Politecnico di Milano

FINANCIAL ENGINEERING

A.Y. 2023/2024

Assignment 4

Group 12

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Exercise: Variance and Covariance method for VaR and ES

We consider the following equally weighted equity portfolio:

- Adidas
- Allianz
- Munich Re
- L'Oréal

We compute the daily ($\Delta = 1$) VaR and ES with a 5y estimation, with confidence level (alpha) equal to 0.99, via a t-student parametric approach, with degrees of freedom (ν) equal to 4. We start by presenting a reduced version of the given data-set, so as to only contain the market prices of the companies constituting our portfolio, up to 5 years before the current date. Any missing data points are addressed through a forward-filling mechanism using the phyton function: **ffill**.

Next, we compute the daily log returns matrix, taking the natural logarithm ratio between the current price value and the previous price value for each day.
First we define the standard values of the Value at Risk and the Expected Shortfall with confidence level α in the case of t-Student distributed losses:

$$VaR_\alpha^{std} = t_\nu^{-1}(\alpha) \qquad ES_\alpha^{std} = \frac{\nu + (t_\nu^{-1}(\alpha))^2}{\nu - 1} \frac{\phi(t_\nu^{-1}(\alpha))}{1 - \alpha}$$

Then, to compute the VaR and the ES values with the variance/covariance method we use the following formulas:

$$VaR_\alpha = \Delta\mu + \sqrt{\Delta} \cdot \sigma \cdot VaR_\alpha^{std} \qquad ES_\alpha = \Delta\mu + \sqrt{\Delta} \cdot \sigma \cdot ES_\alpha^{std}$$

Where Δ, called scaling factor, is the time interval in days of the considered risk measure, and is used to compute VaR (or ES) over the required time span.
Indeed, common practice in the risk measurement world is to first compute the daily VaR (or ES), since all log-returns in the data are recorded daily, and then scale the values to the wanted ones.

The last step is to multiply the obtained VaR and ES by the portfolio value.
In the table below, we display the numerical results:

VaR	**ES**
€563223.316726	€787977.234225

Essentially, VAR indicates the maximum loss that a portfolio or financial position could incur into over a given period of time, at a certain confidence level. In this case a 99% VAR indicates that there is only a 1% probability that the loss will exceed the specified value: €563223.316726.
The Expected Shortfall (ES), or Conditional Value at Risk, is a risk measure that goes beyond VaR, as it considers not only the magnitude of the maximum losses but also the distribution of losses beyond that point. An ES of €787977.234225 indicates that, on average, in extreme situations where the portfolio incurs into losses greater than the VaR, the average loss is expected to be that amount.

Case study: Historical (HS & WHS) Simulation, Bootstrap and PCA for VaR & ES in a linear portfolio

In this exercise, we take on the role of an asset manager that, at the end of the 20th of March 2019, has to compute risk measures (with confidence level equal to 95%) on 3 different portfolios following different approaches.
Our starting point will be the historical records of the shares' prices for the various companies, contained in a given csv file.

The main quantity of interest is the loss of the portfolio, defined as the present value of my portfolio minus the value at a future date, which can be rewritten through a linear expansion as:

$$\underline{L}_{t+\Delta} = -V_t(\ \underline{w}_t X_{t+\Delta} + \underline{c}_t) \tag{1}$$

Where V_t is the portfolio value at time t; $\underline{w}_t$ is the weight vector of the assets in the portfolio; X_t is the matrix of returns at time t (rows) for each company (columns); $\underline{c}_t$ represents the transaction costs, and will be set to zero from here onward.

$$V(t) = \sum_{i=1}^{d} n_i(t)\ S_i(t) \qquad w_i(t) = \frac{n_i(t)\ S_i(t)}{V(t)} \qquad X_i(t) = ln\left(\frac{S_i(t+\Delta)}{S_i(t)}\right)$$

With $n_i(t)$ we refer to the number of shares held in our portfolio for each company; $S_i(t)$ is the price of one share for each company; i goes from 1 to d, which is the number of companies considered in the portfolio.
An evident problem to 1 is that we do not know the future returns $X_{t+\Delta}$. In the following analysis we will see different ways of dealing with this issue, to finally compute the Value at Risk and the Expected Shortfall.
An essential simplifying assumption that will hold throughout this case study is the 'frozen portfolio': we consider the weights constant in time.

