

# Assignment 2

## OPTI 544 Quantum Optics

### University of Arizona

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### Exercise 1

a) The quadrature operators are:

$$\hat{X}_1 = \frac{\hat{a} + \hat{a}^\dagger}{2}, \quad \text{and} \quad \hat{X}_2 = \frac{\hat{a} - \hat{a}^\dagger}{2i}.$$

We can use them to express  $\hat{a}, \hat{a}^\dagger$  in terms of  $\hat{X}_{1,2}$ .

$$\hat{a} + \hat{a}^\dagger = 2\hat{X}_1 \tag{1}$$

$$\hat{a} - \hat{a}^\dagger = 2i\hat{X}_2. \tag{2}$$

Then,

$$(1) + (2): \quad 2\hat{a} = 2(\hat{X}_1 + i\hat{X}_2) \longrightarrow \hat{a} = \hat{X}_1(0) + i\hat{X}_2(0)$$

$$(1) - (2): \quad 2\hat{a}^\dagger = 2(\hat{X}_1 - i\hat{X}_2) \longrightarrow \hat{a}^\dagger = \hat{X}_1(0) - i\hat{X}_2(0)$$

Using these in the time-evolved quadrature operators:

$$\begin{aligned} \hat{X}_1(t) &= \frac{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2} = \frac{[\hat{X}_1(0) + i\hat{X}_2(0)]e^{-i\omega t} + [\hat{X}_1(0) - i\hat{X}_2(0)]e^{i\omega t}}{2} \\ \hat{X}_1(t) &= \frac{e^{i\omega t} + e^{-i\omega t}}{2}\hat{X}_1(0) + \frac{(e^{i\omega t} - e^{-i\omega t})}{2i}\hat{X}_2(0) = \cos(\omega t)\hat{X}_1(0) + \sin(\omega t)\hat{X}_2(0). \end{aligned}$$

Likewise,

$$\begin{aligned} \hat{X}_2(t) &= \frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i} = \frac{[\hat{X}_1(0) + i\hat{X}_2(0)]e^{-i\omega t} - [\hat{X}_1(0) - i\hat{X}_2(0)]e^{i\omega t}}{2i} \\ \hat{X}_2(t) &= -\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\hat{X}_1(0) + i\frac{(e^{i\omega t} + e^{-i\omega t})}{2i}\hat{X}_2(0) = -\sin(\omega t)\hat{X}_1(0) + \cos(\omega t)\hat{X}_2(0). \end{aligned}$$

The commutation relation of the time-evolved quadrature operators is

$$[\hat{X}_1, \hat{X}_2] = \frac{1}{4i}[\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}, \hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}] = \frac{1}{4i} \left\{ -[\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}] \right\} = \frac{1}{4i}(-2) = \frac{i}{2}.$$

b) The expectation value is

$$\begin{aligned}\langle \alpha | \hat{X}_1(t) | \alpha \rangle &= \langle \alpha | \frac{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2} | \alpha \rangle = \frac{1}{2} [\langle \alpha | \hat{a} | \alpha \rangle e^{-i\omega t} + \langle \alpha | \hat{a}^\dagger | \alpha \rangle e^{i\omega t}] = \frac{1}{2} [\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}] \\ &= \frac{1}{2} [(\alpha e^{-i\omega t}) + (\alpha e^{-i\omega t})^*] = \text{Re}(\alpha e^{-i\omega t}). \\ \langle \alpha | \hat{X}_2(t) | \alpha \rangle &= \langle \alpha | \frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i} | \alpha \rangle = \frac{1}{2i} [\langle \alpha | \hat{a} | \alpha \rangle e^{-i\omega t} - \langle \alpha | \hat{a}^\dagger | \alpha \rangle e^{i\omega t}] = \frac{1}{2i} [\alpha e^{-i\omega t} - \alpha^* e^{i\omega t}] \\ &= \frac{1}{2i} [(\alpha e^{-i\omega t}) - (\alpha e^{-i\omega t})^*] = \text{Im}(\alpha e^{-i\omega t}).\end{aligned}$$

The coherent state  $|\alpha e^{-i\omega t}\rangle$  is a complex wave that rotates with a rate given by  $\omega$ .  $|\alpha\rangle$  states the initial position in the  $\hat{X}_1\hat{X}_2$  phase space diagram.

