POLITECNICO
MILANO 1863

Financial Engineering

Assignment 1, Group 16

2023-2024

Alice Vailati - CP: 10683600 - MAT: 222944
Andrea Tarditi - CP: 10728388 - MAT: 251722
Jacopo Stringara - CP: 10687726 - MAT: 222456
Nicolò Toia - CP: 10628899 - MAT: 247208

Contents

1. Introduction and Data

The aim of this assignment is to price a variety of option and to compute some financial instruments using both closed form approaches as well as numerical ones. In particular, these are the data we will be using throughout this project:

Strike price: 1 Euro
value date: 15th of February 2008
time-to-maturity (ttm): 3 month (consider a yearfrac 1/4)
volatility: 22% (per year)
ttm-zero-rate: 3%
Underlying: equity stock
Dividend Yield: 6%
Settlement: physical delivery
Number of contracts: 1 Mln
Underlying price 1 Euro

2. Pricing (Point a)

The request was to price a European Call Option with three different methods: Black's formula, the CRR approach and the MC approach. The value of iterations in Monte Carlo and of steps in CRR is fixed at $M = 100$. Let's consider the first one as the right price of the option and then compare this result with the other ones:

$Black$	CRR	MC
0.0398	0.0398	0.0390
39763.7764	39849.4913	39028.8219

Table 1: Pricing with M=100

In the second row we have included the price for the notional amount of one million. It's notable that the price found with the CRR tree approach is very similar to the one obtained with the closed formula. The same does not hold for the one found with MC. This means that the M=100 is enough to simulate correctly with a binomial tree while it's too little to have a good simulation with MC method, we can conclude that CRR method for basic vanilla options performs better given the same number of iterations. .

3. Errors Rescaling (Point b)

The objective of this point is to select an M such that the error is less than the bid/ask spread (1 bp). In particular the error for the CRR method is defined as the absolute value of the difference between this model and Black's formula, while for the Monte Carlo approach we use the variance of the price as the error.

3.1. CRR method

In this specific case, as we said above, the error is defined as:

$$Error = |ExactPrice - CRRPrice|$$

As already observed before, M = 100 was enough to generate a result that fits the condition on the spread. After doing some calculations, the first M found to satisfy the condition is M = 16: this means that, after sixteen steps, the binomial tree is already sufficiently precise.
Indeed a more accurate search for the optimal M for the CRR model yield 14 time steps as a result (see function *findMCRR*). This can be done since CRR converges much faster.

3.2. MC method

In the MC case, the error is defined as follows:

$$Error = \frac{B}{\sqrt{M}} \cdot std\left([(F_{t,t} - K))]^{+}\right)$$

where B is the discount factor, M is the number of iterations, $F_{t,t}$ is the final value of the forward and std is the unbiased estimator for the standard deviation. We impose that this value must be one order lower than the Bid-Ask spread. As seen before, M = 100 was too little to let the algorithm converge to the exact value; indeed the error at the beginning is around 0.059 which is much bigger than 10^{-4}. After some computations are performed, we found that for the value M=524288 the criterion is met. Graphically we can deduce that the optimal M lies around 425000 iterations.
In conclusion we can see that the CRR model outperforms the MC model in terms of number of iterations.

4. Errors Rescaling (Point c)

Indeed, using a logarithmic scale, we can plot both errors and observe how they rescale as M changes. It can be easily seen that the error for the CRR tree scales as $\frac{1}{M}$ while the error for the MC approach scales as $\frac{1}{\sqrt{(M)}}$.

