OPTI 570, Fall 2025 **Problem Set 8** Prof. C. D. Panda

Due: Friday, Oct 31 (Main Campus); Wed, Nov. 5 (Online section).

Problem I. Consider a particle in an angular momentum state with angular momentum quantum number $j = 1$, and whose state space is spanned by the basis $\{|z+\rangle, \ |z0\rangle, \ |z-\rangle\}$ of three eigenvectors common to $\mathbf{J}^2$ (eigenvalue $2\hbar^2$) and J_z (respective eigenvalues $+\hbar$, 0, and $-\hbar$). The state of the particle is:

$$|\psi\rangle = a\,|z+\rangle + b\,|z0\rangle + c\,|z-\rangle$$

where a, b, c are three *complex* parameters.

(a) Calculate $\langle J_x \rangle$ in terms of a, b, c using two different methods: (i) the state vector formalism, and (ii) the density operator formalism. You should get the same answer in both approaches. For the remainder of the problem, you may stick with the state vector formalism.

(b) Calculate $\langle \mathbf{J}^2 \rangle$ in terms of a, b, and c.

(c) Write expressions for $\langle J_x^2 \rangle$, $\langle J_y^2 \rangle$, and $\langle J_z^2 \rangle$ in terms of a, b, c.

(d) Write out an expression for $\langle J_x \rangle^2$ and use it to calculate the variance $(\Delta J_x)^2$ in terms of a, b, and c. The variance is the square of the standard (or r.m.s.) deviation.

(e) Use your results from above to determine one combination of allowed values for a, b, and c that minimizes $(\Delta J_x)^2$ ($a = b = c = 0$ is not allowed, since $|\psi\rangle$ is then not a physical state). If your result from part (d) is correct, this should not be too difficult if you consider the case $b = 0$ (although there are also correct answers for $b \neq 0$).

(f) Write a different combination of values of a, b and c that minimizes $(\Delta J_z)^2$. Note the subscript! This should be a one-liner!

(g) Without calculating (ΔJ_y) or (ΔJ_z), write the uncertainty product for $(\Delta J_y)(\Delta J_z)$ in terms of any a, b, and c. Does your answer make sense for the two combinations of a, b, and c that you determined in parts (e) and (f)?

Problem II. Here we again consider a particle with angular momentum quantum number $j = 1$, and we will use the two bases $\{|z+\rangle, \ |z0\rangle, \ |z-\rangle\}$ and $\{|x+\rangle, \ |x0\rangle, \ |x-\rangle\}$ defined in the previous problem. We will examine the effect of measurements on the state of the particle; specifically, assume that we can measure two quantities (one at a time), the component of angular momentum in the x direction (associated with operator J_x) and the component of angular momentum along the z axis (associated with J_z). Suppose that the quantum state of the particle at time $t = 0$ is $|\psi(t = 0)\rangle = |z0\rangle$, and that no forces act on the particle after $t = 0$ other than the measurement processes themselves.

(a) What is the probability that a measurement of the x component of angular momentum on the state $|\psi(0)\rangle$ will give $\hbar$?

(b) What is the probability that a measurement of the x component of angular momentum on the state $|\psi(0)\rangle$ will give 0?

(c) What is the probability that a measurement of the x component of angular momentum on the state $|\psi(0)\rangle$ will give $-\hbar$?

(d) What is the probability that a measurement of the z component of angular momentum on the state $|\psi(0)\rangle$ will give $\hbar$?

(e) Suppose that the x component of angular momentum of the state $|\psi(0)\rangle$ is measured, but the result of the measurement is unknown. What are each of the probabilities that a subsequent measurement of the z component of angular momentum will give $\hbar$, 0, or $-\hbar$?

(f) Regarding the results of the first measurement of part (e): immediately after the measurement, should we describe the particle as being in a pure state or a statistical mixture?

(g) Write down the density matrix ρ for the particle immediately after the measurement of the x component of angular momentum for the situation described in part (e). Use the $\{|x+\rangle,\ |x0\rangle,\ |x-\rangle\}$ representation for ρ. We will designate this matrix as $\rho^{(x)}$ to explicitly indicate that it is written in terms of the eigenvectors of J_x.

