Assignment 2

OPTI 544 Quantum Optics

University of Arizona

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Exercise 1

a) The quadrature operators are:

$$\hat{X}_1 = \frac{\hat{a} + \hat{a}^\dagger}{2}, \quad \text{and} \quad \hat{X}_2 = \frac{\hat{a} - \hat{a}^\dagger}{2i}.$$

We can use them to express $\hat{a}, \hat{a}^\dagger$ in terms of $\hat{X}_{1,2}$.

$$\hat{a} + \hat{a}^\dagger = 2\hat{X}_1 \tag{1}$$
$$\hat{a} - \hat{a}^\dagger = 2i\hat{X}_2. \tag{2}$$

Then,

$$(1) + (2): \quad 2\hat{a} = 2(\hat{X}_1 + i\hat{X}_2) \longrightarrow \hat{a} = \hat{X}_1(0) + i\hat{X}_2(0)$$
$$(1) - (2): \quad 2\hat{a}^\dagger = 2(\hat{X}_1 - i\hat{X}_2) \longrightarrow \hat{a}^\dagger = \hat{X}_1(0) - i\hat{X}_2(0)$$

Using these in the time-evolved quadrature operators:

$$\hat{X}_1(t) = \frac{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2} = \frac{[\hat{X}_1(0) + i\hat{X}_2(0)]e^{-i\omega t} + [\hat{X}_1(0) - i\hat{X}_2(0)]e^{i\omega t}}{2}$$
$$\hat{X}_1(t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}\hat{X}_1(0) + \frac{(e^{i\omega t} - e^{-i\omega t})}{2i}\hat{X}_2(0) = \cos(\omega t)\hat{X}_1(0) + \sin(\omega t)\hat{X}_2(0).$$

Likewise,

$$\hat{X}_2(t) = \frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i} = \frac{[\hat{X}_1(0) + i\hat{X}_2(0)]e^{-i\omega t} - [\hat{X}_1(0) - i\hat{X}_2(0)]e^{i\omega t}}{2i}$$
$$\hat{X}_2(t) = -\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\hat{X}_1(0) + i\frac{(e^{i\omega t} + e^{-i\omega t})}{2i}\hat{X}_2(0) = -\sin(\omega t)\hat{X}_1(0) + \cos(\omega t)\hat{X}_2(0).$$

The commutation relation of the time-evolved quadrature operators is

$$[\hat{X}_1, \hat{X}_2] = \frac{1}{4i}[\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}, \hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}] = \frac{1}{4i}\left\{-[\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}]\right\} = \frac{1}{4i}(-2) = \frac{i}{2}.$$

b) The expectation value is

$$
\begin{aligned}
\langle\alpha|\hat{X}_1(t)|\alpha\rangle &= \langle\alpha|\frac{\hat{a}e^{-i\omega t}+\hat{a}^\dagger e^{i\omega t}}{2}|\alpha\rangle = \frac{1}{2}\left[\langle\alpha|\hat{a}|\alpha\rangle e^{-i\omega t}+\langle\alpha|\hat{a}^\dagger|\alpha\rangle e^{i\omega t}\right] = \frac{1}{2}\left[\alpha e^{-i\omega t}+\alpha^* e^{i\omega t}\right] \\
&= \frac{1}{2}\left[(\alpha e^{-i\omega t})+(\alpha e^{-i\omega t})^*\right] = \mathrm{Re}\left(\alpha e^{-i\omega t}\right). \\
\langle\alpha|\hat{X}_2(t)|\alpha\rangle &= \langle\alpha|\frac{\hat{a}e^{-i\omega t}-\hat{a}^\dagger e^{i\omega t}}{2i}|\alpha\rangle = \frac{1}{2i}\left[\langle\alpha|\hat{a}|\alpha\rangle e^{-i\omega t}-\langle\alpha|\hat{a}^\dagger|\alpha\rangle e^{i\omega t}\right] = \frac{1}{2i}\left[\alpha e^{-i\omega t}-\alpha^* e^{i\omega t}\right] \\
&= \frac{1}{2i}\left[(\alpha e^{-i\omega t})-(\alpha e^{-i\omega t})^*\right] = \mathrm{Im}\left(\alpha e^{-i\omega t}\right).
\end{aligned}
$$

The coherent state $\left|\alpha e^{-i\omega t}\right\rangle$ is a complex wave that rotates with a rate given by ω. $|\alpha\rangle$ states the initial position in the $\hat{X}_1\hat{X}_2$ phase space diagram.

$$
\alpha e^{-i\omega t} = \langle\hat{X}_1\rangle + i\langle\hat{X}_2\rangle.
$$

