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# Multivariate asset models using Lévy processes and applications

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In this paper, we propose a multivariate asset model based on Lévy processes for pricing of products written on more than one underlying asset. Our construction is based on a two-factor representation of the dynamics of the asset log-returns. We investigate the properties of the model and introduce a multivariate generalization of some processes which are quite common in financial applications, such as subordinated Brownian motions, jump-diffusion processes and time-changed Lévy processes. Finally, we explore the issue of model calibration for the proposed setting and illustrate its robustness on a number of numerical examples.

**Keywords:** jump-diffusion process; Lévy processes; model calibration; multi-names derivative contracts; subordinated Brownian motions; time-changed Lévy processes

*JEL Classification*: G13, G12, C63, D52

## 1. Introduction

The aim of this paper is to introduce a simple, parsimonious and robust model for multivariate Lévy processes with dependence between components, which can be easily implemented for financial applications, such as the pricing of several types of multi-names derivative contracts commonly used, for example, in the credit and the energy markets. The interest in the construction of multidimensional asset models based on Lévy processes is motivated by the importance of capturing market shocks using more refined distribution assumptions compared with the standard Gaussian framework, as highlighted by the recent crisis in the financial markets.

The proposed approach is based on a parsimonious two-factor linear representation of the assets (log)-returns, in the sense that it uses a linear combination of two independent Lévy processes representing, respectively, the systematic factor and the idiosyncratic shock. Hence, the model has a simple and intuitive economic interpretation and retains a high degree of mathematical tractability, as the multivariate characteristic function is always available in closed form. Furthermore, dependence is generated by the chosen construction and the features of the distribution of the processes chosen as systematic and idiosyncratic components. Our construction can be further applied to introduce multidimensional versions of time-changed Lévy processes with dependence between components. This would allow to incorporate stochastic volatility features which are shown to improve the performance of Lévy processes in pricing options across different maturities (see, e.g. Carr et al. 2003; Carr and Wu; 2004; Huang and Wu 2004).

The idea of inducing correlation via a factor approach dates back to Vasicek (1987) for the case of Brownian motions; the application of linear transformations has been extensively adopted in

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the literature for the case of Lévy processes as well. We cite, amongst others, the approaches put forward by Baxter (2007), Moosbrucker (2006a, 2006b), Lindskog and McNeil (2003), Brigo, Pallavicini, and Torresetti (2007), Semeraro (2008) and Luciano and Semeraro (2010). In detail, Baxter (2007) and Moosbrucker (2006a, 2006b) use a factor copula approach for both subordinated Brownian motions and jump-diffusion (JD) processes, whilst Lindskog and McNeil (2003) make use of linear combinations to develop a common Poisson shock process framework in the context of insurance loss modelling and credit risk modelling. This approach is then extended in Brigo, Pallavicini, and Torresetti (2007) to a formulation which avoids repeated defaults at both cluster level and single name level. Semeraro (2008) and Luciano and Semeraro (2010) instead apply the factor approach to build multivariate subordinators from which they derive the multivariate version of several families of subordinated Brownian motions, such as the Variance Gamma (VG) process, in this way generalizing the approaches of Luciano and Schoutens (2006), Cont and Tankov (2004), Leoni and Schoutens (2008) and Eberlein and Madan (2010).

In spite of being in general simple and relatively parsimonious, these approaches present a number of drawbacks, including restrictions on the range of possible dependencies and the set of attainable values for the correlation coefficient. This is also documented, for example, by Wallmeir and Diethelm (2012) whose empirical analysis shows the limited potential to match observed correlations of the multivariate VG models of Leoni and Schoutens (2008) and Semeraro (2008). We note that for the case of subordinated Brownian motions, Semeraro (2008) and Luciano and Semeraro (2010) improve the richness of the correlation structure through an alternative construction which uses correlated Brownian motions. However, as pointed out by the same authors, this is achieved at the cost of increasing the number of parameters required for calibration: the presence of a correlation matrix for the Brownian motion part of the components, in fact, implies that the number of parameters grows with the square of the number of assets included in the basket, whilst market data available for calibration is usually linear in the number of instruments.

Although similar in principle to some of the multivariate constructions discussed above, our model presents some distinctive features. In first place, our construction applies to any type of Lévy process, hence offering a unified treatment, from subordinated Brownian motions to JD processes. In particular, in the case of JD processes, it allows the distribution of the jump sizes to depend on the nature of the underlying shock; in the case of subordinated Brownian motions, instead, the construction does not necessarily rely on the process chosen as subordinator. This is of relevance, for example, in those cases in which the simulation of the process subordinator proves inefficient, as in the case of the Carr–Geman–Madan–Yor (CGMY) process (see, e.g. Ballotta and Kyriacou 2014)). Further, our model is flexible enough to accommodate complete dependence, independence, positive and negative linear correlation, and is relatively parsimonious in terms of the overall number of parameters involved, as this grows linearly with the number of assets, which facilitates its calibration to market data.

Finally, we note that model calibration to market data is an essential step for practical pricing applications; however, calibration of any multivariate model requires the existence of actively traded multi-names derivatives, and this is not the case in general. Hence, in the paper, we explore the implications of this issue on the proposed construction and the potential limitations.

The remaining of the paper is organized as follows. In Section 2, we introduce our class of multivariate Lévy processes, investigate its general properties, and apply it to build multivariate subordinated Brownian motions and JD processes. A financial application focussed on model calibration and testing the robustness and the flexibility of the model is presented in Section 3; in this section, we also consider the pricing of spread options in view of recovering information on the implied correlation matrix. Extensions to the case of time-changed Lévy processes are introduced in Section 4, and Section 5 concludes.

## 2. Multivariate Lévy process via linear transformation

Lévy processes are characterized by independent and stationary increments; they are fully described by their characteristic function which admits Lévy–Khintchine representation:

$$\phi(u;t) = \mathrm{e}^{t\varphi(u)}, \quad u \in \mathbb{R},$$
$$\varphi(u) = \mathrm{i}u\alpha - u^2\frac{\sigma^2}{2} + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}ux} - 1 - \mathrm{i}ux1_{(|x|<1)})\Pi(\mathrm{d}x).$$

The terms in the characteristic exponent, $\varphi(\cdot)$, i.e. $(\alpha, \sigma, \Pi)$, represent the characteristic triple of the Lévy process. The parameter $\alpha \in \mathbb{R}$ describes the drift of the process, $\sigma > 0$ represents its diffusion part, whilst the jumps are fully characterized by the Lévy measure $\Pi$, i.e. a positive measure satisfying $\int_{\mathbb{R}}(1 \wedge |x|^2)\Pi(\mathrm{d}x) < \infty$.

### 2.1 *General framework*

To construct a multivariate Lévy process with dependent components, we use the property that these processes are invariant under linear transformations. The main result is given in the following.

PROPOSITION 1 *Let $Z(t)$, $Y_j(t)$, $j = 1, \ldots, n$, be independent Lévy processes on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with characteristic functions $\phi_Z(u;t)$ and $\phi_{Y_j}(u;t)$, for $j = 1, \ldots, n$, respectively. Then, for $a_j \in \mathbb{R}, j = 1, \ldots, n$,*

$$\boldsymbol{X}(t) = (X_1(t), \ldots X_n(t))^\top = (Y_1(t) + a_1 Z(t), \ldots, Y_n(t) + a_n Z(t))^\top$$

*is a Lévy process on $\mathbb{R}^n$. The resulting characteristic function is*

$$\phi_{\boldsymbol{X}}(\mathbf{u};t) = \phi_Z\left(\sum_{j=1}^{n} a_j u_j; t\right) \prod_{j=1}^{n} \phi_{Y_j}\left(u_j;t\right), \quad \mathbf{u} \in \mathbb{R}^n. \tag{1}$$

COROLLARY 2 *Let $\boldsymbol{X}(t)$ be the multivariate Lévy process introduced in Proposition 1. Then.*

(i) *For $j = 1, \ldots, n$, the mth cumulant, $c_m$, of the jth component of $\boldsymbol{X}(t)$ is*

$$c_m(X_j(t)) = t[c_m(Y_j(1)) + a_j^m c_m(Z(1))]. \tag{2}$$

(ii) *For any $j \neq l$, the covariance between the jth and lth components of $\boldsymbol{X}(t)$ is*

$$\mathbb{C}\mathrm{ov}(X_j(t), X_l(t)) = a_j a_l \mathbb{V}\mathrm{ar}(Z(1))t.$$

The proof of both Proposition 1 and Corollary 2 follows from the properties of Lévy processes (see, e.g., Theorem 4.1 in Cont and Tankov 2004).

The construction given in Proposition 1 offers a simple and intuitive economic interpretation as for each margin, $X_j$, the process $Z$ can be considered as the systematic part of the risk, whilst the process $Y_j$ can be seen as capturing the idiosyncratic shock. Due to the presence of the common factor $Z(t)$, the components of $\mathbf{X}(t)$ may jump together and are dependent. Further, as the model admits computable characteristic function (as in Equation (1)), the joint distribution is given and can be recovered numerically, even in the cases in which the components' distribution is not known analytically. This also implies that the dependence structure is determined by the chosen distributions of $\mathbf{Y}(t)$ and $Z(t)$. Further details on the model dependence are given in the following (the proof is presented in Appendix 1).