$$\alpha e^{-i\omega t} = \langle \hat{X}_1 \rangle + i\langle \hat{X}_2 \rangle.$$

The uncertainties requires also the expectation value of the operator squared:

$$\begin{aligned}\langle \alpha | \hat{X}_1^2(t) | \alpha \rangle &= \frac{1}{4} \langle \alpha | (\hat{a}^2 e^{-i2\omega t} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2} e^{i2\omega t}) | \alpha \rangle = \frac{1}{4} [\alpha^2 e^{-i2\omega t} + 2|\alpha|^2 + 1 + \alpha^{*2} e^{i2\omega t}] \\ &= \frac{1}{4} [(\alpha e^{-i\omega t} + \alpha^* e^{i\omega t})^2 + 1] = \frac{1}{4} \{ [2\text{Re}(\alpha e^{-i\omega t})]^2 + 1 \}. \\ \langle \alpha | \hat{X}_2^2(t) | \alpha \rangle &= -\frac{1}{4} \langle \alpha | (\hat{a}^2 e^{-i2\omega t} - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2} e^{i2\omega t}) | \alpha \rangle = -\frac{1}{4} [\alpha^2 e^{-i2\omega t} - 2|\alpha|^2 - 1 + \alpha^{*2} e^{i2\omega t}] \\ &= \frac{1}{4} [1 - (\alpha e^{-i\omega t} - \alpha^* e^{i\omega t})^2] = \frac{1}{4} \{ 1 - [2i\text{Im}(\alpha e^{-i\omega t})]^2 \}.\end{aligned}$$

Uncertainty is

$$\begin{aligned}\Delta \hat{X}_1(t) &= \sqrt{\langle \hat{X}_1^2(t) \rangle - \langle \hat{X}_1(t) \rangle^2} = \sqrt{\frac{1}{4} \{ [2\text{Re}(\alpha e^{-i\omega t})]^2 + 1 \} - [\text{Re}(\alpha e^{-i\omega t})]^2} = \sqrt{\frac{1}{4}} = \frac{1}{2}. \\ \Delta \hat{X}_2(t) &= \sqrt{\langle \hat{X}_2^2(t) \rangle - \langle \hat{X}_2(t) \rangle^2} = \sqrt{\frac{1}{4} \{ 1 - [2i\text{Im}(\alpha e^{-i\omega t})]^2 \} - [\text{Im}(\alpha e^{-i\omega t})]^2} = \sqrt{\frac{1}{4}} = \frac{1}{2}.\end{aligned}$$

The diagram is shown below.

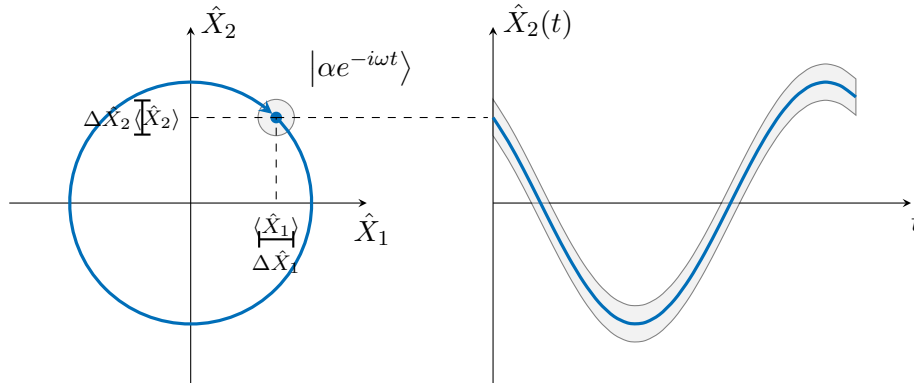


Figure 1: Phase space of the coherent state and time-dependent quadrature operators.

