OPTI 570, Fall 2025 **Practice Midterm Exam 3**

2-hour written exam.
On-campus students: 5-7pm, Nov 19, 2025.
Distance students: available to download starting around 7pm, Nov 19. Download when you are ready to take the exam, then submit your answers as a PDF via D2L **within 2 hours (to take the exam) + up to 30 minutes (for printing, reading instructions, submitting answers)** by 9pm Tuesday evening, Nov 25.

Instructions

- **The 2 hours that you have available for the exam begins as soon as you start working on or reading the problems of the exam.**
- You may consult the following items during the exam: PDF or physical/printed copies of the course notes and the notes from lectures, recap sessions, and recitation sections; QM Field Guide and any textbook including Cohen-Tannoudji; OPTI 570 2024 problem sets and solutions (yours and mine); and any of your own notes or anything you have personally written or typed. **You will not need a calculator, and you are not to use one.** Computers may be used only to access allowed material that is stored on your computer. Distance students may also use the OPTI 570 D2L site to access and return the exam. You must **not** consult other people or AI/chatbots, or accept or provide help to anyone else. You are on your honor to adhere to these rules; violation of these rules will result in a failing grade.
- There are 5 problems on the following 3 pages. 115 points are available. The exam will be graded on a 100-point scale, so 15 extra points are available.
- Use your own paper to solve all problems. Show enough work that I can follow your reasoning and give you partial credit for problems that are not fully correct.
- It is up to you to convince me that you know how to solve the problems, and to write legibly enough that I do not need to struggle to interpret your work. However, I expect you to work quickly, and that the neatness of your solutions might consequently suffer. That's OK as long as I can interpret your solutions. Draw a box around final answers if your final results are not obvious. If you have a mess of equations all over the page, direct my attention to your line of thought if it is not otherwise obvious. If you have obtained an answer that you know is not correct and you do not have enough time to fix the error, please tell me that you know the answer is wrong, why you know that it is wrong, and guess an appropriate answer – this may help you earn significant partial credit.
- If you are convinced that there is a significant mistake in a problem that may affect the answer or interpretation, please ask me about it. Alternatively, if I have made an error that is obvious to you, clearly indicate what you think is wrong, what should be changed to make the problem solvable in the manner that you think I intended, then solve the problem. Make sure that I can understand how you have modified the problem to make it solvable.

Problem 1. [10 pts.] Assume the usual notation systems for three-dimensional Cartesian and spherical coordinates, with $\vec{\mathbf{r}} = (x, y, z)$ and $r = |\vec{\mathbf{r}}|$. Suppose $\psi(r) = f(r) \cdot (z/r) \cdot (x/r + iy/r)$ is an eigenfunction of the orbital angular momentum operators $\hat{L}^2$ and $\hat{L}_z$. **What are the eigenvalues of $\hat{L}^2$ and $\hat{L}_z$ associated with $\psi(r)$?**

Problem 2. [10 pts.] Suppose a vapor of an unknown alkali-metal isotope is present in a vacuum chamber. Since the substance is an alkali metal, it is hydrogenic in the low-energy electronic states: each atom has one unpaired electron with principle quantum number n in addition to a set of completely filled electron orbitals equivalent to the electronic structure of a noble gas. Spectroscopic experiments with this atomic vapor reveal that the $n\,^2P_{3/2}$ hyperfine manifold is associated with total atomic angular momentum quantum numbers $F \in \{5, 4, 3, 2\}$. **What is the nuclear spin quantum number for the atoms of this substance?**

Problem 3. [15 pts.] Consider two particles, designated A and B. Particle A is in a state associated with an angular momentum (magnitude) quantum number $J_A = 2$. Particle B is in a state with angular momentum (magnitude) quantum number $J_B = 3/2$. Let m_A and m_B designate their respective magnetic quantum numbers associated with angular momentum about the $\hat{\mathbf{z}}$ axis. In this problem, we will consider the total angular momentum of the system. The two-particle angular-momentum state space is spanned by the tensor-product basis $\{|J_A = 2, J_B = 3/2, m_A, m_B\rangle\}$ and the total-angular-momentum basis $\{|J_T, m_T\rangle\}$ (the subscript T refers to *total*).

(a) Express the state $|J_A = 2, J_B = 3/2, m_A = 1, m_B = -1/2\rangle$ as a superposition of the total-angular-momentum basis elements.

(d) Express the state $|J_T = 3/2, m_T = 3/2\rangle$ as a superposition of the tensor-product basis elements.

