$\boxed{\text{OPTI 570 LECTURE 6 Th Sep 4}}$

Last time: $[\hat{A}, \hat{B}] = \hat{C}$ and $[\hat{A}, \hat{C}] = [\hat{B}, \hat{C}] = 0$

$$\boxed{\begin{aligned} \#1.\quad [\hat{A}, F(\hat{B})] &= [\hat{A}, \hat{B}] \frac{dF(\hat{B})}{d\hat{B}} \\ [F(\hat{A}), \hat{B}] &= [\hat{A}, \hat{B}] \frac{dF(\hat{A})}{d\hat{A}} \end{aligned}}$$

Recap

$\mathcal{E}$	$\mathcal{F}_x$	$\mathcal{F}_p$
$\|\psi\rangle \in \mathcal{E}$	$\psi(x)$	$\bar{\psi}(p)$
$\|\varphi\rangle \in \mathcal{E}$	$\varphi(x)$	$\bar{\varphi}(p)$
$\langle\varphi\|\psi\rangle =$	$\int_{-\infty}^{\infty} \varphi^*(x)\psi(x)\,dx =$	$\int_{-\infty}^{\infty} \bar{\varphi}^*(p)\bar{\psi}(p)\,dp$
$\{\|x'\rangle\} \notin \mathcal{E}$	$\{\delta(x-x')\}$	$\left\{\frac{1}{\sqrt{2\pi\hbar}} e^{-ipx'/\hbar}\right\}$
$\{\|p'\rangle\} \notin \mathcal{E}$	$\left\{\frac{1}{\sqrt{2\pi\hbar}} e^{ip'x/\hbar}\right\}$	$\{\delta(p-p')\}$

Orthonormality

$$\langle x|x'\rangle = \delta(x-x')$$

$$\langle p|p'\rangle = \delta(p-p')$$

$$\mathbb{1} = \int_{-\infty}^{\infty} |x\rangle\langle x|\,dx \quad \Rightarrow \quad \mathbb{1} = \int_{-\infty}^{\infty} |p\rangle\langle p|\,dp$$

Representations in continuous bases

$$\langle x | \psi \rangle = \psi(x)$$

$$\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{i xp/\hbar}$$ ⇐ plane waves

$$\langle p | x \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-i xp/\hbar}$$

$$\langle p | \psi \rangle = \langle p | \mathbb{1}_x | \psi \rangle =$$

$$= \langle p | \left(\int_{-\infty}^{\infty} |x\rangle\langle x| dx \right) | \psi \rangle =$$

$$= \int_{-\infty}^{\infty} \langle p | x \rangle\langle x | \psi \rangle \, dx =$$

$$\boxed{\langle p | \psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ipx/\hbar} \psi(x) \, dx}$$

$$\tilde{\psi}(p) = FT[\psi(x)]$$

New: define operator $\hat{x}$

s.t. for $\hat{x} | \psi \rangle = b | \varphi \rangle$

$$\varphi(x) = \frac{x}{b} \psi(x)$$

$$\langle x | \varphi \rangle = \varphi(x) = \frac{x}{b} \psi(x) = \frac{x}{b} \langle x | \psi \rangle$$

$$\Rightarrow \frac{x}{\cancel{b}} \langle x | \varphi \rangle = \frac{1}{\cancel{b}} \langle x | \hat{x} | \psi \rangle$$

$$\Rightarrow x \langle x | \psi \rangle = \langle x | \hat{x} | \psi \rangle \quad \text{for any } |\psi\rangle$$

$$x \langle x | = \langle x | \hat{x} \quad \Big| \text{ H.C.}$$

$$|x\rangle x^{\dagger} = \hat{x}^{*} |x\rangle$$

$\hat{x}$ – Hermitian conjugate $\Rightarrow x = x^{*}$

$$\Rightarrow \boxed{\hat{x}|x\rangle = \underbrace{x}_{\text{eigenvalues of } \hat{x}}|x\rangle}$$ so $\hat{x}$ is the position operator

