OPTI 570, Fall 2025 **Midterm Exam 2**

2-hour written exam.

On-campus students: exam time and location: 5 - 7pm (Tucson time), Oct 22 in Meinel 305.

Distance students: exam will be available on the D2L **Assignments** page, starting 7pm (Tucson time), Oct 22, after submitting a **Quiz** acknowledging that you have read these instructions. You must complete and return by 9pm (Tucson time), Oct 28, unless you have made other arrangements with me. Distance students have 2 hours to work on the exam questions, plus *up to* 30 minutes for downloading and printing the exam, changing locations, scanning answers, and uploading answers to D2L as a single PDF (or email to me if there's a problem uploading). These extra 30 minutes are *not* to be used to work on the exam.

Instructions

- **The 2 hours that you have available for the exam begin *after* you finish reading this instructions page and as soon as you start reading the problems of the exam.**
- You may consult the following items during the exam: PDF or physical/printed copies of the course notes and the notes from lectures, recap sessions, and recitation sections; QM Field Guide; OPTI 570 problem sets and solutions (yours and mine); and any of your own notes or anything you have personally written or typed. You may use a calculator for numerical calculations only (no graphic functions or symbolic manipulation). Computers may be used only to access allowed material that is stored on your computer, or allowed internet resources. Allowed internet resources are only the OPTI 570 D2L site for distance students to access and return the exam, and email to communicate with me if needed. You must **not** consult other people, or accept or provide help to anyone else in the class. You are on your honor to adhere to these rules; violation of these rules will result in a failing grade.
- There are 5 problems on the following 3 pages. **120 points are available**, although the exam is graded out of 100 points, so there are **20 extra points** available. It is possible that the exam scores may also be further scaled up. The highest final grade that will be recorded will be 100, even for those who earn more than 100 points.
- Use your own paper to solve all problems. Show enough work that I can follow your reasoning and give you partial credit for problems that are not fully correct.
- It is up to you to convince me that you know how to solve the problems, and to write legibly enough that I do not need to struggle to interpret your work. However, I expect you to work quickly, and that the neatness of your solutions might consequently suffer. That's OK as long as I can interpret your solutions. Draw a box around final answers if your final results are not obvious. If you have a mess of equations all over the page, direct my attention to your line of thought if it is not otherwise obvious. If you have obtained an answer that you know is not correct and you do not have enough time to fix the error, please tell me that you know the answer is wrong, why you know that it is wrong, and guess an appropriate answer – this may help you earn significant partial credit.
- If you are convinced that there is a significant mistake in a problem that may affect the answer or interpretation, please ask about it. Or if a mistake is obvious to you, you may indicate what you think is wrong, what should be changed to make the problem solvable in the manner that you think I intended, then solve the problem. Make sure that I can understand how you have modified the problem to make it solvable. Part of the challenge of learning a new subject is to try to identify mistakes and speculate about the original intention of given problems!

1. [10 pts.] Consider a 3D isotropic harmonic oscillator with angular frequency ω. A particle of mass m exists at time $t = 0$ in the state $|\psi(0)\rangle$, defined in the usual way in terms of the energy eigenstate basis $\{|n_x, n_y, n_z\rangle\}$ as

$$|\psi(0)\rangle = \frac{1}{\sqrt{3}}|0\,0\,0\rangle + \frac{1}{3}|1\,0\,0\rangle - \frac{i}{3}|0\,1\,0\rangle + \frac{1}{3}|0\,0\,2\rangle - \frac{i}{\sqrt{3}}|1\,0\,1\rangle.$$

(a) What are the possible outcomes and associated probabilities of measuring the total energy of the particle at $t = 0$? Will this answer be different if the energy is to be measured at a later time instead of at $t = 0$, and if so, what are the possible outcomes and probabilities?

(b) If the $t = 0$ energy measurement of part (a) yields the result $\frac{5}{2}\hbar\omega$, specify the normalized state that the particle is left in immediately after the measurement.

(c) Evaluate $|\psi(\pi/\omega)\rangle$, the state of the particle at time $t = \pi/\omega$, and simplify your answer as much as possible. As usual, you can neglect (or choose your own) global phase factors.

