Assignment 1
OPTI 544 Quantum Optics
University of Arizona

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Exercise 1

The Hamiltonian in position space of the quantum harmonic oscillator is:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \quad [\hat{x}, \hat{p}] = i\hbar.$$

Defining the *ladder operators*

$$\begin{aligned}
\hat{a} &= \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{\sigma} + i\frac{\sigma\hat{p}}{\hbar}\right) \\
\hat{a}^\dagger &= \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{\sigma} - i\frac{\sigma\hat{p}}{\hbar}\right)
\end{aligned} \quad , \quad \text{with} \quad
\begin{aligned}
\hat{x} &= \frac{\sigma}{\sqrt{2}}(\hat{a}^\dagger + \hat{a}) \\
\hat{p} &= \frac{i\hbar}{\sigma\sqrt{2}}(\hat{a}^\dagger - \hat{a})
\end{aligned}$$

, with σ the oscillation length

$$\sigma = \sqrt{\frac{\hbar}{m\omega}},$$

allow us to express the Hamiltonian in a simpler form. First, we notice that the commutation of $\hat{x}$ and $\hat{p}$ is inherited for $\hat{a}$ and $\hat{a}^\dagger$:

$$[\hat{a}, \hat{a}^\dagger] = \frac{1}{2}\left[\frac{\hat{x}}{\sigma} + i\frac{\sigma\hat{p}}{\hbar}, \frac{\hat{x}}{\sigma} - i\frac{\sigma\hat{p}}{\hbar}\right] = \frac{1}{2}\left(-\frac{2i}{\hbar}[\hat{x}, \hat{p}]\right) = -\frac{i}{\hbar}[\hat{x}, \hat{p}] = -\frac{i}{\hbar}(i\hbar) = 1.$$

If we replace the $\hat{x}, \hat{p}$ in terms of $\hat{a}, \hat{a}^\dagger$ in the Hamiltonian, we have:

$$\begin{aligned}
\hat{H} &= \frac{1}{2m}\left[-\frac{\hbar^2}{2\sigma^2}(\hat{a}^\dagger - \hat{a})^2\right] + \frac{1}{2}m\omega^2\left[\frac{\sigma^2}{2}(\hat{a}^\dagger + \hat{a})^2\right] \\
&= \frac{\hbar\omega}{2}\left[\hat{a}^{\dagger 2} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^2 - \hat{a}^{\dagger 2} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger - \hat{a}^2\right] \\
&= \frac{\hbar\omega}{2}\left[2(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)\right] \\
&= \hbar\omega(\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a} + 1) \\
&= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \\
\hat{H} &= \hbar\omega\left(\hat{N} + \frac{1}{2}\right),
\end{aligned}$$

where $\hat{N} = \hat{a}^\dagger \hat{a}$ is the *number operator*. If we know the eigenequation of $\hat{N}$ we automatically know the eigenequation of $\hat{H}$. Assuming the following equation for $\hat{N}$, then:

$$\hat{N} \ket{n} = n \ket{n} \Longrightarrow \hat{H} \ket{n} = \left(n + \frac{1}{2} \right) \ket{n}.$$

This means that the energy engenvalues are

$$E_n = \left(n + \frac{1}{2} \right) \hbar\omega.$$

The nature of n needs to be proved. First, we compute the commutator of $\hat{N}$ with $\hat{a}, \hat{a}^\dagger$:

$$[\hat{N}, \hat{a}] = [\hat{a}^\dagger \hat{a}, \hat{a}] = \hat{a}^\dagger [\hat{a}, \hat{a}] + [\hat{a}^\dagger, \hat{a}]\hat{a} = -\hat{a},$$
$$[\hat{N}, \hat{a}^\dagger] = [\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger [\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}^\dagger]\hat{a} = \hat{a}^\dagger.$$

Using these relations, we can test some actions on $\ket{n}$:

$$\hat{N}\hat{a}^\dagger \ket{n} = ([\hat{N}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{N}) \ket{n} = (\hat{a}^\dagger + \hat{a}^\dagger \hat{N}) \ket{n} = (\hat{a}^\dagger + \hat{a}^\dagger n) \ket{n} = (n+1)\hat{a}^\dagger \ket{n},$$
$$\hat{N}\hat{a} \ket{n} = ([\hat{N}, \hat{a}] + \hat{a}\hat{N}) \ket{n} = (-\hat{a} + \hat{a}\hat{N}) \ket{n} = (\hat{a}n - \hat{a}) \ket{n} = (n-1)\hat{a} \ket{n}.$$

This imply that $\hat{a}^\dagger \ket{n}$ and $\hat{a} \ket{n}$ are also eigenkets of $\hat{N}$, with eigenvalue increased and decreased by one, respectively. With this, we conclude that n must be integer so that $\hat{N}$ and therefore $\hat{H}$ are quantized. A more rigurous proof is given by [1].

