

Notes of Quantum Optics

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Preface

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Chapter 1

Field quantization

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1.1 Electrodynamics review

1.1.1 Plane waves and algebra

1.1.2 Helmholtz and potentials

1.1.3 Wave equations

1.2 Quantization of a single-mode field

1.2.1 Fields in a cavity

Lets consider the following one-dimensional problem, where a cavity of length L is oriented along the z -axis.

A linear polarized E-field is assumed, the medium is free space, perfect conducting walls and there is no free charges nor free current. The scheme is shown in figure 1.1.

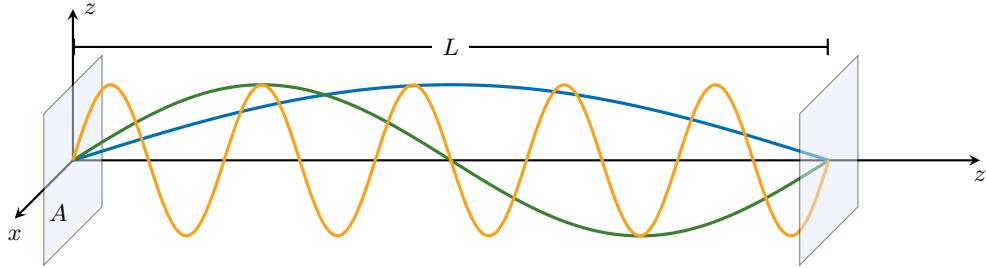


Figure 1.1 One-dimensional cavity problem. Perfect conducting walls.

Our goal is to find the E- and B-field inside the cavity. Maxwell's equations in this case are:

$$\text{Maxwell's equations with free sources} \quad \left\{ \begin{array}{l} \nabla \cdot \mathbf{E} = 0 \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\partial_t \mathbf{B} \\ \nabla \times \mathbf{B} = \frac{1}{c^2} \partial_t \mathbf{E} \end{array} \right. \quad (1.1)$$

The E-field will be assumed to be $\mathbf{E}(z, t) = e\mathbf{E}(z, t)$, where e is the polarization vector. Because fields depends only on z , $\nabla = \hat{z}\partial_z$. First Maxwell equation yields:

$$\nabla \cdot \mathbf{E} = \partial_z(\hat{z} \cdot \mathbf{E}) = \partial_z(\hat{z} \cdot e\mathbf{E}) = 0 \implies e \cdot \hat{z} = 0.$$

This implies that the polarization vector must be unitary in the transverse plane:

$$e = \cos \phi \hat{x} + \sin \phi \hat{y}.$$

Third Maxwell's equation yields

$$\nabla \times \mathbf{E} = (\hat{z}\partial_z) \times (e\mathbf{E}) = (\hat{z} \times e)\partial_z E = -\partial_t \mathbf{B}.$$

Taking the curl of this equation, using Fourth Maxwell's equation and vector identities:

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= -\partial_t(\nabla \times \mathbf{B}) \\ \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\frac{1}{c^2} \partial_t^2 \mathbf{E} \\ -e\partial_z^2 E &= -e\frac{1}{c^2} \partial_t^2 E. \end{aligned}$$

From here, we have the E-field wave equation for this particular problem.

$$\partial_z^2 E - \frac{1}{c^2} \partial_t^2 E = 0. \quad (1.2)$$

Before solving this equation, we need the boundary condition set by the PEC condition. We need to $\hat{n} \times \mathbf{E} = 0$ on the surface. Because the normal surface unit vector is $\hat{n} = \pm \hat{z}$, we have

$$\text{Boundary condition} \quad \hat{n} \times \mathbf{E} = \hat{z} \times (eE) = 0 \implies E(z=0, t) = E(z=L, t) = 0.$$

In order to solve the PDE, we assume a product form $E(z, t) = Z(z)q(t)$. Then, by replacing in it in (1.2):

$$\begin{aligned} \partial_z^2 [Z(z)q(t)] - \frac{1}{c^2} \partial_t^2 [Z(z)q(t)] &= 0 \\ Z''(z)q(t) - \frac{1}{c^2} Z(z)\ddot{q}(t) &= \cancel{[Z(z)q(t)]^{-1}} \\ \frac{Z''(z)}{Z(z)} &= \frac{1}{c^2} \frac{\ddot{q}(z)}{q(z)}. \end{aligned}$$

Left side depends only on z , while the right side only on y . The only way this can be true is if both are a constant, say, $-k^2$. Then,

$$\text{Spatial and temporal differential equations} \quad \left\{ \begin{array}{l} \frac{Z''}{Z} = -k^2 \implies Z'' + k^2 Z = 0 \\ \frac{1}{c^2} \frac{\ddot{T}}{T} = -k^2 \implies \ddot{q} + \omega^2 q(t) = 0 \end{array} \right..$$

For the spatial ODE, we assume a solution of the form

$$Z(z) = A \sin(kz) + B \cos(kz), \quad Z(0) = Z(L) = 0.$$

Setting the boundaries:

$$\begin{aligned} Z(0) = A \sin(0) + B \cos(0) = 0 &\implies B = 0 \\ Z(L) = A \sin(kL) = 0 &\implies k_m = \frac{m\pi}{L}, \quad m \in \mathbb{N}. \end{aligned}$$

We left the temporal ODE unsolved. Finally, putting all together yields the initial E-field:

$$\mathbf{E}_{k,\lambda}(z, t) = e_\lambda \sqrt{\frac{2\omega^2}{V\varepsilon_0}} q_{k,\lambda}(t) \sin(kz), \quad k_m = \frac{m\pi}{L}, \quad \omega = ck.$$

We have includning the subscript λ and k to consider multiple mode k varied with m and λ . Also, the coefficient in red is for better results in the future. Using Faraday's law:

$$\mathbf{B}_{k,\lambda}(z, t) = (\mathbf{k} \times \mathbf{e}_\lambda) \frac{1}{kc^2} \sqrt{\frac{2\omega^2}{V\varepsilon_0}} \dot{q}_{k,\lambda}(t) \sin(kz).$$

The term $\dot{q}(t)$ will play the role of a canonical momentum for a particle of unit mass, $p(t) = \dot{q}(t)$.

