Assignment 2

OPTI 544 Quantum Optics

University of Arizona

Nicolás Hernández Alegría

February 24, 2026

Exercise 1

a) The quadrature operators are:

$$\hat{X}_1 = \frac{\hat{a} + \hat{a}^\dagger}{2}, \quad \text{and} \quad \hat{X}_2 = \frac{\hat{a} - \hat{a}^\dagger}{2i}.$$

We can use them to express $\hat{a}, \hat{a}^\dagger$ in terms of $\hat{X}_{1,2}$.

$$\hat{a} + \hat{a}^\dagger = 2\hat{X}_1 \tag{1}$$
$$\hat{a} - \hat{a}^\dagger = 2i\hat{X}_2. \tag{2}$$

Then,

$$\begin{aligned}
(1) + (2): &\quad 2\hat{a} = 2(\hat{X}_1 + i\hat{X}_2) \longrightarrow \hat{a} = \hat{X}_1(0) + i\hat{X}_2(0) \\
(1) - (2): &\quad 2\hat{a}^\dagger = 2(\hat{X}_1 - i\hat{X}_2) \longrightarrow \hat{a}^\dagger = \hat{X}_1(0) - i\hat{X}_2(0)
\end{aligned}$$

Using these in the time-evolved quadrature operators:

$$\begin{aligned}
\hat{X}_1(t) &= \frac{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2} = \frac{[\hat{X}_1(0) + i\hat{X}_2(0)]e^{-i\omega t} + [\hat{X}_1(0) - i\hat{X}_2(0)]e^{i\omega t}}{2} \\
\hat{X}_1(t) &= \frac{e^{i\omega t} + e^{-i\omega t}}{2}\hat{X}_1(0) + \frac{(e^{i\omega t} - e^{-i\omega t})}{2i}\hat{X}_2(0) = \cos(\omega t)\hat{X}_1(0) + \sin(\omega t)\hat{X}_2(0).
\end{aligned}$$

Likewise,

$$\begin{aligned}
\hat{X}_2(t) &= \frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i} = \frac{[\hat{X}_1(0) + i\hat{X}_2(0)]e^{-i\omega t} - [\hat{X}_1(0) - i\hat{X}_2(0)]e^{i\omega t}}{2i} \\
\hat{X}_2(t) &= -\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\hat{X}_1(0) + i\frac{(e^{i\omega t} + e^{-i\omega t})}{2i}\hat{X}_2(0) = -\sin(\omega t)\hat{X}_1(0) + \cos(\omega t)\hat{X}_2(0).
\end{aligned}$$

The commutation relation of the time-evolved quadrature operators is

$$[\hat{X}_1, \hat{X}_2] = \frac{1}{4i}[\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}, \hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}] = \frac{1}{4i}\left\{-[\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}]\right\} = \frac{1}{4i}(-2) = \frac{i}{2}.$$

b) The expectation value is

$$\begin{aligned}\langle\alpha|\hat{X}_1(t)|\alpha\rangle &= \langle\alpha|\frac{\hat{a}e^{-i\omega t}+\hat{a}^\dagger e^{i\omega t}}{2}|\alpha\rangle = \frac{1}{2}\left[\langle\alpha|\hat{a}|\alpha\rangle e^{-i\omega t}+\langle\alpha|\hat{a}^\dagger|\alpha\rangle e^{i\omega t}\right] = \frac{1}{2}\left[\alpha e^{-i\omega t}+\alpha^* e^{i\omega t}\right]\\ &= \frac{1}{2}\left[(\alpha e^{-i\omega t})+(\alpha e^{-i\omega t})^*\right] = \mathrm{Re}\left(\alpha e^{-i\omega t}\right).\\ \langle\alpha|\hat{X}_2(t)|\alpha\rangle &= \langle\alpha|\frac{\hat{a}e^{-i\omega t}-\hat{a}^\dagger e^{i\omega t}}{2i}|\alpha\rangle = \frac{1}{2i}\left[\langle\alpha|\hat{a}|\alpha\rangle e^{-i\omega t}-\langle\alpha|\hat{a}^\dagger|\alpha\rangle e^{i\omega t}\right] = \frac{1}{2i}\left[\alpha e^{-i\omega t}-\alpha^* e^{i\omega t}\right]\\ &= \frac{1}{2i}\left[(\alpha e^{-i\omega t})-(\alpha e^{-i\omega t})^*\right] = \mathrm{Im}\left(\alpha e^{-i\omega t}\right).\end{aligned}$$