Exercise 2

a) Ill use the definition of the E- and B-fields in terms of the vector potential.

$$\boldsymbol{E} = -\partial_t \boldsymbol{A}, \quad \text{and} \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

Taking the curl of the B-field allows to replace the vector potential and have a triple product, which can be reexpressed using vector identities. Because $\nabla \times \boldsymbol{B}$ is Ampere's law, it also depends on the E-field, where the above definition can be replaced.

$$\nabla \times \boldsymbol{B} = \frac{1}{c^2}\partial_t \boldsymbol{E}$$
$$\nabla \times \nabla \times \boldsymbol{A} = \frac{1}{c^2}\partial_t(-\partial_t \boldsymbol{A})$$
$$\nabla(\cancel{\nabla \cdot \boldsymbol{A}}^{0}) - \nabla^2 \boldsymbol{A} = -\frac{1}{c^2}\partial_t^2 \boldsymbol{A}$$
$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\partial_t^2 \boldsymbol{A} = 0.$$

b) To show that the expression provided is a solution of the A-potential wave equation, we need to substitute it into the wave equation and have an equality. The second time-derivative is:

$$\partial_t \boldsymbol{A}(z,t) = \partial_t \left[\sum_{\boldsymbol{k},\lambda} e_{\boldsymbol{k},\lambda}[A_{\boldsymbol{k},\lambda}e^{-i\omega t} + A_{\boldsymbol{k},\lambda}^* e^{i\omega t}]\sin(kz) \right] = i\sum_{\boldsymbol{k},\lambda} \omega e_{\boldsymbol{k},\lambda}[A_{\boldsymbol{k},\lambda}^* e^{i\omega t} - A_{\boldsymbol{k},\lambda}e^{-i\omega t}]\sin(kz),$$

$$\partial_t^2 \boldsymbol{A}(z,t) = \partial_t \left[i\sum_{\boldsymbol{k},\lambda} \omega e_{\boldsymbol{k},\lambda}[A_{\boldsymbol{k},\lambda}^* e^{i\omega t} - A_{\boldsymbol{k},\lambda}e^{-i\omega t}]\sin(kz) \right] = -\sum_{\boldsymbol{k},\lambda} \omega^2 e_{\boldsymbol{k},\lambda}[A_{\boldsymbol{k},\lambda}^* e^{i\omega t} + A_{\boldsymbol{k},\lambda}e^{-i\omega t}]\sin(kz).$$

For the laplacian, we have:

$$\nabla^2 \boldsymbol{A}(z,t) = -\sum_{\boldsymbol{k},\lambda} k^2 e_{\boldsymbol{k},\lambda}[A_{\boldsymbol{k},\lambda}e^{-i\omega t} + A^*_{\boldsymbol{k},\lambda}e^{i\omega t}]\sin(kz).$$

Substituting both, and using the dispersion relation $\omega = |\boldsymbol{k}|c$:

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\partial_t^2 \boldsymbol{A} = 0$$

$$-\sum_{\boldsymbol{k},\lambda} k^2 e_{\boldsymbol{k},\lambda}[A_{\boldsymbol{k},\lambda}e^{-i\omega t} + A^*_{\boldsymbol{k},\lambda}e^{i\omega t}]\sin(kz) - \frac{1}{c^2}\left[-\sum_{\boldsymbol{k},\lambda}\omega_{\boldsymbol{k}}^2 e_{\boldsymbol{k},\lambda}[A^*_{\boldsymbol{k},\lambda}e^{i\omega t} + A_{\boldsymbol{k},\lambda}e^{-i\omega t}]\sin(kz)\right] =$$

$$\sum_{\boldsymbol{k},\lambda}\left[e_{\boldsymbol{k},\lambda}[A_{\boldsymbol{k},\lambda}e^{-i\omega t} + A^*_{\boldsymbol{k},\lambda}e^{i\omega t}]\sin(kz)\left(\frac{\omega_{\boldsymbol{k}}^2}{c^2} - k^2\right)\right] =$$

