Financial Engineering: Basic concepts for Quantitative Risk Management

NOT FOR ORAL EXAM

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Politecnico Milano, AY 2023-2024

Outline

1. Regulatory Risk Management framework
2. Basic Risk Management Concepts
3. Gaussian (multinomial) parametric approach
4. Non parametric approaches & Derivatives portfolios
5. Other multinomial models
6. Coherent Risk Measures
7. Backtesting VaR

Outline

1. Regulatory Risk Management framework
   - ✓ Basel & Solvency accords
   - ✓ Risk Management *vs* Capital Adequacy
2. Basic Risk Management Concepts
3. Gaussian (multinomial) parametric approach
4. Non parametric approaches & Derivatives portfolios
5. Other multinomial models
6. Coherent Risk Measures
7. Backtesting VaR

Financial risk definition

Any event or action that may:

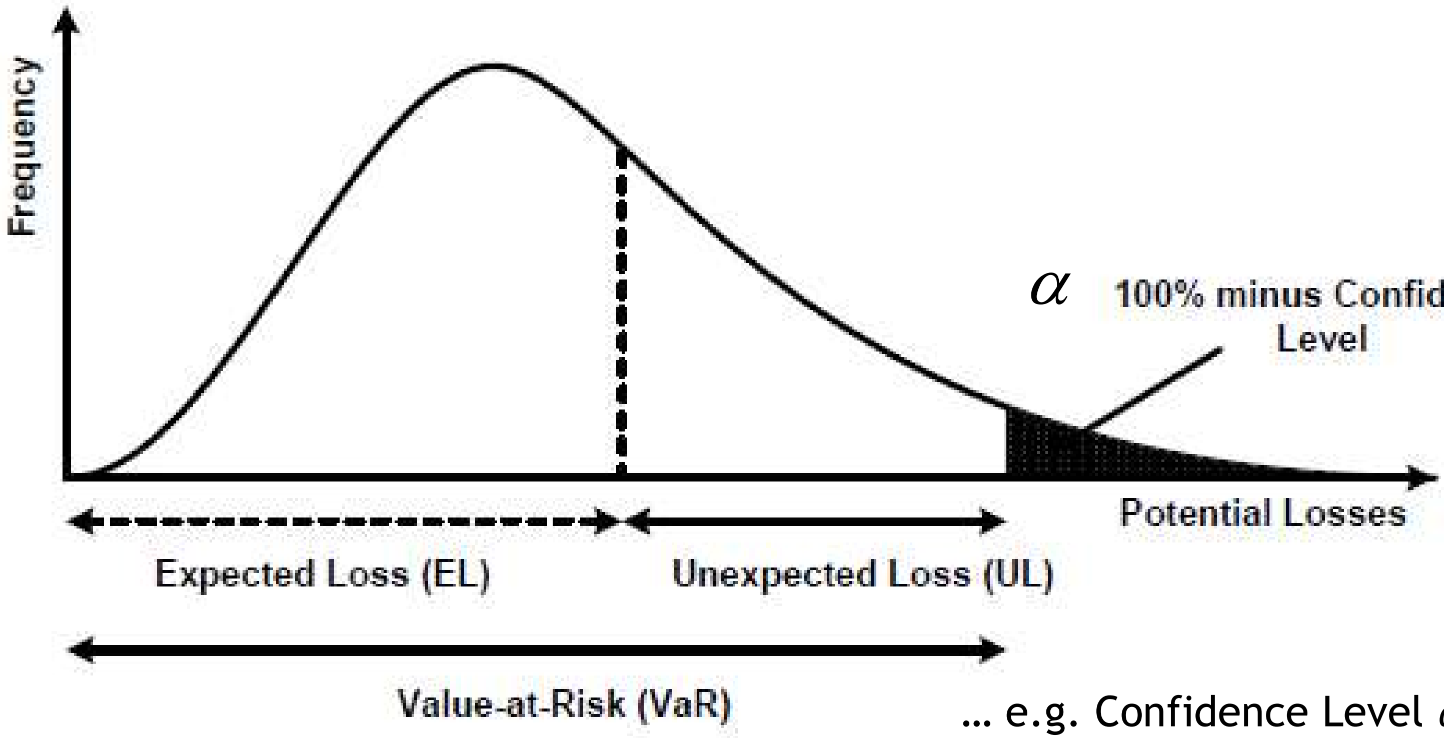
- ✓ adversely affect an organization's ability to achieve its objectives and execute its strategy;
- ✓ generate an unexpected loss or less-than-than-expected returns.

Some factors that have contributed to an increased demand for Financial Risk Management:

- ✓ 1970 abolition of the Bretton-Woods system of fixed exchange rates;
- ✓ worldwide deregulation in the 1980s;
- ✓ a market-oriented accounting practice (IASB –Intern. Accounting Standards Board-, FASB);
- ✓ the "new" regulatory RM framework of Basel and Solvency Accords.

The role of Capital and relation with P&L

Profit&Loss: the relevant variable in firm's "Income Statement"



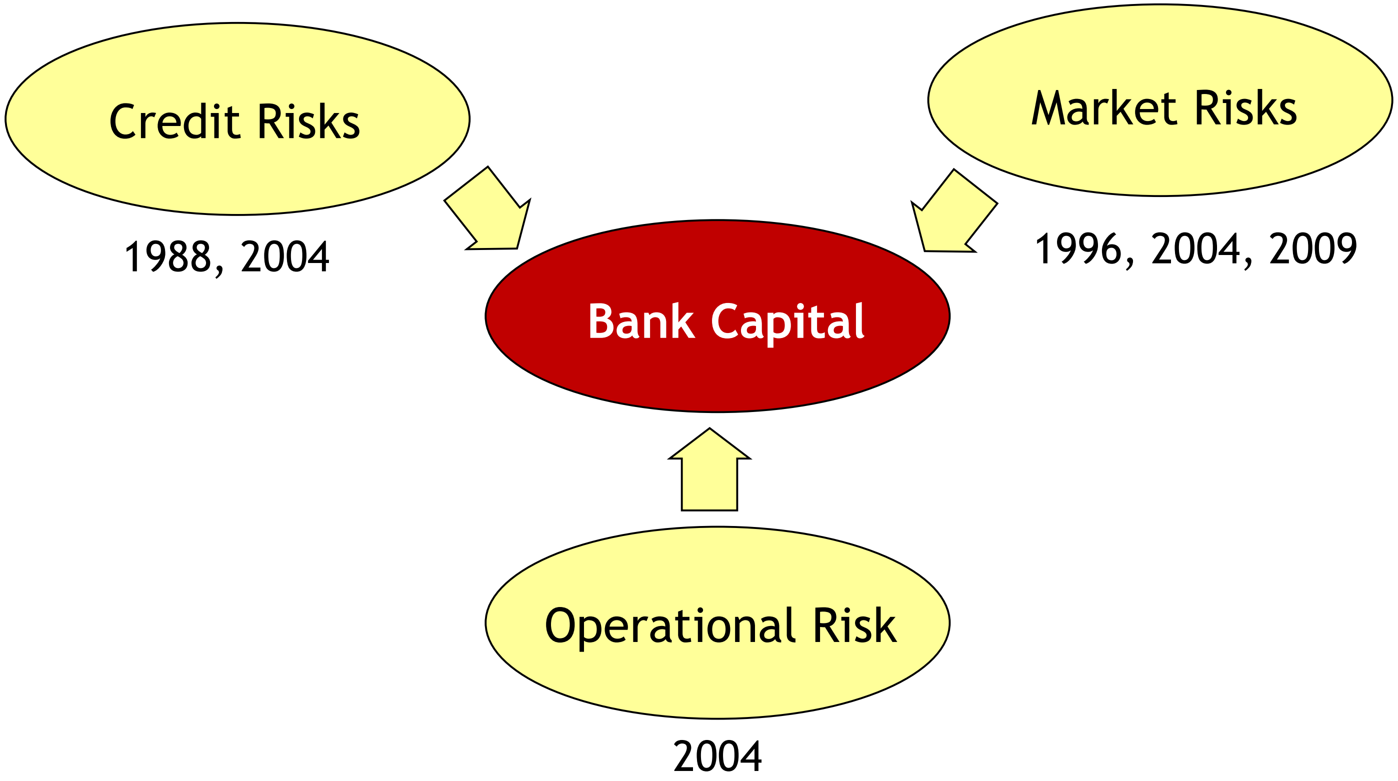
… e.g. Confidence Level α:
99% [Market Risk]
99.9% [Credit Risk]

(BIS 2005)

The framework originates from the Basel Commitee of Banking Supervision at BIS

Accord	Year	Brief description
Basel I	1988	International **minimum capital standard.** Main emphasis on Credit Risk.
Amendment of Basel I	1996	Standardized model for market risk
Basel II	2004	Pillar I: Risk-sensitive minimum capital requirements Pillar II: Supervisors review of banks' capital adequacy Pillar III: Strength market discipline enhancing transparency in banks' financial reporting.
Basel 2.5	2009	Incremental Risk Charge (IRC) Stressed VaR
Basel III	2010-2018	New capital requirements, Liquidity requirements

Bank's main risks



Capital in order to face 3 types of risks:

- ✓ Credit risk: Regulated first in Basel I (1998), reformed in Basel II (2004)
- ✓ Market Risks: Regulated in 1996, revised in Basel II (2004) and Basel 2.5 (2009)
- ✓ Operational Risk: Introduced in Basel II (2004)

Basel I:

✓ Credit risk (<u>only</u>): Risk weighted assets divided in four crude categories (governments, banks, secured loans and others).

$$RC = 8\% \sum_i RW_i \; A_i$$

Risk-weight	Brief description
RW_1 = 0%	Cash or cash equivalents Claims on Central Banks/Governments of OECD countries Governments bonds of OECD countries
RW_2 = 20%	Claims on Supranational banks Claims on public entities in OECD countries Claims on banks in OECD countries
RW_3 = 50%	Mortgages secured by residential properties
RW_4 = 100%	Claims on private sector Equity investments in private companies Claims on banks/Governments outside OECD Other fixed assets

Limits in Basel I Capital Accord:

- ✓ Credit risk only
- ✓ Same risk weight to private companies regardless of credit standing
- ✓ Portfolio diversification/maturity plays no role
- ✓ Hedging not considered (e.g. derivatives protection)

Basel II

- Regulatory capital: Risk-sensitive minimum capital for credit, market and operational risk.

