MATHEMATICS INTERNAL ASSESSMENT

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Determining the optimal impact angle of a tennis serve

Introduction

1.1 Aim of investigation

This mathematical exploration aims to determine the optimal angle of impact for a tennis serve in order to deliver a successful serve that maximizes speed and accuracy. As both a competitive tennis player and LTA-qualified coach, I have been intrigued with both mastering the serve as well as teaching it as it is one of the hardest yet essential skills of tennis. A tennis serve is a weapon once mastered and is crucial in determining a player's overall success in a game. Therefore, a better understanding of the optimal angle of impact will not only improve my tennis play but will make coaching more efficient for my students.

1.2 What is the impact angle?

The impact angle of a serve is the point at which the racquet makes contact with the tennis ball (as displayed in stage 4 of Figure 1).

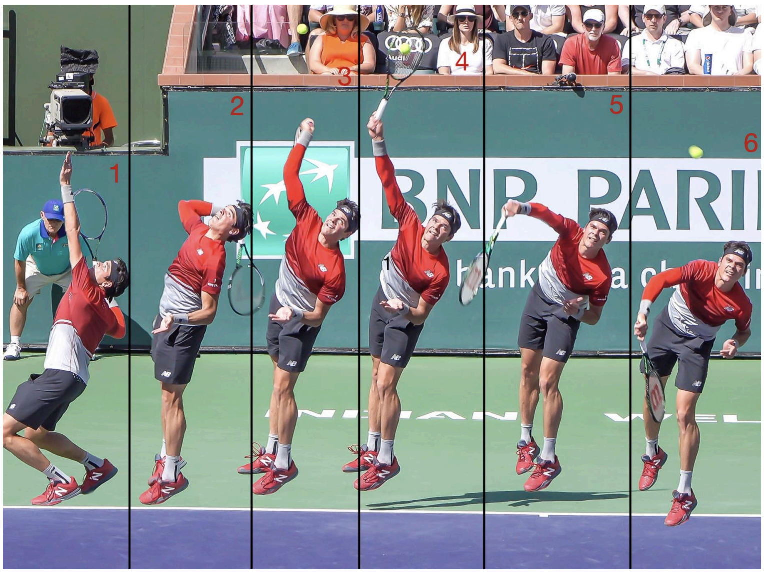

Figure 1 - Frame-by-frame images of Milos Raonic's serve [1].

1.3 Optimum service location

In the game of tennis, the server is allowed to hit anywhere in the opposing diagonal service box, however, there are three specific points the server can aim for to maximize the difficulty of their opponent returning the serve and thus the opportunity for an ace. These placements are the:

- T-Serve (down the middle): The T-serve is often used to minimize the opponent's angle for returns as the T-serve targets the middle of the service box, splitting the service box into two equal parts.

[1] jfawcette.myportfolio.com. (n.d.). *Tennis Photography - Technique*. [online] Available at: https://jfawcette.myportfolio.com/technique

- Wide Serve: Serving wide means hitting the ball towards the sideline. A well-placed wide serve can pull your opponent off the court and create an open space for your next shot.
- Body Serve: A body serve is directed at your opponent's body, making it difficult for them to move and swing freely. This can be particularly effective against players who like to anticipate and take an early swing at the ball as a well-placed body serve can cause discomfort and disrupt the opponent's timing.

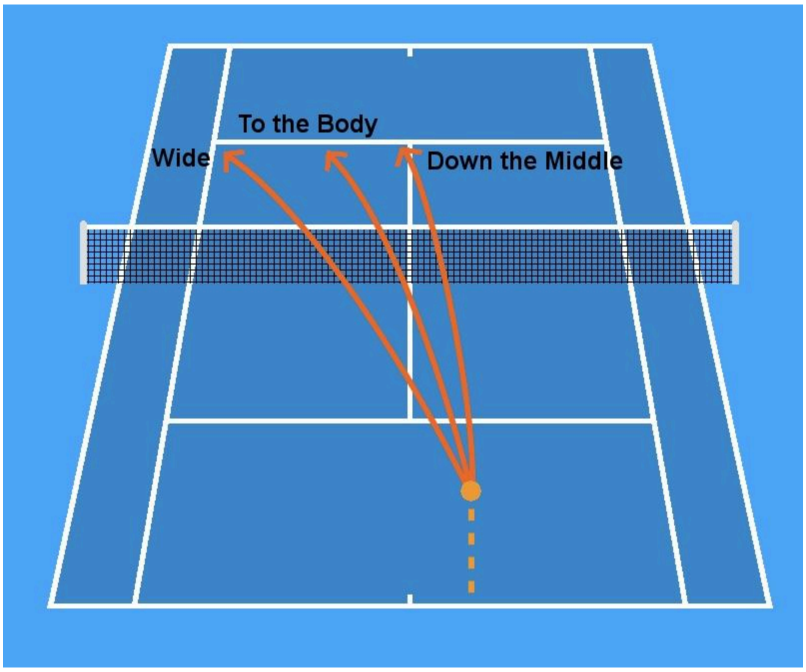

Figure 2 - diagram of the three optimum serve locations[2].

Nevertheless, for simplification purposes, I will use the T point in my calculations as the optimum aim point for my serve.

