Notes of Quantum Optics

Wyant College of Optical Sciences
University of Arizona

Nicolás Hernández Alegría

Preface

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1.2.1   Fields in a cavity

Lets consider the following one-dimensional problem, where a cavity of length L is oriented along the z-axis.

A linear polarized E-field is assumed, the medium is free space, perfect conducting walls and there is no free charges nor free current. The scheme is shown in figure 1.1.
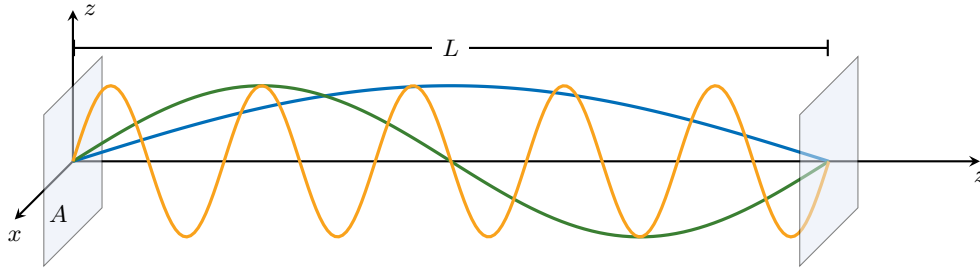
Figure 1.1 One-dimensional cavity problem. Perfect conducting walls.

Our goal is to find the E- and B-field inside the cavity. Maxwell's equations in this case are:

$$\text{Maxwell's equations} \atop \text{with free sources} \quad \begin{cases} \nabla \cdot \boldsymbol{E} = 0 \\ \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B} \\ \nabla \times \boldsymbol{B} = \frac{1}{c^2}\partial_t \boldsymbol{E} \end{cases} \tag{1.1}$$

The E-field will be assumed to be $\boldsymbol{E}(z,t) = \boldsymbol{e}E(z,t)$, where $\boldsymbol{e}$ is the polarization vector. Because fields depends only on z, $\nabla = \hat{\boldsymbol{z}}\partial_z$. First Maxwell equation yields:

$$\nabla \cdot \boldsymbol{E} = \partial_z(\hat{\boldsymbol{z}} \cdot \boldsymbol{E}) = \partial_z(\hat{\boldsymbol{z}} \cdot \boldsymbol{e}E) = 0 \implies \boldsymbol{e} \cdot \hat{\boldsymbol{z}} = 0.$$

This implies that the polarization vector must be unitary in the transverse plane:

$$\boldsymbol{e} = \cos\phi\hat{\boldsymbol{x}} + \sin\phi\hat{\boldsymbol{y}}.$$

Third Maxwell's equation yields

$$\nabla \times \boldsymbol{E} = (\hat{\boldsymbol{z}}\partial_z) \times (\boldsymbol{e}E) = (\hat{\boldsymbol{z}} \times \boldsymbol{e})\partial_z E = -\partial_t \boldsymbol{B}.$$

Taking the curl of this equation, using Fourth Maxwell's equation and vector identities:

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\partial_t(\nabla \times \boldsymbol{B})$$

$$\nabla(\underbrace{\nabla \cdot \boldsymbol{E}}_{0}) - \nabla^2 \boldsymbol{E} = -\frac{1}{c^2}\partial_t^2 \boldsymbol{E}$$

$$-\boldsymbol{e}\partial_z^2 E = -\boldsymbol{e}\frac{1}{c^2}\partial_t^2 E.$$

From here, we have the E-field wave equation for this particular problem.

$$\partial_z^2 E - \frac{1}{c^2}\partial_t^2 E = 0. \tag{1.2}$$

Before solving this equation, we need the boundary condition set by the PEC condition. We need to $\hat{n} \times \boldsymbol{E} = 0$ on the surface. Because the normal surface unit vector is $\hat{n} = \pm\hat{z}$, we have

Boundary condition $\qquad \hat{n} \times \boldsymbol{E} = \hat{z} \times (\boldsymbol{e}E) = 0 \Longrightarrow E(z = 0, t) = E(z = L, t) = 0.$

In order to solve the PDE, we assum a product form $E(z,t) = Z(z)q(t)$. Then, by replacing in it in (1.2):

$$\partial_z^2[Z(z)q(t)] - \frac{1}{c^2}\partial_t^2[Z(z)q(t)] = 0$$

$$Z''(z)q(t) - \frac{1}{c^2}Z(z)\ddot{q}(t) = \Big/ [Z(z)q(t)]^{-1}$$

$$\frac{Z''(z)}{Z(z)} = \frac{1}{c^2}\frac{\ddot{q}(z)}{q(z)}.$$

Left side depends only on z, while the right side only on y. The only way this can be true is if both are a constant, say, $-k^2$. Then,

Spatial and temporal differential equations $\qquad \begin{cases} \dfrac{Z''}{Z} = -k^2 \longrightarrow Z'' + k^2 Z = 0 \\ \dfrac{1}{c^2}\dfrac{\ddot{T}}{T} = -k^2 \longrightarrow \ddot{q} + \omega^2 q(t) = 0 \end{cases}.$

For the spatial ODE, we assume a solution of the form

$$Z(z) = A\sin(kz) + B\cos(kz), \quad Z(0) = Z(L) = 0.$$

Setting the boundaries:

$$Z(0) = A\sin(0) + B\cos(0) = 0 \Longrightarrow B = 0$$

$$Z(L) = A\sin(kL) = 0 \Longrightarrow k_m = \frac{m\pi}{L}, \quad m \in \mathbb{N}.$$

We left the temporal ODE unsolved. Finally, putting all together yields the initial E-field:

$$\boldsymbol{E}_{\boldsymbol{k},\lambda}(z,t) = \boldsymbol{e}_\lambda\sqrt{\frac{2\omega^2}{V\varepsilon_0}}q_{\boldsymbol{k},\lambda}(t)\sin(kz), \quad k_m = \frac{m\pi}{L}, \quad \omega = ck.$$

We have includnig the subscript λ and $\boldsymbol{k}$ to consider multiple mode $\boldsymbol{k}$ varied with m and λ. Also, the coefficient in red is for better results in the future. Using Faraday's law:

$$\boldsymbol{B}_{\boldsymbol{k},\lambda}(z,t) = (\boldsymbol{k} \times \boldsymbol{e}_\lambda)\frac{1}{kc^2}\sqrt{\frac{2\omega^2}{V\varepsilon_0}}\dot{q}_{\boldsymbol{k},\lambda}(t)\sin(kz).$$

The term $\dot{q}(t)$ will play the role of a canonical momentum for a particle of unit mass, $p(t) = \dot{q}(t)$.

1.2.2 Single-mode Hamiltonian

The classical field energy, or Hamiltonian H, of the single-mode field is given by

$$
\begin{aligned}
H_{\boldsymbol{k},\lambda} &= \frac{1}{2} \int dV \left[\varepsilon_0 \boldsymbol{E}_{\boldsymbol{k},\lambda}^2 + \frac{1}{\mu_0} \boldsymbol{B}_{\boldsymbol{k},\lambda}^2 \right] \\
&= \frac{1}{2} A \int_0^L dz \left[\frac{2\omega^2}{V\varepsilon_0} q_{\boldsymbol{k},\lambda}^2(t) \sin^2(kz) + \frac{1}{\mu_0} \frac{1}{k^2 c^4} \frac{2\omega^2}{V\varepsilon_0} p_{\boldsymbol{k},\lambda}^2(t) \cos^2(kz) \right] \\
&= \frac{A}{V} \int_0^L dz \left[\omega^2 q_{\boldsymbol{k},\lambda}^2(t) \sin^2(kz) + p_{\boldsymbol{k},\lambda}^2(t) \cos^2(kz) \right] \\
&= \frac{1}{L} \left[\omega^2 q_{\boldsymbol{k},\lambda}^2(t) \frac{L}{2} + p_{\boldsymbol{k},\lambda}^2(t) \frac{L}{2} \right] \\
H_{\boldsymbol{k},\lambda} &= \frac{1}{2} \left[\omega^2 q_{\boldsymbol{k},\lambda}^2(t) + p_{\boldsymbol{k},\lambda}^2(t) \right].
\end{aligned}
$$

It is apparent that a single-mode field if formally equivalent to a harmonic quantim oscillator of unit mass, where the E- and B-fields play the roles of canonical position and momentum.

$$
H_{\text{harmonic oscillator}} = \frac{1}{2} m\omega^2 x^2 + \frac{p^2}{2m}.
$$

To begin the quantization, we make q, p operators $\hat{q}, \hat{p}$, which needs to satisfy the canonical commutation relations

$$
[\hat{q}_{\boldsymbol{k},\lambda}, \hat{p}_{\boldsymbol{k},\lambda}] = i\hbar. \tag{1.3}
$$

The EM fields with the operators are:

$$
\text{Quantized EM fields} \qquad
\begin{aligned}
\hat{\boldsymbol{E}}_{\boldsymbol{k},\lambda}(z,t) &= \boldsymbol{e}_{\boldsymbol{k},\lambda} \sqrt{\frac{2\omega^2}{V\varepsilon_0}} \hat{q}(t) \sin(kz) \\
\hat{\boldsymbol{B}}_{\boldsymbol{k},\lambda}(z,t) &= (\boldsymbol{k} \times \boldsymbol{e}_{\boldsymbol{k},\lambda}) \frac{1}{kc^2} \sqrt{\frac{2\omega^2}{V\varepsilon_0}} \hat{p}(t) \cos(kz)
\end{aligned}
\qquad . \tag{1.4}
$$

The operators $\hat{q}, \hat{p}$ are Hermitian and therefore correspond to obserable quantites. It is convenient to introduce the ladder operators to simplify the expresison:

$$
\text{Ladder operator} \qquad
\begin{aligned}
\hat{a}_{\boldsymbol{k},\lambda} &= \sqrt{\frac{1}{2\hbar\omega}}(\omega\hat{q}_{\boldsymbol{k},\lambda} + i\hat{p}_{\boldsymbol{k},\lambda}) \\
\hat{a}_{\boldsymbol{k},\lambda}^\dagger &= \sqrt{\frac{1}{2\hbar\omega}}(\omega\hat{q}_{\boldsymbol{k},\lambda} - i\hat{p}_{\boldsymbol{k},\lambda})
\end{aligned}
\longrightarrow
\begin{aligned}
\hat{q}_{\boldsymbol{k},\lambda} &= \sqrt{\frac{\hbar}{2\omega}}(\hat{a}_{\boldsymbol{k},\lambda} + \hat{a}_{\boldsymbol{k},\lambda}^\dagger) \\
\hat{p}_{\boldsymbol{k},\lambda} &= -i\sqrt{\frac{\hbar\omega}{2}}(\hat{a}_{\boldsymbol{k},\lambda} - \hat{a}_{\boldsymbol{k},\lambda}^\dagger)
\end{aligned}
\qquad .
$$