- Internal model: Banks can choose between
  - ✓ standardized approaches or
  - ✓ Internal-Ratings-Based (IRB) approaches.

- Operational risk: Opens the way to the risk of losses resulting from inadequate or failed internal process.

Regulatory Capital (using Internal Model for Market Risk):

$$RC^t(MR) = \max\left\{VaR^{t,10}_{0.99},\, k\,\frac{1}{60}\sum_{i=1}^{60} VaR^{t-i,10}_{0.99}\right\} + C_{SR}$$

where $\begin{cases} k \in [3,4] & \text{is a stress factor} \\ C_{SR} & \text{An added capital requested for specific risk} \end{cases}$

Regulatory Capital (using IRB for Credit Risk) is s.t.:

$$RC_i(CR) \propto LGD_i \times (WCDR_i - PD_i)$$

where
- LGD_i is the Loss Given Default for the ith obligor in bank's Loan portfolio
- PD_i is 1y probability of default for the ith obligor
- $WCDR_i$ is the Worst Case Default 1y probability for the ith obligor

$$WCDR_i = p_i(\hat{y}) = N\left[\frac{N^{-1}(PD_i) + \rho_i\, N^{-1}(99.9\%)}{\sqrt{1-\rho_i^2}}\right]$$

i.e. $WCDR_i$ is the conditional probability $p_i(y)$ given a value $\hat{y}$ in the worst 99.9% case, obtained via the Asymptotic Single Risk Factor (Gordy 2003) model

(Gordy 2003, BIS 2005)

Credit Risk in IRB: Gordy model (I)

Asymptotic Single Risk Factor (ASRF or Gordy) model: $\left\{ \begin{array}{l} \text{inhomogeneous} \\ \text{Large portfolio} \end{array} \right.$

t_0 1y

obligor i defaults iff $\boxed{v_i \leq K_i}$ with v_i std normal r.v.

and $v_i = \rho_i \, y + \sqrt{1-\rho_i^2} \, \epsilon_i$

$$PD_i = N[K_i]$$

$y, \{\epsilon_i\}_{i=1,..I}$ std normal i.i.d. r.v.

Given y ➡ default probability given y :

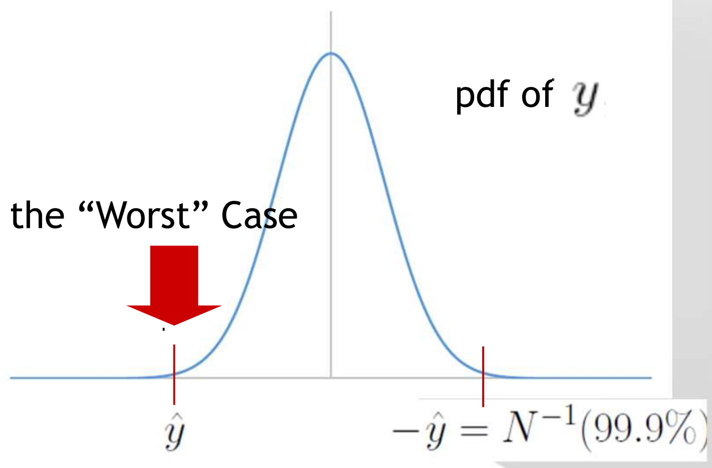
$$p_i(y) = N\left[\frac{N^{-1}(PD_i) - \rho_i \, y}{\sqrt{1-\rho_i^2}}\right]$$

The loss (due to credit) in bank's ptf

$$\begin{cases} L(y) = \sum_i EAD_i \times LGD_i \times p_i(y) \\ UL(y) = L(y) - EL \end{cases}$$

where EAD_i is the Exposure At Default for the ith obligor in bank's Loan portfolio

$$\boxed{RC = VaR_{\alpha=99.9\%}[UL] = UL(\hat{y})}$$



More precisely Regulatory Capital in IRB is:

$$RC = \sum_i EAD_i \times LGD_i \times (WCDR_i - PD_i) \times MA_i$$

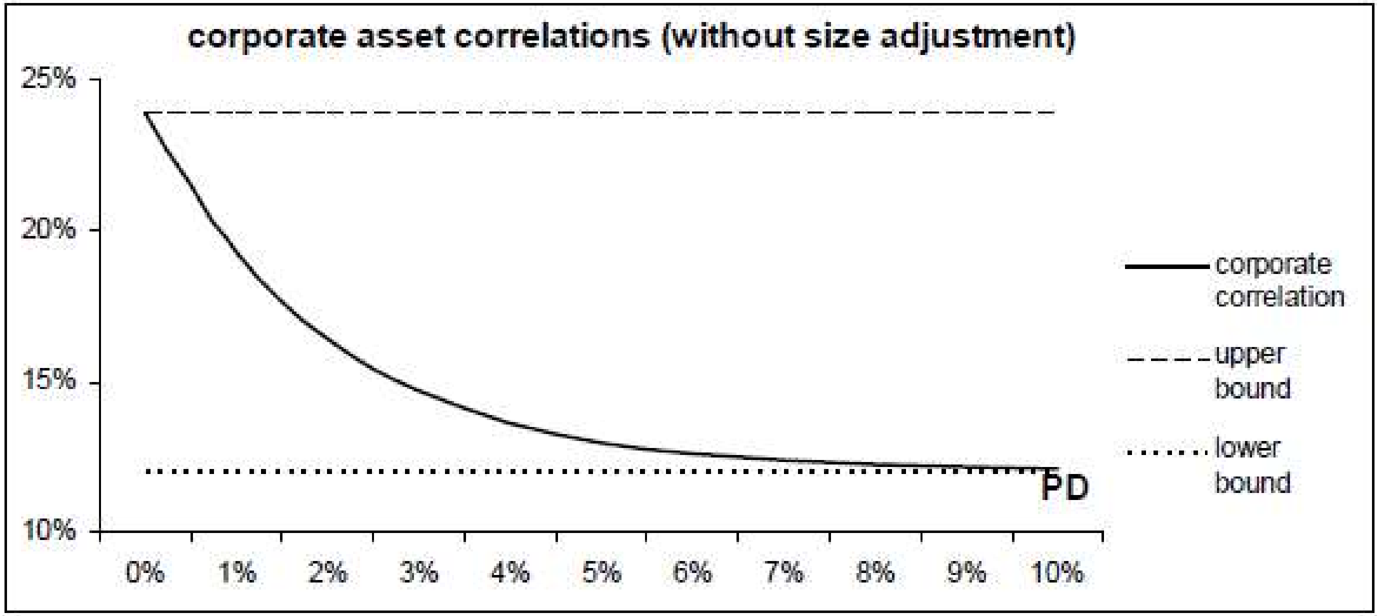
where

- EAD_i is the Exposure At Default for the ith obligor in bank's Loan portfolio (in case of a single loan, approximately loan's principal amount)
- MA_i Maturity adjustment, to correct for loans longer than 1 year
- ρ_i^2 Correlation is a specific function of PD_i

(BIS 2005)

(Hull RM2012, Ch 12 & 11.5)

In particular correlation is a specific function of PD_i , for corporates it is:



(BIS 2005, p.13)

Two different Capital Requirements (CRs) for Credit exposures

Credit CR based on

- 1y time horizon
- a confidence interval of 99.9%

Market CR based on

- 10dd time horizon (60dd average)
- a confidence interval of 99%

Regulatory Arbitrage

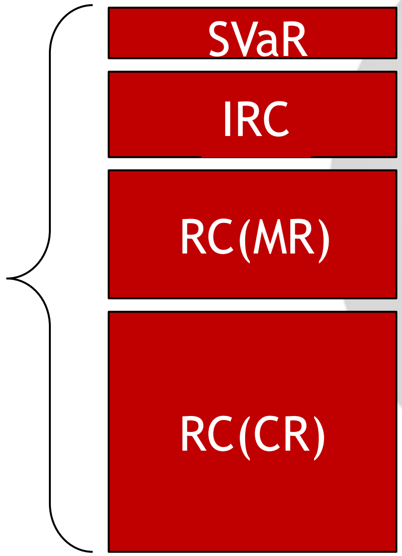
Consequence: banks built large exposures to credits via bonds and credit derivatives in trading portfolio, leveraging on the favourable regulatory treatment compared to the loan book

Two <u>additional</u> Components (effective in 2012 EU, in 2013 US):

- IRC Incremental Risk Charge
  [Default & migration for unsecuritized credit products, based on 1y horizon]

- SVaR Related to 1y observation related to significant losses (e.g. Lehman historical VaR)
  …*vs* last 1y for VaR

Total Capital Requirement:



Basel III requirements:

- More stringent (w.r.t. Basel II) capital requirements as a percentage of Risk Weighted Assets;
  - ✓

	Basel II	Basel III (2010)
Common Equity	2.0%	4.5%
Tier I Capital	4.0%	6.0%

- "additional capital buffers"
  - ✓ "mandatory capital conservation buffer" of 2.5%;
  - ✓ "additional capital buffer requirement" for Systemically Important Fin. Institutions (SIFIs)
  - ✓ "discretionary counter-cyclical buffer", which allows supervisors to require up to another 2.5% of capital during periods of high credit growth;
- strengthen risk coverage of the capital framework (Counterparty risk)
- "leverage ratio" (Tier I capital/Total Assets) larger than 3%;
- Two Liquidity Ratios (over lags of 30 days and 1 year).