1.4 Methodology

In this exploration, in order to determine the optimum impact angle of the server, I will need to obtain the following serve data from professional players and myself:

- The initial speed of the ball (as a scalar value).
- The coordinates on the court where the player makes contact with the ball (note that some players jump into the court during the serve movement so the server contact point has both an x and y coordinate).
- The height at which the player makes contact with the ball to serve.

Once data is collected, I will then determine the optimum impact angle that will hit the aim point through projectile motion equations and will investigate the effects of perturbing the speed.

[2] Dominik (n.d.). *The Perfect Tennis Serve in 8 Simple Steps*. [online] Tennis Uni. Available at: https://tennis-uni.com/en/tennis-serve/.

Finding the length of the serve

2.1 Calculation

In order to calculate the length of the serve, the distance from the server's position at the baseline to the diagonally opposite service box, I must first place a coordinate system (utilising standard court dimensions[3]) over the tennis court where the centre of the server's baseline is (0,0), with meters as the unit of scale.

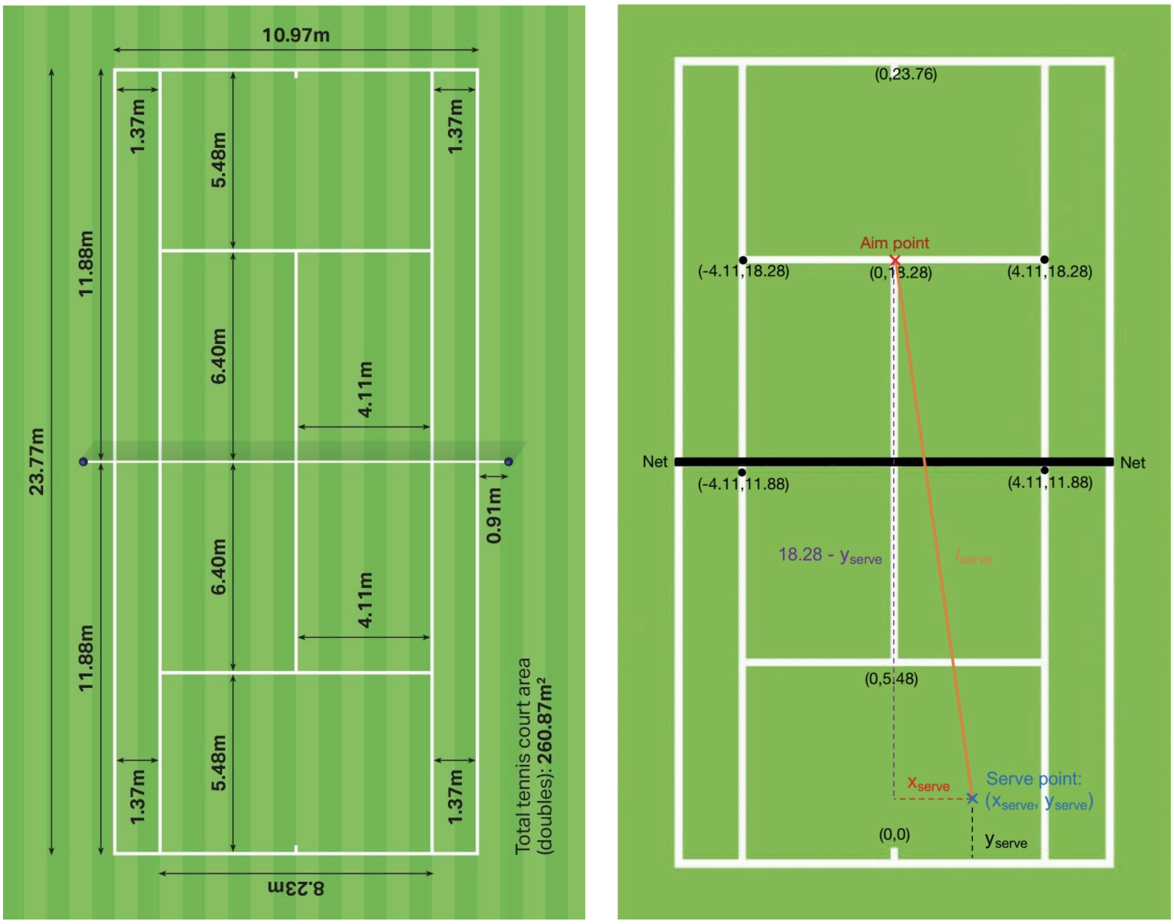

Figure 3 - dimensions of a tennis court translated to coordinate system (made by candidate)

- x_{serve} = x coordinate of service
- y_{server} = y coordinate of service
- (x_{serve}, y_{server}) = Contact point of the serve - not at the baseline as the server jumps into the court during service. However, this remains an educated estimation, as the player's movement onto the court during a serve entirely relies on the extent of their jump.

From the diagram in Figure 3, I can find a formula for l_{serve} through the Pythagorean theorem of $a^2 + b^2 = c^2$:

$$x^2_{serve} + (18.28 - y_{serve})^2 = l^2_{serve}$$

[3] Harrod Sport (2020). *Tennis Court Dimensions & Size | Harrod Sport*. [online] www.harrodsport.com. Available at: https://www.harrodsport.com/advice-and-guides/tennis-court-dimensions.

so

$$l_{\text{serve}} = \sqrt{x_{serve}^{2} + (18.28 - y_{serve})^{2}}$$

As I now have derived a formula for serve length, I will now analyse the serve lengths of professional players to determine their impact angles.

