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MILANO 1863

Portfolio Management

Computational Finance - A.A. 2024-2025

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Introduction

The aim of this work is to investigate Portfolio Management techniques leveraging different allocation strategies and evaluate their performances. The investment universe analyzed comprises 11 sector indices and 5 factor indices derived from the S&P 500. Sector indices can be classified into three main categories based on their response to economic cycles, and factor indices, as introduced by *Fama and French (1992)*, capture systematic drivers of the market:

- **Cyclical sectors** - Consumer Discretionary, Financials, Materials, Real Estate, and Industrials
- **Defensive sectors** - Consumer Staples, Utilities, and Health Care
- **Sensible sectors** - Energy, Information Technology, and Communication Services
- **Factors** - Momentum, Value, Growth, Quality and Low Volatility

The dataset, stored in the file *prices.xlsx*, contains the prices of the eleven sectors and five factors from January 2021 to October 2024. Moreover, the capitalizations of the assets are given in the file *capitalizations.xlsx*. In the first part of the project, the analyses are conducted on the prices from 2023, while in the final part the performances of the built portfolios are tested on the 2024 period. In the computations, the risk-free interest rate is set to 0, as this choice allows us to focus on the return and risk dynamics of the portfolio without the need to account for the risk-free component. As an alternative, the risk-free interest rate could have been chosen as the 10-year US Treasury bond rate. The results presented in the report derive from the application of Modern Portfolio Theory, a practical method to select investments in order to maximize the returns within an acceptable level of risk.

Data Analysis & Normality Test Results

As an initial analysis, we focus on the S&P 500, widely considered as the most reliable benchmark for large-cap US equities and the basis for numerous investment products. The index comprises the top 500 US companies by capitalization and accounts for approximately 80% of the total market capitalization of the US.

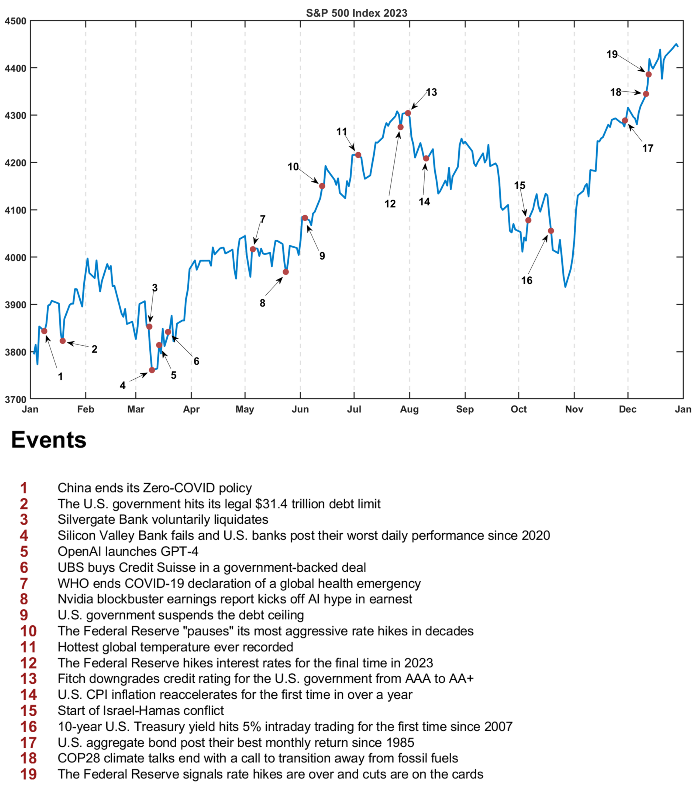

Figure 1: S&P 500 through 2023 from *J.P. Morgan*

Over time, the performance of the S&P 500 has been shaped by several political and economic events. Since its inception, the index has experienced various periods of fluctuations. Taking a glance at the last twenty-five years, the dot-com bubble in the early 2000's, the 2007 crisis, the pandemic in 2020 and lastly the war in Ukraine affected the markets significantly. Nevertheless, the S&P 500 has consistently managed to recover, proving the resilience of the US stock market.

In the graph depicted in Figure 1, we focus our sight on the 2023 S&P 500 performance. In this time period, a few unexpected events that occurred had an important impact in the markets such as the escalation of Israel-Hamas conflict, the Silicon Valley Bank failure and the contiguous US banks worst daily performance since 2020. From a macroeconomic perspective, the quality of US credit has declined, leading to a downgrade of government bond ratings and making the associated interest rates more attractive to investors. Additionally, the suspension of the debt ceiling, coupled with the inflation that impacted the global economy in previous years, has further contributed to raising the interest rates offered on new bond issuances.

As a result, part of the liquidity of the investors has moved to the credit market; indeed, we can see several step-downs of the index corresponding to credit announcements. Nevertheless, the S&P 500 still positively performed during the year, mainly driven by the technology sector, with the significant impact of AI innovation, that brought massive returns on companies such as Nvidia.

Delving into the dataset, the first step performed is to compute the compounded returns exploiting the following relation:

$$r_{t,\tau} = log\left(\frac{p_t}{p_{t-\tau}}\right).$$

This enables to exploit the properties of log-returns, such as time invariance in the equity market, additivity, and the possibility of describing them with a normal distribution. Indeed, Modern Portfolio Theory operates under a few assumptions: perfect market, normally distributed returns and risk-averse investors. In order to assess the normality of the returns, several tests are conducted. The QQ-plot displays the quantiles of the sample data (the log-returns) versus the theoretical quantile values from a normal distribution.

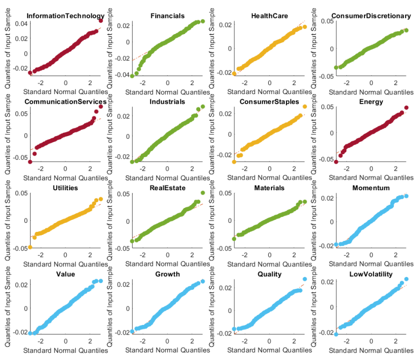

Figure 2: QQ plots of the log-returns, clustered by sectors and factors

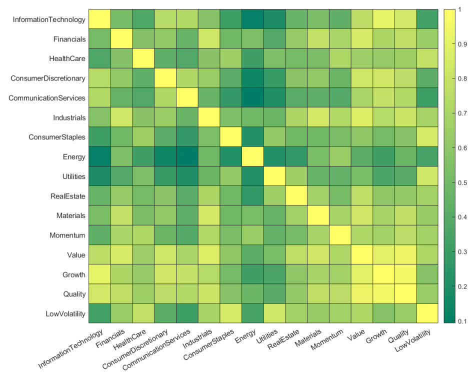

Figure 3: Correlation plots of the log-returns

From these graphs it can be noticed that in general the returns follow the theoretical quantiles, even if the presence of heavy tails can be observed for some assets. To deeply analyze the Normality, we attempt the Shapiro-Wilk test via *swtest* function, where the null hypothesis H_0, assumes that the log-returns are normally distributed.

