Homework 2

Foundations of Quantum Optics

OPTI 544, Spring 2026

1. **Time dynamics of coherent and squeezed states in phase space**:

   Consider the time-evolved quadrature operators:

$$\hat{X}_1(t) \equiv \frac{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2} \tag{1}$$

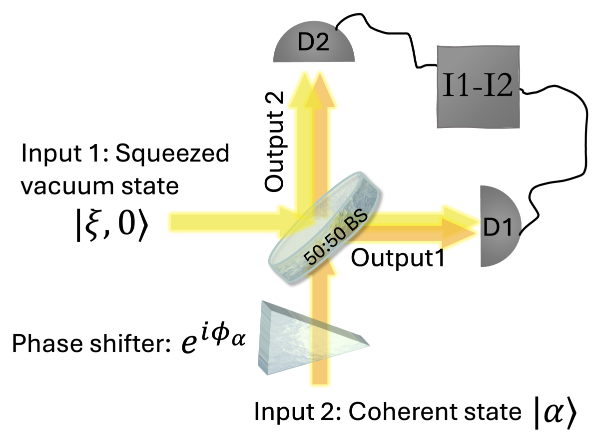
$$\hat{X}_2(t) \equiv \frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i} \tag{2}$$

   (a) Express $\hat{X}_1(t)$ and $\hat{X}_2(t)$ in terms of the quadrature operators $\hat{X}_1(0) = \frac{\hat{a}+\hat{a}^\dagger}{2}$ and $\hat{X}_2(0) = \frac{\hat{a}-\hat{a}^\dagger}{2i}$ at time $t = 0$. Show that the two quadratures follow the canonical commutation relation $\left[\hat{X}_1(t), \hat{X}_2(t)\right] = \frac{i}{2}$ at all times. Note that we are working in the Heisenberg picture such that the operators evolve in time, not the state.

   (b) Find the time-dependent expectation values and uncertainties of the quadrature operators $\hat{X}_{1,2}(t)$ for a coherent state. Depict the evolution of the state (expectation values and uncertainty) on the phase space ($\hat{X}_1$-$\hat{X}_2$ diagram). Plot the time variation of the quadrature $\hat{X}_2(t)$ as a function of time on a separate plot, indicating the uncertainty region with error bars.

   (c) Determine the time-dependent expectation values and uncertainties of quadratures for the displaced squeezed state $|\xi, \alpha\rangle \equiv \hat{D}(\alpha)\hat{S}(\xi)\,|0\rangle$ for $\alpha = |\alpha|$ and $\xi = r$ (α and r being purely real quantities). Depict the evolution of the state (expectation values and uncertainty) on the phase space ($\hat{X}_1$-$\hat{X}_2$ diagram). Plot the time variation of the quadrature $\hat{X}_2(t)$ as a function of time on a separate plot, indicating the uncertainty region with error bars. Discuss the two cases where r is positive and negative.

2. **Balanced homodyne detection of squeezed light:** Consider a squeezed vacuum field mode incident on a balanced homodyne detector setup, as shown in the figure below.



The output modes in the two detector arms are given as a linear combination of the input modes $\hat{a}_{1,2}$:

$$\hat{b}_1 = \frac{1}{\sqrt{2}}\left(\hat{a}_1 + \hat{a}_2\right) \tag{3}$$

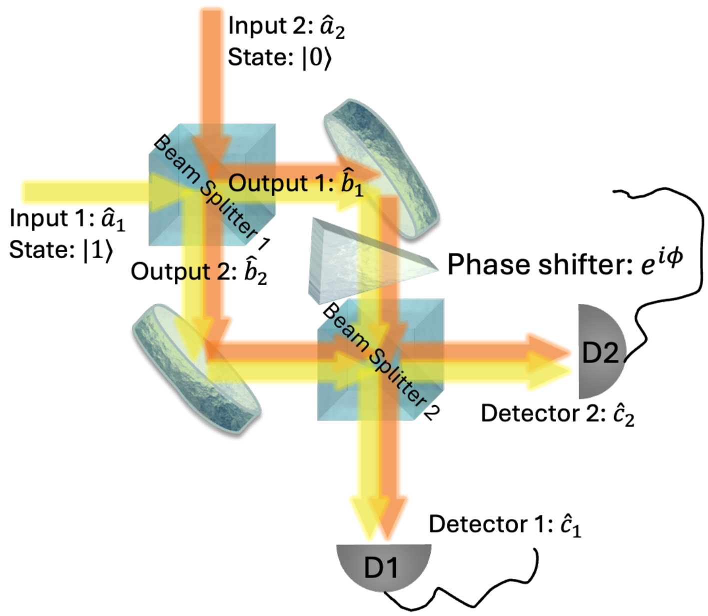
$$\hat{b}_2 = \frac{1}{\sqrt{2}}\left(\hat{a}_1 - \hat{a}_2\right) \tag{4}$$

What is the difference in the photon numbers, $\hat{N}_1 - \hat{N}_2$, where $\hat{N}_{1,2} = \hat{b}_{1,2}^\dagger \hat{b}_{1,2}$, and the variance $\Delta(N_1 - N_2)$ as a function of the variable phase ϕ_α? The phase shifter changes the coherent state input amplitude as follows:

$$\alpha \to \alpha e^{i\phi_\alpha}. \tag{5}$$

You can sssume $|\alpha| \gg 1$ and keep terms to linear order in $|\alpha|$.

3. **Mach-Zehnder interferometer with single photons:** Let us consider the following setup for a Mach-Zehnder interferometer.



(a) Express the output modes $\left\{\hat{b}_1, \hat{b}_2\right\}$ in terms of the input modes $\{\hat{a}_1, \hat{a}_2\}$, assuming the beam splitter 1 to be a 50:50 beam splitter.

(b) What is the expected number of photons in the modes $\hat{b}_1$ and $\hat{b}_2$, i.e., $\left\langle \hat{b}_1^\dagger \hat{b}_1 \right\rangle$ and $\left\langle \hat{b}_2^\dagger \hat{b}_2 \right\rangle$? Note that the average is taken over the initial state of mode 1 having a single photon $|1\rangle$ and mode 2 being in vacuum $|0\rangle$.

(c) The two modes $\hat{b}_1$ and $\hat{b}_2$ travel through two different arms of the interferometer before they are mixed on the second beam splitter. The mode $\hat{b}_1$ can accrue a variable phase shift of $e^{i\phi}$ along its path such that $\hat{b}_1 \to \hat{b}_1 e^{i\phi}$. Determine the output mode operators at the detector positions $\hat{c}_1 = \frac{1}{\sqrt{2}}\left(\hat{b}_1 e^{i\phi} + \hat{b}_2\right)$ and $\hat{c}_2 = \frac{1}{\sqrt{2}}\left(\hat{b}_1 e^{i\phi} - \hat{b}_2\right)$, in terms of the input mode operators $\{\hat{a}_1, \hat{a}_2\}$.

(d) What is the number of photons detected at detector 1 as a function of ϕ?