Historical Simulation and Statistical Bootstrap

We consider the following portfolio:

- Total (25K shares)
- AXA (20K Shares)
- Sanofi (20K Shares)
- Volkswagen (10K Shares)

We compute the daily ($\Delta = 1$) VaR and ES with a 5y estimation, first via a Historical Simulation approach and then with a Bootstrap method with 200 simulations. We start by presenting a reduced version of the given data-set as we did before.

The Historical Simulation approach offers a solution to the aforementioned problem rooted in leveraging the historical market data of returns available to us: using 1 we obtain a vector of losses L_s computed with X_s for s going from the current date back to 5 years before. Then we rearrange this vector in decreasing order, to obtain a new vector $\{L^{[i,n]}\}_i$ for i=1...n, where n is simply the number of historical returns considered. Lastly, considering the confidence level $\alpha = 0.95$, we compute the VaR and the ES as follows:

$$Var_\alpha = \sqrt{\Delta}\ L^{([n(1-\alpha)],n)} \qquad ES_\alpha = \sqrt{\Delta}\ mean[L^{(i,n)}, i = [n(1-\alpha)], ..., 1]$$

The statistical bootstrap technique is based on random sampling: first we sample M=200 integer numbers between 1 and n, and then we select the corresponding rows from the matrix of log-returns obtained at the beginning. We plug these returns in 1, and proceed just as we did in the Historical Simulation approach.

The last step is to multiply the obtained VaR and ES by the portfolio value.
The numerical results are displayed in the table below.

	VaR	ES
Historical Simulation	96039.46615	143630.82908
Statistical Bootstrap	90502.70256	140995.67186
Plausibility check on VaR : 92035.62681		

We can appreciate a consistency of results between the two methods used. Clearly, if we do not fix a seed, we get different results for every execution of the statistical bootstrap method.

Weighted Historical Simulation

We now consider the following portfolio with equally weighted equity:

- Adidas
- Airbus
- BBVA
- BMW
- Deutsche Telekom

After presenting as before a reduced dataset with the new companies, we compute the daily ($\Delta = 1$) VaR and ES with a 5y estimation via a Weighted Historical Simulation approach, where the main idea is that recent losses should have a higher impact in the computation of our risk measures.
We start by computing the vector of losses as in the Historical Simulation approach, but we also associate to each element of L_s a particular weight defined as:

$$w_s = \lambda^{t-s} \ \frac{1-\lambda}{1-\lambda^n}$$

When we rearrange L_s in decreasing order we keep the correspondence between weights and losses, and then define as i^* the largest value such that

$$\sum_{i=1}^{i^*} w_i \leq 1-\alpha$$

Lastly we compute the VaR and the ES using the following formulas:

$$Var_\alpha = \sqrt{\Delta}\ L^{(i^*,n)} \qquad ES_\alpha = \sqrt{\Delta}\ \frac{\sum_{i=1}^{i^*} w_i\ L^{(i,n)}}{\sum_{i=1}^{i^*} w_i}$$

The numerical results are displayed in the table below.