Name	**SW**	**KS**
Information Technology	H_0	H_0
Financials	H_1	H_0
Health Care	H_0	H_0
Consumer Discretionary	H_0	H_0
Communication Services	H_1	H_0
Industrials	H_0	H_0
Consumer Staples	H_1	H_0
Energy	H_1	H_0

Name	**SW**	**KS**
Utilities	H_1	H_0
Real Estate	H_1	H_0
Materials	H_0	H_0
Momentum	H_0	H_0
Value	H_0	H_0
Growth	H_0	H_0
Quality	H_0	H_0
Low Volatility	H_0	H_0

Table 1: Shapiro-Wilk and Kolmogorov-Smirnov Normality tests results for various sectors and factors. H_0: Accept normality VS H_1: Reject normality.

As shown in Table 1, six out of the sixteen indices considered reject the normality assumption under the Shapiro-Wilk test. This is consistent with the previous QQ plots (see Figure 2) and it follows the paradigm of financial time series where often the tails of the distributions appear to be heavier than those of the normal distribution, influenced by the presence of extreme events.
Analyzing the 2023 data, some extreme events stand out. The most significant negative returns coincide with the Silicon Valley Bank bankruptcy around March 10, which led to substantial losses across the US market, as previously discussed. To ensure accordance with the normality assumption, we also perform the Kolmogorov-Smirnov test on the log-returns, which is a less powerful test to check data's Gaussianity, as *Razali and Wah (2011)* suggest. Notably, the test results indicate that the normality assumption is not rejected for any of the assets.

As notable in Figure 3, the correlations between assets are all positive. This result might be quite surprising and might require further analysis; however, since the assets all belong to the S&P 500 index, they tend to move together. This is even more evident in stress periods, such as 2023, where inflation was significantly high as reported by Standard's & Poor's.

1 Efficient frontier under standard constraints

It was firstly required to compute the efficient frontier under the standard constraints and to build the Minimum Variance Portfolio, named *Portfolio A*, and the Maximum Sharpe Ratio Portfolio, named *Portfolio B*, of the frontier.
Given N risky assets, the Portfolio Frontier is defined as the set of all minimum variance portfolios between those with the same expected return. By computing the Minimum Variance Portfolio, we are selecting the portfolio on the frontier with the lowest variance. The Sharpe Ratio is a performance measure that describes how much excess return you receive for the extra volatility you endure by holding a riskier asset.
The standard constraints require that the sum of all weights must be equal to one and no short positions are allowed (i.e. $w_i \geq 0 \; \forall \; i = 1, ..., N$).

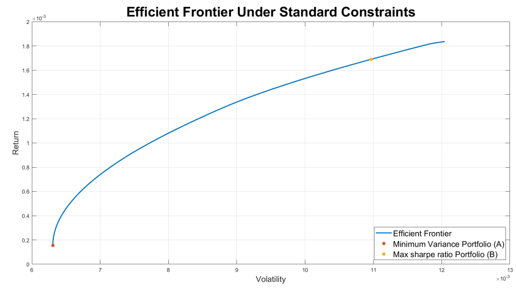

Figure 4: Portfolio Efficient Frontier under standard constraints.

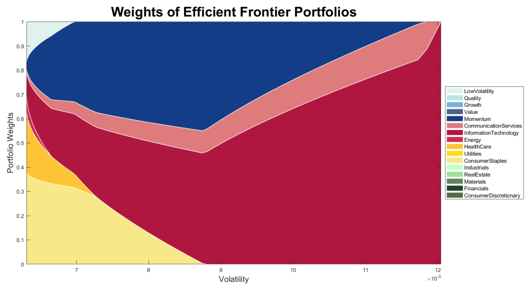

Figure 5: Weights of the Portfolio Efficient Frontier under standard constraints for different volatility levels.

The obtained frontier and the portfolios are plotted in Figure 4, while Table 3 reports the weights of the two Portfolios.

In *Portfolio A* we can observe that we mainly invest in the assets which have the lowest volatility (i.e. Health Care, Consumer Staples, and Low Volatility), which is not surprising since we are trying to minimize the riskiness of our Portfolio. On the other hand, it is more unexpected that we have a positive weight also for Communication Services and Energy, which are the two assets with the highest volatility: this can be justified taking into account the correlation, since these assets have a low level of correlation with the other assets in the portfolio. From the theory, we know that having assets whose correlation tends to zero introduces diversification benefits, reducing the overall risk of the portfolio.

In *Portfolio B*, we only invest in three assets: Information Technology, Communication Services and Momentum. This portfolio guarantees a higher return with respect to the first one, but this comes at the price of having higher volatility. Two out of the three selected assets are those with the highest mean returns during the year under consideration. This selection aligns with the core principle of the Sharpe Ratio: the portfolio's increased exposure to risk is effectively compensated by a corresponding excess return.

2 Efficient frontier under specific constraints

This section examines the issue of the efficient frontier under additional constraints. Specifically, it is required to compute the efficient frontier and to design the Minimum Variance Portfolio, named *Portfolio C*, and the Maximum Sharpe Ratio Portfolio, named *Portfolio D*, under the following constraints:

- Standard constraints
- The total exposure to sensible sectors has to be greater than 10% and the total exposure on cyclical sectors has to be lower than 30%
- The weights of the sector Consumer Staples and the factor Low Volatility have to be set equal to 0
- The maximum exposure on sectors has to be lower than 80%

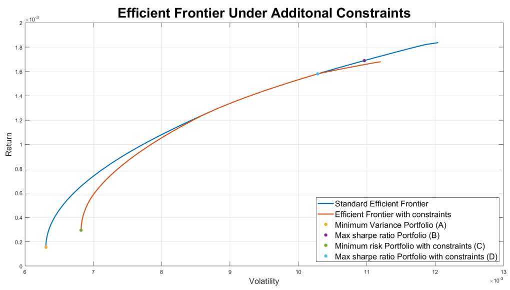

Figure 6: Portfolio Efficient Frontier under standard constraints.

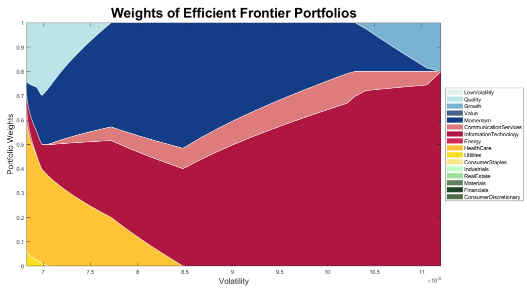

Figure 7: Weights of the Portfolio Efficient Frontier under additional constraints for different volatility levels.

By plotting the Frontier against the one obtained in the previous case (see Figure 6), we observe that we are restricting the set of the attainable portfolios.
One may question the rationale behind adding constraints that restrict the domain of the weights, leading to a sub-optimal portfolio. Diversification and risk management principles may limit the weights on a single asset or a group of assets, but those are not the only reasons to add constraints. For instance, there might be also regulatory requirements to satisfy.

Table 3 reports the weights of *Portfolios C* and *D*, too. For *Portfolio C* we can observe that about 70% of the portfolio is invested in the same assets as the Minimum Variance Portfolio under standard constraints, even if the distribution of the weights is slightly different. *Portfolio D*, instead, picks exactly the same assets of the previous Maximum Sharpe Ratio Portfolio.

3 Robust frontiers

Modern Portfolio Theory (MPT) relies on several key assumptions - such as normally distributed returns - that often break down in practice due to market uncertainties and sudden shocks. To address this, it can be useful to adopt a robust allocation strategy.

