OPTI 570 MIDTERM EXAM I SOLUTIONS

Problem 1

a $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ b $\begin{pmatrix} 1/\sqrt{2} \\ -e^{i\theta}/2 \\ -i/2 \end{pmatrix}$ c $\begin{pmatrix} 1/\sqrt{2} & -e^{-i\theta}/2 & i/2 \end{pmatrix}$

d $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ e $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ f $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

g $P|u_3\rangle = |u_3\rangle\langle u_3| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ h $\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$

i $\hat{P}_\omega = \begin{pmatrix} 1/\sqrt{2} \\ -e^{i\theta}/2 \\ -i/2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -e^{-i\theta}/2 & i/2 \end{pmatrix} = \begin{pmatrix} 1/2 & -\frac{1}{2\sqrt{2}} e^{-i\theta} & \frac{i}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} e^{i\theta} & \frac{1}{4} & -\frac{i e^{i\theta}}{4} \\ -\frac{i}{2\sqrt{2}} & \frac{i e^{-i\theta}}{4} & \frac{1}{4} \end{pmatrix}$

Check: Hermitian ✓

Problem 2

a. $\hat{B}_{\{u\}} = \begin{pmatrix} 0 & 0 & i \\ i & 0 & 0 \\ 0 & i & 0 \end{pmatrix}$

b. $\hat{B}^+_{\{u\}} = \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & -i \\ -i & 0 & 0 \end{pmatrix}$

b. $[\hat{B}, \hat{B}^+]_{\{u\}} = \begin{pmatrix} 0 & 0 & i \\ i & 0 & 0 \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & -i \\ -i & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & -i \\ -i & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & i \\ i & 0 & 0 \\ 0 & i & 0 \end{pmatrix} =$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boxed{= 0}$

c. $\hat{C} = \hat{B} + \hat{B}^+ \rightarrow \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix}$

$\hat{C}$ is Hermitian $\Rightarrow e^{i\hat{C}}$ <u>is unitary</u>

$(e^{i\hat{C}})^+ = e^{-i\hat{C}^+} = e^{-i\hat{C}} \neq e^{i\hat{C}} \Rightarrow e^{i\hat{C}}$ is <u>not Hermitian</u>

d. $e^{i\hat{C}} = e^{i\hat{B}} e^{i\hat{B}^+}$ because $[B, B^+] = 0$

e. $\det(C - \lambda I) = \begin{vmatrix} -\lambda & -i & i \\ i & -\lambda & -i \\ -i & i & -\lambda \end{vmatrix} = 0$

$-\lambda^3 - i + i + \lambda + \lambda + \lambda = 0$

$-\lambda^3 + 3\lambda = 0$

$\boxed{\lambda = \{0, +\sqrt{3}, -\sqrt{3}\}}$

f. eval of $\hat{C}$ is : $c|\varphi\rangle \rightarrow \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\Rightarrow \lambda = 0$

$e^{i\hat{C}}|\varphi\rangle = e^{i0}|\varphi\rangle = \boxed{|\varphi\rangle \rightarrow \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$

3. $|w_3\rangle = a|u_1\rangle + b|u_2\rangle + c|u_3\rangle$

$\langle w_1|w_3\rangle = 0 \qquad \langle w_2|w_3\rangle = 0$

$$\left.\begin{aligned} a + 2b + ic &= 0 \\ 4a - b - 2ic &= 0 \;\Big|\cdot 2 \end{aligned}\right| + \qquad \Big|\cdot 2 \;\Big| +$$

$9a - 3ic = 0 \quad \Rightarrow \quad 3a = ic \quad \Rightarrow c = -3ia$

$6a + 3b = 0 \quad \Rightarrow \quad b = -2a$

$a^2 + b^2 + c^2 = 1 \quad \Rightarrow \quad a^2 + 4a^2 + 9a^2 = 1 \qquad a = \sqrt{\frac{1}{14}}$

$\Rightarrow \; |w_3\rangle = \sqrt{\frac{1}{14}} \cdot \left(|u_1\rangle - 2|u_2\rangle - 3i\,|u_3\rangle \right)$

<u>checks</u>: $\langle w_1|w_3\rangle = N_1 [1 - 4 + i \cdot (-3i)] = N_1 (-3 + 3) = 0$ ✓

$\langle w_2|w_3\rangle = N_2 \cdot [4 + 2 + (-2i)(-3i)] = N_2 \cdot (6 - 6) = 0$ ✓

