

# Assignment 1

## OPTI 544 Quantum Optics

### University of Arizona

Nicolás Hernández Alegria

February 3, 2026

## Exercise 1

The Hamiltonian in position space of the quantum harmonic oscillator is:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \quad [\hat{x}, \hat{p}] = i\hbar.$$

Defining the *ladder operators*

$$\begin{aligned}\hat{a} &= \frac{1}{\sqrt{2}} \left( \frac{\hat{x}}{\sigma} + i \frac{\sigma \hat{p}}{\hbar} \right) & \hat{x} &= \frac{\sigma}{\sqrt{2}}(\hat{a}^\dagger + \hat{a}) \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2}} \left( \frac{\hat{x}}{\sigma} - i \frac{\sigma \hat{p}}{\hbar} \right) & \text{with} & \hat{p} = \frac{i\hbar}{\sigma\sqrt{2}}(\hat{a}^\dagger - \hat{a})\end{aligned}$$

, with  $\sigma$  the oscillation length

$$\sigma = \sqrt{\frac{\hbar}{m\omega}},$$

allow us to express the Hamiltonian in a simpler form. First, we notice that the commutation of  $\hat{x}$  and  $\hat{p}$  is inherited for  $\hat{a}$  and  $\hat{a}^\dagger$ :

$$[\hat{a}, \hat{a}^\dagger] = \frac{1}{2} \left[ \frac{\hat{x}}{\sigma} + i \frac{\sigma \hat{p}}{\hbar}, \frac{\hat{x}}{\sigma} - i \frac{\sigma \hat{p}}{\hbar} \right] = \frac{1}{2} \left( -\frac{2i}{\hbar} [\hat{x}, \hat{p}] \right) = -\frac{i}{\hbar} [\hat{x}, \hat{p}] = -\frac{i}{\hbar}(i\hbar) = 1.$$

If we replace the  $\hat{x}, \hat{p}$  in terms of  $\hat{a}, \hat{a}^\dagger$  in the Hamiltonian, we have:

$$\begin{aligned}\hat{H} &= \frac{1}{2m} \left[ -\frac{\hbar^2}{2\sigma^2} (\hat{a}^\dagger - \hat{a})^2 \right] + \frac{1}{2}m\omega^2 \left[ \frac{\sigma^2}{2} (\hat{a}^\dagger + \hat{a})^2 \right] \\ &= \frac{\hbar\omega}{2} \left[ \hat{a}^{\dagger 2} + \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger + \hat{a}^2 - \hat{a}^{\dagger 2} + \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger - \hat{a}^2 \right] \\ &= \frac{\hbar\omega}{2} \left[ 2(\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) \right] \\ &= \hbar\omega(\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger + 1) \\ &= \hbar\omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \\ \hat{H} &= \hbar\omega \left( \hat{N} + \frac{1}{2} \right),\end{aligned}$$

where  $\hat{N} = \hat{a}^\dagger \hat{a}$  is the *number operator*. If we know the eigenequation of  $\hat{N}$  we automatically know the eigenequation of  $\hat{H}$ . Assuming the following equation for  $\hat{N}$ , then:

$$\hat{N} |n\rangle = n |n\rangle \implies \hat{H} |n\rangle = \left(n + \frac{1}{2}\right) |n\rangle.$$

This means that the energy eigenvalues are

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega.$$

The nature of  $n$  needs to be proved. First, we compute the commutator of  $\hat{N}$  with  $\hat{a}, \hat{a}^\dagger$ :

$$\begin{aligned} [\hat{N}, \hat{a}] &= [\hat{a}^\dagger \hat{a}, \hat{a}] = \hat{a}^\dagger [\hat{a}, \hat{a}] + [\hat{a}^\dagger, \hat{a}] \hat{a} = -\hat{a}, \\ [\hat{N}, \hat{a}^\dagger] &= [\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger [\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}^\dagger] \hat{a} = \hat{a}^\dagger. \end{aligned}$$

Using these relations, we can test some actions on  $|n\rangle$ :

$$\begin{aligned} \hat{N} \hat{a}^\dagger |n\rangle &= ([\hat{N}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{N}) |n\rangle = (\hat{a}^\dagger + \hat{a}^\dagger \hat{N}) |n\rangle = (\hat{a}^\dagger + \hat{a}^\dagger n) |n\rangle = (n+1) \hat{a}^\dagger |n\rangle, \\ \hat{N} \hat{a} |n\rangle &= ([\hat{N}, \hat{a}] + \hat{a} \hat{N}) |n\rangle = (-\hat{a} + \hat{a} \hat{N}) |n\rangle = (\hat{a}n - \hat{a}) |n\rangle = (n-1) \hat{a} |n\rangle. \end{aligned}$$

This imply that  $\hat{a}^\dagger |n\rangle$  and  $\hat{a} |n\rangle$  are also eigenkets of  $\hat{N}$ , with eigenvalue increased and decreased by one, respectively. With this, we conclude that  $n$  must be integer so that  $\hat{N}$  and therefore  $\hat{H}$  are quantized. A more rigorous proof is given by [1].