<u>similarly</u>: $\hat{p}|p\rangle = p|p\rangle$

$\{|x\rangle\}$

$\{|p\rangle\}$

$$\hat{x}|\psi\rangle = b|\psi\rangle$$

$$\langle x|\hat{x}|\psi\rangle = b\langle x|\psi\rangle = b\,\psi(x) =$$

$$= x\,\psi(x)$$

in the position representation, $\hat{x}$ acts as a multiplication by position coordinate x

<u>Q</u>: $\psi(x) = \langle x|\psi\rangle$

evaluate $\langle x|\hat{p}|\psi\rangle =$

$$\langle x|\hat{p}|\psi\rangle = \int_{p=-\infty}^{\infty} \langle x|\hat{p}|p\rangle\langle p|\psi\rangle\,dp =$$

$\hat{p}|p\rangle = p|p\rangle$

$$= \int p\,\langle x|p\rangle\,\bar{\psi}(p)\,dp =$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_p p\,e^{ixp/\hbar}\,\bar{\psi}(p)\,dp =$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_p \frac{\hbar}{i}\frac{\partial}{\partial x}\left[e^{ixp/\hbar}\,\bar{\psi}(p)\right] dp$$

$$= \frac{\hbar}{i}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{2\pi\hbar}} \int e^{ixp/\hbar}\,\bar{\psi}(p)\,dp\right] =$$

$$= -i\hbar \frac{\partial}{\partial x} \left\{ FT^{-1}\left[\bar{\Psi}(p) \right] \right\}$$

$$\boxed{\langle x | \hat{p} | \psi \rangle = -i\hbar \frac{\partial}{\partial x} \psi(x)}$$

"in the pos representation", $\hat{p}$ acts like $-i\hbar \frac{\partial}{\partial x}$

$\hat{x}$ — " — multiply by x

"in the momentum representation", $\hat{p}$ — " — multiply by p

, $\hat{x}$ — " — $i\hbar \frac{\partial}{\partial p}$

language summary:

$\langle x|$ — "what follows is in position rep".

$$\langle x | \psi \rangle = \psi(x)$$

$$\langle x | \hat{x} | \psi \rangle = x\, \psi(x)$$

$$\langle x | \hat{p} | \psi \rangle = -i\hbar \frac{\partial}{\partial x} \psi(x)$$

Ex: Eval $\langle x | [\hat{x}, \hat{p}] | \psi \rangle =$

$$= \langle x | \hat{x}\hat{p} - \hat{p}\hat{x} | \psi \rangle =$$

$$= \langle x | \hat{x}\hat{p} | \psi \rangle - \langle x | \hat{p}\hat{x} | \psi \rangle =$$

$$= x\left(-i\hbar \frac{\partial}{\partial x}\right) \psi(x) + i\hbar \frac{\partial}{\partial x}\left[x\, \psi(x) \right] =$$

$$= \cancel{-i\hbar x \frac{\partial \psi(x)}{\partial x}} + \cancel{i\hbar x \frac{\partial \psi(x)}{\partial x}} + i\hbar\, \psi(x) =$$

$$\boxed{\langle x | [x,p] | \psi \rangle = i\hbar\, \psi(x) = i\hbar \langle x | \psi \rangle = \langle x | i\hbar \mathbb{1} | \psi \rangle}$$

general for any $|\psi\rangle$

$$\Rightarrow \boxed{\begin{aligned}[x,p] &= i\hbar\,\mathbb{1}\\ &= i\hbar\end{aligned}}$$

In 3D: $[\bar{R}, \bar{P}]$

$$\hat{R} = (\hat{x}, \hat{y}, \hat{z})$$
$$\hat{P} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$$

$\boxed{\text{Pset 4}}$: $(\Delta\hat{A})^2(\Delta\hat{B})^2 \geq \frac{1}{4}\left|\langle[\hat{A},\hat{B}]\rangle\right|^2$