c) The process is analogous. We know that

$$\begin{aligned}\tilde{a} &= \hat{S}^\dagger \hat{D}^\dagger \hat{a} \hat{D} \hat{S} = \cosh(r) \hat{a} - e^{i\phi} \sinh(r) \hat{a}^\dagger + \alpha, \quad \text{and} \\ \tilde{a}^\dagger &= \hat{S}^\dagger \hat{D}^\dagger \hat{a}^\dagger \hat{D} \hat{S} = \cosh(r) \hat{a}^\dagger - e^{-i\phi} \sinh(r) \hat{a} + \alpha^*.\end{aligned}$$

The expected value of  $\hat{X}_1(t)$  is

$$\begin{aligned}\langle \hat{X}_1(t) \rangle &= \langle 0 | \hat{S}^\dagger \hat{D}^\dagger \frac{\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2} \hat{D} \hat{S} | 0 \rangle = \frac{1}{2} \langle 0 | \tilde{a} e^{-i\omega t} + \tilde{a}^\dagger e^{i\omega t} | 0 \rangle \\ &= \frac{1}{2} \langle 0 | \left[ \cosh(r) \hat{a} e^{-i\omega t} - e^{i\phi} \sinh(r) \hat{a}^\dagger e^{-i\omega t} + \alpha e^{-i\omega t} + \cosh(r) \hat{a}^\dagger e^{i\omega t} - e^{-i\phi} \sinh(r) \hat{a} e^{i\omega t} + \alpha^* e^{i\omega t} \right] | 0 \rangle \\ &= \frac{1}{2} [\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}] \\ \langle \hat{X}_1(t) \rangle &= \text{Re} (\alpha e^{-i\omega t}).\end{aligned}$$

Similarly,

$$\begin{aligned}\langle \hat{X}_2(t) \rangle &= \langle 0 | \hat{S}^\dagger \hat{D}^\dagger \frac{\hat{a} e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i} \hat{D} \hat{S} | 0 \rangle = \frac{1}{2i} \langle 0 | \tilde{a} e^{-i\omega t} + \tilde{a}^\dagger e^{i\omega t} | 0 \rangle \\ &= \frac{1}{2i} \langle 0 | \left[ \cosh(r) \hat{a} e^{-i\omega t} - e^{i\phi} \sinh(r) \hat{a}^\dagger e^{-i\omega t} + \alpha e^{-i\omega t} - \cosh(r) \hat{a}^\dagger e^{i\omega t} + e^{-i\phi} \sinh(r) \hat{a} e^{i\omega t} - \alpha^* e^{i\omega t} \right] | 0 \rangle \\ &= \frac{1}{2} [\alpha e^{-i\omega t} - \alpha^* e^{i\omega t}] \\ \langle \hat{X}_2(t) \rangle &= \text{Im} (\alpha e^{-i\omega t}).\end{aligned}$$

Doing the same for the operators squared:

$$\begin{aligned}\langle \hat{X}_1^2(t) \rangle &= \langle 0 | \hat{S}^\dagger \hat{D}^\dagger \left( \frac{\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2} \right)^2 \hat{D} \hat{S} | 0 \rangle \\ &= \frac{1}{4} \langle 0 | \hat{S}^\dagger \hat{D}^\dagger \left( \hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t} \right) \hat{D} \hat{S} \hat{S}^\dagger \hat{D}^\dagger \left( \hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t} \right) \hat{D} \hat{S} | 0 \rangle \\ &= \frac{1}{4} \langle 0 | (\tilde{a} e^{-i\omega t} + \tilde{a}^\dagger e^{i\omega t})^2 | 0 \rangle \\ &= \frac{1}{4} \left[ e^{-i2\omega t} \langle 0 | \tilde{a}^2 | 0 \rangle + e^{i2\omega t} \langle 0 | \tilde{a}^{\dagger 2} | 0 \rangle + \langle 0 | \tilde{a} \tilde{a}^\dagger | 0 \rangle + \langle 0 | \tilde{a}^\dagger \tilde{a} | 0 \rangle \right].\end{aligned}$$