(h) Use a transformation matrix to transform ρ into the $\{|z+\rangle,\ |z0\rangle,\ |z-\rangle\}$ representation. We will designate this matrix as $\rho^{(z)}$ to explicitly indicate that it is written in terms of the eigenvectors of J_z. Hint: look on p.70 of the Field Guide, and use the 3 column vectors in the lower-left corner of the green box to construct the transformation matrix.

(i) Use $\rho^{(z)}$ to calculate $\langle J_z\rangle$ and $\langle J_x\rangle$.

(j) Using $\rho^{(z)}$, evaluate the probabilities that a measurement of the z component of angular momentum performed after a measurement of the x component [as in part (e)] will give the values $\hbar$, 0, and $-\hbar$. Compare with your answer to (e).

Problem III.

Find a table of spherical harmonics $Y_l^m(\theta,\phi)$ (Field Guide, p.100), and integrate the following functions F over θ and ϕ (note that the differential element of angular integration is $\sin(\theta)d\theta d\phi$, with the θ integral covering the range 0 to π and the ϕ integral covering the range 0 to 2π.)

Example: let $F = Y_0^0(\theta,\phi)^* \cdot Y_1^0(\theta,\phi)$. Since $Y_0^0(\theta,\phi) = \sqrt{\frac{1}{4\pi}}$ and $Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos(\theta)$, we have $F = \sqrt{\frac{1}{4\pi}} \cdot \sqrt{\frac{3}{4\pi}}\cos(\theta)$. Integrating over θ and ϕ, we find the following:

$$\int_0^{2\pi}\int_0^{\pi} F\sin(\theta)d\theta d\phi = \frac{\sqrt{3}}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi}\cos(\theta)\sin(\theta)d\theta = 0.$$

This result should have been obvious from the beginning, since spherical harmonics are orthogonal to each other. For the following functions however, *three* spherical harmonics are combined to give new functions where orthogonality does not necessarily play an obvious role, and integration is needed. HINTS: (1) try the ϕ integral first, since often it is easiest; (2) don't be surprised if some of your answers are zero; (3) pay attention to the complex conjugate (*) as it sometimes does matter in obtaining a correct answer.

(a) $F = Y_0^0(\theta,\phi)^* \cdot Y_1^0(\theta,\phi) \cdot Y_1^1(\theta,\phi)$.

(b) $F = Y_0^0(\theta,\phi)^* \cdot Y_1^1(\theta,\phi) \cdot Y_1^1(\theta,\phi)$.

(c) $F = Y_0^0(\theta,\phi)^* \cdot Y_1^{-1}(\theta,\phi) \cdot Y_1^1(\theta,\phi)$.

(d) $F = Y_1^0(\theta,\phi)^* \cdot Y_1^0(\theta,\phi) \cdot Y_1^0(\theta,\phi)$.

(e) $F = Y_1^1(\theta,\phi)^* \cdot Y_1^0(\theta,\phi) \cdot Y_2^1(\theta,\phi)$ (you should not get zero for an answer!).

(f) Suppose $F = Y_{l_1}^{m_1}(\theta,\phi)^* \cdot Y_{l_2}^{m_2}(\theta,\phi) \cdot Y_{l_3}^{m_3}(\theta,\phi)$. Determine a rule that will specify the value of m_1, expressed in terms of m_2 and m_3, that will result in a non-zero integral over ϕ. HINT: don't even bother with the θ parts of F, and just group all of the θ terms into some function $g(\theta)$ that can remain unspecified. Only pay attention to the ϕ parts of F and the integral over ϕ.

(g) If l_2 from part (f) is equal to 1, use your result from (f) to list all of the possible values of $\Delta m_{13} \equiv m_1 - m_3$ for which the ϕ integral is non-zero. You will see later that this answer has to do with the conservation of angular momentum in an interaction between an atom and a photon.

Problem IV.
CT Chapter IV, Complement J_{IV}, exercise 2.