In this section, we apply a resampling method to construct efficient frontiers, focusing on two sets of portfolios derived from earlier analyses: the minimum variance portfolios (labeled *Portfolio E* and *Portfolio F*) and the maximum Sharpe ratio portfolios (labeled *Portfolio G* and *Portfolio H*). The central idea behind the resampling approach is to repeat the optimization procedure multiple times to mitigate the impact of estimation errors in the input parameters.

The procedure begins by estimating the initial expected returns, denoted by e_0, and the covariance matrix, denoted by V_0. Using these estimates, we generate M simulated datasets, each containing T observations, drawn from a multivariate normal distribution. For each simulated dataset, we compute an efficient frontier. The final "resampled" frontier is then obtained by averaging the frontiers from all simulations.
By simulating returns in this manner, we relax the assumption that mean returns and covariance matrices are fixed and known exactly. Instead, we acknowledge that they are subject to estimation errors and potential future fluctuations. Thus, the resampling method captures a more realistic range of possible future outcomes, reflecting both uncertainty and variability in the historical data. It helps reduce sensitivity to outliers and rare extreme events - circumstances often overlooked by straightforward historical estimations. In turn, this decreases the risk of overfitting and results in a more robust set of portfolio allocations.

The portfolios we obtained are resumed in Table 3. These are Portfolios that can be defined more conservative as they aim to reduce the overall volatility, and we can observe how we tend to invest in more assets by applying the resampling method.
By plotting the Portfolios and the Frontiers of the previous points as in Figure 8, we can observe that the robust Portfolios lie outside the frontiers. This result might depend on the fact that by simulating some assets overperform the historical ones.

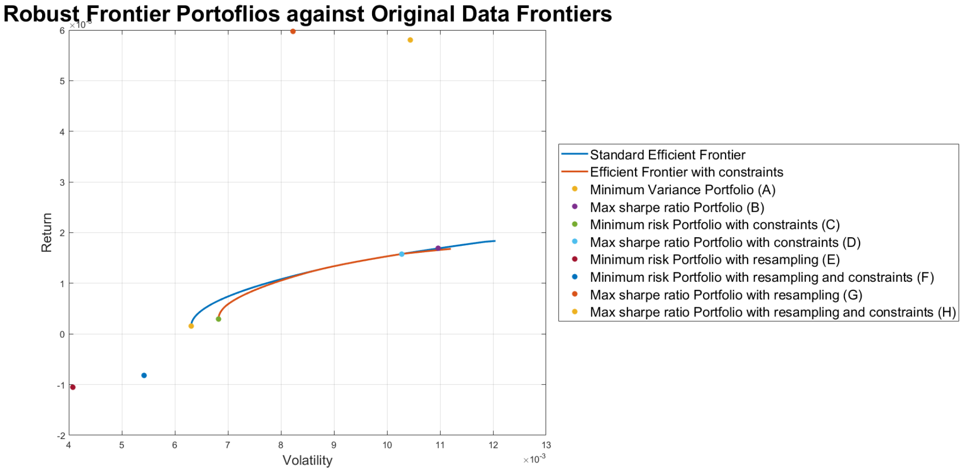

Figure 8: Portfolio Efficient Frontier computed with resampling method

In *Portfolio E*, there is a significant allocation to Consumer Staples and Health Care, which are considered defensive sectors. There is a low or non-existent allocation to cyclical sector (like Industrials and Real Estate assets), which aligns with the portfolio's overall strategy. By introducing additional constraints, *Portfolio F* excludes Consumer Staples entirely while increasing allocations to other assets within the defensive sector.
In contrast, if the primary objective is to maximize the Sharpe ratio with standard constraints (*Portfolio G*), the majority of the portfolio is invested in Information Technology and Communication Services, which belongs to the sensitive sector. This results in a higher expected return, but also greater volatility compared to *Portfolio E*. However, the Sharpe ratio is higher, justifying the increased exposure to riskier assets based on the expected returns. When additional constraints are applied while maintaining the same objective function, *Portfolio H* is created. Although the allocation weights remain similar to those in *Portfolio G*, the constraints reduce the feasible set on the Efficient Frontier, leading to lower levels of both expected return and volatility.

4 Black-Litterman

In this point, we are asked to tackle the optimization problem using the Black-Litterman approach. This method is considered to be more robust because it enables the investor to build a customized portfolio based on the analysis of market data. To find the weights, we start by considering the linear returns in order to combine them with the given market views.
This approach integrates market data with investor views in a Bayesian framework. It starts with a prior distribution derived from the market equilibrium, typically represented by a portfolio weighted according to market capitalizations of the given indices. Investor views, which may deviate from the market's expectations, are then incorporated as additional inputs. The result is an updated posterior distribution that balances the market equilibrium with the investor's subjective views, producing a set of optimized portfolio weights. Given the choices of two parameters we can build our prior distribution as follows

$$\mu_{prior} = \lambda \cdot \Sigma w_{mkt},\ \Sigma_{prior} = \tau \cdot \Sigma$$

with the typical choice of $\lambda = 1.2, \tau = \frac{1}{N_{samples}}$.
Given our idea of the market, the distribution of the views is assumed normal and serves as the likelihood function in the Bayes' theorem. It is built starting from the investor's views, in a linear form

$$Q = P\mu + \epsilon,\ \epsilon \sim N(0, \Omega)$$

where P is a rectangular matrix indicating the assets involved in each view while $\Omega = \tau P \Sigma P'$ represents the covariance matrix filtered on the assets and the factors included in our beliefs.

In this project, we consider two perspectives to the current market situation. First, we believe that the growing importance of the IT sector will lead it to outperform the Financial sector by 2%. Second, we the foresee a bullish market scenario where Momentum will grow 1% more than the Low Volatility factor.
In the following Table 2, we compare the previous beliefs regarding expected returns with the new expectations derived by the posterior distributions. We can notice straight away that the assets directly influenced by our

views now reflect these changes, showing a significant adjustment in their returns. However, also assets not directly targeted by our vision exhibit shifted returns, illustrating how our visions influence the entire Portfolio allocation.
If we take a look at the correlation heat map of the returns of the different assets, we observe that assets and factors more closely correlated with the Financial sector and the Low Volatility factor tend to decrease in expected returns. Conversely, those less correlated with the these segments tend to follow the over performers of our views.

Name	Prior (%)	BL (%)
Information Technology	2.38	3.65
Financials	1.96	1.65
Health Care	1.18	1.12
Consumer Discretionary	2.45	3.07
Communication Services	2.38	3.29
Industrials	1.84	1.83
Consumer Staples	1.00	0.73
Energy	1.42	1.33

Name	Prior (%)	BL (%)
Utilities	1.28	0.70
Real Estate	2.07	1.85
Materials	1.97	1.98
Momentum	1.49	1.75
Value	1.92	2.08
Growth	1.93	2.55
Quality	1.81	2.24
Low Volatility	1.13	0.76

Table 2: Expected Returns (%) Prior beliefs and after applying Black-Litterman.

Analyzing the weights shown in Table 3, *Portfolio I* appears minimally influenced by market views, maintaining a composition very similar to *Portfolio A*. In contrast, the impact of market views is more pronounced in *Portfolio L*, where the Momentum factor gains approximately 20% more weight compared to *Portfolio B*.

