POLITECNICO
MILANO 1863

Financial Engineering

Assignment 1, Group 16

2023-2024

Alice Vailati - CP: 10683600 - MAT: 222944
Andrea Tarditi - CP: 10728388 - MAT: 251722
Jacopo Stringara - CP: 10687726 - MAT: 222456
Nicolò Toia - CP: 10628899 - MAT: 247208

Contents

1. Introduction and Data

In this work our aim is to compute and estimate the curve of Discount Factors and Zero Rates, using a Bootstrapping technique. This entails taking data from the market and reconstructing the discount factors by using what we have computed previously in a cascading effect. Subsequently we also compute the sensitivities of various Interest rates derivatives.

2. Bootstrap

In this section, we apply the Bootstrap technique, with the assumption of a single-curve model, to the Mid-Market rates for depos, futures and swaps.
In particular, owing to the common practice in the financial markets, we chose to use the most liquids assets, that is, we used the first 3 quoted depo rates, the first 7 futures and all swap rates starting from the 2 years one.
Our methodology was the following:

1. **Depo Rates**: this part of the computation was vectorialized since there is a direct relation between Depo rates and Discount Factors.
2. **Futures Rates**: here we have to work in a sequential fashion since each future rate is only directly related to the **forward** discount rate from the settlement date to the expiry date. To obtain the regular discount factor to the expiry date we must multiply the forward discount with the discount at the settlement date. In order to compute the discount at the settlement date we must interpolate the discounts we have already computed. In order to compute the needed discount factor we applied the following methodology:
   (a) If we already have the discount factor for the given date, we simply return the corresponding discount factor
   (b) If we don't find a perfect match but we do have the discount factors for two dates that contain the needed date, we apply a linear interpolation technique to the zero rates and return the computed discount factor.
   (c) in case the needed date is outside the dates we have already computed discounts for we simply apply a flat extrapolation technique. In other words, we simply return the last computed discount.
3. **Swaps Rates**: here, just like for the futures the computation of the new discount factors relies on all previous discount factors. Furthermore we must interpolate the discount factors obtained from the previous dates to compute the factor for the first year swap.

For ease of interpretation we plot the discount factors along with the zero rates curve. The right y-axis we see the zero rates expressed as a percentage, while on the left hand side y-axis we find the value of the discount factors. On the x-axis we find the date.

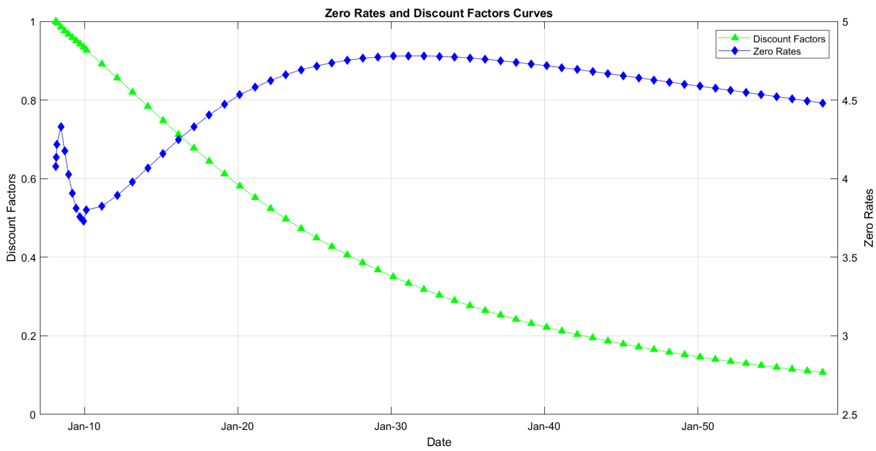


Figure 1: Comparison between Zero rates curve and Discount factors' curve

We can see that the discounts factor curve has the expected inverted exponential shape while the zero rates curve assumes a much more interesting shape.