(Hull RM2012; McNeil & Al. 2005, Ch 1)

A panacea?

- ✓ Cost factor of setting up a well-functioning RM system especially for smaller banks;
- ✓ RM hearding: similar rules and measures ⇒ similar behaviours and exit times in crises;
- ✓ Prociclical: capital requirements rise in times of recession and fall in times of expansion;
- ✓ Misspecification/Model risk.

Solvency I

Completed in 2002, came into force in 2004

Introduces an extra capital buffer against unforeseen events:

- ✓ higher than expected claims
- ✓ unfavorable investment results

Regulatory Capital:

$$RC = 4\% \cdot MR_{Traditional} + 1\% \cdot MR_{UnitLinked} + 0.3\% \cdot MortRiskCapital$$

Neglected issues:

- ✓ guarantees & options embedded in the policies;
- ✓ proper matching of assets and liabilities.

(McNeil & Al. 2005, Ch 1)

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1. Regulatory Risk Management framework
2. Basic Risk Management Concepts
   - ✓ Risk Management (RM) vs Risk measurement (Rm)
   - ✓ Rm approaches: parametric *vs* non parametric
   - ✓ VaR & ES: definitions & some basis examples
3. Gaussian (multinomial) parametric approach
4. Non parametric approaches & Derivatives portfolios
5. Other multinomial models
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Risk Management vs Risk measurement

Often the first step of Risk Management is risk control, that consists in the measurement of some risks for the (financial) institution of interest.

For example:

1. The amount of capital a bank needs to hold as a buffer against unexpected future losses on its assets' portfolio

2. Initial margin requirement from a clearing house

Approaches to Risk measurement (e.g. for a derivatives' portfolio):

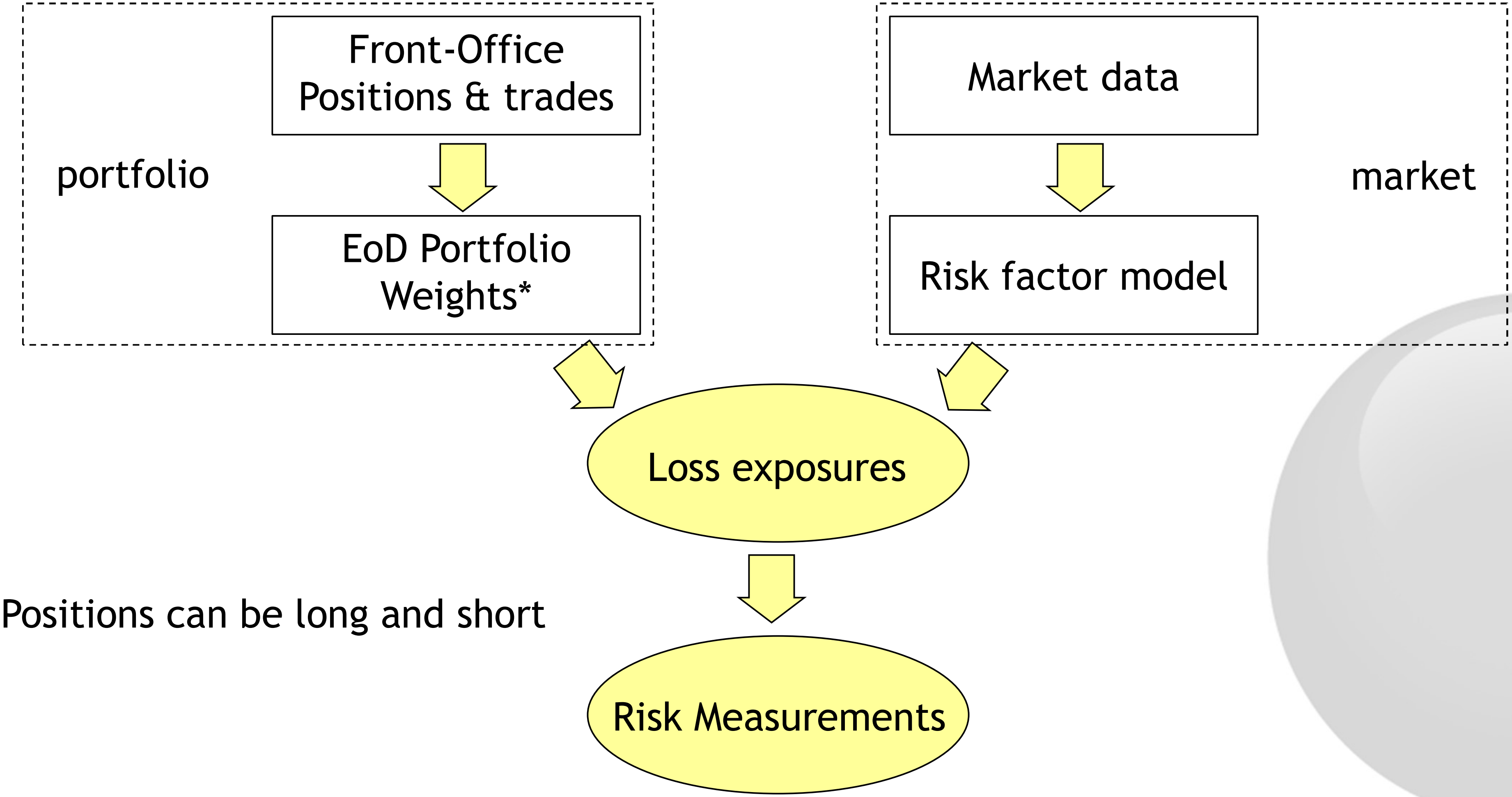
1. Add-on: the weighted sum of the notionals.
   The weight (add-on) is related to asset characteristics.

2. Factor-sensitive measures: Greeks

3. Risk measures based on loss distributions:

- ✓ VaR & ES
- ✓ Maximum loss in a scenario analysis (an approach *à la* Stress Test)

A simplified Risk measurement scheme for a linear portfolio (e.g. Equity Cash)



*Positions can be long and short

Under the "frozen portfolio" condition: i.e. portfolio is considered unchanged over the lag Δ

- Parametric
  - Statist. estimation of distribut. parameters
  - PCA
- Non parametric
  - HS
  - WHS
  - Statistical Bootstrap

P&L: Profit & Loss

$$\begin{cases} P \equiv & V(t+\Delta) - V(t) \\ L \equiv & -[V(t+\Delta) - V(t)] \end{cases}$$

where $V(t)$ is the value of a given portfolio at time t

Δ can be $\begin{cases} \text{1 d or 10d} & \text{for market risk} \\ \text{1y} & \text{for credit risk} \end{cases}$

and the portfolio is (implicitly) considered unchanged over le lag Δ

i.e. $L > 0$ is a loss, while $L < 0$ is a profit

Assumption considered:

The series of risk factor changes $\{X_{t,i}\}_{t\in\mathbb{N};\ i=1,\cdots,m}$ are assumed i.i.d. in time

$$l_\alpha = VaR_\alpha \equiv \inf_{l \in \Re}\{P(L > l) \le 1-\alpha\} = \inf_{l \in \Re}\{F(l) \ge \alpha\}$$

where

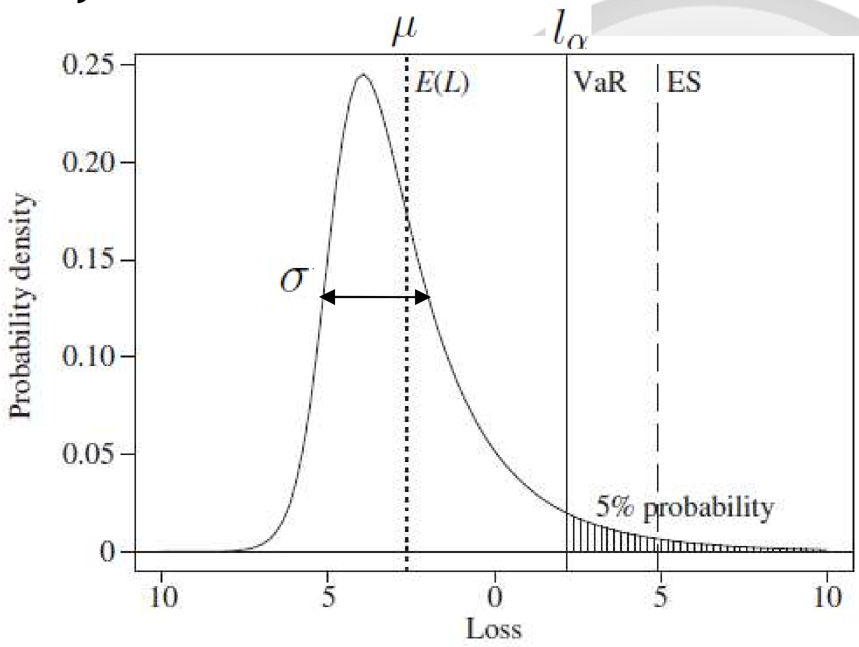
$F(l) = \mathcal{P}(L \le l)$ is the C.D.F. of the Loss