2.2 Professional serve lengths

Professional serve data showcases the following data from three notable tennis players[4]:

Professional player	x_{serve} (meters)	y_{serve} (meters)	l_{serve} (meters)
John Isner	1.0	1.2	17.1
Andy Murray	0.5	0.5	17.79
Serena Williams	0.5	0.25	18.04

Firstly, I chose John Isner as one of my data subjects as he currently holds the ATP's official record for the fastest serve, clocking 253 km/h[5] during a match in the 2016 Davis cup. Therefore, his serve's impact angle presents the most optimal starting point for this investigation due to its exceptional speed and power. In addition, I have chosen Serena Williams as my next data subject as she is one of the most successful tennis players in history and has one of the fastest serves by a female player. Moreover, Serena Williams, being my favourite player, adds a captivating dimension to motivate me further in this investigation. Finally, I have chosen Andy Murray as my final data subject as he represents a different test case to Isner and Williams, as he is considered to have an average professional serve. As a result, these data subjects will provide me with a diverse range of test data to enable me to thoroughly assess and compare impact angles across serve styles and strengths.

With the professional serve length data, I can now attempt to derive the angle of impact utilising trigonometry as shown in my diagram below:

[4] Fawcette, J. (n.d.). Tennis photography - serve impact height. [online] Available at: jfawcette.myportfolio.com

[5] Jonathan (2023). *The Fastest Tennis Serves Ever - peRFect Tennis*. [online] Available at: https://www.perfect-tennis.com/fastest-tennis-serve/.

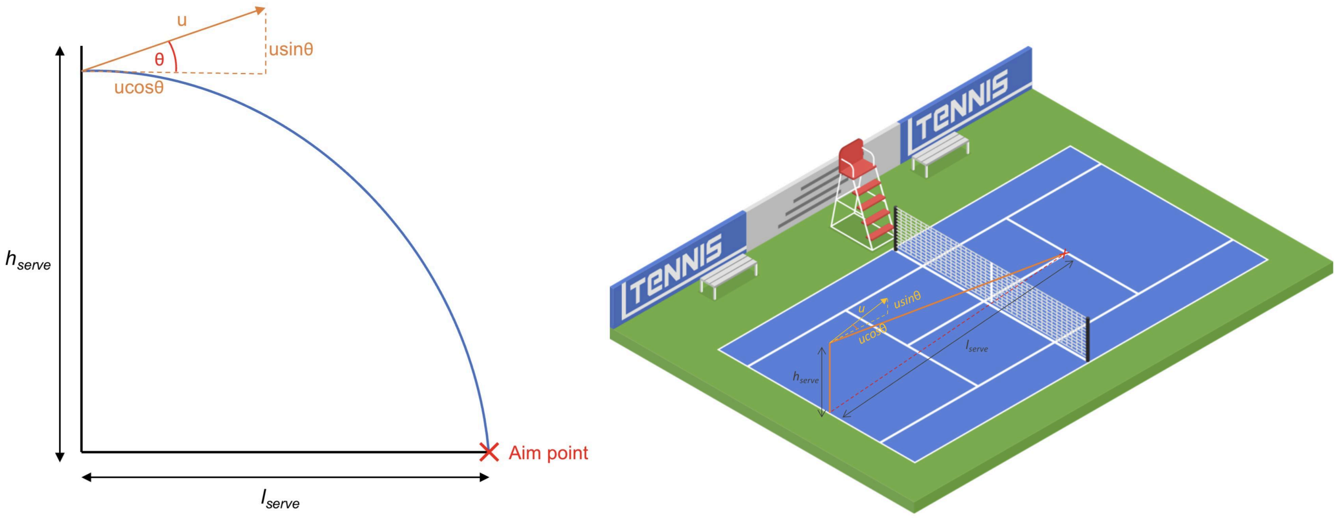

Figure 4 - 2D and 3D diagram of serve trigonometry (made by candidate)

The diagram in Figure 4 displays:

- θ = the impact angle being calculated from the horizontal
- h_{serve} = impact height of serve
- u = impact speed
- u.sin(θ) = the impact vertical speed
- u.cos(θ) = the impact horizontal speed

2.3 Optimum angle of impact with no air resistance

As the serve length has been acquired from online data, I can now look at the horizontal cross section and apply SUVAT mechanics equations to model the motion of the ball. The first part of the calculation is to resolve horizontally. The ball is moving at a constant speed of ucosθ horizontally so we can use:

$$speed = \frac{distance}{time}$$

Through this equation, I can determine t_{serve}, the time taken in seconds for the ball to travel to the aim point, which is l_{serve} meters away. Therefore, substituting the equation as:

$$u.\cos(\theta) = \frac{l_{serve}}{t_{serve}}$$

re-arranging we find that t_{serve} is:

$$t_{serve} = \frac{l_{serve}}{u.cos(\theta)} \tag{1}$$

Now I will resolve vertically using the projectile motion equation:

$$s = ut + \frac{1}{2}at^2$$

Where s = distance (*meters*), u = impact speed (ms^{-1}), t = time taken (*seconds*), and a = acceleration (ms^{-2}). Thus, I can now define the positive direction as downwards as showcased in the diagram below:

