

# Assignment 1

## OPTI 544 Quantum Optics

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## Exercise 1

The Hamiltonian in position space of the quantum harmonic oscillator is:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \quad [\hat{x}, \hat{p}] = i\hbar.$$

Defining the *ladder operators*

$$\begin{aligned}\hat{a} &= \frac{1}{\sqrt{2}} \left( \frac{\hat{x}}{\sigma} + i\frac{\sigma\hat{p}}{\hbar} \right) & \hat{x} &= \frac{\sigma}{\sqrt{2}}(\hat{a}^\dagger + \hat{a}) \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2}} \left( \frac{\hat{x}}{\sigma} - i\frac{\sigma\hat{p}}{\hbar} \right) & \hat{p} &= \frac{i\hbar}{\sigma\sqrt{2}}(\hat{a}^\dagger - \hat{a})\end{aligned}, \quad \text{with}$$

with  $\sigma$  the oscillation length

$$\sigma = \sqrt{\frac{\hbar}{m\omega}},$$

allow us to express the Hamiltonian in a simpler form. First, we notice that the commutation of  $\hat{x}$  and  $\hat{p}$  is inherited for  $\hat{a}$  and  $\hat{a}^\dagger$ :

$$[\hat{a}, \hat{a}^\dagger] = \frac{1}{2} \left[ \frac{\hat{x}}{\sigma} + i\frac{\sigma\hat{p}}{\hbar}, \frac{\hat{x}}{\sigma} - i\frac{\sigma\hat{p}}{\hbar} \right] = \frac{1}{2} \left( -\frac{2i}{\hbar} [\hat{x}, \hat{p}] \right) = -\frac{i}{\hbar} [\hat{x}, \hat{p}] = -\frac{i}{\hbar}(i\hbar) = 1.$$

If we replace the  $\hat{x}, \hat{p}$  in terms of  $\hat{a}, \hat{a}^\dagger$  in the Hamiltonian, we have:

$$\begin{aligned}\hat{H} &= \frac{1}{2m} \left[ -\frac{\hbar^2}{2\sigma^2} (\hat{a}^\dagger - \hat{a})^2 \right] + \frac{1}{2}m\omega^2 \left[ \frac{\sigma^2}{2} (\hat{a}^\dagger + \hat{a})^2 \right] \\ &= \frac{\hbar\omega}{2} \left[ \hat{a}^{\dagger 2} + \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger + \hat{a}^2 - \hat{a}^{\dagger 2} + \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger - \hat{a}^2 \right] \\ &= \frac{\hbar\omega}{2} \left[ 2(\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) \right] \\ &= \hbar\omega(\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger + 1) \\ &= \hbar\omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \\ \hat{H} &= \hbar\omega \left( \hat{N} + \frac{1}{2} \right),\end{aligned}$$

where  $\hat{N} = \hat{a}^\dagger \hat{a}$  is the *number operator*. If we know the eigenequation of  $\hat{N}$  we automatically know the eigenequation of  $\hat{H}$ . Assuming the following eigenequation for  $\hat{N}$ , then:

$$\hat{N} |n\rangle = n |n\rangle \implies \hat{H} |n\rangle = \left(n + \frac{1}{2}\right) |n\rangle.$$

This means that the energy eigenvalues are

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega.$$

The nature of  $n$  needs to be proved. First, we compute the commutator of  $\hat{N}$  with  $\hat{a}, \hat{a}^\dagger$ :

$$\begin{aligned} [\hat{N}, \hat{a}] &= [\hat{a}^\dagger \hat{a}, \hat{a}] = \hat{a}^\dagger [\hat{a}, \hat{a}] + [\hat{a}^\dagger, \hat{a}] \hat{a} = -\hat{a}, \\ [\hat{N}, \hat{a}^\dagger] &= [\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger [\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}^\dagger] \hat{a} = \hat{a}^\dagger. \end{aligned}$$

Using these relations, we can test some actions on  $|n\rangle$ :

$$\begin{aligned} \hat{N} \hat{a}^\dagger |n\rangle &= ([\hat{N}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{N}) |n\rangle = (\hat{a}^\dagger + \hat{a}^\dagger \hat{N}) |n\rangle = (\hat{a}^\dagger + \hat{a}^\dagger n) |n\rangle = (n+1) \hat{a}^\dagger |n\rangle, \\ \hat{N} \hat{a} |n\rangle &= ([\hat{N}, \hat{a}] + \hat{a} \hat{N}) |n\rangle = (-\hat{a} + \hat{a} \hat{N}) |n\rangle = (\hat{a} n - \hat{a}) |n\rangle = (n-1) \hat{a} |n\rangle. \end{aligned}$$

This imply that  $\hat{a}^\dagger |n\rangle$  and  $\hat{a} |n\rangle$  are also eigenkets of  $\hat{N}$ , with eigenvalue increased and decreased by one, respectively. With this, we conclude that  $n$  must be integer so that  $\hat{N}$  and therefore  $\hat{H}$  are quantized. A more rigorous proof is given by [1].