$\alpha \in (0,1)$ e.g. $\alpha = 95\%$ or $\alpha = 99\%$

Hist: Weatherstone 4.15 Report a one-day, one-page summary of JPM's market risk for bank's CEO

Oss: 1. VaR is a Loss (expressed in Euro, Usd, …)

2. VaR is the quantile of the loss distribution



$$ES_\alpha \equiv \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(L)du$$

Oss: 1. $ES_\alpha \ge VaR_\alpha$

2. Average in the tail of the Loss distribution

Main properties:

$$F(l_\alpha) = \alpha$$

and then…

$$\begin{cases} VaR_\alpha = \mu + \sigma VaR_\alpha^{std} \\ \\ ES_\alpha = \mu + \sigma ES_\alpha^{std} \end{cases}$$

Proof:

VaR: $$\mathcal{P}(L \leq l_\alpha) = \mathcal{P}\left(\frac{L-\mu}{\sigma} \leq \frac{l_\alpha - \mu}{\sigma} = l_\alpha^{std}\right)$$

ES: $$ES_\alpha \equiv \frac{1}{1-\alpha}\int_\alpha^1 VaR_u du = \mu + \sigma \frac{1}{1-\alpha}\int_\alpha^1 VaR_u^{std} du$$

Examples on VaR & ES: 1d normal & t-student

	density (std): $\phi(l)$	VaR_α	ES_α^{std}
normal:	$\frac{1}{\sqrt{2\pi}} e^{-l^2/2}$	$\mu + \sigma \mathcal{N}^{-1}(\alpha)$	$\frac{\phi(\mathcal{N}^{-1}(\alpha))}{1-\alpha}$
t-student:	$\underbrace{\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}}_{C_\nu} \left(1+\frac{l^2}{\nu}\right)^{-\frac{\nu+1}{2}}$	$\mu + \sigma t_\nu^{-1}(\alpha)$	$\frac{\nu + (t_\nu^{-1}(\alpha))^2}{\nu - 1} \frac{\phi_\nu(t_\nu^{-1}(\alpha))}{1-\alpha}$

where $\Gamma(\nu) = (\nu-1)\Gamma(\nu-1)$

e.g. for ν even: $\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} = \frac{(\nu-1)(\nu-3)\cdots 5\cdot 3}{2\sqrt{\nu}(\nu-2)(\nu-4)\cdots 4\cdot 2}$

(McNeil & Al. 2005, Ch 2)

Proof: [Hint for t-student ES std]

$$ES_\alpha^{std} = \frac{1}{1-\alpha}\int_{l_\alpha^{std}}^{\infty} l\ \phi_\nu(l)\ dl \quad \text{with} \quad l\ \phi_\nu(l) = -\frac{\nu}{\nu-1}\frac{\partial}{\partial l} C_\nu \left(1+\frac{l^2}{\nu}\right)^{-\frac{\nu-1}{2}}$$

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1. Regulatory Risk Management framework
2. Basic Risk Management Concepts
3. Gaussian (multinomial) parametric approach
   - ✓ Multivariate normal: basic properties, parameter estimations & simulation (Cholesky)
   - ✓ Variance-Covariance method & Scaling
   - ✓ PCA
4. Non parametric approaches & Derivatives portfolios
5. Other multinomial models
6. Coherent Risk Measures
7. Backtesting VaR

The simplest parametric approach: Multinomial Gaussian distribution

The joint multivariate Gaussian density is

$$f(\underline{x}) \equiv \frac{1}{(2\pi)^{d/2}\sqrt{det(\Sigma)}} \exp\left\{-\frac{1}{2}(\underline{x}-\underline{\mu})\cdot\Sigma^{-1}(\underline{x}-\underline{\mu})\right\} \qquad \underline{x} \in \Re^d$$

where $\underline{\mu}$ is the drift and the <u>non singular</u> matrix $\Sigma \in \Re^{dxd}$ is the variance-covariance matrix.

Characteristic function:

$$\phi(\underline{t}) \equiv E\left[e^{i\,\underline{t}\cdot\underline{x}}\right] = \exp\left\{i\,\underline{t}\cdot\underline{\mu} - \frac{\underline{t}\cdot\Sigma\underline{t}}{2}\right\}$$

Cholesky factorization is the only lower triangular matrix A s.t. $AA' = \Sigma$, $A \equiv \Sigma^{1/2}$

A simple way to generate Gaussian r.vs with mean $\underline{\mu}$ and variance Σ:

- ✓ Generate a (column) vector $\underline{y}$ of Gaussian r.vs i.i.d.
- ✓ $\underline{x} = \underline{\mu} + \Sigma^{1/2}\underline{y}$

Proof of characteristic function (using Cholesky decomposition): [Hint]

$$\exp\left\{-\frac{\underline{y}\cdot\underline{y}}{2} - i\underline{t}\cdot A\underline{y} + \frac{\underline{t}\cdot\Sigma\underline{t}}{2} - \frac{\underline{t}\cdot\Sigma\underline{t}}{2}\right\} = \exp\left\{-\frac{(\underline{y}+i\,A'\underline{t})\cdot(\underline{y}+i\,A'\underline{t})}{2} - \frac{\underline{t}\cdot\Sigma\underline{t}}{2}\right\}$$

Remark: chol in Matlab returns A transpose (an upper triangular matrix)

Given a set of n of i.i.d. observations of a d-dimensional Gaussian risk-factor vector

$$\{\underline{X}_t\}_{t=1,\cdots,n}$$

The unbiased estimators are

$$\begin{cases} \overline{\underline{X}} \equiv \dfrac{1}{n}\displaystyle\sum_{t=1}^{n} \underline{X}_t \\ \mathbf{S} \equiv \dfrac{1}{n-1}\displaystyle\sum_{t=1}^{n} (\underline{X}_t - \overline{\underline{X}})(\underline{X}_t - \overline{\underline{X}})' \end{cases}$$

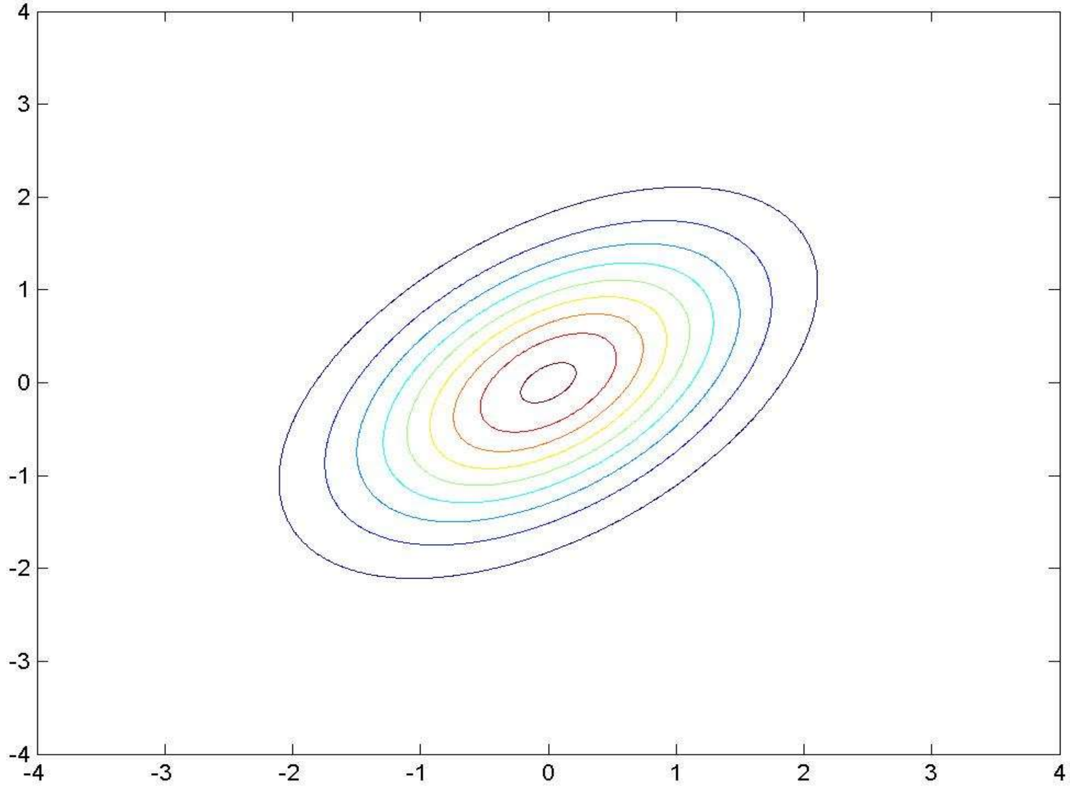
Proof: [Hint]

$$E[\mathbf{S}] = \frac{1}{n-1} E\left[\sum_{t=1}^{n} (\underline{X}_t - \underline{\mu})(\underline{X}_t - \underline{\mu})' - n(\overline{\underline{X}} - \underline{\mu})(\overline{\underline{X}} - \underline{\mu})'\right]$$

(McNeil & Al., Ch 3.1)

A natural extension of multivariate Gaussian distributions are Elliptic distributions.