Using these defintion, the Hamiltonian yields:

$$
\begin{aligned}
\hat{H}_{\boldsymbol{k},\lambda} &= -\frac{1}{2} \frac{\hbar\omega}{2} (\hat{a}_{\boldsymbol{k},\lambda} - \hat{a}_{\boldsymbol{k},\lambda}^\dagger)(\hat{a}_{\boldsymbol{k},\lambda} - \hat{a}_{\boldsymbol{k},\lambda}^\dagger) + \frac{1}{2} \frac{\hbar\omega^2}{2\omega} (\hat{a}_{\boldsymbol{k},\lambda} + \hat{a}_{\boldsymbol{k},\lambda}^\dagger)(\hat{a}_{\boldsymbol{k},\lambda} + \hat{a}_{\boldsymbol{k},\lambda}^\dagger) \\
&= \frac{\hbar\omega}{2} [\hat{a}_{\boldsymbol{k},\lambda}^\dagger \hat{a}_{\boldsymbol{k},\lambda} + \hat{a}_{\boldsymbol{k},\lambda} \hat{a}_{\boldsymbol{k},\lambda}^\dagger] = \hbar\omega \left[\hat{a}_{\boldsymbol{k},\lambda}^\dagger \hat{a}_{\boldsymbol{k},\lambda} + \frac{1}{2} \right] = \hbar\omega \left[n_{\boldsymbol{k},\lambda} + \frac{1}{2} \right].
\end{aligned}
$$

The time dependence of the ladder operator in Heisenberg equation is:

$$
\frac{d\hat{a}_{\boldsymbol{k},\lambda}}{dt} = \frac{i}{\hbar}[\hat{H}_{\boldsymbol{k},\lambda}, \hat{a}_{\boldsymbol{k},\lambda}] = \frac{i}{\hbar} \left[\hbar\omega \left(\hat{a}_{\boldsymbol{k},\lambda}^\dagger \hat{a}_{\boldsymbol{k},\lambda} + \frac{1}{2} \right), \hat{a}_{\boldsymbol{k},\lambda} \right] = -i\omega\hat{a}_{\boldsymbol{k},\lambda} \longrightarrow
\begin{aligned}
\hat{a}_{\boldsymbol{k},\lambda}(t) &= \hat{a}_{\boldsymbol{k},\lambda}(0)e^{-i\omega t} \\
\hat{a}_{\boldsymbol{k},\lambda}^\dagger(t) &= \hat{a}_{\boldsymbol{k},\lambda}^\dagger(0)e^{i\omega t}
\end{aligned}
\quad .
$$

For example, we have the following applications:

$$\hat{a}_{\boldsymbol{k},\lambda}\ket{0} = \ket{1}_{\boldsymbol{k},\lambda}, \quad \hat{a}_{\boldsymbol{k},\lambda}\ket{n}_{\boldsymbol{k},\lambda} = \sqrt{n}\ket{n-1}_{\boldsymbol{k},\lambda}, \quad \hat{a}^{\dagger}_{\boldsymbol{k},\lambda}\ket{n}_{\boldsymbol{k}',\lambda'} = \ket{n}_{\boldsymbol{k},\lambda}\ket{1}_{\boldsymbol{k}',\lambda'}, \quad \hat{a}_{\boldsymbol{k},\lambda}\ket{n}_{\boldsymbol{k}',\lambda'}\ket{0}_{\boldsymbol{k},\omega} = 0.$$

The operator product $\hat{a}^{\dagger}\hat{a}$ has an important significance and is called the number operator $\hat{n}$, whose eigenequation is

$$\hat{n}\ket{n} = n\ket{n}.$$

The state $\ket{n}$ is the energy eigenstte of the single mode field with energy eiganvlaue E_n:

$$\hat{H}\ket{n} = \hbar\omega(\hat{a}^{\dagger}\hat{a} + \frac{1}{2})\ket{n} = E_n\ket{n}$$

The relation between the number state and the raising operator $\hat{a}^{\dagger}$ is:

$$\ket{n} = \frac{(\hat{a}^{\dagger})^n}{\sqrt{n}}\ket{0}.$$

1.2.3 Quantizing the multimode field

We now consider all the modes in the optical cavity:

$$\boldsymbol{E}(z,t) = \sum_{m,\lambda} \hat{\boldsymbol{e}}_{\lambda}\sqrt{\frac{2\omega_m^2}{\varepsilon_0 V}}\hat{q}_{m,\lambda}(t)\sin(k_m z) \tag{1.5}$$

$$\boldsymbol{B}(z,t) = \sum_{m,\lambda} (\hat{\boldsymbol{z}}\times\hat{\boldsymbol{e}}_{\lambda})\frac{1}{k_m c^2}\sqrt{\frac{2\omega_m^2}{\varepsilon_0 V}}\dot{\hat{q}}_{m,\lambda}(t)\cos(k_m z). \tag{1.6}$$

where $kL = m\pi$ and $\hat{\boldsymbol{e}}_{\lambda} = \{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}\}$. The process is anaglogous; we compute the Hamiltonian:

$$H = \int dV \left[\frac{1}{2}\varepsilon_0 \boldsymbol{E}^2(z,t) + \frac{1}{2\mu_0}\boldsymbol{B}(z,t)\right]$$

$$= \int dV \left[\frac{1}{2}\varepsilon_0\left\{\sum_{m,\alpha}\hat{\boldsymbol{e}}_{\alpha}\sqrt{\frac{2\omega_m^2}{\varepsilon_0 V}}q_{m,\alpha}(t)\sin(k_m z)\right\}^2 + \frac{1}{2\mu_0}\varepsilon_0\left\{\sum_{m,\alpha}(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{e}}_{\alpha})\frac{\mu_0}{\varepsilon_0}k_m\sqrt{\frac{2\omega_m^2}{\varepsilon_0 V}}\dot{q}_{m,\alpha}(t)\cos(k_m z)\right\}^2\right]$$

$$\vdots$$

$$H = \frac{1}{2}\sum_{m,\lambda}(\omega_m^2 q_{m,\lambda}^2 + \dot{q}_{m,\lambda}^2).$$

That is,

> Each mode of the EM field is an independent harmonic oscillator.

We can quantize the multimode EM field in a similar way as the sige mode field to obtain the Hamiltonian:

$$\text{Hamiltoninan multimode EM field} \qquad \hat{H} = \sum_{m,\lambda}\hbar\omega_m\left[\hat{a}^{\dagger}_{m,\lambda}a_{m,\lambda} + \frac{1}{2}\right] = \sum_{m,\lambda}\hbar\omega_m\left[\hat{n}_{m,\lambda} + \frac{1}{2}\right],$$

where $\hat{n}_{m,\lambda}$ is the number of excitations in mode m,λ.

1.2.4 Quantizing EM field in free space

We define the scalar and vector potential

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -\partial_t \boldsymbol{A} - \nabla \phi.$$

Using these automatically satisfy the Gauss magnetic equation and Ampere law. In QO, we assume **Coulomb gauge**: $\nabla \cdot \boldsymbol{A} = 0$. With this, the E-field is

$$\boldsymbol{E} = -\partial_t \boldsymbol{A} - \nabla \phi = \boldsymbol{E}^\perp + \boldsymbol{E}^\parallel, \quad \text{where} \begin{cases} \nabla \cdot \boldsymbol{E}^\perp = 0, & \boldsymbol{A} \text{ corresponds to radiation (all transverse)} \\ \boldsymbol{E}'' = -\nabla \phi, & \boldsymbol{A} \text{ corresponds to the field of sources (all longitudinal)} \end{cases}.$$

In Coulomb gauge, in absence of charge $\phi = 0$ such that the vector potential-wave equation is

$$\nabla^2 \boldsymbol{A} - \frac{1}{2}\partial_t^2 \boldsymbol{A} = 0.$$

It can be solved with the following plane wave

$$\boldsymbol{A}(\boldsymbol{r},t) = \sum_{\boldsymbol{k},\lambda} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} \left[A_{\boldsymbol{k},\lambda} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + A_{\boldsymbol{k},\lambda}^* e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \right], \quad \nabla \cdot \boldsymbol{A} = 0 \implies \boldsymbol{k} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} = 0.$$

The vector potential in free space is expressed as a sueprposition of plane waves such that linear dispersion $\omega = |\boldsymbol{k}|c$ is satisfied with $\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda'} = \delta_{\lambda\lambda'}$.

The quantization volume we quantize the field is considered to have finite volume of V and we impose periodic boundary such that each side is equal.

The E- and B-field are then:

$$\boldsymbol{E}(\boldsymbol{r},t) = -\partial_t \boldsymbol{A}(\boldsymbol{r},t) = i\sum_{\boldsymbol{k},\lambda} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} \omega \left[A_{\boldsymbol{k},\lambda} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + A_{\boldsymbol{k},\lambda}^* e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \right],$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \nabla \times \boldsymbol{A}(\boldsymbol{r},t) = i\sum_{\boldsymbol{k},\lambda} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} \left[A_{\boldsymbol{k},\lambda} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + A_{\boldsymbol{k},\lambda}^* e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \right].$$

The energy (Hamiltonian) associated with the fields is:

$$H = -\int dV \left[\frac{1}{2}\varepsilon_0 \boldsymbol{E}^2(\boldsymbol{r},t) + \frac{1}{2\mu_0}\boldsymbol{B}(\boldsymbol{r},t) \right]$$

$$\vdots$$

$$H = 2\varepsilon_0 V \sum_{\boldsymbol{k},\lambda} \omega^2 |A_{\boldsymbol{k},\lambda}|^2.$$

We have used the following relation:

$$\int dV e^{\pm i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} = \delta_{\boldsymbol{k}\boldsymbol{k}'} V.$$

Defining

$$A_{\boldsymbol{k},\lambda} = \frac{1}{2\omega\sqrt{\varepsilon_0 V}}(\omega q_{\boldsymbol{k},\lambda} + ip_{\boldsymbol{k},\lambda}), \quad A_{\boldsymbol{k},\lambda}^* = \frac{1}{2\omega\sqrt{\varepsilon_0 V}}(\omega q_{\boldsymbol{k},\lambda} - ip_{\boldsymbol{k},\lambda}).$$

The Hamiltonian is:

$$H = \frac{1}{2} \sum_{\boldsymbol{k},\lambda} (\omega^2 q_{\boldsymbol{k},\lambda}^2 + p_{\boldsymbol{k},\lambda}^2) \xrightarrow{\text{quantizing}} \hat{H} = \sum_{\boldsymbol{k},\lambda} \hbar\omega \left(\hat{a}_{\boldsymbol{k},\lambda}^\dagger \hat{a}_{\boldsymbol{k},\lambda} + \frac{1}{2} \right),$$

where we have defined the ladder operators:

