Financial Risk Laboratory:
2. Default and Migration Risk

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Overview

Credit Ratings

Credit Ratings allow to order obligors according to their credit risk and map it to an estimate of default probability.
We express credit ratings according to an ordered categorical scale:

$$S = \{1, ..., K\} \tag{1}$$

where 1 represents the highest credit quality (e.g. S&P AAA) and K represents default.
In all practical cases credit ratings are associated to real-world default probabilities measured in $\mathbb{P}$.

Rating Migrations

To describe the dynamics of rating changes, we introduce the *transition probability matrix* (Q), defined over the time horizon (t, T):

$$Q = \begin{pmatrix} q_{11}(t,T) & q_{12}(t,T) & \dots & q_{1K}(t,T) \\ q_{21}(t,T) & q_{22}(t,T) & \dots & q_{2K}(t,T) \\ \vdots & \vdots & \ddots & \vdots \\ q_{K1}(t,T) & q_{K2}(t,T) & \dots & q_{KK}(t,T) \end{pmatrix} \tag{2}$$

Each element q_{ij} of Q is the probability for an obligor with rating i at time t to have rating j at time T:

$$q_{ij}(t_0, \omega; t, T) = P[R(T) = j | R(t) = i \wedge \mathcal{F}_{t_0}] \forall i, j \in S, t_0 \leq t \leq T \tag{3}$$

where we introduced the adapted stochastic process (*rating process*) $R : \Omega \times [0, T] \to S$ as the rating of the obligor at time t

Fundamental (intuitive) properties of rating matrices

Conservation of probability:

$$\sum_{j=1}^{K} q_{ij}(t, T) = 1 \tag{4}$$

Hint: each row of the transition matrix corresponds to an initial rating, each column to a final rating.

No transition happens as $T \to t$:

$$Q(t, t) = \mathbb{I} \tag{5}$$

As the time horizon length becomes short, the off-diagonal terms of the transition matrix become smaller.

The Markov Property

The rating process has no memory so that the current rating represents all the information needed to forecast future rating distributions:

$$P[R(T) = i|\mathcal{F}_t] = P[R(T) = i|R(t)]\forall i \in S \tag{6}$$

In terms of the transition matrix:

$$Q(t_0, t, T) = Q(t, T), \forall t_0 < t \leq T \tag{7}$$

Markov Chain

We say that the rating process follows a Markov chain (i.e. a Markov process with countable state space).

The Chapman Kolmogorov Equation

Following the assumption that the rating process follows a Markov chain, we are able to write the most important equation for rating migration matrices, the *Chapman-Kolmogorov equation* (CK):

$$Q(t, T) = Q(t, s) \cdot Q(s, T), \forall t < s < T \tag{8}$$

Written in terms of the elements of Q, the CK equation reads:

$$q_{ij}(t, T) = \sum_{k=1}^{K} q_{ik}(t, s) \cdot q_{kj}(s, T), \forall t < s < T, \forall i, j \in S \tag{9}$$

Critics to the Markov Hypothesis

There is evidence against the Markov hypothesis in empirical measures of rating changes. Hovever it is widely accepted, since it leads to tractable models with well understood mathematics.

Discrete Time Markov Chain

We also assume the rating process has the *time homogeneity* property:

$$Q(t, T) = Q(T - t), \forall t \leq T \tag{10}$$

We appreciate the utility of the time homogeneity assumption if we focus on *discrete-time Markov chains*.
In particular we define a one-period time interval[1], discretize time: $t = t_1, t_2, t_3, \dots, t_N$ and simplify the notation: $Q = Q(t_n, t_{n+1})$.
The CK equation reads:

$$Q(t_n, t_m) = Q^{n-m}, \forall n \leq m \leq N \tag{11}$$

[1] common: one year

Estimating the Transition Matrix

Maximum likelihood estimator of the elements of Q:

$$\hat{q}_{ij} = \frac{\sum_{n=1}^{N-1} N_{t_n ij}}{\sum_{n=1}^{N} N_{t_n i}} \tag{12}$$

where, given the time index $t_1, ..., t_N$, $N_{t_n i}$ denote the number of firm that are rated i at time t_n for which a rating is available at time t_{n+1} and $N_{t_n ij}$ denote the subset of those firms which are rated j at time t_{n+1}.

