OPTI 570, Fall 2025 **Practice Midterm Exam 1**

2-hour written exam.
On-campus students:
Distance students: exam will be available on the D2L **Assignments** page, starting 7pm (Tucson time), Sep 24. You must complete and return by 9pm (Tucson time), Sep 29, unless you have made other arrangements with me. Distance students have 2 hours to work on the exam questions, plus *up to* 30 minutes for downloading and printing the exam (if desired), changing locations, scanning answers, and uploading answers to D2L as a single PDF (or email to me if there's a problem uploading). These extra 30 minutes are *not* to be used to work on the exam.

Instructions

- **The 2 hours that you have available for the exam begin *after* you finish reading this instructions page and as soon as you start reading the problems of the exam.**
- You may consult the following items during the exam: PDF or physical/printed copies of the course notes and the notes from lectures, recap sessions, and recitation sections; QM Field Guide; OPTI 570 problem sets and solutions (yours and mine); and any of your own notes or anything you have personally written or typed. You do not need a calculator, and you must not use one. Computers may be used only to access allowed material that is stored on your computer, or allowed internet resources. Allowed internet resources are only the OPTI 570 D2L site for distance students to access and return the exam, and email to communicate with me if needed. You must **not** consult other people, or accept or provide help to anyone else in the class. You are on your honor to adhere to these rules; violation of these rules will result in a failing grade.
- There are 5 problems on the following 3 pages. **120 points are available**, although the exam is graded out of 100 points, so there are **20 extra points** available. It is possible that the exam scores may also be further scaled up. The highest final grade that will be recorded will be 100, even for those who earn more than 100 points.
- Use your own paper to solve all problems. Show enough work that I can follow your reasoning and give you partial credit for problems that are not fully correct.
- It is up to you to convince me that you know how to solve the problems, and to write legibly enough that I do not need to struggle to interpret your work. However, I expect you to work quickly, and that the neatness of your solutions might consequently suffer. That's OK as long as I can interpret your solutions. Draw a box around final answers if your final results are not obvious. If you have a mess of equations all over the page, direct my attention to your line of thought if it is not otherwise obvious. If you have obtained an answer that you know is not correct and you do not have enough time to fix the error, please tell me that you know the answer is wrong, why you know that it is wrong, and guess an appropriate answer – this may help you earn significant partial credit.
- If you are convinced that there is a significant mistake in a problem that may affect the answer or interpretation, please ask about it. Or if a mistake is obvious to you, you may indicate what you think is wrong, what should be changed to make the problem solvable in the manner that you think I intended, then solve the problem. Make sure that I can understand how you have modified the problem to make it solvable. Part of the challenge of learning a new subject is to try to identify mistakes and speculate about the original intention of given problems!

1. [20 pts. total, (a)-(h) together are worth a total of 10 pts.] This problem involves two different discrete bases for an arbitrary system's three-dimensional state space $\mathcal{E}$. One basis is $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$. The elements of this basis are the eigenkets of the system's time-independent Hamiltonian $\hat{H}$, where $\hat{H}|u_n\rangle = E_n|u_n\rangle$ for $n \in \{1, 2, 3\}$. The energy eigenvalues are $E_1 = -\hbar\omega$, $E_2 = 0$, $E_3 = \hbar\omega$, where ω is an angular frequency. We will label this basis $\{|u_n\rangle\}$.

A second basis is $\{|a_1\rangle, |a_2\rangle, |a_3\rangle\}$. The elements of this basis are eigenkets of a dimensionless observable $\hat{A}$, where $\hat{A}|a_m\rangle = a_m|a_m\rangle$ for $m \in \{1, 2, 3\}$. The eigenvalues of $\hat{A}$ are $a_1 = 1$, $a_2 = 0$, $a_3 = -1$. We label this basis $\{|a_m\rangle\}$. These basis elements are expanded in the $\{|u_n\rangle\}$ basis as follows:

$$\begin{aligned}
|a_1\rangle &= \tfrac{1}{2}|u_1\rangle + \tfrac{1}{\sqrt{2}}|u_2\rangle + \tfrac{1}{2}|u_3\rangle \\
|a_2\rangle &= \tfrac{1}{\sqrt{2}}|u_1\rangle + 0|u_2\rangle - \tfrac{1}{\sqrt{2}}|u_3\rangle \\
|a_3\rangle &= \tfrac{1}{2}|u_1\rangle - \tfrac{1}{\sqrt{2}}|u_2\rangle + \tfrac{1}{2}|u_3\rangle
\end{aligned}$$

For the questions below, give the scalar, column vector, row vector, or matrix that corresponds to each one of the following items in the representation defined by the $\{\,|u_n\rangle\}$ basis.

