Assignment 2

OPTI 544 Quantum Optics

University of Arizona

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Exercise 1

a) The quadrature operators are:

$$\hat{X}_1 = \frac{\hat{a}+\hat{a}^\dagger}{2}, \quad \text{and} \quad \hat{X}_2 = \frac{\hat{a}-\hat{a}^\dagger}{2i}.$$

We can use them to express $\hat{a}, \hat{a}^\dagger$ in terms of $\hat{X}_{1,2}$.

$$\hat{a}+\hat{a}^\dagger = 2\hat{X}_1 \tag{1}$$
$$\hat{a}-\hat{a}^\dagger = 2i\hat{X}_2. \tag{2}$$

Then,

$$\begin{aligned}
(1)+(2): &\quad 2\hat{a} = 2(\hat{X}_1 + i\hat{X}_2) \longrightarrow \hat{a} = \hat{X}_1(0) + i\hat{X}_2(0) \\
(1)-(2): &\quad 2\hat{a}^\dagger = 2(\hat{X}_1 - i\hat{X}_2) \longrightarrow \hat{a}^\dagger = \hat{X}_1(0) - i\hat{X}_2(0)
\end{aligned}$$

Using these in the time-evolved quadrature operators:

$$\begin{aligned}
\hat{X}_1(t) &= \frac{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2} = \frac{[\hat{X}_1(0) + i\hat{X}_2(0)]e^{-i\omega t} + [\hat{X}_1(0) - i\hat{X}_2(0)]e^{i\omega t}}{2} \\
\hat{X}_1(t) &= \frac{e^{i\omega t}+e^{-i\omega t}}{2}\hat{X}_1(0) + \frac{(e^{i\omega t}-e^{-i\omega t})}{2i}\hat{X}_2(0) = \cos(\omega t)\hat{X}_1(0) + \sin(\omega t)\hat{X}_2(0).
\end{aligned}$$

Likewise,

$$\begin{aligned}
\hat{X}_2(t) &= \frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i} = \frac{[\hat{X}_1(0) + i\hat{X}_2(0)]e^{-i\omega t} - [\hat{X}_1(0) - i\hat{X}_2(0)]e^{i\omega t}}{2i} \\
\hat{X}_2(t) &= -\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\hat{X}_1(0) + i\frac{(e^{i\omega t}+e^{-i\omega t})}{2i}\hat{X}_2(0) = -\sin(\omega t)\hat{X}_1(0) + \cos(\omega t)\hat{X}_2(0).
\end{aligned}$$

The commutation relation of the time-evolved quadrature operators is

$$[\hat{X}_1, \hat{X}_2] = \frac{1}{4i}[\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}, \hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}] = \frac{1}{4i}\left\{-[\hat{a},\hat{a}^\dagger] + [\hat{a}^\dagger,\hat{a}]\right\} = \frac{1}{4i}(-2) = \frac{i}{2}.$$

b) The expectation value is

$$\begin{aligned}
\langle\alpha|\hat{X}_1(t)|\alpha\rangle &= \langle\alpha|\frac{\hat{a}e^{-i\omega t}+\hat{a}^\dagger e^{i\omega t}}{2}|\alpha\rangle = \frac{1}{2}\left[\langle\alpha|\hat{a}|\alpha\rangle e^{-i\omega t} + \langle\alpha|\hat{a}^\dagger|\alpha\rangle e^{i\omega t}\right] = \frac{1}{2}\left[\alpha e^{-i\omega t}+\alpha^* e^{i\omega t}\right] \\
&= \frac{1}{2}\left[(\alpha e^{-i\omega t}) + (\alpha e^{-i\omega t})^*\right] = \mathrm{Re}\left(\alpha e^{-i\omega t}\right). \\
\langle\alpha|\hat{X}_2(t)|\alpha\rangle &= \langle\alpha|\frac{\hat{a}e^{-i\omega t}-\hat{a}^\dagger e^{i\omega t}}{2i}|\alpha\rangle = \frac{1}{2i}\left[\langle\alpha|\hat{a}|\alpha\rangle e^{-i\omega t} - \langle\alpha|\hat{a}^\dagger|\alpha\rangle e^{i\omega t}\right] = \frac{1}{2i}\left[\alpha e^{-i\omega t}-\alpha^* e^{i\omega t}\right] \\
&= \frac{1}{2i}\left[(\alpha e^{-i\omega t}) - (\alpha e^{-i\omega t})^*\right] = \mathrm{Im}\left(\alpha e^{-i\omega t}\right).
\end{aligned}$$