5 Diversification

In the previous points, assumptions on the expected returns are needed to obtain results. However, incorrect assumptions may affect the efficiency of the portfolio allocation. This limitation made investment strategies that ignore assumptions on the expected returns fundamental in portfolio management applications.
The creation of the Maximum Diversified portfolio, *Portfolio M*, and the Maximum Entropy portfolio in asset allocation, *Portfolio N*, relies exclusively on the risk model, avoiding reliance on expected returns. As a general statement, we can say that a portfolio is diversified when no single asset or group of assets disproportionately influences the portfolio's overall performance.
Diversification in a portfolio can be measured via several diversification metrics, such as the Gini coefficient or the Herfindal index. In this project, the Maximum Diversified Portfolio is computed maximizing the logarithm of the Diversification Ratio as a function of the weights as follows:

$$w^* = \underset{w}{max}\left[log\left(\frac{\sum_{i=1}^{N} w_i\sigma_i}{\sigma_P}\right)\right] ,$$

where $\sigma_P = \sqrt{w' \cdot \Sigma \cdot w}$ and σ_i is the volatility of the i^{th} asset.
An alternative method to assess diversification in a portfolio involves evaluating its entropy via entropy metrics. This procedure enables analysts to build the Maximum Entropy Portfolio. This project focuses on maximizing the entropy of asset volatility, indeed the exploited metric is the following:

$$w^* = \underset{w}{max}\left[-\sum_{i=1}^{N} x_i \cdot log(x_i)\right], \text{ where } x_i = \frac{w_i^2 \cdot \sigma_i^2}{\sum_{i=1}^{N} w_i^2 \cdot \sigma_i^2}$$

Building *Portfolio M* and *Portfolio N*, additional constraints need to be considered:

- Standard Constraints
- The total exposure on cyclicals has to be greater than 20%
- The sum of the absolute value of the difference between the weights in the capitalization weighted portfolio and the optimal weights has to be greater than 20%.

As shown in Table 3, the Most Diversified Portfolio weights are not uniformly distributed as one could expect. However, this can be justified considering that this diversification metric is inversely proportional to the volatility of the portfolio σ_P, thus we maximize the diversification ratio by minimizing σ_p. Conversely, *Portfolio N* is almost equally weighted as the Equally Weighted Portfolio maximizes the entropy under standard constraints.

6 Principal Component Analysis

Principal Component Analysis (PCA) is a statistical technique from the family of Statistical Factor Models, used to reduce the influence of noisy factors that can affect effective portfolio optimization. In this approach, the factors are represented by principal components, which capture the largest portion of the variance in asset returns. A target threshold for cumulative variance is typically set to determine the number of components to use. The initial step of the procedure is to standardize the data before performing PCA.
Cumulative variance is then calculated by normalizing the eigenvalues of the covariance matrix, enabling the selection of components based on the percentage of variance they explain. Once the number of principal components is determined, PCA extracts these components and projects the returns into a lower-dimensional space.

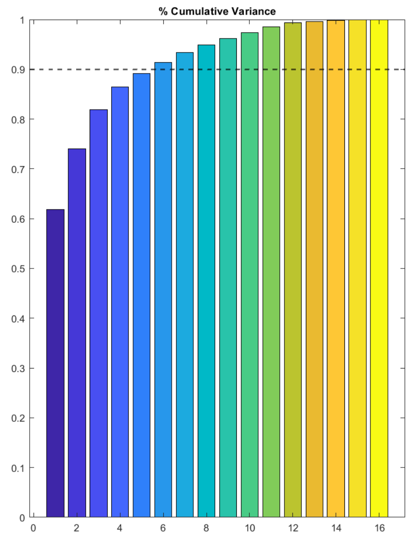

Figure 9: Percentage of cumulative variance explained by the principal components.

We identify the target number of components to consider by looking at the graph in Figure 9 where we can notice the first breakthrough over the 90% threshold at 6 components. Afterwards, we perform the PCA over the number of components selected and rebuild the returns. We can express the asset returns as

$$r_a = \mu_a + F \cdot r_f + \epsilon_a,$$

where μ_a represents the mean return, r_f the factor returns and F the factor loading; ϵ_a is the unexplained portion of the data. With this formulation we can give a new shape to the variance of the portfolio:

$$\sigma_p^2 = Var[r_a^T w_a] = w_f^T \Sigma_f w_f + w_a^T D w_a$$

with $\Sigma_f = cov(r_f),\ D = cov(\epsilon_f)$.

With these writings we can formulate the following problem under standard constraints and by setting an additional constraint on the level of risk of the portfolio.

$$\max\ \mu_a^T w_a \quad \text{s.t.} \quad \begin{cases} w_a^T \mathbf{1} = 1 \\ 0 \leq w_i \leq 1,\ \forall i = 1, \dots, n \\ w_f^T \Sigma_f w_f + w_a^T D w_a \leq \sigma_*^2, \quad \sigma_* = 0.7 \end{cases}$$

The resulting *Portfolio P* is presented in Table 3. It shows investments in five assets, all belonging to the Defensive or Sensible Sectors; however, it is challenging to correlate these with the selected principal components. This highlights one of the primary limitations of PCA: the difficulty in interpreting the principal components and identifying the underlying sources of risk.
An interesting observation is that the portfolio reaches the maximum volatility threshold imposed by the constraints. This outcome is not surprising, as it aligns with the principles of the Efficient Frontier: since the objective function's goal is to maximize returns, the portfolio must also attain the highest possible level of volatility within the given constraints.

7 Value at Risk in portfolio optimization

Assessing the risk of a portfolio is not an easy task and we have already seen several indicators which capture different shades of the risk a portfolio is carrying. In this point, we try to modify a classical metric of performance for a portfolio adding a different measure of risk, the VaR. The VaR represents a quantile of the losses' distribution; it can be a good measure of risk since it permits to manage tail risk, which may be crucial during periods of market stress, and to build customized portfolios based on specific risk thresholds.
The optimization problem aims at maximizing the VaR-modified Sharpe Ratio, and can be written as follows:

$$\max\ \frac{w^T \mu - r_f}{VaR^{0.05}(w)} \quad \text{s.t.} \quad \begin{cases} w^T \mathbf{1} = 1 \\ 0 \leq w_i \leq 1,\ \forall i = 1, \dots, n \end{cases}$$

The VaR can be calculated in different ways, for this exercise we follow the assumption of Normality of the returns as previously justified. We have the following formula:

$$VaR^{0.05}(w) = -\mu' w + \sqrt{w'\Sigma w} \cdot \Phi^{-1}(0.95),$$

where $\Phi(\cdot)$ is the cdf of a Standard Gaussian and we put the minus sign in front of μ to have the mean of the losses. Moreover, we select a value of $p = 0.05$, since this is one of the more recurrent values for p in the literature and in the industry.

By observing the weights of *Portfolio Q* in Table 3, we can notice that the obtained weights are the same of *Portfolio B* up to $1bps$; *Portfolio B* is the one that maximizes the Sharpe Ratio under standard constraints. This result is not surprising since the VaR-modified Sharpe Ratio substitutes the Volatility of the "classic" Sharpe Ratio formula with the VaR expression: this change simply rescales the variance by a positive coefficient and subtracts a term which is close to zero ($o(10^{-3})$).