We compute each term separately:

$$\begin{aligned}\langle 0 | \tilde{a}^2 | 0 \rangle &= \langle 0 | (\cosh(r) \hat{a} - e^{i\phi} \sinh(r) \hat{a}^\dagger + \alpha) (\cosh(r) \hat{a} - e^{i\phi} \sinh(r) \hat{a}^\dagger + \alpha) | 0 \rangle \\ &= \cosh^2(r) \langle \hat{a}^2 \rangle - e^{i\phi} \sinh(r) \cosh(r) \langle \hat{a} \hat{a}^\dagger \rangle + \alpha \cosh(r) \langle \hat{a} \rangle - e^{i\phi} \sinh(r) \cosh(r) \langle \hat{a}^\dagger \hat{a} \rangle \\ &\quad + e^{i2\phi} \sinh^2(r) \langle \hat{a}^{\dagger 2} \rangle - \alpha e^{i\phi} \sinh(r) \langle \hat{a}^\dagger \rangle + \alpha \cosh(r) \langle \hat{a} \rangle - \alpha e^{i\phi} \sinh(r) \langle \hat{a}^\dagger \rangle + \alpha^2 \\ &= \alpha^2 - e^{i\phi} \sinh(r) \cosh(r) \\ \langle 0 | \tilde{a}^{\dagger 2} | 0 \rangle &= \alpha^2 - \frac{1}{2} e^{i\phi} \sinh(2r).\end{aligned}$$

$$\begin{aligned}\langle 0 | \tilde{a} \tilde{a}^\dagger | 0 \rangle &= \langle 0 | (\cosh(r) \hat{a}^\dagger - e^{-i\phi} \sinh(r) \hat{a} + \alpha^*) (\cosh(r) \hat{a}^\dagger - e^{-i\phi} \sinh(r) \hat{a} + \alpha^*) | 0 \rangle \\ &= \cosh^2(r) \langle \hat{a}^{\dagger 2} \rangle - e^{-i\phi} \cosh(r) \sinh(r) \langle \hat{a}^\dagger \hat{a} \rangle + \alpha^* \cosh(r) \langle \hat{a}^\dagger \rangle - e^{-i\phi} \sinh(r) \cosh(r) \langle \hat{a} \hat{a}^\dagger \rangle \\ &\quad + e^{-i2\phi} \sinh^2(r) \langle \hat{a}^2 \rangle - \alpha^* e^{-i\phi} \sinh(r) \langle \hat{a} \rangle + \alpha^* \cosh(r) \langle \hat{a}^\dagger \rangle - \alpha^* e^{-i\phi} \sinh(r) \langle \hat{a}^\dagger \rangle + \alpha^{*2} \\ &= \alpha^{*2} - e^{-i\phi} \sinh(r) \cosh(r)\end{aligned}$$

$$\langle 0 | \tilde{a}^\dagger \tilde{a} | 0 \rangle = \alpha^{*2} - \frac{1}{2} e^{-i\phi} \sinh(2r).$$

$$\begin{aligned}\langle 0|\tilde{a}\tilde{a}^\dagger|0\rangle &= \langle 0|(\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha)(\cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*)|0\rangle \\ \langle 0|\tilde{a}\tilde{a}^\dagger|0\rangle &= \cosh^2(r) + |\alpha|^2.\end{aligned}$$

$$\begin{aligned}\langle 0|\tilde{a}^\dagger\tilde{a}|0\rangle &= \langle 0|(\cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*)(\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha)|0\rangle \\ \langle 0|\tilde{a}^\dagger\tilde{a}|0\rangle &= \sinh^2(r) + |\alpha|^2.\end{aligned}$$

Putting all together:

$$\begin{aligned}\langle \hat{X}_1^2(t) \rangle &= \frac{1}{4} \left\{ [\alpha^2 - \frac{1}{2}e^{i\phi}\sinh(2r)]e^{-i2\omega t} + [\alpha^{*2} - \frac{1}{2}e^{-i\phi}\sinh(2r)]e^{i2\omega t} + \cosh^2(r) + |\alpha|^2 + \sinh^2(r) + |\alpha|^2 \right\} \\ &= \frac{1}{4} [(\alpha e^{-i\omega t} + \alpha^* e^{i\omega t})^2 + \cosh(2r) - \sinh(2r) \cos(\phi - 2\omega t)] \\ \langle \hat{X}_1(t) \rangle &= [\text{Re}(\alpha e^{-i\omega t})]^2 + \frac{1}{4} [\cosh(2r) - \sinh(2r) \cos(\phi - 2\omega t)].\end{aligned}$$

