Notes of Quantum Optics

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Preface

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1.2.1 Fields in a cavity

Lets consider the following one-dimensional problem, where a cavity of length L is oriented along the z-axis.

A linear polarized E-field is assumed, the medium is free space, perfect conducting walls and there is no free charges nor free current. The scheme is shown in figure 1.1.

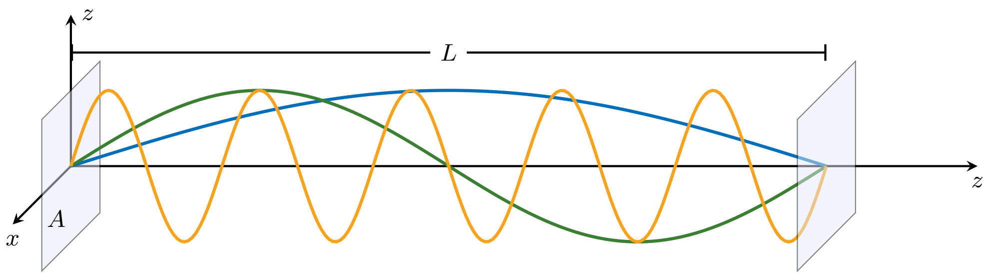


Figure 1.1 One-dimensional cavity problem. Perfect conducting walls.

Our goal is to find the E- and B-field inside the cavity. Maxwell's equations in this case are:

$$\text{Maxwell's equations with free sources} \quad \begin{cases} \nabla\cdot\boldsymbol{E}=0 \\ \nabla\cdot\boldsymbol{B}=0 \\ \nabla\times\boldsymbol{E}=-\partial_t\boldsymbol{B} \\ \nabla\times\boldsymbol{B}=\frac{1}{c^2}\partial_t\boldsymbol{E} \end{cases} \tag{1.1}$$

The E-field will be assumed to be $\boldsymbol{E}(z,t)=\boldsymbol{e}E(z,t)$, where $\boldsymbol{e}$ is the polarization vector. Because fields depends only on z, $\nabla=\hat{\boldsymbol{z}}\partial_z$. First Maxwell equation yields:

$$\nabla\cdot\boldsymbol{E}=\partial_z(\hat{\boldsymbol{z}}\cdot\boldsymbol{E})=\partial_z(\hat{\boldsymbol{z}}\cdot\boldsymbol{e}E)=0\Longrightarrow\boldsymbol{e}\cdot\hat{\boldsymbol{z}}=0.$$

This implies that the polarization vector must be unitary in the transverse plane:

$$\boldsymbol{e}=\cos\phi\hat{\boldsymbol{x}}+\sin\phi\hat{\boldsymbol{y}}.$$

Third Maxwell's equation yields

$$\nabla\times\boldsymbol{E}=(\hat{\boldsymbol{z}}\partial_z)\times(\boldsymbol{e}E)=(\hat{\boldsymbol{z}}\times\boldsymbol{e})\partial_zE=-\partial_t\boldsymbol{B}.$$

Taking the curl of this equation, using Fourth Maxwell's equation and vector identities:

$$\begin{aligned}\nabla\times(\nabla\times\boldsymbol{E})&=-\partial_t(\nabla\times\boldsymbol{B})\\ \nabla(\cancelto{0}{\nabla\cdot\boldsymbol{E}})-\nabla^2\boldsymbol{E}&=-\frac{1}{c^2}\partial_t^2\boldsymbol{E}\\ -\boldsymbol{e}\partial_z^2E&=-\boldsymbol{e}\frac{1}{c^2}\partial_t^2E.\end{aligned}$$

From here, we have the E-field wave equation for this particular problem.

$$\partial_z^2 E - \frac{1}{c^2}\partial_t^2 E = 0. \tag{1.2}$$

Before solving this equation, we need the boundary condition set by the PEC condition. We need to $\hat{n} \times \boldsymbol{E} = 0$ on the surface. Because the normal surface unit vector is $\hat{n} = \pm\hat{\boldsymbol{z}}$, we have

$$\text{Boundary condition} \qquad \hat{n} \times \boldsymbol{E} = \hat{\boldsymbol{z}} \times (\boldsymbol{e}E) = 0 \Longrightarrow E(z=0,t) = E(z=L,t) = 0.$$

In order to solve the PDE, we assum a product form $E(z,t) = Z(z)q(t)$. Then, by replacing in it in (1.2):

$$\begin{aligned} \partial_z^2[Z(z)q(t)] - \frac{1}{c^2}\partial_t^2[Z(z)q(t)] &= 0 \\ Z''(z)q(t) - \frac{1}{c^2}Z(z)\ddot{q}(t) &= \Big/ [Z(z)q(t)]^{-1} \\ \frac{Z''(z)}{Z(z)} &= \frac{1}{c^2}\frac{\ddot{q}(z)}{q(z)}. \end{aligned}$$

Left side depends only on z, while the right side only on y. The only way this can be true is if both are a constant, say, $-k^2$. Then,

$$\text{Spatial and temporal differential equations} \qquad \begin{cases} \dfrac{Z''}{Z} = -k^2 \longrightarrow Z'' + k^2 Z = 0 \\ \dfrac{1}{c^2}\dfrac{\ddot{T}}{T} = -k^2 \longrightarrow \ddot{q} + \omega^2 q(t) = 0 \end{cases}.$$

For the spatial ODE, we assume a solution of the form

$$Z(z) = A\sin(kz) + B\cos(kz), \quad Z(0) = Z(L) = 0.$$

Setting the boundaries:

$$\begin{aligned} Z(0) &= A\sin(0) + B\cos(0) = 0 \Longrightarrow B = 0 \\ Z(L) &= A\sin(kL) = 0 \Longrightarrow k_m = \frac{m\pi}{L}, \quad m \in \mathbb{N}. \end{aligned}$$

We left the temporal ODE unsolved. Finally, putting all together yields the initial E-field:

$$\boldsymbol{E}_{\boldsymbol{k},\lambda}(z,t) = \boldsymbol{e}_\lambda \sqrt{\frac{2\omega^2}{V\varepsilon_0}}\, q_{\boldsymbol{k},\lambda}(t)\sin(kz), \quad k_m = \frac{m\pi}{L}, \quad \omega = ck.$$

We have includnig the subscript λ and $\boldsymbol{k}$ to consider multiple mode $\boldsymbol{k}$ varied with m and λ. Also, the coefficient in red is for better results in the future. Using Faraday's law:

$$\boldsymbol{B}_{\boldsymbol{k},\lambda}(z,t) = (\boldsymbol{k} \times \boldsymbol{e}_\lambda)\frac{1}{kc^2}\sqrt{\frac{2\omega^2}{V\varepsilon_0}}\, \dot{q}_{\boldsymbol{k},\lambda}(t)\sin(kz).$$

The term $\dot{q}(t)$ will play the role of a canonical momentum for a particle of unit mass, $p(t) = \dot{q}(t)$.

1.2.2 Single-mode Hamiltonian

The classical field energy, or Hamiltonian H, of the single-mode field is given by

$$
\begin{aligned}
H_{\boldsymbol{k},\lambda} &= \frac{1}{2}\int dV \left[\varepsilon_0 \boldsymbol{E}^2_{\boldsymbol{k},\lambda} + \frac{1}{\mu_0}\boldsymbol{B}^2_{\boldsymbol{k},\lambda}\right] \\
&= \frac{1}{2}A\int_0^L dz \left[\frac{2\omega^2}{V\varepsilon_0} q^2_{\boldsymbol{k},\lambda}(t)\sin^2(kz) + \frac{1}{\mu_0}\frac{1}{k^2c^4}\frac{2\omega^2}{V\varepsilon_0} p^2_{\boldsymbol{k},\lambda}(t)\cos^2(kz)\right] \\
&= \frac{A}{V}\int_0^L dz \left[\omega^2 q^2_{\boldsymbol{k},\lambda}(t)\sin^2(kz) + p^2_{\boldsymbol{k},\lambda}(t)\cos^2(kz)\right] \\
&= \frac{1}{L}\left[\omega^2 q^2_{\boldsymbol{k},\lambda}(t)\frac{L}{2} + p^2_{\boldsymbol{k},\lambda}(t)\frac{L}{2}\right] \\
H_{\boldsymbol{k},\lambda} &= \frac{1}{2}\left[\omega^2 q^2_{\boldsymbol{k},\lambda}(t) + p^2_{\boldsymbol{k},\lambda}(t)\right].
\end{aligned}
$$

It is apparent that a single-mode field if formally equivalent to a harmonic quantim oscillator of unit mass, where the E- and B-fields play the roles of canonical position and momentum.