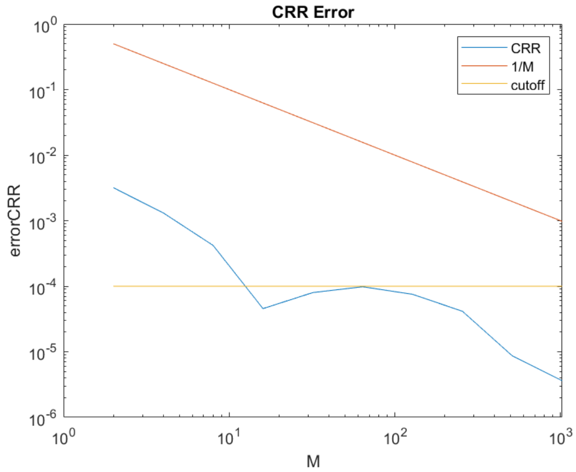

Figure 1: Comparison between CRR error and 1/M

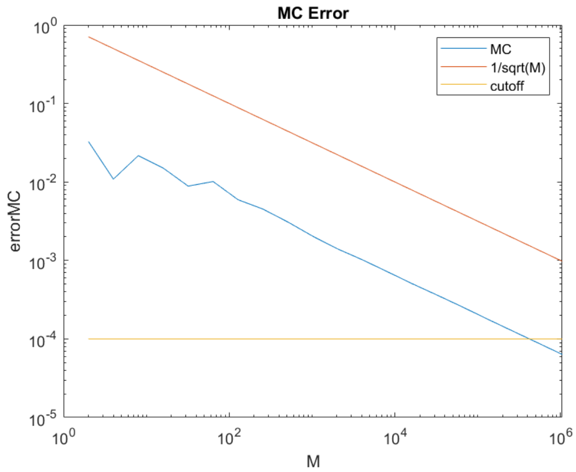

Figure 2: Comparison between MC error and $1/\sqrt{M}$

We can also see that the cutoff line of the error confirms what we found above about the M that satisfy the requirement on the error.

5. KI Option (Point d)

In this section we will pass from pricing a vanilla European call option to pricing an Up & In call option with European barrier. Such an option has the following payoff:

$$\Phi(S_T) = [S_T - K]^+ \cdot \mathbb{I}_{S_T > KI}$$

where K is the strike price of the call option while KI is the option's barrier.

We can see this payoff as the composition of two other options. A vanilla call with strike in KI and a money-or-nothing option of value $(KI - K)$ and strike KI. Indeed:

$$\Phi(S_T) = [S_T - K]^+ \cdot \mathbb{I}_{S_T > KI} = [S_T - KI]^+ + (KI - K) \cdot \mathbb{I}_{S_T > KI}$$

Since both options admit a closed formula under Black's model we can thus derive a closed formula for this option's price:

$$P = B \cdot (F_0 \cdot N(d_1) - KI \cdot N(d_2)) + B \cdot (KI - K)N(d_2)$$

where P is the price at t_0, B is the discount factor, T is the time to maturity, F_0 is the price of the forward with same final value as the underlying and d_1 and d_2 are calculated as:

$$d_{1,2} = \frac{ln(\frac{F_0}{KI}) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

Hence the closed formula of the price of such an option is:

$$P = B \cdot (F_0 \cdot N(d_1) - K \cdot N(d_2))$$

with the d_1 and d_2 computed above using KI where in a vanilla call we would find K.
Of course we can also compute such a price using both the CRR and MC approach simply changing the payoff function accordingly. In particular with $M = 1000$ for both methods we obtained the following results:

$Black$	CRR	MC
0.0021	0.0022	0.0011

Table 2: Pricing with M=1000

We can observe that the CRR is a bit off the mark but not too far off from the closed formula result. On the other hand, the MC result has a much larger error with respect to the closed formula. Indeed MC can only achieve a similar result if the number of simulations is increased substantially. This is of course related to how the error of the two methods rescales differently as M changes.
Indeed according to theory, under Black's dynamics the Monte Carlo method perform poorly when pricing exotic option such as this one.