The coherent state $\left|\alpha e^{-i\omega t}\right\rangle$ is a complex wave that rotates with a rate given by ω. $|\alpha\rangle$ states the initial position in the $\hat{X}_1\hat{X}_2$ phase space diagram.

$$\alpha e^{-i\omega t} = \langle\hat{X}_1\rangle + i\langle\hat{X}_2\rangle.$$

The uncertainties requires also the expectation value of the operator squared:

$$\begin{aligned}\langle\alpha|\hat{X}_1^2(t)|\alpha\rangle &= \frac{1}{4}\langle\alpha|(\hat{a}^2e^{-i2\omega t}+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger 2}e^{i2\omega t})|\alpha\rangle = \frac{1}{4}\left[\alpha^2e^{-i2\omega t}+2|\alpha|^2+1+\alpha^{*2}e^{i2\omega t}\right]\\ &= \frac{1}{4}\left[(\alpha e^{-i\omega t}+\alpha^* e^{i\omega t})^2+1\right] = \frac{1}{4}\left\{[2\mathrm{Re}\left(\alpha e^{-i\omega t}\right)]^2+1\right\}.\\ \langle\alpha|\hat{X}_2^2(t)|\alpha\rangle &= -\frac{1}{4}\langle\alpha|(\hat{a}^2e^{-i2\omega t}-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger 2}e^{i2\omega t})|\alpha\rangle = -\frac{1}{4}\left[\alpha^2e^{-i2\omega t}-2|\alpha|^2-1+\alpha^{*2}e^{i2\omega t}\right]\\ &= \frac{1}{4}\left[1-(\alpha e^{-i\omega t}-\alpha^* e^{i\omega t})^2\right] = \frac{1}{4}\left\{1-[2i\mathrm{Im}\left(\alpha e^{-i\omega t}\right)]^2\right\}.\end{aligned}$$

Uncertainty is

$$\begin{aligned}\Delta\hat{X}_1(t) &= \sqrt{\langle\hat{X}_1^2(t)\rangle-\langle\hat{X}_1(t)\rangle^2} = \sqrt{\frac{1}{4}\{[2\mathrm{Re}\,(\alpha e^{-i\omega t})]^2+1\}-[\mathrm{Re}\,(\alpha e^{-i\omega t})]^2} = \sqrt{\frac{1}{4}} = \frac{1}{2}.\\ \Delta\hat{X}_2(t) &= \sqrt{\langle\hat{X}_2^2(t)\rangle-\langle\hat{X}_2(t)\rangle^2} = \sqrt{\frac{1}{4}\{1-[2i\mathrm{Im}\,(\alpha e^{-i\omega t})]^2\}-[\mathrm{Im}\,(\alpha e^{-i\omega t})]^2} = \sqrt{\frac{1}{4}} = \frac{1}{2}.\end{aligned}$$

The diagram is shown below. Two distances are shown in red to verify that the uncertainty is maintained.

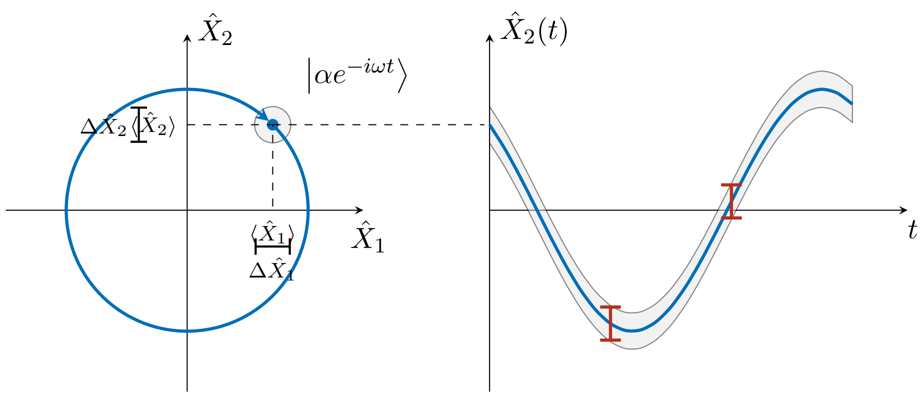

Figure 1: Phase space of a coherent state. Distance in red is shown to verify the uncertainty is maintained.