	VaR	ES
Weighted Historical Simulation	0.015937	0.017470
Plausibility check on VaR : 0.019219		

Principal Component Analysis

We now consider a portfolio with equally weighted equity containing the first 18 companies in the provided csv file "_indexes.csv" . After presenting as before a reduced dataset with the new companies, we compute the 10 days ($\Delta = 10$) VaR and ES with a 5y estimation via a Gaussian parametric PCA approach using the first n principal components, with the parameter n =1,..,5.
Since we are assuming that the loss distribution is gaussian, we are in the case of continuous cumulative distribution function, and we can use the following formulas:

$$VaR_\alpha = \Delta\ \mu_{pca} + \sqrt{\Delta}\ \sigma_{pca}\ N^{-1}(\alpha) \qquad ES_\alpha = \Delta\ \mu_{pca} + \sqrt{\Delta}\ \sigma_{pca}\ \frac{\Phi(N^{-1}(\alpha))}{1-\alpha} \tag{2}$$

Where μ_{pca} and σ_{pca} are specifically computed mean and standard deviation of the returns; N is the gaussian cumulative distribution function of the loss; Φ is the gaussian density function.
The calculations of the mean and volatility follow the PCA approach and are executed within the function 'PrincCompAnalysis'.

From our matrix of past market returns, we obtain the yearly mean vector and the yearly covariance matrix by multiplying the daily values for the number of business days in a year (256). Then we compute the mean and the standard deviation of the Gaussian loss as follows:

$$L \sim \mathcal{N}(\underline{w} \cdot \underline{\mu} \ , \ \underline{w} \cdot \Sigma \ \underline{w})$$

As for the reduced form portfolio mean and the variance we use the following equations:

$$\sigma^2_{pca} = \sum_{i=1}^{k} \hat{w}_i^2 \lambda_i \qquad \mu_{pca} = \sum_{i=1}^{k} \hat{w}_i^2 \hat{\mu}_i$$

Where λ is the ordered vector of decreasing eigenvalues of the yearly covariance matrix of the returns, while $\underline{\hat{\mu}}$ and $\underline{\hat{w}}$ are obtained through the eigenvectors matrix Γ, which follows the order given to the eigenvalues, as follows:

$$\underline{\hat{\mu}} = \Gamma' \underline{\mu} \qquad \underline{\hat{w}} = \Gamma' \underline{w}$$

With $\underline{\mu}$ we call the mean vector of the returns and $\underline{w}$ is the vector of weights of our portfolio.
The numerical results obtained are reported in the table below.

Components	VaR	ES
1	0.057869	0.650877
2	0.058212	0.651878
3	0.058192	0.651866
4	0.058148	0.651972
5	0.058165	0.651996
Plausibility check on VaR : 0.054632		

We can see that there's not a big difference in the VaR values changing the number of principal components used. We could say that any value between 3 and 5 could be a good choice, to achieve a good VaR result without having an overly complex and expensive model: indeed by using this method with only 3/5 principal components I have significantly reduced the dimensionality of the problem, and therefore its overall cost.

Plausibility check

As engineers, we are interested in a quick and easy method of computation of the VaR to serve as a rule of thumb, to know the order of magnitude of the VaR and subsequently assess whether my results are plausible or not.
This is precisely what we implemented in the function **'plausibilityCheck'**.
First we found the 2 quantiles of the returns of order $1-\alpha$ (lower: l_i) and α (upper: u_i) (considered confidence level is 0.95) for each company in the portfolio; then we computed the $sVaR_i$, using the weights of the portfolio as sensitivities, as follows:

$$sVaR_i = sens_i \frac{|l_i|+|u_i|}{2}$$

Finally, using the correlation matrix C of the returns, we obtain the VaR of our portfolio for the Δ time interval:

$$VaR^{ptf} = \sqrt{\Delta}\sqrt{sVaR'\sigma sVaR}$$

In general, we can say that this approximation is equivalent to the Analytical Valuation via a Variance-Covariance method (discussed above) if the returns are driftless and Gaussian in a linear portfolio. In our cases, we see above that all our results seem reasonable when we compare them with the VaR plausibility check.

Exercise: Full Montecarlo and Delta-normal VaR

In this exercise, we are working with a portfolio composed by a long position of value €$1,186,680$ on BMW stocks and a short position of Call options on this underlying. The number of Calls η_C is the same as the number η_S of BMW shares held, the expiry is the 18^{th} of April 2017, with strike €25 and volatility equal to 15.4%. Furthermore, we are assuming a dividend yield of 3.1% and fixed interest rate equal to 0.5%.