1.2.2 Single-mode Hamiltonian

The classical field energy, or Hamiltonian H , of the single-mode field is given by

$$\begin{aligned}
H_{\mathbf{k},\lambda} &= \frac{1}{2} \int dV \left[\varepsilon_0 \mathbf{E}_{\mathbf{k},\lambda}^2 + \frac{1}{\mu_0} \mathbf{B}_{\mathbf{k},\lambda}^2 \right] \\
&= \frac{1}{2} A \int_0^L dz \left[\frac{2\omega^2}{V\varepsilon_0} q_{\mathbf{k},\lambda}^2(t) \sin^2(kz) + \frac{1}{\mu_0} \frac{1}{k^2 c^4} \frac{2\omega^2}{V\varepsilon_0} p_{\mathbf{k},\lambda}^2(t) \cos^2(kz) \right] \\
&= \frac{A}{V} \int_0^L dz [\omega^2 q_{\mathbf{k},\lambda}^2(t) \sin^2(kz) + p_{\mathbf{k},\lambda}^2(t) \cos^2(kz)] \\
&= \frac{1}{L} \left[\omega^2 q_{\mathbf{k},\lambda}^2(t) \frac{L}{2} + p_{\mathbf{k},\lambda}^2(t) \frac{L}{2} \right] \\
H_{\mathbf{k},\lambda} &= \frac{1}{2} [\omega^2 q_{\mathbf{k},\lambda}^2(t) + p_{\mathbf{k},\lambda}^2(t)].
\end{aligned}$$

It is apparent that a single-mode field is formally equivalent to a harmonic quantum oscillator of unit mass, where the E- and B-fields play the roles of canonical position and momentum.

$$H_{\text{harmonic oscillator}} = \frac{1}{2} m \omega^2 x^2 + \frac{p^2}{2m}.$$

To begin the quantization, we make q, p operators \hat{q}, \hat{p} , which needs to satisfy the canonical commutation relations

$$[\hat{q}_{\mathbf{k},\lambda}, \hat{p}_{\mathbf{k},\lambda}] = i\hbar. \quad (1.3)$$

The EM fields with the operators are:

$$\begin{aligned}
\text{Quantized EM fields} \quad \hat{\mathbf{E}}_{\mathbf{k},\lambda}(z, t) &= \mathbf{e}_{\mathbf{k},\lambda} \sqrt{\frac{2\omega^2}{V\varepsilon_0}} \hat{q}(t) \sin(kz) \\
\hat{\mathbf{B}}_{\mathbf{k},\lambda}(z, t) &= (\mathbf{k} \times \mathbf{e}_{\mathbf{k},\lambda}) \frac{1}{kc^2} \sqrt{\frac{2\omega^2}{V\varepsilon_0}} \hat{p}(t) \cos(kz).
\end{aligned} \quad (1.4)$$

The operators \hat{q}, \hat{p} are Hermitian and therefore correspond to observable quantities. It is convenient to introduce the ladder operators to simplify the expression:

$$\begin{aligned}
\text{Ladder operator} \quad \hat{a}_{\mathbf{k},\lambda} &= \sqrt{\frac{1}{2\hbar\omega}} (\omega \hat{q}_{\mathbf{k},\lambda} + i \hat{p}_{\mathbf{k},\lambda}) \quad \rightarrow \quad \hat{q}_{\mathbf{k},\lambda} = \sqrt{\frac{\hbar}{2\omega}} (\hat{a}_{\mathbf{k},\lambda} + \hat{a}_{\mathbf{k},\lambda}^\dagger) \\
\hat{a}_{\mathbf{k},\lambda}^\dagger &= \sqrt{\frac{1}{2\hbar\omega}} (\omega \hat{q}_{\mathbf{k},\lambda} - i \hat{p}_{\mathbf{k},\lambda}) \quad \rightarrow \quad \hat{p}_{\mathbf{k},\lambda} = -i \sqrt{\frac{\hbar\omega}{2}} (\hat{a}_{\mathbf{k},\lambda} - \hat{a}_{\mathbf{k},\lambda}^\dagger).
\end{aligned}$$

Using these definitions, the Hamiltonian yields:

$$\begin{aligned}
\hat{H}_{\mathbf{k},\lambda} &= -\frac{1}{2} \frac{\hbar\omega}{2} (\hat{a}_{\mathbf{k},\lambda} - \hat{a}_{\mathbf{k},\lambda}^\dagger)(\hat{a}_{\mathbf{k},\lambda} - \hat{a}_{\mathbf{k},\lambda}^\dagger) + \frac{1}{2} \frac{\hbar\omega^2}{2\omega} (\hat{a}_{\mathbf{k},\lambda} + \hat{a}_{\mathbf{k},\lambda}^\dagger)(\hat{a}_{\mathbf{k},\lambda} + \hat{a}_{\mathbf{k},\lambda}^\dagger) \\
&= \frac{\hbar\omega}{2} [\hat{a}_{\mathbf{k},\lambda}^\dagger \hat{a}_{\mathbf{k},\lambda} + \hat{a}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda}^\dagger] = \hbar\omega \left[\hat{a}_{\mathbf{k},\lambda}^\dagger \hat{a}_{\mathbf{k},\lambda} + \frac{1}{2} \right] = \hbar\omega \left[n_{\mathbf{k},\lambda} + \frac{1}{2} \right].
\end{aligned}$$

The time dependence of the ladder operator in Heisenberg equation is:

$$\frac{d\hat{a}_{\mathbf{k},\lambda}}{dt} = \frac{i}{\hbar} [\hat{H}_{\mathbf{k},\lambda}, \hat{a}_{\mathbf{k},\lambda}] = \frac{i}{\hbar} \left[\hbar\omega \left(\hat{a}_{\mathbf{k},\lambda}^\dagger \hat{a}_{\mathbf{k},\lambda} + \frac{1}{2} \right), \hat{a}_{\mathbf{k},\lambda} \right] = -i\omega \hat{a}_{\mathbf{k},\lambda} \rightarrow \begin{aligned} \hat{a}_{\mathbf{k},\lambda}(t) &= \hat{a}_{\mathbf{k},\lambda}(0) e^{-i\omega t} \\ \hat{a}_{\mathbf{k},\lambda}^\dagger(t) &= \hat{a}_{\mathbf{k},\lambda}^\dagger(0) e^{i\omega t} \end{aligned}.$$

For example, we have the following applications:

$$\hat{a}_{\mathbf{k},\lambda} |0\rangle = |1\rangle_{\mathbf{k},\lambda}, \quad \hat{a}_{\mathbf{k},\lambda} |n\rangle_{\mathbf{k},\lambda} = \sqrt{n} |n-1\rangle_{\mathbf{k},\lambda}, \quad \hat{a}_{\mathbf{k},\lambda}^\dagger |n\rangle_{\mathbf{k}',\lambda'} = |n\rangle_{\mathbf{k},\lambda} |1\rangle_{\mathbf{k}',\lambda'}, \quad \hat{a}_{\mathbf{k},\lambda} |n\rangle_{\mathbf{k}',\lambda'} |0\rangle_{\mathbf{k},\omega} = 0.$$