c) The process is analogous. We know that

$$\tilde{a} = \hat{S}^\dagger \hat{D}^\dagger \hat{a} \hat{D} \hat{S} = \cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha, \quad \text{and}$$
$$\tilde{a}^\dagger = \hat{S}^\dagger \hat{D}^\dagger \hat{a}^\dagger \hat{D} \hat{S} = \cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*.$$

The expected value of $\hat{X}_1(t)$ is

$$\begin{aligned}
\langle \hat{X}_1(t)\rangle &= \langle 0|\hat{S}^\dagger \hat{D}^\dagger \frac{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2} \hat{D}\hat{S}|0\rangle = \frac{1}{2}\langle 0|\tilde{a}e^{-i\omega t} + \tilde{a}^\dagger e^{i\omega t}|0\rangle \\
&= \frac{1}{2}\langle 0|\left[\cosh(r)\hat{a}e^{-i\omega t} - e^{i\phi}\sinh(r)\hat{a}^\dagger e^{-i\omega t} + \alpha e^{-i\omega t} + \cosh(r)\hat{a}^\dagger e^{i\omega t} - e^{-i\phi}\sinh(r)\hat{a}e^{i\omega t} + \alpha^* e^{i\omega t}\right]|0\rangle \\
&= \frac{1}{2}[\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}] \\
\langle \hat{X}_1(t)\rangle &= \mathrm{Re}\left(\alpha e^{-i\omega t}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle \hat{X}_2(t)\rangle &= \langle 0|\hat{S}^\dagger \hat{D}^\dagger \frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i} \hat{D}\hat{S}|0\rangle = \frac{1}{2i}\langle 0|\tilde{a}e^{-i\omega t} + \tilde{a}^\dagger e^{i\omega t}|0\rangle \\
&= \frac{1}{2i}\langle 0|\left[\cosh(r)\hat{a}e^{-i\omega t} - e^{i\phi}\sinh(r)\hat{a}^\dagger e^{-i\omega t} + \alpha e^{-i\omega t} - \cosh(r)\hat{a}^\dagger e^{i\omega t} + e^{-i\phi}\sinh(r)\hat{a}e^{i\omega t} - \alpha^* e^{i\omega t}\right]|0\rangle \\
&= \frac{1}{2}[\alpha e^{-i\omega t} - \alpha^* e^{i\omega t}] \\
\langle \hat{X}_2(t)\rangle &= \mathrm{Im}\left(\alpha e^{-i\omega t}\right).
\end{aligned}$$

Doing the same for the operators squared:

$$\begin{aligned}
\langle \hat{X}_1^2(t)\rangle &= \langle 0|\hat{S}^\dagger \hat{D}^\dagger \left(\frac{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}}{2}\right)^2 \hat{D}\hat{S}|0\rangle \\
&= \frac{1}{4}\langle 0|\hat{S}^\dagger \hat{D}^\dagger \left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right)\hat{D}\hat{S}\hat{S}^\dagger \hat{D}^\dagger \left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right)\hat{D}\hat{S}|0\rangle \\
&= \frac{1}{4}\langle 0|(\tilde{a}e^{-i\omega t} + \tilde{a}^\dagger e^{i\omega t})^2|0\rangle \\
&= \frac{1}{4}\left[e^{-i2\omega t}\langle 0|\tilde{a}^2|0\rangle + e^{i2\omega t}\langle 0|\tilde{a}^{\dagger 2}|0\rangle + \langle 0|\tilde{a}\tilde{a}^\dagger|0\rangle + \langle 0|\tilde{a}^\dagger\tilde{a}|0\rangle\right].
\end{aligned}$$