4. $|\psi\rangle = \frac{1}{3}|u_1\rangle + \frac{i}{\sqrt{3}}|u_2^1\rangle + \frac{1}{3}|u_2^2\rangle + \frac{\sqrt{2}}{3}|u_3\rangle - \frac{\sqrt{2}}{3}|u_4^2\rangle$

a

2ϵ	$P_{|\psi\rangle} = \frac{1}{9}$
5ϵ	$\frac{3}{\cancel{1}}\frac{1}{3} + \frac{1}{9} = \frac{4}{9}$
8ϵ	$\frac{2}{9}$
10ϵ	$\frac{2}{9}$

check: $\frac{1}{9} + \frac{4}{9} + \frac{2}{9} + \frac{2}{9} = \frac{9}{9} = 1$ ✓

b. $5\epsilon \Rightarrow \quad |\psi\rangle_{after} = N \cdot \left(\frac{i}{\sqrt{3}}|u_2^1\rangle - \frac{1}{3}|u_2^2\rangle\right)$

$N^2 \cdot \left(\frac{3}{\cancel{1}}\frac{1}{3} + \frac{1}{9}\right) = 1 \qquad N = \frac{3}{2}$

$\Rightarrow \boxed{|\psi\rangle_{after} = \frac{\sqrt{3}}{2} i |u_2^1\rangle - \frac{1}{2}|u_2^2\rangle}$

c. $\langle \hat{H} \rangle = \langle \psi | \hat{H} | \psi \rangle =$

$= (2\epsilon)\cdot\frac{1}{9} + (5\epsilon)\cdot\frac{4}{9} + (8\epsilon)\cdot\frac{2}{9} + (10\epsilon)\cdot\frac{2}{9} =$

$= \frac{2+20+16+20}{9}\epsilon \boxed{= \frac{58}{9}\epsilon}$

5. $J(x,t) = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)$

a $\hbar \to$ distance $\cdot$ momentum = distance $\cdot$ mass $\cdot$ velocity

$\int \psi^2 dx = 1 \Rightarrow \psi \to 1/\sqrt{\text{distance}}$

$\Rightarrow J \to \cancel{\text{distance}} \cdot \cancel{\text{mass}} \cdot \frac{\text{dist}}{\text{time}} \cdot \cancel{(1/\text{mass})} \cdot \cancel{\left(\frac{1}{\sqrt{\text{distance}}}\right)^2} \cdot \frac{1}{\text{distance}}$

$\to \frac{1}{\text{time}}, \frac{1}{\text{second}}$

b $P_{ab} = \int_a^b \psi^* \psi \, dx$

$$\frac{dP_{ab}}{dt} = \int_a^b \psi^* \frac{\partial}{\partial t}\psi \, dx + \int_a^b \left(\frac{\partial}{\partial t}\psi^*\right)\psi \, dx$$

Time derivatives $\Rightarrow$ use SE:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi$$

$$-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + V\psi^*$$

$$\frac{dP_{ab}}{dt} = \frac{1}{i\hbar}\int_a^b \left[\psi^*\left(-\frac{\hbar}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi\right) + \left(+\frac{\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2} - V\psi^*\right)\psi\right]dx$$

$$= \frac{1}{i\hbar}\int_a^b \left[-\frac{\hbar^2}{2m}\left(\psi^*\frac{\partial^2\psi}{\partial x^2} - \psi\frac{\partial^2\psi^*}{\partial x}\right) + \cancel{\psi^* V\psi} - \cancel{V\psi^*}\psi \, dx\right.$$

Integration by parts: $\int_a^b f\frac{\partial g}{\partial x}dx = fg\Big|_a^b - \int_a^b \frac{\partial f}{\partial x} g \, dx$

$$= \frac{1}{i\cancel{\hbar}}\left(-\frac{\hbar^{\cancel{2}}}{2m}\right)\left[\psi^*\frac{\partial\psi}{\partial x}\Big|_a^b - \psi\frac{\partial\psi^*}{\partial x}\Big|_a^b - \int_a^b\left[\cancel{\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x}} - \right.\right.$$

$$\left.+ \cancel{\frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial x}}\right]dx =$$

$$= \frac{i}{2m} \cdot \left[\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right) \Big|_b - \left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right) \Big|_a \right] =$$

$$= J(a,t) - J(b,t) \quad \checkmark$$

c $\quad \psi(x,t) = |\psi(x,t)| \, e^{i\phi(x,t)}$

$\phi(x,t) = x p'/\hbar \qquad p'$ – units of momentum

$$\psi(x,t) = |\psi| \, e^{ixp'/\hbar} \qquad \frac{\partial \psi}{\partial x} = \frac{\partial |\psi|}{\partial x} \cdot e^{ixp'/\hbar} + |\psi| \cdot e^{ixp'/\hbar} \cdot (ip'/\hbar)$$

$$J = \frac{\hbar}{2mi} \left\{ |\psi| \cancel{e^{-ixp'/\hbar}} \cdot \left[\frac{\partial |\psi|}{\partial x} \cancel{e^{ixp'/\hbar}} + (ip'/\hbar) |\psi| \cancel{e^{ixp'/\hbar}} \right] - \right.$$

$$\left. - |\psi| \cancel{e^{ixp'/\hbar}} \cdot \left[\frac{\partial |\psi|}{\partial x} \cancel{e^{-ixp'/\hbar}} - (ip'/\hbar) |\psi| \cancel{e^{-ixp'/\hbar}} \right] \right\} =$$

$$= \frac{\hbar}{2mi} \cdot \left[\cancel{|\psi| \frac{\partial |\psi|}{\partial x}} + ip'/\hbar \, |\psi|^2 - \cancel{|\psi| \frac{\partial |\psi|}{\partial x}} + ip'/\hbar \, |\psi|^2 \right]$$

$$= \frac{\cancel{\hbar}}{\cancel{2}m\cancel{i}} \cdot \cancel{2} \cancel{i} p' / \cancel{\hbar} \, |\psi|^2 =$$

