{\sqrt{2}} |\varphi_2\rangle$$

$$\langle H\rangle = \frac{1}{2}\left(\frac{\hbar\omega}{2}\right) + \frac{1}{2}\left(\frac{5\hbar\omega}{2}\right) = \frac{3}{2}\hbar\omega$$

$$\langle H^2\rangle = \frac{1}{2}\left(\frac{\hbar\omega}{2}\right)^2 + \frac{1}{2}\left(\frac{5\hbar\omega}{2}\right)^2 = \frac{13}{4}(\hbar\omega)^2$$

$\Delta H = \hbar\omega$



$$\langle x \rangle = \langle \varphi_n | x | \varphi_n \rangle$$

$$\langle x^2 \rangle = \langle \varphi_n | x^2 | \varphi_n \rangle$$

$$\langle x \rangle = \int_{-\infty}^{+\infty} dx \, \underbrace{x}_{\text{odd}} \, \underbrace{|\varphi_n(x)|^2}_{\text{even}} = \underline{0}$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 \, \frac{1}{2^n n!} \, |H_n(x/\sigma)|^2 \, (\varphi_0(x))^2 \, dx$$

$$\langle x^2 \rangle = \frac{\sigma^2}{2} \langle \varphi_n | (a^\dagger + a)^2 | \varphi_n \rangle =$$

$$= \frac{\sigma^2}{2} \langle \underbrace{a^{\dagger 2}}_{0} + a^\dagger a + a a^\dagger + \underbrace{a^2}_{0} \rangle =$$

$$= \frac{\sigma^2}{2} \langle a^\dagger a + a a^\dagger \rangle =$$

$$= \frac{\sigma^2}{2} \langle 1 + 2\hat{N} \rangle =$$

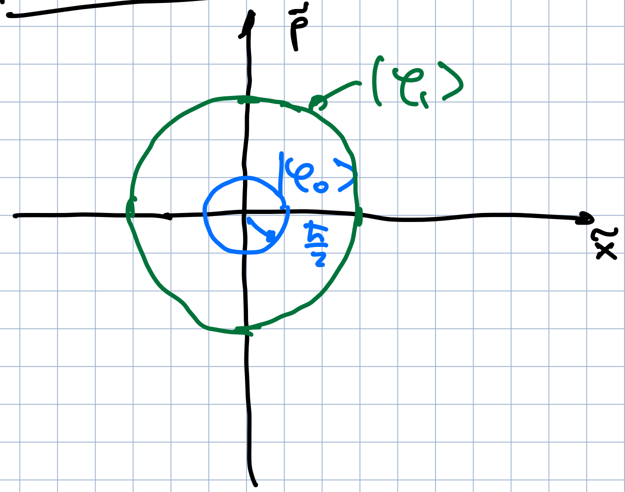
$$= \frac{\sigma^2}{2} (1 + 2n)$$

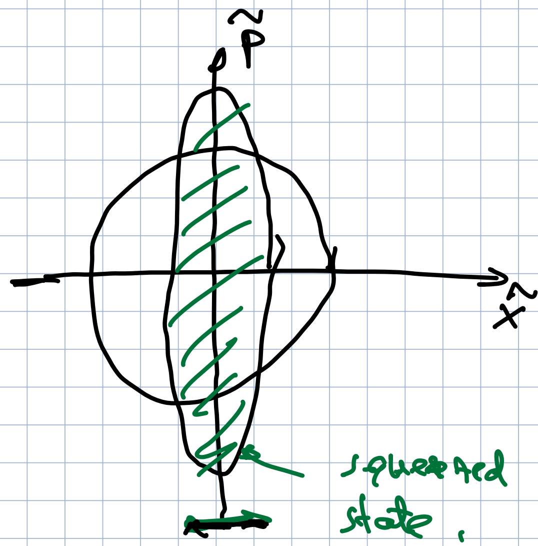
$$\boxed{\Delta x = \frac{\sigma}{2} \sqrt{1 + 2n} = \sigma \sqrt{n + \frac{1}{2}}}$$

Similarly: $\langle p \rangle = 0 \qquad \Delta p = \frac{\hbar}{\sigma} \sqrt{n + \frac{1}{2}}$

so: $\Delta x \cdot \Delta p = \hbar \left(n + \frac{1}{2}\right) \geq \frac{\hbar}{2}$ for all n

$|\varphi_0\rangle$: $\Delta x \cdot \Delta p = \frac{\hbar}{2}$ – "minimum uncertainty state" phase space


$\tilde{p}$
$|\varphi_1\rangle$
$|\varphi_0\rangle$
$\frac{\hbar}{2}$
$\tilde{x}$


$\tilde{p}$
$\tilde{x}$
squeezed state.