$$
H_{\text{harmonic oscillator}} = \frac{1}{2}m\omega^2 x^2 + \frac{p^2}{2m}.
$$

To begin the quantization, we make q, p operators $\hat{q}, \hat{p}$, which needs to satisfy the canonical commutation relations

$$
[\hat{q}_{\boldsymbol{k},\lambda}, \hat{p}_{\boldsymbol{k},\lambda}] = i\hbar. \tag{1.3}
$$

The EM fields with the operators are:

$$
\text{Quantized EM fields} \qquad
\begin{aligned}
\hat{\boldsymbol{E}}_{\boldsymbol{k},\lambda}(z,t) &= \boldsymbol{e}_{\boldsymbol{k},\lambda}\sqrt{\frac{2\omega^2}{V\varepsilon_0}}\hat{q}(t)\sin(kz) \\
\hat{\boldsymbol{B}}_{\boldsymbol{k},\lambda}(z,t) &= (\boldsymbol{k}\times\boldsymbol{e}_{\boldsymbol{k},\lambda})\frac{1}{kc^2}\sqrt{\frac{2\omega^2}{V\varepsilon_0}}\hat{p}(t)\cos(kz)
\end{aligned}. \tag{1.4}
$$

The operators $\hat{q}, \hat{p}$ are Hermitian and therefore correspond to obserable quantites. It is convenient to introduce the ladder operators to simplify the expresion:

$$
\text{Ladder operator} \qquad
\begin{aligned}
\hat{a}_{\boldsymbol{k},\lambda} &= \sqrt{\tfrac{1}{2\hbar\omega}}(\omega\hat{q}_{\boldsymbol{k},\lambda} + i\hat{p}_{\boldsymbol{k},\lambda}) \\
\hat{a}^\dagger_{\boldsymbol{k},\lambda} &= \sqrt{\tfrac{1}{2\hbar\omega}}(\omega\hat{q}_{\boldsymbol{k},\lambda} - i\hat{p}_{\boldsymbol{k},\lambda})
\end{aligned}
\longrightarrow
\begin{aligned}
\hat{q}_{\boldsymbol{k},\lambda} &= \sqrt{\tfrac{\hbar}{2\omega}}(\hat{a}_{\boldsymbol{k},\lambda} + \hat{a}^\dagger_{\boldsymbol{k},\lambda}) \\
\hat{p}_{\boldsymbol{k},\lambda} &= -i\sqrt{\tfrac{\hbar\omega}{2}}(\hat{a}_{\boldsymbol{k},\lambda} - \hat{a}^\dagger_{\boldsymbol{k},\lambda})
\end{aligned}.
$$

Using these defintion, the Hamiltonian yields:

$$
\begin{aligned}
\hat{H}_{\boldsymbol{k},\lambda} &= -\frac{1}{2}\frac{\hbar\omega}{2}(\hat{a}_{\boldsymbol{k},\lambda} - \hat{a}^\dagger_{\boldsymbol{k},\lambda})(\hat{a}_{\boldsymbol{k},\lambda} - \hat{a}^\dagger_{\boldsymbol{k},\lambda}) + \frac{1}{2}\frac{\hbar\omega^2}{2\omega}(\hat{a}_{\boldsymbol{k},\lambda} + \hat{a}^\dagger_{\boldsymbol{k},\lambda})(\hat{a}_{\boldsymbol{k},\lambda} + \hat{a}^\dagger_{\boldsymbol{k},\lambda}) \\
&= \frac{\hbar\omega}{2}[\hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} + \hat{a}_{\boldsymbol{k},\lambda}\hat{a}^\dagger_{\boldsymbol{k},\lambda}] = \hbar\omega\left[\hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} + \frac{1}{2}\right] = \hbar\omega\left[n_{\boldsymbol{k},\lambda} + \frac{1}{2}\right].
\end{aligned}
$$

The time dependence of the ladder operator in Heisenberg equation is:

$$
\frac{d\hat{a}_{\boldsymbol{k},\lambda}}{dt} = \frac{i}{\hbar}[\hat{H}_{\boldsymbol{k},\lambda}, \hat{a}_{\boldsymbol{k},\lambda}] = \frac{i}{\hbar}\left[\hbar\omega\left(\hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} + \frac{1}{2}\right), \hat{a}_{\boldsymbol{k},\lambda}\right] = -i\omega\hat{a}_{\boldsymbol{k},\lambda} \longrightarrow
\begin{aligned}
\hat{a}_{\boldsymbol{k},\lambda}(t) &= \hat{a}_{\boldsymbol{k},\lambda}(0)e^{-i\omega t} \\
\hat{a}^\dagger_{\boldsymbol{k},\lambda}(t) &= \hat{a}^\dagger_{\boldsymbol{k},\lambda}(0)e^{i\omega t}
\end{aligned}.
$$

For example, we have the following applications:

$$\hat{a}_{\boldsymbol{k},\lambda}\left|0\right\rangle = \left|1\right\rangle_{\boldsymbol{k},\lambda}, \quad \hat{a}_{\boldsymbol{k},\lambda}\left|n\right\rangle_{\boldsymbol{k},\lambda} = \sqrt{n}\left|n-1\right\rangle_{\boldsymbol{k},\lambda}, \quad \hat{a}^{\dagger}_{\boldsymbol{k},\lambda}\left|n\right\rangle_{\boldsymbol{k}',\lambda'} = \left|n\right\rangle_{\boldsymbol{k},\lambda}\left|1\right\rangle_{\boldsymbol{k}',\lambda'}, \quad \hat{a}_{\boldsymbol{k},\lambda}\left|n\right\rangle_{\boldsymbol{k}',\lambda'}\left|0\right\rangle_{\boldsymbol{k},\omega} = 0.$$

The operator product $\hat{a}^{\dagger}\hat{a}$ has an important significance and is called the number operator $\hat{n}$, whose eigenequation is

$$\hat{n}\left|n\right\rangle = n\left|n\right\rangle.$$

The state $\left|n\right\rangle$ is the energy eigenstte of the single mode field with energy eiganvlaue E_n:

$$\hat{H}\left|n\right\rangle = \hbar\omega(\hat{a}^{\dagger}\hat{a} + \frac{1}{2})\left|n\right\rangle = E_n\left|n\right\rangle$$

The relation between the number state and the raising operator $\hat{a}^{\dagger}$ is:

$$\left|n\right\rangle = \frac{(\hat{a}^{\dagger})^n}{\sqrt{n}}\left|0\right\rangle.$$

1.2.3 Quantizing the multimode field

We now consider all the modes in the optical cavity:

$$\boldsymbol{E}(z,t) = \sum_{m,\lambda}\hat{\boldsymbol{e}}_{\lambda}\sqrt{\frac{2\omega_m^2}{\varepsilon_0 V}}\hat{q}_{m,\lambda}(t)\sin(k_m z) \tag{1.5}$$

$$\boldsymbol{B}(z,t) = \sum_{m,\lambda}(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{e}}_{\lambda})\frac{1}{k_m c^2}\sqrt{\frac{2\omega_m^2}{\varepsilon_0 V}}\dot{\hat{q}}_{m,\lambda}(t)\cos(k_m z). \tag{1.6}$$

where $kL = m\pi$ and $\hat{\boldsymbol{e}}_{\lambda} = \{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}\}$. The process is anaglogous; we compute the Hamiltonian:

$$H = \int dV\left[\frac{1}{2}\varepsilon_0\boldsymbol{E}^2(z,t) + \frac{1}{2\mu_0}\boldsymbol{B}(z,t)\right]$$

$$= \int dV\left[\frac{1}{2}\varepsilon_0\left\{\sum_{m,\alpha}\hat{\boldsymbol{e}}_{\alpha}\sqrt{\frac{2\omega_m^2}{\varepsilon_0 V}}q_{m,\alpha}(t)\sin(k_m z)\right\}^2 + \frac{1}{2\mu_0}\varepsilon_0\left\{\sum_{m,\alpha}(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{e}}_{\alpha})\frac{\mu_0}{\varepsilon_0}k_m\sqrt{\frac{2\omega_m^2}{\varepsilon_0 V}}\dot{q}_{m,\alpha}(t)\cos(k_m z)\right\}^2\right]$$

$$\vdots$$

$$H = \frac{1}{2}\sum_{m,\lambda}(\omega_m^2 q_{m,\lambda}^2 + \dot{q}_{m,\lambda}^2).$$

That is,

> Each mode of the EM field is an independent harmonic oscillator.