(a) $|u_3\rangle$

(b) $\langle u_2|$

(c) $|u_3\rangle\langle u_2|$

(d) $|a_3\rangle$

(e) $\langle u_2|a_3\rangle$

(f) $\displaystyle\sum_{m=1}^{m=3} |a_m\rangle\langle a_m|$

(g) $\hat{H}$

(h) $\hat{\mathbb{U}}(t) \equiv e^{-i\hat{H}t/\hbar}$. This is the time evolution operator for a time duration t.

(i) [3 pts.] Calculate $\hat{\mathbb{U}}^\dagger(t)|a_3\rangle = e^{i\hat{H}t/\hbar}|a_3\rangle$, and express your answer as a column vector in the $\{\,|u_n\rangle\}$ representation.

(j) [7 pts.] Let $\hat{\mathbb{P}}_{a_3,S} = |a_3\rangle\langle a_3|$ be the Schrödinger-picture projector onto $|a_3\rangle$. Using your answer from **(i)**, or by other methods, calculate the $\{\,|u_n\rangle\}$ rep. matrix that corresponds to the projector onto $|a_3\rangle$ in the Heisenberg-picture, defined as $\hat{\mathbb{P}}_{a_3,H} = \hat{\mathbb{U}}^\dagger\hat{\mathbb{P}}_{a_3,S}\hat{\mathbb{U}}$.

---

2. [5 pts.] Suppose $\psi(x)$ is a wavefunction over the one-dimensional position coordinate x, and is neither even nor odd about any position. $\tilde{\psi}(p)$ is the Fourier transform of $\psi(x)$, where p is the momentum coordinate. Let x_0 be an arbitrary position along x. What is the Fourier transform of $\psi(x - x_0)$ in terms of the quantities given above? This should be very quick to answer!

3. [12 pts] A state space $\mathcal{E}$ corresponds to the physically realizable quantum states of a particle in one-dimensional coordinate space over positions x, associated with position operator $\hat{X}$. $\hat{P}$ is the corresponding momentum operator. Give interpretations for the following operators by describing in words what they do or how they act on elements of $\mathcal{E}$ or an associated function space.

(a) $\hat{A} = (e^{-ix_0\hat{P}/\hbar}e^{ip_0\hat{X}/\hbar}e^{ix_0\hat{P}/\hbar}e^{-ip_0\hat{X}/\hbar})^n$, where the superscript exponent n is any integer greater than 0, and x_0 and p_0 are constant position and momentum scalars, respectively. For this operator, you may also give an alternate way of writing the operator, using whatever global phase factor you wish to use.

(b) $\hat{B} = \int_{x_0-\sigma}^{x_0+\sigma} dx'|x'\rangle\langle x'|$, where $\hat{X}|x'\rangle = x'|x'\rangle$, and x_0 and σ are real scalars with dimensional units of length.

(c) $\hat{C} = \int_{-\infty}^{\infty} dx'|x'\rangle\langle x'|F\rangle\langle x'|$, where $\hat{X}|x'\rangle = x'|x'\rangle$, and $\langle x|F\rangle = F(x)$ is a function of position x. What does $\hat{C}$ do in the position representation? What are the general restrictions on $F(x)$ if $\hat{C}$ is to be a projector?

---

4. [38 pts.] A state space $\mathcal{E}$ is spanned by the basis $\{|n{=}2\rangle, |n{=}1\rangle, |n{=}0\rangle, |n{=}-1\rangle, |n{=}-2\rangle\}$, where these basis elements are eigenkets of an operator $\hat{N}$ such that

$$\hat{N}|n\rangle = n|n\rangle.$$

An operator $\hat{B}$ is defined as

$$\hat{B} = \sum_{n=-2}^{2} |n|\,|-n\rangle\langle n|$$

where $|n|$ is the absolute value of the quantum number (and summation index) n; make sure you distinguish $|n|$ from the ket that immediately follows it in the definition of $\hat{B}$. Consider a properly normalized quantum state $|\psi\rangle \in \mathcal{E}$ that is expanded in the above basis as

$$|\psi\rangle = \frac{1}{2}|n=2\rangle + \frac{1}{\sqrt{2}}|n=1\rangle + \frac{1}{2}|n=0\rangle.$$

(a) Evaluate $\langle\psi|\hat{N}|\psi\rangle$ in terms of the quantities given above, and simplify as much as possible.