COROLLARY 3 *For each $t \geq 0$, $\boldsymbol{X}(t)$ is positive associated, i.e.*

$$\mathbb{C}\mathrm{ov}(f(\boldsymbol{X}(t)), g(\boldsymbol{X}(t))) \geq 0$$

*for all non-decreasing function $f, g : \mathbb{R}^n \to \mathbb{R}$ for which the covariance is well-defined, if either $a_j \geq 0$ for $j = 1, \ldots, n$ or $a_j \leq 0$ for $j = 1, \ldots, n$.*

In the more general case in which the coefficients $a_j$ do not have the same sign for $j = 1, \ldots, n$, the components $X_j(t)$ and $X_l(t)$, $j \neq l$, are pairwise negative quadrant dependent if $a_j a_l < 0$. Moreover, it follows directly from the construction of $\mathbf{X}(t)$ that the components are conditionally independent; further, if $\mathbf{Y}(t)$ is degenerate, the components of $\mathbf{X}(t)$ are perfectly (linear) dependent; on the other hand, if $Z(t)$ is degenerate, the components of $\mathbf{X}(t)$ are independent.

For the case of the proposed construction, the dependence between components of the multivariate Lévy process $\mathbf{X}(t)$ is correctly described by the pairwise linear correlation coefficient

$$\rho_{jl}^{\mathbf{X}} = \mathbb{C}\mathrm{orr}(X_j(t), X_l(t)) = \frac{a_j a_l \mathbb{V}\mathrm{ar}(Z(1))}{\sqrt{\mathbb{V}\mathrm{ar}(X_j(1))}\sqrt{\mathbb{V}\mathrm{ar}(X_l(1))}} \tag{3}$$

(on this issue, see, e.g. Embrechts, McNeil, and Straumann 2002). Indeed, $\rho_{jl}^{\mathbf{X}} = 0$ if and only if either $a_j a_l = 0$ or $\mathbb{V}\mathrm{ar}(Z(1)) = 0$, i.e. $Z$ is degenerate and the margins are independent. Moreover, $|\rho_{jl}^{\mathbf{X}}| = 1$ if and only if $\mathbf{Y}(t)$ is degenerate and there is no idiosyncratic factor in the margins. Further, $\mathrm{sign}(\rho_{jl}^{\mathbf{X}}) = \mathrm{sign}(a_j a_l)$ and therefore both positive and negative correlation can be accommodated. Finally, for fixed $a_j = \bar{a} > 0$ (resp. $a_j = \bar{a} < 0$), $\rho_{jl}^{\mathbf{X}}$ is a monotone increasing (resp. decreasing) function of $a_l$, which can take any value from $-1$ to 1 (resp. from 1 to $-1$). In particular, $\rho_{jl}^{\mathbf{X}} = 0$ if either $\bar{a} = 0$, or $a_l = 0$ or both, whilst $|\rho_{jl}^{\mathbf{X}}| = 1$ as a limit case for $\bar{a} \to \infty$ and $a_l \to \infty$.

The previous results highlight an advantage of our model compared to the multivariate subordinator approach of Semeraro (2008) and Luciano and Semeraro (2010), and the factor copula approach of Baxter (2007) and Moosbrucker (2006a, 2006b). All these constructions, in fact, can only accommodate strictly positive correlation values due to restrictions on the parameter controlling the correlation coefficient, which are required to ensure the existence of the characteristic function of the processes involved. Moreover, in the case of Semeraro (2008) and Luciano and Semeraro (2010), the correlation coefficient can be zero (for symmetric subordinated Brownian motions) even though the processes are still dependent.

Finally, the pairwise linear correlation between the margin processes can be expressed in terms of the correlation between each margin and the systematic component as

$$\mathbb{C}\mathrm{orr}(X_j(t), Z(t)) = a_j \sqrt{\frac{\mathbb{V}\mathrm{ar}(Z(1))}{\mathbb{V}\mathrm{ar}(X_j(1))}} \quad \forall j = 1, \ldots, n, \tag{4}$$

implying that $\rho_{jl}^{\mathbf{X}} = \mathbb{C}\mathrm{orr}(X_j(t), Z(t))\mathbb{C}\mathrm{orr}(X_l(t), Z(t))$.

The multidimensional modelling approach put forward in this section is quite flexible as it allows to specify any univariate Lévy process for $\mathbf{Y}(t)$ and $Z(t)$; the resulting distribution of the margin might not be known analytically, but it is still accessible via the corresponding characteristic function. On the other hand, for any chosen distribution for the margin process $\mathbf{X}(t)$, it is possible to impose convolution conditions on the processes $\mathbf{Y}(t)$ and $Z(t)$ so that the linear combination $\mathbf{Y}(t) + \mathbf{a}Z(t)$ has the same given distribution of $\mathbf{X}(t)$. This could be particularly convenient in the case in which the multivariate Lévy process $\mathbf{X}(t)$ is used to build a model for financial assets which is consistent with the information provided by traded vanilla (univariate) options. As in general correlation cannot be directly observed in the market due to lack of sufficiently liquid multinames derivative contracts (and therefore reliable quotes for these instruments), by imposing convolution the calibration of the marginal distribution to observable market data would be independent of the fitting of the correlation matrix, and therefore the parameters governing the idiosyncratic and the systematic processes. These parameters would be recovered at a second stage from any given correlation matrix and the relevant restrictions imposed by the convolution. In more detail, to facilitate the convolution, we choose $\mathbf{X}(t)$, $\mathbf{Y}(t)$, and $Z(t)$ from the same family of processes and, given the margin parameters, we solve

$$\varphi_{Xj}(u) = \varphi_{Yj}(u) + \varphi_Z(a_j u), \quad j = 1, 2, \ldots, n. \tag{5}$$

This implies that, if $m$ is the number of parameters describing the processes $X_j(t)$, $Y_j(t)$, and $Z(t)$, and the parameters of the margin processes are given, for a known correlation matrix the fitting of the joint distribution requires $n(m+1)+m$ parameters. As shown by Equation (3), we can recover the $m$ parameters describing the common process $Z(t)$ and the $n$ loadings $a_j$, $j = 1, 2, \ldots, n$, through the correlation matrix subject to relevant convolution conditions arising from Equation (5). The $nm$ parameters of the idiosyncratic process $Y_j(t)$ would then be obtained by solving Equation (5) directly.

We note the following. In first place, the presence of convolution conditions on the parameters of the idiosyncratic and systematic processes does not restrict the behaviour of the correlation coefficient (3), as their effect would be to ensure that the cumulants $c_m(X_j(t))$ and $c_m(Y_j(t) + a_j Z(t))$ match for $j = 1, \ldots, n$. Further, convolution conditions would not be necessary for applications in which keeping the number of parameters small when dealing with univariate contracts is not of particular relevance, and reliable information on the correlation matrix is available.

Examples illustrating the case of a multivariate subordinated Brownian motions and JD processes are discussed in the following sections, together with the corresponding convolution conditions.

### 2.2 *Multivariate subordinated Brownian motions*

A subordinated Brownian motion $X = (X(t) : t \geq 0)$ is a Lévy process obtained by observing a (arithmetic) Brownian motion on a time scale governed by an independent subordinator, i.e. an increasing, positive Lévy process. Hence, $X(t)$ has general form

$$X(t) = \theta G(t) + \sigma W(G(t)), \quad \theta \in \mathbb{R}, \ \sigma > 0, \tag{6}$$

where $W = (W(t) : t \geq 0)$ is a Brownian motion and $G = (G(t) : t \geq 0)$ is a subordinator independent of $W$. The resulting characteristic function is

$$\phi_X(u; t) = \mathrm{e}^{t\varphi_G(u\theta + \mathrm{i}u^2(\sigma^2/2))}, \quad u \in \mathbb{R}, \tag{7}$$

where $\varphi_G(\cdot)$ denotes the characteristic exponent of the subordinator.

Constructing Lévy processes by subordination has particular economic appeal as, in first place, empirical evidence shows that stock log-returns are Gaussian but only under trade time, rather than standard calendar time (see, e.g. Geman and Ané 1996). Further, the time-change construction recognizes that stock prices are largely driven by news, and the time between one piece of news and the next is random as is its impact.

In general, the parameters of the distribution of the subordinator are chosen so that $\mathbb{E}G(t) = t$, in order to guarantee that the stochastic clock $G(t)$ is an unbiased reflection of calendar time (see, e.g. Madan, Carr, and Chang 1998). The law of the increments of $G(t)$ allows us to characterize the resulting process. There are different methods for choosing a subordinator which is suitable for financial modelling; one class of such processes which proves to be quite popular due to its mathematical tractability is the family of tempered stable subordinators, which have characteristic exponent

$$\varphi_G(u) = \frac{\alpha - 1}{\alpha k}\left[\left(1 - \frac{\mathrm{i}uk}{1-\alpha}\right)^{\alpha} - 1\right], \quad u \in \mathbb{R}, \tag{8}$$

where $k > 0$ is the variance rate of $G(t)$ and $\alpha \in [0, 1)$ is the index of stability. In particular, if $\alpha = 0$, expression (8) is to be understood in a limiting sense and $G(t)$ is a Gamma process so that $X(t)$ is a (asymmetric) VG process (see, e.g. Madan, Carr, and Chang 1998). If, instead, $\alpha = \frac{1}{2}$, the subordinator follows an inverse Gaussian process and $X(t)$ is the normal inverse Gaussian (NIG) process introduced by Barndorff-Nielsen (1995). We note that the probability density of tempered stable processes is known in explicit form only for these values of the stability index (i.e. $\alpha = 0$ and $\alpha = \frac{1}{2}$); however, through Equations (7) and (8), it is possible to construct subordinated Brownian motions for any value of $\alpha \in [0, 1)$.

To build the multivariate version of a subordinated Brownian motion of the form (6), we follow Proposition 1 and let $Y_j(t)$ and $Z(t)$ be independent subordinated Brownian motions chosen from the same family of distributions and obtained by subordinating, respectively, a Brownian motion with drift $\beta_j \in \mathbb{R}$ and volatility $\gamma_j > 0$ by an unbiased subordinator $G_{Yj}$, and a Brownian motion with drift $\beta_Z \in \mathbb{R}$ and volatility $\gamma_Z > 0$ by an unbiased subordinator $G_Z$. Then, $\mathbf{X}(t)$ is a multivariate subordinated Brownian motion with margins of the same distribution's class as $Y_j(t)$ and $Z(t)$ if the convolution condition (5) is satisfied.