We compute each term separately:

$$\begin{aligned}
\langle 0|\tilde{a}^2|0\rangle =& \langle 0|(\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha)(\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha)|0\rangle \\
=& \cosh^2(r)\langle\hat{a}^2\rangle - e^{i\phi}\sinh(r)\cosh(r)\langle\hat{a}\hat{a}^\dagger\rangle + \alpha\cosh(r)\langle\hat{a}\rangle - e^{i\phi}\sinh(r)\cosh(r)\langle\hat{a}^\dagger\hat{a}\rangle \\
&+ e^{i2\phi}\sinh^2(r)\langle\hat{a}^{\dagger 2}\rangle - \alpha e^{i\phi}\sinh(r)\langle\hat{a}^\dagger\rangle + \alpha\cosh(r)\langle\hat{a}\rangle - \alpha e^{i\phi}\sinh(r)\langle\hat{a}^\dagger\rangle + \alpha^2 \\
&= \alpha^2 - e^{i\phi}\sinh(r)\cosh(r) \\
\langle 0|\tilde{a}^2|0\rangle &= \alpha^2 - \frac{1}{2}e^{i\phi}\sinh(2r).
\end{aligned}$$

$$\begin{aligned}
\langle 0|\tilde{a}^{\dagger 2}|0\rangle =& \langle 0|(\cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*)(\cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*)|0\rangle \\
=& \cosh^2(r)\langle\hat{a}^{\dagger 2}\rangle - e^{-i\phi}\cosh(r)\sinh(r)\langle\hat{a}^\dagger\hat{a}\rangle + \alpha^*\cosh(r)\langle\hat{a}^\dagger\rangle - e^{-i\phi}\sinh(r)\cosh(r)\langle\hat{a}\hat{a}^\dagger\rangle \\
&+ e^{-i2\phi}\sinh^2(r)\langle\hat{a}^2\rangle - \alpha^* e^{-i\phi}\sinh(r)\langle\hat{a}\rangle + \alpha^*\cosh(r)\langle\hat{a}^\dagger\rangle - \alpha^* e^{-i\phi}\sinh(r)\langle\hat{a}^\dagger\rangle + \alpha^{*2} \\
=& \alpha^{*2} - e^{-i\phi}\sinh(r)\cosh(r) \\
\langle 0|\tilde{a}^{\dagger 2}|0\rangle =& \alpha^{*2} - \frac{1}{2}e^{-i\phi}\sinh(2r).
\end{aligned}$$

$$\begin{aligned}\langle 0|\tilde{a}\tilde{a}^\dagger|0\rangle &= \langle 0|(\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha)(\cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*)|0\rangle \\ \langle 0|\tilde{a}\tilde{a}^\dagger|0\rangle &= \cosh^2(r) + |\alpha|^2.\end{aligned}$$

$$\begin{aligned}\langle 0|\tilde{a}^\dagger\tilde{a}|0\rangle &= \langle 0|(\cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a} + \alpha^*)(\cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha)|0\rangle \\ \langle 0|\tilde{a}^\dagger\tilde{a}|0\rangle &= \sinh^2(r) + |\alpha|^2.\end{aligned}$$

Putting all together:

$$\begin{aligned}\langle \hat{X}_1^2(t)\rangle &= \frac{1}{4}\left\{[\alpha^2 - \frac{1}{2}e^{i\phi}\sinh(2r)]e^{-i2\omega t} + [\alpha^{*2} - \frac{1}{2}e^{-i\phi}\sinh(2r)]e^{i2\omega t} + \cosh^2(r) + |\alpha|^2 + \sinh^2(r) + |\alpha|^2\right\} \\ &= \frac{1}{4}\left[(\alpha e^{-i\omega t} + \alpha^* e^{i\omega t})^2 + \cosh(2r) - \sinh(2r)\cos(\phi - 2\omega t)\right] \\ \langle \hat{X}_1^2(t)\rangle &= [\mathrm{Re}\left(\alpha e^{-i\omega t}\right)]^2 + \frac{1}{4}[\cosh(2r) - \sinh(2r)\cos(\phi - 2\omega t)].\end{aligned}$$

The uncertainty is:

$$\Delta\hat{X}_1(t) = \sqrt{\langle \hat{X}_1^2(t)\rangle - \langle \hat{X}_1(t)\rangle^2} = \frac{1}{2}\sqrt{\cosh(2r) - \sinh(2r)\cos(\phi - 2\omega t)}.$$