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p}). \tag{1.7}$$

So, the coefficients are expressed in term of the ladder:

$$A_{\boldsymbol{k},\lambda} = \sqrt{\frac{\hbar}{2\omega\varepsilon_0 V}} \hat{a}_{\boldsymbol{k},\lambda}, \quad A_{\boldsymbol{k},\lambda}^* = \sqrt{\frac{\hbar}{2\omega\varepsilon_0 V}} \hat{a}_{\boldsymbol{k},\lambda}^\dagger.$$

Thus the fields are:

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{r},t) &= \sum_{\boldsymbol{k},\lambda} \sqrt{\frac{\hbar}{2\omega\varepsilon_0 V}} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} \left[\hat{a}_{\boldsymbol{k},\lambda} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \hat{a}_{\boldsymbol{k},\lambda}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \right], \\
\boldsymbol{E}(\boldsymbol{r},t) &= i \sum_{\boldsymbol{k},\lambda} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} \left[\hat{a}_{\boldsymbol{k},\lambda} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \hat{a}_{\boldsymbol{k},\lambda}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \right], \\
\boldsymbol{B}(\boldsymbol{r},t) &= \frac{i}{c} \sum_{\boldsymbol{k},\lambda} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} (\boldsymbol{k} \times \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}) \left[\hat{a}_{\boldsymbol{k},\lambda} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \hat{a}_{\boldsymbol{k},\lambda}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \right].
\end{aligned} \tag{1.8}$$

1.2.5 Electric field expectation and uncertainty

Lets conside one mode of the E-field in a number state: $|n\rangle_{\boldsymbol{k},\lambda}$. Lets add the time dependence to the ladder operators:

$$\hat{a}_{\boldsymbol{k},\lambda} e^{-i\omega t} \to \hat{a}_{\boldsymbol{k},\lambda}(t), \quad \hat{a}_{\boldsymbol{k},\lambda}^\dagger e^{i\omega t} \to \hat{a}_{\boldsymbol{k},\lambda}^\dagger(t).$$

The E-field is:

$$\hat{\boldsymbol{E}}(\boldsymbol{r},t) = i \sum_{\boldsymbol{k},\lambda} e_{\boldsymbol{k},\lambda} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \left[\hat{a}_{\boldsymbol{k},\lambda}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{a}_{\boldsymbol{k},\lambda}^\dagger(t) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \right].$$

Expectation value of the E-field is:

$$\begin{aligned}
{}_{\boldsymbol{k}_0,\lambda_0}\langle n|\hat{\boldsymbol{E}}(\boldsymbol{r},t)|n\rangle_{\boldsymbol{k}_0,\lambda_0} &= i_{\boldsymbol{k},\lambda}\langle n|\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \left[\hat{a}_{\boldsymbol{k},\lambda}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{a}_{\boldsymbol{k},\lambda}^\dagger(t) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \right] |n\rangle_{\boldsymbol{k}_0,\lambda_0} \\
&= i_{\boldsymbol{k}_0,\lambda_0} \langle n| \hat{\boldsymbol{e}}_{\boldsymbol{k}_0,\lambda_0} \sqrt{\frac{\hbar\omega_0}{2\varepsilon_0 V}} \left[\sqrt{n} |n-1\rangle_{\boldsymbol{k}_0,\lambda_0} e^{i\boldsymbol{k}_0\cdot\boldsymbol{r}} - \sqrt{n+1} |n+1\rangle_{\boldsymbol{k}_0,\lambda_0} e^{-i\boldsymbol{k}_0\cdot\boldsymbol{r}} \right] \\
&= 0.
\end{aligned}$$

Thus, the average E-field is zero.

Lets consider the uncertainty of the E-field:

$$\begin{aligned}
\langle \hat{\boldsymbol{E}} \rangle = {}_{\boldsymbol{k}_0,\lambda_0} \langle n| \sum_{\boldsymbol{k},\lambda} \sum_{\boldsymbol{k}',\lambda'} & \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \sqrt{\frac{\hbar\omega'}{2\varepsilon_0 V}} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{k}',\lambda'} \left[\hat{a}_{\boldsymbol{k},\lambda} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{a}_{\boldsymbol{k},\lambda}^\dagger e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \right] \left[\hat{a}_{\boldsymbol{k}',\lambda'} e^{i\boldsymbol{k}'\cdot\boldsymbol{r}} - \hat{a}_{\boldsymbol{k}',\lambda'}^\dagger e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}} \right] |n\rangle_{\boldsymbol{k}_0,\lambda_0} \\
= \sum_{\boldsymbol{k},\lambda} \sum_{\boldsymbol{k}',\lambda'} & \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \sqrt{\frac{\hbar\omega'}{2\varepsilon_V}}_{\boldsymbol{k},\lambda} \langle 1|1\rangle_{\boldsymbol{k}',\lambda'} e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} + \frac{\hbar\omega_0}{2\varepsilon_0 V} [{}_{\boldsymbol{k}_0,\lambda_0} \langle n+1| \sqrt{n+1} + {}_{\boldsymbol{k}_0,\lambda_0} \langle n-1| \sqrt{n}][\sqrt{n+1} |n+1\rangle_{\boldsymbol{k}_0,\lambda_0} + \\
= \sum_{\{\boldsymbol{k},\lambda\} \neq \{\boldsymbol{k}_0,\lambda_0\}} & \frac{\hbar\omega}{2\varepsilon_0 V} + \frac{\hbar\omega_0}{2\varepsilon_0 V}(2n+1).
\end{aligned}$$

So, the uncertainty is:

E-field uncertainty $\qquad \langle \Delta E^2 \rangle = \underbrace{\sum_{\{\boldsymbol{k},\lambda\} \neq \{\boldsymbol{k}_0,\lambda_0\}} \frac{\hbar\omega}{2\varepsilon_0 V}}_{\text{All modes in vaccum state}} + \underbrace{\frac{\hbar\omega_0}{2\varepsilon_0 V}(2n+1)}_{\text{Mode } (\boldsymbol{k}_0,\,\lambda_0)}.$

1.3 Coherent states

The coherent state is defined as the eigenstate of the annihilator operator:

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle, \quad \alpha \in \mathbb{C}.$$

The expected number of photons in the coherent state is:

$$\langle\alpha|\hat{n}|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2.$$

Coherent states are the most classical states of the quantized EM field.

It has the following results:

$$\langle\alpha|\hat{a}^2|\alpha\rangle = \alpha^2, \quad \langle\alpha|\hat{a}^{\dagger 2}|\alpha\rangle = \alpha^{*2}, \quad \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2, \quad \langle\alpha|\hat{a}\hat{a}^\dagger|\alpha\rangle = |\alpha|^2 + 1.$$

For example, computing the expectation of the E-field (1.8) is:

$$\langle\boldsymbol{E}\rangle = \langle\alpha|i\sum_{\boldsymbol{k},\lambda}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}[\hat{a}_{\boldsymbol{k},\lambda}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \hat{a}^\dagger_{\boldsymbol{k},\lambda}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}]|\alpha\rangle$$

$$= i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}[\alpha e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \alpha^* e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}]$$

$$\langle\boldsymbol{E}\rangle = \boldsymbol{\mathcal{E}}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \boldsymbol{\mathcal{E}}^* e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \longrightarrow \boldsymbol{\mathcal{E}} = i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\alpha\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}.$$

The expectation value of the E-field for a coherent state corresponds to a classical E-field.

Let us now consider the uncertainty of the E-field for $|\alpha\rangle$:

$$\langle\Delta E^2\rangle = \langle\boldsymbol{E}^2\rangle - \langle\boldsymbol{E}\rangle^2$$

$$= \langle\alpha|i\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}[\hat{a}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \hat{a}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}] \cdot i\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}[\hat{a}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \hat{a}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}]|\alpha\rangle - \langle\boldsymbol{E}\rangle^2$$

$$= -\frac{\hbar\omega}{2\varepsilon_0 V}[\alpha^2 e^{i2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \alpha^{*2}e^{-i2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - |\alpha|^2 - (|\alpha|^2+1)] - \boldsymbol{\mathcal{E}}^2 e^{i2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \boldsymbol{\mathcal{E}}^{*2}e^{-i2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - 2|\boldsymbol{\mathcal{E}}|^2$$

$$= \boldsymbol{\mathcal{E}}^2 e^{i2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \boldsymbol{\mathcal{E}}^{*2}e^{-i2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + 2|\boldsymbol{\mathcal{E}}|^2 + \frac{\hbar\omega}{2\varepsilon_0 V} - \boldsymbol{\mathcal{E}}^2 e^{i2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \boldsymbol{\mathcal{E}}^{*2}e^{-2i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - 2|\boldsymbol{\mathcal{E}}|^2$$

$$\langle\Delta E^2\rangle = \frac{\hbar\omega}{2\varepsilon_0 V}.$$

This is the uncertainty in E-field limited by quantum noise.

Also, we know that the eigenstates of $|\alpha\rangle$ do not form an orthonormal basis. However, we know that the number state $|n\rangle$ form an orthonormal basis and we can expand the coherent state in terms of the number states, as follows,

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$

The probability of finding n photons in a coherent state is:

$$P_n = |\langle n|\alpha\rangle|^2 = \left| \langle n| e^{-|\alpha|^2/2} \sum_m \frac{\alpha^m}{\sqrt{m!}} |m\rangle \right|^2 = \left| e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \right|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}.$$

Recalling that $\langle \hat{n} \rangle = |\alpha|^2 = \overline{n}$:

$$\text{Probability of finding } n \text{ photons in a coherent state} \qquad P_n = \frac{e^{-\overline{n}} \overline{n}^n}{n!}.$$

1.4 Quadrature operators

Recall the E-field operator for a singel mode is:

$$\hat{\boldsymbol{E}}(\boldsymbol{r}, t) = i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} [\hat{a} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} - \hat{a}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}]$$

$$= i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} [\hat{a} e^{-i\omega t} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{a}^\dagger e^{i\omega t} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}]$$

$$= i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} [\hat{a}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{a}^\dagger(t) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}]$$

$$\hat{\boldsymbol{E}}(\boldsymbol{r}, t) = i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} [\hat{a}(t) \cos(\boldsymbol{k}\cdot\boldsymbol{r}) + i\hat{a}(t) \sin(\boldsymbol{k}\cdot\boldsymbol{r}) - \hat{a}^\dagger(t) \cos(\boldsymbol{k}\cdot\boldsymbol{r}) + i\hat{a}^\dagger(t) \sin(\boldsymbol{k}\cdot\boldsymbol{r})].$$