$$\sum_{\boldsymbol{k},\lambda}\left[e_{\boldsymbol{k},\lambda}[A_{\boldsymbol{k},\lambda}e^{-i\omega t} + A^*_{\boldsymbol{k},\lambda}e^{i\omega t}]\sin(kz)\left(k^2 - k^2\right)\right] = \quad (\omega^2 = |\boldsymbol{k}|^2 c^2)$$

$$0 = 0.$$

We could also propose a standard harmonic solution, substitute it into the vector equation and impose boundary conditions. The one I did is easier. Also, I denoted $\omega_{\boldsymbol{k}}$ to remember that ω depends on $\boldsymbol{k}$. The dispersion relation holds for each mode $\boldsymbol{k}$.

c) Assuming a x polarized vector potential

$$\boldsymbol{A}(z,t) = \hat{x}[Ae^{-i\omega t} + A^*e^{i\omega t}]\sin(kz).$$

The E- and B-fields are:

$$\boldsymbol{E}(z,t) = -\partial_t \boldsymbol{A}(z,t) = i\omega\hat{x}[Ae^{-i\omega t} - A^*e^{i\omega t}]\sin(kz) \equiv 2\omega|A|\hat{x}\sin(\omega t - \phi)\sin(kz),$$

$$\boldsymbol{B}(z,t) = \nabla \times \boldsymbol{A}(z,t) = \hat{y}\partial_z A_x = \hat{y}k[Ae^{-i\omega t} + A^*e^{i\omega t}]\cos(kz) \equiv 2k|A|\hat{y}\cos(\omega t - \phi)\cos(kz),$$

where we have assumed a phasor form of A: $A = |A|e^{i\phi}$.

d) We use the expression from previous part. We first compute $\boldsymbol{E}^2_{\boldsymbol{k},\lambda}$ and $\boldsymbol{B}^2_{\boldsymbol{k},\lambda}$

$$\boldsymbol{E}^2 = 4\omega^2|A|^2\sin^2(\omega t - \phi)\sin^2(kz),$$

$$\boldsymbol{B}^2 = 4k^2|A|^2\cos^2(\omega t - \phi)\cos^2(kz).$$

Putting these two in the volume integral, and recalling that

$$\int_0^L dz\,\sin^2(kz) = \int_0^L dz\,\cos^2(kz) = \frac{L}{2},$$

yields

$$H_\omega = \int_0^L dV\left\{2\varepsilon_0\omega^2|A|^2\sin^2(\omega t - \phi)\sin^2(kz) + 2\frac{k^2}{\mu_0}|A|^2\cos^2(\omega t - \phi)\cos^2(kz)\right\}$$

$$= 2\varepsilon_0\omega^2|A|^2\sin^2(\omega t - \phi)S\int_0^L dz\,\sin^2(kz) + 2\frac{k^2}{\mu_0}|A|^2\cos^2(\omega t - \phi)S\int_0^L dz\,\cos^2(kz)$$

$$H_\omega = V\left[\varepsilon_0\omega^2|A|^2\sin^2(\omega t - \phi) + \frac{k^2}{\mu_0}|A|^2\cos^2(\omega t - \phi)\right].$$

Using the fact that $\omega = kc$, $1/\mu_0 = \varepsilon_0 c^2$, then $k^2/\mu_0 = (\omega/c)^2(\varepsilon_0 c^2) = \varepsilon_0\omega^2$, which is the same as the E-field factor term. Therefore,

$$H_\omega = V\varepsilon_0\omega^2|A|[\sin^2(\omega t - \phi) + \cos^2(\omega t - \phi)] = \varepsilon_0 V\omega^2|A|^2.$$

e) Equating the last result with $\hbar\omega$ and solving for A:

$$\varepsilon_0 V \omega^2 |A|^2 = \hbar\omega$$

$$|A| = \sqrt{\frac{\hbar}{\varepsilon_0 V \omega}}.$$

Its phase, ϕ, remains unknown, but it acts as a global phase factor so it has no physical meaning. The E- and B-fields are:

$$\boldsymbol{E}(z,t) = 2\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\,\hat{x}\sin(\omega t - \phi)\sin(kz),$$

$$\boldsymbol{B}(z,t) = \frac{2}{c}\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\,\hat{y}\cos(\omega t - \phi)\cos(kz).$$