## Exercise 2

- a) Ill use the definition of the E- and B-fields in terms of the vector potential.

$$\mathbf{E} = -\partial_t \mathbf{A}, \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

Taking the curl of the B-field allows to replace the vector potential and have a triple product, which can be reexpressed using vector identities. Because  $\nabla \times \mathbf{B}$  is Ampere's law, it also depends on the E-field, where the above definition can be replaced.

$$\begin{aligned} \nabla \times \mathbf{B} &= \frac{1}{c^2} \partial_t \mathbf{E} \\ \nabla \times \nabla \times \mathbf{A} &= \frac{1}{c^2} \partial_t (-\partial_t \mathbf{A}) \\ \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} &= -\frac{1}{c^2} \partial_t^2 \mathbf{A} \\ \nabla^2 \mathbf{A} - \frac{1}{c^2} \partial_t^2 \mathbf{A} &= 0. \end{aligned}$$

- b) To show that the expression provided is a solution of the A-potential wave equation, we need to substitute it into the wave equation and have an equality. The second time-derivative is:

$$\begin{aligned} \partial_t \mathbf{A}(z, t) &= \partial_t \left[ \sum_{\mathbf{k}, \lambda} e_{\mathbf{k}, \lambda} [A_{\mathbf{k}, \lambda} e^{-i\omega t} + A_{\mathbf{k}, \lambda}^* e^{i\omega t}] \sin(kz) \right] = i \sum_{\mathbf{k}, \lambda} \omega e_{\mathbf{k}, \lambda} [A_{\mathbf{k}, \lambda}^* e^{i\omega t} - A_{\mathbf{k}, \lambda} e^{-i\omega t}] \sin(kz), \\ \partial_t^2 \mathbf{A}(z, t) &= \partial_t \left[ i \sum_{\mathbf{k}, \lambda} \omega e_{\mathbf{k}, \lambda} [A_{\mathbf{k}, \lambda}^* e^{i\omega t} - A_{\mathbf{k}, \lambda} e^{-i\omega t}] \sin(kz) \right] = -\sum_{\mathbf{k}, \lambda} \omega^2 e_{\mathbf{k}, \lambda} [A_{\mathbf{k}, \lambda}^* e^{i\omega t} + A_{\mathbf{k}, \lambda} e^{-i\omega t}] \sin(kz). \end{aligned}$$

For the laplacian, we have:

$$\nabla^2 \mathbf{A}(z, t) = - \sum_{\mathbf{k}, \lambda} k^2 e_{\mathbf{k}, \lambda} [A_{\mathbf{k}, \lambda} e^{-i\omega t} + A_{\mathbf{k}, \lambda}^* e^{i\omega t}] \sin(kz).$$

Substituting both, and using the dispersion relation  $\omega = |\mathbf{k}|c$ :

$$\begin{aligned} & \nabla^2 \mathbf{A} - \frac{1}{c^2} \partial_t^2 \mathbf{A} = 0 \\ & - \sum_{\mathbf{k}, \lambda} k^2 e_{\mathbf{k}, \lambda} [A_{\mathbf{k}, \lambda} e^{-i\omega t} + A_{\mathbf{k}, \lambda}^* e^{i\omega t}] \sin(kz) - \frac{1}{c^2} \left[ - \sum_{\mathbf{k}, \lambda} \omega_{\mathbf{k}}^2 e_{\mathbf{k}, \lambda} [A_{\mathbf{k}, \lambda}^* e^{i\omega t} + A_{\mathbf{k}, \lambda} e^{-i\omega t}] \sin(kz) \right] = \\ & \sum_{\mathbf{k}, \lambda} \left[ e_{\mathbf{k}, \lambda} [A_{\mathbf{k}, \lambda} e^{-i\omega t} + A_{\mathbf{k}, \lambda}^* e^{i\omega t}] \sin(kz) \left( \frac{\omega_{\mathbf{k}}^2}{c^2} - k^2 \right) \right] = \\ & \sum_{\mathbf{k}, \lambda} [e_{\mathbf{k}, \lambda} [A_{\mathbf{k}, \lambda} e^{-i\omega t} + A_{\mathbf{k}, \lambda}^* e^{i\omega t}] \sin(kz) (k^2 - k^2)] = \quad (\omega^2 = |\mathbf{k}|^2 c^2) \\ & 0 = 0. \end{aligned}$$