We can quantize the multimode EM field in a similar way as the sige mode field to obtain the Hamiltonian:

Hamiltoninan multimode EM field $$\hat{H} = \sum_{m,\lambda}\hbar\omega_m\left[\hat{a}^{\dagger}_{m,\lambda}a_{m,\lambda} + \frac{1}{2}\right] = \sum_{m,\lambda}\hbar\omega_m\left[\hat{n}_{m,\lambda} + \frac{1}{2}\right],$$

where $\hat{n}_{m,\lambda}$ is the number of excitations in mode m, λ.

1.2.4 Quantizing EM field in free space

We define the scalar and vector potential

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -\partial_t \boldsymbol{A} - \nabla \phi.$$

Using these automatically satisfy the Gauss magnetic equation and Ampere law. In QO, we assume **Coulomb gauge**: $\nabla \cdot \boldsymbol{A} = 0$. With this, the E-field is

$$\boldsymbol{E} = -\partial_t \boldsymbol{A} - \nabla \phi = \boldsymbol{E}^{\perp} + \boldsymbol{E}^{\parallel}, \quad \text{where} \begin{cases} \nabla \cdot \boldsymbol{E}^{\perp} = 0, & \boldsymbol{A} \text{ corresponds to radiation (all transverse)} \\ \boldsymbol{E}'' = -\nabla \phi, & \boldsymbol{A} \text{ corresponds to the field of sources (all longitudinal)} \end{cases}.$$

In Coulomb gauge, in absence of charge $\phi = 0$ such that the vector potential-wave equation is

$$\nabla^2 \boldsymbol{A} - \frac{1}{2} \partial_t^2 \boldsymbol{A} = 0.$$

It can be solved with the following plane wave

$$\boldsymbol{A}(\boldsymbol{r}, t) = \sum_{\boldsymbol{k},\lambda} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} \left[A_{\boldsymbol{k},\lambda} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} + A^*_{\boldsymbol{k},\lambda} e^{-i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \right], \quad \nabla \cdot \boldsymbol{A} = 0 \Longrightarrow \boldsymbol{k} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} = 0.$$

The vector potential in free space is expressed as a sueprposition of plane waves such that linear dispersion $\omega = |\boldsymbol{k}|c$ is satisfied with $\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda'} = \delta_{\lambda\lambda'}$.

The quantization volume we quantize the field is considered to have finite volume of V and we impose periodic boundary such that each side is equal.

The E- and B-field are then:

$$\boldsymbol{E}(\boldsymbol{r}, t) = -\partial_t \boldsymbol{A}(\boldsymbol{r}, t) = i \sum_{\boldsymbol{k},\lambda} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} \omega \left[A_{\boldsymbol{k},\lambda} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} + A^*_{\boldsymbol{k},\lambda} e^{-i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \right],$$
$$\boldsymbol{B}(\boldsymbol{r}, t) = \nabla \times \boldsymbol{A}(\boldsymbol{r}, t) = i \sum_{\boldsymbol{k},\lambda} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} \left[A_{\boldsymbol{k},\lambda} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} + A^*_{\boldsymbol{k},\lambda} e^{-i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \right].$$

The energy (Hamiltonian) associated with the fields is:

$$H = -\int dV \left[\frac{1}{2} \varepsilon_0 \boldsymbol{E}^2(\boldsymbol{r}, t) + \frac{1}{2\mu_0} \boldsymbol{B}(\boldsymbol{r}, t) \right]$$
$$\vdots$$
$$H = 2\varepsilon_0 V \sum_{\boldsymbol{k},\lambda} \omega^2 |A_{\boldsymbol{k},\lambda}|^2.$$

We have used the following relation:

$$\int dV e^{\pm i(\boldsymbol{k} - \boldsymbol{k}')\cdot\boldsymbol{r}} = \delta_{\boldsymbol{k}\boldsymbol{k}'} V.$$

Defining

$$A_{\boldsymbol{k},\lambda} = \frac{1}{2\omega\sqrt{\varepsilon_0 V}} (\omega q_{\boldsymbol{k},\lambda} + i p_{\boldsymbol{k},\lambda}), \quad A^*_{\boldsymbol{k},\lambda} = \frac{1}{2\omega\sqrt{\varepsilon_0 V}} (\omega q_{\boldsymbol{k},\lambda} - i p_{\boldsymbol{k},\lambda}).$$

The Hamiltonian is:

$$H = \frac{1}{2}\sum_{\boldsymbol{k},\lambda}(\omega^2 q_{\boldsymbol{k},\lambda}^2 + p_{\boldsymbol{k},\lambda}^2) \xrightarrow{\text{quantizing}} \hat{H} = \sum_{\boldsymbol{k},\lambda} \hbar\omega\left(\hat{a}_{\boldsymbol{k},\lambda}^\dagger \hat{a}_{\boldsymbol{k},\lambda} + \frac{1}{2}\right),$$

where we have defined the ladder operators:

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p}). \tag{1.7}$$

So, the coefficients are expressed in term of the ladder:

$$A_{\boldsymbol{k},\lambda} = \sqrt{\frac{\hbar}{2\omega\varepsilon_0 V}}\hat{a}_{\boldsymbol{k},\lambda}, \quad A_{\boldsymbol{k},\lambda}^* = \sqrt{\frac{\hbar}{2\omega\varepsilon_0 V}}\hat{a}_{\boldsymbol{k},\lambda}^\dagger.$$

Thus the fields are:

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{r},t) &= \textstyle\sum_{\boldsymbol{k},\lambda}\sqrt{\frac{\hbar}{2\omega\varepsilon_0 V}}\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}\left[\hat{a}_{\boldsymbol{k},\lambda}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \hat{a}_{\boldsymbol{k},\lambda}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right],\\
\boldsymbol{E}(\boldsymbol{r},t) &= i\textstyle\sum_{\boldsymbol{k},\lambda}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}\left[\hat{a}_{\boldsymbol{k},\lambda}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \hat{a}_{\boldsymbol{k},\lambda}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right],\\
\boldsymbol{B}(\boldsymbol{r},t) &= \tfrac{i}{c}\textstyle\sum_{\boldsymbol{k},\lambda}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}(\boldsymbol{k}\times\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda})\left[\hat{a}_{\boldsymbol{k},\lambda}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \hat{a}_{\boldsymbol{k},\lambda}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right].
\end{aligned} \tag{1.8}$$

1.2.5 Electric field expectation and uncertainty

Lets conside one mode of the E-field in a number state: $|n\rangle_{\boldsymbol{k},\lambda}$. Lets add the time dependence to the ladder operators:

$$\hat{a}_{\boldsymbol{k},\lambda}e^{-i\omega t} \to \hat{a}_{\boldsymbol{k},\lambda}(t), \quad \hat{a}_{\boldsymbol{k},\lambda}^\dagger e^{i\omega t} \to \hat{a}_{\boldsymbol{k},\lambda}^\dagger(t).$$

The E-field is:

$$\hat{\boldsymbol{E}}(\boldsymbol{r},t) = i\sum_{\boldsymbol{k},\lambda} e_{\boldsymbol{k},\lambda}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left[\hat{a}_{\boldsymbol{k},\lambda}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{a}_{\boldsymbol{k},\lambda}^\dagger(t)e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right].$$

Expectation value of the E-field is:

$$\begin{aligned}
{}_{\boldsymbol{k}_0,\lambda_0}\langle n|\hat{\boldsymbol{E}}(\boldsymbol{r},t)|n\rangle_{\boldsymbol{k}_0,\lambda_0} &= i_{\boldsymbol{k},\lambda}\langle n|\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left[\hat{a}_{\boldsymbol{k},\lambda}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{a}_{\boldsymbol{k},\lambda}^\dagger(t)e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right]|n\rangle_{\boldsymbol{k}_0,\lambda_0}\\
&= i_{\boldsymbol{k}_0,\lambda_0}\langle n|\,\hat{\boldsymbol{e}}_{\boldsymbol{k}_0,\lambda_0}\sqrt{\frac{\hbar\omega_0}{2\varepsilon_0 V}}\left[\sqrt{n}\,|n-1\rangle_{\boldsymbol{k}_0,\lambda_0}e^{i\boldsymbol{k}_0\cdot\boldsymbol{r}} - \sqrt{n+1}\,|n+1\rangle_{\boldsymbol{k}_0,\lambda_0}e^{-i\boldsymbol{k}_0\cdot\boldsymbol{r}}\right]\\
&= 0.
\end{aligned}$$

Thus, the average E-field is zero.