These distributions are characterized by contour plots of the following type:



A typical example are multivariate t-Student distributions, for which similar properties hold.

Variance-Covariance (VC) method

The linearized loss is: $$L(\underline{X}_t) = -V_t(c_t + \underline{w}_t \cdot \underline{X}_t)$$

e.g. Equity ptf: $n_{i,t}$ number of shares in the ith asset and $V_t = \sum_i n_{i,t} S_i(t)$ pft value, then

$$w_{i,t} = n_{i,t} S_i(t)/V_t$$

If the returns are Gaussian

$$\frac{L(\underline{X}_t)}{V_t} \sim \mathcal{N}\left[-(c_t + \underline{w}_t \cdot \underline{\mu}), \underline{w}_t \cdot \Sigma \underline{w}_t\right]$$

<u>Contribution</u> to VaR & ES for elliptic distributions:

$$\begin{cases} CVaR_i \equiv w_i \beta_i \, VaR \\ CES_i \equiv w_i \beta_i \, ES \end{cases}$$

where the weights are:

$$\beta = \frac{\Sigma \underline{w}(t)}{\underline{w}(t) \cdot \Sigma \underline{w}(t)}$$

Scaling: Losses over time lags

Generally we are interested in risk measures over a time lag Δ having daily iid r.v.

In practice it used the so called "squared-root-of-time scaling rule" for Gaussian i.i.d. returns, i.e.

$$\boxed{\underline{\mu} \rightarrow \Delta \underline{\mu} \qquad \Sigma \rightarrow \Delta \Sigma}$$

e.g. in the 1-d case the risk measurements become

$$\begin{cases} VaR_\alpha = \Delta\mu + \sqrt{\Delta}\sigma VaR_\alpha^{std} \\ ES_\alpha = \Delta\mu + \sqrt{\Delta}\sigma ES_\alpha^{std} \end{cases}$$

Idea:

$$\underline{X}_t \sim \mathcal{N}_d(\underline{\mu}, \Sigma) \quad \Rightarrow \quad \sum_{i=1}^{\Delta} \underline{X}_{t+i} \sim \mathcal{N}_d(\Delta\underline{\mu}, \Delta\Sigma)$$

Principal Component Analysis (PCA)

Aim: reduce the dimensionality of the problem

Ordered Eigenvalues: $\Lambda = diag(\lambda_1, \cdots, \lambda_d)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_d > 0$

$$\Sigma = \Gamma \Lambda \Gamma'$$

Eigenvectors: $\Gamma = \left(\underline{\gamma}_1 \ \underline{\gamma}_2 \ \cdots \underline{\gamma}_d \right)$ s.t. $\Gamma'\Gamma = \Gamma\Gamma' = I_d$

Factor Model: Principals as factors

$$\underline{x} = \underline{\mu} + \Gamma \, \Lambda^{1/2} \, \underline{y} = \underline{\mu} + \sum_{i=1}^{k} \sqrt{\lambda_i} \, \underline{\gamma}_i \, y_i + \epsilon$$

with ϵ an idiosyncratic error term

Defining:

$$\underline{\hat{\mu}} := \Gamma' \, \underline{\mu}$$

The projected portfolio on Principals is:

$$\underline{\hat{\omega}} = \Gamma' \, \underline{\omega}$$

Original portfolio:

$$\begin{cases} \sigma^2_{ptf} &= \underline{\omega} \cdot \Sigma \underline{\omega} = \underline{\hat{\omega}} \cdot \Lambda \underline{\hat{\omega}} = \sum_{i=1}^{d} \hat{\omega}_i^2 \lambda_i \\ \mu_{ptf} &= \underline{\mu} \cdot \underline{\omega} = \underline{\hat{\mu}} \cdot \underline{\hat{\omega}} = \sum_{i=1}^{d} \hat{\omega}_i \hat{\mu}_i \end{cases}$$

$\Rightarrow$

Reduced form portfolio:

$$\begin{cases} \sigma^2_{red} &= \sum_{i=1}^{k} \hat{\omega}_i^2 \lambda_i \\ \mu_{red} &= \sum_{i=1}^{k} \hat{\omega}_i \hat{\mu}_i \end{cases}$$

Remark: If assets volatilities differ significantly, PCA is applied to correlation matrices

(McNeil, Ch 3.4)

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1. Regulatory Risk Management framework
2. Basic Risk Management Concepts
3. Gaussian (multinomial) parametric approach
4. Non parametric approaches & Derivatives portfolios
   - ✓ Historical Simulation, Bootstrap & WHS
   - ✓ Plausibility Check
   - ✓ Full MonteCarlo for a derivatives portfolio
   - ✓ Delta normal & Delta-Gamma approaches
5. Other multinomial models
6. Coherent Risk Measures
7. Backtesting VaR

Returns are not Gaussian and then… let we use the realized returns

1. Value losses under the empirical set of realized values for factors

$$L_s = \{L(\underline{X}_s)\}_{s=t-n+1,\cdots,t}$$

2. Consider the (decreasing) ordered sequence $L^{(i,n)}$: $L^{(n,n)} \leq \cdots \leq L^{(1,n)}$

3. Risk mesurements

VaR corresponds to: $VaR_\alpha = L^{([n(1-\alpha)],n)}$

ES corresponds to: $ES_\alpha = \text{mean}\left\{L^{(i,n)}, i = [n(1-\alpha)], \cdots, 1\right\}$

where $[n(1-\alpha)]$ is the largest integer not exceeding $n(1-\alpha)$

(Ferguson, 1996)

Remark:

1. HS preserves dependences among factors
2. The technique can be applied to an interval using the scaling factor

Historical Simulation (2)

Example: n = 500 (2y) and $\alpha = 99\%$

- VaR corresponds to the 5th highest value;
- ES corresponds to the average of the 5 largest values.

Main advantages:

- No statistical estimation of the parameters;
- No assumptions about the dependence structure of risk factors.

Improvements:

If tails data are not enough, tail estimation can be improved formulating extreme scenarios

Weighted Historical Simulation (WHS)

In WHS some weights decreasing in the past are associated to the loss sequence, i.e. we consider the sequence

$$\{w_s\}_{s=t-n+1,\cdots,t}$$ instead of considering uniform weights

where:

$$w_s = C\lambda^{t-s}$$

with 0 < λ < 1 (typically within 0.95,0.99) an C a normalization factor s.t. $C = \frac{1-\lambda}{1-\lambda^n}$

After having ordered losses' sequence risk measurements as in the HS case

- $VaR_\alpha = L^{(i^*,n)}$ and i^* the largest value s.t. $\sum_{i=1}^{i^*} w_i \leq 1-\alpha$
- $ES_\alpha = \frac{\sum_{i=1}^{i^*} w_i L^{(i,n)}}{\sum_{i=1}^{i^*} w_i}$

Remark: WHS is more responsive to recent conditions but generate more volatile VaRs.

(Statistical – non parametric) Bootstrap

Random sampling with replacement from the original dataset (e.g. composed of n sets of returns).

1. Sample an integer number between 1 and n, M times;

2. Select the corresponding set of returns;

3. Value the loss distribution on the set and risk measures as in the HS case;

4. Adjust with the scaling factor.

Remark:
The method preserves dependences among factors since it is applied at the same time to all factors.

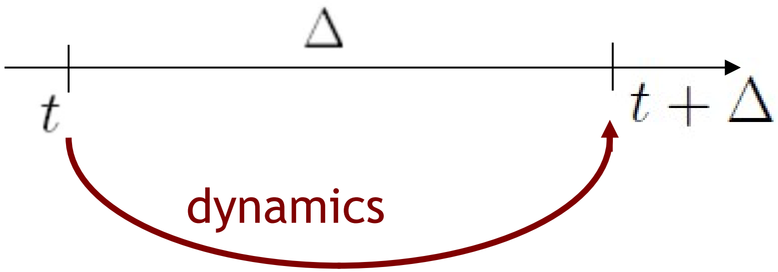
In presence of a derivative portfolio (often a non-linear derivative portfolio), the loss is valued via a Monte Carlo approach with starting value in t

$$L^{der}(\underline{X}_t;\Delta) = -\sum_i \left[\mathcal{C}_i(t+\Delta) - \mathcal{C}_i(t)\right]$$

where $\mathcal{C}_i(t)$ is the value in t of the ith derivative.

Then Loss distribution can be obtained via a Monte Carlo approach considering a simulated set of values for the risk factors.

Full-Valuation Molte-Carlo method: example



Let us consider the case where the derivatives' portfolio is just composed by one Call

In a simple model the underlying value in $t+\Delta$ can be seen as the value in t multiplied by the exponential return between t and $t+\Delta$

$$S_{t+\Delta} = S_t e^{X_t}$$

Option value should be computed just considering the new starting value and updating the ttm (that is a Δ shorter).

One obtains the Loss distribution simulating several values the returns up to $t+\Delta$.

Delta-normal method

For a derivatives portfolio an expansion up to the first order Loss derivatives:

$$L(\underline{X}_t) = -\sum_{i=1}^{d} \mathrm{sens}_i(t) X_{t;i}$$

where $\mathrm{sens}_i(t)$ is portfolio sensitivity (in dollar terms) w.r.t. ith risk factor.