6. KI Option Vega (Point e)

In order to evaluate the Vega with the closed formula, we derive the price (as written in question d) with respect to the volatility σ. We obtain the following expression:

$$\nu = B \cdot \left[F_0 \cdot \frac{e^{\frac{-d_1^2}{2}}}{\sqrt{2\pi}} \cdot \left(-\frac{\ln \frac{F_0}{KI}}{\sigma^2\sqrt{t-t_0}} + \frac{\sqrt{t-t_0}}{2} \right) - K \cdot \frac{e^{\frac{-d_2^2}{2}}}{\sqrt{2\pi}} \cdot \left(-\frac{\ln \frac{F_0}{KI}}{\sigma^2\sqrt{t-t_0}} - \frac{\sqrt{t-t_0}}{2} \right) \right]$$

We also evaluated the Vega using both numerical approaches. For the numerical approaches we used the center scheme to approximate the derivative with respect to σ.

$$\nu = \frac{\partial P}{\partial \sigma} \approx \frac{P(\sigma + \epsilon) - P(\sigma - \epsilon)}{2\epsilon}$$

Where $P(\sigma + \epsilon)$ represents the Price computed with a volatility of $\sigma + \epsilon$.
For $M = 10000$ these yield the following graphs:

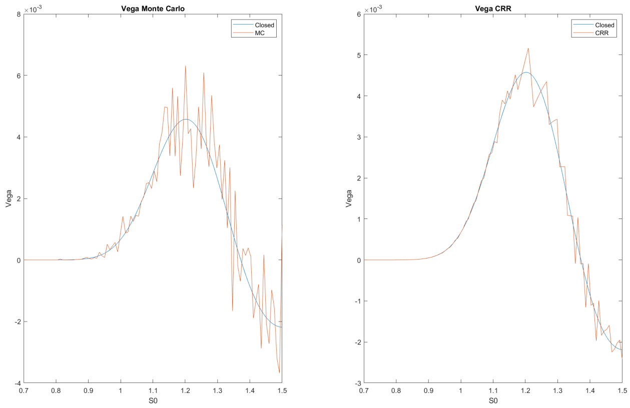

Figure 3: Comparison between Vega with the closed formula and with numerical approaches

In all of these cases, when we are deep out of the money, it's unlikely to exercise the option. Thus, changing the volatility of the underlying does not influence the price much. Indeed in this region the Vega is exactly equal to zero.
It is interesting to notice that in the MC case the error in the Vega seems to be quite large while it is quite small in the CRR case. The reason for this lies in the fact that the center scheme relies directly on the price to compute the derivative numerically. Hence the error in the Vega decreases as it did for the prices.
Indeed if we increase the number of simulations for Monte Carlo to be the same as the best one for the European option ($M = 524288$) we can observe a marked improvement in the vega's accuracy.

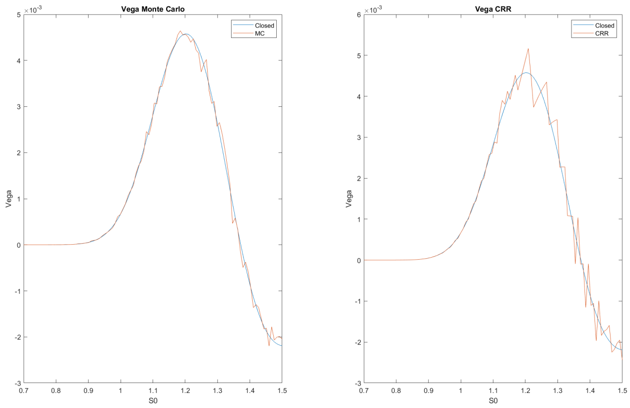

Figure 4: Comparison with closed formula with Optimal M for MC and unchanged for CRR

In general we can observe that while usually the vega of a vanilla call option is a bell curve, similar in shape to a gaussian distribution, indeed the derivative of the price is a transformation of the gaussian distribution. On the other hand the vega of a money or nothing is in general negative as the value of the underlying increases, as the volatility rises we are still rewarded with the same amount of money. Therefore for high values of the underlying the vega of the money or nothing influences the vega of our Up & In option more and more, making it negative.