For sake of illustration, in the following we consider the case in which the subordinators $G_j$, $G_{Yj}$, for $j = 1, \ldots, n$, and $G_Z$ are unbiased tempered stable processes with variance rates $k_j > 0$, $\nu_j > 0$, and $\nu_Z > 0$, respectively. Then, Equations (5) and (8) imply

$$\begin{aligned} k_j\theta_j &= \nu_Z a_j \beta_Z, \quad j = 1, \ldots, n, \\ k_j\sigma_j^2 &= \nu_Z a_j^2 \gamma_Z^2, \quad j = 1, \ldots, n, \end{aligned} \tag{9}$$

consequently

$$\left(\theta_j = \beta_j + a_j\beta_Z,\ \sigma_j^2 = \gamma_j^2 + a_j^2\gamma_Z^2,\ k_j = \frac{\nu_j\nu_Z}{(\nu_j + \nu_Z)}\right). \tag{10}$$

The subordinators, in fact, are assumed to have the same stability index $\alpha$ in order to guarantee that $X_j(t)$, $Y_j(t)$, and $Z(t)$ belong to the same family of distributions.

We note the following. Firstly, the application of Proposition 1 only requires knowledge of the characteristic function of the subordinated Brownian motions, whilst the exact features of the subordinator processes are not necessary (see, e.g. Ballotta and Kyriacou 2014, for the construction based on Proposition 1 of a multivariate CGMY process, which is a subordinated Brownian motion whose subordinator's distribution is not available in explicit form). Secondly, in this construction, dependence stems from both the subordinator and the associated Wiener process. In particular, somehow similar to the model of Luciano and Semeraro (2010), our approach allows for the activity of the (margin) stochastic clock to be governed by a systematic component and a component which is instead asset specific, as supported by the empirical analysis performed by Lo and Wang (2000).

*Example 1 (The VG process)* Let $G(t)$ be a gamma process, i.e. a tempered stable process with scale parameter $\alpha = 0$; further, assume that $G(t)$ is unbiased and has variance rate $k > 0$. Then, $X(t)$ is a VG process with characteristic function

$$\phi_X(u;t) = \left(1 - \mathrm{i}u\theta k + u^2\frac{\sigma^2}{2}k\right)^{-t/k}, \quad u \in \mathbb{R}.$$

Under the restrictions imposed by Equation (9), $\mathbf{X}(t)$ is a multivariate VG process with margins' parameters $(\theta_j, \sigma_j, k_j)$ constructed as in Equation (10) and characteristic function

$$\phi_{\mathbf{X}}(\mathbf{u};t) = \left(1 - \mathrm{i}\beta_Z\nu_Z\sum_{j=1}^n a_ju_j + \frac{\gamma_Z^2}{2}\nu_Z\left(\sum_{j=1}^n a_ju_j\right)^2\right)^{-t/\nu_Z}\prod_{j=1}^n\left(1 - \mathrm{i}u_j\beta_j\nu_j + u_j^2\frac{\gamma_j^2}{2}\nu_j\right)^{-t/\nu_j}.$$

The coefficient of pairwise correlation given by Equation (3) in this case reads

$$\rho_{jl}^{\mathbf{X}} = \frac{a_ja_l(\gamma_Z^2 + \beta_Z^2\nu_Z)}{\sqrt{\sigma_j^2 + \theta_j^2k_j}\sqrt{\sigma_l^2 + \theta_l^2k_l}}. \tag{11}$$

*Example 2 (The NIG process)* In the case in which the (unbiased) tempered stable subordinator $G(t)$ has scale parameter $\alpha = \frac{1}{2}$, i.e. is an inverse Gaussian process, and variance rate $k > 0$, $X(t)$ is a NIG process with characteristic function

$$\phi_X(u;t) = \mathrm{e}^{(t/k)(1-\sqrt{1-2\mathrm{i}u\theta k+u^2\sigma^2k})}, \quad u \in \mathbb{R}.$$

Under the convolution restrictions (9), the margins $X_j(t)$ are NIG processes with parameters $(\theta_j, \sigma_j, k_j)$ as in Equation (10). The resulting characteristic function of the multivariate NIG process is

$$\begin{aligned}\phi_{\mathbf{X}}(\mathbf{u};t) &= \mathrm{e}^{t\varphi(\mathbf{u})},\\ \varphi(\mathbf{u}) &= \frac{1}{\nu_Z}\left(1 - \sqrt{1 - 2\mathrm{i}\beta_Z\nu_Z\sum_{j=1}^n a_ju_j + \gamma_Z^2\nu_Z\left(\sum_{j=1}^n a_ju_j\right)^2}\right)\\ &\quad + \sum_{j=1}^n\frac{1}{\nu_j}(1 - \sqrt{1 - 2\mathrm{i}u_j\beta_j\nu_j + u_j^2\gamma_j^2\nu_j}).\end{aligned}$$

Equation (11) describes the pairwise correlation coefficient also in this case.

As both the VG and NIG are three-parameter processes, the number of parameters required for the joint fit, given the margins, is $(4n + 3)$, of which $3 + n$ are observed from the correlation matrix subject to conditions (9), and $3n$ are obtained from Equation (10).

### 2.3 *Multivariate JD process*

An alternative representation of Lévy processes quite common in financial applications relies on the observation that stock prices appear to have small continuous movements most of the time (due, for example, to a temporary imbalance between demand and supply); sometimes though they experience large jumps upon the arrival of important information with more than just a marginal impact. By its very nature, important information arrives only at discrete points in time and the jumps it causes have finite activity. A motion portraying such a dynamic is a JD process, which can be decomposed as the sum of a Brownian motion with drift and an independent compound Poisson process. Hence, a Lévy process in the JD class has the form

$$X(t) = \mu t + \sigma W(t) + \sum_{k=1}^{N(t)} \xi(k), \quad \mu \in \mathbb{R}, \sigma > 0,$$

where $W = (W(t) : t \geq 0)$ is a Brownian motion, $N = (N(t) : t \geq 0)$ is a Poisson process counting the jumps of $X$, and $\xi(k)$ are i.i.d. random variables capturing the jump sizes (severities). $W$, $N$ and $\xi$ are independent of each other.

We assume that the rate of arrival of the Poisson process is $\lambda > 0$. In this case, we say that the process $X(t)$ has parameters $(\mu, \sigma, \lambda)$ and jump sizes distributed as a random variable $\xi$; the resulting characteristic function is

$$\phi_X(u;t) = \mathrm{e}^{t(\mathrm{i}u\mu - u^2(\sigma^2/2) + \lambda(\phi_\xi(u)-1))},$$

$$\phi_\xi(u) = \mathbb{E}(\mathrm{e}^{\mathrm{i}u\xi}), \quad u \in \mathbb{R}.$$

Popular examples of JD processes used in finance are the so-called Merton (1976) process, for which the jump sizes are Gaussian, and the Kou (2002) process in which case the jump sizes follow an asymmetric double exponential distribution.

In order to construct the multivariate version of the JD process, we follow the same steps as in the previous sections and let the idiosyncratic factor, $Y_j$, and the global factor, $Z$, to be two independent JD processes, respectively, with parameters $(\beta_j, \gamma_j, \delta_j)$ and jump sizes distributed as a random variable $\eta_j$, and $(\beta_Z, \gamma_Z, \delta_Z)$ and jump sizes distributed as a random variable $\eta_Z$. The corresponding pairwise correlation coefficient is

$$\rho_{jl}^{\mathbf{X}} = \frac{a_j a_l(\gamma_Z^2 + \delta_Z \mathbb{E}(\eta_Z^2))}{\sqrt{\sigma_j^2 + \lambda_j \mathbb{E}(\xi_j^2)}\sqrt{\sigma_l^2 + \lambda_l \mathbb{E}(\xi_l^2)}}. \tag{12}$$

Further, for the convolution condition (5) to hold, i.e. for the process $\mathbf{X}(t) = \mathbf{Y}(t) + \mathbf{a}Z(t)$ to be a multivariate JD process, whose margins have parameters $(\mu_j, \sigma_j, \lambda_j)$ and jump sizes distributed as a random variable $\xi_j$, we require

$$(\mu_j = \beta_j + a_j\beta_Z,\ \sigma_j^2 = \gamma_j^2 + a_j^2\gamma_Z^2); \tag{13}$$

further, as the Poisson process is closed under convolution, we also impose

$$\lambda_j = \delta_j + \delta_Z, \tag{14}$$

from which it follows that Equation (5) reduces to the following convolution on the distribution of the jump sizes:

$$\phi_{\xi_j}(u) = \frac{\delta_j \phi_{\eta_j}(u) + \delta_Z \phi_{\eta_Z}(a_j u)}{\delta_j + \delta_Z}, \quad u \in \mathbb{R}. \tag{15}$$

We note the following. Firstly, under the proposed construction, the compound Poisson process components are allowed to jump at different points in time. Secondly, the convolution conditions reported above show the decomposition of both the continuous part of the risk and the pure jump one into their corresponding asset specific part and the one common to the entire basket under consideration. Further, the proposed construction of multivariate JD processes falls in the more general common Poisson shock framework, reviewed in Lindskog and McNeil (2003) and further extended by Brigo, Pallavicini, and Torresetti (2007). In our case, we use only two different types of shock (systematic and idiosyncratic); however, the distribution of the jump sizes depends on the nature of the underlying shock.