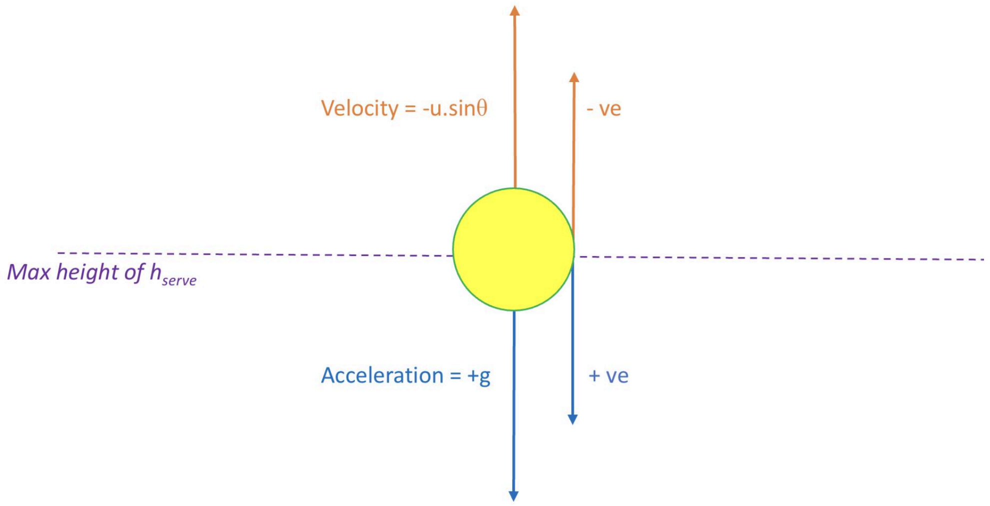

Figure 5 - diagram of the ball moving in the vertical plane, at impact height, falling with gravity (made by candidate)

Therefore:

- acceleration (a) = g (gravitational constant)
- initial speed (u) = - u.sin(θ) -> the negative sign is a result of the ball initially moving upwards
- distance (s) = h_{serve} (the impact height to the floor)

I acquired this mathematical knowledge through the topics of kinematics and volumes of revolution in Maths AA; I hope the vast exploration of these topics will enhance my understanding as I progress through this IA.

Through this, h_{serve} can be defined as:

$$h_{serve} = - u.sin(\theta)t_{serve} + \frac{gt_{serve}^2}{2}$$

Re-arranging this equation to be in a quadratic form:

$$\frac{gt_{serve}^2}{2} - u.sin(\theta)t_{serve} - h_{serve} = 0$$

Now I will substitute in equation 1 (the previously resolved equation of t_{serve}):

$$\frac{g}{2}\left(\frac{l_{serve}}{u.cos(\theta)}\right)^2 - u \cdot sin(\theta)t_{serve} - h_{serve} = 0$$

Resulting in:

$$\frac{gl_{serve}^2}{2u^2} \cdot sec^2(\theta) - l_{serve} \cdot tan(\theta) - h_{serve} = 0$$

I can now use the trig identity:

$$sec^2(\theta) = 1 + tan^2(\theta)$$

Substituting this identity into the previous equation, I find that:

$$\frac{gl^2_{serve}}{2u^2} \cdot (1 + tan^2(\theta)) - l_{serve}tan(\theta) - h_{serve} = 0$$

Rearranging this equation into the quadratic form, I find that:

$$\frac{gl^2_{serve}}{2u^2} \cdot tan^2(\theta) - l_{serve}tan(\theta) + (\frac{gl^2_{serve}}{2u^2} - h_{serve}) = 0$$

I can now use the quadratic formula to solve this equation:

$$tan(\theta) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence:

$$tan(\theta) = \frac{l_{serve} \pm \sqrt{l^2_{serve} - 4\frac{gl^2_{serve}}{2u^2}(\frac{gl^2_{serve}}{2u^2} - h_{serve})}}{2\frac{gl^2_{serve}}{2u^2}}$$

Resulting in the final equation:

$$\theta = tan^{-1}(\frac{l_{serve} \pm \sqrt{l^2_{serve} - 2\frac{gl^2_{serve}}{u^2} \bullet (\frac{gl^2_{serve}}{2u^2} - h_{serve})}}{\frac{gl^2_{serve}}{u^2}})$$

Candidate data to determine optimum impact angle

3.1 Candidate data

Now that I have found an equation for impact angle, I am intrigued to determine the impact angle of myself, an amateur tennis player. I measured from the images that I served with an impact height (h_{serve}) of 2m, and remained on the baseline for my serve as jumping into the court is a more advanced technique developed with more skill. Therefore, my length of serve (l_{serve}) is 18.28m; values do not have to be precise as the change in trajectory with speed is what I am interested in. To collect data on my average serve speed, I used the app ‘Tennis Serve Tracker’ which calculates the speed of the serve by processing video footage frame by frame and highlighting the court and ball on the image frame. I filmed myself with the help of my sister using this app, noting down the speed of all serves that hit the T aim point and disregarding the speeds of serves that didn't hit the aim point. Through this method, I collected 10 results of speed.