Lets consider the uncertainty of the E-field:

$$\begin{aligned}
\langle\hat{\boldsymbol{E}}\rangle &=_{\boldsymbol{k}_0,\lambda_0}\langle n|\sum_{\boldsymbol{k},\lambda}\sum_{\boldsymbol{k}',\lambda'}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\sqrt{\frac{\hbar\omega'}{2\varepsilon_0 V}}\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}\cdot\hat{\boldsymbol{e}}_{\boldsymbol{k}',\lambda'}\left[\hat{a}_{\boldsymbol{k},\lambda}e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{a}_{\boldsymbol{k},\lambda}^\dagger e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right]\left[\hat{a}_{\boldsymbol{k}',\lambda'}e^{i\boldsymbol{k}'\cdot\boldsymbol{r}} - \hat{a}_{\boldsymbol{k}',\lambda'}^\dagger e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}}\right]|n\rangle_{\boldsymbol{k}_0,\lambda_0}\\
&= \sum_{\boldsymbol{k},\lambda}\sum_{\boldsymbol{k}',\lambda'}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\sqrt{\frac{\hbar\omega'}{2\varepsilon_V}}_{\boldsymbol{k},\lambda}\langle 1|1\rangle_{\boldsymbol{k}',\lambda'}e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} + \frac{\hbar\omega_0}{2\varepsilon_0 V}[_{\boldsymbol{k}_0,\lambda_0}\langle n+1|\sqrt{n+1} +_{\boldsymbol{k}_0,\lambda_0}\langle n-1|\sqrt{n}][\sqrt{n+1}\,|n+1\rangle_{\boldsymbol{k}_0,\lambda_0} +\\
&= \sum_{\{\boldsymbol{k},\lambda\}\neq\{\boldsymbol{k}_0,\lambda_0\}}\frac{\hbar\omega}{2\varepsilon_0 V} + \frac{\hbar\omega_0}{2\varepsilon_0 V}(2n+1).
\end{aligned}$$

So, the uncertainty is:

E-field uncertainty $$\langle \Delta E^2 \rangle = \underbrace{\sum_{\{\boldsymbol{k},\lambda\}\neq\{\boldsymbol{k}_0,\lambda_0\}} \frac{\hbar\omega}{2\varepsilon_0 V}}_{\text{All modes in vaccum state}} + \underbrace{\frac{\hbar\omega_0}{2\varepsilon_0 V}(2n+1)}_{\text{Mode }(\boldsymbol{k}_0,\lambda_0)}.$$

1.3 Coherent states

The coherent state is defined as the eigenstate of the annihilator operator:

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,, \quad \alpha \in \mathbb{C}.$$

The expected number of photons in the coherent state is:

$$\langle\alpha|\hat{n}|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2.$$

Coherent states are the most classical states of the quantized EM field.

It has the following results:

$$\langle\alpha|\hat{a}^2|\alpha\rangle = \alpha^2, \quad \langle\alpha|\hat{a}^{\dagger 2}|\alpha\rangle = \alpha^{*2}, \quad \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2, \quad \langle\alpha|\hat{a}\hat{a}^\dagger|\alpha\rangle = |\alpha|^2 + 1.$$

For example, computing the expectation of the E-field (1.8) is:

$$\begin{aligned}
\langle \boldsymbol{E} \rangle &= \langle\alpha| i \sum_{\boldsymbol{k},\lambda} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} [\hat{a}_{\boldsymbol{k},\lambda} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \hat{a}^\dagger_{\boldsymbol{k},\lambda} e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}]|\alpha\rangle \\
&= i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} [\alpha e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \alpha^* e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}] \\
\langle \boldsymbol{E} \rangle &= \boldsymbol{\mathcal{E}} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \boldsymbol{\mathcal{E}}^* e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \longrightarrow \boldsymbol{\mathcal{E}} = i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \alpha \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}.
\end{aligned}$$

The expectation value of the E-field for a coherent state corresponds to a classical E-field.

Let us now consider the uncertainty of the E-field for $|\alpha\rangle$:

$$\begin{aligned}
\langle \Delta E^2 \rangle &= \langle \boldsymbol{E}^2 \rangle - \langle \boldsymbol{E} \rangle^2 \\
&= \langle\alpha| i\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} [\hat{a} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \hat{a}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}] \cdot i\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} [\hat{a} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \hat{a}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}]|\alpha\rangle - \langle \boldsymbol{E} \rangle^2 \\
&= -\frac{\hbar\omega}{2\varepsilon_0 V} [\alpha^2 e^{i2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \alpha^{*2} e^{-i2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - |\alpha|^2 - (|\alpha|^2+1)] - \boldsymbol{\mathcal{E}}^2 e^{i2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \boldsymbol{\mathcal{E}}^{*2} e^{-i2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - 2|\boldsymbol{\mathcal{E}}|^2 \\
&= \boldsymbol{\mathcal{E}}^2 e^{i2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \boldsymbol{\mathcal{E}}^{*2} e^{-i2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + 2|\boldsymbol{\mathcal{E}}|^2 + \frac{\hbar\omega}{2\varepsilon_0 V} - \boldsymbol{\mathcal{E}}^2 e^{i2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \boldsymbol{\mathcal{E}}^{*2} e^{-2i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - 2|\boldsymbol{\mathcal{E}}|^2 \\
\langle \Delta E^2 \rangle &= \frac{\hbar\omega}{2\varepsilon_0 V}.
\end{aligned}$$

This is the uncertainty in E-field limited by quantum noise.

Also, we know that the eigenstates of $|\alpha\rangle$ do not form an orthonormal basis. However, we know that the number state $|n\rangle$ form an orthonormal basis and we can expand the coherent state in terms of the number states, as follows,

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,.$$

The probability of finding n photons in a coherent state is:

$$P_n = |\langle n|\alpha\rangle|^2 = \left| \langle n| e^{-|\alpha|^2/2} \sum_m \frac{\alpha^m}{\sqrt{m!}} |m\rangle \right|^2 = \left| e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \right|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}.$$

Recalling that $\langle \hat{n} \rangle = |\alpha|^2 = \overline{n}$:

$$\text{Probability of finding } n \text{ photons in a coherent state} \qquad \boxed{P_n = \frac{e^{-\overline{n}}\overline{n}^n}{n!}}\,.$$

1.4 Quadrature operators

Recall the E-field operator for a singel mode is:

$$\begin{aligned}
\hat{\boldsymbol{E}}(\boldsymbol{r}, t) &= i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} [\hat{a} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} - \hat{a}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}] \\
&= i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} [\hat{a} e^{-i\omega t} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{a}^\dagger e^{i\omega t} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}] \\
&= i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} [\hat{a}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{a}^\dagger(t) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}] \\
\hat{\boldsymbol{E}}(\boldsymbol{r}, t) &= i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} [\hat{a}(t)\cos(\boldsymbol{k}\cdot\boldsymbol{r}) + i\hat{a}(t)\sin(\boldsymbol{k}\cdot\boldsymbol{r}) - \hat{a}^\dagger(t)\cos(\boldsymbol{k}\cdot\boldsymbol{r}) + i\hat{a}^\dagger(t)\sin(\boldsymbol{k}\cdot\boldsymbol{r})].
\end{aligned}$$

Lets define the quadrature operators:

$$\text{Quadrature operators} \qquad \boxed{\begin{aligned} \hat{X}_1 &= \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \\ \hat{X}_2 &= \frac{1}{2i}(\hat{a} - \hat{a}^\dagger) \end{aligned}}\,, \tag{1.9}$$

where $[\hat{X}_1, \hat{X}_2] = \frac{i}{2}$. In the E-field, if we collect the cosine and sine, we get

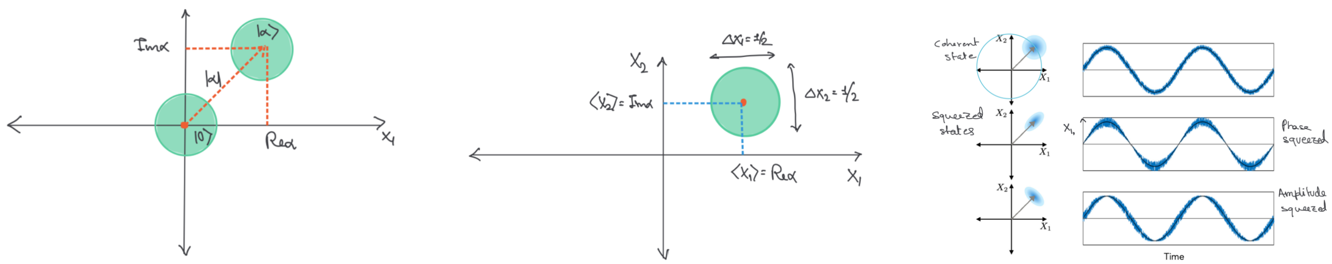
$$\hat{\boldsymbol{E}}(\boldsymbol{r}, t) = -\sqrt{\frac{2\hbar\omega}{\varepsilon_0 V}} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} [\hat{X}_1(t)\sin(\boldsymbol{k}\cdot\boldsymbol{r}) + \hat{X}_2(t)\cos(\boldsymbol{k}\cdot\boldsymbol{r})].$$