We want to compute a 10 days 95% VaR: in the Full MC we will simulate the evolution of the underlying for a time interval $\Delta = 10d$, while for the Delta-Normal we will compute a daily VaR and then rescale it multiplying it by $\sqrt{\Delta}$.

The highest sensitivity with relation to the price for the Call is the Gamma, i.e. the second order sensitivity, making the Call a non-linear derivative. We are therefore in presence of a non-linear derivative portfolio, and we cannot use anymore the techniques applied in the previous exercises.

Full Monte-Carlo VaR

The Full Valuation Monte-Carlo is a technique used to compute the VaR for non-linear derivative portfolios.
Firstly, we evaluate the option's price at time $t = 16^{th}$ of January 2017, obtained via a simple application of the Black&Scholes formula for the Call.
For this exercise we use a 2 year WHS with Statistical Bootstrap, extracting ten log-returns for each simulation and summing them, to obtain a simulated value of the stock in $t+\Delta$ as $S_{t+\Delta} = S_t \cdot e^{X_{t+\Delta}}$.
We do not sample all the indexes with the same probability, but we use the function **'np.random.choice'** assigning a higher probability to the indexes with bigger weights. In this way, we make sure that we use with higher probability log-returns that are closer to the valuation date of the VaR, as they are more statistically significant.
Then, for each simulation we compute its index as mean of the ten indexes, and we compute its weight to be used for the continuation of the WHS method. We also perform a re-scaling of the weights to make sure that they add up to 1.
Afterwards, we price the Call option in $t+\Delta$, obtaining one value for each simulation. We can therefore obtain the vector of simulated portfolio loss L_π as the sum of contributions of the derivative part and the stock part:

$$
\begin{aligned}
L_C(X_t, \Delta) &= \eta_C \, \left(C(t+\Delta) - C(t)\right) \\
L_S(X_t, \Delta) &= -\eta_S \, \left(S_{t+\Delta} - S_t\right) \\
L_\pi(X_t, \Delta) &= L_C(X_t, \Delta) + L_S(X_t, \Delta)
\end{aligned}
$$

where L_C is taken with a plus sign because we are in a short position. Now, following the WHS approach already described, we sort the losses in descending order, keeping the correspondence with the weights, and we find a VaR = €1494.32640.
We can observe that the VaR is quite small if we compare it to the value of our portfolio. This probably means that we are performing a good hedging by shorting the Calls.

Q: Why can the Full Monte-Carlo be numerical intensive for an exotic derivative that cannot be priced via a closed formula?
The reason is that, if we cannot price our exotic derivative with a closed formula, at time t we will need to perform some sort of MC simulation to price it. Then, for each simulated underlying value in $t+\Delta$, we would again perform a MC simulation to price the option, thus entering in a situation of "Nested" Monte Carlo, which can prove to be extremely expensive computationally. Since large banks can have portfolios with plenty of exotic derivatives, we see why they cannot use the Full Valuation method. Indeed, some less expensive techniques are put in place for the risk management of non-linear derivatives portfolios, as we will see in the next subsection.

Delta-Normal VaR

The Delta-Normal method reduces the computational cost of the VaR by using an expansion up to the first order sensitivities for the losses of a derivatives portfolio.

$$L(X_t) = -\sum_{i=1}^{d} sens_i(t)\ X_{t,i}$$

where $sens_i(t) = \delta_i(t) S_i(t)$ is the portfolio sensitivity with relation to the i^{th} risk factor. For an European Call, $\delta_i(t) = e^{-q(T-t)} \cdot N(d_1)$ and for a Stock $\delta_i = 1$.
As before, we extract some random indexes to select from the vector of past log-returns and we assign to each simulation an historical weight, making sure that the weights sum up to 1.
As can be seen from the formula, it is not necessary to simulate the underlying's dynamics in this method. We just compute the Call's Delta value at the time t when the VaR is evaluated, and then, plugging in the values and adjusting for the scaling factor, we obtain a VaR = €626.31002.
The result we obtain is not particularly precise, but this is to be expected as this method is less complex than the Full MC. We recap here below the two values obtained:

