Notes of Quantum Optics

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Preface

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1.2.1 Fields in a cavity

Lets consider the following one-dimensional problem, where a cavity of length L is oriented along the z-axis.

A linear polarized E-field is assumed, the medium is free space, perfect conducting walls and there is no free charges nor free current. The scheme is shown in figure 1.1.

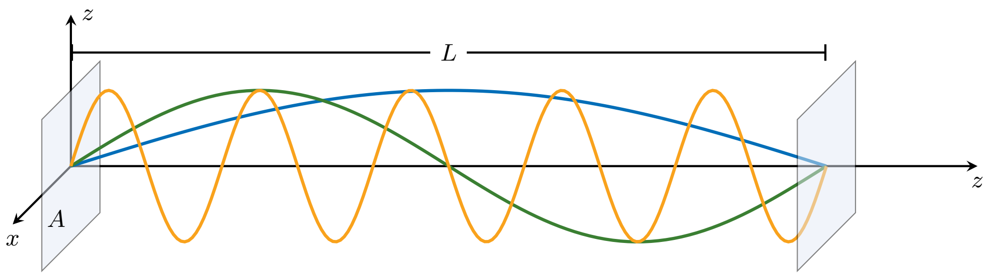

Figure 1.1 One-dimensional cavity problem. Perfect conducting walls.

Our goal is to find the E- and B-field inside the cavity. Maxwell's equations in this case are:

$$\text{Maxwell's equations with free sources} \quad \begin{cases} \nabla \cdot \boldsymbol{E} = 0 \\ \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B} \\ \nabla \times \boldsymbol{B} = \frac{1}{c^2}\partial_t \boldsymbol{E} \end{cases} \tag{1.1}$$

The E-field will be assumed to be $\boldsymbol{E}(z,t) = \boldsymbol{e}E(z,t)$, where $\boldsymbol{e}$ is the polarization vector. Because fields depends only on z, $\nabla = \hat{\boldsymbol{z}}\partial_z$. First Maxwell equation yields:

$$\nabla \cdot \boldsymbol{E} = \partial_z(\hat{\boldsymbol{z}} \cdot \boldsymbol{E}) = \partial_z(\hat{\boldsymbol{z}} \cdot \boldsymbol{e}E) = 0 \Longrightarrow \boldsymbol{e} \cdot \hat{\boldsymbol{z}} = 0.$$

This implies that the polarization vector must be unitary in the transverse plane:

$$\boldsymbol{e} = \cos\phi\hat{\boldsymbol{x}} + \sin\phi\hat{\boldsymbol{y}}.$$

Third Maxwell's equation yields

$$\nabla \times \boldsymbol{E} = (\hat{\boldsymbol{z}}\partial_z) \times (\boldsymbol{e}E) = (\hat{\boldsymbol{z}} \times \boldsymbol{e})\partial_z E = -\partial_t \boldsymbol{B}.$$

Taking the curl of this equation, using Fourth Maxwell's equation and vector identities:

$$\begin{aligned} \nabla \times (\nabla \times \boldsymbol{E}) &= -\partial_t(\nabla \times \boldsymbol{B}) \\ \nabla\cancelto{0}{(\nabla \cdot \boldsymbol{E})} - \nabla^2 \boldsymbol{E} &= -\frac{1}{c^2}\partial_t^2 \boldsymbol{E} \\ -\boldsymbol{e}\partial_z^2 E &= -\boldsymbol{e}\frac{1}{c^2}\partial_t^2 E. \end{aligned}$$

From here, we have the E-field wave equation for this particular problem.

$$\partial_z^2 E - \frac{1}{c^2}\partial_t^2 E = 0. \tag{1.2}$$

Before solving this equation, we need the boundary condition set by the PEC condition. We need to $\hat{\boldsymbol{n}} \times \boldsymbol{E} = 0$ on the surface. Because the normal surface unit vector is $\hat{\boldsymbol{n}} = \pm\hat{\boldsymbol{z}}$, we have

$$\text{Boundary condition} \qquad \hat{\boldsymbol{n}} \times \boldsymbol{E} = \hat{\boldsymbol{z}} \times (\boldsymbol{e}E) = 0 \Longrightarrow E(z=0,t) = E(z=L,t) = 0.$$

In order to solve the PDE, we assum a product form $E(z,t) = Z(z)q(t)$. Then, by replacing in it in (1.2):

$$\begin{aligned}
\partial_z^2[Z(z)q(t)] - \frac{1}{c^2}\partial_t^2[Z(z)q(t)] &= 0 \\
Z''(z)q(t) - \frac{1}{c^2}Z(z)\ddot{q}(t) &= \Big/ [Z(z)q(t)]^{-1} \\
\frac{Z''(z)}{Z(z)} &= \frac{1}{c^2}\frac{\ddot{q}(z)}{q(z)}.
\end{aligned}$$