We note that a simple solution to Equation (15) can be obtained by assuming that $\xi_j, \eta_j$, and $a_j\eta_Z$ are identically distributed. This is the case discussed by Moosbrucker (2006b). However, in the following we do not consider this alternative as it imposes the unrealistic restriction that the jump sizes of each margin and the ones of its idiosyncratic component are identically distributed. Therefore, we make use of the (numerical) solution of Equation (15). In particular, as this condition indicates that the margins' jump size distribution is given by a mixture of the distributions of the components' jump sizes, we solve the resulting missing data problem by moment matching.

*Example 3 (The Merton process)* Assume that the distribution of the jump sizes is Gaussian. Then, if $\eta_j \sim N(\vartheta_{Yj}, \upsilon_{Yj}^2)$ and $\eta_Z \sim N(\vartheta_Z, \upsilon_Z^2)$, the process $X_j(t) = Y_j(t) + a_j Z(t)$ is a Merton JD process with parameters $(\mu_j, \sigma_j, \lambda_j)$ as given in Equations (13) and (14), and jump sizes $\xi_j \sim N(\vartheta_j, \upsilon_j^2)$, where $\vartheta_j$ and $\upsilon_j$ are the solutions of

$$\mathrm{e}^{\mathrm{i}u\vartheta_j - u^2(\upsilon_j^2/2)} = \frac{\delta_j \mathrm{e}^{\mathrm{i}u\vartheta_{Yj} - u^2(\upsilon_{Yj}^2/2)} + \delta_Z \mathrm{e}^{\mathrm{i}ua_j\vartheta_Z - u^2(a_j^2\upsilon_Z^2/2)}}{\delta_j + \delta_Z}, \quad u \in \mathbb{R}. \tag{16}$$

The above implies

$$\vartheta_j = \frac{\delta_j\vartheta_{Yj} + \delta_Z a_j \vartheta_Z}{\delta_j + \delta_Z},$$

$$\upsilon_j^2 = \frac{\delta_j(\vartheta_{Yj}^2 + \upsilon_{Yj}^2) + \delta_Z a_j^2(\vartheta_Z^2 + \upsilon_Z^2)}{\delta_j + \delta_Z} - \vartheta_j^2.$$

The coefficient of correlation is given by Equation (12), with

$$\mathbb{E}(\xi_j^2) = \vartheta_j^2 + \upsilon_j^2 \quad \forall j = 1, \ldots, n,$$

$$\mathbb{E}(\eta_Z^2) = \vartheta_Z^2 + \upsilon_Z^2.$$

*Example 4 (The Kou process)* In the case of the Kou process, the jump sizes follow a double exponential distribution with parameters $(p, \alpha^+, \alpha^-)$, i.e. their density function is given by

$$p\alpha^+ \mathrm{e}^{-\alpha^+ y} 1_{(y \geq 0)} + (1-p)\alpha^- \mathrm{e}^{\alpha^- y} 1_{(y<0)}, \quad \alpha^+, \alpha^- \in \mathbb{R}^{++}, \ p \in [0, 1].$$

Thus, in addition to Equations (13) and (14), if $\eta_j$, $\eta_Z$, $\xi_j$ have a double exponential distribution, respectively, with parameters $(p_{Yj}, \alpha_{Yj}^+, \alpha_{Yj}^-)$, $(p_Z, \alpha_Z^+, \alpha_Z^-)$, and $(p_j, \alpha_j^+, \alpha_j^-)$, then, for the convolution condition (15) to hold, these parameters must satisfy the following:

$$p_j \frac{\alpha_j^+}{\alpha_j^+ - \mathrm{i}u} + (1-p_j)\frac{\alpha_j^-}{\alpha_j^- + \mathrm{i}u} = \frac{1}{\delta_j + \delta_Z}\left[p_{Yj}\frac{\delta_j \alpha_{Yj}^+}{\alpha_{Yj}^+ - \mathrm{i}u} + (1-p_{Yj})\frac{\delta_j \alpha_{Yj}^-}{\alpha_{Yj}^- + \mathrm{i}u}\right.$$
$$\left. + p_Z \frac{\delta_Z \alpha_Z^+}{\alpha_Z^+ - \mathrm{i}a_j u} + (1-p_Z)\frac{\delta_Z \alpha_Z^-}{\alpha_Z^- + \mathrm{i}a_j u}\right]. \tag{17}$$

The correlation coefficient is obtained from Equation (12) for

$$\mathbb{E}(\xi_j^2) = 2\left(\frac{p_j}{(\alpha_j^+)^2} + \frac{1-p_j}{(\alpha_j^-)^2}\right) \quad \forall j = 1, \dots, n,$$
$$\mathbb{E}(\eta_Z^2) = 2\left(\frac{p_Z}{(\alpha_Z^+)^2} + \frac{1-p_Z}{(\alpha_Z^-)^2}\right).$$

Finally, we note that for the multivariate Kou model, the reconstruction of the margin parameters $(p, \alpha^+, \alpha^-)$ from the component parameters can only be performed numerically.

In the following, we consider applications of our multivariate approach to option pricing problems; therefore, without loss of generality, we consider the case of a JD process with no drift, i.e. we set $\mu_j = \beta_j = \beta_Z = 0$ for $j = 1, \dots, n$. This implies that for the joint fit, given the margins, we require $(5n + 4)$ parameters in the case of the Merton process and $(6n + 5)$ in the case of the Kou process.

## 3. Multivariate asset modelling: calibration and derivative pricing

In this section, we analyse the calibration of the multivariate Lévy process model to market data in view of applications to the problem of pricing multi-assets products.

To this purpose, we consider a frictionless market in which (equity) asset log-returns are modelled by the multivariate Lévy process defined in Proposition 1, so that under any risk-neutral martingale measure asset prices are given by

$$S_j(t) = S_j(0)\,\mathrm{e}^{(r - q_j - \varphi_{Xj}(-\mathrm{i}))t + X_j(t)}, \quad j = 1, \dots, n,$$

where $r > 0$ is the risk-free rate of interest, $S_j(0)$ and $q_j$ denote, respectively, the spot price and the dividend yield of the $j$th asset, and $\varphi_{Xj}(-\mathrm{i})$ is the exponential compensator of the $j$th component of the multivariate Lévy process, $X_j(t)$. As in general the given market is incomplete, there are infinitely many risk-neutral martingale measures; the availability of market prices for European vanilla options, though, allows us to 'complete' the market and extract the pricing measure by calibration.

As outlined in the previous sections, the full calibration procedure should use both single-name and multi-names derivatives in order to access information on the log-returns correlation matrix as well. However, in general suitable multi-names contracts are not sufficiently liquid to generate reliable estimates. Therefore, we assume that option traders views about this correlation is strongly based on observed asset prices; we further explore a procedure with which information about the market consensus on correlation could be recovered, in a way similar to the one used to extract implied volatility from vanilla options, if a suitable number of prices of exotic options (that are sensitive to correlation) is available. Finally, we note that the calibration of the model can only be solved numerically via constrained least square; hence, we analyse the resulting approximation error by quantifying the difference between the moments of the distribution of the processes $\mathbf{X}(t)$, calculated using the component parameters in conjunction with Equation (2), and the same moments calculated instead using the margin parameters and the corresponding model exact formulae (reported in Appendix 1 for the case of subordinated Brownian motions and Appendix A.2 for the case of JD processes).

The analysis is organized as follows. In Section 3.1, we consider the case of three assets, as this is a relatively common situation, for example, when assessing bilateral counterparty credit risk of contracts on a distinct reference name (see, e.g. Ballotta and Fusai 2013 and references therein). In Section 3.2, we present a procedure aimed at recovering 'implied' correlation. Finally, in Section 3.3, we offer some further comments on the performance of the model and its robustness by considering different combinations of higher dimensional cases.

### 3.1 *Model calibration: a three-asset case*

We test the flexibility of the model by calibrating it to option prices on Ford Motor Company, Abbott Laboratories, and Baxter International Inc. We use Bloomberg quotes at three different valuation dates, 30 September 2008, 27 February 2009, and 30 September 2009, in order to explore the behaviour of the proposed model when fitting different correlation values. A synopsis of the three assets is reported in Table 1. The risk-free rate of interest is taken from Bloomberg as well in correspondence with the relevant dates. Correlation between assets log-returns has been estimated on a time window of 125 days up to (and including) the valuation date.

The three assets considered in this analysis are constituents of the S&P100 index and represent three different industries: automotive, drug manufacturers, and medical instruments and supplies, respectively. Abbott Laboratories and Baxter International Inc. are part of the same healthcare sector. Further, from Table 1 we observe that in September 2008 the three assets exhibit positive correlation, at a level which is fairly similar between Ford and the remaining two assets, whilst it is significantly higher between Abbott and Baxter. This date, in fact, coincides with the peak

Table 1. Synopsis of market data for Ford Motor Company, Abbott Laboratories, and Baxter International Inc.

| | | | | 125-day correlation | | |
|---|---|---|---|---|---|---|
| Valuation date | Asset | $S(0)$ | $q$ (%) | F (Ford) (%) | ABT (Abbott Lab.) (%) | BAX (Baxter) (%) |
| 30/09/2008 | F | 5.20 | 0.0 | 100 | | |
| | ABT | 57.58 | 2.8 | 25 | 100 | |
| | BAX | 65.67 | 1.5 | 30 | 64 | 100 |
| 27/02/2009 | F | 2.00 | 0.0 | 100 | | |
| | ABT | 47.34 | 3.0 | 37 | 100 | |
| | BAX | 50.91 | 1.8 | 34 | 83 | 100 |
| 30/09/2009 | F | 7.21 | 0.0 | 100 | | |
| | ABT | 49.47 | 3.0 | −22 | 100 | |
| | BAX | 57.02 | 1.7 | −15 | 45 | 100 |

Source: Bloomberg.
Note: Correlation matrix estimated using historical log-returns of the three assets over a 125-day time window, up to (and including) the valuation date.

of the financial crisis which led to the collapse of Lehman Brothers; the car industry was also experiencing a particularly difficult period following the General Motors liquidity crisis and the sales fall also reported by its main competitors. Correlation values further increase in February 2009, when the effects of the credit crisis are fully captured by the estimation procedure used in this analysis. These observations lead us to expect the common component $Z(t)$ to play a significant role in the prices of Abbott Lab. and Baxter, whilst we expect it to have a smaller impact on Ford prices. The same consideration holds especially for the September 2009 valuation date, when Ford exhibits negative correlation with the other two assets considered in this analysis.