	Full Valuation MC	Delta-Normal
VaR	€1494.32640	€626.31002

Q: Can you improve the Delta normal VaR? How?
We can improve this method involving also the second order Greek, i.e. the Gamma. The Delta-Gamma method should give a more accurate result, closer to the one obtained via the Full Valuation technique. However, we can observe that the option is deep in-the-money ($S0 =$ €86.53 and $K =$ €25), so the Call's Gamma will be really small and therefore the value obtained with the Delta-Gamma will not differ largely from the Delta method one.

Case Study: Pricing in presence of counterparty risk

In this case study we analyze a Cliquet option on an equity stock with no dividends and constant volatility 20%, issued by ISP.
The option has a time to maturity of 7 years, and offers yearly payoff equal to $[L \cdot S(t_i) - S(t_{i-1})]^+$ with L participation coefficient equal to 0.99. In this case, we are in a situation where the buyer of the Cliquet is the only party with counterparty risk: it always has a positive NPV because it is expecting to receive annual flows until the maturity of the contract. For ISP, instead, there's clearly no counterparty risk, because once they sell the Cliquet option, their NPV will be negative.

Firstly, we price this option with an analytical approach.
Our strategy consists of pricing a new portfolio that replicates the cash-flows generated by the Cliquet option: the key lies in recognizing that the Cliquet option is simply constituted by a series of Forward Starting Calls, each starting 1 year after the other and with time to maturity 1 year.
Let us rewrite the payoff of the Cliquet at the end of the year i+1:

$$\begin{aligned}
\Phi_{cliquet}(t_i + 1) &= [L \cdot S(t_{i+1}) - S(t_i)]^+ \\
&= L \cdot \left[S(t_{i+1}) - \frac{S(t_i)}{L} \right]^+ \\
&= L \cdot [S(t_{i+1}) - K_i]^+ \\
&= L \cdot \Phi_{fwd\ start\ call}(t_i + 1)
\end{aligned}$$

It is clear that we can replicate our option with a portfolio based on seven Forward Starting Calls FSC_i, each beginning at t_i and expiring at t_{i+1} with strike K_i for $i = 0, ..., 6$.

$$Portfolio = L\ [FSC_0,\ FSC_1,\ FSC_2,\ FSC_3,\ FSC_4,\ FSC_5,\ FSC_6]$$

So now our objective is to price these options, and we will do so by rewriting the Black-Scholes formula for Calls.

$$\begin{aligned}\Pi_{FSC\_i}(t_i) &= S(t_i)\ N(d_1) - \frac{S(t_i)}{L}\ e^{-r_{i;i+1}\ (t_{i+1}-t_i)}\ N(d_2)\\ &= S(t_i)\ \left(N(d_1) - \frac{1}{L}\ e^{-r_{i;i+1}\ (t_{i+1}-t_i)}\ N(d_2)\right)\\ &= S(t_i)\ \Pi_{Call\_i}(t_i)\end{aligned}$$

Where with $\Pi_{Calli}(t_i)$ we are referring to the BS closed formula for calls with underlying starting price equal to 1, strike equal to $1/L$ and time to maturity 1 year; N is the standard Gaussian cumulative distribution function; $r_{i;i+1}$ is the forward risk free rate that matches the bootstrapped discount factor at maturity in t_{i+1}.
Now that we have the prices of the forward starting calls at t_i, we have to discount them to find the prices at t_0, i.e. today.