The operator product $\hat{a}^\dagger \hat{a}$ has an important significance and is called the number operator \hat{n} , whose eigenequation is

$$\hat{n} |n\rangle = n |n\rangle.$$

The state $|n\rangle$ is the energy eigenstate of the single mode field with energy eigenvalue E_n :

$$\hat{H} |n\rangle = \hbar\omega (\hat{a}^\dagger \hat{a} + \frac{1}{2}) |n\rangle = E_n |n\rangle$$

The relation between the number state and the raising operator \hat{a}^\dagger is:

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n}} |0\rangle.$$

1.2.3 Quantizing the multimode field

We now consider all the modes in the optical cavity:

$$\mathbf{E}(z, t) = \sum_{m,\lambda} \hat{\mathbf{e}}_\lambda \sqrt{\frac{2\omega_m^2}{\varepsilon_0 V}} \hat{q}_{m,\lambda}(t) \sin(k_m z) \quad (1.5)$$

$$\mathbf{B}(z, t) = \sum_{m,\lambda} (\hat{\mathbf{z}} \times \hat{\mathbf{e}}_\lambda) \frac{1}{k_m c^2} \sqrt{\frac{2\omega_m^2}{\varepsilon_0 V}} \dot{\hat{q}}_{m,\lambda}(t) \cos(k_m z). \quad (1.6)$$

where $kL = m\pi$ and $\hat{\mathbf{e}}_\lambda = \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$. The process is analogous; we compute the Hamiltonian:

$$\begin{aligned} H &= \int dV \left[\frac{1}{2} \varepsilon_0 \mathbf{E}^2(z, t) + \frac{1}{2\mu_0} \mathbf{B}^2(z, t) \right] \\ &= \int dV \left[\frac{1}{2} \varepsilon_0 \left\{ \sum_{m,\alpha} \hat{\mathbf{e}}_\alpha \sqrt{\frac{2\omega_m^2}{\varepsilon_0 V}} q_{m,\alpha}(t) \sin(k_m z) \right\}^2 + \frac{1}{2\mu_0} \varepsilon_0 \left\{ \sum_{m,\alpha} (\hat{\mathbf{z}} \times \hat{\mathbf{e}}_\alpha) \frac{\mu_0}{\varepsilon_0} k_m \sqrt{\frac{2\omega_m^2}{\varepsilon_0 V}} \dot{q}_{m,\alpha}(t) \cos(k_m z) \right\}^2 \right] \\ &\vdots \\ H &= \frac{1}{2} \sum_{m,\lambda} (\omega_m^2 q_{m,\lambda}^2 + \dot{q}_{m,\lambda}^2). \end{aligned}$$

That is,

Each mode of the EM field is an independent harmonic oscillator.

We can quantize the multimode EM field in a similar way as the single mode field to obtain the Hamiltonian:

$$\text{Hamiltonian multimode EM field} \quad \hat{H} = \sum_{m,\lambda} \hbar\omega_m \left[\hat{a}_{m,\lambda}^\dagger a_{m,\lambda} + \frac{1}{2} \right] = \sum_{m,\lambda} \hbar\omega_m \left[\hat{n}_{m,\lambda} + \frac{1}{2} \right],$$

where $\hat{n}_{m,\lambda}$ is the number of excitations in mode m, λ .

1.2.4 Quantizing EM field in free space

We define the scalar and vector potential

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{E} = -\partial_t \mathbf{A} - \nabla \phi.$$

Using these automatically satisfy the Gauss magnetic equation and Ampere law. In QO, we assume **Coulomb gauge**: $\nabla \cdot \mathbf{A} = 0$. With this, the E-field is

$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla \phi = \mathbf{E}^\perp + \mathbf{E}^\parallel, \quad \text{where } \begin{cases} \nabla \cdot \mathbf{E}^\perp = 0, & \mathbf{A} \text{ corresponds to radiation (all transverse)} \\ \mathbf{E}^\parallel = -\nabla \phi, & \mathbf{A} \text{ corresponds to the field of sources (all longitudinal)} \end{cases}.$$

In Coulomb gauge, in absence of charge $\phi = 0$ such that the vector potential-wave equation is

$$\nabla^2 \mathbf{A} - \frac{1}{2} \partial_t^2 \mathbf{A} = 0.$$

It can be solved with the following plane wave

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}, \lambda} \hat{\mathbf{e}}_{\mathbf{k}, \lambda} \left[A_{\mathbf{k}, \lambda} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + A_{\mathbf{k}, \lambda}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \right], \quad \nabla \cdot \mathbf{A} = 0 \implies \mathbf{k} \cdot \hat{\mathbf{e}}_{\mathbf{k}, \lambda} = 0.$$

The vector potential in free space is expressed as a superposition of plane waves such that linear dispersion $\omega = |\mathbf{k}|c$ is satisfied with $\hat{\mathbf{e}}_{\mathbf{k}, \lambda} \cdot \hat{\mathbf{e}}_{\mathbf{k}, \lambda'} = \delta_{\lambda \lambda'}$.

The quantization volume we quantize the field is considered to have finite volume of V and we impose periodic boundary such that each side is equal.

The E- and B-field are then:

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= -\partial_t \mathbf{A}(\mathbf{r}, t) = i \sum_{\mathbf{k}, \lambda} \hat{\mathbf{e}}_{\mathbf{k}, \lambda} \omega \left[A_{\mathbf{k}, \lambda} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + A_{\mathbf{k}, \lambda}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \right], \\ \mathbf{B}(\mathbf{r}, t) &= \nabla \times \mathbf{A}(\mathbf{r}, t) = i \sum_{\mathbf{k}, \lambda} \hat{\mathbf{e}}_{\mathbf{k}, \lambda} \left[A_{\mathbf{k}, \lambda} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + A_{\mathbf{k}, \lambda}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \right]. \end{aligned}$$

The energy (Hamiltonian) associated with the fields is:

$$\begin{aligned} H &= - \int dV \left[\frac{1}{2} \varepsilon_0 \mathbf{E}^2(\mathbf{r}, t) + \frac{1}{2\mu_0} \mathbf{B}^2(\mathbf{r}, t) \right] \\ &\quad \vdots \\ H &= 2\varepsilon_0 V \sum_{\mathbf{k}, \lambda} \omega^2 |A_{\mathbf{k}, \lambda}|^2. \end{aligned}$$

We have used the following relation:

$$\int dV e^{\pm i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}} = \delta_{\mathbf{k} \mathbf{k}'} V.$$