The calibration of the proposed multivariate asset model is performed in steps, as described in Section 2.1. In first place, we extract the parameters of the margin processes, $\mathbf{X}(t)$, using market quotes of European options written on each asset, by minimizing the weighted root-mean-square error. In particular, we follow Huang and Wu (2004) for the choice of both the error function and the weights. Thus, we consider the pricing error outside the bid–ask spread, in the sense that the error is calculated as the difference between the model price and the bid–ask quotes only if the model prices fall outside the market bid–ask spread. As argued by Huang and Wu (2004) and Dumas, Fleming, and Whaley (1998), this choice is aimed at measuring the exactness with which the model fits within the trading cost bounds. As for the choice of the weights, we use an optimal weighting approach based on the variance of option prices (see Huang and Wu 2004, for further details). Model prices are computed using the Fourier inversion procedures of Carr and Madan (1999); out-of-the-money options are dealt with the time value approach. The second step consists of the calibration of the parameters of the idiosyncratic process, $\mathbf{Y}(t)$, and the systematic component, $Z(t)$, by fitting the correlation matrix using least squares, and imposing the relevant convolution conditions.

The calibrated parameters of the margins, the idiosyncratic components, and the systematic process are reported in Tables 2–5 for all the valuation dates considered and models analysed in this paper. The tables also report the accuracy with which the postulated linear combination reproduces the margin process distribution (which we quantify under the heading 'Moment matching error'), and the error originated in fitting the given correlation matrix (heading 'Correlation error'). Further, Figure 1 shows the QQ plots of the (simulated) samples of the margin process obtained by direct calibration to European vanilla options and the same process obtained, instead, by linear combination of the idiosyncratic process and the systematic process. In particular, in these plots we consider the case of the multivariate VG model (similar results have been obtained for the other models presented in this paper and are available from the authors). These results illustrate the goodness of the convolution provided by the fitting procedure, although the accuracy of the approximation tends to deteriorate at the very far end of the tails. Also, the full range of observed correlations is captured with a satisfactory degree of accuracy.

As a further test, we re-calculate the prices of the European vanilla options using the joint characteristic function and quantify the error against the corresponding market data, as reported in Tables 2–5. The (weighted) root-mean-squared errors are very close to the ones generated by direct calibration of the marginal distribution, which shows that any potential approximation error introduced by the joint fitting procedure is relatively negligible for this type of application. We note though that the higher the number of parameters in the joint distribution, the less flexible the fitting of the multivariate model, which highlights the importance of having a parsimonious margin model for the fitting procedure to converge efficiently. Finally, Figure 2 shows the volatilities recovered by the standard Black–Scholes formula in the case in which the input prices are generated by the multidimensional VG process. The plot also reports the original bid–ask volatilities obtained from market data.

Table 2. Calibration of the multivariate VG model.

| VG model | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Valuation date | | | | | | | | |
| | 30/09/2008 | | | 27/02/2009 | | | 30/09/2009 | | |
| Margins | F | ABT | BAX | F | ABT | BAX | F | ABT | BAX |
| $\theta$ | −2.6871 | −0.6373 | −0.5286 | −6.3009 | −0.8664 | −0.7969 | 0.4058 | −0.2283 | −0.5425 |
| $\sigma$ | 0.8537 | 0.2259 | 0.2296 | 0.5354 | 0.1509 | 0.2613 | 0.6040 | 0.2352 | 0.2129 |
| $k$ | 0.0264 | 0.0928 | 0.0897 | 0.0588 | 0.1555 | 0.0805 | 0.0104 | 0.2339 | 0.0944 |
| RMSE | 3.75E−02 | 1.23E−01 | 1.49E−01 | 4.54E−02 | 2.72E−01 | 3.60E−01 | 4.78E−02 | 7.15E−02 | 1.01E−01 |
| (w)RMSE | 2.21E−03 | 9.94E−03 | 1.18E−02 | 4.64E−03 | 1.51E−02 | 1.61E−02 | 1.86E−03 | 4.09E−03 | 3.89E−03 |
| Idiosyncratic part | | | | | | | | | |
| $\beta$ | −2.1117 | −0.2552 | −0.1467 | −4.9115 | −0.0838 | −0.1316 | 0.2888 | −0.1168 | −0.4356 |
| $\gamma$ | 0.8120 | 0.1429 | 0.1488 | 0.4710 | 0.0469 | 0.2311 | 0.5788 | 0.1682 | 0.1431 |
| $\nu$ | 0.0318 | 0.2316 | 0.2137 | 0.0892 | 1.6068 | 0.1512 | 0.0106 | 0.4570 | 0.1176 |
| $a$ | 1.1564 | 0.7678 | 0.7675 | 1.4550 | 0.8197 | 0.6969 | −0.9348 | 0.8903 | 0.8541 |
| Systemic part | | | | | | | | | |
| $\beta_Z$ | | −0.4976 | | | −0.9547 | | | −0.1252 | |
| $\gamma_Z$ | | 0.2278 | | | 0.1750 | | | 0.1846 | |
| $\nu_Z$ | | 0.1547 | | | 0.1721 | | | 0.4790 | |

(*Continued*)

Table 2. Continued

| VG model | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Valuation date | | | | | | | | |
| | 30/09/2008 | | | 27/02/2009 | | | 30/09/2009 | | |
| Margins | F | ABT | BAX | F | ABT | BAX | F | ABT | BAX |
| Moment matching error | | | | | | | | | |
| $\mathbb{E}X(1)$ | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 |
| $stdX(1)$ | −1.34E−03 | 0.00E+00 | −3.72E−03 | −4.62E−02 | −1.08E−14 | −3.80E−02 | −4.72E−03 | −2.36E−14 | −1.33E−14 |
| $\gamma_1(1)$ | −3.61E−03 | 2.11E−15 | 7.77E−03 | 3.41E−02 | 6.64E−14 | −1.91E−02 | −1.80E−02 | 1.11E−12 | −1.58E−07 |
| $\gamma_2(1)$ | −5.32E−05 | −5.27E−15 | −2.25E−02 | −6.16E−02 | −2.08E−13 | −5.84E−02 | −1.28E−02 | −4.05E−12 | −8.48E−02 |
| Calibration error | | | | | | | | | |
| RMSE | 9.49E−03 | 1.70E−07 | 1.18E−07 | 9.71E−09 | −5.14E−08 | −1.30E−08 | 3.69E−09 | 0.00E+00 | 1.11E−09 |
| (w)RMSE | 8.81E−04 | −1.17E−08 | 2.45E−09 | 1.17E−09 | −3.61E−10 | −7.81E−10 | 9.43E−11 | 0.00E+00 | 5.45E−11 |
| Correlation error | | | | | | | | | |
| F | – | | | – | | | – | | |
| ABT | 3.05E−02 | – | | 5.43E−08 | – | | 4.60E−07 | – | |
| BAX | −5.84E−04 | 5.47E−08 | – | 1.39E−07 | 1.22E−07 | – | −5.28E−02 | −9.41E−07 | – |

Notes: Parameters of the margins, the systemic part, and the idiosyncratic components as at 30/09/2009, 27/02/2009, and 30/09/2009. Parameters of the marginal distributions $(\theta_j, \sigma_j, k_j)$ are obtained by direct calibration to market prices. Parameters governing the idiosyncratic risk process, $(\beta_j, \gamma_j, \nu_j, a_j)$, and the systematic risk process, $(\beta_Z, \gamma_Z, \nu_Z)$, are obtained by fitting the correlation matrix and then solving the parameter conditions given in Example 1. Moment matching error: the difference between the exact moments provided in Appendix 1 (calculated using the parameters of the marginal process) and the moments reconstructed using Equation (2). Calibration error: difference between the errors produced by the calibration to market option prices of the margin processes, $\mathbf{X}(t)$, and the linear transformation $\mathbf{Y}(t) + \mathbf{a}Z(t)$. Correlation error: difference between the model and the sample correlation. RMSE, root-mean-square error.