Problem 4. [45 pts.] Consider a particle with a spin quantum number $s = 1/2$ and a negative gyromagnetic ratio γ. For times $t \geq 0$, the particle's spin state evolves under the presence of a spatially uniform magnetic field that points in the $\hat{\mathbf{x}}$ direction. The magnetic field has an exponentially decaying amplitude given by $B_0 \exp\{-t/\tau_0\}$. Both B_0 and τ_0 are real and positive. At time $t = 0$, the particle is in a spin state $|\psi(0)\rangle = |+\rangle_z$ corresponding to spin up along the $\hat{\mathbf{z}}$ direction.

(a) There are a couple of ways in which it is easy to make a mistake calculating the $t > 0$ time-evolution operator $\mathbb{U}(t)$ for this problem, which you will do in the next part. So first take a moment to visualize how the Bloch vector will evolve under the influence of the given magnetic field, and write down **brief** answers to the following three questions *before* moving on to part (b). **(i)** What must $\mathbb{U}(t)$ be equivalent to when evaluated at $t = 0$? **(ii)** State qualitatively how the Bloch vector precession rate (ie, the Larmor frequency) depends on time. **(iii)** For times $t \gg \tau_0$ (ie, for times when the magnetic field strength is near zero), the Bloch vector precession rate should approach zero, and the state stops evolving. We can call this the $t = \infty$ limit, and evaluate $\mathbb{U}(t)$ in this limit by choosing $t = \infty$ within our expression for $\mathbb{U}(t)$. In this limit, for initial state $|+\rangle_z$, how much do you know about what the final state $\mathbb{U}(t = \infty)|+\rangle_z$ will be? In other words, is it necessarily the case that the system will always end up in a specific state such as $|+\rangle_z$ in the $t = \infty$ limit, or can the final state only be determined by given information such as B_0 and τ_0?

(b) Calculate the $t \geq 0$ time-evolution operator $\mathbb{U}(t)$, and simplify your expression as much as possible (including by having a Pauli spin operator show up in your expression). Check whether or not your calculated $\mathbb{U}(t)$ corresponds to your answers to (a) by considering $\mathbb{U}(t)$ in the corresponding limits of t. If it does not, then either fix the error in your calculation of $\mathbb{U}(t)$, or determine why your answers for (a) might not be correct. To see how $\mathbb{U}(t)$ corresponds to your answer to (a)(ii), you may need to recognize that the term in the exponential of $\mathbb{U}(t)$ (aside from the operator part and $i = \sqrt{-1}$) is proportional to a time-dependent Bloch vector precession angle. (If you are having trouble figuring out the source of errors, you might be able to proceed, either with the rest of this problem knowing that there could be errors, or with Problem 5 and then return to this one later).

(c) For $t \geq 0$, calculate $P_-(t)$, the time-dependent probability of finding the particle in the state $|-\rangle_z$, which corresponds to spin down along the $\hat{\mathbf{z}}$ direction. There are various ways to solve this problem, choose whichever approach makes sense to you, and show all work.

(d) Carefully (and methodically) sketch $P_-(t)$ for the case $\tau_0 = -2\pi/(\gamma B_0)$. Remember, $\gamma < 0$.

(e) For $\tau_0 = -2\pi/(\gamma B_0)$, at what time is P_- maximized, and what is this maximum value?

Problem 5. [35 pts.] This problem introduces a new unitary operator $\mathbb{R}$, called the rotation operator. Consider a system with a generalized angular momentum quantum number j, and the associated pair of angular momentum eigenvalue equations

$$\hat{J}^2 \, |j,\, m_u\rangle = j(j+1)\hbar^2 \, |j,\, m_u\rangle \qquad \text{and} \qquad \hat{J}_u \, |j,\, m_u\rangle = \hbar \, m_u \, |j,\, m_u\rangle$$

where $\hat{J}_u$ is the operator for angular momentum about the $\hat{\mathbf{u}}$ direction. The **operator for rotations** about the $\hat{\mathbf{u}}$ direction is defined as

$$\mathbb{R}_u^{(j)}(\Phi) \equiv e^{-i\Phi \hat{J}_u/\hbar}$$

where Φ is an angle of rotation, and the superscript (j) indicates that this operator is to act on states associated with generalized angular momentum quantum number j. Although rotation operators act on kets, we can visualize their actions by making rotations of the corresponding $\langle \vec{J} \rangle$ in real space. For example, for a spin-1/2 system, the operator for a rotation through angle $\pi/2$ about the $\hat{\mathbf{z}}$ axis is

$$\mathbb{R}_z^{(1/2)}(\pi/2) = e^{-i\frac{\pi}{2}\,\hat{S}_z/\hbar} = e^{-i\frac{\pi}{4}\,\hat{\sigma}_z}$$

where $\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z$, and $\hat{\sigma}_z$ is the Pauli spin operator for the $\hat{\mathbf{z}}$ direction. To interpret this operator's action for this spin-1/2 case, first visualize any arbitrary Bloch vector on the Bloch sphere. The action of $\mathbb{R}_z^{(1/2)}(\pi/2) = e^{-i\frac{\pi}{4}\,\hat{\sigma}_z}$ can be interpreted as precessing that Bloch vector about the $\hat{\mathbf{z}}$ axis through an angle $\pi/2$. (For positive angles Φ, rotations are done in a right-handed direction.) The next few parts of this problem help clarify this concept.