Defining

$$A_{\mathbf{k}, \lambda} = \frac{1}{2\omega\sqrt{\varepsilon_0 V}} (\omega q_{\mathbf{k}, \lambda} + ip_{\mathbf{k}, \lambda}), \quad A_{\mathbf{k}, \lambda}^* = \frac{1}{2\omega\sqrt{\varepsilon_0 V}} (\omega q_{\mathbf{k}, \lambda} - ip_{\mathbf{k}, \lambda}).$$

The Hamiltonian is:

$$H = \frac{1}{2} \sum_{\mathbf{k}, \lambda} (\omega^2 q_{\mathbf{k}, \lambda}^2 + p_{\mathbf{k}, \lambda}^2) \xrightarrow{\text{quantizing}} \hat{H} = \sum_{\mathbf{k}, \lambda} \hbar\omega \left(\hat{a}_{\mathbf{k}, \lambda}^\dagger \hat{a}_{\mathbf{k}, \lambda} + \frac{1}{2} \right),$$

where we have defined the ladder operators:

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}} (\omega\hat{q} + i\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}} (\omega\hat{q} - i\hat{p}). \quad (1.7)$$

So, the coefficients are expressed in term of the ladder:

$$A_{\mathbf{k}, \lambda} = \sqrt{\frac{\hbar}{2\omega\varepsilon_0 V}} \hat{a}_{\mathbf{k}, \lambda}, \quad A_{\mathbf{k}, \lambda}^* = \sqrt{\frac{\hbar}{2\omega\varepsilon_0 V}} \hat{a}_{\mathbf{k}, \lambda}^\dagger.$$

Thus the fields are:

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \sum_{\mathbf{k}, \lambda} \sqrt{\frac{\hbar}{2\omega\varepsilon_0 V}} \hat{e}_{\mathbf{k}, \lambda} \left[\hat{a}_{\mathbf{k}, \lambda} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \hat{a}_{\mathbf{k}, \lambda}^\dagger e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \right], \\ \mathbf{E}(\mathbf{r}, t) &= i \sum_{\mathbf{k}, \lambda} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{e}_{\mathbf{k}, \lambda} \left[\hat{a}_{\mathbf{k}, \lambda} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \hat{a}_{\mathbf{k}, \lambda}^\dagger e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \right], \\ \mathbf{B}(\mathbf{r}, t) &= \frac{i}{c} \sum_{\mathbf{k}, \lambda} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} (\mathbf{k} \times \hat{e}_{\mathbf{k}, \lambda}) \left[\hat{a}_{\mathbf{k}, \lambda} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \hat{a}_{\mathbf{k}, \lambda}^\dagger e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \right]. \end{aligned} \quad (1.8)$$

1.2.5 Electric field expectation and uncertainty

Lets consider one mode of the E-field in a number state: $|n\rangle_{\mathbf{k}, \lambda}$. Lets add the time dependence to the ladder operators:

$$\hat{a}_{\mathbf{k}, \lambda} e^{-i\omega t} \rightarrow \hat{a}_{\mathbf{k}, \lambda}(t), \quad \hat{a}_{\mathbf{k}, \lambda}^\dagger e^{i\omega t} \rightarrow \hat{a}_{\mathbf{k}, \lambda}^\dagger(t).$$

The E-field is:

$$\hat{\mathbf{E}}(\mathbf{r}, t) = i \sum_{\mathbf{k}, \lambda} e_{\mathbf{k}, \lambda} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \left[\hat{a}_{\mathbf{k}, \lambda}(t) e^{i\mathbf{k} \cdot \mathbf{r}} - \hat{a}_{\mathbf{k}, \lambda}^\dagger(t) e^{-i\mathbf{k} \cdot \mathbf{r}} \right].$$

Expectation value of the E-field is:

$$\begin{aligned} {}_{\mathbf{k}_0, \lambda_0} \langle n | \hat{\mathbf{E}}(\mathbf{r}, t) | n \rangle_{\mathbf{k}_0, \lambda_0} &= i_{\mathbf{k}, \lambda} \langle n | \hat{e}_{\mathbf{k}, \lambda} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \left[\hat{a}_{\mathbf{k}, \lambda}(t) e^{i\mathbf{k} \cdot \mathbf{r}} - \hat{a}_{\mathbf{k}, \lambda}^\dagger(t) e^{-i\mathbf{k} \cdot \mathbf{r}} \right] | n \rangle_{\mathbf{k}_0, \lambda_0} \\ &= i_{\mathbf{k}_0, \lambda_0} \langle n | \hat{e}_{\mathbf{k}_0, \lambda_0} \sqrt{\frac{\hbar\omega_0}{2\varepsilon_0 V}} \left[\sqrt{n} | n-1 \rangle_{\mathbf{k}_0, \lambda_0} e^{i\mathbf{k}_0 \cdot \mathbf{r}} - \sqrt{n+1} | n+1 \rangle_{\mathbf{k}_0, \lambda_0} e^{-i\mathbf{k}_0 \cdot \mathbf{r}} \right] \\ &= 0. \end{aligned}$$

Thus, the average E-field is zero.

Lets consider the uncertainty of the E-field:

$$\begin{aligned} \langle \hat{\mathbf{E}} \rangle &= {}_{\mathbf{k}_0, \lambda_0} \langle n | \sum_{\mathbf{k}, \lambda} \sum_{\mathbf{k}', \lambda'} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \sqrt{\frac{\hbar\omega'}{2\varepsilon_0 V}} \hat{e}_{\mathbf{k}, \lambda} \cdot \hat{e}_{\mathbf{k}', \lambda'} \left[\hat{a}_{\mathbf{k}, \lambda} e^{i\mathbf{k} \cdot \mathbf{r}} - \hat{a}_{\mathbf{k}, \lambda}^\dagger e^{-i\mathbf{k} \cdot \mathbf{r}} \right] \left[\hat{a}_{\mathbf{k}', \lambda'} e^{i\mathbf{k}' \cdot \mathbf{r}} - \hat{a}_{\mathbf{k}', \lambda'}^\dagger e^{-i\mathbf{k}' \cdot \mathbf{r}} \right] | n \rangle_{\mathbf{k}_0, \lambda_0} \\ &= \sum_{\mathbf{k}, \lambda} \sum_{\mathbf{k}', \lambda'} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \sqrt{\frac{\hbar\omega'}{2\varepsilon_0 V}} \langle 1 | 1 \rangle_{\mathbf{k}', \lambda'} e^{i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}} + \frac{\hbar\omega_0}{2\varepsilon_0 V} [{}_{\mathbf{k}_0, \lambda_0} \langle n+1 | \sqrt{n+1} + {}_{\mathbf{k}_0, \lambda_0} \langle n-1 | \sqrt{n}] [\sqrt{n+1} | n+1 \rangle_{\mathbf{k}_0, \lambda_0} + \\ &\quad \sum_{\{\mathbf{k}, \lambda\} \neq \{\mathbf{k}_0, \lambda_0\}} \frac{\hbar\omega}{2\varepsilon_0 V} + \frac{\hbar\omega_0}{2\varepsilon_0 V} (2n+1)]. \end{aligned}$$

