OPTI 570 LECTURE Th OCT 16

QM of angular momentum

Two types of AM: • orbital AM → classical analogue
• spin AM → instrinsic
→ QM, no classical analog

Both described by general theory in QM

Angular momentum in QM

$\vec{L}$: Orbital angular momentum (OAM)
$\vec{S}$: Spin angular momentum
$\bar{I}$: nuclear spin angular momentum

$\bar{J}$: sum of all $\bar{e}$ OAM and SAM, L+S

$\bar{F}$: sum of all AM of atom: sum of $\bar{L}, \bar{S}, \bar{I}$
$\bar{F} = \bar{L} + \bar{S} + \bar{I}$

for now $\underline{\bar{J}}$: general angular momentum in QM

Angular momentum in QM:

$\bar{J}$ ≡ any ordered set of three observables $\bar{J} = (J_x, J_y, J_z)$ that satisfy:

$$[J_x, J_y] = i\hbar J_z$$
$$[J_y, J_z] = i\hbar J_x$$
$$[J_z, J_x] = i\hbar J_y$$

$$\bar{J}^2 = \bar{J}\cdot\bar{J} = J_x^2 + J_y^2 + J_z^2$$

Based on definition, J_x, J_y, J_z are not compatible observables

$\bar{J}$ does not have eigenstates

$[J^2, J_x] = 0$
$[J^2, J_y] = 0$
$[J^2, J_z] = 0$

$\Big\}$ $[J^2, J_u] = 0$
$\hat{u}$ is any direction in space
CSCO : $\{J^2, J_u\}$

J^2 : squared magnitude of AM in the system.
J_u : amount of AM about $\hat{u}$

Find the eigenstates and ass. eigenvalues of J^2 and J_u ?

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New operators :

$J_+ \equiv J_x + i J_y$
$J_- \equiv J_x - i J_y$
$\rightarrow$
$J_x = \frac{1}{2}(J_+ + J_-)$
$J_y = -\frac{i}{2}(J_+ - J_-)$

Results : eigenstates + eigenvalues of $\{J^2, J_u\}$

$$J^2 |j\ m_u\rangle = j(j+1)\hbar^2\ |j\ m_u\rangle$$
$$J_u |j\ m_u\rangle = m_u\hbar\ |j\ m_u\rangle$$

o AM or spin AM
spin only
$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

j : any $\boxed{\text{integer}}$ or $\boxed{\text{half integer}}$ ≥ 0

m_u : for any given j, $m_u \in \underbrace{\{j, j-1, \ldots -j+1, -j\}}_{2j+1}$

magnetic quantum number

Convention $\hat{u} = \hat{z}$. Choose a basis $\{|j, m_z\rangle$

magnitude of AM
magnetic quantum number about z

$J_\pm |j, m_z\rangle = \hbar\sqrt{j(j+1) - m_z(m_z \pm 1)}\ |j, m_z \pm 1\rangle$ if $m_z \neq \{j, -j\}$

$J_- \ |j, m_z = -j\rangle = 0$

$J_+ \ |j, m_z = j\rangle = 0$

$\mathcal{E}_j$: basis $\{|j\ m_z\rangle\}$

standard representation: order elements of basis from $m_z = j$, $m_z = j-1$, ... $m_z = -j$

$$\{|j, m_z = j\rangle, |j, m_z = j-1\rangle, |j, m_z = j-2\rangle, \ldots |j, m_z = -j\rangle\}$$

2j+1 elements $\Rightarrow$ $\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$

<u>Ex:</u> $\mathcal{E}_j = 3/2$: corresponds to nuclear of R_b^{87} atom

$$m_z \in \{3/2, 1/2, -1/2, -3/2\}$$

<u>standard rep.:</u> $\{|3/2\ \ 3/2\rangle, |3/2\ \ 1/2\rangle, |3/2\ \ -1/2\rangle, |3/2, -3/2\rangle\}$
$|j, m_z\rangle$

any $|\psi\rangle \in \underline{\mathcal{E}_j} = 3/2$ $\quad j = 3/2$

- meas J^2 what are pos. results? $\frac{3}{2} \cdot \frac{5}{2} \cdot \hbar^2 = \frac{15}{4}\hbar^2$
  J magnitude of ang. momentum $\sqrt{\frac{15}{4}}\,\hbar$
- meas J_x - x component of AM, what are possible results?
  $\frac{3}{2}\hbar$, $\frac{1}{2}\hbar$, $-\frac{1}{2}\hbar$, $-\frac{3}{2}\hbar$
- meas J_z - z component of AM, — // — ?