Lets define the quadrature operators:

$$\text{Quadrature operators} \qquad \begin{aligned} \hat{X}_1 &= \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \\ \hat{X}_2 &= \frac{1}{2i}(\hat{a} - \hat{a}^\dagger) \end{aligned}, \qquad (1.9)$$

where $[\hat{X}_1, \hat{X}_2] = \frac{i}{2}$. In the E-field, if we collect the cosine and sine, we get

$$\hat{\boldsymbol{E}}(\boldsymbol{r}, t) = -\sqrt{\frac{2\hbar\omega}{\varepsilon_0 V}} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} [\hat{X}_1(t) \sin(\boldsymbol{k}\cdot\boldsymbol{r}) + \hat{X}_2(t) \cos(\boldsymbol{k}\cdot\boldsymbol{r})].$$
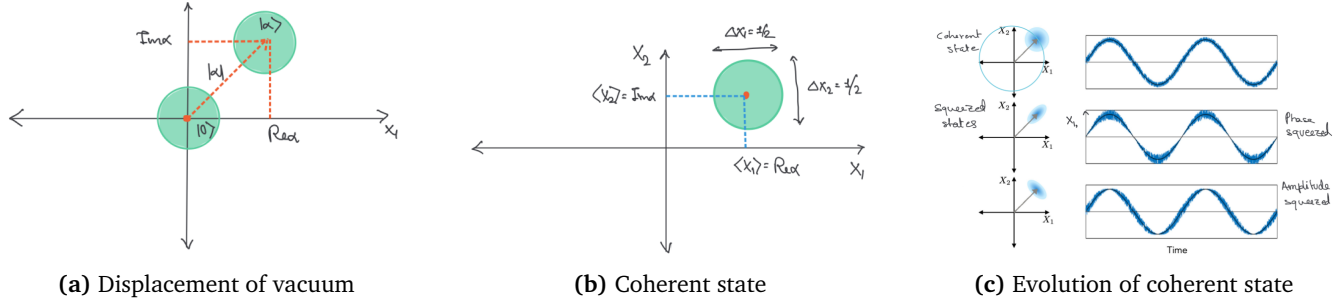
The expectation value of these operators is:

$$\langle \hat{X}_1 \rangle = \langle \alpha|\hat{X}_1(0)|\alpha\rangle = \langle \alpha|\frac{\hat{a} + \hat{a}^\dagger}{2}|\alpha\rangle = \text{Re}\,(\alpha), \quad \langle \hat{X}_2 \rangle = \langle \alpha|\hat{X}_2(0)|\alpha\rangle = \langle \alpha|\frac{\hat{a} - \hat{a}^\dagger}{2i}|\alpha\rangle = \text{Im}\,(\alpha),$$

$$\langle \Delta \hat{X}_1^2 \rangle = \langle \alpha|\frac{1}{4}(\hat{a} + \hat{a}^\dagger)^2|\alpha\rangle - \frac{1}{4}(\alpha + \alpha^*)^2 = \frac{1}{4}, \quad \langle \Delta \hat{X}_2^2 \rangle = \frac{1}{4}.$$

That is, there is a minimum uncertainty that a state can possible have in the two quadrature $\hat{X}_1, \hat{X}_2$, such a state os refferred as *minimum uncertainty state*. When $\alpha = 0$ we have the vaccum state and therefore $\langle \hat{X}_1(0) \rangle = \langle \hat{X}_2(0) \rangle = 0$, $\langle \hat{X}_1^2(0) \rangle = \langle \hat{X}_2^2(0) \rangle = 1/4$. Coherent states are also refferees to as displaced vaccum states. Specifically,

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle, \quad \hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}.$$

(a) Displacement of vacuum

(b) Coherent state

(c) Evolution of coherent state

The phase space representation of states allows one to access information aabout the noise/variance associated with any state when a specific E-field quadrature is measured.

1.4.1 Time dependence of quadrature operators

Using (1.9) and the inclusion of time-dependence yields

Time-dependent Quadrature operators

$$\begin{aligned} \hat{X}_1(t) &= \frac{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2} \\ \hat{X}_2(t) &= \frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i} \end{aligned}. \tag{1.10}$$

We can use them to express them in terms of $\hat{X}_1(0)$ and $\hat{X}_2(0)$.

$$\hat{a} + \hat{a}^\dagger = 2\hat{X}_1(0), \tag{1.11}$$

$$\hat{a} - \hat{a}^\dagger = 2i\hat{X}_2(0). \tag{1.12}$$

$$(1.11) + (1.12): \quad 2\hat{a} = 2[\hat{X}_1(0) + i\hat{X}_2(0)] \longrightarrow \hat{a} = \hat{X}_1(0) + i\hat{X}_2(0)$$

$$(1.11) - (1.12): \quad 2\hat{a}^\dagger = 2[\hat{X}_1(0) - i\hat{X}_2(0)] \longrightarrow \hat{a}^\dagger = \hat{X}_1(0) - i\hat{X}_2(0)$$

Using these in the time-evolved quadrature operators:

$$\hat{X}_1(t) = \frac{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2} = \frac{[\hat{X}_1(0) + i\hat{X}_2(0)]e^{-i\omega t} + [\hat{X}_1(0) - i\hat{X}_2(0)]e^{i\omega t}}{2}$$

$$\hat{X}_1(t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}\hat{X}_1(0) + \frac{(e^{i\omega t} - e^{-i\omega t})}{2i}\hat{X}_2(0).$$

Likewise,

$$\hat{X}_2(t) = \frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i} = \frac{[\hat{X}_1(0) + i\hat{X}_2(0)]e^{-i\omega t} - [\hat{X}_1(0) - i\hat{X}_2(0)]e^{i\omega t}}{2i}$$

$$\hat{X}_2(t) = -\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\hat{X}_1(0) + i\frac{(e^{i\omega t} + e^{-i\omega t})}{2i}\hat{X}_2(0).$$

Then, we have

$$\hat{X}_1(t) = \cos(\omega t)\hat{X}_1(0) + \sin(\omega t)\hat{X}_2(0),$$
$$\hat{X}_2(t) = -\sin(\omega t)\hat{X}_1(0) + \cos(\omega t)\hat{X}_2(0).$$

$$(1.13)$$

1.4.2 Taking the continuum limit

In free space the sum over all k becomes an integral:

$$\sum_{k} \to N \int d^2\mathbf{k} = N \int_0^\infty dk k^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi,$$

where N is the normalization equal to volume/mode. We consider a volume of quantization that is the region in free space where we are quantizing the field $V = L_x L_y L_z$. Allowed k vectors are $k_x = (2\pi/L_x)m_x$, $k_y = (2\pi/L_y)m_y$ y $k_z = (2\pi/L_z)m_z$, that is, we ensure that the lenths fit an integer number of wavelengths.

The number of modes in between k and $k + \Delta k$ is:

$$\Delta m_x \Delta m_y \Delta m_z = \frac{L_x L_y L_z}{(2\pi)^2}\Delta\mathbf{k}_x\Delta\mathbf{k}_y\Delta\mathbf{k}_z = \frac{V}{(2\pi)^3}\Delta\mathbf{k}_x\Delta\mathbf{k}_y\Delta\mathbf{k}_z.$$

Using the above normalization:

$$\sum_{k,\lambda} \to \sum_{\lambda} \int dk k^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \frac{V}{8\pi^2}.$$

Thus, any continuum limit involves

$$\text{Continuum limit} \qquad \sum_{k} \to \frac{V}{8\pi^2}\int d^3\mathbf{k}.$$

1.5 Squeezed states

Lets define the squeeze operator

$$\text{Squeezing state} \qquad |\xi, 0\rangle = \hat{S}(\xi)|0\rangle, \quad \text{with} \quad \hat{S}(\xi) = e^{\frac{1}{2}(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2})}, \quad \xi \in \mathbb{C}.$$

Lets use it to compute the expectation value for the quadrature operators.

$$\langle\xi, 0|\hat{X}_1|\xi, 0\rangle = \langle 0|\hat{S}^\dagger(\xi)\left(\frac{\hat{a} + \hat{a}^\dagger}{2}\right)\hat{S}(\xi)|0\rangle = \frac{1}{2}\langle 0|(\hat{S}^\dagger\hat{a}\hat{S} + \hat{S}^\dagger\hat{a}^\dagger\hat{S})|0\rangle.$$

We define displaced ladder operators as

$$\tilde{a} = \hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi), \quad \xi = re^{i\phi}.$$

Using the BCH formula

$$e^L A e^{-L} = A + [L, A] + \frac{1}{2!}[L, [L, A]] + \cdots$$

and $A = \hat{a}$ with $L = -\frac{1}{2}(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2})$ allow us to express the squeezing ladder operator

$$\text{Squeezing ladder operators} \qquad \begin{aligned} \tilde{a} &= \hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi) = \hat{a}\cosh(r) - \hat{a}^\dagger e^{i\phi}\sinh(r) \\ \tilde{a}^\dagger &= S(\xi)^\dagger\hat{a}^\dagger\hat{S}(\xi) = \hat{a}^\dagger\cosh(r) - \hat{a}e^{i\phi}\sinh(r). \end{aligned} \qquad (1.14)$$

We have used several orders. Its derivative wrt r is:

$$\frac{d\hat{a}}{dr} = \hat{a}\sinh(r) - \hat{a}^\dagger e^{i\phi}\cosh(r) = -e^{i\phi}\tilde{a}^\dagger, \quad \frac{d\tilde{a}^\dagger}{dr} = -e^{-i\phi}\tilde{a}.$$

Taking another derivative yields:

$$\frac{d^2\tilde{a}}{dr^2} = \tilde{a}.$$

Now we can continue the expectation of the quadrature:

$$\begin{aligned} \langle\hat{X}_1\rangle &= \frac{1}{2}\langle 0|(\hat{S}^\dagger\hat{a}\hat{S} + \hat{S}^\dagger\hat{a}^\dagger\hat{S})|0\rangle = \frac{1}{2}\langle 0|\left[\tilde{a} + \tilde{a}^\dagger\right]|0\rangle \\ &= \frac{1}{2}\langle 0|[\hat{a}\cosh(r) - e^{i\phi}\hat{a}^\dagger\sinh(r) + \hat{a}^\dagger\cosh(r) - e^{-i\phi}\sinh(r)]|0\rangle = 0. \end{aligned}$$
$$\langle\hat{X}_2\rangle = \text{similarly} = 0.$$

We can do the same for the operator square:

$$\begin{aligned} \langle\hat{X}_1^2\rangle &= \frac{1}{4}\langle 0|\hat{S}^\dagger(\xi)[(\hat{a} + \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger)]\hat{S}(\xi)|0\rangle = \frac{1}{4}\langle 0|\hat{S}^\dagger(\hat{a} + \hat{a}^\dagger)\hat{S}\hat{S}^\dagger(\hat{a} + \hat{a}^\dagger)\hat{S}|0\rangle \\ &= \frac{1}{4}\langle 0|(\tilde{a} + \tilde{a}^\dagger)(\tilde{a} + \tilde{a}^\dagger)|0\rangle = \frac{1}{4}\langle 0|e^{-2r}(\hat{a} + \hat{a}^\dagger)^2|0\rangle = \frac{e^{-2r}}{4}. \end{aligned}$$