Exercise 3

a) We can simplify the problem using the BCH formula

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \cdots$$

In this case, $\hat{A} = \alpha^*\hat{a} - \alpha\hat{a}^\dagger$ and $\hat{B} = \hat{a}$. The first two commutators are:

$$[\hat{A}, \hat{B}] = [(\alpha^*\hat{a} - \alpha\hat{a}^\dagger), \hat{a}] = \alpha^*[\hat{a}, \hat{a}] - \alpha[\hat{a}^\dagger, \hat{a}] = \alpha,$$

$$[\hat{A}, [\hat{A}, \hat{B}]] = [(\alpha^*\hat{a} - \alpha\hat{a}^\dagger), \alpha] = 0,$$

$$\vdots$$

Now, we can express the displacement operator as:

$$\tilde{a} = \mathcal{D}^\dagger(\alpha)\hat{a}\mathcal{D}(\alpha) = e^{\alpha^*\hat{a} - \alpha\hat{a}^\dagger}\hat{a}e^{-(\alpha^*\hat{a} - \alpha\hat{a}^\dagger)} = \hat{a} + \alpha, \quad \text{and}$$

$$\tilde{a}^\dagger = \mathcal{D}^\dagger(\alpha)\hat{a}^\dagger\mathcal{D}(\alpha) = \hat{a}^\dagger + \alpha^*.$$

Application of both in $|0\rangle$ (just to test them) yields:

$$\tilde{a}|0\rangle = (\hat{a} + \alpha)|0\rangle = \alpha|0\rangle, \quad \text{and} \quad \tilde{a}^\dagger|0\rangle = (\hat{a}^\dagger + \alpha^*)|0\rangle = |1\rangle + \alpha^*|0\rangle.$$

b) We use algabra of operators and the fact that the coherent state $|\alpha\rangle$ is got from $|0\rangle$ by appliying a displacement operator $\mathcal{D}(\alpha)$:

$$\begin{aligned}
\langle\hat{X}_1\rangle &= \frac{1}{2}\langle\alpha|(\hat{a} + \hat{a}^\dagger)|\alpha\rangle \\
&= \frac{1}{2}\langle 0|\mathcal{D}^\dagger(\alpha)(\hat{a} + \hat{a}^\dagger)\mathcal{D}(\alpha)|0\rangle \\
&= \frac{1}{2}\left[\langle 0|\mathcal{D}^\dagger(\alpha)\hat{a}\mathcal{D}(\alpha)|0\rangle + \langle 0|\mathcal{D}^\dagger(\alpha)\hat{a}^\dagger\mathcal{D}(\alpha)|0\rangle\right] \\
&= \frac{1}{2}\left[\langle 0|\tilde{a}|0\rangle + \langle 0|\tilde{a}^\dagger|0\rangle\right] \\
&= \frac{1}{2}\left[\langle 0|(\hat{a} + \alpha)|0\rangle + \langle 0|(\hat{a}^\dagger + \alpha^*)|0\rangle\right] \\
&= \frac{1}{2}\left[\langle 0|\hat{a}|0\rangle + \alpha + \langle 0|\hat{a}^\dagger|0\rangle + \alpha^*\right] \\
&= \frac{1}{2}[\alpha + \alpha^*] \\
\langle\hat{X}_1\rangle &= \mathrm{Re}(\alpha).
\end{aligned}$$

For $\hat{X}_2$, and using the developmente from above:

$$\langle \hat{X}_2 \rangle = \frac{1}{2i} \left[\langle 0|(\hat{a} + \alpha)|0\rangle - \langle 0|(\hat{a}^\dagger + \alpha^*)|0\rangle \right]$$
$$= \frac{1}{2i} \left[\langle 0|\hat{a}|0\rangle + \alpha - \langle 0|\hat{a}^\dagger|0\rangle - \alpha^* \right]$$
$$= \frac{1}{2i} \left[\alpha - \alpha^* \right]$$
$$\langle \hat{X}_2 \rangle = \mathrm{Im}\,(\alpha)\,.$$