$$\begin{aligned}\Pi_{FSC\_i}(t_0) &= \mathbb{E}[D(t_0,t_i)\ S(t_i)\ \Pi_{Calli}(t_i)] && \Pi_{Calli}(t_i) \text{ is not stochastic}\\ &= \mathbb{E}[D(t_0,t_i)\ S(t_i)]\ \Pi_{Calli}(t_i) && D(t_0,t_i)\ S(t_i) \text{ is a martingale}\\ &= S(t_0)\ \Pi_{Call\_i}(t_i)\end{aligned}$$

At last, to obtain the price in case of no default the only step missing is adding these prices and multiplying them by L:

$$\Pi_{Cliquet}(t_0) = L\ \sum_{i=0}^{6}\ \Pi_{FSC\_i}(t_0) = L\ \sum_{i=0}^{6}\ S(t_0)\ \Pi_{Call\_i}(t_i)$$

We also compute the price in the case of counterparty risk by considering the survival and default probabilities. From the file 'CDS bootstrap.csv' we upload the survival probabilities of each year going from 1 to 7, and from these we find the default probabilities. At this point, we just weigh the default-free discounted cash-flows by survival/default probability for each possible scenario:

$$\overline{\Pi_{Cliquet}}(t_0) = L\ \sum_{i=0}^{6}\ S(t_0)\ \Pi_{Call\_i}(t_i)\ P(t_0,t_i) + R\sum_{i=1}^{7}\ (P(t_0,t_{i-1}) - P(t_0,t_i))\ CF_{future}(t_i) \quad (3)$$

where R=0.4 is the recovery value, and $CF_{future}(t_i)$ indicates the sum of the future expected cash flows computed in t_i, in case of default of ISP between t_{i-1} and t_i.
In the table below we report the values obtained, multiplied by the €$30Mln$ notional:

	No-Default	Default
Analytical Price	19732627.11624	19424418.94319

Now, we consider a numerical method.
We chose to follow a hybrid approach, which consists of first using a Monte Carlo simulation to obtain some times to default, and then using the analytical formula to obtain the final price. This can be done because in this particular case we found an easy analytical formula, and it makes the process way lighter, since otherwise we would have had to simulate also the underlying's evolution in time with another MC to price the option.

The first step of this approach is then to simulate the times to default τ. We extract values from a uniform distribution on [0,1] and then we invert the piece-wise constant probability function obtained from the CDS bootstrap of the previous assignment to obtain the times to default.
For the simulations in which there was no default event before the Cliquet's maturity, we simply take as final price the one already computed analytically. Then, for the remaining ones, we sum up the cash flows accordingly to their τ value, considering both the full cash flows and the recovery part

contributions.
Finally, to obtain the MC price we compute the mean of the values just found. Then, we also compute the confidence interval for this price, as $\hat{\mu} \pm z_{\alpha/2}\ \frac{\hat{\sigma}}{\sqrt{n}}$, where $\hat{\mu}$ is the mean of the simulations, i.e the MC price, $\hat{\sigma}$ is the sample variance and n is the number of simulations.
In the table below we report the values obtained:

	Analytical Price	MC Price	MC Confidence Interval
No-Default	19732627.11624718	-	-
Default	19424418.943189755	19411862.647972014	[19381027.863531947, 19442697.43241208]

Here we see we do not compute a MC price and CI for the No-Default case because clearly, since we are just simulating the time to defaults, without the default probability's contribution we just get the price obtained with the analytical formula.

For the sake of completeness, we also tried to compute the prices using the "full" Monte-Carlo approach. Indeed, we first sampled the times to default as above and then we simulated the underlying's evolution in time using Black&Scholes dynamics.

We were then able to price the Cliquet's option both in the No-Default case and in the Default one, obtaining mean values and confidence intervals that were coherent with the analytical prices.
In general, we can see that the prices in the defaultable case are a bit lower than the ones computed neglecting the default risk, as is expected. Clearly, the correct price at which the Cliquet option should be sold is the analytical Default one, but ISP may want to leave this out and try to sell this option at the No-Default price.