Similarly we can obtain the respective quantities for $\hat{X}_2$ and the uncertainties are:

$$\langle\hat{X}_1\rangle = \langle\hat{X}_2\rangle = 0 \implies \Delta\hat{X}_1 = \frac{e^{-r}}{2}, \quad \Delta\hat{X}_2 = \frac{e^r}{2}.$$

1.5.1 Displaced squeezing state

Now,

$$\text{Displaced squeezing state} \qquad |\xi,\alpha\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle.$$

Now, the operators are:

$$\begin{aligned}\text{Displaced squeezing} \\ \text{operator}\end{aligned} \qquad \begin{aligned} \tilde{a} &= \hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha)\hat{S}(\xi) = \cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha, \\ \tilde{a}^\dagger &= \hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha)\hat{S}(\xi) = \cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*. \end{aligned} \qquad (1.15)$$

The expected number of photons in a displaced squeezed state is:

$$\langle 0|\tilde{a}^\dagger\tilde{a}|0\rangle = \langle 0|[\cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*][\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha]|0\rangle = \sinh^2|\xi| + |\alpha|^2.$$

The expected value in the quadrature value is:

$$\begin{aligned} \langle\xi,\alpha|\hat{X}_1|\xi,\alpha\rangle &= \langle 0|\hat{S}^\dagger\hat{D}^\dagger\hat{X}_1\hat{D}\hat{S}|0\rangle = \langle 0|\hat{S}^\dagger\hat{D}^\dagger(\frac{\hat{a} + \hat{a}^\dagger}{2})\hat{D}\hat{S}|0\rangle = \frac{1}{2}\langle 0|(\tilde{a} + \tilde{a}^\dagger)|0\rangle \\ &= \frac{1}{2}\langle 0|[\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha + \cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*]|0\rangle = \text{Re}\,(\alpha). \end{aligned}$$
$$\langle\xi,\alpha|\hat{X}_2|\xi,\alpha\rangle = \text{Im}\,(\alpha).$$

On the other hand, the quadrature uncertainties are:

$$\Delta \hat{X}_1^2 = \langle \xi, \alpha | \hat{X}_1^2 | \xi, \alpha \rangle - \langle \xi, \alpha | \hat{X}_1 | \xi, \alpha \rangle^2 = \langle 0 | \hat{S}^\dagger \hat{D}^\dagger \hat{X}_1 \hat{D} \hat{S} \hat{S}^\dagger \hat{D}^\dagger \hat{X}_1 \hat{D} \hat{S} | 0 \rangle - \text{Re} \left(\alpha \right)^2$$

$$= \langle 0 | \left(\frac{\tilde{a} + \tilde{a}^\dagger}{2} \right) \left(\frac{\tilde{a} + \tilde{a}^\dagger}{2} \right) | 0 \rangle - \text{Re} \left(\alpha \right)^2$$

$$= \frac{1}{4} \langle 0 | [\cosh(r)\hat{a} - e^{i\phi} \sinh(r)\hat{a}^\dagger + \alpha + \cosh(r)\hat{a}^\dagger - e^{-i\phi} \sinh(r)\hat{a} + \alpha^*]^2 | 0 \rangle - \text{Re} \left(\alpha \right)^2$$

$$= \frac{1}{4} (\cosh^2(r) + \sinh^2(r) - 2\cosh(r)\sinh(r)\cos\phi) + \text{Re} \left(\alpha \right)^2 - \text{Re} \left(\alpha \right)^2$$

$$\Delta \hat{X}_1^2 = \frac{1}{4} [\cosh^2(r) + \sinh^2(r) - 2\cosh(r)\sinh(r)\cos(\phi)].$$

$$\Delta \hat{X}_2^2 = \frac{1}{4} [\cosh^2(r) + \sinh^2(r) + 2\cosh(r)\sinh(r)\cos(\phi)].$$

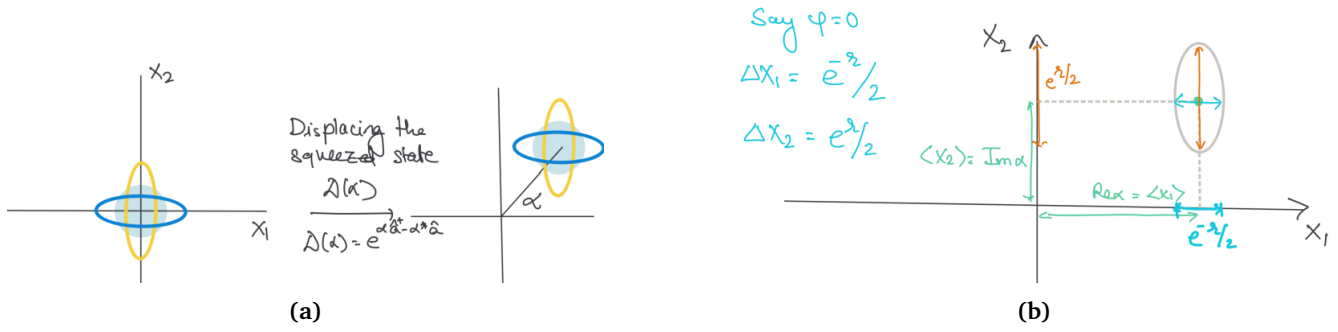**(a)**   **(b)**

Figure 1.1

1.5.2   Displaced squeezed temporal state

This is the most general case of the two above.

The expected value of $\hat{X}_1(t)$ is

$$\langle \hat{X}_1(t) \rangle = \langle 0 | \hat{S}^\dagger \hat{D}^\dagger \frac{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2} \hat{D} \hat{S} | 0 \rangle = \frac{1}{2} \langle 0 | \tilde{a}e^{-i\omega t} + \tilde{a}^\dagger e^{i\omega t} | 0 \rangle$$

$$= \frac{1}{2} \langle 0 | \left[\cosh(r)\hat{a}e^{-i\omega t} - e^{i\phi} \sinh(r)\hat{a}^\dagger e^{-i\omega t} + \alpha e^{-i\omega t} + \cosh(r)\hat{a}^\dagger e^{i\omega t} - e^{-i\phi} \sinh(r)\hat{a}e^{i\omega t} + \alpha^* e^{i\omega t} \right] | 0 \rangle$$

$$= \frac{1}{2} [\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}]$$

$$\langle \hat{X}_1(t) \rangle = \text{Re} \left(\alpha e^{-i\omega t} \right).$$

Similarly,

$$\langle \hat{X}_2(t) \rangle = \langle 0 | \hat{S}^\dagger \hat{D}^\dagger \frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i} \hat{D} \hat{S} | 0 \rangle = \frac{1}{2i} \langle 0 | \tilde{a}e^{-i\omega t} + \tilde{a}^\dagger e^{i\omega t} | 0 \rangle$$

$$= \frac{1}{2i} \langle 0 | \left[\cosh(r)\hat{a}e^{-i\omega t} - e^{i\phi} \sinh(r)\hat{a}^\dagger e^{-i\omega t} + \alpha e^{-i\omega t} - \cosh(r)\hat{a}^\dagger e^{i\omega t} + e^{-i\phi} \sinh(r)\hat{a}e^{i\omega t} - \alpha^* e^{i\omega t} \right] | 0 \rangle$$

$$= \frac{1}{2} [\alpha e^{-i\omega t} - \alpha^* e^{i\omega t}]$$

$$\langle \hat{X}_2(t) \rangle = \text{Im} \left(\alpha e^{-i\omega t} \right).$$

Doing the same for the operators squared:

$$\langle \hat{X}_1^2(t)\rangle = \langle 0|\hat{S}^\dagger \hat{D}^\dagger \left(\frac{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2}\right)^2 \hat{D}\hat{S}|0\rangle$$

$$= \frac{1}{4}\langle 0|\hat{S}^\dagger \hat{D}^\dagger \left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right)\hat{D}\hat{S}\hat{S}^\dagger\hat{D}^\dagger \left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right)\hat{D}\hat{S}|0\rangle$$

$$= \frac{1}{4}\langle 0|(\tilde{a}e^{-i\omega t} + \tilde{a}^\dagger e^{i\omega t})^2|0\rangle$$

$$= \frac{1}{4}\left[e^{-i2\omega t}\langle 0|\tilde{a}^2|0\rangle + e^{i2\omega t}\langle 0|\tilde{a}^{\dagger 2}|0\rangle + \langle 0|\tilde{a}\tilde{a}^\dagger|0\rangle + \langle 0|\tilde{a}^\dagger\tilde{a}|0\rangle\right].$$

We compute each term separately:

$$\langle 0|\tilde{a}^2|0\rangle = \langle 0|(\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha)(\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha)|0\rangle$$

$$= \cosh^2(r)\langle \hat{a}^2\rangle - e^{i\phi}\sinh(r)\cosh(r)\langle \hat{a}\hat{a}^\dagger\rangle + \alpha\cosh(r)\langle \hat{a}\rangle - e^{i\phi}\sinh(r)\cosh(r)\langle \hat{a}^\dagger\hat{a}\rangle$$

$$+ e^{i2\phi}\sinh^2(r)\langle \hat{a}^{\dagger 2}\rangle - \alpha e^{i\phi}\sinh(r)\langle \hat{a}^\dagger\rangle + \alpha\cosh(r)\langle \hat{a}\rangle - \alpha e^{i\phi}\sinh(r)\langle \hat{a}^\dagger\rangle + \alpha^2$$

$$= \alpha^2 - e^{i\phi}\sinh(r)\cosh(r)$$

$$\langle 0|\tilde{a}^2|0\rangle = \alpha^2 - \frac{1}{2}e^{i\phi}\sinh(2r).$$

$$\langle 0|\tilde{a}^{\dagger 2}|0\rangle = \langle 0|(\cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*)(\cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*)|0\rangle$$

$$= \cosh^2(r)\langle \hat{a}^{\dagger 2}\rangle - e^{-i\phi}\cosh(r)\sinh(r)\langle \hat{a}^\dagger\hat{a}\rangle + \alpha^*\cosh(r)\langle \hat{a}^\dagger\rangle - e^{-i\phi}\sinh(r)\cosh(r)\langle \hat{a}\hat{a}^\dagger\rangle$$

