Assignment 9

OPTI 570 Quantum Mechanics

University of Arizona

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Total time: 8 hours

Problem I

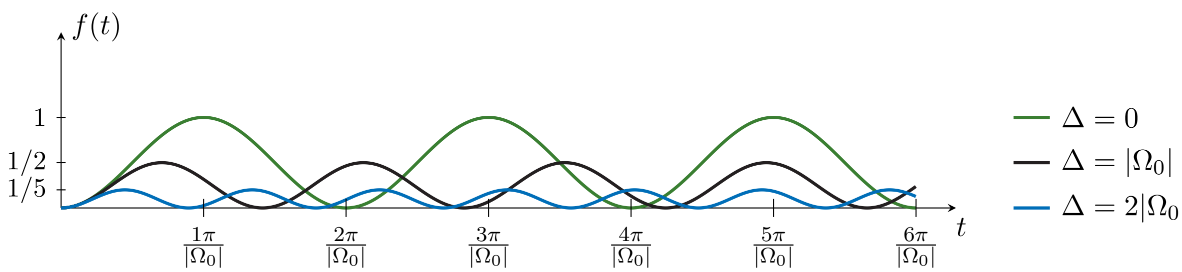
a) The general expression for the transition probability is:

$$P_{|+\rangle\to|-\rangle}(t) = \left|\frac{\Omega_0}{\Omega}\right|^2 \sin^2\frac{\Omega t}{2}, \quad \Omega = \sqrt{\Omega_0^2 + \Delta^2}.$$

By evaluating the different detuning given, we have:

$$\begin{aligned}
\Delta = 0 : &\qquad \Omega = \Omega_0 \Longrightarrow P_{|+\rangle\to|-\rangle}(t) = \sin^2\frac{\Omega_0 t}{2} \\
\Delta = |\Omega_0| : &\qquad \Omega = \sqrt{2}\Omega_0 \Longrightarrow P_{|+\rangle\to|-\rangle}(t) = \frac{1}{2}\sin^2\frac{\sqrt{2}\Omega_0 t}{2} \\
\Delta = 2|\Omega_0| : &\qquad \Omega = \sqrt{5}\Omega_0 \Longrightarrow P_{|+\rangle\to|-\rangle}(t) = \frac{1}{5}\sin^2\frac{\sqrt{5}\Omega_0 t}{2}
\end{aligned}$$

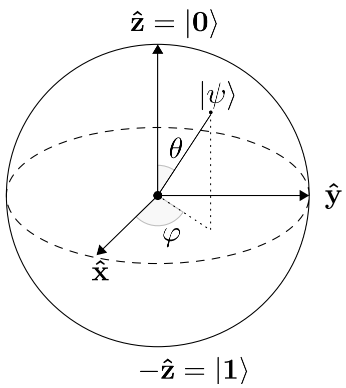
For visualization, we will set $\Omega_0 = 1$.



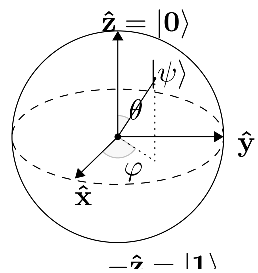
b) The Hamiltonian can be written in term of the Pauli matrices when $\Omega_0 = |\Omega_0|e^{i\beta}$:

$$H = \frac{\hbar}{2}[\Delta\sigma_z + |\Omega_0|\cos\beta\sigma_x - |\Omega_0|\sin\beta\sigma_y] = \frac{\hbar}{2}\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}.$$

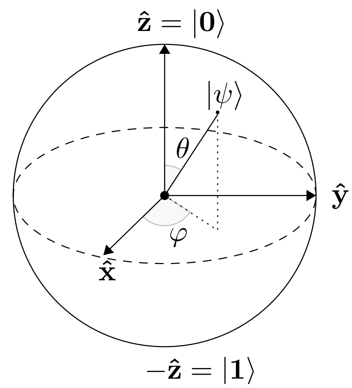
The last is called effective field, or Rabi vector. The Bloch sphere is $\boldsymbol{r}(t) = \langle\psi(t)|\boldsymbol{\sigma}|\psi(t)\rangle$, with $\boldsymbol{R}(0) = (0, 0, 1)$.