For the uncertainties, we now need to get the mean value of $\hat{X}_1^2$ and $\hat{X}_2^2$, with the analogous process:

$$\langle \hat{X}_1^2 \rangle = \frac{1}{4} \langle 0|\mathcal{D}^\dagger(\alpha)[\hat{a} + \hat{a}^\dagger]^2 \mathcal{D}(\alpha)|0\rangle$$
$$= \frac{1}{4} \langle 0| \left[\mathcal{D}^\dagger(\alpha)[\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2}]\mathcal{D}(\alpha) \right] |0\rangle$$
$$= \frac{1}{4} \langle 0| \left[\mathcal{D}^\dagger\hat{a}^2\mathcal{D}(\alpha) + 2\mathcal{D}^\dagger(\alpha)\hat{a}^\dagger\hat{a}\mathcal{D}(\alpha) + 1 + \mathcal{D}^\dagger(\alpha)\hat{a}^{\dagger 2}\mathcal{D}(\alpha) \right] |0\rangle$$
$$= \frac{1}{4} \langle 0| \left[(\mathcal{D}^\dagger\hat{a}\mathcal{D})(\mathcal{D}^\dagger\hat{a}\mathcal{D}) + 2(\mathcal{D}^\dagger\hat{a}^\dagger\mathcal{D})(\mathcal{D}^\dagger\hat{a}\mathcal{D}) + 1 + (\mathcal{D}^\dagger\hat{a}^\dagger\mathcal{D})(\mathcal{D}^\dagger\hat{a}^\dagger\mathcal{D}) \right] |0\rangle$$
$$= \frac{1}{4} \langle 0| \left[(\hat{a} + \alpha)^2 + 2(\hat{a}^\dagger + \alpha^*)(\hat{a} + \alpha) + 1 + (\hat{a}^\dagger + \alpha^*)^2 \right] |0\rangle$$
$$= \frac{1}{4}[\hat{a}^2 + 2\alpha\hat{a} + \alpha^2 + 2\hat{a}^\dagger\hat{a} + 2\hat{a}^\dagger\alpha + 2\alpha^*\hat{a} + 2|\alpha|^2 + 1 + \hat{a}^{\dagger 2} + 2\alpha^*\hat{a}^\dagger + \alpha^{*2}]$$
$$\langle \hat{X}_1^2 \rangle = \frac{1}{4} \left[(\alpha + \alpha^*)^2 + 1 \right].$$

Then,

$$\Delta \hat{X}_1 = \sqrt{\langle \hat{X}_1^2 \rangle - \langle \hat{X}_1 \rangle^2}$$
$$= \sqrt{\frac{1}{4} \left[(\alpha + \alpha^*)^2 + 1 \right] - \frac{1}{4}(\alpha + \alpha^*)^2}$$
$$\Delta \hat{X}_1 = \frac{1}{2}.$$

The same procedure is done for the uncertainty of $\hat{X}_2$:

$$\langle \hat{X}_2 \rangle = \frac{1}{4} \langle 0| \left[(\mathcal{D}^\dagger\hat{a}\mathcal{D})(\mathcal{D}^\dagger\hat{a}\mathcal{D}) - 2(\mathcal{D}^\dagger\hat{a}^\dagger\mathcal{D})(\mathcal{D}^\dagger\hat{a}\mathcal{D}) - 1 + (\mathcal{D}^\dagger\hat{a}^\dagger\mathcal{D})(\mathcal{D}^\dagger\hat{a}^\dagger\mathcal{D}) \right] |0\rangle$$
$$= -\frac{1}{4} \langle 0| \left[(\hat{a} + \alpha)^2 - 2(\hat{a}^\dagger + \alpha^*)(\hat{a} + \alpha) - 1 + (\hat{a}^\dagger + \alpha^*)^2 \right] |0\rangle$$
$$= -\frac{1}{4}[\hat{a}^2 + 2\alpha\hat{a} + \alpha^2 - 2\hat{a}^\dagger\hat{a} - 2\hat{a}^\dagger\alpha - 2\alpha^*\hat{a} - 2|\alpha|^2 - 1 + \hat{a}^{\dagger 2} + 2\alpha^*\hat{a}^\dagger + \alpha^{*2}]$$
$$\langle \hat{X}_2^2 \rangle = \frac{1}{4} \left[1 - (\alpha - \alpha^*)^2 \right].$$

Then,

$$\Delta \hat{X}_2 = \sqrt{\langle \hat{X}_2^2 \rangle - \langle \hat{X}_2 \rangle^2}$$
$$= \sqrt{\frac{1}{4}[1 - (\alpha - \alpha^*)^2] + \frac{1}{4}(\alpha - \alpha^*)^2}$$
$$\Delta \hat{X}_2 = \frac{1}{2}.$$

References

[1] Claude Cohen-Tannoudji, Bernard Diu, and Frank Laloe. Quantum mechanics, volume 1. *Quantum Mechanics*, 1:898, 1986.