$$(\Delta x)^2(\Delta p)^2 \geq \frac{1}{4}\left|(i\hbar)^2\right|$$

$$\boxed{\Delta x\,\Delta p \geq \frac{\hbar}{2}}$$

Grand table of math

Σ	$\mathcal{F}_x$	$\mathcal{F}_p$	$\{\lvert u_n\rangle\}$ rep
$\lvert x_0\rangle$	$\delta(x-x_0)$	$\frac{1}{\sqrt{2\pi\hbar}}\, e^{-i x_0 p/\hbar}$	$\langle u_n\lvert x_0\rangle = \underbrace{\langle x_0\lvert u_n\rangle}_{u_n(x_0)}^*$
$\lvert p_0\rangle$	$\frac{1}{\sqrt{2\pi\hbar}}\, e^{i x p_0/\hbar}$	$\delta(p-p_0)$	$\langle u_n\lvert p_0\rangle = \tilde{u}_n^*(p_0)$
$\lvert u_n\rangle$	$\langle x\lvert u_n\rangle = u_n(x)$	$\langle p\lvert u_n\rangle = \tilde{u}_n(p)$	$\lvert u_n\rangle_{\{u\}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix}$ nth pos
$\lvert\psi\rangle$	$\psi(x)$	$\tilde{\psi}(p)$	$\lvert\psi\rangle_{\{u\}} = \begin{pmatrix} \langle u_1\lvert\psi\rangle \\ \langle u_2\lvert\psi\rangle \\ \vdots \end{pmatrix}$
$\langle\varphi\lvert\psi\rangle$	$\int_{-\infty}^{\infty} \varphi^*(x)\,\psi(x)\,dx$	$\int_{-\infty}^{\infty} \tilde{\varphi}^*(p)\,\tilde{\psi}(p)\,dp$	$\left(\langle\varphi\lvert u_1\rangle, \langle\varphi\lvert u_2\rangle \dots\right) \cdot \begin{pmatrix} \langle u_1\lvert\psi\rangle \\ \langle u_2\lvert\psi\rangle \\ \vdots \end{pmatrix}$
$\hat{x}$	x	$i\hbar \frac{\partial}{\partial p}$	?
$\hat{p}$	$-i\hbar \frac{\partial}{\partial x}$	p	

$$\hat{x} = \mathbb{1}\,\hat{x}\,\mathbb{1} = \sum_n \lvert u_n\rangle\langle u_n\rvert\, \hat{x} \sum_m \lvert u_m\rangle\langle u_m\rvert =$$

$$= \sum_{n,m} \underbrace{\langle u_n\lvert \hat{x}\rvert u_m\rangle}_{\text{matrix elements}} \underbrace{\lvert u_n\rangle\langle u_m\rvert}$$

matrix elements ($\lvert u_n\rangle\langle u_n\rvert$) projector

connectors $m \neq n$

$$\hat{X}_{\{|u_m\rangle\}} = \begin{pmatrix} \int x\,|u_1(x)|^2 dx & \int x\,u_1^*(x)\,u_2(x)\,dx & \dots \\ \int x\,u_2^*(x)\,u_1(x)\,dx & \int x\,|u_2(x)|^2 dx & \\ \vdots & & \ddots \end{pmatrix}$$

Postulates of Quantum

$\boxed{I}$ $|\psi(t_0)\rangle \in \mathcal{E}$

$\boxed{II}$ Physical quantity $A \to A^\dagger$, $\underset{\text{Hermitian}}{A = A^\dagger}$, $\hat{A}|\psi\rangle = |\psi'\rangle \in \mathcal{E}$

- implies the possibility of solving for eigenvalues

$\boxed{III}$ $\boxed{\text{Measurement outcomes of } A \Leftrightarrow \text{eigenvalues of } \hat{A}}$

- discrete set of eigenvalues is possible.

$\boxed{IV}$ Probability postulate - $P(\text{eigenvalue of } \hat{A}) = ?$

$\boxed{V}$ Collapse postulate

measure A on a system in state $|\psi\rangle$,
the result is a state that is projection of $|\psi\rangle$ onto $|u_m\rangle$

$\hat{A}|u_m\rangle = a_m|u\rangle$

$$|\psi_{\substack{\text{after}\\\text{meas}}}\rangle = \frac{|u_m\rangle\langle u_m|\psi\rangle}{\sqrt{\langle\psi|u_m\rangle\langle u_m|\psi\rangle}} = \frac{\hat{P}_{u_m}|\psi\rangle}{\sqrt{\langle\psi|\hat{P}_{u_m}|\psi_m\rangle}}$$

VI Schrodinger Eq:

$$\boxed{i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H}(t) |\Psi(x\rangle}$$