TTC vs. PIT

Through the Business Cycle (TTC): Input of the estimation are series sampled across multiple business cycles.
Point in Time (PIT): Adjustment to the TTC estimation is made reflecting the current macro-economic environment.

Rating Changes

Summary Of Annual Corporate Rating Changes* (%) (cont.)

	Number of issuers as of Jan. 1	Upgrades	Downgrades¶	Defaults	Withdrawn ratings	Changed ratings	Unchanged ratings	Downgrade/upgrade ratio
2007	5,651	13.54	9.29	0.37	10.62	33.82	66.18	0.69
2008	5,733	7.94	16.01	1.80	7.71	33.46	66.54	2.02
2009	5,620	4.80	19.13	4.18	8.88	36.99	63.01	3.98
2010	5,312	11.90	8.79	1.20	6.51	28.41	71.59	0.74
2011	5,633	12.23	12.00	0.80	7.85	32.88	67.12	0.98
2012	5,807	8.40	12.31	1.14	6.91	28.76	71.24	1.47
2013	6,044	11.43	9.40	1.06	6.72	28.61	71.39	0.82
2014	6,489	9.20	8.48	0.69	7.15	25.52	74.48	0.92
2015	6,912	7.38	11.78	1.36	8.33	28.85	71.15	1.60
2016	6,928	7.92	12.18	2.06	8.34	30.51	69.49	1.54

*This table compares the net change in ratings from the first to the last day of each year. All intermediate ratings are disregarded. ¶Excludes downgrades to 'D', shown separately in the default column. Sources: S&P Global Fixed Income Research and S&P CreditPro®.

Rating Transition Matrix

Global Corporate Average Transition Rates (1981 - 2016) (%)									
From/to	**AAA**	**AA**	**A**	**BBB**	**BB**	**B**	**CCC/C**	**D**	**NR**
One-year									
AAA	87.05	9.03	0.53	0.05	0.08	0.03	0.05	0.00	3.17
	(7.14)	(7.16)	(0.83)	(0.25)	(0.25)	(0.17)	(0.35)	(0.00)	(2.42)
AA	0.52	86.82	8.00	0.51	0.05	0.07	0.02	0.02	3.99
	(0.52)	(5.25)	(4.21)	(0.69)	(0.20)	(0.21)	(0.07)	(0.08)	(1.89)
A	0.03	1.77	87.79	5.33	0.32	0.13	0.02	0.06	4.55
	(0.09)	(1.02)	(3.53)	(2.11)	(0.39)	(0.27)	(0.07)	(0.11)	(1.77)
BBB	0.01	0.10	3.51	85.56	3.79	0.51	0.12	0.18	6.23
	(0.04)	(0.16)	(1.65)	(3.73)	(1.52)	(0.70)	(0.22)	(0.26)	(1.59)
BB	0.01	0.03	0.12	4.97	76.98	6.92	0.61	0.72	9.63
	(0.06)	(0.09)	(0.26)	(1.89)	(4.41)	(3.12)	(0.76)	(0.85)	(2.38)
B	0.00	0.03	0.09	0.19	5.15	74.26	4.46	3.76	12.06
	(0.00)	(0.09)	(0.21)	(0.22)	(2.04)	(4.22)	(2.19)	(3.25)	(2.19)
CCC/C	0.00	0.00	0.13	0.19	0.63	12.91	43.97	26.78	15.39
	(0.00)	(0.00)	(0.45)	(0.69)	(0.97)	(8.02)	(9.03)	(11.48)	(5.43)

Numbers in parentheses are weighted standard deviations, weighted by the issuer base. Sources: S&P Global Fixed Income Research and S&P CreditPro®.