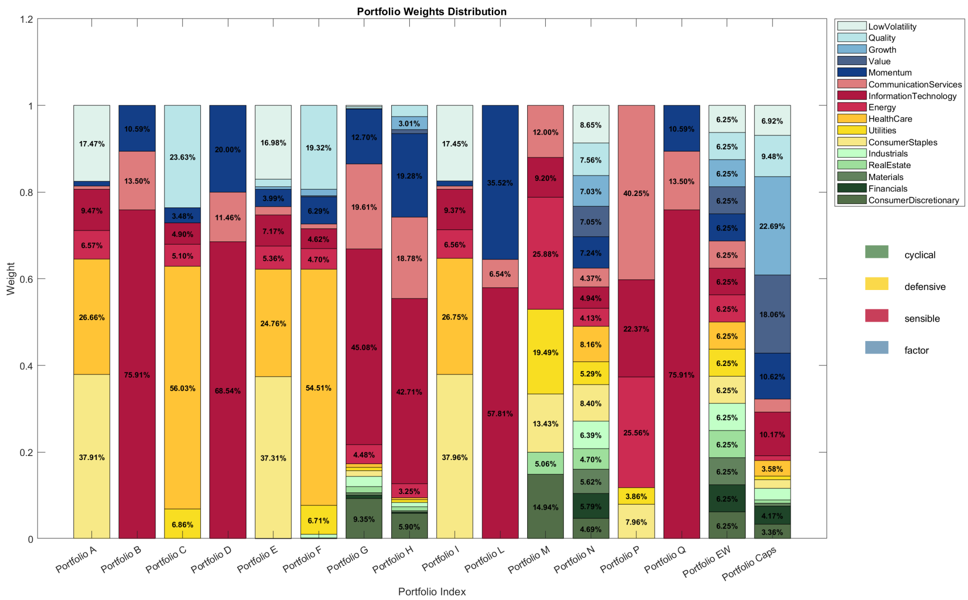

Figure 10: Portfolios weights distributions.

8 Discussion on the portfolios

To provide a comprehensive understanding of the portfolio performance, we first examine their allocation strategies. In Figure 10, the portfolios' weights distributions are illustrated. The plot clearly shows that when constructing portfolios aimed at reducing volatility (i.e. *Portfolios A*, *C*, *E*, *F*, and *I*), investments tend to favor the defensive sector, particularly Consumer Staples and Health Care assets. Conversely, when the goal is to maximize metrics that focus on a risk-reward trade-off, such as the Sharpe Ratio, the sensible sector is prioritized. Notably, Information Technology, being one of the assets with the highest Sharpe Ratio, dominates the composition of these portfolios.

Having built all the portfolios, we now focus on analyzing and evaluating their performances during the 2023 year, the same time interval used to retrieve the weights. This analysis is conducted relying on quantitative metrics estimating the reward, the risk, the trade-off among the two, and the diversification.

To begin with, we set as a benchmark the *Equally Weighted (EW) Portfolio*, which represents a naive description of the whole market. However, it presents some limits; indeed, by investing the same percentage in each asset, it does not take into account the capitalization of the different sectors and factors, which is the building block of the S&P 500 index. We can see that that this naive portfolio distinctly segments the portfolios into two groups: the first group demonstrates superior performance relative to the other portfolios, including the *EW Portfolio*, based on reward-oriented metrics; while the second group prioritizes achieving and maintaining a lower level of risk. Moreover, the *EW Portfolio* performs well in terms of diversification, especially when the diversification is measured via the Entropy measure in asset volatility. Let's now discuss the portfolios one at the time with the help of the Performance Metrics reported in Table 3 and of the graph displayed in Figure 11.

Portfolio A represents the Minimum Variance Portfolio on the Efficient Frontier under standard constraints. It achieves the lowest Annual Volatility among all portfolios, though its Annual Return is lower than that of the *EW Portfolio* and most other portfolios. Additionally, it underperforms in terms of the Sharpe Ratio, Calmar Ratio, and diversification metrics. In summary, *Portfolio A* excels at minimizing risk but does so at the expense of other performance measures.

Portfolio B was designed to maximize the Sharpe Ratio, and it achieves this goal, outperforming all other portfolios in this metric. It is also the best portfolio in terms of Annual Return and it performs strongly in terms of Calmar Ratio. Although it exhibits relatively high Annual Volatility and Maximum Drawdown, the metrics evaluating the trade-off between risk and return indicate that its exposure to risky assets is well-compensated by substantial returns. However, *Portfolio B* falls short in terms of diversification, with results below those of the *EW Portfolio*.

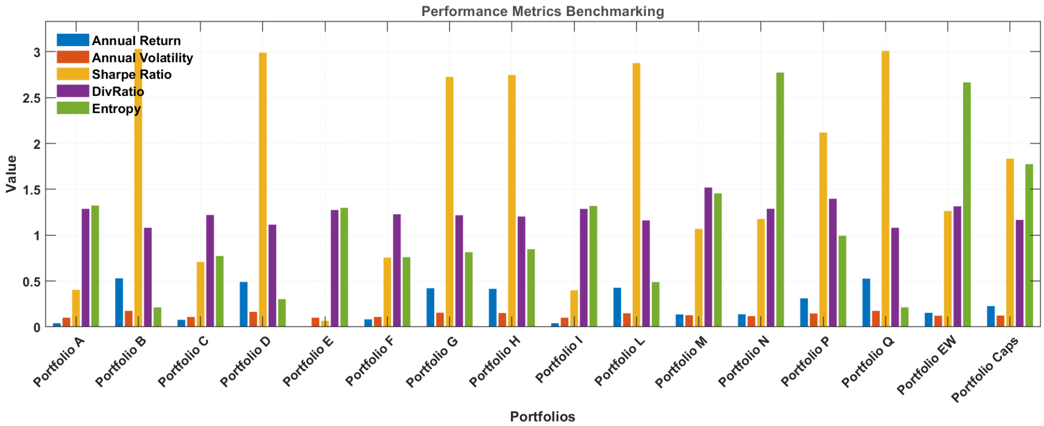

Figure 11: Comparison of Portfolios performance metrics in 2023.

The observations regarding the Minimum Variance Portfolio with additional constraints, *Portfolio C*, are similar to those for *Portfolio A*. While the added constraints prevent us from minimizing risk to the same extent as in the standard-constrained case, *Portfolio C* still achieves lower volatility than most other portfolios. Notably, its exposure to risk, as measured by both Annual Volatility and Maximum Drawdown, remains below that of the *EW Portfolio*.

The portfolio built to maximize the Sharpe Ratio under additional constraints, *Portfolio D*, behaves similarly to the less constrained *Portfolio B*. Indeed, *Portfolio D* performs well across key metrics such as Annual Return, Annual Sharpe Ratio, and Calmar Ratio (the latter being the highest among all portfolios analyzed). However, similarly to *Portfolio B*, *Portfolio D* underperforms in risk and diversification metrics in comparison to most of other portfolios, including the *EW Portfolio*. Remarkably, the additional constraints narrow the efficient frontier, resulting in lower risk levels as reflected in the Annual Volatility and Maximum Drawdown metrics when compared to *Portfolio B*, together with a slightly better diversification level.