Table 3. Calibration of the multivariate NIG model.

| | NIG model | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Valuation date | | | | | | | | |
| | 30/09/2008 | | | 27/02/2009 | | | 30/09/2009 | | |
| Margins | F | ABT | BAX | F | ABT | BAX | F | ABT | BAX |
| $\theta$ | −2.0985 | −0.3917 | −0.3879 | −6.2583 | −0.8635 | −0.8041 | 0.5358 | −0.2567 | −0.5414 |
| $\sigma$ | 0.8082 | 0.2206 | 0.2141 | 0.9382 | 0.2350 | 0.2570 | 0.5968 | 0.2303 | 0.2167 |
| $k$ | 0.0175 | 0.0698 | 0.0559 | 0.0397 | 0.1140 | 0.0881 | 0.0196 | 0.2536 | 0.0937 |
| RMSE | 3.63E−02 | 1.20E−01 | 1.40E−01 | 4.25E−02 | 2.72E−01 | 3.60E−01 | 5.00E−02 | 7.15E−02 | 9.79E−02 |
| (w)RMSE | 2.11E−03 | 9.70E−03 | 1.13E−02 | 4.44E−03 | 1.50E−02 | 1.60E−02 | 1.94E−03 | 4.09E−03 | 3.76E−03 |
| Idiosyncratic part | | | | | | | | | |
| $\beta$ | −1.7346 | −0.1828 | −0.1483 | −4.9265 | −0.0836 | −0.1328 | 0.4072 | −0.0783 | −0.4024 |
| $\gamma$ | 0.7579 | 0.1507 | 0.1081 | 0.8572 | 0.0731 | 0.1706 | 0.5806 | 0.1271 | 0.1569 |
| $\nu$ | 0.0201 | 0.1495 | 0.0976 | 0.0580 | 1.1777 | 0.2917 | 0.0207 | 0.8316 | 0.1260 |
| $a$ | 1.1480 | 0.6591 | 0.7559 | 1.3965 | 0.8178 | 0.7039 | −0.6866 | 0.9523 | 0.7420 |
| Systemic part | | | | | | | | | |
| $\beta_Z$ | | −0.3170 | | | −0.9537 | | | −0.1874 | |
| $\gamma_Z$ | | 0.2445 | | | 0.2731 | | | 0.2016 | |
| $\nu_Z$ | | 0.1308 | | | 0.1262 | | | 0.3648 | |

(*Continued*)

Table 3. Continued

| NIG model | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Valuation date | | | | | | | | |
| | 30/09/2008 | | | 27/02/2009 | | | 30/09/2009 | | |
| Margins | F | ABT | BAX | F | ABT | BAX | F | ABT | BAX |
| Moment matching error | | | | | | | | | |
| $\mathbb{E}X(1)$ | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 |
| $stdX(1)$ | −6.22E−04 | 0.00E+00 | −2.66E−03 | −2.39E−02 | −6.48E−13 | −7.14E−03 | −3.18E−03 | −3.35E−13 | −8.07E−14 |
| $\gamma_1(1)$ | −1.81E−04 | 4.44E−16 | 6.64E−02 | 3.02E−02 | 6.29E−12 | 1.23E−03 | −2.61E−03 | 1.54E−11 | −2.24E−07 |
| $\gamma_2(1)$ | −2.40E−04 | −1.39E−15 | −1.39E−01 | −6.35E−02 | −2.36E−11 | −4.23E−02 | −3.76E−03 | −7.30E−11 | −2.83E−02 |
| Calibration error | | | | | | | | | |
| RMSE | 3.82E−02 | 2.86E−01 | 2.95E−01 | −1.85E−09 | 4.90E−09 | −3.91E−09 | 4.16E−09 | 0.00E+00 | −4.86E−11 |
| (w)RMSE | 3.00E−03 | 2.27E−02 | 3.04E−02 | −2.35E−10 | 1.19E−11 | 9.51E−10 | 9.68E−11 | 0.00E+00 | 3.12E−10 |
| Correlation error | | | | | | | | | |
| F | – | | | – | | | – | | |
| ABT | 1.50E−02 | – | | 6.43E−07 | – | | 9.88E−07 | – | |
| BAX | 1.80E−02 | −5.48E−08 | – | 1.80E−07 | 4.38E−07 | – | −1.59E−02 | 7.42E−02 | – |

Notes: Parameters of the margins, the systemic part, and the idiosyncratic components as at 30/09/2009, 27/02/2009, and 30/09/2009. Parameters of the marginal distributions $(\theta_j, \sigma_j, k_j)$ are obtained by direct calibration to market prices. Parameters governing the idiosyncratic risk process, $(\beta_j, \gamma_j, \nu_j, a_j)$, and the systematic risk process, $(\beta_Z, \gamma_Z, \nu_Z)$, are obtained by fitting the correlation matrix and then solving the parameter conditions given in Example 2. Moment matching error: difference between the exact moments provided in Appendix 1 (calculated using the parameters of the marginal process) and the moments reconstructed using Equation (2). Calibration error: difference between the errors produced by the calibration to market option prices of the margin processes, $\mathbf{X}(t)$, and the linear transformation $\mathbf{Y}(t) + \mathbf{a}Z(t)$. Correlation error: difference between the model and the sample correlation. RMSE, root-mean-square error.

Table 4. Calibration of the multivariate Merton JD model.

| | Merton model | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Valuation date | | | | | | | | |
| | 30/09/2008 | | | 27/02/2009 | | | 30/09/2009 | | |
| Margins | F | ABT | BAX | F | ABT | BAX | F | ABT | BAX |
| $\sigma$ | 0.8232 | 0.2553 | 0.2353 | 0.9462 | 0.2193 | 0.2551 | 0.5858 | 0.2085 | 0.2219 |
| $\lambda$ | 0.6969 | 0.1974 | 0.2017 | 2.3781 | 0.5619 | 0.5674 | 0.2708 | 0.2055 | 0.2186 |
| $\vartheta$ | −0.4738 | −0.2600 | −0.2837 | −0.6313 | −0.3574 | −0.2644 | 0.0177 | −0.2099 | −0.3031 |
| $\upsilon$ | 0.2770 | 0.1998 | 0.1867 | 0.4650 | 0.2251 | 0.1800 | 0.3120 | 0.2772 | 0.1500 |
| RMSE | 5.03E−02 | 1.53E−01 | 1.46E−01 | 3.69E−02 | 2.74E−01 | 3.60E−01 | 4.67E−02 | 6.97E−02 | 9.79E−02 |
| (w)RMSE | 3.37E−03 | 1.23E−02 | 1.20E−02 | 3.87E−03 | 1.51E−02 | 1.60E−02 | 1.82E−03 | 3.10E−03 | 3.88E−03 |
| Idiosyncratic part | | | | | | | | | |
| $\gamma$ | 0.7748 | 0.1751 | 0.1018 | 0.8909 | 0.1000 | 0.1960 | 0.5682 | 0.1000 | 0.1791 |
| $\delta$ | 0.5183 | 0.0187 | 0.0231 | 1.9165 | 0.1003 | 0.1057 | 0.1683 | 0.1030 | 0.1161 |
| $\vartheta_Y$ | −0.5267 | −0.5807 | −0.3502 | −0.6257 | 0.1445 | 0.0491 | −0.0578 | −0.2320 | −0.4288 |
| $\upsilon_Y$ | 0.2736 | 0.1252 | 0.2651 | 0.4881 | 0.1469 | 0.0100 | 0.3352 | 0.2792 | 0.0784 |
| $a$ | 0.6467 | 0.4320 | 0.4936 | 0.8863 | 0.5428 | 0.4544 | −0.5606 | 0.7188 | 0.5148 |
| Systemic part | | | | | | | | | |
| $\gamma_Z$ | | 0.4299 | | | 0.3595 | | | 0.2546 | |
| $\delta_Z$ | | 0.1787 | | | 0.4617 | | | 0.1025 | |
| $\vartheta_Z$ | | −0.5073 | | | −0.7437 | | | −0.2695 | |
| $\upsilon_Z$ | | 0.3062 | | | 0.3900 | | | 0.3738 | |

(*Continued*)

Table 4. Continued

| | Merton model | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Valuation date | | | | | | | | |
| | 30/09/2008 | | | 27/02/2009 | | | 30/09/2009 | | |
| Margins | F | ABT | BAX | F | ABT | BAX | F | ABT | BAX |
| Moment matching error | | | | | | | | | |
| $\mathbb{E}X(1)$ | 1.40E−03 | −1.30E−03 | −4.42E−03 | 2.09E−03 | −3.37E−06 | 8.19E−04 | −9.76E−04 | 6.04E−04 | −2.27E−03 |
| $stdX(1)$ | 5.89E−04 | 4.99E−03 | 6.38E−03 | −1.74E−04 | 1.26E−04 | −1.32E−02 | 1.09E−04 | 6.30E−05 | −5.12E−03 |
| $\gamma_1(1)$ | 4.16E−04 | −4.23E−02 | −6.42E−02 | −1.41E−03 | 9.06E−03 | 9.32E−02 | 1.03E−02 | 4.41E−03 | 3.51E−02 |
| $\gamma_2(1)$ | 1.83E−03 | 7.07E−02 | 9.96E−02 | −2.24E−04 | −1.98E−02 | −1.76E−01 | −2.02E−03 | 5.89E−03 | −5.94E−02 |
| Calibration error | | | | | | | | | |
| RMSE | 3.92E−05 | 6.05E−03 | 6.69E−04 | −9.62E−06 | −7.44E−05 | 2.63E−02 | 5.80E−05 | −6.14E−04 | −1.11E−03 |
| (w)RMSE | 3.92E−06 | 5.20E−04 | 5.97E−04 | −1.80E−06 | −4.99E−07 | 8.55E−04 | 2.05E−06 | −3.40E−05 | −8.55E−05 |
| Correlation error | | | | | | | | | |
| F | – | | | – | | | – | | |
| ABT | −2.08E−04 | – | | −3.87E−07 | – | | 5.03E−04 | – | |
| BAX | −9.11E−04 | 3.56E−04 | – | −3.56E−07 | −8.68E−07 | – | −7.91E−04 | −3.47E−04 | – |

Notes: Parameters of the margins, the systemic part, and the idiosyncratic components as at 30/09/2009, 27/02/2009, and 30/09/2009. Parameters of the marginal distributions $(\sigma_j, \lambda_j, \vartheta_j, \upsilon_j)$ are obtained by direct calibration to market prices. Parameters governing the idiosyncratic risk process, $(\gamma_j, \delta_j, \vartheta_{Yj}, \upsilon_{Yj}, a_j)$, and the systematic risk process, $(\gamma_Z, \delta_Z, \vartheta_Z, \upsilon_Z)$, are obtained by fitting the correlation matrix and then solving the parameters conditions given in Example 3. Moment matching error: difference between the exact moments provided in Appendix A.2 (calculated using the parameters of the marginal process) and the moments reconstructed using Equation (2). Calibration error: difference between the errors produced by the calibration to market option prices of the margin processes, $\mathbf{X}(t)$, and the linear transformation $\mathbf{Y}(t) + \mathbf{a}Z(t)$. Correlation error: difference between the model and the sample correlation. RMSE, root-mean-square error.