The expectation value of these operators is:

$$\begin{aligned}
&\langle \hat{X}_1 \rangle = \langle \alpha | \hat{X}_1(0) | \alpha \rangle = \langle \alpha | \frac{\hat{a} + \hat{a}^\dagger}{2} | \alpha \rangle = \mathrm{Re}\,(\alpha)\,, \quad \langle \hat{X}_2 \rangle = \langle \alpha | \hat{X}_2(0) | \alpha \rangle = \langle \alpha | \frac{\hat{a} - \hat{a}^\dagger}{2i} | \alpha \rangle = \mathrm{Im}\,(\alpha)\,, \\
&\langle \Delta \hat{X}_1^2 \rangle = \langle \alpha | \frac{1}{4}(\hat{a} + \hat{a}^\dagger)^2 | \alpha \rangle - \frac{1}{4}(\alpha + \alpha^*)^2 = \frac{1}{4}, \quad \langle \Delta \hat{X}_2^2 \rangle = \frac{1}{4}.
\end{aligned}$$

That is, there is a minimum uncertainty that a state can possible have in the two quadrature $\hat{X}_1, \hat{X}_2$, such a state os refferred as *minimum uncertainty state*. When $\alpha = 0$ we have the vaccum state and therefore $\langle \hat{X}_1(0) \rangle = \langle \hat{X}_2(0) \rangle = 0$, $\langle \hat{X}_1^2(0) \rangle = \langle \hat{X}_2^2(0) \rangle = 1/4$. Coherent states are also reffereees to as displaced vaccum states. Specifically,

$$|\alpha\rangle = \hat{D}(\alpha)\,|0\rangle\,, \quad \hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}.$$



(a) Displacement of vacuum **(b)** Coherent state **(c)** Evolution of coherent state

If we include the time dependence:

$$\langle\alpha|\hat{X}_1(t)|\alpha\rangle = \frac{\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}}{2} = \mathrm{Re}\left(\alpha e^{-i\omega t}\right), \quad \langle\alpha|\hat{X}_2(t)|\alpha\rangle = \frac{\alpha e^{-i\omega t} - \alpha^* e^{i\omega t}}{2i} = \mathrm{Im}\left(\alpha e^{-i\omega t}\right).$$

The phase space representation of states allows one to acces information aabout the noise/variance associated with any state when a specific E-field quadrature is measured.

1.4.1 Time dependence of quadrature operators

Using (1.9) and the inclusion of time-dependence yields

Time-dependent Quadrature operators

$$\boxed{\begin{aligned} \hat{X}_1(t) &= \frac{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2} \\ \hat{X}_2(t) &= \frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i} \end{aligned}}\,. \tag{1.10}$$

We can use them to express them in terms of $\hat{X}_1(0)$ and $\hat{X}_2(0)$.

$$\hat{a} + \hat{a}^\dagger = 2\hat{X}_1(0), \tag{1.11}$$
$$\hat{a} - \hat{a}^\dagger = 2i\hat{X}_2(0). \tag{1.12}$$

$$\begin{aligned} (1.11) + (1.12):&\quad 2\hat{a} = 2[\hat{X}_1(0) + i\hat{X}_2(0)] \longrightarrow \hat{a} = \hat{X}_1(0) + i\hat{X}_2(0) \\ (1.11) - (1.12):&\quad 2\hat{a}^\dagger = 2[\hat{X}_1(0) - i\hat{X}_2(0)] \longrightarrow \hat{a}^\dagger = \hat{X}_1(0) - i\hat{X}_2(0) \end{aligned}$$

Using these in the time-evolved quadrature operators:

$$\begin{aligned} \hat{X}_1(t) &= \frac{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2} = \frac{[\hat{X}_1(0) + i\hat{X}_2(0)]e^{-i\omega t} + [\hat{X}_1(0) - i\hat{X}_2(0)]e^{i\omega t}}{2} \\ \hat{X}_1(t) &= \frac{e^{i\omega t} + e^{-i\omega t}}{2}\hat{X}_1(0) + \frac{(e^{i\omega t} - e^{-i\omega t})}{2i}\hat{X}_2(0). \end{aligned}$$

Likewise,

$$\hat{X}_2(t) = \frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i} = \frac{[\hat{X}_1(0) + i\hat{X}_2(0)]e^{-i\omega t} - [\hat{X}_1(0) - i\hat{X}_2(0)]e^{i\omega t}}{2i}$$
$$\hat{X}_2(t) = -\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\hat{X}_1(0) + i\frac{(e^{i\omega t} + e^{-i\omega t})}{2i}\hat{X}_2(0).$$

Then, we have

$$\begin{aligned} \hat{X}_1(t) &= \cos(\omega t)\hat{X}_1(0) + \sin(\omega t)\hat{X}_2(0), \\ \hat{X}_2(t) &= -\sin(\omega t)\hat{X}_1(0) + \cos(\omega t)\hat{X}_2(0). \end{aligned} \tag{1.13}$$

1.4.2 Taking the continuum limit

In free space the sum over all $\boldsymbol{k}$ becomes an integral:

$$\sum_{\boldsymbol{k}} \to N \int d^2\boldsymbol{k} = N \int_0^\infty dk k^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi,$$

where N is the normalization equal to volume/mode. We consider a volume of quantization that is the region in free space where we are quantizing the field $V = L_x L_y L_z$. Allowed $\boldsymbol{k}$ vectors are $k_x = (2\pi/L_x)m_x$, $k_y = (2\pi/L_y)m_y$ y $k_z = (2\pi/L_z)m_z$, that is, we ensure that the lenths fit an integer number of wavelengths.

The number of modes in between $\boldsymbol{k}$ and $\boldsymbol{k} + \Delta\boldsymbol{k}$ is:

$$\Delta m_x \Delta m_y \Delta m_z = \frac{L_x L_y L_z}{(2\pi)^2}\Delta\boldsymbol{k}_x\Delta\boldsymbol{k}_y\Delta\boldsymbol{k}_z = \frac{V}{(2\pi)^3}\Delta\boldsymbol{k}_x\Delta\boldsymbol{k}_y\Delta\boldsymbol{k}_z.$$

Using the above normalization:

$$\sum_{\boldsymbol{k},\lambda} \to \sum_\lambda \int dk k^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \frac{V}{8\pi^2}.$$

Thus, any continuum limit involves

$$\text{Continuum limit} \qquad \sum_{\boldsymbol{k}} \to \frac{V}{8\pi^2}\int d^3\boldsymbol{k}.$$

1.5 Squeezed states

Lets define the squeeze operator

$$\text{Squeezing state} \qquad |\xi, 0\rangle = \hat{S}(\xi)\,|0\rangle\,, \quad \text{with} \quad \hat{S}(\xi) = e^{\frac{1}{2}(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2})}, \quad \xi \in \mathbb{C}.$$

Lets use it to compute the expectation value for the quadrature operators.

$$\langle\xi, 0|\hat{X}_1|\xi, 0\rangle = \langle 0|\hat{S}^\dagger(\xi)\left(\frac{\hat{a} + \hat{a}^\dagger}{2}\right)\hat{S}(\xi)|0\rangle = \frac{1}{2}\langle 0|(\hat{S}^\dagger\hat{a}\hat{S} + \hat{S}^\dagger\hat{a}^\dagger\hat{S})|0\rangle.$$

We define displaced ladder operators as

$$\tilde{a} = \hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi), \quad \xi = re^{i\phi}.$$

Using the BCH formula

$$e^L A e^{-L} = A + [L, A] + \frac{1}{2!}[L, [L, A]] + \cdots$$

and $A = \hat{a}$ with $L = -\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2})$ allow us to express the squeezing ladder operator

Squeezing ladder operators
$$\begin{aligned} \tilde{a} &= \hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi) = \hat{a}\cosh(r) - \hat{a}^\dagger e^{i\phi}\sinh(r) \\ \tilde{a}^\dagger &= S(\xi)^\dagger \hat{a}^\dagger \hat{S}(\xi) = \hat{a}^\dagger\cosh(r) - \hat{a}e^{i\phi}\sinh(r). \end{aligned} \tag{1.14}$$

We have used several orders. Its derivative wrt r is:

$$\frac{d\hat{a}}{dr} = \hat{a}\sinh(r) - \hat{a}^\dagger e^{i\phi}\cosh(r) = -e^{i\phi}\tilde{a}^\dagger, \quad \frac{d\tilde{a}^\dagger}{dr} = -e^{-i\phi}\tilde{a}.$$