The coherent state $\left|\alpha e^{-i\omega t}\right\rangle$ is a complex wave that rotates with a rate given by ω. $|\alpha\rangle$ states the initial position in the $\hat{X}_1\hat{X}_2$ phase space diagram.

$$\alpha e^{-i\omega t} = \langle\hat{X}_1\rangle + i\langle\hat{X}_2\rangle.$$

The uncertainties requires also the expectation value of the operator squared:

$$\begin{aligned}
\langle\alpha|\hat{X}_1^2(t)|\alpha\rangle &= \frac{1}{4}\langle\alpha|(\hat{a}^2 e^{-i2\omega t} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2}e^{i2\omega t})|\alpha\rangle = \frac{1}{4}\left[\alpha^2 e^{-i2\omega t} + 2|\alpha|^2 + 1 + \alpha^{*2}e^{i2\omega t}\right] \\
&= \frac{1}{4}\left[(\alpha e^{-i\omega t}+\alpha^* e^{i\omega t})^2 + 1\right] = \frac{1}{4}\left\{[2\mathrm{Re}\left(\alpha e^{-i\omega t}\right)]^2 + 1\right\}. \\
\langle\alpha|\hat{X}_2^2(t)|\alpha\rangle &= -\frac{1}{4}\langle\alpha|(\hat{a}^2 e^{-i2\omega t} - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2}e^{i2\omega t})|\alpha\rangle = -\frac{1}{4}\left[\alpha^2 e^{-i2\omega t} - 2|\alpha|^2 - 1 + \alpha^{*2}e^{i2\omega t}\right] \\
&= \frac{1}{4}\left[1 - (\alpha e^{-i\omega t}-\alpha^* e^{i\omega t})^2\right] = \frac{1}{4}\left\{1 - [2i\mathrm{Im}\left(\alpha e^{-i\omega t}\right)]^2\right\}.
\end{aligned}$$

Uncertainty is

$$\begin{aligned}
\Delta\hat{X}_1(t) &= \sqrt{\langle\hat{X}_1^2(t)\rangle - \langle\hat{X}_1(t)\rangle^2} = \sqrt{\frac{1}{4}\{[2\mathrm{Re}\,(\alpha e^{-i\omega t})]^2+1\} - [\mathrm{Re}\,(\alpha e^{-i\omega t})]^2} = \sqrt{\frac{1}{4}} = \frac{1}{2}. \\
\Delta\hat{X}_2(t) &= \sqrt{\langle\hat{X}_2^2(t)\rangle - \langle\hat{X}_2(t)\rangle^2} = \sqrt{\frac{1}{4}\{1-[2i\mathrm{Im}\,(\alpha e^{-i\omega t})]^2\} - [\mathrm{Im}\,(\alpha e^{-i\omega t})]^2} = \sqrt{\frac{1}{4}} = \frac{1}{2}.
\end{aligned}$$

The diagram is shown below.

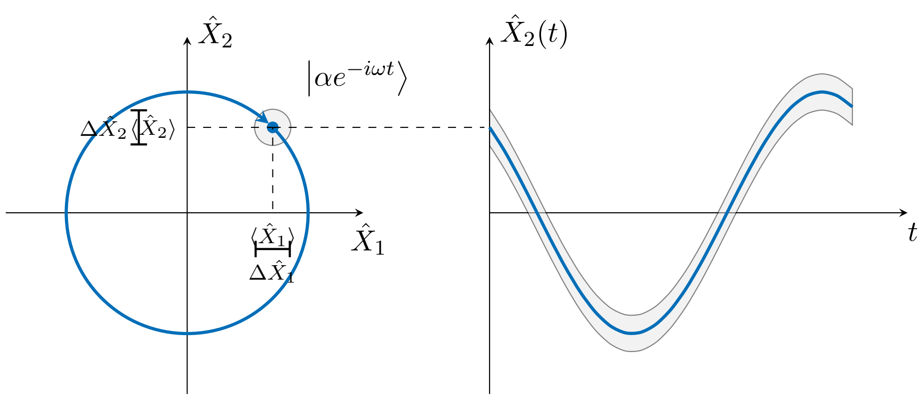

Figure 1: Phase space of the coherent state and time-dependent quadrature operators.