The uncertainty is:

$$\Delta \hat{X}_1(t) = \sqrt{\langle \hat{X}_1^2(t) \rangle - \langle \hat{X}_1(t) \rangle^2} = \frac{1}{2} \sqrt{\cosh(2r) - \sinh(2r) \cos(\phi - 2\omega t)}.$$

Similarly,

$$\begin{aligned}\langle \hat{X}_2^2(t) \rangle &= \langle 0|\hat{S}^\dagger \hat{D}^\dagger \left( \frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i} \right)^2 \hat{D} \hat{S}|0\rangle \\ &= -\frac{1}{4} \langle 0|\hat{S}^\dagger \hat{D}^\dagger (\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}) \hat{D} \hat{S} \hat{S}^\dagger \hat{D}^\dagger (\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}) \hat{D} \hat{S}|0\rangle \\ &= -\frac{1}{4} \langle 0|(\tilde{a}e^{-i\omega t} - \tilde{a}^\dagger e^{i\omega t})^2|0\rangle \\ &= -\frac{1}{4} [e^{-i2\omega t} \langle 0|\tilde{a}^2|0\rangle + e^{i2\omega t} \langle 0|\tilde{a}^{\dagger 2}|0\rangle - \langle 0|\tilde{a}\tilde{a}^\dagger|0\rangle - \langle 0|\tilde{a}^\dagger\tilde{a}|0\rangle].\end{aligned}$$

We use the previous results to get

$$\begin{aligned}\langle \hat{X}_2^2(t) \rangle &= -\frac{1}{4} \left\{ [\alpha^2 - \frac{1}{2}e^{i\phi}\sinh(2r)]e^{-i2\omega t} + [\alpha^{*2} - \frac{1}{2}e^{-i\phi}\sinh(2r)]e^{i2\omega t} - \cosh^2(r) - |\alpha|^2 - \sinh^2(r) - |\alpha|^2 \right\} \\ &= -\frac{1}{4} [(\alpha e^{-i\omega t} - \alpha^* e^{i\omega t})^2 - \cosh(2r) - \sinh(2r) \cos(\phi - 2\omega t)] \\ \langle \hat{X}_2(t) \rangle &= [\text{Im}(\alpha e^{-i\omega t})]^2 + \frac{1}{4} [\cosh(2r) + \sinh(2r) \cos(\phi - 2\omega t)].\end{aligned}$$

And the uncertainty is:

$$\Delta \hat{X}_2(t) = \sqrt{\langle \hat{X}_2^2(t) \rangle - \langle \hat{X}_2(t) \rangle^2} = \frac{1}{2} \sqrt{\cosh(2r) + \sinh(2r) \cos(\phi - 2\omega t)}$$

We can verify that  $\Delta \hat{X}_1(0) = e^{-r}/2$  and  $\Delta \hat{X}_2(0) = e^r/2$  when  $\phi = t = 0$ .  
The phase space representation is shown below.

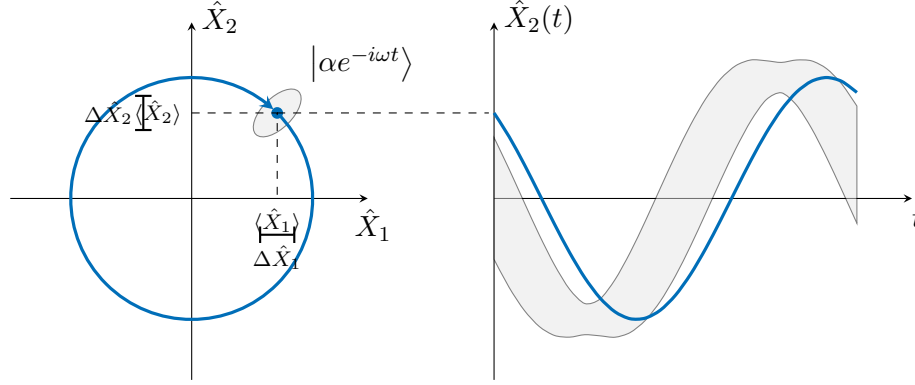


Figure 2: Phase space of the coherent state and time-dependent quadrature operators.