Let us highlight how the rates computed using the depo and swap rates both follow an increasing trend, while the ones computed using the futures follow a decreasing trend. Indeed the lowest zero rate can be observed for

the last future used in the computation.
As is well know, 2008 is infamously know for the Global Economic Crisis, which saw Mortgage-based securities and other related derivatives take a nose dive, reaching its apex in the Bankruptcy of Lehman Brothers (on September 15 2008) and the subsequent banking crisis.
We can interpret the aforementioned trends as a sign of the oncoming crisis.
Indeed, to quote the Financial Times:

“ The yield curve is usually upward sloping, whereby a higher fixed rate of return is earned from lending money for longer periods of time. Shorter-term yields tend to represent what investors believe will happen to central bank policies in the near future. Longer-dated maturities represent investors’ best guess at where inflation, growth and interest rates are headed over the medium to long term. “

In our case we can interpret the short term rates as an expectation by the market that Central Banks will raise rates to encourage people to save more rather than spend. While, on the other hand, the high values of zero rates for long-term maturities (from 10 years onwards) are influenced by the uncertainty of futures rates and how the economy will perform.

2.1. Why Bootstrap?

There are many alternatives techniques to derive discount factors from market rates. Some notable mentions are the Nelson-Siegel Method, the Svensson Method or Cubic Spline Interpolation.
All these models take into consideration various factors that bootstrap does not consider, and thus render it sometimes less accurate.
What truly makes bootstrap so convenient is its rather ease of use and consistency. As the name suggests we need no prior knowledge of the behaviour of interest rates and no model, all we need is the market rates. Indeed we are ”building following discount factors by standing on top of the previous ones”.

Furthermore we also ensure that our curve perfectly matches the discount factors observed on the market, thus yielding a higher consistency and stability for a rather simple method.
On a final note it is also a very explainable method. There is no estimation of coefficients, involved unlike other more complex methods such as regression based methods.

3. Sensitivities

In this section we take into consideration a portfolio comprised of just one Interest Rates Swap, a plain vanilla 6 year IR swap with fixed rate of 2.8173% and a notional value of €10 millions.
In particular for such a portfolio we also computed the following quantities:

1. $DV01$: for a parallel shift in the market rates.
2. $DV01^Z$ for a parallel shift in the zero rates
3. BPV which is the NPV of 1 Basis Point

Performing the needed calculations yields the following results:

