Homework 1

Foundations of Quantum Optics

OPTI 544, Spring 2026

1. **Review of Quantum Harmonic Oscillators:** The quantum harmonic oscillator is one of the most important quantum systems in quantum physics, and as we have seen, quantized EM field is a collection of quantum harmonic oscillators. The purpose of this problem is to overview the quantization of energy for a quantum harmonic oscillator, following Section 2.3 from the "Modern Quantum Mechanics" textbook by J. J. Sakurai. To quantize a simple harmonic oscillator, we assume only two things:

   (a) The position and momentum of the harmonic oscillator, $\hat{x}$ and $\hat{p}$, are *quantum operators*, or *observables*.

   (b) These operators satisfy the *canonical commutation relation*: $[\hat{x}, \hat{p}] = i\hbar$.

   Show that the above two conditions imply that the energy levels of the quantum harmonic oscillator are discrete, or *quantized* with eigenvalues $\hbar\omega\,(n + 1/2)$, with n as an integer. You can follow this derivation from Sakurai Eq. 2.3.6 to Eq. 2.3.19. This should establish that imposing a commutation relation on certain observables, we arrive at the fact that an oscillator's energy can only take specific values.

2. **Classical electromagnetic (EM) field in a cavity:** Consider the Maxwell equations in the absence of charges and currents describing the classical electromagnetic

(EM) field:

$$\nabla \cdot \mathbf{E} = 0 \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{4}$$

(a) Defining the vector potential as (no charges, therefore the scalar potential $\phi = 0$)

$$\mathbf{B} = \nabla \times \mathbf{A}; \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \tag{5}$$

and working in the Coulomb gauge where $\nabla \cdot \mathbf{A} = 0$, show that the vector potential follows the wave equation:

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \tag{6}$$

The above wave equation can be solved for different boundary conditions.

(b) Show that the solution of the wave equation for the vector potential for an optical cavity, with perfect mirrors at $z = 0$ and $z = L$, is given by:

$$\mathbf{A}(z,t) = \sum_{\mathbf{k},\lambda} \mathbf{e}_{\mathbf{k}\lambda} A_{\mathbf{k}\lambda} e^{-i\omega t} \sin(kz) + c.c., \tag{7}$$

where $\omega = |\mathbf{k}|\, c$.

(c) Consider one specific mode with wavevector $\mathbf{k}$ and polarization index λ such that $\mathbf{e}_{\mathbf{k}\lambda} = \mathbf{x}$. For this single mode of the total field, determine the corresponding electric and magnetic fields, $\mathbf{E}_{\mathbf{k}\lambda}$ and $\mathbf{B}_{\mathbf{k}\lambda}$, respectively (use Eq.(5)).

(d) From the single mode electric and magnetic fields, obtain the energy contained in the mode inside the cavity volume, using:

$$H_\omega = \int \mathrm{d}V \left[\frac{1}{2}\epsilon_0 \mathbf{E}^2_{\mathbf{k}\lambda} + \frac{1}{2\mu_0}\mathbf{B}^2_{\mathbf{k}\lambda}\right]. \tag{8}$$

(e) If this energy is set equal to $\hbar\omega$, what is the normalization constant $A_{\mathbf{k}\lambda}$? What are the corresponding electric and magnetic fields?

3. **Coherent states:** The coherent state of the quantized EM field can be defined as the displaced vacuum state:

$$\hat{\mathcal{D}}(\alpha)\,|0\rangle = |\alpha\rangle\,, \tag{9}$$

with $\hat{\mathcal{D}}(\alpha) \equiv e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$ as the displacement operator.

(a) Using this definition of coherent states determine the displaced annihilation and creation operators $\tilde{a} \equiv \hat{\mathcal{D}}^\dagger(\alpha)\hat{a}\hat{\mathcal{D}}(\alpha)$ and $\tilde{a}^\dagger \equiv \hat{\mathcal{D}}^\dagger(\alpha)\hat{a}^\dagger\hat{\mathcal{D}}(\alpha)$.

(b) Using the above displaced operators, determine the expectation values and uncertainties of the quadrature operators for a coherent state:

$$\hat{X}_1 \equiv \frac{\hat{a} + \hat{a}^\dagger}{2}, \tag{10}$$

$$\hat{X}_2 \equiv \frac{\hat{a} - \hat{a}^\dagger}{2i} \tag{11}$$

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