c) The process is analogous. We know that

$$\tilde{a} = \hat{S}^\dagger \hat{D}^\dagger \hat{a} \hat{D} \hat{S} = \cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha, \quad \text{and}$$
$$\tilde{a}^\dagger = \hat{S}^\dagger \hat{D}^\dagger \hat{a}^\dagger \hat{D} \hat{S} = \cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*.$$

The expected value of $\hat{X}_1(t)$ is

$$\begin{aligned}
\langle \hat{X}_1(t)\rangle &= \langle 0|\hat{S}^\dagger \hat{D}^\dagger \frac{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2} \hat{D}\hat{S}|0\rangle = \frac{1}{2}\langle 0|\tilde{a}e^{-i\omega t} + \tilde{a}^\dagger e^{i\omega t}|0\rangle \\
&= \frac{1}{2}\langle 0| \left[\cosh(r)\hat{a}e^{-i\omega t} - e^{i\phi}\sinh(r)\hat{a}^\dagger e^{-i\omega t} + \alpha e^{-i\omega t} + \cosh(r)\hat{a}^\dagger e^{i\omega t} - e^{-i\phi}\sinh(r)\hat{a}e^{i\omega t} + \alpha^* e^{i\omega t}\right]|0\rangle \\
&= \frac{1}{2}[\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}] \\
\langle \hat{X}_1(t)\rangle &= \mathrm{Re}\left(\alpha e^{-i\omega t}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle \hat{X}_2(t)\rangle &= \langle 0|\hat{S}^\dagger \hat{D}^\dagger \frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i} \hat{D}\hat{S}|0\rangle = \frac{1}{2i}\langle 0|\tilde{a}e^{-i\omega t} + \tilde{a}^\dagger e^{i\omega t}|0\rangle \\
&= \frac{1}{2i}\langle 0| \left[\cosh(r)\hat{a}e^{-i\omega t} - e^{i\phi}\sinh(r)\hat{a}^\dagger e^{-i\omega t} + \alpha e^{-i\omega t} - \cosh(r)\hat{a}^\dagger e^{i\omega t} + e^{-i\phi}\sinh(r)\hat{a}e^{i\omega t} - \alpha^* e^{i\omega t}\right]|0\rangle \\
&= \frac{1}{2}[\alpha e^{-i\omega t} - \alpha^* e^{i\omega t}] \\
\langle \hat{X}_2(t)\rangle &= \mathrm{Im}\left(\alpha e^{-i\omega t}\right).
\end{aligned}$$

Doing the same for the operators squared:

$$\begin{aligned}
\langle \hat{X}_1^2(t)\rangle &= \langle 0|\hat{S}^\dagger \hat{D}^\dagger \left(\frac{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2}\right)^2 \hat{D}\hat{S}|0\rangle \\
&= \frac{1}{4}\langle 0|\hat{S}^\dagger \hat{D}^\dagger \left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right)\hat{D}\hat{S}\hat{S}^\dagger \hat{D}^\dagger \left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right)\hat{D}\hat{S}|0\rangle \\
&= \frac{1}{4}\langle 0|(\tilde{a}e^{-i\omega t} + \tilde{a}^\dagger e^{i\omega t})^2|0\rangle \\
&= \frac{1}{4}\left[e^{-i2\omega t}\langle 0|\tilde{a}^2|0\rangle + e^{i2\omega t}\langle 0|\tilde{a}^{\dagger 2}|0\rangle + \langle 0|\tilde{a}\tilde{a}^\dagger|0\rangle + \langle 0|\tilde{a}^\dagger\tilde{a}|0\rangle\right].
\end{aligned}$$