$DV01$	$DV01^Z$	BPV
0.0501%	0.0519%	0.0524%

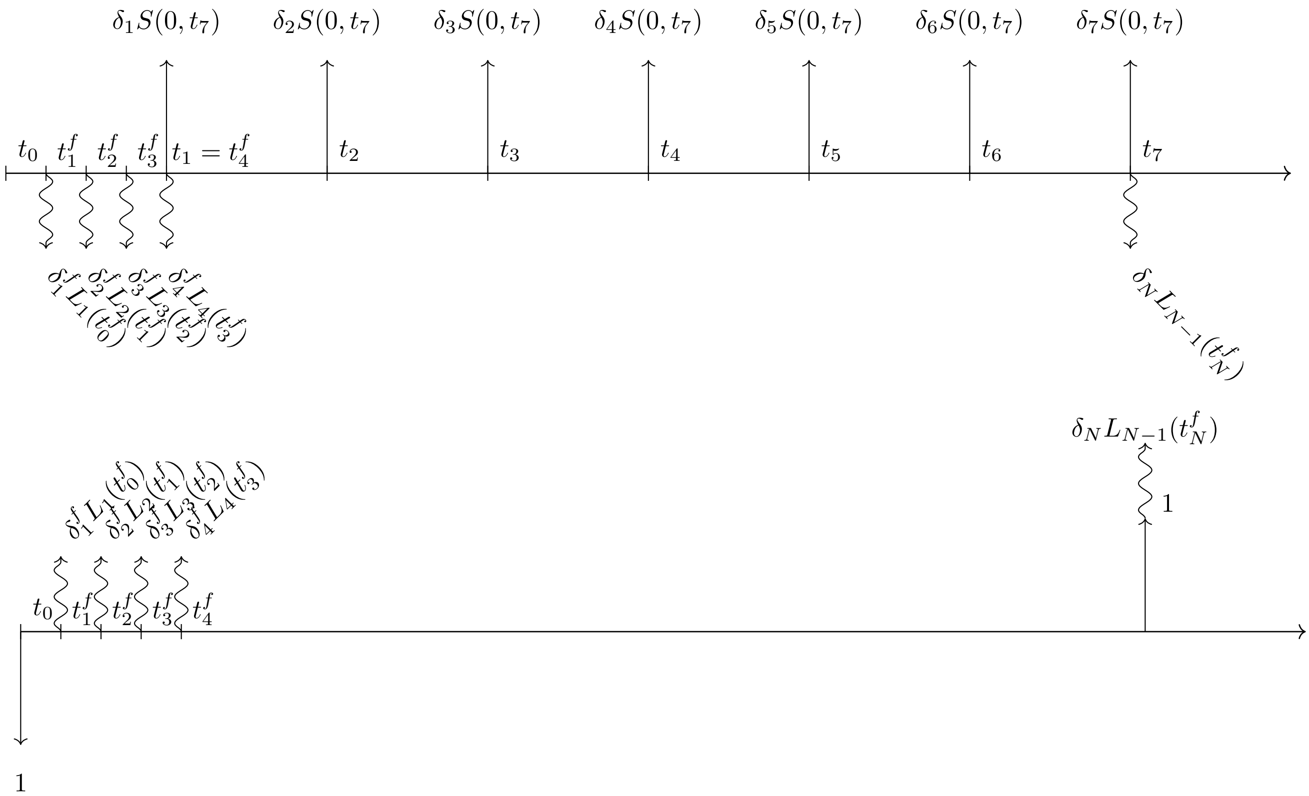
We also computed the Macaulay Duration for an InterBank coupon bond with same expiry, fixed rate and reset dates as the aforementioned IRS. This computations yielded a result of **5.5849** years.

The sensitivities we have computed above can be used to gauge how our portfolio would react to a change in market rates and how much exposure to the interest rate we are taking on. In particular, we can observe that in both cases (either a parallel shift in market rates or in the zero rates) for a change of 1 bp the value of our portfolio would move by 0.05%. This seems relatively high, but overall it depends on the context of our portfolio.
This is probably due to the fact that our swap has a fixed rate of 2.8173%, which a bit off of the par rate we have for the bootstrap. Moreover, another factor may be the curve structure of the interest rates which presents a rather steep segment in the first 6 years. This denotes uncertainty from the market’s perspective.
On the other hand, the Duration of the Coupon Bond tells the average time it takes to receive the present value of its cash flows. The value we have obtained is to be expected.
Indeed the duration is greater than the one of a similar contract with the par swap rate. Furthermore we can

observe that the Duration is more than the $DV01_Z$, this due to the fact that our fixed rate is less than the par swap rate.

4. Theoretical Exercise 3

We are asked to price a 6 year "IB Coupon Bond" issued on the 15th of Feb '2008 with coupon rate equal to the corresponding mid-market 7y swap rate. For the coupons we assume a 30/360 European day count.
Thanks to the Arbitrage Pricing Theorem we can price such an asset by replicating its payoff.
In order to do so we start by considering a receiver Vanilla Mid-Market IRS with maturity 7 years and an I.B. floater with same maturity. These two asset can be graphically represented as follows:



Were we have omitted most of the Libor payments. It is quite clear from the graph that the libor payments cancel out between the IRS and the floater. This is replicating a coupon bond for 7 years with coupon rate equal to the 7 year swap rate. Thus, to reach the desired payoff we must cancel out the payment of the last coupon as well as shift the payment of the principal amount to one year before.
This can be achieved by shorting a zero coupon bond with maturity 7 years and nominal value of $1 + \delta(t_6, t_7)S(t_0, t_7)$ and by going long on a zero coupon bond with maturity 6 years. In other words our portfolio is the following:

$$IRS(t_0, t_7) + Floater(t_0, t_7) - [1 + \delta(t_6, t_7)S(t_0, t_7)]ZCB(t_0, t_7) + ZCB(t_0, t_6)$$