So, the uncertainty is:

$$\text{E-field uncertainty} \quad \langle \Delta E^2 \rangle = \underbrace{\sum_{\{\mathbf{k}, \lambda\} \neq \{\mathbf{k}_0, \lambda_0\}} \frac{\hbar\omega}{2\varepsilon_0 V}}_{\text{All modes in vacuum state}} + \underbrace{\frac{\hbar\omega_0}{2\varepsilon_0 V}(2n+1)}_{\text{Mode } (\mathbf{k}_0, \lambda_0)}.$$

1.3 Coherent states

The coherent state is defined as the eigenstate of the annihilator operator:

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle, \quad \alpha \in \mathbb{C}.$$

The expected number of photons in the coherent state is:

$$\langle \alpha | \hat{n} | \alpha \rangle = \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = |\alpha|^2.$$

Coherent states are the most classical states of the quantized EM field.

It has the following results:

$$\langle \alpha | \hat{a}^2 | \alpha \rangle = \alpha^2, \quad \langle \alpha | \hat{a}^{\dagger 2} | \alpha \rangle = \alpha^{*2}, \quad \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = |\alpha|^2, \quad \langle \alpha | \hat{a} \hat{a}^\dagger | \alpha \rangle = |\alpha|^2 + 1.$$

For example, computing the expectation of the E-field (1.8) is:

$$\begin{aligned} \langle \mathbf{E} \rangle &= \langle \alpha | i \sum_{\mathbf{k}, \lambda} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{\mathbf{e}}_{\mathbf{k}, \lambda} [\hat{a}_{\mathbf{k}, \lambda} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} - \hat{a}_{\mathbf{k}, \lambda}^\dagger e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)}] | \alpha \rangle \\ &= i \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{\mathbf{e}}_{\mathbf{k}, \lambda} [\alpha e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} - \alpha^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)}] \\ \langle \mathbf{E} \rangle &= \mathcal{E} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \mathcal{E}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \longrightarrow \mathcal{E} = i \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \alpha \hat{\mathbf{e}}_{\mathbf{k}, \lambda}. \end{aligned}$$

The expectation value of the E-field for a coherent state corresponds to a classical E-field.

Let us now consider the uncertainty of the E-field for $|\alpha\rangle$:

$$\begin{aligned} \langle \Delta E^2 \rangle &= \langle \mathbf{E}^2 \rangle - \langle \mathbf{E} \rangle^2 \\ &= \langle \alpha | i \hat{\mathbf{e}}_{\mathbf{k}, \lambda} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} [\hat{a} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} - \hat{a}^\dagger e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)}] \cdot i \hat{\mathbf{e}}_{\mathbf{k}, \lambda}^\dagger \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} [\hat{a} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} - \hat{a}^\dagger e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)}] | \alpha \rangle - \langle \mathbf{E} \rangle^2 \\ &= -\frac{\hbar\omega}{2\varepsilon_0 V} [\alpha^2 e^{i2(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \alpha^{*2} e^{-i2(\mathbf{k} \cdot \mathbf{r} - \omega t)} - |\alpha|^2 - (|\alpha|^2 + 1)] - \mathcal{E}^2 e^{i2(\mathbf{k} \cdot \mathbf{r} - \omega t)} - \mathcal{E}^{*2} e^{-i2(\mathbf{k} \cdot \mathbf{r} - \omega t)} - 2|\mathcal{E}|^2 \\ &= \mathcal{E}^2 e^{i2(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \mathcal{E}^{*2} e^{-i2(\mathbf{k} \cdot \mathbf{r} - \omega t)} + 2|\mathcal{E}|^2 + \frac{\hbar\omega}{2\varepsilon_0 V} - \mathcal{E}^2 e^{i2(\mathbf{k} \cdot \mathbf{r} - \omega t)} - \mathcal{E}^{*2} e^{-2i(\mathbf{k} \cdot \mathbf{r} - \omega t)} - 2|\mathcal{E}|^2 \\ \langle \Delta E^2 \rangle &= \frac{\hbar\omega}{2\varepsilon_0 V}. \end{aligned}$$

This is the uncertainty in E-field limited by quantum noise.

Also, we know that the eigenstates of $|\alpha\rangle$ do not form an orthonormal basis. However, we know that the number state $|n\rangle$ form an orthonormal basis and we can expand the coherent state in terms of the number states, as follows,

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$

The probability of finding n photons in a coherent state is:

$$P_n = |\langle n|\alpha\rangle|^2 = \left| \langle n|e^{-|\alpha|^2/2} \sum_m \frac{\alpha^m}{\sqrt{m!}} |m\rangle \right|^2 = \left| e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \right|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}.$$

Recalling that $\langle \hat{n} \rangle = |\alpha|^2 = \bar{n}$:

Probability of finding n photons in a coherent state

$$P_n = \frac{e^{-\bar{n}} \bar{n}^n}{n!}.$$

1.4 Quadrature operators

Recall the E-field operator for a singel mode is:

$$\begin{aligned} \hat{E}(\mathbf{r}, t) &= i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{\mathbf{e}}_{\mathbf{k},\lambda} [\hat{a}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} - \hat{a}^\dagger e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}] \\ &= i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{\mathbf{e}}_{\mathbf{k},\lambda} [\hat{a}e^{-i\omega t} e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{a}^\dagger e^{i\omega t} e^{-i\mathbf{k}\cdot\mathbf{r}}] \\ &= i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{\mathbf{e}}_{\mathbf{k},\lambda} [\hat{a}(t)e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{a}^\dagger(t)e^{-i\mathbf{k}\cdot\mathbf{r}}] \\ \hat{E}(\mathbf{r}, t) &= i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \hat{\mathbf{e}}_{\mathbf{k},\lambda} [\hat{a}(t)\cos(\mathbf{k}\cdot\mathbf{r}) + i\hat{a}(t)\sin(\mathbf{k}\cdot\mathbf{r}) - \hat{a}^\dagger(t)\cos(\mathbf{k}\cdot\mathbf{r}) + i\hat{a}^\dagger(t)\sin(\mathbf{k}\cdot\mathbf{r})]. \end{aligned}$$