The portfolios obtained from the robust frontiers (*Portfolio E*, *Portfolio F*, *Portfolio G*, and *Portfolio H*) display moderate performance, broadly comparable to their previously analyzed counterparts. For the minimum variance strategy, the observations closely align with those derived from the standard approach, indicating no significant performance improvements. In contrast, the maximum Sharpe ratio portfolios derived from the robust frontiers exhibit a notable enhancement in the Entropy metric. These results align with the underlying philosophy of robust frontiers, which seek to mitigate estimation errors and model uncertainties by prioritizing diversification and stability.

As previously highlighted, Black Litterman's Minimum Variance Portfolio, *Portfolio I*, presents weights that are very similar to those of *Portofolio A*; as a consequence also the considerations on the performance metrics remain the same.

In contrast, *Portfolio L*, the Maximum Sharpe Ratio portfolio influenced by market views, shows some differences compared to the similar *Portfolio B*. *Portfolio L* exhibits slightly lower Annual Return and weaker risk-return trade-off metrics but achieves reduced risk through lower Annual Volatility and Maximum Drawdown. This aligns with the theoretical framework of the Black-Litterman model, which aims to produce more stable estimates of expected returns. In comparison with the *EW Portfolio*, *Portfolio L* performs better in terms of return and risk-return trade-off metrics, while it underperforms on risk and diversification metrics.

Focusing on the diversification metrics we analyze *Portfolio M*, constructed under additional constraints, prioritizes the maximization of the Diversification Ratio. Compared to the *EW Portfolio*, *Portfolio M* generally under-performs both in terms of pure reward metrics (Annual Return) and risk-adjusted measures (Sharpe, Calmar Ratios). However, *Portfolio M* surpasses the *EW Portfolio* in the Diversification Ratio, reflecting its design objective and the effects of the constraints that encourage broader risk distribution. Although *Portfolio M* did not excel in return-oriented metrics and showed a relatively low Entropy score, it fulfilled its primary objective of enhancing certain aspects of diversification under the given constraints.
Similar performance patterns emerge for *Portfolio N* in terms of return and risk metrics. However, a notable distinction arises in the diversification measures. While *Portfolio N* is designed to maximize Entropy in asset volatility contributions, it remains subject to the additional constraints and ultimately succeed in achieving the highest Entropy level. Thus, *Portfolio N* offers a more balanced risk distribution than most constrained portfolios, but does not reach the benchmark performance represented by the *EW Portfolio* in terms of return on investment.

Portfolio P, generated via Principal Component Analysis (PCA) and capped by a volatility threshold, ends up having relatively good scores. It surpasses the *EW Portfolio* in returns or risk-adjusted metrics, but it does not distinguish itself through exceptional diversification scores. *Portfolio P* presents a measured compromise: it remains well within its targeted risk limit and avoids excessive volatility, yet lacks standout performance characteristics. This outcome highlights the difficulty of translating sophisticated factor-based approaches (like PCA) directly into superior returns or diversification gains.
No additional analysis is required for *Portfolio Q* since, as we have already highlighted, it closely resembles *Portfolio B*; therefore, the same observations and conclusions apply.

Figure 12 displays the Equity Curves for all portfolios, highlighting the reward segmentation achieved by the *EW Portfolio*. The graph reveals that *Portfolios B*, *Q*, and *D* consistently deliver the highest Equity values throughout the observed period, while *Portfolios C* and *I* exhibit the weakest performance in terms of Equity. A notable observation from the Equity curves is that all portfolios tend to move together, reflecting similar responses to market conditions. During periods of market expansion, all portfolios experience growth in Equity value, while negative jumps affect them simultaneously. As expected, portfolios with higher Equity levels also exhibit the largest fluctuations in those periods.

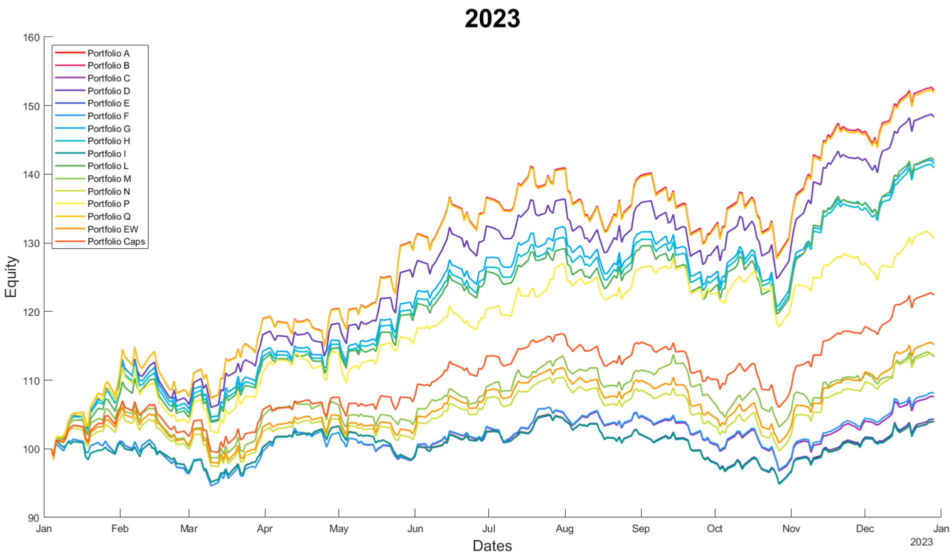

Figure 12: 2023 Equity curves for all the portfolios.

9 Portfolios comparison - 2024 Out of sample performances

To evaluate the robustness of the previously determined portfolio allocations, we now turn our attention to their out-of-sample performance in 2024.
This year has continued the positive market momentum established in 2023, with the S&P 500 surging by approximately 26% as of October 25, 2024. This substantial rally follows a similarly strong gain of around 24% the previous year, resulting in one of the most bullish two-year periods in recent history. A combination of factors including the widespread diffusion of artificial intelligence (AI), supportive monetary policy via interest rate cuts, and a resilient economic landscape fueled investor confidence. Against this dynamic backdrop, examining how our strategies have performed out of sample offers valuable insights into their stability, adaptability, and potential vulnerabilities under evolving market conditions.
The benchmarks considered for this evaluation are both the *EW Portfolio* and the 2023 performances. By looking at Table 4, it can be appreciated that in 2024 the *EW Portfolio* overperforms the previous year results in all the metrics considered. This observation reinforces the notion that 2024 was indeed an exceptionally favorable period for the S&P 500.

Starting with the minimum variance strategy under standard and additional constraints (*Portfolio A* and *Portfolio C*), the portfolios showed a notable improvement in every considered measure compared to the previous year. Their annual returns increased in line with the favorable market conditions, while volatility remained effectively contained, staying consistent to their underlying objective. As a result, these portfolios achieved a substantially higher Sharpe ratio. This pattern mirrors the strong performance observed in the *EW Portfolio*, which also benefited from the positive market environment. In conclusion, both *Portfolio A* and *Portfolio C* preserved their low-volatility profile while successfully adapting to the evolving market dynamics, ultimately matching the Sharpe ratios seen in portfolios explicitly designed to maximize this metric.