We can simply discount each of these assets and obtain the final price. The IRS has by construction of our bootstrap curve an NPV of 0, while the floater has a price of exactly 1. Adding the price of the ZCBs we get the final price of a 6 year "IB Coupon Bond" with coupon rate equal to the corresponding mid-market 7y swap rate as follows:

$$1 + B(t_0, t_6) - [1 + \delta(t_6, t_7)S(t_0, t_7)]B(t_0, t_7)$$

If we compute this with MATLAB we obtain a price of 1.0045€.

5. Theoretical Exercise 4

Now, we pass to show that the Garman-Kohlhagen formula holds for a European Call option with interest rates, continuous dividends and volatilities that are all a function of time. To be more exact, the formula is the same as the standard one (i.e. with constants) but the rates are substituted with their average values over the time-to-maturity.
We start by writing out the SDE of the Garman-Kohlhagen model:

$$\begin{cases} dS(t) = [r(t) - q(t)] \cdot S(t)\ dt + \sigma(t)S(t)\ dW_t \\ S(0) = S_0 \end{cases}$$

Just for the Black and Scholes formulation, we can apply Itô's formula taking $f(x) = \ln(x)$. In our case we obtain:

$$\begin{aligned} d\left(\ln(S_t)\right) &= \frac{1}{S_t} \cdot \left[(r(t) - q(t))\, S(t)\ dt + \sigma(t)S(t)\ dW_t\right] - \frac{1}{2S_t^2}\left[\sigma^2(t)\ dt\right] \\ &= \left[(r(t) - q(t) - \frac{1}{2}\sigma^2(t)\right]\ dt + \sigma(t)\ dW_t \end{aligned}$$

We now integrate as usual between 0 and the maturity T and obtain:

$$\begin{aligned} \ln\left(\frac{S_T}{S_0}\right) &= \int_0^T \left[r(t) - q(t) - \frac{1}{2}\sigma^2(t)\right]\ dt + \int_0^T \sigma(t)\ dW_t \\ \implies S_T &= S0 \cdot \exp\left\{\int_0^T \left[r(t) - q(t) - \frac{1}{2}\sigma^2(t)\right]\ dt + \int_0^T \sigma(t)\ dW_t\right\} \end{aligned}$$

Now, in order to derive a pricing formula (in our case, just for the plain vanilla EU call) based on this model we must take the expected value of the payoff $[S_T - K]^+$ with respect to the filtration $\mathcal{F}_0$. First off, let us define a few quantities to ease the notation:

$$\begin{aligned} R(T) &= \int_0^T r(t)\ dt \\ Q(T) &= \int_0^T q(t)\ dt \\ S^2(T) &= \int_0^T \sigma^2(t)\ dt \end{aligned}$$

Just like in the Black and Scholes case we can rewrite the expected value of the payoff as:

$$C(0,T) = \mathbb{E}_0\left[D(0,T)\left[S_T - K\right]^+\right] = \mathbb{E}_0\left[D(0,T)S_T \cdot \mathbb{I}_{S_T > K}\right] - \mathbb{E}_0\left[D(0,T)K \cdot \mathbb{I}_{S_T > K}\right]$$