---

2. [30 pts.] Let $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle\}$ be an orthonormal basis for a state space $\mathcal{E}$. We define a Hamiltonian H_0 as

$$H_0 = \frac{1}{2}\hbar\omega\,(\,|\phi_1\rangle\langle\phi_1| \;+\; 3\,|\phi_2\rangle\langle\phi_2| \;+\; 5\,|\phi_3\rangle\langle\phi_3|\,),$$

where ω is the angular frequency. We will examine the effect of adding a new component to the Hamiltonian on the states of the system. We define a new Hamiltonian $H = H_0 + W$ where

$$W = \frac{1}{2}\hbar\omega\,(\,i\sqrt{3}|\phi_2\rangle\langle\phi_3| \;-\; i\sqrt{3}\,|\phi_3\rangle\langle\phi_2|\,).$$

(a) Find the eigenvalues of H and the associated eigenstates. Using Dirac notation, clearly label the eigenstates of H, and express them in terms of the $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle\}$ basis elements.

(b) Suppose we define a quantum state at $t = 0$ to be $|\psi(0)\rangle = |\phi_2\rangle$. Calculate $|\psi(t)\rangle$ for any later time t, and express your answer in whatever basis you prefer.

(c) Given $|\psi(0)\rangle = |\phi_2\rangle$ (same as (b)), calculate the probability $\mathcal{P}_3(t)$ that at time t, the system will be found in state $|\phi_3\rangle$. Simplify your answer to trigonometric functions.

(d) Sketch $\mathcal{P}_3(t)$, label the axes, and indicate important points: namely, maximum and minimum values of $\mathcal{P}_3(t)$, and the time that it takes to reach the first maximum of $\mathcal{P}_3(t)$. Interpret this result by explaining in a sentence or two what is the effect of adding W to H_0.

---

3. [30 pts.] Consider a toy model of a linear triatomic molecule: two atomic nuclei each have an effective net positive charge that is given by $Q = 1.6 \times 10^{-19}$ C. These two nuclei are fixed at points $x = d$ and $x = -d$ and cannot move relative to each other. A third atomic nucleus, having an effective net charge of $2Q$ and with a mass m, is free to move in the x-direction (only) but is confined to the region between the two fixed nuclei, $-d < x < d$. The potential energy (along x) for this central nucleus due to the Coulomb electrostatic interaction is therefore

$$V(x) = 2k_e\,Q^2\left(\frac{1}{x+d} - \frac{1}{x-d}\right),$$

where $k_e = (4\pi\epsilon_0)^{-1} \approx 9 \times 10^9\ \mathrm{N\,m^2\,C^{-2}}$.

(a) By approximating the potential well in the region near $x = 0$ as a harmonic oscillator, give an expression for the oscillator's angular frequency ω in terms of the various quantities given above.

(b) The frequency ω indicates the order of magnitude of the vibrational frequencies of the linear triatomic molecule. In the harmonic oscillator approximation, $\hbar\omega$ is the energy spacing between adjacent vibrational energy levels. Now let $d = 10^{-10}$ m, approximately correct for a water molecule (although H_2O is not actually a linear molecule). Also assume the middle nucleus is that of an oxygen atom, so that $m \approx 2.7 \times 10^{-26}$ kg. Give a number for ω.

(c) Suppose that this model water molecule is initially at rest and the central oxygen nucleus is in the ground state of the approximatelly harmonic oscillator potential well. A room-temperature ($T = 300$ K) CO_2 molecule moving in the x direction then collides with the water molecule. In the coordinate frame of the water molecule, this collision gives a momentum kick of magnitude p_0 to the central nucleus.

- Use $k_B T = M v^2$ to find the momentum of the CO_2 molecule, where $k_B = 1.4 \times 10^{-23}$ J/K is Boltzmann's constant, and $M \approx 7.3 \times 10^{-26}$ kg is the mass of the CO_2 molecule.
- What is the maximum displacement of the expectation value of the central nucleus position? Divide this number by d to estimate the amplitude of vibrations of the central nucleus as a fraction of the distance to one of the end nuclei. From this calculation we can guess that molecular collisions at room temperature may easily excite molecular vibrations, but do not rip molecules apart (check your work if your answer suggests otherwise).
- Suppose that we could measure the total energy of the central nucleus in this harmonic oscillator. What is the probability that immediately after the collision, the central nucleus is found in the ground state of the model molecule? Give a number.

---

4. [25 pts.] This problem explores a model of a "beamsplitter" for the wavefunction of an atom in a 1D quantum harmonic oscillator in the $\hat{x}$ direction. Consider an atom in a stationary state of the potential well. If the atom absorbs a photon from a pulse of light from a laser beam that is propagating in the $+\hat{x}$ direction, the atom receives an $\hat{x}$-direction momentum kick of p_0. The momentum kick is mathematically represented by the momentum translation operator $\hat{T}(p_0)$.