Similarly,

$$\begin{aligned}\langle \hat{X}_2^2(t)\rangle &= \langle 0|\hat{S}^\dagger\hat{D}^\dagger\left(\frac{\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}}{2i}\right)^2\hat{D}\hat{S}|0\rangle \\ &= -\frac{1}{4}\langle 0|\hat{S}^\dagger\hat{D}^\dagger(\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t})\hat{D}\hat{S}\hat{S}^\dagger\hat{D}^\dagger(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t})\hat{D}\hat{S}|0\rangle \\ &= -\frac{1}{4}\langle 0|(\tilde{a}e^{-i\omega t} - \tilde{a}^\dagger e^{i\omega t})^2|0\rangle \\ &= -\frac{1}{4}[e^{-i2\omega t}\langle 0|\tilde{a}^2|0\rangle + e^{i2\omega t}\langle 0|\tilde{a}^{\dagger 2}|0\rangle - \langle 0|\tilde{a}\tilde{a}^\dagger|0\rangle - \langle 0|\tilde{a}^\dagger\tilde{a}|0\rangle].\end{aligned}$$

We use the previous results to get

$$\begin{aligned}\langle \hat{X}_2^2(t)\rangle &= -\frac{1}{4}\left\{[\alpha^2 - \frac{1}{2}e^{i\phi}\sinh(2r)]e^{-i2\omega t} + [\alpha^{*2} - \frac{1}{2}e^{-i\phi}\sinh(2r)]e^{i2\omega t} - \cosh^2(r) - |\alpha|^2 - \sinh^2(r) - |\alpha|^2\right\} \\ &= -\frac{1}{4}[(\alpha e^{-i\omega t} - \alpha^* e^{i\omega t})^2 - \cosh(2r) - \sinh(2r)\cos(\phi - 2\omega t)] \\ \langle \hat{X}_2^2(t)\rangle &= [\mathrm{Im}\left(\alpha e^{-i\omega t}\right)]^2 + \frac{1}{4}[\cosh(2r) + \sinh(2r)\cos(\phi - 2\omega t)].\end{aligned}$$

And the uncertainty is:

$$\Delta\hat{X}_2(t) = \sqrt{\langle \hat{X}_2^2(t)\rangle - \langle \hat{X}_2(t)\rangle^2} = \frac{1}{2}\sqrt{\cosh(2r) + \sinh(2r)\cos(\phi - 2\omega t)}$$

We can verify that $\Delta\hat{X}_1(0) = e^{-r}/2$ and $\Delta\hat{X}_2(0) = e^{r}/2$ when $\phi = t = 0$.
The phase space representation is shown below for $t = \phi = 0$ and $r = 0.6$.

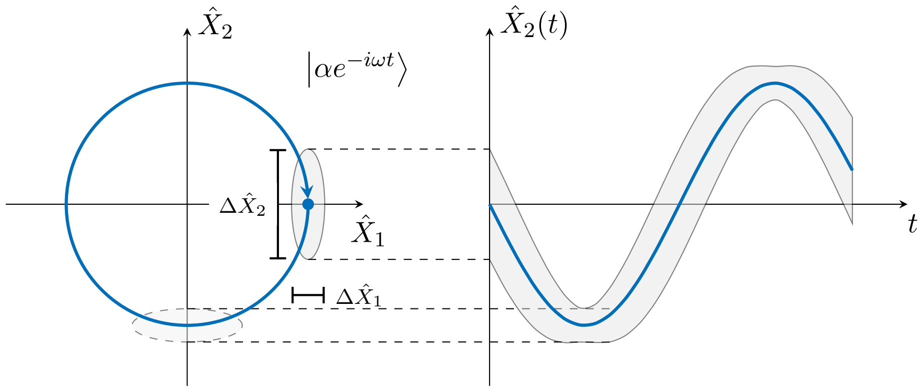

Figure 2: Phase space of the squeezed displaced state for $r = 0.6$.