$$+ e^{-i2\phi}\sinh^2(r)\langle \hat{a}^2\rangle - \alpha^* e^{-i\phi}\sinh(r)\langle \hat{a}\rangle + \alpha^*\cosh(r)\langle \hat{a}^\dagger\rangle - \alpha^* e^{-i\phi}\sinh(r)\langle \hat{a}^\dagger\rangle + \alpha^{*2}$$

$$= \alpha^{*2} - e^{-i\phi}\sinh(r)\cosh(r)$$

$$\langle 0|\tilde{a}^{\dagger 2}|0\rangle = \alpha^{*2} - \frac{1}{2}e^{-i\phi}\sinh(2r).$$

$$\langle 0|\tilde{a}\tilde{a}^\dagger|0\rangle = \langle 0|(\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha)(\cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*)|0\rangle$$

$$\langle 0|\tilde{a}\tilde{a}^\dagger|0\rangle = \cosh^2(r) + |\alpha|^2.$$

$$\langle 0|\tilde{a}^\dagger\tilde{a}|0\rangle = \langle 0|(\cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*)(\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha)|0\rangle$$

$$\langle 0|\tilde{a}^\dagger\tilde{a}|0\rangle = \sinh^2(r) + |\alpha|^2.$$

Putting all together:

$$\langle \hat{X}_1^2(t)\rangle = \frac{1}{4}\left\{[\alpha^2 - \frac{1}{2}e^{i\phi}\sinh(2r)]e^{-i2\omega t} + [\alpha^{*2} - \frac{1}{2}e^{-i\phi}\sinh(2r)]e^{i2\omega t} + \cosh^2(r) + |\alpha|^2 + \sinh^2(r) + |\alpha|^2\right\}$$

$$= \frac{1}{4}\left[(\alpha e^{-i\omega t} + \alpha^* e^{i\omega t})^2 + \cosh(2r) - \sinh(2r)\cos(\phi - 2\omega t)\right]$$

$$\langle \hat{X}_1^2(t)\rangle = [\text{Re}\left(\alpha e^{-i\omega t}\right)]^2 + \frac{1}{4}[\cosh(2r) - \sinh(2r)\cos(\phi - 2\omega t)].$$

The uncertainty is:

$$\Delta \hat{X}_1(t) = \sqrt{\langle \hat{X}_1^2(t)\rangle - \langle \hat{X}_1(t)\rangle^2} = \frac{1}{2}\sqrt{\cosh(2r) - \sinh(2r)\cos(\phi - 2\omega t)}.$$

Similarly,

$$\langle \hat{X}_2^2(t) \rangle = \langle 0| \hat{S}^\dagger \hat{D}^\dagger \left(\frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i} \right)^2 \hat{D}\hat{S}|0\rangle$$

$$= -\frac{1}{4}\langle 0|\hat{S}^\dagger \hat{D}^\dagger (\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t})\hat{D}\hat{S}\hat{S}^\dagger \hat{D}^\dagger (\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t})\hat{D}\hat{S}|0\rangle$$

$$= -\frac{1}{4}\langle 0|(\tilde{a}e^{-i\omega t} - \tilde{a}^\dagger e^{i\omega t})^2|0\rangle$$

$$= -\frac{1}{4}[e^{-i2\omega t}\langle 0|\tilde{a}^2|0\rangle + e^{i2\omega t}\langle 0|\tilde{a}^{\dagger 2}|0\rangle - \langle 0|\tilde{a}\tilde{a}^\dagger|0\rangle - \langle 0|\tilde{a}^\dagger \tilde{a}|0\rangle].$$

We use the previous results to get

$$\langle \hat{X}_2^2(t) \rangle = -\frac{1}{4}\left\{ [\alpha^2 - \frac{1}{2}e^{i\phi}\sinh(2r)]e^{-i2\omega t} + [\alpha^{*2} - \frac{1}{2}e^{-i\phi}\sinh(2r)]e^{i2\omega t} - \cosh^2(r) - |\alpha|^2 - \sinh^2(r) - |\alpha|^2 \right\}$$

$$= -\frac{1}{4}[(\alpha e^{-i\omega t} - \alpha^* e^{i\omega t})^2 - \cosh(2r) - \sinh(2r)\cos(\phi - 2\omega t)]$$

$$\langle \hat{X}_2^2(t) \rangle = [\mathrm{Im}\left(\alpha e^{-i\omega t}\right)]^2 + \frac{1}{4}[\cosh(2r) + \sinh(2r)\cos(\phi - 2\omega t)].$$

And the uncertainty is:

$$\Delta \hat{X}_2(t) = \sqrt{\langle \hat{X}_2^2(t) \rangle - \langle \hat{X}_2(t) \rangle^2} = \frac{1}{2}\sqrt{\cosh(2r) + \sinh(2r)\cos(\phi - 2\omega t)}$$

We can verify that $\Delta \hat{X}_1(0) = e^{-r}/2$ and $\Delta \hat{X}_2(0) = e^r/2$ when $\phi = t = 0$. The following table sumarizes the most improtant results for the most general case from which we can get the squeezed or displaced states.

Table 1.1 Summary of the displaced squeezed temporal state $|\xi, \alpha\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle$ with $\xi = r\underline{/\phi}$.

BCH formula	$e^L A e^{-L} = A + [L, A] + \frac{1}{2!}[L, [L, A]] + \cdots$
Displaced and Squeezed operator	$\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} \qquad \hat{S}(\xi) = e^{\frac{1}{2}(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2})}$
Quadrature operators	$\hat{X}_1(t) = \frac{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2} \qquad \hat{X}_2(t) = \frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega}}{2i}$
DS Operator	$\tilde{a} = \hat{S}^\dagger \hat{D}^\dagger \hat{a}\hat{D}\hat{S} = \cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha$
Mean value	$\langle \hat{X}_1(t) \rangle = \mathrm{Re}\left(\alpha e^{-i\omega t}\right) \qquad \langle \hat{X}_2(t) \rangle = \mathrm{Im}\left(\alpha e^{-i\omega t}\right)$
Variance	$\langle \hat{X}_1^2(t) \rangle = \langle \hat{X}_1(t) \rangle^2 + \frac{1}{4}[\cosh(2r) - \sinh(2r)\cos(\phi - 2\omega t)]$
	$\langle \hat{X}_2^2(t) \rangle = \langle \hat{X}_2(t) \rangle^2 + \frac{1}{4}[\cosh(2r) + \sinh(2r)\cos(\phi - 2\omega t)]$

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Chapter 2

Optical devices

2.1 **Beam splitter**

Let us consider the action of a beam splitter (BS) on two input modes of same frequency and polarization. Clasically, the output E-field can be defined in terms of the input E-fields:

$$\begin{bmatrix} E_1' \\ E_2' \end{bmatrix} = \begin{bmatrix} t' & r \\ r' & t \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}.$$

$\{r, t\}$ and $\{r', t'\}$ are reflection/transmission coefficient associated with the two interfaces of the BS. From energy conservation, we must have

$$\text{Energy conservation in the BS} \qquad |E_1|^2 + |E_2|^2 = |E_1'|^2 + |E_2'|^2.$$

If we assume $E_2 = 0$, then

$$|E_1|^2 = |t'E_1|^2 + |r'E_1|^2 \longrightarrow |r'|^2 + |t'|^2 = 1. \tag{2.1}$$

Similarly, setting $E_1 = 0$ yields

$$|E_2|^2 = |rE_2|^2 + |tE_2|^2 \longrightarrow |r|^2 + |t|^2 = 1. \tag{2.2}$$

In general,

$$
\begin{aligned}
|E_1|^2 + |E_2|^2 &= |E_1'|^2 + |E_2'|^2 \\
&= (t'E_1 + rE_2)(t'^*E_1^* + r^*E_2^*) + (r'E_1 + tE_2)(r'^*E_1^* + t^*E_2^*) \\
&= |t'|^2|E_1|^2 + |r|^2|E_2|^2 + |r'|^2|E_1|^2 + |t|^2|E_2|^2 + rt'^*E_1^*E_2 + t'r^*E_1E_2^* + r't^*E_1E_2^* + r'^*tE_1^*E_2 \\
|E_1|^2 + |E_2|^2 &= |E_1|^2(|r'|^2 + |t'|^2) + |E_2|^2(|r|^2 + |t|^2) + E_1^*E_2(rt'^* + r'^*t) + E_1E_2^*(r't^* + r^*t') \\
0 &= E_1^*E_2(rt'^* + r'^*t) + c.c.
\end{aligned}
$$

Thus,

$$rt'^* + r'^*t = 0. \tag{2.3}$$

For a $50 : 50$ BS which as $r' = 1/\sqrt{2}$, $t' = 1/\sqrt{2}$, and $r = 1/\sqrt{2}$, (2.1), (2.2) and (2.3) implies that $t = -1/\sqrt{2}$. Similarly, the quantum description of the BS is

$$\begin{bmatrix} \hat{a}_1' \\ \hat{a}_2' \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix}. \tag{2.4}$$

The ladder operators are equal to the E-fields. In addition, the output modes are written as a linear combination of the input modes:
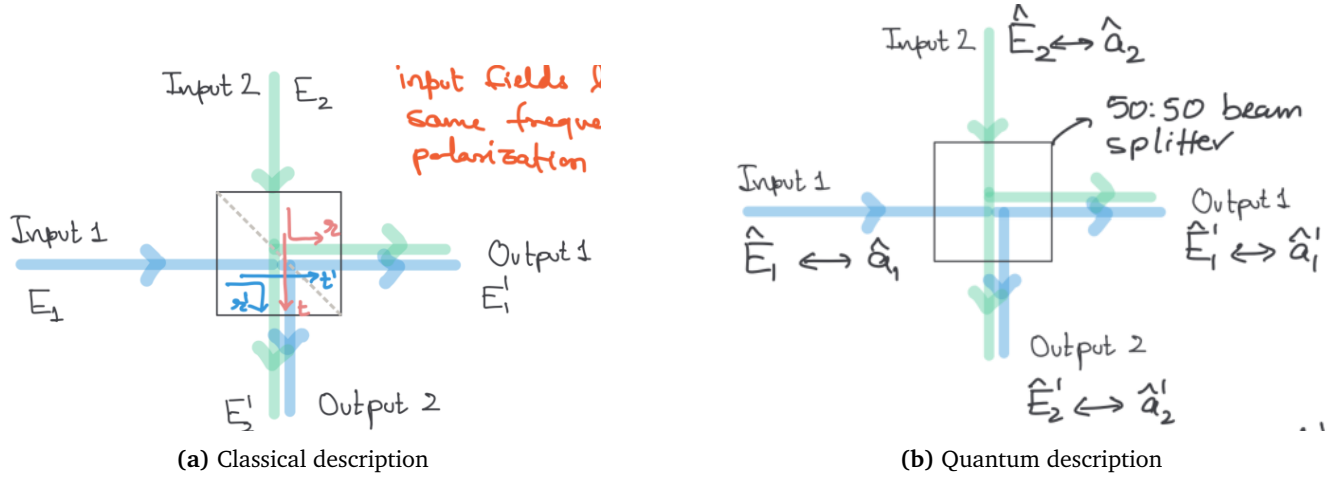
$$
\begin{aligned}
\hat{a}_1' &= \tfrac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2) \\
\hat{a}_2' &= \tfrac{1}{\sqrt{2}}(\hat{a}_1 - \hat{a}_2)
\end{aligned}
\quad \longrightarrow \quad
\begin{aligned}
\hat{a}_1 &= \tfrac{1}{\sqrt{2}}(\hat{a}_1' + \hat{a}_2') \\
\hat{a}_2 &= \tfrac{1}{\sqrt{2}}(\hat{a}_1' - \hat{a}_2')
\end{aligned}
$$

Ejemplo 2.1   **Single photon incident in one input**

The input is:

$$\text{Input} = |1\rangle_1 |0\rangle_2 = (\hat{a}_1^\dagger |0\rangle_1) |0\rangle_2 = |1\rangle_1 |0\rangle_2.$$

(a) Classical description   **(b)** Quantum description

Figure 2.1 Description of a beamsplitter (BS).