We could also propose a standard harmonic solution, substitute it into the vector equation and impose boundary conditions. The one I did is easier. Also, I denoted  $\omega_{\mathbf{k}}$  to remember that  $\omega$  depends on  $\mathbf{k}$ . The dispersion relation holds for each mode  $\mathbf{k}$ .

- c) Assuming a x polarized vector potential

$$\mathbf{A}(z, t) = \hat{x} [A e^{-i\omega t} + A^* e^{i\omega t}] \sin(kz).$$

The E- and B-fields are:

$$\begin{aligned} \mathbf{E}(z, t) &= -\partial_t \mathbf{A}(z, t) = i\omega \hat{x} [A e^{-i\omega t} - A^* e^{i\omega t}] \sin(kz) \equiv 2\omega |A| \hat{x} \sin(\omega t - \phi) \sin(kz), \\ \mathbf{B}(z, t) &= \nabla \times \mathbf{A}(z, t) = \hat{y} \partial_z A_x = \hat{y} k [A e^{-i\omega t} + A^* e^{i\omega t}] \cos(kz) \equiv 2k |A| \hat{y} \cos(\omega t - \phi) \cos(kz), \end{aligned}$$

where we have assumed a phasor form of  $A$ :  $A = |A| e^{i\phi}$ .

- d) We use the expression from previous part. We first compute  $\mathbf{E}_{\mathbf{k}, \lambda}^2$  and  $\mathbf{B}_{\mathbf{k}, \lambda}^2$

$$\begin{aligned} \mathbf{E}^2 &= 4\omega^2 |A|^2 \sin^2(\omega t - \phi) \sin^2(kz), \\ \mathbf{B}^2 &= 4k^2 |A|^2 \cos^2(\omega t - \phi) \cos^2(kz). \end{aligned}$$

Putting these two in the volume integral, and recalling that

$$\int_0^L dz \sin^2(kz) = \int_0^L dz \cos^2(kz) = \frac{L}{2},$$

yields

$$\begin{aligned} H_{\omega} &= \int_0^L dV \left\{ 2\varepsilon_0 \omega^2 |A|^2 \sin^2(\omega t - \phi) \sin^2(kz) + 2\frac{k^2}{\mu_0} |A|^2 \cos^2(\omega t - \phi) \cos^2(kz) \right\} \\ &= 2\varepsilon_0 \omega^2 |A|^2 \sin^2(\omega t - \phi) S \int_0^L dz \sin^2(kz) + 2\frac{k^2}{\mu_0} |A|^2 \cos^2(\omega t - \phi) S \int_0^L dz \cos^2(kz) \\ H_{\omega} &= V \left[ \varepsilon_0 \omega^2 |A|^2 \sin^2(\omega t - \phi) + \frac{k^2}{\mu_0} |A|^2 \cos^2(\omega t - \phi) \right]. \end{aligned}$$

Using the fact that  $\omega = kc$ ,  $1/\mu_0 = \varepsilon_0 c^2$ , then  $k^2/\mu_0 = (\omega/c)^2 (\varepsilon_0 c^2) = \varepsilon_0 \omega^2$ , which is the same as the E-field factor term. Therefore,

$$H_{\omega} = V \varepsilon_0 \omega^2 |A| [\sin^2(\omega t - \phi) + \cos^2(\omega t - \phi)] = \varepsilon_0 V \omega^2 |A|^2.$$

e) Equating the last result with  $\hbar\omega$  and solving for  $A$ :

$$\varepsilon_0 V \omega^2 |A|^2 = \hbar\omega$$

$$|A| = \sqrt{\frac{\hbar}{\varepsilon_0 V \omega}}.$$

Its phase,  $\phi$ , remains unknown, but it acts as a global phase factor so it has no physical meaning. The E- and B-fields are:

$$\mathbf{E}(z, t) = 2\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \hat{x} \sin(\omega t - \phi) \sin(kz),$$

$$\mathbf{B}(z, t) = \frac{2}{c} \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \hat{y} \cos(\omega t - \phi) \cos(kz).$$