Lets define the quadrature operators:

| | | |
|----------------------|---|---------|
| Quadrature operators | $\hat{X}_1 = \frac{1}{2}(\hat{a} + i\hat{a}^\dagger)$ | (1.9) |
| | $\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$ | |

where $[\hat{X}_1, \hat{X}_2] = \frac{i}{2}$. In the E-field, if we collect the cosine and sine, we get

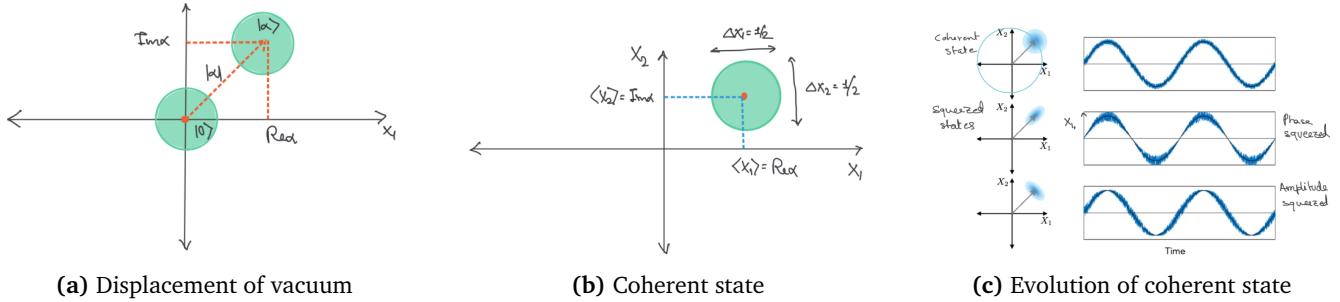
$$\hat{E}(\mathbf{r}, t) = -\sqrt{\frac{2\hbar\omega}{\varepsilon_0 V}} \hat{\mathbf{e}}_{\mathbf{k},\lambda} [\hat{X}_1(t)\sin(\mathbf{k}\cdot\mathbf{r}) + \hat{X}_2(t)\cos(\mathbf{k}\cdot\mathbf{r})].$$

The expectation value of these operators is:

$$\begin{aligned} \langle \alpha | \hat{X}_1(0) | \alpha \rangle &= \langle \alpha | \frac{\hat{a} + \hat{a}^\dagger}{2} | \alpha \rangle = \operatorname{Re}(\alpha), \quad \langle \alpha | \hat{X}_2(0) | \alpha \rangle = \langle \alpha | \frac{\hat{a} - \hat{a}^\dagger}{2i} | \alpha \rangle = \operatorname{Im}(\alpha), \\ \langle \Delta \hat{X}_1^2 \rangle &= \langle \alpha | \frac{1}{4}(\hat{a} + \hat{a}^\dagger)^2 | \alpha \rangle - \frac{1}{4}(\alpha + \alpha^*)^2 = \frac{1}{4}, \quad \langle \Delta \hat{X}_2^2 \rangle = \frac{1}{4}. \end{aligned}$$

That is, there is a minimum uncertainty that a state can possibly have in the two quadrature \hat{X}_1, \hat{X}_2 , such a state os refferred as *minimum uncertainty state*. When $\alpha = 0$ we have the vaccum state and therefore $\langle \hat{X}_1(0) \rangle = \langle \hat{X}_2(0) \rangle = 0$, $\langle \hat{X}_1^2(0) \rangle = \langle \hat{X}_2^2(0) \rangle = 1/4$. Coherent states are also refferees to as displaced vaccum states. Specifically,

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle, \quad \hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}.$$



If we include the time dependence:

$$\langle \alpha | \hat{X}_1(t) | \alpha \rangle = \frac{\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}}{2} = \text{Re}(\alpha e^{-i\omega t}), \quad \langle \alpha | \hat{X}_2(t) | \alpha \rangle = \frac{\alpha e^{-i\omega t} - \alpha^* e^{i\omega t}}{2i} = \text{Im}(\alpha e^{-i\omega t}).$$

The phase space representation of states allows one to acces information aabout the noise/variance associated with any state when a specific E-field quadrature is measured.

1.4.1 Taking the continuum limit

In free space the sum over all \mathbf{k} becomes an integral:

$$\sum_{\mathbf{k}} \rightarrow N \int d^2 \mathbf{k} = N \int_0^\infty dk k^2 \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi,$$

where N is the normalization equal to volume/mode. We consider a volume of quantization that is the region in free space where we are quantizing the field $V = L_x L_y L_z$. Allowed \mathbf{k} vectors are $k_x = (2\pi/L_x)m_x$, $k_y = (2\pi/L_y)m_y$ y $k_z = (2\pi/L_z)m_z$, that is, we ensure that the lengths fit an integer number of wavelengths.

The number of modes in between \mathbf{k} and $\mathbf{k} + \Delta\mathbf{k}$ is:

$$\Delta m_x \Delta m_y \Delta m_z = \frac{L_x L_y L_z}{(2\pi)^2} \Delta k_x \Delta k_y \Delta k_z = \frac{V}{(2\pi)^3} \Delta \mathbf{k}_x \Delta \mathbf{k}_y \Delta \mathbf{k}_z.$$

Using the above normalization:

$$\sum_{\mathbf{k}, \lambda} \rightarrow \sum_{\lambda} \int dk k^2 \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \frac{V}{8\pi^2}.$$

Thus, any continuum limit involves

$$\text{Continuum limit} \quad \sum_{\mathbf{k}} \rightarrow \frac{V}{8\pi^2} \int d^3 \mathbf{k}.$$

1.5 Squeezed states

Lets define the squeeze operator

$$\text{Squeezing operator } |\xi, 0\rangle = \hat{S}(\xi) |0\rangle, \quad \text{with } \hat{S}(\xi) = e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^\dagger)^2}, \quad \xi \in \mathbb{C}.$$

Lets use it to compute the expectation value for the quadrature operators.

$$\langle \xi, 0 | \hat{X}_1 | \xi, 0 \rangle = \langle 0 | \hat{S}^\dagger(\xi) \left(\frac{\hat{a} + \hat{a}^\dagger}{2} \right) \hat{S}(\xi) | 0 \rangle = \frac{1}{2} \langle 0 | (\hat{S}^\dagger \hat{a} \hat{S} + \hat{S}^\dagger \hat{a}^\dagger \hat{S}) | 0 \rangle.$$

We define displaced ladder operators as

$$\text{Squeezing ladder operators } \tilde{a} = \hat{S}^\dagger(\xi) \hat{a} \hat{S}(\xi), \quad \xi = r e^{i\phi}.$$

