

Assignment 1

OPTI 544 Quantum Optics

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Total time: 12 hours

Exercise 1

1. Use the definition of the E- and B-fields in terms of the vector potential.

$$\mathbf{E} = -\partial_t \mathbf{A}, \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

Taking the curl of the B-field allows to replace the vector potential and have a triple product, which can be reexpressed using vector identities. Because $\nabla \times \mathbf{B}$ is Ampere's law, it also depends on the E-field, where the above definition can be replaced.

$$\begin{aligned}\nabla \times \mathbf{B} &= \frac{1}{c^2} \partial_t \mathbf{E} \\ \nabla \times \nabla \times \mathbf{A} &= \frac{1}{c^2} \partial_t (-\partial_t \mathbf{A}) \\ \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} &= -\frac{1}{c^2} \partial_t^2 \mathbf{A} \\ \nabla^2 \mathbf{A}(\mathbf{r}, t) - \frac{1}{c^2} \partial_t^2 \mathbf{A}(\mathbf{r}, t) &= 0.\end{aligned}$$

2. fsf
3. asgsag
4. asgsa
5. sagasg

Exercise 2

Exercise 3

- a) We can simplify the problem using the BCH formula

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \dots$$

In this case, $\hat{A} = \alpha^* \hat{a} - \alpha \hat{a}^\dagger$ and $\hat{B} = \hat{a}$. The first two commutators are:

$$\begin{aligned} [\hat{A}, \hat{B}] &= [(\alpha^* \hat{a} - \alpha \hat{a}^\dagger), \hat{a}] = \alpha^* [\hat{a}, \hat{a}] - \alpha [\hat{a}^\dagger, \hat{a}] = \alpha, \\ [\hat{A}, [\hat{A}, \hat{B}]] &= [(\alpha^* \hat{a} - \alpha \hat{a}^\dagger), \alpha] = 0, \\ &\vdots \end{aligned}$$

Now, we can express the displacement operator as:

$$\begin{aligned} \tilde{a} \mathcal{D}^\dagger(\alpha) \hat{a} \mathcal{D}(\alpha) &= e^{\alpha^* \hat{a} - \alpha \hat{a}^\dagger} \hat{a} e^{-(\alpha^* \hat{a} - \alpha \hat{a}^\dagger)} = \hat{a} + \alpha, \quad \text{and} \\ \tilde{a}^\dagger &= \mathcal{D}^\dagger(\alpha) \hat{a}^\dagger \mathcal{D}(\alpha) = \hat{a}^\dagger + \alpha^*. \end{aligned}$$

Application of both in $|0\rangle$ (just to test them) yields:

$$\tilde{a} |0\rangle = (\hat{a} + \alpha) |0\rangle = \alpha |0\rangle, \quad \text{and} \quad \tilde{a}^\dagger |0\rangle = (\hat{a}^\dagger + \alpha^*) |0\rangle = |1\rangle + \alpha^* |0\rangle.$$

- b) We use algebra of operators and the fact that the coherent state $|\alpha\rangle$ is got from $|0\rangle$ by applying a displacement operator $\mathcal{D}(\alpha)$:

$$\begin{aligned} \langle \hat{X}_1 \rangle &= \frac{1}{2} \langle \alpha | (\hat{a} + \hat{a}^\dagger) | \alpha \rangle \\ &= \frac{1}{2} \langle 0 | \mathcal{D}^\dagger(\alpha) (\hat{a} + \hat{a}^\dagger) \mathcal{D}(\alpha) | 0 \rangle \\ &= \frac{1}{2} \left[\langle 0 | \mathcal{D}^\dagger(\alpha) \hat{a} \mathcal{D}(\alpha) | 0 \rangle + \langle 0 | \mathcal{D}^\dagger(\alpha) \hat{a}^\dagger \mathcal{D}(\alpha) | 0 \rangle \right] \\ &= \frac{1}{2} \left[\langle 0 | \tilde{a} | 0 \rangle + \langle 0 | \tilde{a}^\dagger | 0 \rangle \right] \\ &= \frac{1}{2} \left[\langle 0 | (\hat{a} + \alpha) | 0 \rangle + \langle 0 | (\hat{a}^\dagger + \alpha^*) | 0 \rangle \right] \\ &= \frac{1}{2} \left[\langle 0 | \hat{a} | 0 \rangle + \alpha + \langle 0 | \hat{a}^\dagger | 0 \rangle + \alpha^* \right] \\ &= \frac{1}{2} [\alpha + \alpha^*] \\ \langle \hat{X}_1 \rangle &= \text{Re}(\alpha). \end{aligned}$$