Exercise 2

In general, the photon number for output i is $\hat{N}_i = \hat{b}_i^\dagger \hat{b}_i$. For each output we have:

$$\hat{N}_1 = \frac{1}{2}(\hat{a}_1^\dagger + \hat{a}_2^\dagger)(\hat{a}_1 + \hat{a}_2) = \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2),$$
$$\hat{N}_2 = \frac{1}{2}(\hat{a}_1^\dagger - \hat{a}_2^\dagger)(\hat{a}_1 - \hat{a}_2) = \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2).$$

Substraction gives

$$\hat{N}_1 - \hat{N}_2 = \frac{1}{2}\left[(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2) - (\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2)\right] = \hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1.$$

With the phase shift, the input 2 is:

$$\alpha \to \alpha e^{i\phi_\alpha}, \quad \text{and} \quad \hat{a}_2 \approx \alpha e^{i\phi_\alpha}.$$

The number photon difference is then:

$$\hat{N}_1 - \hat{N}_2 = \hat{a}_1^\dagger \alpha e^{i\phi_\alpha} + \hat{a}_1\alpha^* e^{-i\phi_\alpha}.$$

If $\alpha = |\alpha|$, then

$$\hat{N}_1 - \hat{N}_2 = |\alpha|\left(\hat{a}_1^\dagger e^{i\phi_\alpha} + \hat{a}_1 e^{-i\phi_\alpha}\right) = 2|\alpha|\hat{X}_{\phi_\alpha}, \quad \hat{X}_{\phi_\alpha} = \frac{\hat{a}_1^\dagger e^{i\phi_\alpha} + \hat{a}_1 e^{-i\phi_\alpha}}{2},$$

where $\hat{X}_{\phi_\alpha}$ is the quadrature of mode 1 modulated by the phase shift ϕ_α.

To get the uncertainty $\Delta(\hat{N}_1 - \hat{N}_2)$, we compute the expected alue of the difference and the expected value of the difference squared. We can use the results from the previoous exercise as the input 1 is a squeezed state of the vacuum.

$${}_1\langle\xi,0|_2\langle\alpha|\,(\hat{N}_1 - \hat{N}_2)\,|\alpha\rangle_2\,|\xi,0\rangle_1 = 2|\alpha|\langle\hat{X}_{\phi_\alpha}\rangle = 0,$$
$${}_1\langle\xi,0|_2\langle\alpha|\,(\hat{N}_1 - \hat{N}_2)^2\,|\alpha\rangle_2\,|\xi,0\rangle_1 = 4|\alpha|^2\langle\hat{X}_{\phi_\alpha}^2\rangle = |\alpha|^2[\cosh(2r) - \sinh(2r)\cos(\theta - 2\phi_\alpha)],$$

where $\xi = re^{i\theta}$ is the squeezing factor, and ϕ_α the phase shift imposed by the input 2. Uncertainty is therefore:

$$\Delta(\hat{N}_1 - \hat{N}_2) = |\alpha|\sqrt{\cosh(2r) - \sinh(2r)\cos(\theta - 2\phi_\alpha)}.$$

Exercise 3

(a) For the $50:50$ beamsplitter 1, we have:

$$\begin{bmatrix}\hat{b}_1\\ \hat{b}_2\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\hat{a}_1\\ \hat{a}_2\end{bmatrix}, \quad \text{with} \quad \begin{bmatrix}\hat{a}_1\\ \hat{a}_2\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\hat{b}_1\\ \hat{b}_2\end{bmatrix}.$$

(b) From the last part, we know that

$$\hat{a}_1 = \frac{1}{\sqrt{2}}(\hat{b}_1+\hat{b}_2) \xrightarrow{h.c.} \hat{a}_1^\dagger = \frac{1}{\sqrt{2}}(\hat{b}_1^\dagger+\hat{b}_2^\dagger), \quad \text{and} \quad \hat{a}_2 = \frac{1}{\sqrt{2}}(\hat{b}_1-\hat{b}_2) \xrightarrow{h.c.} \hat{a}_2^\dagger = \frac{1}{\sqrt{2}}(\hat{b}_1^\dagger-\hat{b}_2^\dagger).$$

The input is then:

$$\text{Input} = |1\rangle_1|0\rangle_2 = \hat{a}_1^\dagger|0\rangle_1|0\rangle_2.$$

Output is:

$$\text{Output} = \frac{1}{\sqrt{2}}(\hat{b}_1^\dagger+\hat{b}_2^\dagger)|0\rangle_1|0\rangle_2 = \frac{1}{\sqrt{2}}[|1\rangle_{1'}|0\rangle_{2'}+|0\rangle_{1'}|1\rangle_{2'}].$$