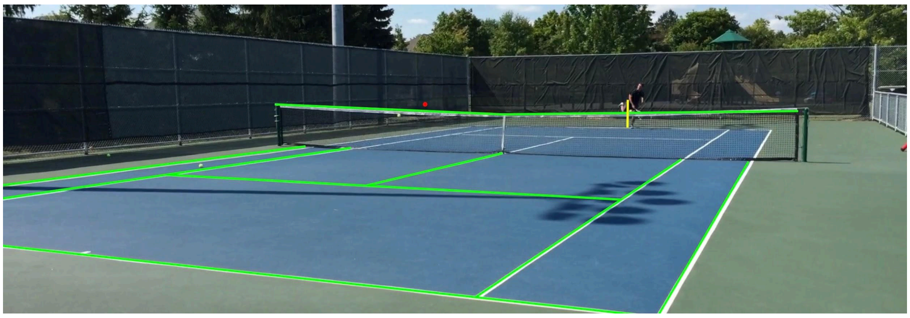

Figure 6 - Images of the candidate collecting data through the 'Tennis Serve Tracker' app, highlighted lines on the court and the net is the app recognition software (made by candidate).

<u>Results table</u>

Using the impact angle formula previously derived, I calculated the impact angle from the speed of my serve at a height of 2m and length of 18.28m. As the impact angle formula has a square root and thus a $\pm$ solution, I will use the smaller solution of the two, as it is the shortest path between the server and the target.

Speed (ms^{-1})	**Smaller impact angle (degrees)**
30.3	-0.672
29.6	-0.405
31.1	-0.956
28.8	-0.0749
32.2	-1.31
27.9	0.331
15.6	16.2
19.8	7.00
25.2	1.83
13.5	27.1
Avg: 22.5	**Not applicable**

With these results, I plotted the graph below:

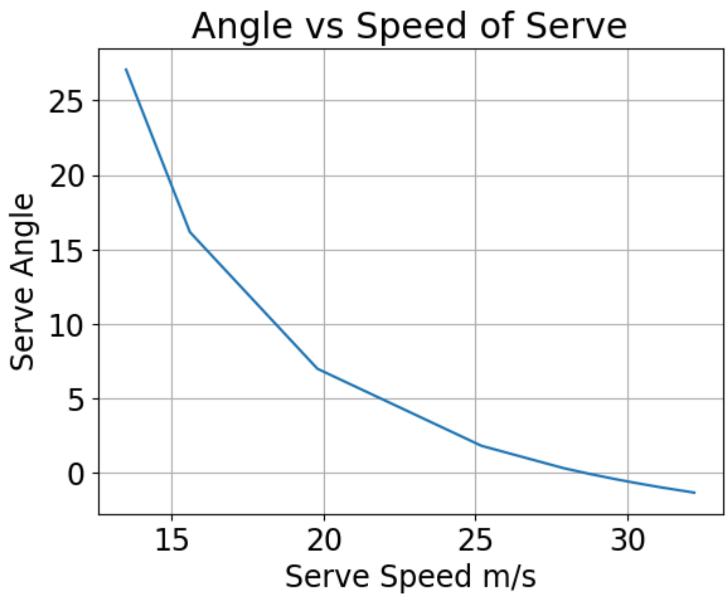

Figure 7 - graph to show the relationship between my serve speed and resulting impact angle

Figure 7 highlights how there is a relationship between serve speed and serve angle. Because of this, I am curious to see what trajectories are possible for myself if I had a faster or slower serve. Plotting this data using Python, I can evidently see the path to which each serve speed will take, as shown below:

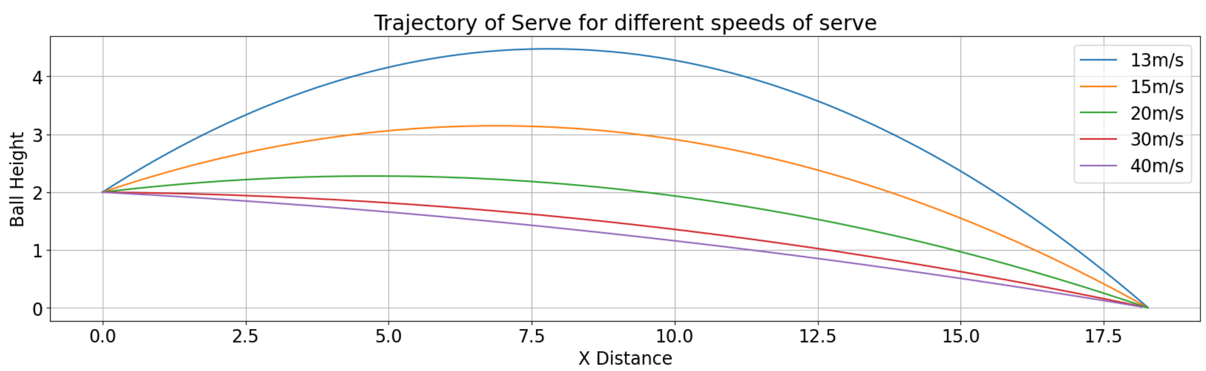

Figure 8 - graph to show trajectory/side view of my serves at differing speeds

I opted to utilise Python for generating these graphs due to my familiarity with its formatting and my previous experience using it in various projects. As depicted in Figure 8, a clear trend emerges: when serving the ball at a lower speed, the trajectory and impact angle tend to be higher. However, it becomes apparent that increasing the speed of my serve while keeping the jump height constant would yield a significantly flatter trajectory, resulting in a faster and more direct flight path