For \hat{X}_2 , and using the developpement from above:

$$\begin{aligned} \langle \hat{X}_2 \rangle &= \frac{1}{2i} \left[\langle 0 | (\hat{a} + \alpha) | 0 \rangle - \langle 0 | (\hat{a}^\dagger + \alpha^*) | 0 \rangle \right] \\ &= \frac{1}{2i} \left[\langle 0 | \hat{a} | 0 \rangle + \alpha - \langle 0 | \hat{a}^\dagger | 0 \rangle - \alpha^* \right] \\ &= \frac{1}{2i} [\alpha - \alpha^*] \\ \langle \hat{X}_2 \rangle &= \text{Im}(\alpha). \end{aligned}$$

For the uncertainties, we now need to get the mean value of \hat{X}_1^2 and \hat{X}_2^2 , with the analogous process:

$$\begin{aligned}
\langle \hat{X}_1^2 \rangle &= \frac{1}{4} \langle 0 | \mathcal{D}^\dagger(\alpha) [\hat{a} + \hat{a}^\dagger]^2 \mathcal{D}(\alpha) | 0 \rangle \\
&= \frac{1}{4} \langle 0 | \left[\mathcal{D}^\dagger(\alpha) [\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2}] \mathcal{D}(\alpha) \right] | 0 \rangle \\
&= \frac{1}{4} \langle 0 | \left[\mathcal{D}^\dagger \hat{a}^2 \mathcal{D}(\alpha) + 2\mathcal{D}^\dagger(\alpha) \hat{a}^\dagger \hat{a} \mathcal{D}(\alpha) + 1 + \mathcal{D}^\dagger(\alpha) \hat{a}^{\dagger 2} \mathcal{D}(\alpha) \right] | 0 \rangle \\
&= \frac{1}{4} \langle 0 | \left[(\mathcal{D}^\dagger \hat{a} \mathcal{D})(\mathcal{D}^\dagger \hat{a} \mathcal{D}) + 2(\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D})(\mathcal{D}^\dagger \hat{a} \mathcal{D}) + 1 + (\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D})(\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D}) \right] | 0 \rangle \\
&= \frac{1}{4} \langle 0 | \left[(\hat{a} + \alpha)^2 + 2(\hat{a}^\dagger + \alpha^*)(\hat{a} + \alpha) + 1 + (\hat{a}^\dagger + \alpha^*)^2 \right] | 0 \rangle \\
&= \frac{1}{4} [\hat{a}^2 + 2\alpha\hat{a} + \alpha^2 + 2\hat{a}^\dagger\hat{a} + 2\hat{a}^\dagger\alpha + 2\alpha^*\hat{a} + 2|\alpha|^2 + 1 + \hat{a}^{\dagger 2} + 2\alpha^*\hat{a}^\dagger + \alpha^{*2}] \\
\langle \hat{X}_1^2 \rangle &= \frac{1}{4} [(\alpha + \alpha^*)^2 + 1] .
\end{aligned}$$

Then,

$$\begin{aligned}
\Delta \hat{X}_1 &= \sqrt{\langle \hat{X}_1^2 \rangle - \langle \hat{X}_1 \rangle^2} \\
&= \sqrt{\frac{1}{4} [(\alpha + \alpha^*)^2 + 1] - \frac{1}{4} (\alpha + \alpha^*)^2} \\
\Delta \hat{X}_1 &= \frac{1}{2} .
\end{aligned}$$

The same procedure is done for the uncertainty of \hat{X}_2 :

$$\begin{aligned}
\langle \hat{X}_2 \rangle &= \frac{1}{4} \langle 0 | \left[(\mathcal{D}^\dagger \hat{a} \mathcal{D})(\mathcal{D}^\dagger \hat{a} \mathcal{D}) - 2(\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D})(\mathcal{D}^\dagger \hat{a} \mathcal{D}) - 1 + (\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D})(\mathcal{D}^\dagger \hat{a}^\dagger \mathcal{D}) \right] | 0 \rangle \\
&= -\frac{1}{4} \langle 0 | \left[(\hat{a} + \alpha)^2 - 2(\hat{a}^\dagger + \alpha^*)(\hat{a} + \alpha) - 1 + (\hat{a}^\dagger + \alpha^*)^2 \right] | 0 \rangle \\
&= -\frac{1}{4} [\hat{a}^2 + 2\alpha\hat{a} + \alpha^2 - 2\hat{a}^\dagger\hat{a} - 2\hat{a}^\dagger\alpha - 2\alpha^*\hat{a} - 2|\alpha|^2 - 1 + \hat{a}^{\dagger 2} + 2\alpha^*\hat{a}^\dagger + \alpha^{*2}] \\
\langle \hat{X}_2^2 \rangle &= \frac{1}{4} [1 - (\alpha - \alpha^*)^2] .
\end{aligned}$$

Then,

$$\begin{aligned}
\Delta \hat{X}_2 &= \sqrt{\langle \hat{X}_2^2 \rangle - \langle \hat{X}_2 \rangle^2} \\
&= \sqrt{\frac{1}{4} [1 - (\alpha - \alpha^*)^2] + \frac{1}{4} (\alpha - \alpha^*)^2} \\
\Delta \hat{X}_2 &= \frac{1}{2} .
\end{aligned}$$