(a) Use your intuition here and make a guess: Let $|\psi\rangle = \mathbb{R}_z^{(1/2)}(\pi/2)\,|+\rangle_x$. $|\psi\rangle$ is spin-up along what direction in space? In other words, if you start with the Bloch vector corresponding to $|+\rangle_x$, then rotate the end point of the vector through an angle $\pi/2$ about $\hat{\mathbf{z}}$ (in a right-handed sense), you will get a Bloch vector that corresponds to what particular spin state?

(b) Check your guess for **(a)** by now calculating $|\psi\rangle = \mathbb{R}_z^{(1/2)}(\pi/2)\,|+\rangle_x = e^{-i\frac{\pi}{4}\sigma_z}|+\rangle_x$. Express $|\psi\rangle$ as a state vector that is spin up along some direction that you are to determine and indicate. You may choose/omit whatever global phase factor you wish when expressing your answer.

Now use these next two examples of spin-1/2 rotation operators acting on spin-1/2 states to check your understanding (note all subscripts and angles!):

Ex. 1: What spin state is $\mathbb{R}_z^{(1/2)}(\pi/4)\,|+\rangle_x$ equal to? Answer: $|+\rangle_u$, where $\hat{\mathbf{u}} = \frac{1}{\sqrt{2}}\hat{\mathbf{x}} + \frac{1}{\sqrt{2}}\hat{\mathbf{y}}$.

Ex. 2: What spin state is $\mathbb{R}_x^{(1/2)}(\pi)\,|+\rangle_z$ equal to? Answer: $|-\rangle_z$.

Without undertaking calculations, write the resulting states in parts (c) through (g) as spin-up or spin-down along some direction that you are to indicate and define if needed, similar to the above examples. **Pay close attention to all subscripts, angles, and other notation.**

(c) $\mathbb{R}_z^{(1/2)}(2\pi)\,|+\rangle_y$

(d) $\mathbb{R}_z^{(1/2)}(\pi/4)\,|-\rangle_z$

(e) $\mathbb{R}_x^{(1/2)}(\pi/4)\,|+\rangle_y$

(f) $\mathbb{R}_y^{(1/2)}(\pi/2)\,\mathbb{R}_z^{(1/2)}(\pi/2)\,\mathbb{R}_x^{(1/2)}(\pi/2)\,|+\rangle_z$

(g) $\mathbb{R}_z^{(1/2)}(\pi/2)\,\mathbb{R}_y^{(1/2)}(\pi/2)\,\mathbb{R}_x^{(1/2)}(\pi/2)\,|+\rangle_z$

(By comparing (f) with (g), you can see that rotations about different axes do not commute.)

(h) We will now extend the use of rotation operators to states with angular momentum quantum numbers $j > 1/2$. Calculate

$$|\psi\rangle = \mathbb{R}_z^{(1)}(\pi/2)\,|j=1,\,m_x=1\rangle = e^{-i\frac{\pi}{2}\hat{J}_z/\hbar}\,|j=1,\,m_x=1\rangle$$

and express the answer as a superposition of the $\{|j=1,\,m_z\rangle\}$ basis elements. As a check of your understanding, refer to page 70 of the Field Guide to confirm that your calculation produced the state $|j=1,\,m_y=1\rangle$. Conceptually, it may also help to realize that rotation operators do not change the value of $|\langle\vec{J}\rangle|$.

(i) Now consider angular momentum states with $j=2$. As seen in the practice exam, the state $|j=2,\,m_x=2\rangle$ is expressed in the $\{|j=2, m_z\rangle\}$ basis as

$$|m_x\!=\!2\rangle = \frac{1}{4}|m_z\!=\!2\rangle + \frac{1}{2}|m_z\!=\!1\rangle + \sqrt{\frac{3}{8}}|m_z\!=\!0\rangle + \frac{1}{2}|m_z\!=\!-1\rangle + \frac{1}{4}|m_z\!=\!-2\rangle$$

where the $j=2$ index has been suppressed in each ket above for ease of notation. Use rotation operators applied to this result to figure out how to express both $|j=2,\,m_y=2\rangle$ and $|j=2,\,m_x=-2\rangle$ as superpositions of the elements of the $\{|j=2, m_z\rangle\}$ basis. You may choose whatever convenient global phase factors that you wish when giving your answers.

End of exam. Staple the exam questions to your answer pages.