## Exercise 2

In general, the photon number for output  $i$  is  $\hat{N}_i = \hat{b}_i^\dagger \hat{b}_i$ . For each output we have:

$$\begin{aligned}\hat{N}_1 &= \frac{1}{2}(\hat{a}_1^\dagger + \hat{a}_2^\dagger)(\hat{a}_1 + \hat{a}_2) = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2), \\ \hat{N}_2 &= \frac{1}{2}(\hat{a}_1^\dagger - \hat{a}_2^\dagger)(\hat{a}_1 - \hat{a}_2) = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2).\end{aligned}$$

Substraction gives

$$\hat{N}_1 - \hat{N}_2 = \frac{1}{2} \left[ (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) - (\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) \right] = \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1.$$

With the phase shift, the input 2 is:

$$\alpha \rightarrow \alpha e^{i\phi_\alpha}, \quad \text{and} \quad \hat{a}_2 \approx \alpha e^{i\phi_\alpha}.$$

The number photon difference is then:

$$\hat{N}_1 - \hat{N}_2 = \hat{a}_1^\dagger \alpha e^{i\phi_\alpha} + \hat{a}_1 \alpha^* e^{-i\phi_\alpha}.$$

If  $\alpha = |\alpha|$ , then

$$\hat{N}_1 - \hat{N}_2 = |\alpha| \left( \hat{a}_1^\dagger e^{i\phi_\alpha} + \hat{a}_1 e^{-i\phi_\alpha} \right) = 2|\alpha| \hat{X}_{\phi_\alpha}, \quad \hat{X}_{\phi_\alpha} = \frac{\hat{a}_1^\dagger e^{i\phi_\alpha} + \hat{a}_1 e^{-i\phi_\alpha}}{2},$$

where  $\hat{X}_{\phi_\alpha}$  is the quadrature of mode 1 modulated by the phase shift  $\phi_\alpha$ .

To get the uncertainty  $\Delta(\hat{N}_1 - \hat{N}_2)$ , we compute the expected value of the difference and the expected value of the difference squared. We can use the results from the previous exercise as the input 1 is a squeezed state of the vacuum.

$$\begin{aligned}{}_1 \langle \xi, 0 |_2 \langle \alpha | (\hat{N}_1 - \hat{N}_2) | \alpha \rangle_2 | \xi, 0 \rangle_1 &= 2|\alpha| \langle \hat{X}_{\phi_\alpha} \rangle = 0, \\ {}_1 \langle \xi, 0 |_2 \langle \alpha | (\hat{N}_1 - \hat{N}_2)^2 | \alpha \rangle_2 | \xi, 0 \rangle_1 &= 4|\alpha|^2 \langle \hat{X}_{\phi_\alpha}^2 \rangle = |\alpha|^2 [\cosh(2r) - \sinh(2r) \cos(\theta - 2\phi_\alpha)],\end{aligned}$$

where  $\xi = r e^{i\theta}$  is the squeezing factor, and  $\phi_\alpha$  the phase shift imposed by the input 2. Uncertainty is therefore:

$$\Delta(\hat{N}_1 - \hat{N}_2) = |\alpha| \sqrt{\cosh(2r) - \sinh(2r) \cos(\theta - 2\phi_\alpha)}.$$

### Exercise 3

(a) For the 50 : 50 beamsplitter 1, we have:

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix}, \quad \text{with} \quad \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix}.$$

(b) From the last part, we know that

$$\hat{a}_1 = \frac{1}{\sqrt{2}}(\hat{b}_1 + \hat{b}_2) \xrightarrow{h.c.} \hat{a}_1^\dagger = \frac{1}{\sqrt{2}}(\hat{b}_1^\dagger + \hat{b}_2^\dagger), \quad \text{and} \quad \hat{a}_2 = \frac{1}{\sqrt{2}}(\hat{b}_1 - \hat{b}_2) \xrightarrow{h.c.} \hat{a}_2^\dagger = \frac{1}{\sqrt{2}}(\hat{b}_1^\dagger - \hat{b}_2^\dagger).$$

The input is then:

$$\text{Input} = |1\rangle_1 |0\rangle_2 = \hat{a}_1^\dagger |0\rangle_1 |0\rangle_2.$$

Output is:

$$\text{Output} = \frac{1}{\sqrt{2}}(\hat{b}_1^\dagger + \hat{b}_2^\dagger) |0\rangle_1 |0\rangle_2 = \frac{1}{\sqrt{2}} [|1\rangle_{1'} |0\rangle_{2'} + |0\rangle_{1'} |1\rangle_{2'}].$$

We expect then to have the same probability of getting one photon at each output.