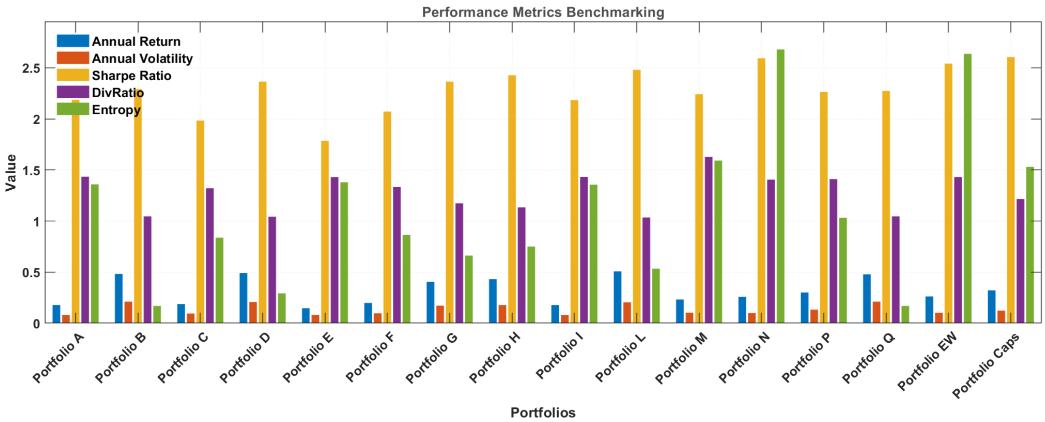

Figure 13: Comparison of Portfolios performance metrics computed on 2024 returns.

Shifting our focus to *Portfolio B* and *Portfolio D*, both designed to maximize the Sharpe ratio, albeit one without and one with additional constraints, we encounter a slightly less robust out-of-sample performance. Although they continue to deliver strong results across the evaluated metrics, these portfolios no longer hold the leading position in terms of risk-adjusted returns. Despite achieving two of the highest annual returns, their increased volatility translates into a sharper reduction in the Sharpe ratio. In fact, under this measure, they are now surpassed even by the *EW Portfolio*.

It is worth noting that under this limited data sample, all portfolios designed to maximize the Sharpe ratio experienced a decline in this metric's out-of-sample performance. This pattern indicates that when trying to optimize returns relative to volatility, the resulting allocations (as seen in *Portfolios B*, *D*, *L*, and *Q*) tend to become more concentrated Figure 10. While this approach may yield impressive results in the training sample, it can lose effectiveness once applied to new market conditions, a dynamic reminiscent of overfitting. A more robust approach, such as the one employed by *Portfolios G* and *H* (designed via resampling method), can partially address this issue by offering greater resilience to changing datasets and improved diversification.
Another practical consideration involves the real-world management of a portfolio built to maximize the Sharpe ratio. Even if these strategies perform exceptionally well on the initial dataset, they may quickly show deterioration in their Sharpe ratios, despite maintaining some of the highest annual returns, due to increased volatility in out-of-sample periods. To sustain the optimal Sharpe ratio, such portfolios would likely require more frequent rebalancing, which in turn drives up operational costs. These added expenses could erode a portion of

the outstanding margins that the portfolios initially displayed and thereby make the portfolios less attractive, bringing their absolute performance closer to that of other strategies.

The robust strategy, as previously noted, demonstrated notable resilience in both the minimum variance and maximum Sharpe ratio scenarios. A similar pattern emerges in the portfolios designed to minimize risk, indeed, despite maintaining relatively low annual volatility, these portfolios still manage a substantial improvement in their Sharpe ratios. Once again, the positive market conditions are leveraged effectively, and these portfolios' enhanced performance aligns with the overall improvement observed in the *EW Portfolio*.

The Black-Litterman portfolios, *Portfolio I* and *Portfolio L*, align with the observed trend: the minimum variance portfolio surpasses the previous year's performance, while the maximum Sharpe portfolio increases its Annual Return and Entropy but it falls short across all other metrics compared to 2023.

Portfolio M and *Portfolio N*, the diversification oriented portfolios, demonstrate a significant improvement in 2024 performances. Across all considered metrics, these portfolios outperform their 2023 results, with one exception: the Entropy measure of *Portfolio N*, which, while slightly lower than in 2023, remains the highest recorded value in this out-of-sample dataset. The superior performance of these portfolios can be attributed to their robust construction principles: by emphasizing diversification, these portfolios effectively mitigate concentration risk, leading to more stable returns across a variety of market conditions.

The portfolio built via the PCA, *Portfolio P*, is consistent to 2023 performances. Metrics such as Annual Volatility, Annual Sharpe Ratio, Diversification Ratio, and Entropy show slight improvements, while the remaining metrics experience a modest decline. This consistency may be attributed to the built-in stability of PCA methodology: by focusing on principal components, this approach captures the dominant sources of variance in the asset returns, effectively reducing noise and aligning the portfolio with the key drivers of market dynamics. This ensures that the portfolio performs reliably even when tested in a different market period.
For *Portfolio Q*, as already highlighted in 2023 analyses, there is no need to add further comments since the asset allocation is almost equal to *Portfolio B*, hence the same performance metrics are attained.

By observing the Equity curves of the different portfolios reported in Figure 14, it is remarkable that the highest performances in terms of Equity are achieved by *Portfolios L*, *D*, *B* and *Q*, which is not surprising since they were the portfolios with greater Annual Return. Although the various portfolios continue to exhibit similar movement across the graph, the volatility in portfolios with higher equity is notably more pronounced compared to the previous year. This is especially evident during the period at the end of July 2024, when the Cboe Volatility Index (VIX)[1] also reached its peak for the entire period under consideration.

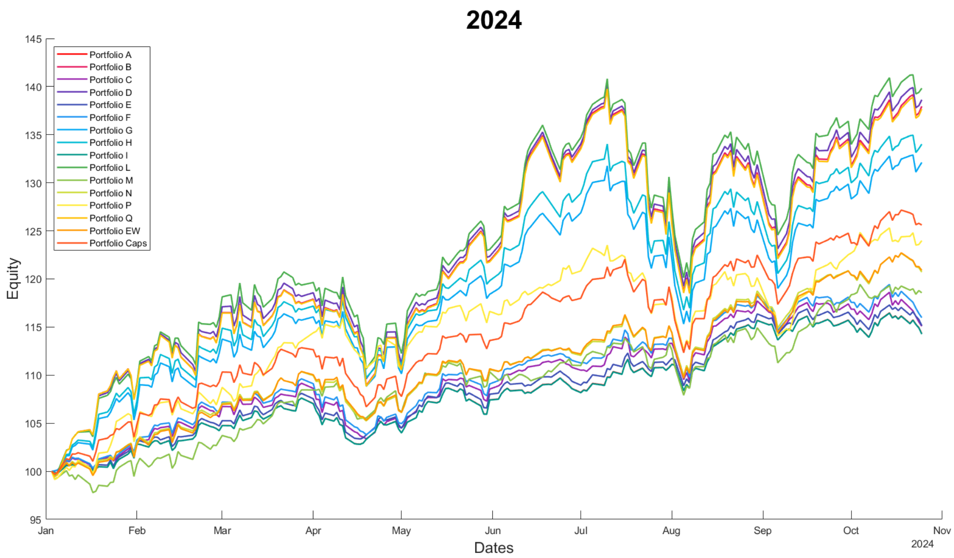

Figure 14: 2024 Equity curves for all the portfolios.