Note that

$$\hat{a}_1 = \frac{1}{\sqrt{2}}(\hat{a}_1' + \hat{a}_2') \quad \text{and} \quad \hat{a}_1^\dagger = \frac{1}{\sqrt{2}}(\hat{a}_1'^\dagger + \hat{a}_2'^\dagger).$$

The output is then

$$\text{Output} = \frac{1}{\sqrt{2}}(\hat{a}_1'^\dagger + \hat{a}_2'^\dagger)\left|0\right\rangle_{1'}\left|0\right\rangle_{2'} = \frac{1}{\sqrt{2}}[\left|1\right\rangle_{1'}\left|0\right\rangle_{2'} + \left|0\right\rangle_{1'}\left|1\right\rangle_{2'}].$$

Thus, there is a $50:50$ probability of detecting the photon in one of the ouput ports.

Ejemplo 2.2   **Both input with photons**

The input is:

$$\text{Input} = \left|1\right\rangle_1\left|1\right\rangle_2 = \hat{a}_1^\dagger\left|0\right\rangle_1\,\hat{a}_2^\dagger\left|0\right\rangle_2.$$

The output is:
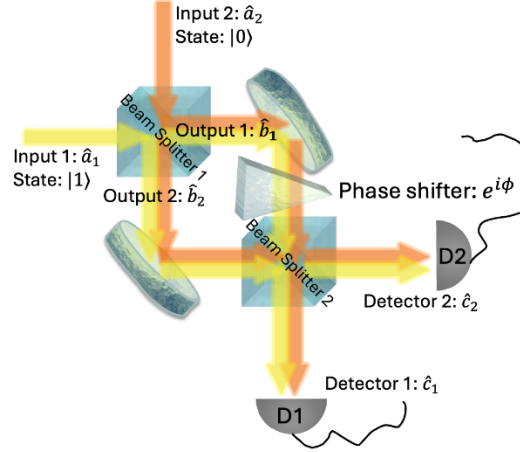
$$\text{Output} = \frac{1}{2}[\hat{a}_1'^\dagger + \hat{a}_2'^\dagger][\hat{a}_1'^\dagger - \hat{a}_2'^\dagger]\left|0\right\rangle_{1'}\left|0\right\rangle_{2'}$$

$$= \frac{1}{2}[\hat{a}_1'^\dagger\hat{a}_1'^\dagger - \hat{a}_2'^\dagger\hat{a}_2'^\dagger + \hat{a}_1'^\dagger\hat{a}_2'^\dagger - \hat{a}_2'^\dagger\hat{a}_1'^\dagger]\left|0\right\rangle_{1'}\left|0\right\rangle_{2'}$$

$$= \frac{1}{2}[\sqrt{2}\left|2\right\rangle_{1'}\left|0\right\rangle_{2'} - \sqrt{2}\left|0\right\rangle_{1'}\left|2\right\rangle_{2'}]$$

$$\text{Output} = \frac{1}{\sqrt{2}}[\left|2\right\rangle_{1'}\left|0\right\rangle_{2'} - \left|0\right\rangle_{1'}\left|2\right\rangle_{2'}].$$

Thus, when two photons are simultaneously incident on each input, both either go into output 1 or output 2 but never in both a the same time.

Ejemplo 2.3   **Mach-Zehnder**

a) Express the output modes $\{\hat{b}_1, \hat{b}_2\}$ in terms of the input modes $\{\hat{a}_1, \hat{a}_2\}$ assuming the first BS is $50:50$.

b) What is the expected number of photons $\langle \hat{b}_i^\dagger \hat{b}_i \rangle$ in each mode?

c) If $\hat{b}_1 \to \hat{b}_1 e^{i\phi}$, determine the output mode operators are the detector positions $\hat{c}_1 = \frac{1}{\sqrt{2}}(\hat{b}_1 e^{i\phi} + \hat{b}_2)$ and $\hat{b}_2 = \frac{1}{\sqrt{2}}(\hat{b}_1 e^{i\phi} - \hat{b}_2)$ in terms of the input mode operators $\{\hat{a}_1, \hat{a}_2\}$.

d) What is the number of photons detected at detector 1 as a function of ϕ?

Solution

(a) For the $50 : 50$ beamsplitter 1, we have:

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix}, \quad \text{with} \quad \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix}.$$

(b) From the last part, we know that

$$\hat{a}_1 = \frac{1}{\sqrt{2}}(\hat{b}_1 + \hat{b}_2) \xrightarrow{h.c.} \hat{a}_1^\dagger = \frac{1}{\sqrt{2}}(\hat{b}_1^\dagger + \hat{b}_2^\dagger), \quad \text{and} \quad \hat{a}_2 = \frac{1}{\sqrt{2}}(\hat{b}_1 - \hat{b}_2) \xrightarrow{h.c.} \hat{a}_2^\dagger = \frac{1}{\sqrt{2}}(\hat{b}_1^\dagger - \hat{b}_2^\dagger).$$

The input is then:

$$\text{Input} = |1\rangle_1 |0\rangle_2 = \hat{a}_1^\dagger |0\rangle_1 |0\rangle_2.$$

Output is:

$$\text{Output} = \frac{1}{\sqrt{2}}(\hat{b}_1^\dagger + \hat{b}_2^\dagger) |0\rangle_1 |0\rangle_2 = \frac{1}{\sqrt{2}}[|1\rangle_{1'} |0\rangle_{2'} + |0\rangle_{1'} |1\rangle_{2'}].$$

We expect then to have the same probability of getting one photon at each output.
The expected value at output 1 is:

$$\begin{aligned} {}_2\langle 0|_1 \langle 1| \hat{b}_1^\dagger \hat{b}_1 |1\rangle_1 |0\rangle_2 &= \frac{1}{2}\langle 0,1|(\hat{a}_1^\dagger + \hat{a}_2^\dagger)(\hat{a}_1 + \hat{a}_2)|1,0\rangle \\ &= \frac{1}{2}\langle 0,1|(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2)|1,0\rangle \end{aligned}$$

$${}_2\langle 0|_1 \langle 1| \hat{b}_1^\dagger \hat{b}_1 |1\rangle_1 |0\rangle_2 = \frac{1}{2}.$$

Likewise, the expected value at output 2 is:

$$\begin{aligned} {}_2\langle 0|_1 \langle 1| \hat{b}_2^\dagger \hat{b}_2 |1\rangle_1 |0\rangle_2 &= \frac{1}{2}\langle 0,1|(\hat{a}_1^\dagger - \hat{a}_2^\dagger)(\hat{a}_1 - \hat{a}_2)|1,0\rangle \\ &= \frac{1}{2}\langle 0,1|(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2)|1,0\rangle \end{aligned}$$

$${}_2\langle 0|_1 \langle 1| \hat{b}_2^\dagger \hat{b}_2 |1\rangle_1 |0\rangle_2 = \frac{1}{2}.$$

(c) We replace the first output as the input of the second output:

$$\hat{c}_1 = \frac{1}{\sqrt{2}}[\hat{b}_1 e^{i\phi} + \hat{b}_2] = \frac{1}{2}\left[(\hat{a}_1 + \hat{a}_2)e^{i\phi} + (\hat{a}_1 - \hat{a}_2)\right] = \frac{1}{2}[\hat{a}_1(1 + e^{i\phi}) + \hat{a}_2(e^{i\phi} - 1)],$$

$$\hat{c}_2 = \frac{1}{\sqrt{2}}[\hat{b}_1 e^{i\phi} - \hat{b}_2] = \frac{1}{2}\left[(\hat{a}_1 + \hat{a}_2)e^{i\phi} - (\hat{a}_1 - \hat{a}_2)\right] = \frac{1}{2}[\hat{a}_1(e^{i\phi} - 1) + \hat{a}_2(1 + e^{i\phi})].$$

(d) The expected number of photons at detector 1 is:

$$\langle 0, 1|\hat{c}_1^\dagger \hat{c}_1|1, 0\rangle = \frac{1}{4}\left\{[\hat{a}_1^\dagger(1 + e^{-i\phi}) + \hat{a}_2^\dagger(e^{-i\phi} - 1)][\hat{a}_1(1 + e^{i\phi}) + \hat{a}_2(e^{i\phi} - 1)]\right\}$$

$$= \frac{1}{4}\langle 0, 1|\left\{\hat{a}_1^\dagger \hat{a}_1(\cdots) + \hat{a}_1^\dagger \hat{a}_2(\cdots) + \hat{a}_2^\dagger \hat{a}_1(\cdots) + \hat{a}_2^\dagger \hat{a}_2(\cdots)\right\}|1, 0\rangle$$

$$= \frac{1}{4}(1 + e^{-i\phi})(1 + e^{i\phi})$$

$$\langle 0, 1|\hat{c}_1^\dagger \hat{c}_1|1, 0\rangle = \frac{1}{2}[1 + \cos\phi] = \cos^2\frac{\phi}{2}.$$

By conservation of energy (or probability), the expected number of photon at detector 2 is:

$$\langle 0, 1|\hat{c}_2^\dagger \hat{c}_2|1, 0\rangle = \sin^2\frac{\phi}{2}.$$
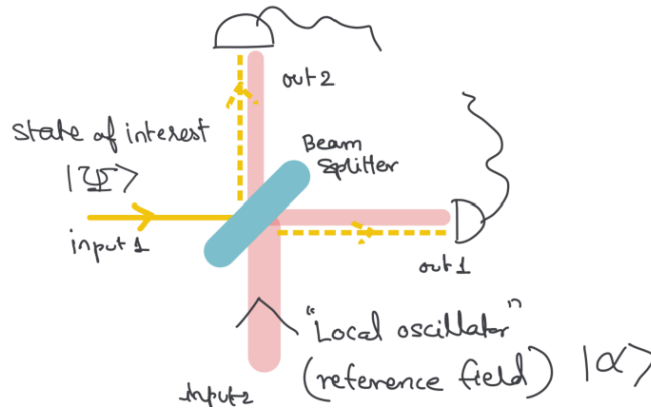
Chapter 3

Nonclassical light

3.1   Detection of quadrature states

Modes at the beam splitter output are:

$$\hat{A}_1 = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2), \quad \hat{A}_2 = \frac{1}{\sqrt{2}}(\hat{a}_1 - \hat{a}_2).$$

Detectors D_1, D_2 measure the intensity at each output:

$$\langle \hat{I}_k \rangle = \langle \hat{A}_k^\dagger \hat{A}_k \rangle.$$

Figure 3.1 Balanced homodyne detection.