The uncertainties requires also the expectation value of the operator squared:

$$
\begin{aligned}
\langle\alpha|\hat{X}_1^2(t)|\alpha\rangle &= \frac{1}{4}\langle\alpha|(\hat{a}^2e^{-i2\omega t}+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger 2}e^{i2\omega t})|\alpha\rangle = \frac{1}{4}\left[\alpha^2e^{-i2\omega t}+2|\alpha|^2+1+\alpha^{*2}e^{i2\omega t}\right] \\
&= \frac{1}{4}\left[(\alpha e^{-i\omega t}+\alpha^* e^{i\omega t})^2+1\right] = \frac{1}{4}\left\{[2\mathrm{Re}\left(\alpha e^{-i\omega t}\right)]^2+1\right\}. \\
\langle\alpha|\hat{X}_2^2(t)|\alpha\rangle &= -\frac{1}{4}\langle\alpha|(\hat{a}^2e^{-i2\omega t}-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger 2}e^{i2\omega t})|\alpha\rangle = -\frac{1}{4}\left[\alpha^2e^{-i2\omega t}-2|\alpha|^2-1+\alpha^{*2}e^{i2\omega t}\right] \\
&= \frac{1}{4}\left[1-(\alpha e^{-i\omega t}-\alpha^* e^{i\omega t})^2\right] = \frac{1}{4}\left\{1-[2i\mathrm{Im}\left(\alpha e^{-i\omega t}\right)]^2\right\}.
\end{aligned}
$$

Uncertainty is

$$
\begin{aligned}
\Delta\hat{X}_1(t) &= \sqrt{\langle\hat{X}_1^2(t)\rangle-\langle\hat{X}_1(t)\rangle^2} = \sqrt{\frac{1}{4}\{[2\mathrm{Re}\,(\alpha e^{-i\omega t})]^2+1\}-[\mathrm{Re}\,(\alpha e^{-i\omega t})]^2} = \sqrt{\frac{1}{4}} = \frac{1}{2}. \\
\Delta\hat{X}_2(t) &= \sqrt{\langle\hat{X}_2^2(t)\rangle-\langle\hat{X}_2(t)\rangle^2} = \sqrt{\frac{1}{4}\{1-[2i\mathrm{Im}\,(\alpha e^{-i\omega t})]^2\}-[\mathrm{Im}\,(\alpha e^{-i\omega t})]^2} = \sqrt{\frac{1}{4}} = \frac{1}{2}.
\end{aligned}
$$

The diagram is shown below.

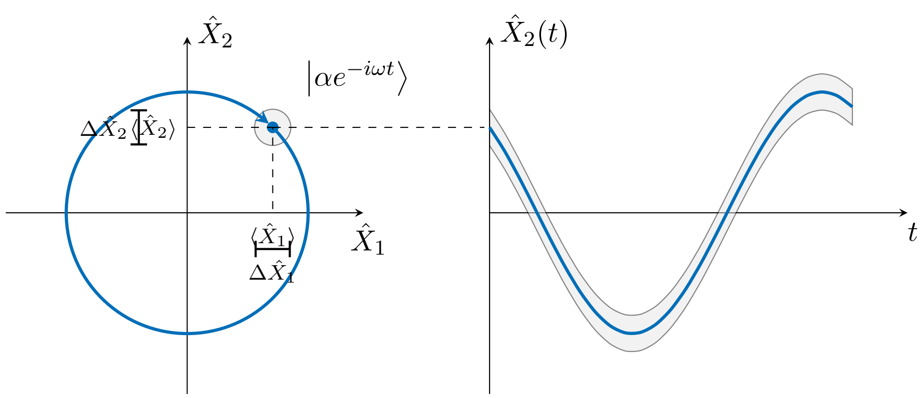

Figure 1: Phase space of the coherent state and time-dependent quadrature operators.

c) The process is analogous. We have the following operators:

$$\tilde{a}(t) = \cosh(r)\hat{a}e^{-i\omega t} - e^{-i\phi}\sinh(r)\hat{a}^\dagger e^{i\omega t} + \alpha, \quad \text{and} \quad \tilde{a}^\dagger(t) = \cosh(r)\hat{a}^\dagger e^{i\omega t} - e^{-i\phi}\sinh(r)\hat{a}e^{-i\omega t} + \alpha^*$$