[1]As per S&P Global "the Cboe Volatility Index, better known as VIX, projects the probable range of movement in the U.S. equity markets, above and below their current level, in the immediate future. Specifically, VIX measures the implied volatility of the S&P 500 for the next 30 days".

10 Numeric Results

Asset Name	Portfolios A	B	C	D	E	F	G	H	I	L	M	N	P	Q	EW	Caps
Information Technology	0.0947	0.7591	0.0490	0.6854	0.0717	0.0462	0.4508	0.4271	0.0937	0.5781	0.0920	0.0494	0.2237	0.7591	0.0625	0.1017
Financials	-	-	-	-	-	0.0008	0.0073	0.0026	-	-	-	0.0579	-	-	0.0625	0.0417
Health Care	0.2667	-	0.5603	-	0.2476	0.5451	0.0080	0.0041	0.2675	-	-	0.0816	-	-	0.0625	0.0358
Consumer Discretionary	-	-	-	-	0.0003	0.0005	0.0935	0.0590	-	-	0.1494	0.0469	-	-	0.0625	0.0336
Communication Services	0.0081	0.1350	-	0.1146	0.0195	0.0109	0.1961	0.1878	0.0080	0.0654	0.1200	0.0437	0.4025	0.1350	0.0625	0.0295
Industrials	-	-	-	-	0.0005	0.0083	0.0234	0.0100	-	-	-	0.0639	-	-	0.0625	0.0270
Consumer Staples	0.3791	-	-	-	0.3731	-	0.0136	-	0.3796	-	0.1343	0.0840	0.0796	-	0.0625	0.0204
Energy	0.0657	-	0.0510	-	0.0536	0.0470	0.0448	0.0325	0.0656	0.0013	0.2588	0.0413	0.2556	-	0.0625	0.0107
Utilities	-	-	0.0686	-	-	0.0671	0.0074	0.0064	-	-	0.1949	0.0529	0.0386	-	0.0625	0.0078
Real Estate	-	-	-	-	-	-	0.0141	0.0094	-	-	0.0506	0.0470	-	-	0.0625	0.0073
Materials	-	-	-	-	-	0.0005	0.0057	0.0032	-	-	-	0.0562	-	-	0.0625	0.0069
Momentum	0.0110	0.1059	0.0348	0.2000	0.0399	0.0629	0.1270	0.1928	0.0111	0.3552	-	0.0724	-	0.1059	0.0625	0.1062
Value	-	-	-	-	-	0.0020	-	0.0097	-	-	-	0.0705	-	-	0.0625	0.1806
Growth	-	-	-	-	0.0060	0.0153	0.0007	0.0301	-	-	-	0.0703	-	-	0.0625	0.2269
Quality	-	-	0.2363	-	0.0177	0.1933	0.0043	0.0255	-	-	-	0.0756	-	-	0.0625	0.0948
Low Volatility	0.1747	-	-	-	0.1698	-	0.0034	-	0.1745	-	-	0.0865	-	-	0.0625	0.0692
Expected Return	0.0002	0.0017	0.0003	0.0016	0.0002	0.0003	0.0018	0.0017	0.0000	0.0001	0.0005	0.0005	0.0011	0.0017	0.0006	0.0008
Volatility	0.0063	0.0110	0.0068	0.0103	0.0062	0.0074	0.0102	0.0099	0.0063	0.0093	0.0080	0.0074	0.7000	0.0110	0.0077	0.0078
Sharpe Ratio	0.0246	0.1541	0.0433	0.1536	0.0271	0.0444	0.1709	0.1683	0.0074	0.0125	0.0612	0.0710	0.0015	0.1541	0.0753	0.1059
Annual Return	0.0405	0.5300	0.0768	0.4903	0.0067	0.0821	0.4212	0.4152	0.0400	0.4262	0.1358	0.1381	0.3109	0.5262	0.1536	0.2271
Annual Volatility	0.1003	0.1749	0.1085	0.1640	0.1004	0.1087	0.1546	0.1512	0.1003	0.1482	0.1271	0.1173	0.1467	0.1749	0.1216	0.1238
Annual Sharpe Ratio	0.4041	3.0296	0.7082	2.9901	0.0665	0.7557	2.7255	2.7466	0.3991	2.8754	1.0681	1.1770	2.1191	3.0081	1.2634	1.8339
Max Drawdown	-0.0965	-0.0936	-0.0875	-0.0850	-0.1033	-0.0859	-0.0882	-0.0820	-0.0965	-0.0771	-0.0929	-0.0974	-0.0864	-0.0942	-0.0981	-0.0919
Calmar Ratio	0.4199	5.6624	0.8781	5.7698	0.0646	0.9558	4.7776	5.0662	0.4146	5.5300	1.4612	1.4185	3.5987	5.5845	1.5647	2.4703
DivRatio	1.2873	1.0808	1.2203	1.1147	1.2746	1.2280	1.2174	1.2032	1.2863	1.1607	1.5193	1.2881	1.3966	1.0808	1.3150	1.1666
Entropy in assets Vol	1.3227	0.2127	0.7733	0.3023	1.2985	0.7601	0.8145	0.8474	1.3192	0.4885	1.4555	2.7726	0.9939	0.2127	2.6654	1.7735

Table 3: Comparison of Portfolios A, B, C, D, E, F, G, H, I, L, M, N, P, and Q under Different Constraints [2] and metrics on 2023 returns.

Metric	Portfolios A	B	C	D	E	F	G	H	I	L	M	N	P	Q	EW	Caps
Annual Return	0.1776	0.4823	0.1875	0.4912	0.1460	0.1988	0.4058	0.4306	0.1772	0.5069	0.2315	0.2589	0.3007	0.4786	0.2612	0.3217
Annual Volatility	0.0812	0.2106	0.0945	0.2077	0.0818	0.0960	0.1716	0.1775	0.0812	0.2044	0.1033	0.0999	0.1329	0.2106	0.1028	0.1235
Annual Sharpe Ratio	2.1865	2.2902	1.9829	2.3649	1.7840	2.0717	2.3647	2.4267	2.1826	2.4798	2.2416	2.5928	2.2637	2.2730	2.5409	2.6049
Max Drawdown	-0.0446	-0.1520	-0.0529	-0.1497	-0.0456	-0.0530	-0.1247	-0.1288	-0.0446	-0.1469	-0.0520	-0.0536	-0.0884	-0.1522	-0.0569	-0.0797
Calmar Ratio	3.9858	3.1728	3.5433	3.2819	3.2001	3.2507	3.2533	3.3428	3.9764	3.4503	4.4564	4.8343	3.4038	3.1454	4.5936	4.0376
DivRatio	1.4344	1.0455	1.3202	1.0434	1.4294	1.3327	1.1727	1.1329	1.4329	1.0348	1.6270	1.4049	1.4101	1.0455	1.4296	1.2149
Entropy in assets Vol	1.3588	0.1688	0.8376	0.2919	1.3792	0.8640	0.6614	0.7509	1.3559	0.5338	1.5921	2.6792	1.0318	0.1688	2.6363	1.5298

Table 4: Comparison of Portfolios A, B, C, D, E, F, G, H, I, L, M, N, P, and Q applied to 2024 data.

[2]The values are truncated to the 4^{th} decimal place.

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