We expect then to have the same probability of getting one photon at each output. The expected value at output 1 is:

$$\begin{aligned}
{}_2\langle 0|_1\langle 1|\hat{b}_1^\dagger\hat{b}_1|1\rangle_1|0\rangle_2 &= \frac{1}{2}\langle 0,1|(\hat{a}_1^\dagger+\hat{a}_2^\dagger)(\hat{a}_1+\hat{a}_2)|1,0\rangle\\
&= \frac{1}{2}\langle 0,1|(\hat{a}_1^\dagger\hat{a}_1+\hat{a}_1^\dagger\hat{a}_2+\hat{a}_2^\dagger\hat{a}_1+\hat{a}_2^\dagger\hat{a}_2)|1,0\rangle\\
{}_2\langle 0|_1\langle 1|\hat{b}_1^\dagger\hat{b}_1|1\rangle_1|0\rangle_2 &= \frac{1}{2}.
\end{aligned}$$

Likewise, the expected value at output 2 is:

$$\begin{aligned}
{}_2\langle 0|_1\langle 1|\hat{b}_2^\dagger\hat{b}_2|1\rangle_1|0\rangle_2 &= \frac{1}{2}\langle 0,1|(\hat{a}_1^\dagger-\hat{a}_2^\dagger)(\hat{a}_1-\hat{a}_2)|1,0\rangle\\
&= \frac{1}{2}\langle 0,1|(\hat{a}_1^\dagger\hat{a}_1-\hat{a}_1^\dagger\hat{a}_2-\hat{a}_2^\dagger\hat{a}_1+\hat{a}_2^\dagger\hat{a}_2)|1,0\rangle\\
{}_2\langle 0|_1\langle 1|\hat{b}_2^\dagger\hat{b}_2|1\rangle_1|0\rangle_2 &= \frac{1}{2}.
\end{aligned}$$

(c) We replace the first output as the input of the second output:

$$\begin{aligned}
\hat{c}_1 &= \frac{1}{\sqrt{2}}[\hat{b}_1e^{i\phi}+\hat{b}_2] = \frac{1}{2}\Big[(\hat{a}_1+\hat{a}_2)e^{i\phi}+(\hat{a}_1-\hat{a}_2)\Big] = \frac{1}{2}[\hat{a}_1(1+e^{i\phi})+\hat{a}_2(e^{i\phi}-1)],\\
\hat{c}_2 &= \frac{1}{\sqrt{2}}[\hat{b}_1e^{i\phi}-\hat{b}_2] = \frac{1}{2}\Big[(\hat{a}_1+\hat{a}_2)e^{i\phi}-(\hat{a}_1-\hat{a}_2)\Big] = \frac{1}{2}[\hat{a}_1(e^{i\phi}-1)+\hat{a}_2(1+e^{i\phi})].
\end{aligned}$$

(d) The expected number of photons at detector 1 is:

$$\begin{aligned}
\langle 0,1|\hat{c}_1^\dagger\hat{c}_1|1,0\rangle &= \frac{1}{4}\Big\{[\hat{a}_1^\dagger(1+e^{-i\phi})+\hat{a}_2^\dagger(e^{-i\phi}-1)][\hat{a}_1(1+e^{i\phi})+\hat{a}_2(e^{i\phi}-1)]\Big\}\\
&= \frac{1}{4}\langle 0,1|\Big\{\hat{a}_1^\dagger\hat{a}_1(\cdots)+\hat{a}_1^\dagger\hat{a}_2(\cdots)+\hat{a}_2^\dagger\hat{a}_1(\cdots)+\hat{a}_2^\dagger\hat{a}_2(\cdots)\Big\}|1,0\rangle\\
&= \frac{1}{4}(1+e^{-i\phi})(1+e^{i\phi})\\
\langle 0,1|\hat{c}_1^\dagger\hat{c}_1|1,0\rangle &= \frac{1}{2}[1+\cos\phi] = \cos^2\frac{\phi}{2}.
\end{aligned}$$

By conservation of energy (or probability), the expected number of photon at detector 2 is:

$$\langle 0,1|\hat{c}_2^\dagger \hat{c}_2|1,0\rangle = \sin^2\frac{\phi}{2}.$$