In this case, $\phi = 0$. The expected value of $\hat{X}_1$ and $\hat{X}_2$ were calculated in the lectures so I list them here with the time inclusion:

$$\langle \xi, \alpha|\hat{X}_1|\xi, \alpha\rangle = \mathrm{Re}\left(\alpha e^{-i\omega t}\right), \quad \langle \xi, \alpha|\hat{X}_2|\xi, \alpha\rangle = \mathrm{Im}\left(\alpha e^{i\omega t}\right).$$

The expectation of the operators squares is computed now:

$$\begin{aligned}\langle \xi, \alpha|\hat{X}_1^2(t)|\xi, \alpha\rangle &= \frac{1}{4}\langle 0|S^\dagger D^\dagger(\hat{a}^2 e^{-i2\omega t} + 2\hat{a}^\dagger\hat{a} + 1 + \hat{a}^{\dagger 2}e^{i2\omega t})DS|0\rangle \\ &= \frac{1}{4}\left[\langle 0|S^\dagger D^\dagger \hat{a}^2 DS|0\rangle e^{-i2\omega t} + 2\langle 0|S^\dagger D^\dagger \hat{a}^\dagger \hat{a} DS|0\rangle + 1 + \langle 0|S^\dagger D^\dagger \hat{a}^{\dagger 2} DS|0\rangle e^{i2\omega t}\right]\end{aligned}$$

$$\hat{X}_1(t) = \frac{1}{2}\left[\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right], \quad \hat{X}_1^2(t) = \frac{1}{4}\left[\hat{a}^2 e^{-i2\omega t} + \hat{a}^{\dagger 2}e^{i2\omega t} + 2\hat{a}^\dagger\hat{a} + 1\right].$$

The expectation is:

$$\begin{aligned}\langle \hat{X}_1(t)\rangle &= \frac{1}{2}\langle 0|S^\dagger D^\dagger\left[\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right]DS|0\rangle = \frac{1}{2}\left[\langle 0|S^\dagger D^\dagger \hat{a} DS|0\rangle e^{-i\omega t} + \langle 0|S^\dagger D^\dagger \hat{a}^\dagger DS|0\rangle e^{i\omega t}\right] \\ &= \frac{1}{2}\left[[\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a} + \alpha]e^{-i\omega t} + [\cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*]e^{i\omega t}\right]\end{aligned}$$

$$\begin{aligned}\langle \hat{X}_1^2(t)\rangle &= \frac{1}{4}\langle 0|S^\dagger D^\dagger[\hat{a}^2 e^{-i2\omega t} + \hat{a}^{\dagger 2}e^{i2\omega t} + 2\hat{a}^\dagger\hat{a} + 1]DS|0\rangle \\ &= \frac{1}{4}\left[\langle 0|S^\dagger D^\dagger \hat{a}^2 DS|0\rangle e^{-i2\omega t} + \langle 0|S^\dagger D^\dagger \hat{a}^{\dagger 2} DS|0\rangle e^{i2\omega t} + 2\langle 0|S^\dagger D^\dagger \hat{a}^\dagger\hat{a} DS|0\rangle + \langle 0|S^\dagger D^\dagger DS|0\rangle\right] \\ &= \frac{1}{4}\left[\cosh(r)(\hat{a} + \alpha)^2 - e^{i\phi}\sinh(r)(\hat{a}^\dagger + \alpha^*)^2 + \alpha\right]\end{aligned}$$

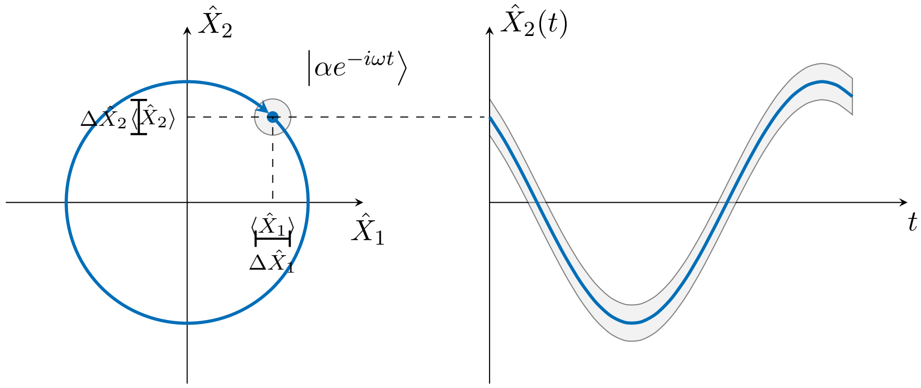

Figure 2: Phase space of the coherent state and time-dependent quadrature operators.

References

[1] Claude Cohen-Tannoudji, Bernard Diu, and Frank Laloe. Quantum mechanics, volume 1. *Quantum Mechanics*, 1:898, 1986.