7. Antithetic Variables (Point f)

The idea of the antithetic variable is to consider two variables with the same distribution but with negative covariance, created as follows:

$$F_{t,t} = a \cdot e^{-0.5*\sigma^2*T+\sigma*\sqrt{T}*g}$$

$$F_{t,t}^{AV} = a \cdot e^{-0.5*\sigma^2*T-\sigma*\sqrt{T}*g}$$

where g is distributed as a standard Gaussian. In the antithetic variable technique we compute the payoff on both variables and then sum the two. Thus we create the final variable on which we have to evaluate the error as follows:

$$Payoff = B(t0,T) \cdot [F_{t,t} - K]^+$$
$$Payoff_{AV} = B(t0,T) \cdot [F_{t,t}^{AV} - K]^+$$
$$Price = \mathbb{E}_0 \left[\frac{Payoff + Payoff_{AV}}{2} \right]$$

Evaluating the error, it results to decrease as $\frac{1}{\sqrt{M}}$. The number of iterations applying the antithetic variable technique decreases since the core idea behind this method is to reduce the variance of the estimator used to compute the error for Monte Carlo. The methods both converge to the same price but by analyzing the graph it is possible to deduce that with the AV technique it converges in 132000 iterations (instead of 524288) yielding the same accuracy.

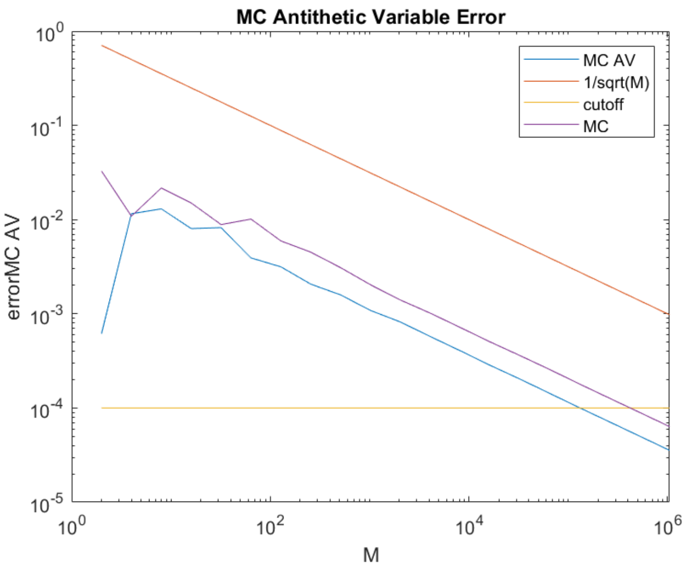

Figure 5: Comparison between MC and AV

8. Bermudan Option (Point g)

In this section we price a Bermudan option with the same maturity as the call option in a, where the holder has the right to exercise at the end of each month.
To price this option we used the CRR tree approach. In particular for non-exercise dates we regress the tree as before, for exercise dates we computed the Intrinsic and Continuation value. The continuation values is simply the value regressed up to that point, the intrinsic value is computed as follows:

$$S_t = \frac{F(t,T)}{e^{(r-d)\cdot(T-t)}}$$
$$IT = [S_t - K]^+$$

where d is the dividend yield, and IT is the intrinsic value of the option. Computing the price for the Bermudan Option with same strike price and the same dividend yield used in point a we obtained a price of 0.0401.
This tracks well with what we now the behaviour of the Bermudan call option should be.

9. Vary the Dividend Yield (Point h)

To better understand the behaviour of the Bermudan call option with respect to the Vanilla Call we compared and plotted their prices for different values of d ranging from 0% to 6%.
To confirm the correctness of our results we also computed the prices of a Pseudo-American option, that is a Bermudan option with the possibility of exercising it at any time step rather than only at the exercise dates.

d	0	0.0050	0.0100	0.0150	0.0200	0.0250	0.0300	0.0350	0.0400	0.0450	0.0500	0.0550	0.0600
EU	0.0475	0.0468	0.0462	0.0455	0.0449	0.0442	0.0435	0.0429	0.0423	0.0416	0.0410	0.0404	0.0398
Berm	0.0475	0.0469	0.0462	0.0455	0.0448	0.0442	0.0436	0.0430	0.0424	0.0418	0.0412	0.0407	0.0401
Pseudo-AM	0.0475	0.0469	0.0462	0.0455	0.0448	0.0442	0.0436	0.0430	0.0425	0.0419	0.0414	0.0409	0.0404