Then Loss distribution can be obtained either via an analytic approach or via a “simulation” (e.g. HS, WHS, Bootstrap) approach.

Remark on the name (Delta-normal):
Delta means just the first order expansion in a Greek (same arguments can be used for Vega)

Delta-Gamma (Monte-Carlo) method

For a derivatives' portfolio derivatives loss can be also obtained via expansion in its Greeks:

$$L\left(\underline{X}_t\right) = -\left\{\sum_{i=1}^{d} \mathrm{sens_i}(t) X_{t;i} + \frac{1}{2}\sum_{i=1}^{d} \gamma_{\mathrm{ij}}(t)\, X_{t;i} X_{t;j}\right\}$$

where $\gamma_{\mathrm{ij}}(t)$ is (portfolio) cross-gamma (in dollar terms).

Then, Loss distribution can be obtained via a Monte Carlo (e.g. HS, WHS, Bootstrap) approach.

Remark:
In the case of null cross Gamma case (for $i \neq j$),
an analytical formula for VaR could be obtained via the Cornish-Fisher approximation.

Plausibility Check

Sometimes one needs just an estimation of the order of magnitude of portfolio VaR.
In this case:

1. Value risk-factors correlation C and lower/upper percentiles

$$l_i \equiv VaR_{X_i}(1-\alpha) \qquad\qquad u_i \equiv VaR_{X_i}(\alpha)$$

2. Compute signed-VaR for each risk factor

$$sVaR_i = \text{sens}_i \frac{|l_i| + |u_i|}{2}$$

3. Portfolio VaR can be estimated by

$$\boxed{VaR^{pft} = \sqrt{sVaR \cdot C\, sVaR}}$$

Remark:
If returns are driftless & Gaussian in a linear portfolio the described approach is equivalent to the Analytical valuation via a Variance-Covariance method

Outline

1. Regulatory Risk Management framework
2. Basic Risk Management Concepts
3. Gaussian (multinomial) parametric approach
4. Non parametric approaches & Derivatives portfolios
5. Other multinomial models
   - ✓ Normal Mixture & Normal Mean-Variance Mixture distributions
   - ✓ Spherical & Elliptical distributions
6. Coherent Risk Measures
7. Backtesting VaR

Normal Mixture (NM, also called Normal Variance Mixture)

model:

$$\underline{x} = \underline{\mu} + \sqrt{W} A \underline{y}$$

where: $\begin{cases} A = \Sigma^{1/2} \in \Re^{dxd} & \text{s.t. } AA' = \Sigma \text{ : dispersion matrix} \\ \underline{y} \sim N_d(0, I_d) \\ W \geq 0 & \text{r.v. called mixing variable ind. from } \underline{y} \text{ with } E[W] < +\infty \end{cases}$

Remarks:

- $\underline{x}|W = w \sim N_d(\underline{\mu}, w\Sigma)$
- $E[\underline{x}] = \underline{\mu}$
- $cov[\underline{x}] = E[W]\Sigma$
- $\phi(\underline{t}) \equiv E[e^{i\, \underline{t}\cdot\underline{x}}] = e^{i\, \underline{t}\cdot\underline{\mu}}\, \mathcal{L}\left[\underline{t}\cdot\Sigma\underline{t}/2\right]$ with Laplace transform $\mathcal{L}[s] \equiv \int_0^\infty dW\, f(W) e^{-W\, s}$

Normal Mixture: Properties & example

Lemma (in the bivariate case):

$x_1, \; x_2$ uncorrelated n. r.v.s
if W is not a.s. a constant $\Rightarrow$ $x_1, \; x_2$ not independent

Proof: [Hint] (in the driftless case, with var-covar matrix equal to the identity matrix)

- $E[x_1 x_2] = E[W y_1 y_2] = 0$
- $E[|x_1| \; |x_2|] = E[W|y_1||y_2|] \overset{\text{Jensen ineq.}}{\geq} E[\sqrt{W}]^2 E[|y_1||y_2|] = E[|x_1|]E[|x_2|]$

  where the equality is true iff W is a.s. a constant

Example: t-Student

$W \sim$ Inverse gamma (Ig), for $\beta = \dfrac{1}{2}$ and $\alpha = \dfrac{\nu}{2}$ with ν the number of d.o.f.

$$f(W; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} W^{-\alpha-1} \exp\left(-\frac{\beta}{W}\right) \qquad W > 0$$

model:

$$\underline{x} = \underline{\mu} + W\,\underline{\gamma} + \sqrt{W}A\underline{y}$$

where in addition to normal mixture parameters we have introduced the constant vector $\underline{\gamma}$ and $W \geq 0$ r.v. called mixing variable i. from $\underline{y}$ with $VaR[W] < +\infty$

Remarks:

- $\underline{x}|W = w \sim N_d(\underline{\mu} + w\underline{\gamma}, w\Sigma)$
- $E[\underline{x}] = \underline{\mu} + E[W]\underline{\gamma}$
- $cov[\underline{x}] = Var(W)\,\underline{\gamma}\,\underline{\gamma}' + E[W]\Sigma$
- $\phi(\underline{t}) = e^{i\,\underline{t}\cdot\underline{\mu}}\,\mathcal{L}\left[-i\,\underline{t}\cdot\underline{\gamma} + \underline{t}\cdot\Sigma\underline{t}/2\right]$

Normal Mean-Variance-Mixture (MVM) example

Generalized Inverse Gaussian:

$W \sim$ GIG

$$\boxed{f(W;p,a,b) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} W^{(p-1)} e^{-(aW+b/W)/2}, \qquad W > 0,}$$

where $a, b \in \Re^+, p \in \Re$ and $K_p(\lambda)$ is a Bessel function of the second kind, i.e.

$$K_p(\lambda) \equiv \frac{1}{2}\int_0^{+\infty} W^{p-1} e^{-\lambda(W+1/W)/2} dW$$

<u>special cases</u>

W	condition	normal MVM
➢ IG:	$p = -\frac{1}{2}$	NIG
➢ gamma:	" $b = 0$ "	VG
➢ Ig:	" $a = 0$ "	skewed t

Definition: $\underline{x} \overset{\text{law}}{=} U\underline{x} \qquad \forall U \in \Re^{dxd} \text{ s.t. } U'U = UU' = I_d$

i.e. spherical random vectors are distributionally invariant under “rotation”

Theorem: spherical distributions properties (McNeil, Th 3.19)

1. $\underline{x}$ is spherical

2. $\phi_{\underline{x}}(\underline{t}) \equiv E\left[e^{i\underline{t}\cdot\underline{x}}\right] = \psi(||\underline{t}||^2)$ with ψ a scalar function

3. $a'\underline{x} \overset{\text{law}}{=} ||a||x_1 \qquad \forall a \in \Re^d$

Example: Normal Mixture with $\Sigma = I_d; \underline{\mu} = 0$ (via characteristic function…)

Elliptical distributions

$\underline{x} = \underline{\mu} + A\underline{y}$ with $\underline{y} \in \Re^d$ a Spherical distribution

$$\begin{cases} \underline{\mu} \in \Re^d \\ AA' = \Sigma \in \Re^{d\mathrm{x}d} \quad \text{positive definite} \end{cases}$$

i.e. Elliptical distributions: multivariate affine transform of Spherical distributions

Characteristic function: $\phi_{\underline{x}}(\underline{t}) = e^{i\underline{t}\cdot\underline{\mu}}\,\psi(\underline{t}\cdot\Sigma\underline{t})$

Example: Normal Mixture

(McNeil, Ch 3)

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   - ✓ Is VaR a coherent risk measure?
   - ✓ Is ES a coherent risk measure?
   - ✓ Capital Allocation & Euler Principle
7. Backtesting VaR

Definition: Convex cone $\mathcal{M}$ s.t. rvs $L_1, L_2 \in \mathcal{M}$ $\Rightarrow$ $\begin{cases} L_1 + L_2 \in \mathcal{M} \\ \lambda L_1 \in \mathcal{M}, \forall \lambda \in \Re^+ \end{cases}$

where $\Re^+$ is the set on non-negative real numbers.

Axioms for a Coherent measure of risk $\rho : \mathcal{M} \to \Re$:

1. translation invariance $\rho(L + l) = \rho(L) + l \qquad \forall L \in \mathcal{M},\ l \in \Re$
2. subadditivity $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2) \qquad \forall L_1, L_2 \in \mathcal{M}$
3. positive homogeneity $\rho(\lambda\, L) = \lambda \rho(L) \qquad \forall \lambda \in \Re^+$
4. monotonicity $L_1 \leq L_2 \text{ a.s. } \forall L_1, L_2 \in \mathcal{M} \Rightarrow \rho(L_1) \leq \rho(L_2)$

Remarks:

1. Given 2 & 3,
   Axiom 4 $\Leftrightarrow$ $\{L \leq 0 \text{ a.s. } \forall L \in \mathcal{M} \Rightarrow \rho(L) \leq 0\}$
2. Convexity axiom holds (relaxes axioms 2 & 3) $\Rightarrow$ Convex measure of risk (liquidity…)

VaR is a Coherent measures of risk?

VaR Quantile representation ⇒ axioms 1. 3. 4.