Taking another derivative yields:

$$\frac{d^2\tilde{a}}{dr^2} = \tilde{a}.$$

Now we can continue the expectation of the quadrature:

$$\begin{aligned} \langle \hat{X}_1 \rangle &= \frac{1}{2}\langle 0|(\hat{S}^\dagger\hat{a}\hat{S} + \hat{S}^\dagger\hat{a}^\dagger\hat{S})|0\rangle = \frac{1}{2}\langle 0| \left[\tilde{a} + \tilde{a}^\dagger\right] |0\rangle \\ &= \frac{1}{2}\langle 0|[\hat{a}\cosh(r) - e^{i\phi}\hat{a}^\dagger\sinh(r) + \hat{a}^\dagger\cosh(r) - e^{-i\phi}\sinh(r)]|0\rangle = 0. \\ \langle \hat{X}_2 \rangle &= \text{similarly} = 0. \end{aligned}$$

We can do the same for the operator square:

$$\begin{aligned} \langle \hat{X}_1^2 \rangle &= \frac{1}{4}\langle 0|\hat{S}^\dagger(\xi)[(\hat{a}+\hat{a}^\dagger)(\hat{a}+\hat{a}^\dagger)]\hat{S}(\xi)|0\rangle = \frac{1}{4}\langle 0|\hat{S}^\dagger(\hat{a}+\hat{a}^\dagger)\hat{S}\hat{S}^\dagger(\hat{a}+\hat{a}^\dagger)\hat{S}|0\rangle \\ &= \frac{1}{4}\langle 0|(\tilde{a}+\tilde{a}^\dagger)(\tilde{a}+\tilde{a}^\dagger)|0\rangle = \frac{1}{4}\langle 0|e^{-2r}(\hat{a}+\hat{a}^\dagger)^2|0\rangle = \frac{e^{-2r}}{4}. \end{aligned}$$

Similarly we can obtain the respective quantities for $\hat{X}_2$ and the uncertainties are:

$$\langle \hat{X}_1 \rangle = \langle \hat{X}_2 \rangle = 0 \Longrightarrow \Delta\hat{X}_1 = \frac{e^{-r}}{2}, \quad \Delta\hat{X}_2 = \frac{e^r}{2}.$$

1.5.1 Displaced squeezing state

Now,

Displaced squeezing state $\quad |\xi, \alpha\rangle = \hat{D}(\alpha)\hat{S}(\xi)\,|0\rangle\,.$

Now, the operators are:

Displaced squeezing operator
$$\begin{aligned} \tilde{a} &= \hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha)\hat{S}(\xi) = \cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha, \\ \tilde{a}^\dagger &= \hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha)\hat{S}(\xi) = \cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*. \end{aligned} \tag{1.15}$$

The expected number of photons in a displaced squeezed state is:

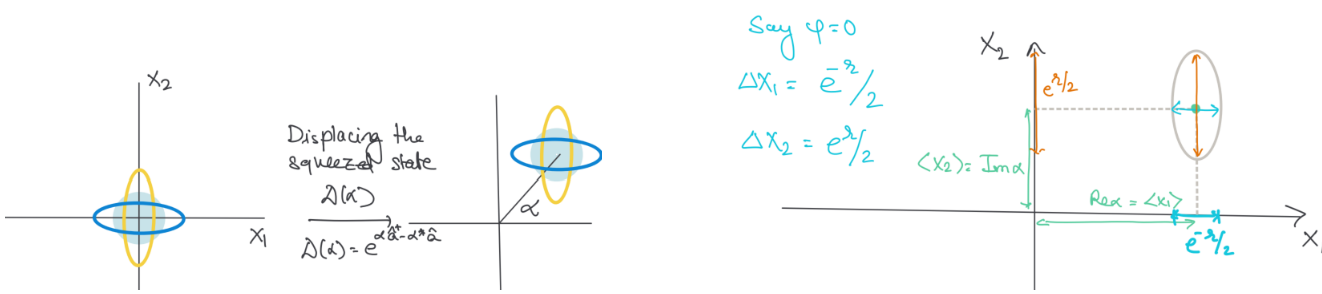
$$\langle 0|\tilde{a}^\dagger \tilde{a}|0\rangle = \langle 0|[\cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*][\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha]|0\rangle = \sinh^2|\xi| + |\alpha|^2.$$

The expected value in the quadrature value is:

$$\begin{aligned}
\langle \xi,\alpha|\hat{X}_1|\xi,\alpha\rangle &= \langle 0|\hat{S}^\dagger\hat{D}^\dagger\hat{X}_1\hat{D}\hat{S}|0\rangle = \langle 0|\hat{S}^\dagger\hat{D}^\dagger(\frac{\hat{a}+\hat{a}^\dagger}{2})\hat{D}\hat{S}|0\rangle = \frac{1}{2}\langle 0|(\tilde{a}+\tilde{a}^\dagger)|0\rangle \\
&= \frac{1}{2}\langle 0|[\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha + \cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*]|0\rangle = \mathrm{Re}\,(\alpha)\,. \\
\langle \xi,\alpha|\hat{X}_2|\xi,\alpha\rangle &= \mathrm{Im}\,(\alpha)\,.
\end{aligned}$$

On the other hand, the quadrature uncertainties are:

$$\begin{aligned}
\Delta\hat{X}_1^2 &= \langle \xi,\alpha|\hat{X}_1^2|\xi,\alpha\rangle - \langle \xi,\alpha|\hat{X}_1|\xi,\alpha\rangle^2 = \langle 0|\hat{S}^\dagger\hat{D}^\dagger\hat{X}_1\hat{D}\hat{S}\hat{S}^\dagger\hat{D}^\dagger\hat{X}_1\hat{D}\hat{S}|0\rangle - \mathrm{Re}\,(\alpha)^2 \\
&= \langle 0|\left(\frac{\tilde{a}+\tilde{a}^\dagger}{2}\right)\left(\frac{\tilde{a}+\tilde{a}^\dagger}{2}\right)|0\rangle - \mathrm{Re}\,(\alpha)^2 \\
&= \frac{1}{4}\langle 0|[\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha + \cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*]^2|0\rangle - \mathrm{Re}\,(\alpha)^2 \\
&= \frac{1}{4}(\cosh^2(r) + \sinh^2(r) - 2\cosh(r)\sinh(r)\cos\phi) + \mathrm{Re}\,(\alpha)^2 - \mathrm{Re}\,(\alpha)^2 \\
\Delta\hat{X}_1^2 &= \frac{1}{4}[\cosh^2(r) + \sinh^2(r) - 2\cosh(r)\sinh(r)\cos(\phi)]. \\
\Delta\hat{X}_2^2 &= \frac{1}{4}[\cosh^2(r) + \sinh^2(r) + 2\cosh(r)\sinh(r)\cos(\phi)].
\end{aligned}$$



(a) (b)

Figure 1.1

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Chapter 2

Optical devices

2.1 Beam splitter

Let us consider the action of a beam splitter (BS) on two input modes of same frequency and polarization. Clasically, the output E-field can be defined in terms of the input E-fields:

$$\begin{bmatrix} E_1' \\ E_2 \end{bmatrix} = \begin{bmatrix} t' & r \\ r' & t \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}.$$

$\{r, t\}$ and $\{r', t'\}$ are reflection/transmission coefficient associated with the two interfaces of the BS. From energy conservation, we must have

$$\text{Energy conservation in the BS} \qquad \boxed{|E_1|^2 + |E_2|^2 = |E_1'|^2 + |E_2'|^2}.$$

If we assume $E_2 = 0$, then

$$|E_1|^2 = |t'E_1|^2 + |r'E_1|^2 \longrightarrow |r'|^2 + |t'|^2 = 1. \tag{2.1}$$

Similarly, setting $E_1 = 0$ yields

$$|E_2|^2 = |rE_2|^2 + |tE_2|^2 \longrightarrow |r|^2 + |t|^2 = 1. \tag{2.2}$$

In general,

$$\begin{aligned} |E_1|^2 + |E_2|^2 &= |E_1'|^2 + |E_2'|^2 \\ &= (t'E_1 + rE_2)(t^{'*}E_1^* + r^*E_2^*) + (r'E_1 + tE_2)(r^{'*}E_1^* + t^*E_2^*) \\ &= |t'|^2|E_1|^2 + |r|^2|E_2|^2 + |r'|^2|E_1|^2 + |t|^2|E_2|^2 + rt^{'*}E_1^*E_2 + t'r^*E_1E_2^* + r't^*E_1E_2^* + r^{'*}tE_1^*E_2 \\ |E_1|^2 + |E_2|^2 &= |E_1|^2(|r'|^2 + |t'|^2) + |E_2|^2(|r|^2 + |t|^2) + E_1^*E_2(rt^{'*} + r^{'*}t) + E_1E_2^*(r't^* + r^*t') \\ 0 &= E_1^*E_2(rt^{'*} + r^{'*}t) + c.c. \end{aligned}$$