## Exercise 2

- a) Ill use the definition of the E- and B-fields in terms of the vector potential.

$$\mathbf{E} = -\partial_t \mathbf{A}, \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

Taking the curl of the B-field allows to replace the vector potential and have a triple product, which can be reexpressed using vector identities. Because  $\nabla \times \mathbf{B}$  is Ampere's law, it also depends on the E-field, where the above definition can be replaced.

$$\begin{aligned} \nabla \times \mathbf{B} &= \frac{1}{c^2} \partial_t \mathbf{E} \\ \nabla \times \nabla \times \mathbf{A} &= \frac{1}{c^2} \partial_t (-\partial_t \mathbf{A}) \\ \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} &= -\frac{1}{c^2} \partial_t^2 \mathbf{A} \\ \nabla^2 \mathbf{A}(\mathbf{r}, t) - \frac{1}{c^2} \partial_t^2 \mathbf{A}(\mathbf{r}, t) &= 0. \end{aligned}$$

- b) fsfaf
- c) asgsag
- d) asgsa
- e) sagasg

## Exercise 3

a) We can simplify the problem using the BCH formula

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \dots$$

In this case,  $\hat{A} = \alpha^* \hat{a} - \alpha \hat{a}^\dagger$  and  $\hat{B} = \hat{a}$ . The first two commutators are:

$$\begin{aligned} [\hat{A}, \hat{B}] &= [(\alpha^* \hat{a} - \alpha \hat{a}^\dagger), \hat{a}] = \alpha^* [\hat{a}, \hat{a}] - \alpha [\hat{a}^\dagger, \hat{a}] = \alpha, \\ [\hat{A}, [\hat{A}, \hat{B}]] &= [(\alpha^* \hat{a} - \alpha \hat{a}^\dagger), \alpha] = 0, \\ &\vdots \end{aligned}$$

Now, we can express the displacement operator as:

$$\begin{aligned} \tilde{a} &= \mathcal{D}^\dagger(\alpha) \hat{a} \mathcal{D}(\alpha) = e^{\alpha^* \hat{a} - \alpha \hat{a}^\dagger} \hat{a} e^{-(\alpha^* \hat{a} - \alpha \hat{a}^\dagger)} = \hat{a} + \alpha, \quad \text{and} \\ \tilde{a}^\dagger &= \mathcal{D}^\dagger(\alpha) \hat{a}^\dagger \mathcal{D}(\alpha) = \hat{a}^\dagger + \alpha^*. \end{aligned}$$

Application of both in  $|0\rangle$  (just to test them) yields:

$$\tilde{a}|0\rangle = (\hat{a} + \alpha)|0\rangle = \alpha|0\rangle, \quad \text{and} \quad \tilde{a}^\dagger|0\rangle = (\hat{a}^\dagger + \alpha^*)|0\rangle = |1\rangle + \alpha^*|0\rangle.$$

b) We use algabra of operators and the fact that the coherent state  $|\alpha\rangle$  is got from  $|0\rangle$  by applying a displacement operator  $\mathcal{D}(\alpha)$ :

$$\begin{aligned} \langle \hat{X}_1 \rangle &= \frac{1}{2} \langle \alpha | (\hat{a} + \hat{a}^\dagger) | \alpha \rangle \\ &= \frac{1}{2} \langle 0 | \mathcal{D}^\dagger(\alpha) (\hat{a} + \hat{a}^\dagger) \mathcal{D}(\alpha) | 0 \rangle \\ &= \frac{1}{2} \left[ \langle 0 | \mathcal{D}^\dagger(\alpha) \hat{a} \mathcal{D}(\alpha) | 0 \rangle + \langle 0 | \mathcal{D}^\dagger(\alpha) \hat{a}^\dagger \mathcal{D}(\alpha) | 0 \rangle \right] \\ &= \frac{1}{2} \left[ \langle 0 | \tilde{a} | 0 \rangle + \langle 0 | \tilde{a}^\dagger | 0 \rangle \right] \\ &= \frac{1}{2} \left[ \langle 0 | (\hat{a} + \alpha) | 0 \rangle + \langle 0 | (\hat{a}^\dagger + \alpha^*) | 0 \rangle \right] \\ &= \frac{1}{2} \left[ \langle 0 | \hat{a} | 0 \rangle + \alpha + \langle 0 | \hat{a}^\dagger | 0 \rangle + \alpha^* \right] \\ &= \frac{1}{2} [\alpha + \alpha^*] \\ \langle \hat{X}_1 \rangle &= \text{Re}(\alpha). \end{aligned}$$

For  $\hat{X}_2$ , and using the developmente from above:

$$\begin{aligned} \langle \hat{X}_2 \rangle &= \frac{1}{2i} \left[ \langle 0 | (\hat{a} + \alpha) | 0 \rangle - \langle 0 | (\hat{a}^\dagger + \alpha^*) | 0 \rangle \right] \\ &= \frac{1}{2i} \left[ \langle 0 | \hat{a} | 0 \rangle + \alpha - \langle 0 | \hat{a}^\dagger | 0 \rangle - \alpha^* \right] \\ &= \frac{1}{2i} [\alpha - \alpha^*] \\ \langle \hat{X}_2 \rangle &= \text{Im}(\alpha). \end{aligned}$$