(b) Is $\hat{B}$ Hermitian? Is $\hat{B}$ unitary? (Simple yes/no answers are fine, no justification needed, but if you are not sure then some brief justification or showing work might give you some partial credit if you are not correct.)

(c) Evaluate the trace of $\hat{B}$ in whatever way is most comfortable to you. If you are sure of your answer, you can just list it without showing work.

(d) Define the ket $|\varphi\rangle$ as $|\varphi\rangle \equiv C\hat{B}|\psi\rangle$, where C is a positive and real scalar such that $|\varphi\rangle$ is properly normalized. First, determine C in terms of information given above. Next, determine $\langle\varphi|\hat{N}|\varphi\rangle$, simplifying your answer to a rational number (do not give your answer in decimal notation).

(e) Define $|\beta_1\rangle \equiv \mu|n=-1\rangle + \nu|n=1\rangle$. Give values for μ and ν such that $|\beta_1\rangle$ is a properly normalized eigenstate of $\hat{B}$, and such that the associated eigenvalue is 1. In other words, find μ and ν so that $\hat{B}|\beta_1\rangle = |\beta_1\rangle$.

(f) The other eigenvalues $\hat{B}$ are -2,-1,0 and 2. Specify the corresponding eigenstates of $\hat{B}$ that are all properly normalized, and orthogonal to each other and to $|\beta_1\rangle$ of part (e). Give suitable labels to these new new eigenstates. These four kets, along with $|\beta_1\rangle$ from part (e), should now define a new basis for $\mathcal{E}$.

(g) Express $|\psi\rangle$, defined at the beginning of this problem, in the new basis of eigenstates of $\hat{B}$. Use Dirac notation, do not answer this question by representing $|\psi\rangle$ as a column vector.

(h) Evaluate and simplify as much as possible: $\langle n{=}1|e^{i\theta\hat{B}}|n{=}1\rangle$, for real scalar θ.

---

5. [45 pts.] Using the angular frequency ω, we define a Hamiltonian $\hat{H}_0$ for a three-dimensional state space $\mathcal{E}$ as

$$\hat{H}_0 = \frac{\hbar\omega}{2}\left(\,|\phi_1\rangle\langle\phi_1| \;+\; 3\,|\phi_2\rangle\langle\phi_2| \;+\; 5\,|\phi_3\rangle\langle\phi_3|\,\right),$$

such that $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle\}$ is an orthonormal basis that spans $\mathcal{E}$. If a particle is in state $|\phi_2\rangle$ (for example), it will remain in this state as long as $\hat{H}_0$ remains the system Hamiltonian. However, if another component to the Hamiltonian is present, $|\phi_2\rangle$ might not be a stationary state of the new Hamiltonian. To examine this idea, we define a new Hamiltonian $\hat{H} = \hat{H}_0 + \hat{W}$ where

$$\hat{W} = \frac{\hbar\omega}{2}\left(-i\sqrt{3}|\phi_2\rangle\langle\phi_3| \;+\; i\sqrt{3}\,|\phi_3\rangle\langle\phi_2|\right).$$

(a) Find the eigenvalues of $\hat{H}$ and the associated eigenstates. Write the properly normalized eigenstates of $\hat{H}$ in the $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle\}$ basis.

(b) Calculate $\hat{U}(t,0)$, the time evolution operator for $\hat{H}$ that evolves for a duration t, and write your answer as a matrix in the $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle\}$ representation. You do not need to simplify your answer; it is fine to leave the matrix elements in terms of exponentials.

(c) Suppose we define a quantum state at $t=0$ to be $|\psi(0)\rangle = |\phi_2\rangle$. The state of the system evolves under the Hamiltonian $\hat{H}$. Calculate $|\psi(t)\rangle$ for any later time t. Use this $|\psi(t)\rangle$ in the remainder of this problem. **Note: you can solve this and the following parts even if you did not answer (b).**

(d) Calculate the probability $\mathcal{P}_3(t)$ that at any time $t>0$, the system will be found in state $|\phi_3\rangle$ (or stated correctly, that a measurement of the quantity associated with $\hat{H}_0$ will yield the result $5E_1$). Simplify your answer.

(e) Sketch $\mathcal{P}_3(t)$, label the axes, and indicate important points: namely, maximum and minimum values of $\mathcal{P}_3(t)$, and any significant time scales that appear in the plot.

END OF EXAM