Results analysis with example data

4.1 Professional players against candidate

Utilising the professional tennis data from section 2.2, I will use the final equation alongside technology to determine the impact angle of Isner, Murray, Williams and myself with g being 9.81 ms^{-2}. Inserting Isner's data to the final equation:

$$\theta = tan^{-1}\left(\frac{l_{serve} \pm \sqrt{l^2_{serve} - 2\frac{gl^2_{serve}}{u^2} \bullet \left(\frac{gl^2_{serve}}{2u^2} - h_{serve}\right)}}{\frac{gl^2_{serve}}{u^2}}\right)$$

$$\theta = tan^{-1}\left(\frac{17.1 \pm \sqrt{(17.1)^2 - 2\frac{9.81\,x\,(17.1)^2}{(60.35)^2} \bullet \left(\frac{9.81\,x\,(17.1)^2}{2(60.35)^2} - 3.28\right)}}{\frac{9.81\,x\,(17.1)^2}{(60.35)^2}}\right)$$

$$\theta = tan^{-1}\left(\frac{17.1 \pm \sqrt{296.956351}}{0.7876017624}\right)$$

$$\theta = \text{-9.54 degrees or } 88.68583859$$

I will take the smaller answer as it will arrive at the target in the fastest time. I proceeded to replicate these steps for all remaining datasets, which resulted in:

<u>Results table</u>

Name	l_{serve} (metres)	u (ms^{-1})	h_{serve} (metres)	**θ (degrees)**
Isner	17.1	60.35	3.28	**-9.54**
Murray	17.79	58.12	2.82	**-7.53**
Williams	18.04	57.36	2.51	**-6.39**
Candidate	18.28	22.5	2.00	**3.920**

Plotting these results using Python, I find that the impact angle for professional players aim down and their trajectory is almost a straight line, whereas my impact angle is positive and the trajectory aims up as shown below:

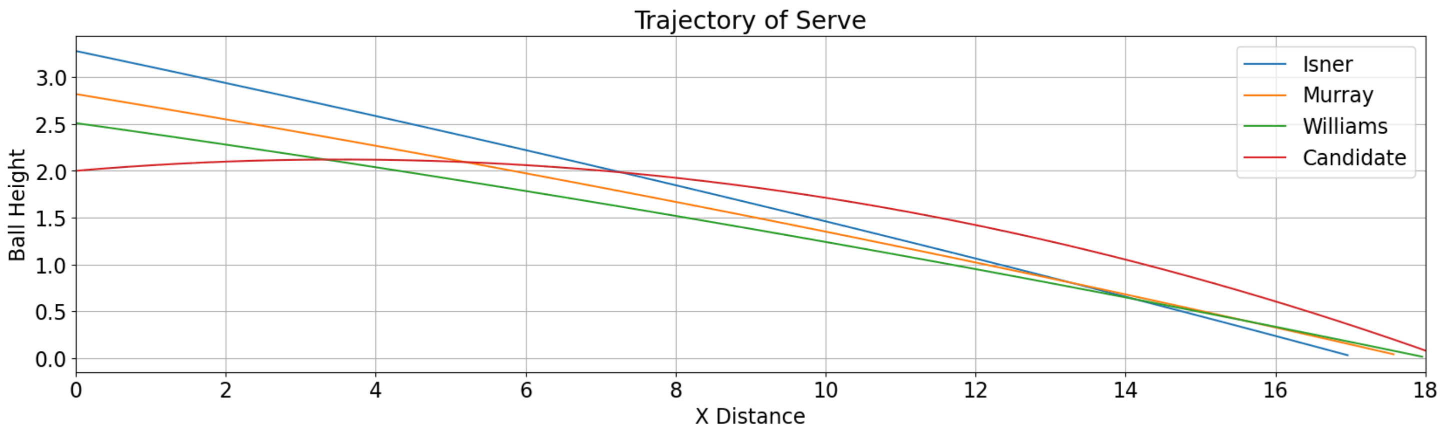

Figure 9 - Graph to show serve trajectory of professional and amateur players (made by candidate)

Figure 9 distinctly underscores the pronounced disparities in serve trajectory between professional players and my own performance. Most notably, there is a clear difference in shape, signifying considerable room for improvement to align more closely with the standards set by professionals. Subsequently, upon closer examination, I noticed Figure 7 has a domain that stops at approximately 12.5. This observation has piqued my curiosity, and will conduct more investigation to ascertain both the underlying cause of this limitation and the precise discriminative value associated with it."