$\boxed{J(x,t) = \frac{p'}{m} |\psi|^2} \qquad p'/m = v'. \quad$ flow of probability associated w/ a spatial gradient in phase

6. $\hat{\pi} = \int dp' \, |-p'\rangle\langle p'|$

$\hat{\pi}|p_0\rangle = |-p_0\rangle$

a $\hat{\pi}^2|p_0\rangle = \hat{\pi}\left(\hat{\pi}|p_0\rangle\right) = \hat{\pi}\,|-p_0\rangle = |p_0\rangle$

b $\langle p|\hat{\pi}|p_0\rangle = \int_{-\infty}^{+\infty} dp' \underbrace{\langle p|-p'\rangle}_{\delta(p+p')} \langle p'|p_0\rangle =$

$= \langle -p|p_0\rangle = \delta(p+p_0)$

c $\langle p|\hat{\pi}|\psi\rangle = \int \langle p|\pi|p'\rangle\langle p'|\psi\rangle \, dp' =$

$= \int \delta(p+p') \langle p'|\psi\rangle \, dp' =$

$= \langle -p|\psi\rangle = \tilde{\psi}(-p)$

d $\hat{F}_+ = \frac{1}{2}(\hat{\mathbb{1}} + \hat{\pi})$

$\langle p|\hat{F}_+|\psi\rangle = \frac{1}{2}\langle p|\mathbb{1}|\psi\rangle + \frac{1}{2}\tilde{\psi}(-p) =$

$= \frac{1}{2}\left[\tilde{\psi}(p) + \tilde{\psi}(-p)\right]$

e $\tilde{\psi}(-p) = \tilde{\psi}(p)$

$\langle p|\hat{F}_+|\psi\rangle = \psi(p)$

f. $\tilde{\psi}(-p) = -\psi(p)$ $\quad\langle p|\hat{F}_+|\psi\rangle = 0$

g. $\hat{\pi}\,|p_0\rangle = |-p_0\rangle$

$\hat{\pi}\,|v\rangle = e.\,|v\rangle \Rightarrow |v\rangle = |p\rangle + |-p\rangle$

$\hat{\pi}\,|v\rangle = |-p\rangle + |p\rangle$ ✓

eigen value 1

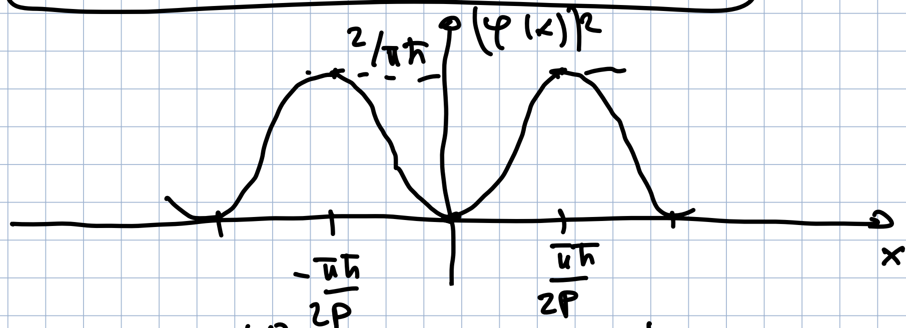
b. $|\varphi\rangle = |p\rangle + |-p\rangle$

$$\varphi(x) = \langle x|\varphi\rangle = \langle x|p\rangle - \langle x,-p\rangle =$$
$$= \frac{1}{\sqrt{2\pi\hbar}}\left(e^{ixp/\hbar} + e^{-ixp/\hbar}\right)$$

$$\underbrace{|\varphi(x)|^2 = \frac{2}{\pi\hbar}\cos^2(xp/\hbar)}$$

$xp/\hbar = \pi/2$

$x = \frac{\pi\hbar}{2p}$



i $\hat{\pi}^\dagger = \left(\int_{-\infty}^{+\infty} dp' \, |-p'\rangle\langle p'|\right)^\dagger =$

$= \int_{-\infty}^{+\infty} dp' \, |p'\rangle\langle -p'| =$ $\quad u = -p$

$= -\int_{\infty}^{-\infty} du \, |-u\rangle\langle u| =$

$= \int_{-\infty}^{+\infty} du \, |-u\rangle\langle u| = \hat{\pi}$ $\Rightarrow$ yes, Hermitian ✓