We compute each term separately:

$$\begin{aligned}
\langle 0|\tilde{a}^2|0\rangle =& \langle 0|(\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha)(\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha)|0\rangle \\
=& \cosh^2(r)\langle\hat{a}^2\rangle - e^{i\phi}\sinh(r)\cosh(r)\langle\hat{a}\hat{a}^\dagger\rangle + \alpha\cosh(r)\langle\hat{a}\rangle - e^{i\phi}\sinh(r)\cosh(r)\langle\hat{a}^\dagger\hat{a}\rangle \\
&+ e^{i2\phi}\sinh^2(r)\langle\hat{a}^{\dagger 2}\rangle - \alpha e^{i\phi}\sinh(r)\langle\hat{a}^\dagger\rangle + \alpha\cosh(r)\langle\hat{a}\rangle - \alpha e^{i\phi}\sinh(r)\langle\hat{a}^\dagger\rangle + \alpha^2 \\
&= \alpha^2 - e^{i\phi}\sinh(r)\cosh(r) \\
\langle 0|\tilde{a}^2|0\rangle &= \alpha^2 - \frac{1}{2}e^{i\phi}\sinh(2r).
\end{aligned}$$

$$\begin{aligned}
\langle 0|\tilde{a}^{\dagger 2}|0\rangle =& \langle 0|(\cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*)(\cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*)|0\rangle \\
=& \cosh^2(r)\langle\hat{a}^{\dagger 2}\rangle - e^{-i\phi}\cosh(r)\sinh(r)\langle\hat{a}^\dagger\hat{a}\rangle + \alpha^*\cosh(r)\langle\hat{a}^\dagger\rangle - e^{-i\phi}\sinh(r)\cosh(r)\langle\hat{a}\hat{a}^\dagger\rangle \\
&+ e^{-i2\phi}\sinh^2(r)\langle\hat{a}^2\rangle - \alpha^* e^{-i\phi}\sinh(r)\langle\hat{a}\rangle + \alpha^*\cosh(r)\langle\hat{a}^\dagger\rangle - \alpha^* e^{-i\phi}\sinh(r)\langle\hat{a}^\dagger\rangle + \alpha^{*2} \\
=& \alpha^{*2} - e^{-i\phi}\sinh(r)\cosh(r) \\
\langle 0|\tilde{a}^{\dagger 2}|0\rangle =& \alpha^{*2} - \frac{1}{2}e^{-i\phi}\sinh(2r).
\end{aligned}$$

$$\langle 0|\tilde{a}\tilde{a}^\dagger|0\rangle = \langle 0|(\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha)(\cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*)|0\rangle$$
$$\langle 0|\tilde{a}\tilde{a}^\dagger|0\rangle = \cosh^2(r) + |\alpha|^2.$$

$$\langle 0|\tilde{a}^\dagger\tilde{a}|0\rangle = \langle 0|(\cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*)(\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha)|0\rangle$$
$$\langle 0|\tilde{a}^\dagger\tilde{a}|0\rangle = \sinh^2(r) + |\alpha|^2.$$

Putting all together:

$$\begin{aligned}\langle \hat{X}_1^2(t)\rangle &= \frac{1}{4}\left\{[\alpha^2 - \frac{1}{2}e^{i\phi}\sinh(2r)]e^{-i2\omega t} + [\alpha^{*2} - \frac{1}{2}e^{-i\phi}\sinh(2r)]e^{i2\omega t} + \cosh^2(r) + |\alpha|^2 + \sinh^2(r) + |\alpha|^2\right\}\\ &= \frac{1}{4}\left[(\alpha e^{-i\omega t} + \alpha^* e^{i\omega t})^2 + \cosh(2r) - \sinh(2r)\cos(\phi - 2\omega t)\right]\\ \langle \hat{X}_1^2(t)\rangle &= [\mathrm{Re}\left(\alpha e^{-i\omega t}\right)]^2 + \frac{1}{4}[\cosh(2r) - \sinh(2r)\cos(\phi - 2\omega t)].\end{aligned}$$

The uncertainty is:

$$\Delta\hat{X}_1(t) = \sqrt{\langle \hat{X}_1^2(t)\rangle - \langle \hat{X}_1(t)\rangle^2} = \frac{1}{2}\sqrt{\cosh(2r) - \sinh(2r)\cos(\phi - 2\omega t)}.$$