4.2 The slowest speed that provides an optimum angle

In the previous section, I found that my impact angle equation did not solve for a speed less than approximately 12.5ms^{-2}. Because of this, I will find the maximum possible impact of the angle and minimum possible speed using algebra. Going back to my formula for finding θ:

$$\theta = tan^{-1}\left(\frac{l_{serve} \pm \sqrt{l^2_{serve} - 2\frac{gl^2_{serve}}{u^2}\left(\frac{gl^2_{serve}}{2u^2} - h_{serve}\right)}}{\frac{gl^2_{serve}}{u^2}}\right)$$

The square root term normally has two solutions ($\pm$) for which I have used the smaller solution as explained in 3.1 results table due to the smaller solution providing a shorter path. Therefore, I see that there will only be one solution to the equation when the square-root term goes to 0:

$$l^2_{serve} - 2\frac{gl^2_{serve}}{u^2}\left(\frac{gl^2_{serve}}{2u^2} - h_{serve}\right) = 0 \qquad (2)$$

And when the value falls below 0 there will be no solutions (due to the square toot). With this in mind the maximum value of θ must therefore be:

$$\theta_{max} = tan^{-1}\left(\frac{l_{serve}}{\frac{gl_{serve}^2}{u_{min}^2}}\right) = tan^{-1}\left(\frac{u_{min}^2}{gl_{serve}}\right)$$

To find u_{min} I must solve equation (2) for u by multiplying out the equation:

$$l_{serve}^2 - \left(2\frac{gl_{serve}^2}{u^2}\frac{gl_{serve}^2}{2u^2}\right) + \left(2\frac{gl_{serve}^2}{u^2}h_{serve}\right) = 0$$

Simplifying:

$$1 - \frac{g^2 l_{serve}^2}{u^4} + 2\frac{g}{u^2}h_{serve} = 0$$

And then by multiplying through by u^4 and collecting terms for u:

$$u^4 + (2g\,h_{serve})u^2 - g^2 l_{serve}^2 = 0$$

Substituting:

$$x = u^2$$

The equation simplifies to a quadratic:

$$x^2 + (2gh_{serve})x - g^2 l_{serve}^2 = 0$$

x is now solvable using the quadratic formula of:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus:

$$x = \frac{-2gh_{serve} \pm \sqrt{4g^2(l_{serve}^2 + h_{serve}^2)}}{2} = -\,gh_{serve} \pm g\sqrt{l_{serve}^2 + h_{serve}^2}$$

Substituting back x = u_{min}^2 and simplifying:

$$u_{min} = \sqrt{-\,gh_{serve} \pm g\sqrt{l_{serve}^2 + h_{serve}^2}}$$

Thus, ignoring the negative roots as all variables are positive and so negative square roots do not solve, the final equation u_{min} is:

$$u_{min} = \sqrt{-\,gh_{serve} - g\sqrt{l_{serve}^2 + h_{serve}^2}}$$

Inserting the values of my data: l_{serve} = 18.28 and h_{serve} = 2.0. My personal slowest speed that will make it over the net is:

$$u_{min} = 12.68$$

Now that I have found u_{min} for my serve, I will determine the angle that this converges on, through the equation:

$$\theta_{max} = tan^{-1}(\frac{u_{min}^2}{gl_{serve}})$$

Experimenting with varying data heights at intervals of 0.25 meters, I've determined the minimum speed and optimal angle for each height by inputting the values into the equations and employing technology, as demonstrated below:

Height (meters)	Min speed (m/s)	**Angle (degrees)**
2	12.7	**41.9**
2.25	12.6	**41.5**
2.5	12.5	**41.1**
2.75	12.4	**40.7**
3	12.3	**40.3**
3.25	12.3	**40**

I have plotted these results on a graph to visualise the relationship:

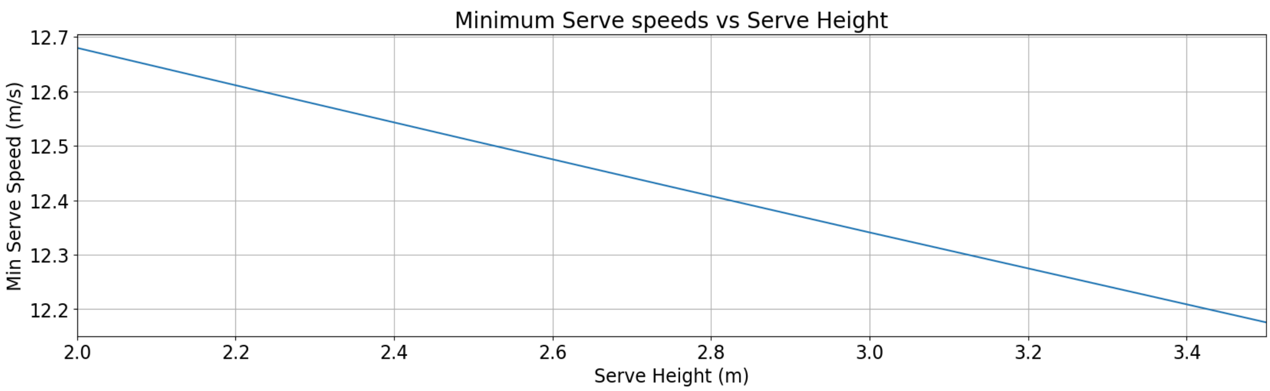

Figure 10 - graph to show the relationship between serve height against the minimum speed to reach the aim point.

Overall, despite the complexity of the u_{min} equation, the results prove to have a linear type relationship. Illustrated in Figure 10, a greater height corresponds to a lower minimum speed requirement. The principle is clear: the higher you leap up to strike the ball, the lesser power needed to propel it towards the aim point, and verse verse. Consequently, there is no one true optimal angle of impact as it is a true function of height, speed and serve length.