### Exercise 3

a) We can simplify the problem using the BCH formula

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \dots$$

In this case,  $\hat{A} = \alpha^* \hat{a} - \alpha \hat{a}^\dagger$  and  $\hat{B} = \hat{a}$ . The first two commutators are:

$$[\hat{A}, \hat{B}] = [(\alpha^* \hat{a} - \alpha \hat{a}^\dagger), \hat{a}] = \alpha^* [\hat{a}, \hat{a}] - \alpha [\hat{a}^\dagger, \hat{a}] = \alpha,$$

$$[\hat{A}, [\hat{A}, \hat{B}]] = [(\alpha^* \hat{a} - \alpha \hat{a}^\dagger), \alpha] = 0,$$

$$\vdots$$

Now, we can express the displacement operator as:

$$\tilde{a} = \mathcal{D}^\dagger(\alpha) \hat{a} \mathcal{D}(\alpha) = e^{\alpha^* \hat{a} - \alpha \hat{a}^\dagger} \hat{a} e^{-(\alpha^* \hat{a} - \alpha \hat{a}^\dagger)} = \hat{a} + \alpha, \quad \text{and}$$

$$\tilde{a}^\dagger = \mathcal{D}^\dagger(\alpha) \hat{a}^\dagger \mathcal{D}(\alpha) = \hat{a}^\dagger + \alpha^*.$$

Application of both in  $|0\rangle$  (just to test them) yields:

$$\tilde{a}|0\rangle = (\hat{a} + \alpha)|0\rangle = \alpha|0\rangle, \quad \text{and} \quad \tilde{a}^\dagger|0\rangle = (\hat{a}^\dagger + \alpha^*)|0\rangle = |1\rangle + \alpha^*|0\rangle.$$

b) We use algebra of operators and the fact that the coherent state  $|\alpha\rangle$  is got from  $|0\rangle$  by applying a displacement operator  $\mathcal{D}(\alpha)$ :

$$\begin{aligned} \langle \hat{X}_1 \rangle &= \frac{1}{2} \langle \alpha | (\hat{a} + \hat{a}^\dagger) | \alpha \rangle \\ &= \frac{1}{2} \langle 0 | \mathcal{D}^\dagger(\alpha) (\hat{a} + \hat{a}^\dagger) \mathcal{D}(\alpha) | 0 \rangle \\ &= \frac{1}{2} \left[ \langle 0 | \mathcal{D}^\dagger(\alpha) \hat{a} \mathcal{D}(\alpha) | 0 \rangle + \langle 0 | \mathcal{D}^\dagger(\alpha) \hat{a}^\dagger \mathcal{D}(\alpha) | 0 \rangle \right] \\ &= \frac{1}{2} \left[ \langle 0 | \tilde{a} | 0 \rangle + \langle 0 | \tilde{a}^\dagger | 0 \rangle \right] \\ &= \frac{1}{2} \left[ \langle 0 | (\hat{a} + \alpha) | 0 \rangle + \langle 0 | (\hat{a}^\dagger + \alpha^*) | 0 \rangle \right] \\ &= \frac{1}{2} \left[ \langle 0 | \hat{a} | 0 \rangle + \alpha + \langle 0 | \hat{a}^\dagger | 0 \rangle + \alpha^* \right] \\ &= \frac{1}{2} [\alpha + \alpha^*] \\ \langle \hat{X}_1 \rangle &= \text{Re}(\alpha). \end{aligned}$$

For  $\hat{X}_2$ , and using the developments from above:

$$\begin{aligned}\langle \hat{X}_2 \rangle &= \frac{1}{2i} \left[ \langle 0 | (\hat{a} + \alpha) | 0 \rangle - \langle 0 | (\hat{a}^\dagger + \alpha^*) | 0 \rangle \right] \\ &= \frac{1}{2i} \left[ \langle 0 | \hat{a} | 0 \rangle + \alpha - \langle 0 | \hat{a}^\dagger | 0 \rangle - \alpha^* \right] \\ &= \frac{1}{2i} [\alpha - \alpha^*] \\ \langle \hat{X}_2 \rangle &= \text{Im}(\alpha).\end{aligned}$$