Theorem: VaR subadditivity for elliptical risk factors and linear portfolio (McNeil, Th 6.8)

$$L_1, L_2 \in \mathcal{M}, \; 0.5 \leq \alpha \leq 1 \quad \Rightarrow \quad VaR_\alpha(L_1 + L_2) \leq VaR_\alpha(L_1) + VaR_\alpha(L_2)$$

with $\mathcal{M} = \left\{ L : L = \lambda_0 + \sum_{i=1}^{d} \lambda_i x_i, \; \lambda_i \in \Re \right\}$

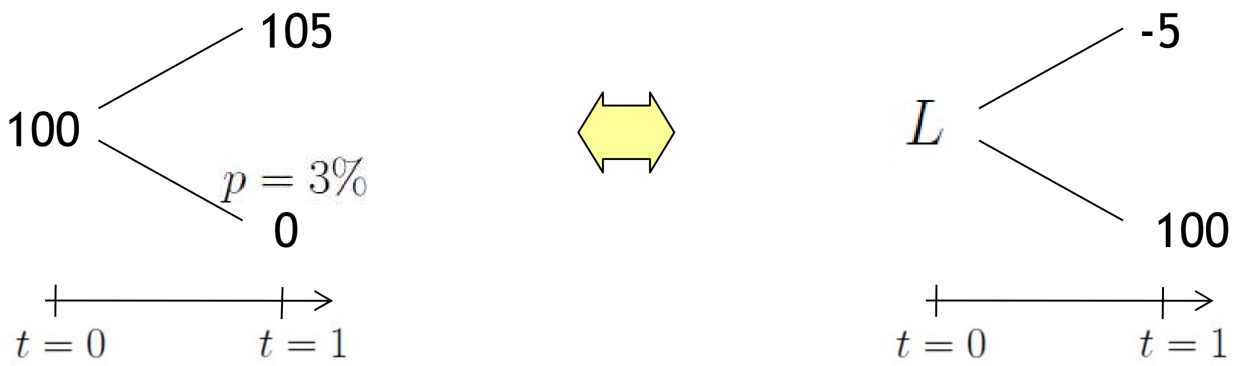
$\underline{x} = \underline{\mu} + A\underline{y}$ with $\underline{y} \in \Re^d$ a Spherical distribution

Proof: [Hint]

- $L = \lambda_0 + \underline{\lambda}'\underline{\mu} + \underline{\lambda}'A\underline{y} \overset{\text{law}}{=} \lambda_0 + \underline{\lambda}'\underline{\mu} + ||\underline{\lambda}'A|| \; y_1$
- $VaR_\alpha(L) = EL + ||\underline{\lambda}'A|| \; VaR_\alpha(y_1)$
- $VaR_\alpha(y_1) \geq 0$ and triangular inequality

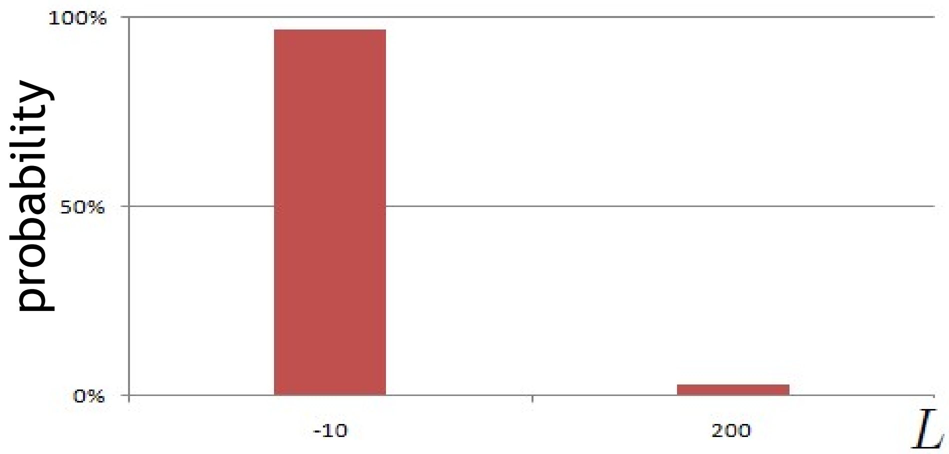
Counterexample: VaR not-coherent risk measure

assets: 2 independent defaultable bonds (B_i), discount factors equal to 1, zero recovery

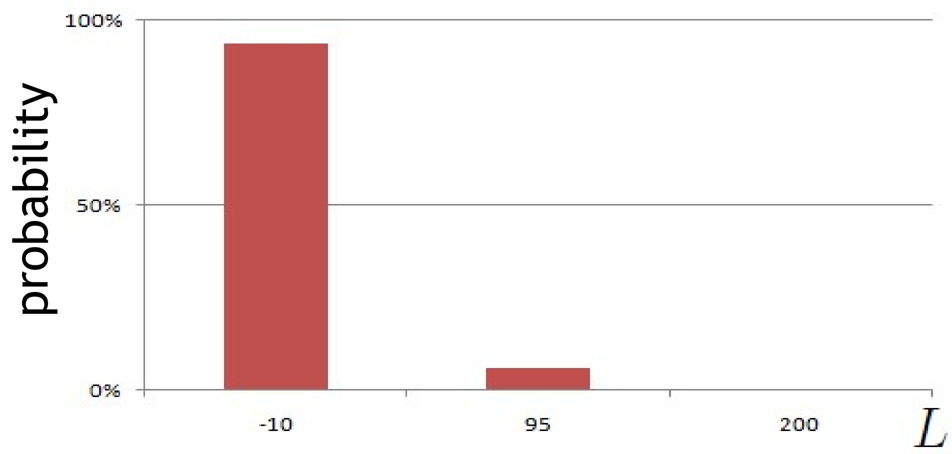


2 portfolios:

undiversified: 200 invested in the first bond



diversified: 100 invested in each bond



Example VaR not-coherent risk measure

undiversified (1)

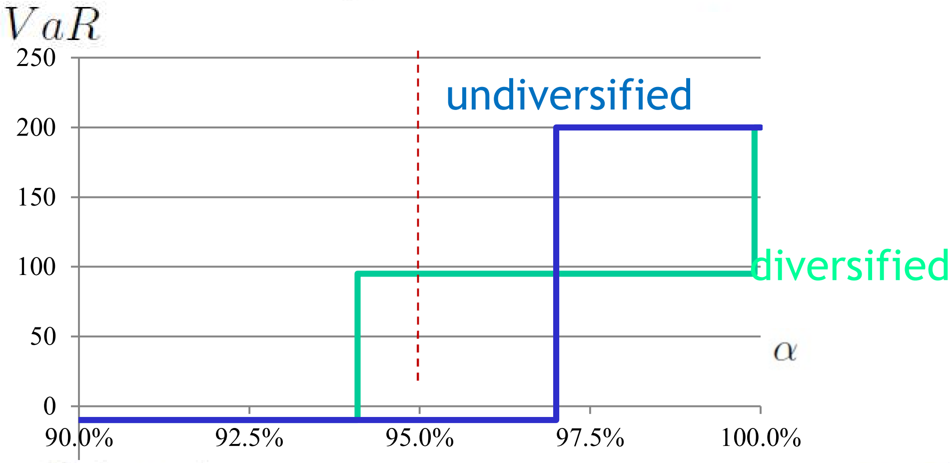
$$\begin{cases} \mathcal{P}(L > l) &= 100\%, l < -10 \\ \mathcal{P}(L > -10) &= 3\% \\ \mathcal{P}(L > 200) &= 0\% \end{cases}$$

diversified (2)

$$\begin{cases} \mathcal{P}(L > l) &= 100\%, l < -10 \\ \mathcal{P}(L > -10) &= 5.82\% \\ \mathcal{P}(L > 95) &= 0.09\% \\ \mathcal{P}(L > 200) &= 0\% \end{cases}$$

$VaR(\alpha = 95\%)$: the lowest loss s.t. probability $\leq 1 - \alpha = 5\%$

$$VaR_1 \quad < \quad VaR_2$$



$$ES_1(95\%) = \frac{3\% \cdot 200 + 2\%(-10)}{5\%} = 116.0 \quad > \quad ES_2(95\%) = \frac{0.09\% \cdot 200 + 4.91\%\ 95}{5\%} = 96.9$$

Theorem: ES is a coherent risk measure (Th. 6.9 McNeil)

Proof: [Hint]

- ES representation & axioms 1. 3. 4. for VaR $\Rightarrow$ axioms 1. 3. 4. for ES
- ES sub-additivity

$$\forall n, m \in \mathbb{N} \text{ s.t. } 1 \le m \le n \,; \quad \sum_{i=1}^{m} L^{(i,n)} = \sup_{\{i_j\}} \{ L_{i_1} + \cdots + L_{i_m} : 1 \le i_1, \cdots, i_m \le n \}$$

(ordered) (sampling)

$$\sum_{i=1}^{m} (L + \tilde{L})^{(i,n)} = \sup_{\{i_j\}} \left\{ (L_{i_1} + \tilde{L}_{i_1}) + \cdots + (L_{i_m} + \tilde{L}_{i_m}) \right\}$$

$$\le \sup_{\{i_j\}} \{ L_{i_1} + \cdots + L_{i_m} \} + \sup_{\{i_j\}} \{ \tilde{L}_{i_1} + \cdots + \tilde{L}_{i_m} \}$$

we can normalize for m and then we choose $m = [n(1-\alpha)]$

Contribution to a Risk Measure (RM): decomposition of the RM to each i_{th} risk factor...

$$\boxed{\rho(L) = \underline{\lambda}'\pi^\rho(\underline{\lambda})}$$

...and then decomposition of the RC

Possible for a positive homogeneous (ph) risk measure (e.g. VaR & ES).