In output 1, we have

$$I_1 = \langle \psi_1 \alpha_2 | \hat{A}_1^\dagger \hat{A}_1 | \alpha_2 \Psi_1 \rangle = \langle \psi_1 \alpha_2 | \frac{\hat{a}_1^\dagger + \hat{a}_2^\dagger}{\sqrt{2}} \frac{\hat{a}_1 + \hat{a}_2}{\sqrt{2}} | \alpha_2 \Psi_1 \rangle = \frac{1}{2} \langle \Psi_1 \alpha_2 | (\hat{a}_1^\dagger + \alpha^*)(\hat{a}_1 + \alpha) | \alpha_2 \Psi_1 \rangle$$

$$= \frac{1}{2} \langle \Psi_1 | (\hat{a}^\dagger \hat{a}_1 + \hat{a}_1 \alpha^* + \hat{a}_1^\dagger \alpha + |\alpha|^2) | \Psi_1 \rangle$$

We see that the irradiance is just the average of the number operator counting photons at each output. In particular, we can change the nature of α_2 to have different quadratures:

$$I_1 - I_2 = \langle \Psi_1 | \hat{a}_1 \alpha_2^* + \hat{a}_1^\dagger \alpha_2 | \Psi_1 \rangle = \begin{cases} 2\alpha_2 \langle \Psi_1 | \dfrac{\hat{a}_1 + \hat{a}_1^\dagger}{2} | \Psi_1 \rangle = 2\alpha_2 \langle \Psi_1 | \hat{X}_1 | \Psi_1 \rangle, & \alpha_2 = \alpha_2^*, \\ -2i\alpha_2 \langle \Psi_1 | \dfrac{\hat{a}_1 - \hat{a}_1^\dagger}{2i} | \Psi_1 \rangle = -2i\alpha_2 \langle \Psi_1 | \hat{X}_2 | \Psi_1 \rangle, & \alpha_2 = -\alpha_2^*. \end{cases}$$

In general by changing the phase of input 2 we can measure a general quadrature. Noise in intensity corresponds to noise in specific quadratures. If a quadrature measurement has uncertainty $\Delta \hat{X}^2 < 1/4$.

3.2   Generation of squeezed states

The following is a scheme of a nonlinear medium. Radiation from oscillating dipoles is of different frequency as the input stimulation. Nonlinear polarization induced in a medium by an E-field $\boldsymbol{E}$ is:
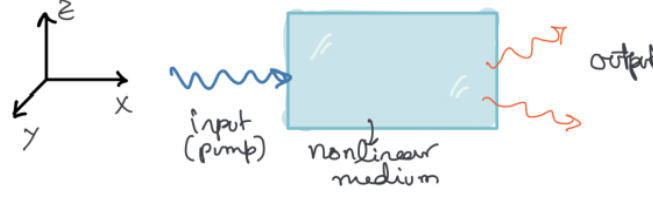
Figure 3.1 Nonlinear medium

$$\boldsymbol{P}_i = \varepsilon_0 \left[\chi^{(1)}_{jk} \boldsymbol{E}_k + \varepsilon_0 \chi^{(2)}_{jkl} \boldsymbol{E}_k \boldsymbol{E}_l + \cdots \right].$$

This polarization is for one dipole, the total would be a sum over all of them. Thses quantities are tensor. In particular, we have a linear and second-order polarization terms:

$$\boldsymbol{P}_{Lj} = \text{Linear polarization} = \varepsilon_0 \chi^{(1)}_{jk} \boldsymbol{E}_k$$

$$\boldsymbol{P}_{NLj} = \text{Second-order polarization} = \varepsilon_0 \chi^{(2)}_{jkl} \boldsymbol{E}_k \boldsymbol{E}_l.$$

The interaction (dipole-field) energy density associated with the field becomes:

$$U = -\boldsymbol{P} \cdot \boldsymbol{E} = -\varepsilon_0 [\chi^{(1)}_{jk} \boldsymbol{E}_k + \chi^{(2)}_{jkl} \boldsymbol{E}_j \boldsymbol{E}_k] \boldsymbol{E}_l = \underbrace{-\varepsilon_0 \chi^{(1)}_{jk} \boldsymbol{E}_k \boldsymbol{E}_l}_{U_L} + \underbrace{-\varepsilon_0 \chi^{(2)}_{jkl} \boldsymbol{E}_j \boldsymbol{E}_k \boldsymbol{E}_l}_{U^{(2)}_{NL} \ \text{(Squeezed output)}}.$$

Nonlinear-Hamiltonian is therefore

$$H^{(2)}_{NL} = -\int dV U^{(2)}_{NL}, \tag{3.1}$$

where the integration region is the medium. We will analyze the above Hamoltonian for the specific interaction where we have an incident blue photon spontaneously down-converted to two red photons.

The total E-field is:

$$\hat{\boldsymbol{E}}_{tot} = \boldsymbol{E}_B + \hat{\boldsymbol{E}}, \qquad \begin{matrix} \boldsymbol{E}_B = \text{Classical blue field} \\ \hat{\boldsymbol{E}} = \text{Quantized E-field} \end{matrix}.$$

Hamiltonian becomes

$$\hat{H}^{(2)}_{NL} = -\varepsilon_0 \int dV \, \hat{H}^{(2)}_{jkl} (\boldsymbol{E}_B + \hat{\boldsymbol{E}})_j (\boldsymbol{E}_B + \hat{\boldsymbol{E}})_k (\boldsymbol{E}_B + \hat{\boldsymbol{E}})_l$$

For further simplification let us assume that only $H^{(2)}_{zzz} \neq 0 = H^{(2)}$:

$$H^{(2)}_{NL} = -3\varepsilon_0 \chi^{(2)} \int dV \, E_{Bz} \hat{E}_z \hat{E}_z,$$

where

$$\boldsymbol{E}_{Bz} = \hat{\boldsymbol{z}} [\mathcal{E}_B e^{i(\boldsymbol{k}_B \cdot \boldsymbol{r} - \omega_B t)} + \mathcal{E}^*_B e^{-i(\boldsymbol{k}_B \cdot \boldsymbol{r} - \omega_B t)}].$$

and

$$\hat{\boldsymbol{E}}_z = i \sum_{\boldsymbol{k}} \hat{\boldsymbol{z}} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} [\hat{a}_{\boldsymbol{k}} e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} + \hat{a}^\dagger_{\boldsymbol{k}} e^{-i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}]$$

Substituting the E-fields in the nonlinear Hamiltonian yields

$$\hat{H}_{NL}^{(2)} = -\frac{3}{2V}\varepsilon_0\chi^{(2)}\hbar \int dV[\mathcal{E}_B e^{i(\boldsymbol{k}_B\cdot\boldsymbol{r}-\omega_B t)} + \mathcal{E}_B^* e^{-i(\boldsymbol{k}_B\cdot\boldsymbol{r}-\omega_B t)}]$$
$$\sum_{\boldsymbol{k}_1}\sqrt{\omega_1}[\hat{a}_{\boldsymbol{k}}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \hat{a}_{\boldsymbol{k}}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}]\sum_{\boldsymbol{k}_2}\sqrt{\omega_2}[\hat{a}_{\boldsymbol{k}}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \hat{a}_{\boldsymbol{k}}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}].$$

Consider the time-dependent terms:

$$e^{i(\omega_B\pm\omega_1\pm\omega_2)t} \to 0$$

which averages to zero because they are all fast oscilatting. To get a non-zero contribution we choose $\omega_B = \omega_1 + \omega_2$.

Similarly, by energy conservation we have

$$\int dV e^{i(\boldsymbol{k}_B-\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}}$$

For this integral to not vanish we must have $\boldsymbol{k}_B = \boldsymbol{k}_1 + \boldsymbol{k}_2$ which iplies **momentum conservation**.
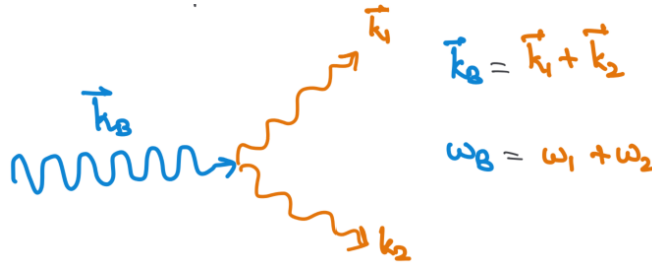

Figure 3.2 Momentum conservation.

This is the **spontaneous parametric downconservsion process** (SPDC).

Thus, keeping only the terms that conserve energy and momentum:

$$\hat{H}_{NL}^{(2)} = -\frac{3}{2}\hbar\chi^{(2)}\sqrt{\omega_1\omega_2}\left[\mathcal{E}_B\hat{a}_{\boldsymbol{k}_1}^\dagger\hat{a}_{\boldsymbol{k}_2}^\dagger + \mathcal{E}_B^*\hat{a}_{\boldsymbol{k}_1}\hat{a}_{\boldsymbol{k}_2}\right].$$

If the two modes are the same:

$$\hat{H}_{NL}^{(2)} = -\frac{1}{2}[\xi^*\hat{a}^2 + \xi\hat{a}^{\dagger 2}], \quad \xi = 3\hbar\chi^{(2)}\omega\mathcal{E}_B.$$

and

$$\omega_B = 2\omega, \quad \boldsymbol{k}_B = 2\boldsymbol{k}.$$

The corresponding evolution operator becomes

$$U(t,0) = e^{-i\hat{H}_{NL}^{(2)}t/\hbar} = e^{\frac{1}{2}[\xi^*\hat{a}^2-\xi^*\hat{a}^{\dagger 2}]}, \quad \xi = 3i\chi^{(2)}\mathcal{E}_B\omega t.$$

Thus starting with an initial vacuum state in the red modes the system one obtains a squeezed vacuum output.

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