Table 5. Calibration of the multivariate Kou JD model.

| | Kou model | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Valuation date | | | | | | | | |
| | 30/09/2008 | | | 27/02/2009 | | | 30/09/2009 | | |
| Margins | F | ABT | BAX | F | ABT | BAX | F | ABT | BAX |
| $\sigma$ | 0.8116 | 0.2549 | 0.2500 | 1.1776 | 0.2234 | 0.2794 | 0.5828 | 0.2300 | 0.2432 |
| $\lambda$ | 0.2549 | 0.2000 | 0.2170 | 0.9873 | 0.9939 | 0.5105 | 0.3624 | 0.2033 | 0.2115 |
| $p$ | 0.0254 | 0.3000 | 0.6184 | 0.0579 | 0.0772 | 0.0658 | 0.2385 | 0.4600 | 0.2300 |
| $\alpha^+$ | 21.3413 | 22.1345 | 26.9924 | 5.5704 | 19.7977 | 11.7134 | 3.5667 | 8.0000 | 22.4224 |
| $\alpha^-$ | 2.4050 | 3.5001 | 2.7864 | 2.0662 | 4.1458 | 4.5614 | 7.3701 | 4.3438 | 4.8337 |
| RMSE | 5.11E−02 | 1.63E−01 | 1.55E−01 | 3.77E−02 | 2.71E−01 | 3.56E−01 | 4.81E−02 | 1.25E−01 | 1.25E−01 |
| (w)RMSE | 3.57E−03 | 1.31E−02 | 1.29E−02 | 4.51E−03 | 1.49E−02 | 1.59E−02 | 1.87E−03 | 6.66E−03 | 5.17E−03 |
| Idiosyncratic part | | | | | | | | | |
| $\gamma$ | 0.7667 | 0.1639 | 0.1169 | 1.1446 | 0.1042 | 0.2278 | 0.5629 | 0.1216 | 0.2000 |
| $\delta$ | 0.0668 | 0.0119 | 0.0289 | 0.4967 | 0.5033 | 0.0198 | 0.2424 | 0.0833 | 0.0915 |
| $p_Y$ | 0.0115 | 0.0568 | 0.4937 | 0.0100 | 0.9000 | 0.1000 | 0.2681 | 0.2185 | 0.2031 |
| $\alpha_Y^+$ | 6.3406 | 6.4452 | 41.0497 | 1.9113 | 4.7301 | 38.6608 | 3.4307 | 4.5397 | 8.8803 |
| $\alpha_Y^-$ | 1.7582 | 1.9443 | 1.8969 | 2.0154 | 5.0739 | 87.2172 | 6.3361 | 4.5398 | 4.1503 |
| $a$ | 0.5455 | 0.4003 | 0.4530 | 0.9538 | 0.6813 | 0.5581 | −0.5226 | 0.6754 | 0.4788 |
| Systemic part | | | | | | | | | |
| $\gamma_Z$ | | 0.4878 | | | 0.2900 | | | 0.2891 | |
| $\delta_Z$ | | 0.1881 | | | 0.4906 | | | 0.1200 | |
| $p_Z$ | | 0.1908 | | | 0.0204 | | | 0.3709 | |
| $\alpha_Z^+$ | | 2.3883 | | | 6.7138 | | | 22.9985 | |
| $\alpha_Z^-$ | | 2.3863 | | | 2.0884 | | | 3.2841 | |

(*Continued*)

Table 5. Continued

| | Kou model | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Valuation date | | | | | | | | |
| | 30/09/2008 | | | 27/02/2009 | | | 30/09/2009 | | |
| Margins | F | ABT | BAX | F | ABT | BAX | F | ABT | BAX |
| Moment matching error | | | | | | | | | |
| $\mathbb{E}X(1)$ | −3.90E−02 | −1.21E−02 | 4.70E−03 | 1.96E−02 | −1.47E−01 | 2.61E−02 | −1.52E−02 | 1.10E−02 | −5.97E−03 |
| $stdX(1)$ | 1.37E−02 | 1.13E−02 | −1.41E−06 | −2.34E−03 | −4.74E−02 | −3.05E−02 | 0.00E+00 | 2.55E−05 | 3.62E−03 |
| $\gamma_1(1)$ | −3.41E−02 | −2.14E−01 | −2.16E−01 | −1.32E−03 | −3.62E−01 | 2.93E−01 | 0.00E+00 | 1.20E−04 | −4.84E−02 |
| $\gamma_2(1)$ | −2.13E−02 | −4.83E−01 | 1.12E−04 | 2.53E−03 | −9.59E−01 | −1.02E+00 | 0.00E+00 | −6.71E−04 | −3.02E−04 |
| Calibration error | | | | | | | | | |
| RMSE | 2.56E−02 | 5.37E−02 | 6.71E−02 | −6.49E−04 | 4.68E−01 | 6.07E−02 | −1.35E−12 | −5.20E−05 | 8.08E−03 |
| (w)RMSE | 1.72E−03 | 4.32E−03 | 5.75E−03 | −7.79E−05 | 2.35E−02 | 2.07E−03 | 2.12E−12 | −3.44E−06 | 3.83E−04 |
| Correlation error | | | | | | | | | |
| F | – | | | – | | | – | | |
| ABT | 9.16E−03 | – | | 1.06E−04 | – | | −2.57E−04 | – | |
| BAX | 0.00E+00 | 2.89E−08 | – | 6.25E−04 | −2.44E−04 | – | 1.90E−04 | −5.49E−04 | – |

Notes: Parameters of the margins, the systemic part, and the idiosyncratic components as at 30/09/2009, 27/02/2009, and 30/09/2009. Parameters of the marginal distributions $(\sigma_j, \lambda_j, p_j, \alpha_j^+, \alpha_j^-)$ are obtained by direct calibration to market prices. Parameters governing the idiosyncratic risk process, $(\gamma_j, \delta_j, p_{Yj}, \alpha_{Yj}^+, \alpha_{Yj}^-, a_j)$, and the systematic risk process, $(\gamma_Z, \delta_Z, p_Z, \alpha_Z^+, \alpha_Z^-)$, obtained by fitting the correlation matrix and then solving the parameter conditions given in Example 4. Moment matching error: difference between the exact moments provided in Appendix A.2 (calculated using the parameters of the marginal process) and the moments reconstructed using Equation (2). Calibration error: difference between the errors produced by the calibration to market option prices of the margin processes, $\mathbf{X}(t)$, and the linear transformation $\mathbf{Y}(t) + \mathbf{a}Z(t)$. Correlation error: difference between the model and the sample correlation. RMSE, root-mean-square error.

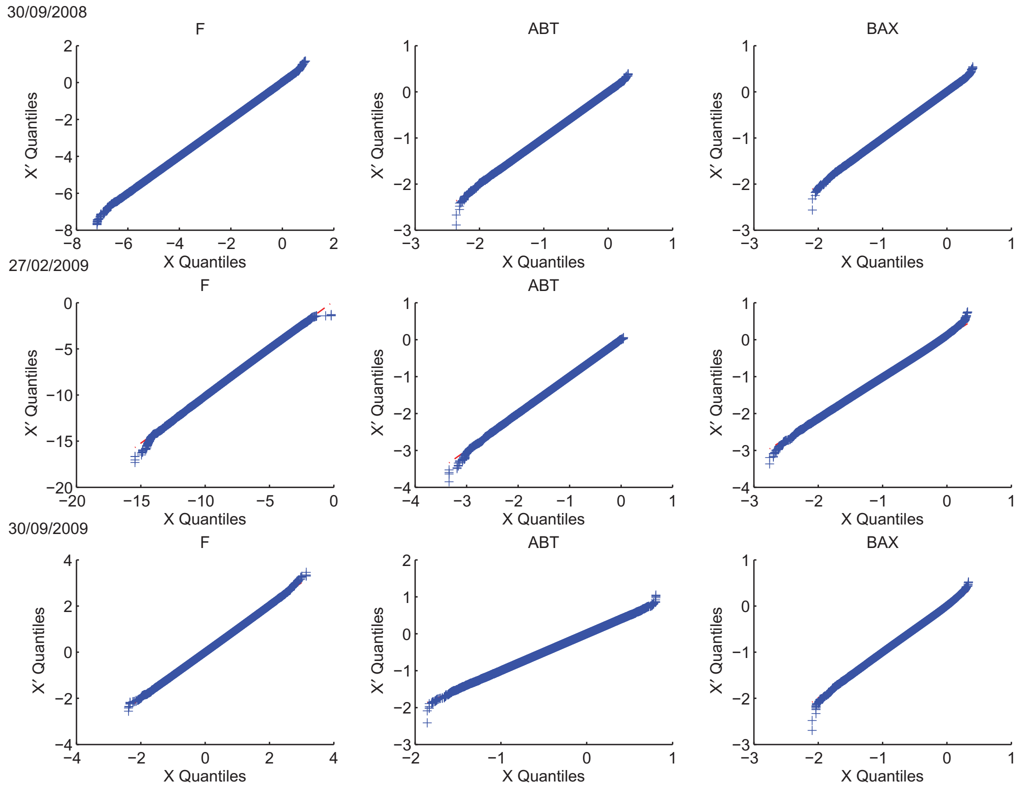


Figure 1. Convolution error: recovering the VG distribution.
Notes: QQ plots of a Monte Carlo sample of the margin VG process, $X(t)$, and the linear transformation process, $X'(t) = Y(t) + aZ(t)$, for Ford, Abbott Lab., and Baxter at 30 September 2008, 27 February 2009, and 30 September 2009. Monte Carlo simulation based on 1,000,000 iterations.

