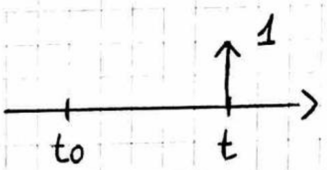
DIGITAL OPTION

Payoff $\begin{cases} 1, \text{ if } S_t \geq K \\ 0, \text{ otherwise} \end{cases}$



Before, we had $[S_t - K]^+ = S_t \mathbb{1}_{S_t \geq K} - K \mathbb{1}_{S_t \geq K}$.

Now, the price for a digital option becomes:

$$d c_B = B(t_0,t)\, \mathbb{E}\left[\mathbb{1}_{S_t \geq K}\right] = B(t_0,t)\, N(d_2)$$

BLACK DIGITAL OPTION

It can be proven that:

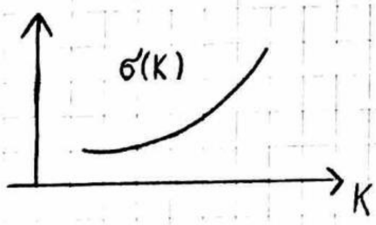
$$d c_B (K) = - \frac{\partial c_B(K)}{\partial K}$$

IMPLIED VOLATILITY APPROACH

A volatility $\sigma(K)$ s.t.

$$C(K, \sigma(K)) = \text{Quoted Price}$$

Once calibrated implied volatility, I can use this model only for plain vanillas.



"SMILE": I don't have the same volatility with ≠ strikes, so the B&S model isn't correct.

Implied volatility is the main tool to price PLAIN VANILLAS.
We can't use the implied volatility directly for a digital option.

How to price a digital option with an implied volatility approach?

A digital option is well approximated by a CALL SPREAD / BULL SPREAD.

CORRECTED BLACK FORMULA : $dC = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [C(\overset{K}{K}, \sigma(K)) - C(\overset{K+\varepsilon}{K+\varepsilon}, \sigma(K+\varepsilon))] =$

$$= - \frac{d}{dK} C[K, \sigma(K)] =$$

$$= \underbrace{- \frac{\partial}{\partial K} C[K, \sigma(K)]}_{\text{Black}} \underbrace{\frac{\partial \sigma(K)}{\partial K}}_{\text{slope impact}} \underbrace{\frac{\partial C[K, \sigma(K)]}{\partial \sigma}}_{\text{Vega PV}}$$

slope impact : digital "directs" the difference Black vs. Implied.

If I consider the Black model, I can't see the digital risk.

ASS.6 : $dC = dC_B - m\,V$

price of a digital option taking into account the smile in the curve of the implied vol.

$m = \frac{\partial \sigma(K)}{\partial K}$: slope of the tangent line at the curve of the implied volatility passing exactly from the point (K, σ).

computed as the slope of the straight line passing for the two closest point at K that we have in the dataset

$$m = \frac{\sigma_2 - \sigma_1}{K_2 - K_1}$$