OPTI 570, Fall 2025 **Problem Set 5** Prof. C. D. Panda

Due: Mon, Sept. 29 (Main Campus); Tues, Sep 30 (Online section).

Problem I.

Part 1. In this part you are to derive the main results on page 42 of the Field Guide to Quantum Mechanics. Start by writing the Schrödinger equation for the state $|\psi(t)\rangle$ subject to a time-dependent Hamiltonian $\hat{H}(t)$:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle.$$

Now define the state $|\psi_E(t)\rangle = \hat{\mathbb{F}}(t)|\psi(t)\rangle$, as on p.42, where $\hat{\mathbb{F}}(t)$ is some unitary time-dependent operator. Substitute $|\psi(t)\rangle = \hat{\mathbb{F}}^\dagger(t)|\psi_E(t)\rangle$ into the Schrödinger equation to obtain what are called the **effective Schrödinger equation** and the **effective Hamiltonian** $\hat{H}_E(t)$ on p.42.

Part 2. In this part you will be doing something very similar to Part 1, but starting with a time-dependent Hamiltonian of the form

$$\hat{H}(t) = \hat{H}_0 + \hat{W}(t),$$

where $\hat{H}_0$ is time-independent and $\hat{W}(t)$ is time-dependent. Obtain the **effective Schrödinger Equation in the Interaction Picture** and the **effective Hamiltonian for the Interaction Picture** as given on p.46 using the transformations defined on p.46. This part of the problem is quick and straightforward if you make use of the result derived in Part 1.

Problem II.

CT V, complement $\mathbf{M}_V$, problem 1. Solve this problem **without making use of either the position or momentum representations.** Don't use any wavefunctions, and stick with Dirac notation only.

Problem III.

Part 1. Read CT complement $\mathrm{E_{III}}$. We will not discuss this topic in our OPTI 570 lectures, but you should try your best to understand the topic by reading this section without the aid of a lecture (you're practicing the skill of "learning how to learn" on your own with this problem). You will also use some of these ideas in an upcoming homework assignment.

Part 2. Work CT III, complement $\mathrm{L_{III}}$, problem 17.

Problem IV.

Let $\{\, |u_1\rangle, |u_2\rangle, |u_3\rangle \,\}$ be an orthonormal basis for a state space $\mathcal{E}$. The elements of this basis are non-degenerate eigenstates of a time-independent Hamiltonian $\hat{H}$, with associated energy eigenvalues E_n:

$$\hat{H}\,|u_n\rangle = E_n\,|u_n\rangle.$$

For this Hamiltonian, the time-evolution operator is

$$\hat{U}(t,0) = \exp\left\{ \frac{-i\hat{H}\cdot t}{\hbar} \right\},$$

so that

$$|\psi(t)\rangle = \hat{U}(t,0)\,|\psi(0)\rangle = \hat{U}(t,0)\sum_{n=1}^{3} c_n(0)\,|u_n\rangle = \sum_{n=1}^{3} c_n(t)\,|u_n\rangle$$

for any arbitrary state $|\psi(0)\rangle$ given at time $t=0$. The expansion coefficients $c_n(0)$ asumed to be known at $t=0$ are such that $|\psi(0)\rangle$ is normalized to 1. The problem below introduces the concept of a *time-dependent* projector. Let $\hat{\mathbb{P}}_{\psi(t)} \equiv |\psi(t)\rangle\langle\psi(t)|$ be the time-dependent projector onto $|\psi(t)\rangle$.

(a) *Without* using a matrix to represent $\hat{\mathbb{P}}_{\psi(t)}$, show that the trace of $\hat{\mathbb{P}}_{\psi(t)}$ is equal to 1 for any time t. Recall that the trace of an arbitrary operator $\hat{A}$ can be expressed for **any** discrete basis $\{|\phi_j\rangle\}$ as a sum of the diagonal matrix elements:

$$\mathrm{Tr}\{\hat{A}\} = \sum_{\text{all}\, j} \langle\phi_j|\hat{A}|\phi_j\rangle.$$

(b) Now give the matrix representing $\hat{\mathbb{P}}_{\psi(t)}$ at time $t=0$ using the $\{|u_n\rangle\}$ representation. (Check that the trace of the matrix must be equal to 1.)

(c) Suppose that there is an orthonormal basis for which $|\psi(0)\rangle$ is the first element of the basis. Using this basis to define a representation, give the matrix representing $\hat{\mathbb{P}}_{\psi(t)}$ at time $t=0$. (Check that the trace of the matrix is 1.) You can answer this question even though you do not know what the other two basis elements are.

(d) Consider a measurement of an observable $\hat{Q}$ that has a discrete non-degenerate spectrum of eigenvalues $\{\lambda_m\}$ associated with eigenstates $\{|w_m\rangle\}$, where $m \in \{1,2,3\}$. These are not necessarily energy eigenstates, nor are they necessarily orthonormal to the energy eigenstates, so be careful if you make use of orthonormality relationships. Show that the time-dependent probability of obtaining the measurement result λ_m is equal to

$$\mathrm{Tr}\{\hat{\mathbb{P}}_{\psi(t)}\hat{\mathbb{P}}_m\},$$

where $\hat{\mathbb{P}}_m$ is the projector onto $|w_m\rangle$. Remember that according to the definition of trace in part **(a)**, it is up to you to decide which discrete basis to use when constructing the trace, so you might want to consider different options if you get stuck.

(e) Starting by working out an expression for $\frac{d}{dt}\hat{\mathbb{P}}_{\psi(t)}$, derive the following equation:

$$\frac{d}{dt}\hat{\mathbb{P}}_{\psi(t)} = \frac{1}{i\hbar}\,[\hat{H},\,\hat{\mathbb{P}}_{\psi(t)}].$$