For the uncertainties, we now need to get the mean value of  $\hat{X}_1^2$  and  $\hat{X}_2^2$ , with the analogous process:

$$\begin{aligned}\langle \hat{X}_1^2 \rangle &= \frac{1}{4} \langle 0 | \mathcal{D}^\dagger(\alpha) [\hat{a} + \hat{a}^\dagger]^2 \mathcal{D}(\alpha) | 0 \rangle \\ &= \frac{1}{4} \langle 0 | \left[ \mathcal{D}^\dagger(\alpha) [\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2}] \mathcal{D}(\alpha) \right] | 0 \rangle \\ &= \frac{1}{4} \langle 0 | \left[ \mathcal{D}^\dagger \hat{a}^2 \mathcal{D}(\alpha) + 2\mathcal{D}^\dagger(\alpha) \hat{a}^\dagger \hat{a} \mathcal{D}(\alpha) + 1 + \mathcal{D}^\dagger(\alpha) \hat{a}^{\dagger 2} \mathcal{D}(\alpha) \right] | 0 \rangle \\ &= \frac{1}{4} \langle 0 | \left[ (\mathcal{D}^\dagger \hat{a} \mathcal{D})(\mathcal{D}^\dagger \hat{a} \mathcal{D}) + 2(\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D})(\mathcal{D}^\dagger \hat{a} \mathcal{D}) + 1 + (\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D})(\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D}) \right] | 0 \rangle \\ &= \frac{1}{4} \langle 0 | \left[ (\hat{a} + \alpha)^2 + 2(\hat{a}^\dagger + \alpha^*)(\hat{a} + \alpha) + 1 + (\hat{a}^\dagger + \alpha^*)^2 \right] | 0 \rangle \\ &= \frac{1}{4} [\hat{a}^2 + 2\alpha\hat{a} + \alpha^2 + 2\hat{a}^\dagger\hat{a} + 2\hat{a}^\dagger\alpha + 2\alpha^*\hat{a} + 2|\alpha|^2 + 1 + \hat{a}^{\dagger 2} + 2\alpha^*\hat{a}^\dagger + \alpha^{*2}] \\ \langle \hat{X}_1^2 \rangle &= \frac{1}{4} [(\alpha + \alpha^*)^2 + 1].\end{aligned}$$

Then,

$$\langle \Delta \hat{X}_1 \rangle = \sqrt{\langle \hat{X}_1^2 \rangle - \langle \hat{X}_1 \rangle^2} = \sqrt{\frac{1}{4} [(\alpha + \alpha^*)^2 + 1] - \frac{1}{4} (\alpha + \alpha^*)^2} = \frac{1}{2}.$$

The same procedure is done for the uncertainty of  $\hat{X}_2$ :

$$\begin{aligned}\langle \hat{X}_2 \rangle &= \frac{1}{4} \langle 0 | \left[ (\mathcal{D}^\dagger \hat{a} \mathcal{D})(\mathcal{D}^\dagger \hat{a} \mathcal{D}) - 2(\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D})(\mathcal{D}^\dagger \hat{a} \mathcal{D}) - 1 + (\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D})(\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D}) \right] | 0 \rangle \\ &= -\frac{1}{4} \langle 0 | \left[ (\hat{a} + \alpha)^2 - 2(\hat{a}^\dagger + \alpha^*)(\hat{a} + \alpha) - 1 + (\hat{a}^\dagger + \alpha^*)^2 \right] | 0 \rangle \\ &= -\frac{1}{4} [\hat{a}^2 + 2\alpha\hat{a} + \alpha^2 - 2\hat{a}^\dagger\hat{a} - 2\hat{a}^\dagger\alpha - 2\alpha^*\hat{a} - 2|\alpha|^2 - 1 + \hat{a}^{\dagger 2} + 2\alpha^*\hat{a}^\dagger + \alpha^{*2}] \\ \langle \hat{X}_2^2 \rangle &= \frac{1}{4} [1 - (\alpha - \alpha^*)^2].\end{aligned}$$

Then,

$$\langle \Delta \hat{X}_2 \rangle = \sqrt{\langle \hat{X}_2^2 \rangle - \langle \hat{X}_2 \rangle^2} = \sqrt{\frac{1}{4} [1 - (\alpha - \alpha^*)^2] + \frac{1}{4} (\alpha - \alpha^*)^2} = \frac{1}{2}.$$

## References

- [1] Claude Cohen-Tannoudji, Bernard Diu, and Frank Laloe. Quantum mechanics, volume 1. *Quantum Mechanics*, 1:898, 1986.