$$\frac{3}{2}\hbar, \frac{1}{2}\hbar, -\frac{1}{2}\hbar, -\frac{3}{2}\hbar$$

- $|j=\frac{3}{2}, m_y=\frac{1}{2}\rangle$ - what do I know?
  - $J^2 = \frac{15}{4}\hbar^2$

$\frac{1}{2}\hbar$ about y axis

$$|j=\frac{3}{2}, m_y=\frac{1}{2}\rangle \neq |j=\frac{3}{2}, m_z=\frac{1}{2}\rangle$$

$$\underline{|j, m_x\rangle} = \sum_{m_z=-j}^{j} \underline{c_{m_x}} \; m_z \underline{|j, m_z\rangle}$$

<u>Ex:</u> ε $j=1$ 2p state in Hydrogen.

<u>One basis</u> $\{|j=1, m_z=1\rangle, |j=1, m_z=0\rangle, |j=1, m_z=-1\rangle\}$
$\underline{|z_+\rangle}$ $|z_0\rangle$ $|z_-\rangle$

<u>Another basis</u> $\{|j=1, m_x=1\rangle, |j=1, m_x=0\rangle, |j=1, m_x=-1\rangle\}$
$|x_+\rangle$ $|x_0\rangle$ $|x_-\rangle$

<u>Consider</u> $|j=1, m_z=0\rangle = |z_0\rangle$

$$\langle \bar{J} \rangle = (\langle \bar{J}_x\rangle, \langle \bar{J}_y\rangle, \langle \hat{J}_z\rangle)$$

$$\langle z_0| \underline{J_z} |z_0\rangle = 0$$

$$\langle z_0| J_x |z_0\rangle = \frac{1}{2}\langle z_0| J_+ + J_- |z_0\rangle = 0$$

$$\langle J_y\rangle = 0$$

$$\langle \bar{J}\rangle = (0, 0, 0)$$

Consider $|z_+\rangle = |j=1, m_z=1\rangle$

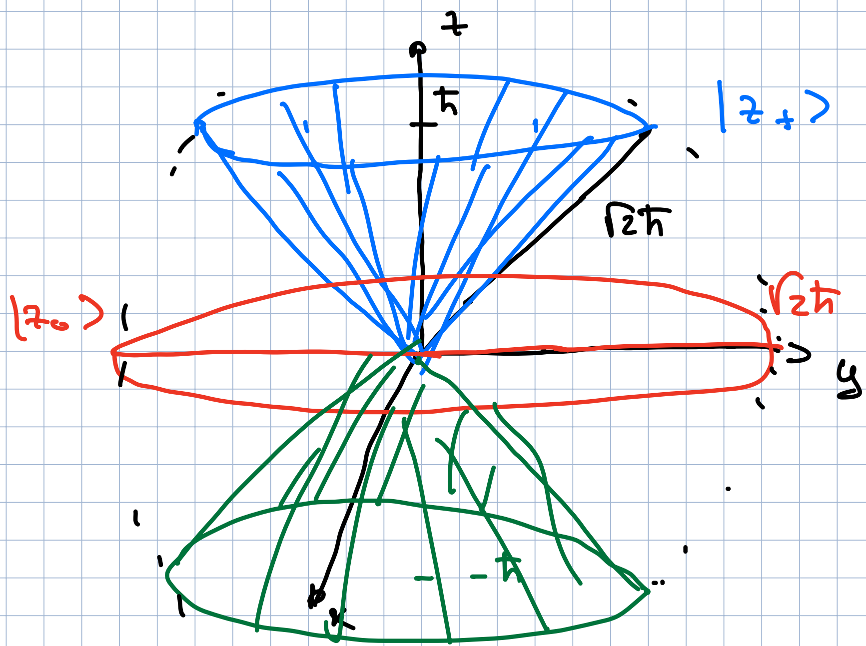
$\langle J_z \rangle = \hbar$

$\langle J_x \rangle = 0$

$\langle J_y \rangle = 0$

$\langle \bar{J} \rangle = (0, 0, \hbar) = (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle)$

$|z_+\rangle$, $J_z = \hbar$, magnitude of AM is $\sqrt{j(j+1)}\,\hbar = \sqrt{2}\,\hbar$ for $j=1$



$\sqrt{j(j+1)\hbar^2} = \sqrt{2}\,\hbar$