Left side depends only on z, while the right side only on y. The only way this can be true is if both are a constant, say, $-k^2$. Then,

$$\text{Spatial and temporal differential equations} \qquad \begin{cases} \dfrac{Z''}{Z} = -k^2 \longrightarrow Z'' + k^2 Z = 0 \\ \dfrac{1}{c^2}\dfrac{\ddot{T}}{T} = -k^2 \longrightarrow \ddot{q} + \omega^2 q(t) = 0 \end{cases}.$$

For the spatial ODE, we assume a solution of the form

$$Z(z) = A\sin(kz) + B\cos(kz), \quad Z(0) = Z(L) = 0.$$

Setting the boundaries:

$$\begin{aligned}
Z(0) &= A\sin(0) + B\cos(0) = 0 \Longrightarrow B = 0 \\
Z(L) &= A\sin(kL) = 0 \Longrightarrow k_m = \frac{m\pi}{L}, \quad m \in \mathbb{N}.
\end{aligned}$$

We left the temporal ODE unsolved. Finally, putting all together yields the initial E-field:

$$\boldsymbol{E}_{\boldsymbol{k},\lambda}(z,t) = \boldsymbol{e}_\lambda \sqrt{\frac{2\omega^2}{V\varepsilon_0}}\, q_{\boldsymbol{k},\lambda}(t)\sin(kz), \quad k_m = \frac{m\pi}{L}, \quad \omega = ck.$$

We have includnig the subscript λ and $\boldsymbol{k}$ to consider multiple mode $\boldsymbol{k}$ varied with m and λ. Also, the coefficient in red is for better results in the future.

Using Faraday's law:

$$\boldsymbol{B}(z,t) = (\boldsymbol{k} \times \boldsymbol{e}_\lambda)\frac{1}{kc^2}\sqrt{\frac{2\omega^2}{V\varepsilon_0}}\, \dot{q}_{\boldsymbol{k},\lambda}(t)\sin(kz).$$

The term $\dot{q}(t)$ will play the role of a canonical momentum for a particle of unit mass, $p(t) = \dot{q}(t)$.

1.2.2 Single-mode Hamiltonian

The classical field energy, or Hamiltonian H, of the single-mode field is given by

$$
\begin{aligned}
H &= \frac{1}{2}\int dV \left[\varepsilon_0 \boldsymbol{E}^2 + \frac{1}{\mu_0}\boldsymbol{B}^2\right] \\
&= \frac{1}{2}A\int_0^L dz \left[\frac{2\omega^2}{V\varepsilon_0} q^2(t)\sin^2(kz) + \frac{1}{\mu_0}\frac{1}{k^2c^4}\frac{2\omega^2}{V\varepsilon_0} p^2(t)\cos^2(kz)\right] \\
&= \frac{A}{V}\int_0^L dz \left[\omega^2 q^2(t)\sin^2(kz) + p^2(t)\cos^2(kz)\right] \\
&= \frac{1}{L}\left[\omega^2 q^2(t)\frac{L}{2} + p^2(t)\frac{L}{2}\right] \\
H &= \frac{1}{2}\left[\omega^2 q^2(t) + p^2(t)\right].
\end{aligned}
$$

It is apparent that a single-mode field if formally equivalent to a harmonic quantim oscillator of unit mass, where the E- and B-fields play the roles of canonical position and momentum. To begin the quantization, we make q, p operators $\hat{q}, \hat{p}$, which needs to satisfy the canonical commutation relations

$$[\hat{q}, \hat{p}] = i\hbar. \tag{1.3}$$

The EM fields with the operators are:

$$
\text{Quantized EM fields} \qquad
\begin{aligned}
\hat{\boldsymbol{E}}_{\boldsymbol{k},\lambda}(z,t) &= \boldsymbol{e}_{\boldsymbol{k},\lambda}\sqrt{\frac{2\omega^2}{V\varepsilon_0}}\hat{q}(t)\sin(kz) \\
\hat{\boldsymbol{B}}_{\boldsymbol{k},\lambda}(z,t) &= (\boldsymbol{k}\times\boldsymbol{e}_{\boldsymbol{k},\lambda})\frac{1}{kc^2}\sqrt{\frac{2\omega^2}{V\varepsilon_0}}\hat{p}(t)\cos(kz)
\end{aligned}
. \tag{1.4}
$$

1.3 Section Two

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Mathematics

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