Conclusion

5.1 Evaluation

In conclusion, the optimal impact angle for a tennis serve differs significantly by the factors of speed and height. This optimum angle is effectively a function of a player's highest attainable jump height and maximum power output, therefore a taller player will have a natural advantage on the optimum tennis serve. It was truly fascinating to determine that when a player serves at a speed below 30 m/s to the aim point, the impact angle is positive, whereas speeds exceeding 30m/s lead to a negative impact angle. In addition, I also found out how the optimal angles differ as speed increases by plotting differing values and questioning the domain and range of the resulting graph, leading to my conclusion that 12.68 m/s is the lowest serve speed to reach the aim point at a serving height of 2m. Armed with this knowledge, I will be able to know the optimum angle I should serve the ball with as my serve speed improves with training. I am genuinely excited to apply this knowledge to my tennis progression as well as coaching the progression of others.

5.2 Limitations - Sources of error

1. The coordinates of (x_{serve}, y_{server}) were determined through an informed estimation drawing from player images. Players typically throw the ball in front of the baseline before making their leap into the court. However, the extent of this leap can significantly deviate due to factors such as a player's fatigue, skill level and whether it is the player's first or second serve. To enhance the accuracy of this estimation, I could gather data from professional and amateur athletes and separately model the locations of their serve coordinates. By doing so, I can ascertain the average (x_{serve}, y_{server}) values across various player categories and track how these coordinates evolve throughout a match, providing a comprehensive understanding of the deviations of the coordinates.

2. In Section 3.1 'Candidate data', I gauged the speed of my serves using a mobile application. However, it became apparent that this approach yielded certain discrepancies in the results as the app relied on my phone's camera and tracked the serve's trajectory of my shot from a single angle, leading to somewhat anomalous results. With funding to support this investigation, I would have gone to a tennis centre equipped with radar gun technology akin to the ones used by professional match officials. This would have ensured my candidate results matched the same high level of accuracy and precision as those obtained for professional players' serve speeds.

3. All my calculations have been made with the assumption of negligible air resistance. However, this choice stems from the inherent complexity of the air resistance equation, which demands a more extensive investment of time and resources. It is important to acknowledge that in the real world, a tennis serve is indeed subject to air resistance, making my results approximate rather than entirely

precise. Given additional time, I would apply the drag equation - an understanding I acquired beyond my IB curriculum through independent research [6]:

$$F_{drag} = -\frac{pv^2 C_{drag} A}{2}$$

F_{drag} = drag

p = density of air

v = speed of ball relative to the air

C_{drag} = drag coefficient

A = Cross-sectional area

I would then equate the drag equation to the fundamental principle F = ma then apply acceleration as the derivative of velocity $\frac{dv}{dt}$ and solve this resulting differential equation:

$$\frac{dv}{dt} = -\frac{pv^2 C_{drag} A}{2m}$$

$$\int \frac{1}{v^2} dv = -\frac{pv^2 C_{drag} A}{2m} \int dt$$

Subsequently, I'd perform another integration to derive the distance, enabling me to plot and conduct a comparative analysis of the obtained results. Through this, I will be able to derive more accurate and refined insights into the optimal impact angles in the presence of air resistance.

5.3 Reflection

On the one hand, given additional time, I would investigate how the serve's impact angle varies with the other two optimum serve locations as discussed in section 1.3. This extended investigation would yield valuable insights, particularly as serve length (l_{serve}) would significantly increase to reach the body and wide serve aim locations. On the other hand, I have thoroughly enjoyed using Python to visually present and validate the mathematical foundations behind my conjecture. Consequently, this has allowed me to gain confidence in my programming abilities and appreciate the role of technology in ensuring precision when inputting data into mathematical functions. Furthermore, this exploration has significantly enhanced my understanding of projectile motion, which I can readily apply to my studies in physics and mechanics. Overall, I have thoroughly enjoyed this investigation and its intriguing and relevant results.

[6] Benson, T. (2021, May 13). The Drag Equation. Retrieved from www.grc.nasa.gov website: https://www.grc.nasa.gov/WWW/K-12/rocket/drageq.html

Bibliography

- Benson, T. (2021, May 13). The Drag Equation. Retrieved from www.grc.nasa.gov website: https://www.grc.nasa.gov/WWW/K-12/rocket/drageq.html
- Dominik (n.d.). *The Perfect Tennis Serve in 8 Simple Steps*. [online] Tennis Uni. Available at: https://tennis-uni.com/en/tennis-serve/.
- Fawcette, J. (n.d.). Tennis photography - serve impact height. [online] Available at: jfawcette.myportfolio.com
- Harrod Sport (2020). *Tennis Court Dimensions & Size | Harrod Sport*. [online] www.harrodsport.com. Available at: https://www.harrodsport.com/advice-and-guides/tennis-court-dimensions.
- jfawcette.myportfolio.com. (n.d.). *Tennis Photography - Technique*. [online] Available at: https://jfawcette.myportfolio.com/technique.
- Jonathan (2023). *The Fastest Tennis Serves Ever - peRFect Tennis*. [online] Available at: https://www.perfect-tennis.com/fastest-tennis-serve/.

All graphs and calculations made by candidate:
https://colab.research.google.com/drive/1BMpgTPQqTIldm3KP5Ga71hXh7puhb4x-?usp=sharing