Using the BCH formula

$$e^L A e^{-L} = A + [L, A] + \frac{1}{2!} [L, [L, A]] + \dots$$

and $A = \hat{a}$ with $L = -\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^\dagger)^2$ allow us to express the squeezing ladder operator

$$\begin{aligned} \tilde{a} &= \hat{S}^\dagger(\xi) \hat{a} \hat{S}(\xi) = \hat{a} \cosh(r) - \hat{a}^\dagger e^{i\phi} \sinh(r) \\ \tilde{a}^\dagger &= \hat{S}(\xi)^\dagger \hat{a}^\dagger \hat{S}(\xi) = \hat{a}^\dagger \cosh(r) - \hat{a} e^{i\phi} \sinh(r). \end{aligned}$$

We have used several orders. Its derivative wrt r is:

$$\frac{d\hat{a}}{dr} = \hat{a} \sinh(r) - \hat{a}^\dagger e^{i\phi} \cosh(r) = -e^{i\phi} \tilde{a}^\dagger, \quad \frac{d\tilde{a}^\dagger}{dr} = -e^{-i\phi} \tilde{a}.$$

Taking another derivative yields:

$$\frac{d^2 \tilde{a}}{dr^2} = \tilde{a}.$$

Now we can continue the expectation of the quadrature:

$$\begin{aligned} \langle \hat{X}_1 \rangle &= \frac{1}{2} \langle 0 | (\hat{S}^\dagger \hat{a} \hat{S} + \hat{S}^\dagger \hat{a}^\dagger \hat{S}) | 0 \rangle = \frac{1}{2} \langle 0 | [\tilde{a} + \tilde{a}^\dagger] | 0 \rangle \\ &= \frac{1}{2} \langle 0 | [\hat{a} \cosh(r) - e^{i\phi} \hat{a}^\dagger \sinh(r) + \hat{a}^\dagger \cosh(r) - e^{-i\phi} \sinh(r)] | 0 \rangle = 0. \\ \langle \hat{X}_2 \rangle &= \text{similarly} = 0. \end{aligned}$$

We can do the same for the operator square:

$$\begin{aligned} \langle \hat{X}_1^2 \rangle &= \frac{1}{4} \langle 0 | \hat{S}^\dagger(\xi) [(\hat{a} + \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger)] \hat{S}(\xi) | 0 \rangle = \frac{1}{4} \langle 0 | \hat{S}^\dagger(\hat{a} + \hat{a}^\dagger) \hat{S} \hat{S}^\dagger(\hat{a} + \hat{a}^\dagger) \hat{S} | 0 \rangle \\ &= \frac{1}{4} \langle 0 | (\tilde{a} + \tilde{a}^\dagger)(\tilde{a} + \tilde{a}^\dagger) | 0 \rangle = \frac{1}{4} \langle 0 | e^{-2r} (\hat{a} + \hat{a}^\dagger)^2 | 0 \rangle = \frac{e^{-2r}}{4}. \end{aligned}$$

Similarly we can obtain the respective quantities for \hat{X}_2 and the uncertainties are:

$$\langle \hat{X}_1 \rangle = \langle \hat{X}_2 \rangle = 0 \implies \Delta \hat{X}_1 = \frac{e^{-r}}{2}, \quad \Delta \hat{X}_2 = \frac{e^r}{2}.$$

1.5.1 Displaced squeezing state

Now,

$$\text{Displaced squeezing state} \quad |\xi, \alpha\rangle = \hat{D}(\alpha) \hat{S}(\xi) |0\rangle.$$

Now, the operators are:

$$\begin{aligned}\tilde{a} &= \hat{S}^\dagger(\xi) \hat{D}^\dagger(\alpha) \hat{a} \hat{D}(\alpha) \hat{S}(\xi) = \cosh(r) \hat{a} - e^{i\phi} \sinh(r) \hat{a}^\dagger + \alpha, \\ \tilde{a}^\dagger &= \hat{S}^\dagger(\xi) \hat{D}^\dagger(\alpha) \hat{a}^\dagger \hat{D}(\alpha) \hat{S}(\xi) = \cosh(r) \hat{a}^\dagger - e^{-i\phi} \sinh(r) \hat{a} + \alpha^*.\end{aligned}$$

The expected number of photons in a displaced squeezed state is:

$$\langle 0 | \tilde{a}^\dagger \tilde{a} | 0 \rangle = \langle 0 | [\cosh(r) \hat{a}^\dagger - e^{-i\phi} \sinh(r) \hat{a} + \alpha^*] [\cosh(r) \hat{a} - e^{i\phi} \sinh(r) \hat{a}^\dagger + \alpha] | 0 \rangle = \sinh^2 |\xi| + |\alpha|^2.$$

The expected value in the quadrature value is:

$$\begin{aligned}\langle \xi, \alpha | \hat{X}_1 | \xi, \alpha \rangle &= \langle 0 | \hat{S}^\dagger \hat{D}^\dagger \hat{X}_1 \hat{D} \hat{S} | 0 \rangle = \langle 0 | \hat{S}^\dagger \hat{D}^\dagger \left(\frac{\hat{a} + \hat{a}^\dagger}{2} \right) \hat{D} \hat{S} | 0 \rangle = \frac{1}{2} \langle 0 | (\tilde{a} + \tilde{a}^\dagger) | 0 \rangle \\ &= \frac{1}{2} \langle 0 | [\cosh(r) \hat{a} - e^{i\phi} \sinh(r) \hat{a}^\dagger + \alpha + \cosh(r) \hat{a}^\dagger - e^{-i\phi} \sinh(r) \hat{a} + \alpha^*] | 0 \rangle = \operatorname{Re}(\alpha). \\ \langle \xi, \alpha | \hat{X}_2 | \xi, \alpha \rangle &= \operatorname{Im}(\alpha).\end{aligned}$$

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Chapter 2

Optical devices

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2.1 Beam splitter

Let us consider the action of a beam splitter (BS) on two input modes of same frequency and polarization. Classically, the output E-field can be defined in terms of the input E-fields:

$$\begin{bmatrix} E'_1 \\ E'_2 \end{bmatrix} = \begin{bmatrix} t' & r \\ r' & t \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}.$$

$\{r, t\}$ and $\{r', t'\}$ are reflection/transmission coefficient associated with the two interfaces of the BS. From energy conservation, we must have

$$\text{Energy conservation in the BS} \quad |E_1|^2 + |E_2|^2 = |E'_1|^2 + |E'_2|^2.$$

If we assume $E_2 = 0$, then

$$|E_1|^2 = |t'E_1|^2 + |r'E_1|^2 \longrightarrow |r'|^2 + |t'|^2 = 1. \quad (2.1)$$