The expected value at output 1 is:

$$\begin{aligned} {}_2 \langle 0|_1 \langle 1| \hat{b}_1^\dagger \hat{b}_1 |1\rangle_1 |0\rangle_2 &= \frac{1}{2} \langle 0, 1 | (\hat{a}_1^\dagger + \hat{a}_2^\dagger) (\hat{a}_1 + \hat{a}_2) |1, 0\rangle \\ &= \frac{1}{2} \langle 0, 1 | (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) |1, 0\rangle \\ {}_2 \langle 0|_1 \langle 1| \hat{b}_1^\dagger \hat{b}_1 |1\rangle_1 |0\rangle_2 &= \frac{1}{2}. \end{aligned}$$

Likewise, the expected value at output 2 is:

$$\begin{aligned} {}_2 \langle 0|_1 \langle 1| \hat{b}_2^\dagger \hat{b}_2 |1\rangle_1 |0\rangle_2 &= \frac{1}{2} \langle 0, 1 | (\hat{a}_1^\dagger - \hat{a}_2^\dagger) (\hat{a}_1 - \hat{a}_2) |1, 0\rangle \\ &= \frac{1}{2} \langle 0, 1 | (\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) |1, 0\rangle \\ {}_2 \langle 0|_1 \langle 1| \hat{b}_2^\dagger \hat{b}_2 |1\rangle_1 |0\rangle_2 &= \frac{1}{2}. \end{aligned}$$

(c) We replace the first output as the input of the second output:

$$\begin{aligned} \hat{c}_1 &= \frac{1}{\sqrt{2}}[\hat{b}_1 e^{i\phi} + \hat{b}_2] = \frac{1}{2} [(\hat{a}_1 + \hat{a}_2) e^{i\phi} + (\hat{a}_1 - \hat{a}_2)] = \frac{1}{2} [\hat{a}_1(1 + e^{i\phi}) + \hat{a}_2(e^{i\phi} - 1)], \\ \hat{c}_2 &= \frac{1}{\sqrt{2}}[\hat{b}_1 e^{i\phi} - \hat{b}_2] = \frac{1}{2} [(\hat{a}_1 + \hat{a}_2) e^{i\phi} - (\hat{a}_1 - \hat{a}_2)] = \frac{1}{2} [\hat{a}_1(e^{i\phi} - 1) + \hat{a}_2(1 + e^{i\phi})]. \end{aligned}$$

(d) The expected number of photons at detector 1 is:

$$\begin{aligned} \langle 0, 1 | \hat{c}_1^\dagger \hat{c}_1 |1, 0\rangle &= \frac{1}{4} \left\{ [\hat{a}_1^\dagger(1 + e^{-i\phi}) + \hat{a}_2^\dagger(e^{-i\phi} - 1)] [\hat{a}_1(1 + e^{i\phi}) + \hat{a}_2(e^{i\phi} - 1)] \right\} \\ &= \frac{1}{4} \langle 0, 1 | \left\{ \hat{a}_1^\dagger \hat{a}_1(\dots) + \hat{a}_1^\dagger \hat{a}_2(\dots) + \hat{a}_2^\dagger \hat{a}_1(\dots) + \hat{a}_2^\dagger \hat{a}_2(\dots) \right\} |1, 0\rangle \\ &= \frac{1}{4} (1 + e^{-i\phi})(1 + e^{i\phi}) \\ \langle 0, 1 | \hat{c}_1^\dagger \hat{c}_1 |1, 0\rangle &= \frac{1}{2} [1 + \cos \phi] = \cos^2 \frac{\phi}{2}. \end{aligned}$$

By conservation of energy (or probability), the expected number of photon at detector 2 is:

$$\langle 0, 1 | \hat{c}_2^\dagger \hat{c}_2 | 1, 0 \rangle = \sin^2 \frac{\phi}{2}.$$