Similarly,

$$\begin{aligned}\langle \hat{X}_2^2(t)\rangle &= \langle 0|\hat{S}^\dagger\hat{D}^\dagger\left(\frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i}\right)^2\hat{D}\hat{S}|0\rangle\\ &= -\frac{1}{4}\langle 0|\hat{S}^\dagger\hat{D}^\dagger(\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t})\hat{D}\hat{S}\hat{S}^\dagger\hat{D}^\dagger(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t})\hat{D}\hat{S}|0\rangle\\ &= -\frac{1}{4}\langle 0|(\tilde{a}e^{-i\omega t} - \tilde{a}^\dagger e^{i\omega t})^2|0\rangle\\ &= -\frac{1}{4}[e^{-i2\omega t}\langle 0|\tilde{a}^2|0\rangle + e^{i2\omega t}\langle 0|\tilde{a}^{\dagger 2}|0\rangle - \langle 0|\tilde{a}\tilde{a}^\dagger|0\rangle - \langle 0|\tilde{a}^\dagger\tilde{a}|0\rangle].\end{aligned}$$

We use the previous results to get

$$\begin{aligned}\langle \hat{X}_2(t)\rangle &= -\frac{1}{4}\left\{[\alpha^2 - \frac{1}{2}e^{i\phi}\sinh(2r)]e^{-i2\omega t} + [\alpha^{*2} - \frac{1}{2}e^{-i\phi}\sinh(2r)]e^{i2\omega t} - \cosh^2(r) - |\alpha|^2 - \sinh^2(r) - |\alpha|^2\right\}\\ &= -\frac{1}{4}[(\alpha e^{-i\omega t} - \alpha^* e^{i\omega t})^2 - \cosh(2r) - \sinh(2r)\cos(\phi - 2\omega t)]\\ \langle \hat{X}_2(t)\rangle &= [\mathrm{Im}\left(\alpha e^{-i\omega t}\right)]^2 + \frac{1}{4}[\cosh(2r) + \sinh(2r)\cos(\phi - 2\omega t)].\end{aligned}$$

And the uncertainty is:

$$\Delta\hat{X}_2(t) = \sqrt{\langle \hat{X}_2^2(t)\rangle - \langle \hat{X}_2(t)\rangle^2} = \frac{1}{2}\sqrt{\cosh(2r) + \sinh(2r)\cos(\phi - 2\omega t)}$$

We can verify that $\Delta\hat{X}_1(0) = e^{-r}/2$ and $\Delta\hat{X}_2(0) = e^{r}/2$ when $\phi = t = 0$.
The phase space representation is shown below.

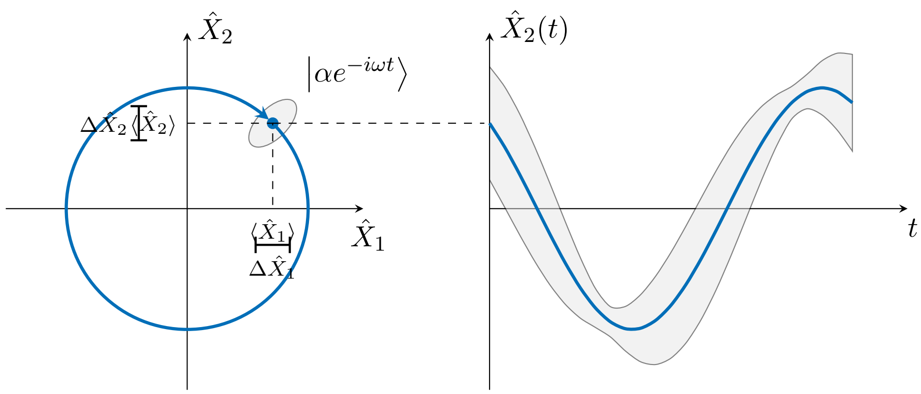

Figure 2: Phase space of the coherent state and time-dependent quadrature operators.

References

[1] Claude Cohen-Tannoudji, Bernard Diu, and Frank Laloe. Quantum mechanics, volume 1. *Quantum Mechanics*, 1:898, 1986.