Thus,

$$rt^{'*} + r^{'*}t = 0. \tag{2.3}$$

For a $50 : 50$ BS which as $r' = 1/\sqrt{2}$, $t' = 1/\sqrt{2}$, and $r = 1/\sqrt{2}$, (2.1), (2.2) and (2.3) implies that $t = -1/\sqrt{2}$. Similarly, the quantum description of the BS is

$$\begin{bmatrix} \hat{a}_1' \\ \hat{a}_2' \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix}. \tag{2.4}$$

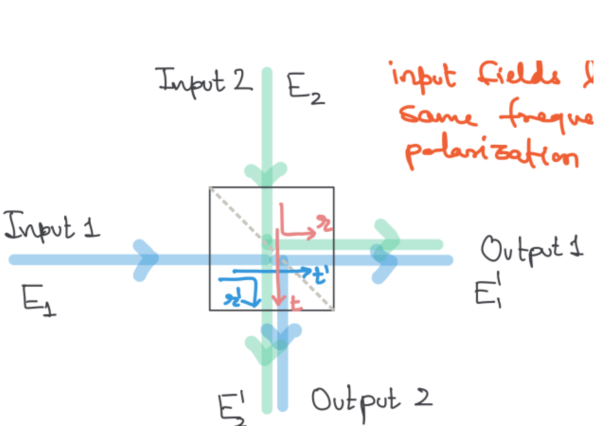
Now, the ladder operators are equal to the E-fields. This equation implies that

$$\hat{a}_1' = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2), \quad \text{and} \quad \hat{a}_2' = \frac{1}{\sqrt{2}}(\hat{a}_1 - \hat{a}_2).$$

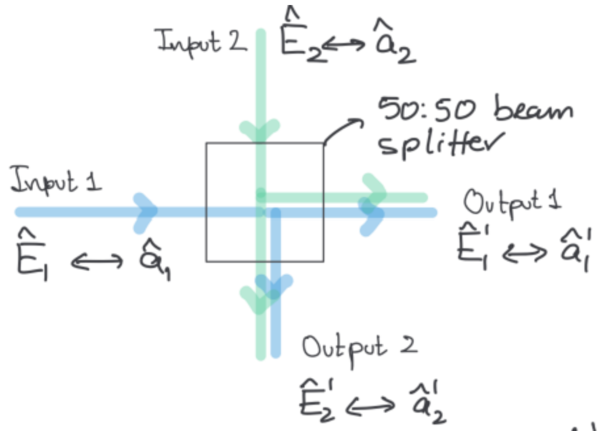
Ejemplo 2.1 **Single photon incident in one input**

The input is:

$$\text{Input} = |1\rangle_1 |0\rangle_2 = (\hat{a}_1^\dagger |0\rangle_1) |0\rangle_2 = |1\rangle_1 |0\rangle_2 .$$



(a) Classical description



(b) Quantum description

Figure 2.1 Description of a beamsplitter (BS).

Note that

$$\hat{a}_1 = \frac{1}{\sqrt{2}}(\hat{a}_1' + \hat{a}_2') \quad \text{and} \quad \hat{a}_1^\dagger = \frac{1}{\sqrt{2}}(\hat{a}_1^{'\dagger} + \hat{a}_2^{'\dagger}).$$

The output is then

$$\text{Output} = \frac{1}{\sqrt{2}}(\hat{a}_1^{'\dagger} + \hat{a}_2^{'\dagger}) \left|0\right\rangle_{1'} \left|0\right\rangle_{2'} = \frac{1}{\sqrt{2}}[\left|1\right\rangle_{1'} \left|0\right\rangle_{2'} + \left|0\right\rangle_{1'} \left|1\right\rangle_{2'}].$$

Thus, there is a $50:50$ probability of detecting the photon in one of the ouput ports.

Ejemplo 2.2 **Both input with photons**

The input is:

$$\text{Input} = \left|1\right\rangle_1 \left|1\right\rangle_2 = \hat{a}_1^\dagger \left|0\right\rangle_1 \hat{a}_2^\dagger \left|0\right\rangle_2 .$$

The output is:

$$\begin{aligned}
\text{Output} &= \frac{1}{2}[\hat{a}_1^{'\dagger} + \hat{a}_2^{'\dagger}][\hat{a}_1^{'\dagger} - \hat{a}_2^{'\dagger}] \left|0\right\rangle_{1'} \left|0\right\rangle_{2'} \\
&= \frac{1}{2}[\hat{a}_1^{'\dagger}\hat{a}_1^{'\dagger} - \hat{a}_2^{'\dagger}\hat{a}_2^{'\dagger} + \cancel{\hat{a}_1^{'\dagger}\hat{a}_2^{'\dagger}} - \cancel{\hat{a}_2^{'\dagger}\hat{a}_1^{'\dagger}}] \left|0\right\rangle_{1'} \left|0\right\rangle_{2'} \\
&= \frac{1}{2}[\sqrt{2}\left|2\right\rangle_{1'} \left|0\right\rangle_{2'} - \sqrt{2}\left|0\right\rangle_{1'} \left|2\right\rangle_{2'}] \\
\text{Output} &= \frac{1}{\sqrt{2}}[\left|2\right\rangle_{1'} \left|0\right\rangle_{2'} - \left|0\right\rangle_{1'} \left|2\right\rangle_{2'}].
\end{aligned}$$

Thus, when two photons are simultaneously incident on each input, both either go into output 1 or output 2 but never in both a the same time.

Chapter 3

Nonclassical light

3.1 Detection of quadrature states

Modes at the beam splitter output are:

$$\hat{A}_1 = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2), \quad \hat{A}_2 = \frac{1}{\sqrt{2}}(\hat{a}_1 - \hat{a}_2).$$

Detectors D_1, D_2 measure the intensity at each output:

$$\langle \hat{I}_k \rangle = \langle \hat{A}_k^\dagger \hat{A}_k \rangle.$$

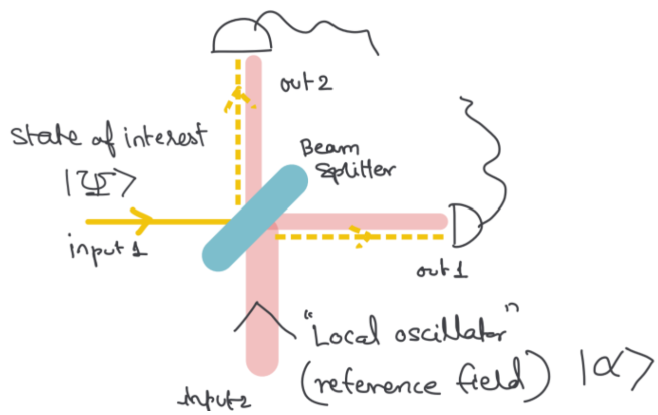


Figure 3.1 Balanced homodyne detection.

In output 1, we have

$$I_1 = \langle \psi_1 \alpha_2 | \hat{A}_1^\dagger \hat{A}_1 | \alpha_2 \Psi_1 \rangle = \langle \psi_1 \alpha_2 | \frac{\hat{a}_1^\dagger + \hat{a}_2^\dagger}{\sqrt{2}} \frac{\hat{a}_1 + \hat{a}_2}{\sqrt{2}} | \alpha_2 \Psi_1 \rangle = \frac{1}{2} \langle \Psi_1 \alpha_2 | (\hat{a}_1^\dagger + \alpha^*)(\hat{a}_1 + \alpha) | \alpha_2 \Psi_1 \rangle$$
$$= \frac{1}{2} \langle \Psi_1 | (\hat{a}^\dagger \hat{a}_1 + \hat{a}_1 \alpha^* + \hat{a}_1^\dagger \alpha + |\alpha|^2) | \Psi_1 \rangle$$

Lets consider that input 2 contains a strong coherent state field that we refer to as the *local oscillator*. Thus the total state over which the average is being taken is

$$\underbrace{|\Psi_1\rangle}_{\text{Input 1}} \quad \underbrace{|\alpha_2\rangle}_{\text{Coherent state input 2}} \quad . \tag{3.1}$$

The averaging if about these two states.