In the beamsplitter model, we consider the same scenario as above, except the power of the laser beam is turned down enough that the atom has only a partial chance of absorption of the photon. In this case, the action of the light on the atom being is characterized by a "beamsplitter" operator

$$\hat{B}(p_0) \equiv \hat{\mathbb{I}} + \hat{T}(p_0)$$

where $\hat{\mathbb{I}}$ is the identity operator.

(a) Suppose that at time t=0 an atom (mass m) in the ground state $|\varphi_0\rangle$ of the harmonic oscillator (frequency ω) is subject to the low-power laser beam pulse described above, producing the state $|\psi(0)\rangle \equiv C \cdot \hat{B}(p_0)|\varphi_0\rangle$ where C is a scalar. Find C such that $|\psi(0)\rangle$ is properly normalized.

(b) Give a complete expression for $\psi(x, t{=}0)$, the position representation of $|\psi(0)\rangle$. You may write expressions here and in the remainder of the problem in terms of C.

(c) Give an expression for $|\psi(x,t{=}0)|^2$, and simplify this function as much as possible by using trigonometric functions rather than complex exponentials. Sketch $|\psi(x,t{=}0)|^2$ vs x for $p_0 = 5\pi\hbar/\sigma$. Label your axes and significant points. Make sure that you clearly illustrate the significant features of the functions plotted.

(d) Assume subsequent evolution of the atomic wavefunction over a duration $t = \frac{\pi}{2\omega}$. Sketch $|\psi(x,t{=}\frac{\pi}{2\omega})|^2$ vs x. Label your axes and significant points. Make sure that you clearly illustrate the significant features of the functions plotted. Hint: Part of the wavefunction may be vanishingly small when plotting.

---

5. [25 pts.] A particle of mass m is trapped by a 2D isotropic harmonic oscillator potential in the $x\!-\!y$ plane with frequency ω. Answer the following questions, supposing that at time $t{=}0$ the system is in a state $|\psi(t{=}0)\rangle$ for which

$$\hat{a}_x|\psi(0)\rangle = 5\,|\psi(0)\rangle \qquad \text{and} \qquad \hat{a}_y|\psi(0)\rangle = 5i\,|\psi(0)\rangle,$$

where $\hat{a}_x$ and $\hat{a}_y$ are the lowering operators in the x and y axes.

(a) Is $|\psi(0)\rangle$ an energy eigenstate of the system? If yes, what is the associated energy eigenvalue? If not, what is the mean energy $\langle\hat{H}\rangle$?

(b) Give values for the uncertainty products

$$\Delta\hat{X}\,\Delta\hat{P}_x \qquad \text{and} \qquad \Delta\hat{Y}\,\Delta\hat{P}_y.$$

(c) Give a normalized expression for $\psi(x,y,t{=}0)$, the position representation of $|\psi(0)\rangle$.

(d) Let $\hat{U}(\tau/4)$ be the time evolution operator that corresponds to evolution over a quarter-period of the oscillation period $\tau \equiv \frac{2\pi}{\omega}$, such that $|\psi(\tau/4)\rangle = \hat{U}(\tau/4)|\psi(0)\rangle$. Is $|\psi(\tau/4)\rangle$ an eigenstate of $\hat{a}_x$ and $\hat{a}_y$? If so what are the associated eigenvalues? If not, what are the expectation values $\langle\hat{a}_x\rangle$ and $\langle\hat{a}_y\rangle$?

(e) Let $\psi(x,y,t)$ be the position-space wavefunction for the evolving state of the system. Describe the dynamics of $|\psi(x,y,t)|^2$. More specifically, quantitatively describe the shape of this probability density distribution, specify its time-dependent mean positions $\langle\hat{X}\rangle(t)$ and $\langle\hat{Y}\rangle(t)$, and accurately describe the trajectory of the center of the wavepacket in the $x\!-\!y$ plane.

(f) Specify the probabilities $\mathcal{P}(E)$ of obtaining the following outcomes of an energy measurement: $E = \hbar\omega$, $E = 2\hbar\omega$, and $E = 3\hbar\omega$. Give your answers in analytical expression, not decimals (i.e., you do not need to evaluate exponential terms). Sketch $\mathcal{P}(E)$ vs E, indicating the width and mean of the probability curve. What is the most likely result of an energy measurement?

END OF EXAM