For the uncertainties, we now need to get the mean value of  $\hat{X}_1^2$  and  $\hat{X}_2^2$ , with the analogous process:

$$\begin{aligned}
 \langle \hat{X}_1^2 \rangle &= \frac{1}{4} \langle 0 | \mathcal{D}^\dagger(\alpha) [\hat{a} + \hat{a}^\dagger]^2 \mathcal{D}(\alpha) | 0 \rangle \\
 &= \frac{1}{4} \langle 0 | \left[ \mathcal{D}^\dagger(\alpha) [\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2}] \mathcal{D}(\alpha) \right] | 0 \rangle \\
 &= \frac{1}{4} \langle 0 | \left[ \mathcal{D}^\dagger \hat{a}^2 \mathcal{D}(\alpha) + 2\mathcal{D}^\dagger(\alpha) \hat{a}^\dagger \hat{a} \mathcal{D}(\alpha) + 1 + \mathcal{D}^\dagger(\alpha) \hat{a}^{\dagger 2} \mathcal{D}(\alpha) \right] | 0 \rangle \\
 &= \frac{1}{4} \langle 0 | \left[ (\mathcal{D}^\dagger \hat{a} \mathcal{D})(\mathcal{D}^\dagger \hat{a} \mathcal{D}) + 2(\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D})(\mathcal{D}^\dagger \hat{a} \mathcal{D}) + 1 + (\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D})(\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D}) \right] | 0 \rangle \\
 &= \frac{1}{4} \langle 0 | \left[ (\hat{a} + \alpha)^2 + 2(\hat{a}^\dagger + \alpha^*)(\hat{a} + \alpha) + 1 + (\hat{a}^\dagger + \alpha^*)^2 \right] | 0 \rangle \\
 &= \frac{1}{4} [\hat{a}^2 + 2\alpha\hat{a} + \alpha^2 + 2\hat{a}^\dagger\hat{a} + 2\hat{a}^\dagger\alpha + 2\alpha^*\hat{a} + 2|\alpha|^2 + 1 + \hat{a}^{\dagger 2} + 2\alpha^*\hat{a}^\dagger + \alpha^{*2}] \\
 \langle \hat{X}_1^2 \rangle &= \frac{1}{4} [(\alpha + \alpha^*)^2 + 1].
 \end{aligned}$$

Then,

$$\begin{aligned}
 \Delta \hat{X}_1 &= \sqrt{\langle \hat{X}_1^2 \rangle - \langle \hat{X}_1 \rangle^2} \\
 &= \sqrt{\frac{1}{4} [(\alpha + \alpha^*)^2 + 1] - \frac{1}{4}(\alpha + \alpha^*)^2} \\
 \Delta \hat{X}_1 &= \frac{1}{2}.
 \end{aligned}$$

The same procedure is done for the uncertainty of  $\hat{X}_2$ :

$$\begin{aligned}
 \langle \hat{X}_2 \rangle &= \frac{1}{4} \langle 0 | \left[ (\mathcal{D}^\dagger \hat{a} \mathcal{D})(\mathcal{D}^\dagger \hat{a} \mathcal{D}) - 2(\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D})(\mathcal{D}^\dagger \hat{a} \mathcal{D}) - 1 + (\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D})(\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D}) \right] | 0 \rangle \\
 &= -\frac{1}{4} \langle 0 | \left[ (\hat{a} + \alpha)^2 - 2(\hat{a}^\dagger + \alpha^*)(\hat{a} + \alpha) - 1 + (\hat{a}^\dagger + \alpha^*)^2 \right] | 0 \rangle \\
 &= -\frac{1}{4} [\hat{a}^2 + 2\alpha\hat{a} + \alpha^2 - 2\hat{a}^\dagger\hat{a} - 2\hat{a}^\dagger\alpha - 2\alpha^*\hat{a} - 2|\alpha|^2 - 1 + \hat{a}^{\dagger 2} + 2\alpha^*\hat{a}^\dagger + \alpha^{*2}] \\
 \langle \hat{X}_2^2 \rangle &= \frac{1}{4} [1 - (\alpha - \alpha^*)^2].
 \end{aligned}$$

Then,

$$\begin{aligned}
 \Delta \hat{X}_2 &= \sqrt{\langle \hat{X}_2^2 \rangle - \langle \hat{X}_2 \rangle^2} \\
 &= \sqrt{\frac{1}{4} [1 - (\alpha - \alpha^*)^2] + \frac{1}{4}(\alpha - \alpha^*)^2} \\
 \Delta \hat{X}_2 &= \frac{1}{2}.
 \end{aligned}$$

## References

- [1] Claude Cohen-Tannoudji, Bernard Diu, and Frank Laloe. Quantum mechanics, volume 1. *Quantum Mechanics*, 1:898, 1986.