We see that the irradiance is just the number operator counting photons at each output. For instance,

$$I_1 - I_2 = \langle \Psi_1 | \hat{a}_1 \alpha_2^* + \hat{a}_1^\dagger \alpha_2 | \Psi_1 \rangle = \begin{cases} 2\alpha_2 \langle \Psi_1 | \frac{\hat{a}_1 + \hat{a}_1^\dagger}{2} | \Psi_1 \rangle = 2\alpha_2 \langle \Psi_1 | \hat{X}_1 | \Psi_1 \rangle, & \alpha_2 = \alpha_2^*, \\ -2i\alpha_2 \langle \Psi_1 | \frac{\hat{a}_1 - \hat{a}_1^\dagger}{2i} | \Psi_1 \rangle = -2i\alpha_2 \langle \Psi_1 | \hat{X}_2 | \Psi_1 \rangle, & \alpha_2 = -\alpha_2^*. \end{cases}$$

In general by changing the phase of input 2 we can measure a general quadrature. Noise in intensity corresponds to noise in specific quadratures. If a quadrature measurement has uncertainty $\Delta \hat{X}^2 < 1/4$.

3.2 Generation of squeezed states

The following is a scheme of a nonlinear medium. Radiation from oscillating dipoles is of different frequency as the input stimulation. Nonlinear polarization induced in a medium by an E-field $\boldsymbol{E}$ is:

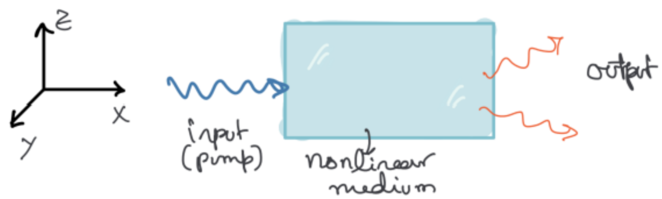


Figure 3.1 Nonlinear medium

$$\boldsymbol{P}_i = \varepsilon_0 \left[\chi^{(1)}_{jk} \boldsymbol{E}_k + \varepsilon_0 \chi^{(2)}_{jkl} \boldsymbol{E}_k \boldsymbol{E}_l + \cdots \right].$$

This polarization is for one dipole, the total would be a sum over all of them. Thses quantities are tensor. In particular, we have a linear and second-order polarization terms:

$$\begin{aligned} \boldsymbol{P}_{Lj} &= \text{Linear polarization} = \varepsilon_0 \chi^{(1)}_{jk} \boldsymbol{E}_k \\ \boldsymbol{P}_{NLj} &= \text{Second-order polarization} = \varepsilon_0 \chi^{(2)}_{jkl} \boldsymbol{E}_k \boldsymbol{E}_l. \end{aligned}$$

The interaction (dipole-field) energy density associated with the field becomes:

$$U = -\boldsymbol{P} \cdot \boldsymbol{E} = -\varepsilon_0 [\chi^{(1)}_{jk} \boldsymbol{E}_k + \chi^{(2)}_{jkl} \boldsymbol{E}_j \boldsymbol{E}_k] \boldsymbol{E}_l = \underbrace{-\varepsilon_0 \chi^{(1)}_{jk} \boldsymbol{E}_k \boldsymbol{E}_l}_{U_L} + \underbrace{-\varepsilon_0 \chi^{(2)}_{jkl} \boldsymbol{E}_j \boldsymbol{E}_k \boldsymbol{E}_l}_{U^{(2)}_{NL} \text{ (Squeezed output)}}.$$

Nonlinear-Hamiltonian is therefore

$$H^{(2)}_{NL} = -\int dV U^{(2)}_{NL}, \tag{3.2}$$

where the integration region is the medium. We will analyze the above Hamoltonian for the specific interaction where we have an incident blue photon spontaneously down-converted to two red photons.

The total E-field is:

$$\hat{\boldsymbol{E}}_{tot} = \boldsymbol{E}_B + \hat{\boldsymbol{E}}, \qquad \begin{aligned} \boldsymbol{E}_B &= \text{Classical blue field} \\ \hat{\boldsymbol{E}} &= \text{Quantized E-field} \end{aligned} \quad .$$

Hamiltonian becomes

$$\hat{H}^{(2)}_{NL} = -\varepsilon_0 \int dV \hat{H}^{(2)}_{jkl} (\boldsymbol{E}_B + \hat{\boldsymbol{E}})_j (\boldsymbol{E}_B + \hat{\boldsymbol{E}})_k (\boldsymbol{E}_B + \hat{\boldsymbol{E}})_l$$

For further simplification let us assume that only $H^{(2)}_{zzz} \neq 0 = H^{(2)}$:

$$H^{(2)}_{NL} = -3\varepsilon_0 \chi^{(2)} \int dV E_{Bz} \hat{\boldsymbol{E}}_z \hat{\boldsymbol{E}}_z,$$

where

$$\boldsymbol{E}_{Bz} = \hat{\boldsymbol{z}}[\mathcal{E}_B e^{i(\boldsymbol{k}_B\cdot\boldsymbol{r}-\omega_B t)} + \mathcal{E}_B^* e^{-i(\boldsymbol{k}_B\cdot\boldsymbol{r}-\omega_B t)}].$$

and

$$\hat{\boldsymbol{E}}_z = i\sum_{\boldsymbol{k}} \hat{\boldsymbol{z}}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}[\hat{a}_{\boldsymbol{k}} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \hat{a}_{\boldsymbol{k}}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}]$$

Substituting the E-fields in the nonlinear Hamiltonian yields

$$\hat{H}_{NL}^{(2)} = -\frac{3}{2V}\varepsilon_0\chi^{(2)}\hbar\int dV[\mathcal{E}_B e^{i(\boldsymbol{k}_B\cdot\boldsymbol{r}-\omega_B t)} + \mathcal{E}_B^* e^{-i(\boldsymbol{k}_B\cdot\boldsymbol{r}-\omega_B t)}]$$
$$\sum_{\boldsymbol{k}_1}\sqrt{\omega_1}[\hat{a}_{\boldsymbol{k}} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \hat{a}_{\boldsymbol{k}}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}]\sum_{\boldsymbol{k}_2}\sqrt{\omega_2}[\hat{a}_{\boldsymbol{k}} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \hat{a}_{\boldsymbol{k}}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}].$$

Consider the time-dependent terms:

$$e^{i(\omega_B\pm\omega_1\pm\omega_2)t} \to 0$$

which averages to zero because they are all fast oscillatting. To get a non-zero contribution we choose $\omega_B = \omega_1 + \omega_2$.

Similarly, by energy conservation we have

$$\int dV e^{i(\boldsymbol{k}_B-\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}}$$

For this integral to not vanish we must have $\boldsymbol{k}_B = \boldsymbol{k}_1 + \boldsymbol{k}_2$ which iplies **momentum conservation**.

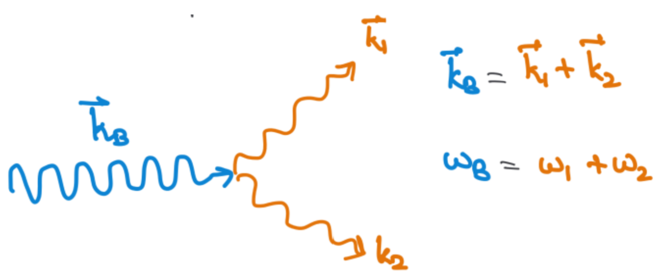


Figure 3.2 Momentum conservation.

This is the **spontaneous parametric downconservsion process** (SPDC).

Thus, keeping only the terms that conserve energy and momentum:

$$\hat{H}_{NL}^{(2)} = -\frac{3}{2}\hbar\chi^{(2)}\sqrt{\omega_1\omega_2}\left[\mathcal{E}_B\hat{a}_{\boldsymbol{k}_1}^\dagger\hat{a}_{\boldsymbol{k}_2}^\dagger + \mathcal{E}_B^*\hat{a}_{\boldsymbol{k}_1}\hat{a}_{\boldsymbol{k}_2}\right].$$

If the two modes are the same:

$$\hat{H}_{NL}^{(2)} = -\frac{1}{2}[\xi^*\hat{a}^2 + \xi\hat{a}^{\dagger 2}], \quad \xi = 3\hbar\chi^{(2)}\omega\mathcal{E}_B.$$

and

$$\omega_B = 2\omega, \quad \boldsymbol{k}_B = 2\boldsymbol{k}.$$

The corresponding evolution operator becomes

$$U(t,0) = e^{-i\hat{H}_{NL}^{(2)}t/\hbar} = e^{\frac{1}{2}[\xi^*\hat{a}^2 - \xi^*\hat{a}^{\dagger 2}]}, \quad \xi = 3i\chi^{(2)}\mathcal{E}_B\omega t.$$

Thus starting with an initial vacuum state in the red modes the system one obtains a squeezed vacuum output.

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