Principle: Euler capital allocation principle

If $\rho(L)$ ph & cont. differentiable $\xRightarrow{\text{Euler Property}}$ $\rho(L) = \sum_{i=1}^{d} \lambda_i \frac{\partial}{\partial \lambda_i}\rho(L) \Leftrightarrow \pi^\rho(\underline{\lambda}) = \nabla_\lambda \rho(L)$

Corollary: For risk factors described by elliptic distributions (& linear pft): $\pi^\rho(\underline{\lambda}) = \frac{\Sigma\underline{\lambda}}{\underline{\lambda}'\Sigma\underline{\lambda}}\rho(UL)$

Proof: [Hint]

$$L = EL + UL = \lambda_0 + \underline{\lambda}'\underline{x} \quad \text{with} \quad \begin{cases} EL = \lambda_0 + \underline{\lambda}'\underline{\mu} \\ UL = \underline{\lambda}' A \underline{y} \end{cases}$$

$$UL \overset{\text{law}}{=} ||\underline{\lambda}' A||\, y_1 \Rightarrow \rho(UL) = \sqrt{\underline{\lambda}'\Sigma\underline{\lambda}}\,\rho(y_1)$$

(see Th 6.8)

... Contribution to VaR & ES

(McNeil, Ch 6)

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   - ✓ Basel requirements
   - ✓ Conditional coverage test
   - ✓ Unconditional coverage test

Evaluation ex-post of the quality of a VaR measurement

Why is it relevant? Basel Committee requires that VaR is regularly tested:
- predictive ability via a comparison of daily estimates and "actual" losses,
- in order to determine market risk capital requirements.

Main idea: Check consistency between number of exceptions and confidence level, e.g.

$VaR(99\%)$ we expect that losses will be higher in 2.5 days each year

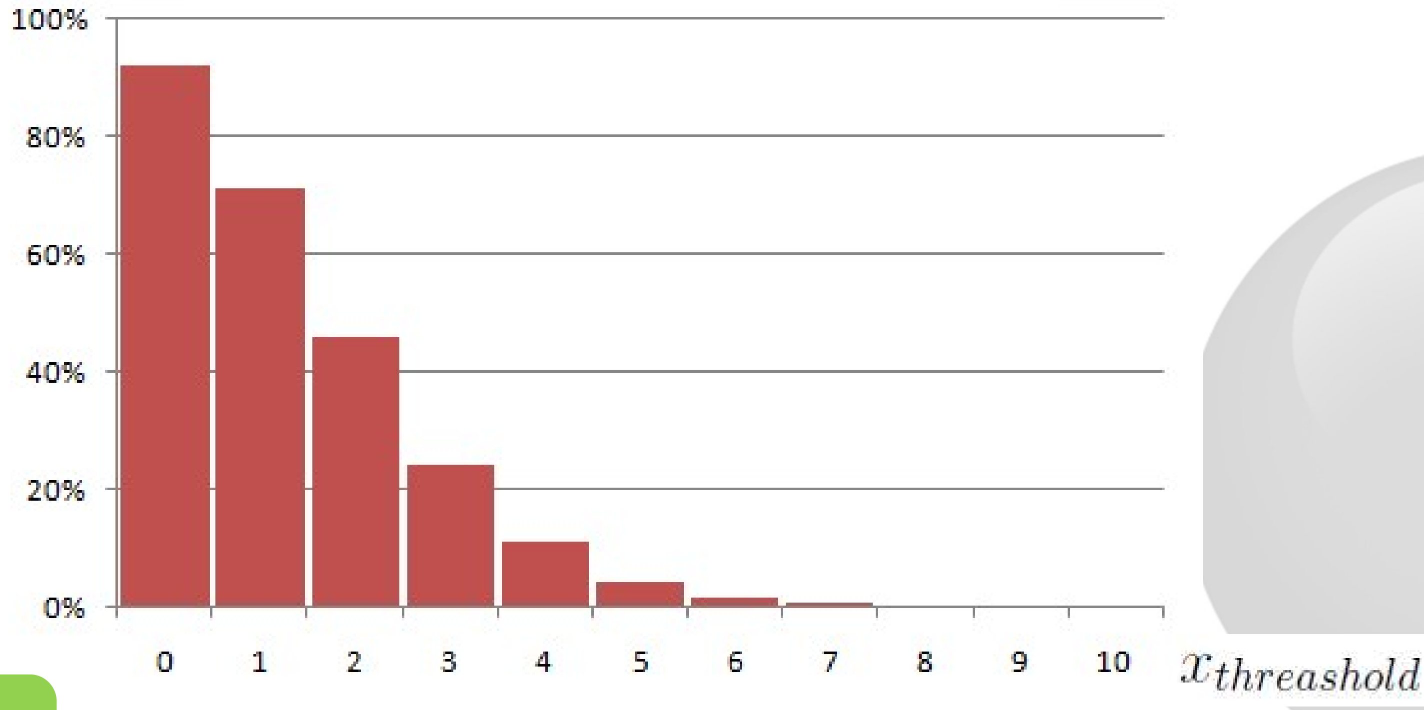
Which P&L should be considered?
- P&L coming from actual positions and trades, detected from Business Control unit (Gestionale).
- P&L obtained revaluing end of previous day positions under new market conditions (static P&L).

Probability to observe a number x of exceptions (if the null hypothesis is true):

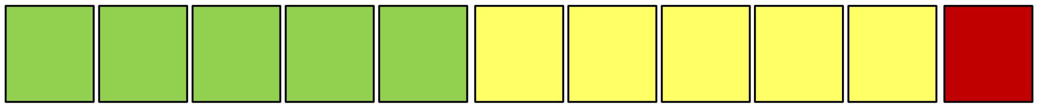
$$\mathcal{P}(x|\alpha, N) = \binom{N}{x} \alpha^x (1-\alpha)^{(N-x)}$$

$x_{threashold}$	$\mathcal{P}(\%)$
0	91.9
1	71.4
2	45.7
3	24.2
4	10.8
5	4.1
6	1.4
7	0.4
8	0.1
9	0.0

$\mathcal{P}(x > x_{threashold}|1-\alpha = 99\%, N = 250)$



Basel Criteria number of exceptions



Likelihood Ratio:

$$LR \equiv -2 \ln \frac{L(x|\text{null hypothesis})}{L(x|\text{model to be tested})}$$

Theorem (Wilks):

$N \to \infty$ $\qquad LR \sim \chi^2_{g_{model}-g_{null}}$ if the null Hypothesis H_0 is correct

Standard approach:

$\text{p} < \beta \Rightarrow H_0$ is rejected

where $p \equiv 1 - \Phi_{\chi^2}(LR)$ ... is the p-value associated to the observed LR

Backtesting VaR: Proportion of failures test or Unconditional coverage test (Kupiec 1995)

Null hypothesis model:

$$L(x|\alpha) = \binom{N}{x} \alpha^x (1-\alpha)^{(N-x)}$$

Model to be tested:

$$L(x|\hat{\alpha}) = \binom{N}{x} \hat{\alpha}^x (1-\hat{\alpha})^{(N-x)}$$

where the frequency of empirical exceptions in the backtest: $\hat{\alpha} \equiv \frac{x}{N}$

Likelihood Ratio (1 degree of freedom):

$$LR_{uc} \equiv -2 \ln \frac{L(x|\alpha)}{L(x|\hat{\alpha})}$$

Example (N = 250):

x	p(N = 250)
5	16.2%
6	5.9%
7	1.9%
8	0.5%
9	1.4‰
10	0.3‰

An elementary Markov chain for exceptions:

$t-1$	t
Yes (1)	Yes (1)
No (0)	No (0)

We define the fractions of exceptions:

$$\begin{cases} \hat{\alpha}_{01} = \dfrac{N_{01}}{N_{01}+N_{00}} \\ \hat{\alpha}_{00} = 1-\hat{\alpha}_{01} \\ \hat{\alpha}_{11} = \dfrac{N_{11}}{N_{11}+N_{10}} \\ \hat{\alpha}_{10} = 1-\hat{\alpha}_{11} \end{cases}$$

Model to be tested:

$$L(x|\{\hat{\alpha}_{ij}\}) = \frac{N!}{N_{00}!\,N_{01}!\,N_{10}!\,N_{11}!}(1-\hat{\alpha}_{01})^{N_{00}}\hat{\alpha}_{01}^{N_{01}}(1-\hat{\alpha}_{11})^{N_{10}}\hat{\alpha}_{11}^{N_{11}}$$

Null hypothesis model:

$$L(x|\alpha) = \frac{N!}{N_{00}!\,N_{01}!\,N_{10}!\,N_{11}!}\alpha^{N_{01}+N_{11}}(1-\alpha)^{N_{00}+N_{10}} = \frac{N!}{N_{00}!\,N_{01}!\,N_{10}!\,N_{11}!}\alpha^{x}(1-\alpha)^{N-x}$$

Likelihood Ratio (2 degrees of freedom):

$$LR_{cc} \equiv -2\ln\frac{L(x|\alpha)}{L(x|\hat{\alpha}_{00},\hat{\alpha}_{01},\hat{\alpha}_{10},\hat{\alpha}_{11})}$$

(Sironi & Resti, Ch 8)

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