Where $D(0,T)$ is the stochastic discount factor. In particular, since here we assume the rate $r(t)$ to be deterministic, we have that $D(0,T) = \exp\left\{-\int_0^T r(t)dt\right\} = e^{-R(T)}$. Thus this factor can be put directly outside of the expected value and we can put it aside for now.
Furthermore, let us also substitute the last term in the exponential by leveraging the properties of the Stochastic integral and Itô's Isometry. In particular thanks to fact that $\sigma(t)$ is a deterministic continuous function on a limited interval, we can observe the following equality in law:

$$\int_0^T \sigma(t)\ dW_t \sim \mathcal{N}(0, S^2(T))$$

Hence we are allowed to substitute the last term with $-\sqrt{S^2(T)} \cdot g$, with $g \sim \mathcal{N}(0,1)$.
Now, just like for Black and Scholes, we try to rewrite the inequality in the indicator function in terms of the gaussian variable g, for which the cumulative distribution function Φ is known:

$$S_T > K \implies S0 \cdot \exp\left\{R(T) - Q(t) - \frac{1}{2}S^2(T) - \sqrt{S^2(T)} \cdot g\right\} > K$$

This can be rewritten to:

$$g < \frac{\ln\left(\frac{S_0}{K}\right) + R(T) - Q(T) - \frac{1}{2}S^2(T)}{\sqrt{S^2(T)}} := d_2(T)$$

Thus:

$$\mathbb{E}_0\left[\mathbb{I}_{S_T>K}\right] = \mathcal{N}(d_2(T))$$

On the other hand, for the first term we must write the expected value as an integral, using the Gaussian probability density function:

$$\mathbb{E}_0\left[S_T \cdot \mathbb{I}_{S_T>K}\right] = \int_{-\infty}^{d_2(T)} S0 \cdot \exp\left\{R(T) - Q(T) - \frac{1}{2}S^2(T) - \sqrt{S^2(T)} \cdot g\right\} \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{g^2}{2}\right\}\, dg$$

As with Black and Scholes we recognize the square of the binomial and define the new gaussian random variable g':

$$g' = \left(g + \sqrt{S^2(T)}\right) \implies dg' = dg$$

Hence the expected value is equal to:

$$\int_{-\infty}^{d1(T)-\sqrt{S^2(T)}:=d_2(T)} S_0 \exp\left\{R(T) - Q(T) - \frac{1}{2}g'^2\right\} dg' = S_0 e^{R(T)-Q(T)} \cdot \mathcal{N}(d_2(T))$$

Hence, putting everything together we obtain the following equation:

$$C(0,T) = S_0 e^{-Q(T)}\Phi(d_2(T)) - e^{-R(T)}K\Phi(d_1(T))$$

At this point we must consider the average value of $R(T)$, $Q(T)$ and $S^2(T)$ over the interval $[0,T]$, which thanks to the continuity of $r(t)$, $q(t)$ and $\sigma(t)$ and the Average value theorem are obtained as:

$$R_{AVG}(T) = \frac{1}{T}\int_0^T r(t)\, dt$$
$$Q_{AVG}(T) = \frac{1}{T}\int_0^T q(t)\, dt$$
$$S^2_{AVG}(T) = \frac{1}{T}\int_0^T \sigma^2(t)\, dt$$

And thus the final formula is obtained:

$$C(0,T) = S_0 e^{-Q_{AVG}(T)\cdot T}\Phi(d_2(T)) - e^{-R_{AVG}(T)\cdot T}K\Phi(d_1(T))$$

where:

$$d_1(T) = d_2(T) - \sqrt{S^2_{AVG}(T) \cdot T}$$
$$d_2(T) = \frac{\ln\left(\frac{S_0}{K}\right) + \left[R_{AVG}(T) - Q_{AVG}(T) - \frac{1}{2}S^2_{AVG}(T)\right] \cdot T}{\sqrt{S^2(T) \cdot T}}$$

6. Appendix

6.1. Script execution time

The execution time is rather small, averaging around 5 seconds on different machines. By hopping into MATLAB's profiler we have notice that the most time-consuming task is reading and loading the data from the excel file. Excluding this task and loading the data from a .mat file instead, elapsed time goes down to less than 2 seconds. In conclusion we suggest passing to this format instead or to keep the original data loaded between calls to the other functions.

6.2. References

An inverted yield curve: why investors are watching closely, by Chelsea Bruce-Lockhart, Emma Lewis and Tommy Stubbington, April 6 2022.