Similarly, setting $E_1 = 0$ yields

$$|E_2|^2 = |rE_2|^2 + |tE_2|^2 \longrightarrow |r|^2 + |t|^2 = 1. \quad (2.2)$$

In general,

$$\begin{aligned} |E_1|^2 + |E_2|^2 &= |E'_1|^2 + |E'_2|^2 \\ &= (t'E_1 + rE_2)(t'^*E_1^* + r^*E_2^*) + (r'E_1 + tE_2)(r'^*E_1^* + t^*E_2^*) \\ &= |t'|^2|E_1|^2 + |r|^2|E_2|^2 + |r'|^2|E_1|^2 + |t|^2|E_2|^2 + rt'^*E_1^*E_2 + t'r^*E_1E_2^* + r't^*E_1E_2^* + r'^*tE_1^*E_2 \\ |E_1|^2 + |E_2|^2 &= |E_1|^2(|r'|^2 + |t'|^2) + |E_2|^2(|r|^2 + |t|^2) + E_1^*E_2(rt'^* + r'^*t) + E_1E_2^*(r't^* + r^*t') \\ 0 &= E_1^*E_2(rt'^* + r'^*t) + c.c. \end{aligned}$$

Thus,

$$rt'^* + r'^*t = 0. \quad (2.3)$$

For a 50 : 50 BS which as $r' = 1/\sqrt{2}$, $t' = 1/\sqrt{2}$, and $r = 1/\sqrt{2}$, (2.1), (2.2) and (2.3) implies that $t = -1/\sqrt{2}$. Similarly, the quantum description of the BS is

$$\begin{bmatrix} \hat{a}'_1 \\ \hat{a}'_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix}. \quad (2.4)$$

Now, the ladder operators are equal to the E-fields. This equation implies that

$$\hat{a}'_1 = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2), \quad \text{and} \quad \hat{a}'_2 = \frac{1}{\sqrt{2}}(\hat{a}_1 - \hat{a}_2).$$

Ejemplo 2.1

The input is:

$$\text{Input} = |1\rangle_1 |0\rangle_2 = (\hat{a}_1^\dagger |0\rangle_1) |0\rangle_2 = |1\rangle_1 |0\rangle_2.$$

Single photon incident in one input

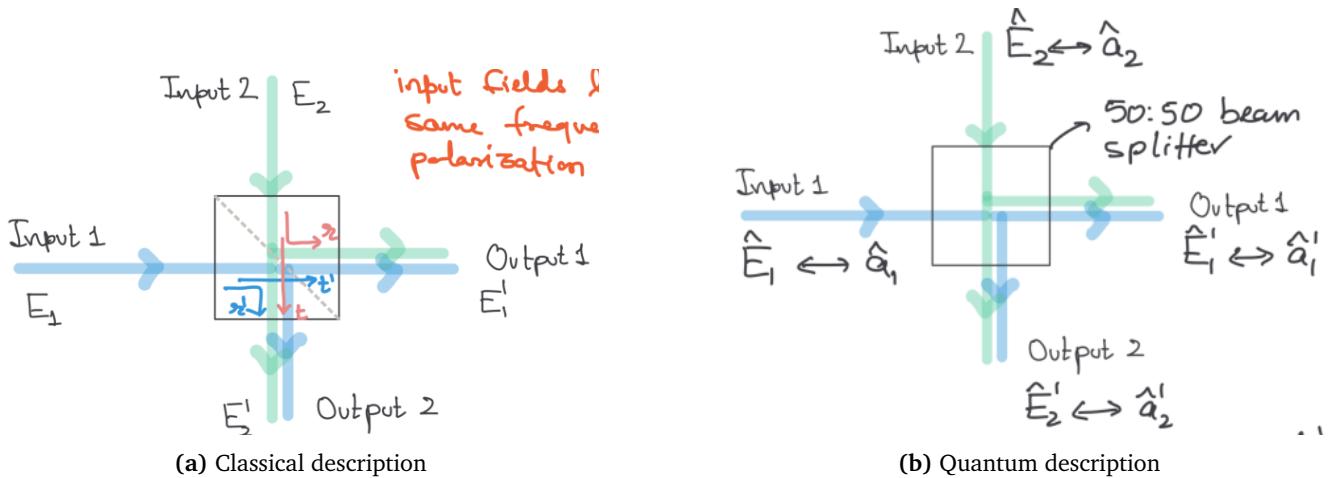


Figure 2.1 Description of a beamsplitter (BS).

Note that

$$\hat{a}_1 = \frac{1}{\sqrt{2}}(\hat{a}'_1 + \hat{a}'_2) \quad \text{and} \quad \hat{a}_1^\dagger = \frac{1}{\sqrt{2}}(\hat{a}'_1^\dagger + \hat{a}'_2^\dagger).$$

The output is then

$$\text{Output} = \frac{1}{\sqrt{2}}(\hat{a}_1^\dagger + \hat{a}_2^\dagger)|0\rangle_{1'}|0\rangle_{2'} = \frac{1}{\sqrt{2}}[|1\rangle_{1'}|0\rangle_{2'} + |0\rangle_{1'}|1\rangle_{2'}].$$

Thus, there is a 50 : 50 probability of detecting the photon in one of the output ports.

Ejemplo 2.2

The input is:

$$\text{Input} = |1\rangle_1 |1\rangle_2 = \hat{a}_1^\dagger |0\rangle_1 \hat{a}_2^\dagger |0\rangle_2.$$

The output is:

$$\begin{aligned}
 \text{Output} &= \frac{1}{2} [\hat{a}_1^{\prime\dagger} + \hat{a}_2^{\prime\dagger}] [\hat{a}_1^{\prime\dagger} - \hat{a}_2^{\prime\dagger}] |0\rangle_1 |0\rangle_2 \\
 &= \frac{1}{2} [\hat{a}_1^{\prime\dagger} \hat{a}_1^{\prime\dagger} - \hat{a}_2^{\prime\dagger} \hat{a}_2^{\prime\dagger} + \hat{a}_1^{\prime\dagger} \hat{a}_2^{\prime\dagger} - \hat{a}_2^{\prime\dagger} \hat{a}_1^{\prime\dagger}] |0\rangle_1 |0\rangle_2 \\
 &= \frac{1}{2} [\sqrt{2} |2\rangle_1 |0\rangle_2 - \sqrt{2} |0\rangle_1 |2\rangle_2] \\
 \text{Output} &= \frac{1}{\sqrt{2}} [|2\rangle_1 |0\rangle_2 - |0\rangle_1 |2\rangle_2].
 \end{aligned}$$

Thus, when two photons are simultaneously incident on each input, both either go into output 1 or output 2 but never in both at the same time.

Both input with photons

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