(a) $\Delta = 0$



(b) $\Delta = |\Omega_0|$



(c) $\Delta = 2|\Omega_0|$

Problem II

a) The unit-vector in cartesian coordinates expressed in terms of the spherical quantities is:

$$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\hat{\boldsymbol{x}} + \sin\theta\sin\phi\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{z}}.$$

We now try to use the table given in the Field guide to substitute these coefficients and express $\hat{\boldsymbol{r}}$ in terms of the Spherical harmonics. The z-direction is the easiest as it only has one quantitiy involved. We now that Y_1^0 has cosine of that angle, so we can use it to say that:

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \longrightarrow \cos\theta = \sqrt{\frac{4\pi}{3}}Y_1^0.$$

For the x-direction, we have the term $\sin\theta\cos\phi$ meaning we need to combine some spherical to have this product form. Using $Y_1^{\pm 1}$ we can create a cosine by considering both sign and collect the expoential:

$$Y_1^{-1} - Y_1^1 = \sqrt{\frac{3}{8\pi}}\sin\theta\left[e^{-i\phi} + e^{i\phi}\right] = \sqrt{\frac{3}{2\pi}}\sin\theta\cos\phi \longrightarrow \sin\theta\cos\phi = \sqrt{\frac{2\pi}{3}}(Y_1^{-1} - Y_1^1).$$

Similarly, we play with $Y_1^{\pm 1}$ to get the $\sin\phi$ term:

$$Y_1^{-1} + Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\sin\theta\left[-e^{-i\phi} + e^{i\phi}\right] = 2i\sqrt{\frac{3}{8\pi}}\sin\theta\sin\phi \longrightarrow \sin\theta\sin\phi = -i\sqrt{\frac{2\pi}{3}}(Y_1^{-1} + Y_1^1).$$

Therefore, we finally have

$$\hat{\boldsymbol{r}} = \sqrt{\frac{2\pi}{3}}(Y_1^{-1} - Y_1^1)\hat{\boldsymbol{x}} - i\sqrt{\frac{2\pi}{3}}(Y_1^{-1} + Y_1^1)\hat{\boldsymbol{y}} + \sqrt{\frac{4\pi}{3}}Y_1^0\hat{\boldsymbol{z}}.$$

b) We have already substituted x for $\sin\theta\cos\phi$ and in terms of the spherical harmonics. The only quantity we need to compute is the r, which is obtained by squaring the compoents.

$$\begin{aligned} r^2 = x^2 + y^2 + z^2 &= \left[\sqrt{\frac{2\pi}{3}}(Y_1^{-1} - Y_1^1)\right]^2 + \left[-i\sqrt{\frac{2\pi}{3}}(Y_1^{-1} + Y_1^1)\right]^2 + \left[\sqrt{\frac{4\pi}{3}}Y_1^0\right]^2 \\ &= \frac{2\pi}{3}\left[(Y_1^{-1})^2 - 2Y_1^{-1}Y_1^1 + (Y_1^1)^2 + (Y_1^{-1})^2 + 2Y_1^{-1}Y_1^1 + (Y_1^1)^2\right] + \frac{4\pi}{3}(Y_1^0)^2 \\ r^2 &= \frac{4\pi}{3}\left[(Y_1^{-1})^2 + (Y_1^1)^2 + (Y_1^0)^2\right]. \end{aligned}$$

Then, we have

$$F(x,y,z) = \frac{(Y_1^{-1} - Y_1^1 - iY_1^{-1} - iY_1^1 + \sqrt{2}Y_1^0)}{\sqrt{2}[(Y_1^{-1})^2 + (Y_1^1)^2 + (Y_1^0)^2]}.$$

Problem III

The initial state is $|+\rangle_z$. The firt pulse is along the y-direction which make a Hamiltonian of the form

$$H = -\gamma\hbar/2\boldsymbol{\sigma}\cdot\boldsymbol{B} \Longrightarrow U(t) = e^{-iHt/\hbar} = e^{-i\alpha\boldsymbol{\sigma}\cdot\boldsymbol{B}/2}, \quad \alpha = -|\gamma|Bt.$$

where α is the rotation angle generated by the pulse along y.

Problem IV