Table 3: Pricing with M=100

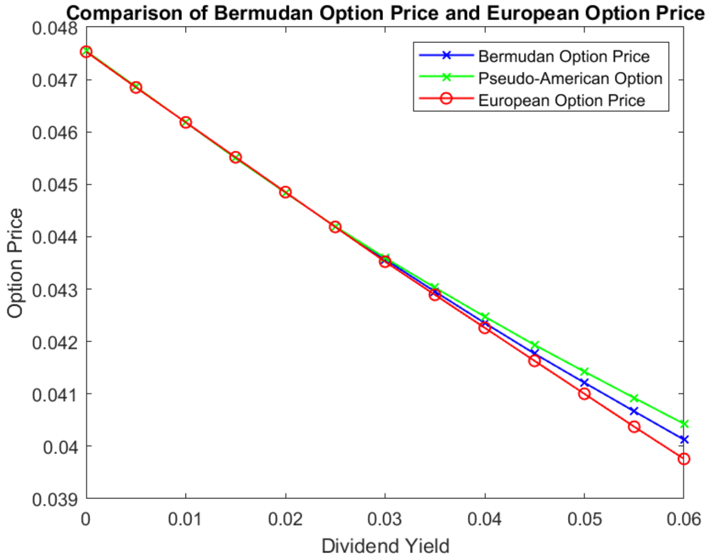

Figure 6: Comparison between MC and AV

For low dividends the three options converge to approximately the same value and the price of the Bermudan dominates the Vanilla call option. In turn the pseudo-American Option dominates the Bermudan we were trying to price.

Indeed raising the dividend yield d makes exercising the option more likely on the whole, thus having the opportunity to exit the contract at a time that is more optimal is more valuable and thus raises the price of the option.

10. Appendix

10.1. ifferent approach for computing Vega

In order to keep a reasonable amount of iterations to guarantee an efficient run time for the code we tried a different approach for computing Vega based on the approach described in Monte Carlo Methods in Financial Engineering by Glasserman . We derived a new estimator for Vega:

$$\text{estimator} = (B + KI - K) \cdot \left(\frac{\log\left(\frac{\text{Ftt}}{F_0}\right) - 0.5 \cdot \sigma^2 \cdot T}{\sigma} \right) \cdot \text{Ftt} \cdot \mathbb{I}(\text{Ftt} > KI)$$

and computed Ftt by using Monte Carlo simulations. Unfortunately this does not lead to a coherent result therefore we decided to discard this method and relay on a higher number of iterations to guarantee a correct computation for Vega.

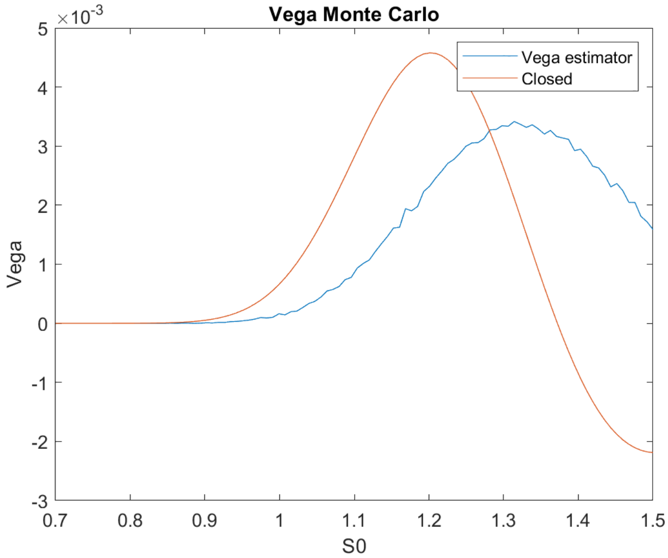

Figure 7: Comparison between MC and AV

11. Reference

Monte Carlo Methods in Financial Engineering by Glasserman
Opzioni, futures e altri derivati J.Hull
Arbitrage Theory in Continuous Time Tomas Bjork