We conclude by noting that, as in the set up proposed in this paper, the correlation coefficient is an explicit function of the model parameters, and market consistent information on the (in general not observable) common component could be recovered directly from the market correlation matrix. The multivariate construction presented in Section 2 would allow us to use this information to observe the impact of these components on each asset through, for example, the correlation coefficient (4). For the case of the assets considered in this study, these results are shown in Table 6. In particular, we observe the very strong impact of the systematic process on the correlation between the log-returns of Abbott Lab. and Baxter; the role of the common factor though is not so relevant in the case of Ford, confirming the economic considerations offered above. Further evidence is provided by the parameters reported in Tables 2–5; for example, in the case of the VG model specification, the systematic component explains only 13% of the total variance of Ford log-returns in September 2008, against 60% of the total variance of Abbot Lab. and 67% of Baxter. This changes in February 2009 to 14% for Ford, 90% for Abbott Lab., and 62% for Baxter. In September 2009, the contribution of $Z$ accounts for 10% of the total variation of Ford, and 49–41% for Abbott Lab. and Baxter, respectively. Similar considerations hold for the other models analysed in this paper.

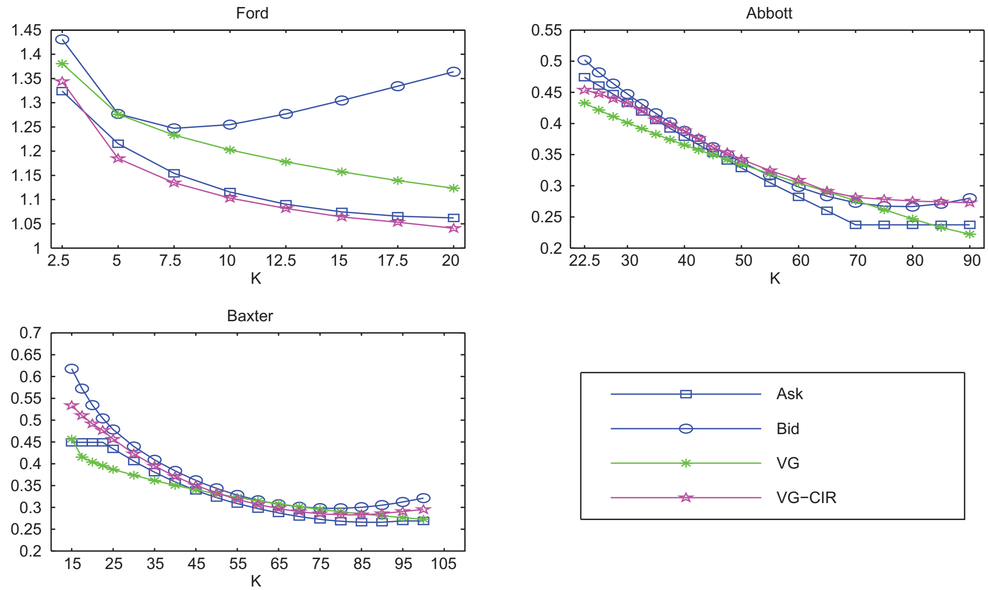


Figure 2. Recovering implied volatilities with multivariate VG processes and VG-CIR processes.
Notes: Implied volatility recovered by inversion for the Black–Scholes formula in correspondence with input vanilla option prices obtained using the given multivariate processes. Parameters: Tables 2 and 8. Maturity: 11 months. Valuation date: 27/02/2009.

### 3.2 *Pricing of exotics and implied correlation*

In this section, we consider the pricing of European style multi-names products in the market model calibrated in Section 3.1. In particular, we consider the case of a spread (call) option with payoff at maturity $T$

$$(S_1(T) - S_2(T) - K)^+.$$

The choice of this contract class is motivated by the fact that they carry information about the market consensus on correlation between the underlying assets.

In this example, we assume a joint VG dynamics for the log-returns of the two assets, with parameters obtained by the joint model calibration reported in Table 2. Further, we assume that the assets considered are Baxter and Abbott Lab. for $j = 1, 2$, respectively; finally, the valuation date is 27/02/2009. All prices are computed using the Fourier inversion method proposed by Hurd and Zhou (2009).

The 'implied' correlation is obtained from the standard model using as input the spread option prices obtained under the multivariate VG model, and the implied volatility of each asset extracted from vanilla option prices computed under the VG model in correspondence with each strike and maturity. The results are presented in Figure 3 – panel (a). We note, in particular, that the implied correlation is higher than the historical correlation (which is fixed at 83% – see Table 1) in the case of in-the-money options (i.e. if $K < A(0)$, for $A(0) = S_1(0) - S_2(0)$) and it decreases as the option moves out-of-the-money and deep out-of-the-money (i.e. $K > A(0)$). This observed 'skew' pattern is consistent with the so-called correlation leverage effect reported, for example, by Da Fonseca, Grasselli, and Tebaldi (2007).

Table 6. Correlation between asset log-returns ($X$) and the common component ($Z$) and the idiosyncratic part ($Y$).

| | Valuation date | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 30/09/2008 | | | | 27/02/2009 | | | | 30/09/2009 | | | |
| | VG model | | NIG model | | VG model | | NIG model | | VG model | | NIG model | |
| | $Z$ | **Y** | $Z$ | **Y** | $Z$ | **Y** | $Z$ | **Y** | $Z$ | **Y** | $Z$ | **Y** |
| F | 0.3622 | 0.9336 | 0.3628 | 0.9326 | 0.3893 | 0.9519 | 0.3893 | 0.9377 | −0.3148 | 0.9573 | −0.2639 | 0.9700 |
| ABT | 0.7713 | 0.6328 | 0.7303 | 0.6831 | 0.9504 | 0.3111 | 0.9504 | 0.3111 | 0.6987 | 0.7154 | 0.8337 | 0.5522 |
| BAX | 0.8265 | 0.5862 | 0.8764 | 0.5050 | 0.8733 | 0.6850 | 0.8733 | 0.5276 | 0.6440 | 0.7650 | 0.6287 | 0.7776 |
| | JD Merton | | JD Kou | | JD Merton | | JD Kou | | JD Merton | | JD Kou | |
| | $Z$ | **Y** | $Z$ | **Y** | $Z$ | **Y** | $Z$ | **Y** | $Z$ | **Y** | $Z$ | **Y** |
| F | 0.3416 | 0.9392 | 0.3485 | 0.9203 | 0.3893 | 0.9212 | 0.3899 | 0.9228 | −0.2713 | 0.9623 | −0.2709 | 0.9626 |
| ABT | 0.7313 | 0.6569 | 0.7436 | 0.6105 | 0.9504 | 0.3100 | 0.9495 | 0.5932 | 0.8090 | 0.5874 | 0.8129 | 0.5822 |
| BAX | 0.8763 | 0.4339 | 0.8607 | 0.5091 | 0.8733 | 0.5604 | 0.8739 | 0.6458 | 0.5558 | 0.8538 | 0.5529 | 0.8171 |

Notes: These values have been obtained using Equation (4) and the parameters of the components as reported in Tables 2–5.

### 3.3 *Model calibration in higher dimensions*

We conclude this section with some additional comments on the performance of the model and the proposed two-step calibration procedure when more than three assets are considered.

For illustration purposes, we discuss only the case of the multidimensional VG process; for the four-asset case, we add to the previous data set the security Harley-Davidson Inc., observed on 30 September 2008. For higher dimension cases, we use part of the data set provided in Fiorani, Luciano, and Semeraro (2010). A synopsis of the results is offered in Table 7, where we report the root-mean-square error resulting from the correlation matrix fit, and the moment matching error (in the interest of space, full results including the parameters of the component processes and relevant QQ plots are omitted, and available from the authors). We note that, as somewhat expected, the dimensionality of the problem affects both the quality of the correlation fit and the robustness of the numerical solution to the convolution conditions, especially as far as the tails of the distributions are concerned.

## 4. Extensions to multivariate time-changed Lévy processes: a simple setting

The multivariate Lévy process introduced in Proposition 1 can also be used as the basis for multivariate time-changed Lévy process constructions, allowing for the introduction of stochastic volatility features. As Lévy processes have independent and stationary increments, in fact, they suffer in terms of fitting performance especially over medium-long maturities. In this respect, time-changed Lévy processes represent a way to simultaneously and parsimoniously capture the fact that not only asset prices jump, but also returns volatilities are stochastic and are correlated to asset returns. These processes have been studied in the context of option pricing by, amongst others, Carr et al. (2003), Carr and Wu (2004), and Huang and Wu (2004) (see Appendix 1).

Table 7. Calibration of the multivariate VG model ($n \geq 3$).

| | | Asset number ($n$) | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Correlation error RMSE | | 1.76E−02 | 1.51E−02 | 7.25E−03 | 6.17E−03 | 5.50E−03 | 6.17E−03 | 6.66E−03 | 7.46E−03 |
| Moment matching error | | | | | | | | | |
| $EX(1)$ | min | 0.00E+00 | 0.00E+00 | −1.11E−16 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | −1.11E−16 |
| | max | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 1.11E−16 | 5.55E−17 | 1.11E−16 | 0.00E+00 |
| | average | 0.00E+00 | 0.00E+00 | −2.22E−17 | 0.00E+00 | 1.59E−17 | 6.94E−18 | 1.23E−17 | −1.11E−17 |
| $stdX(1)$ | min | −3.72E−03 | −1.49E−01 | −8.86E−02 | −9.15E−02 | −8.68E−02 | −8.67E−02 | −7.70E−02 | −8.25E−02 |
| | max | 0.00E+00 | −4.03E−03 | −1.06E−04 | −1.14E−05 | −4.36E−05 | −2.90E−03 | −8.87E−04 | −1.39E−03 |
| | average | −1.69E−03 | −4.17E−02 | −2.50E−02 | −2.31E−02 | −2.39E−02 | −2.44E−02 | −2.08E−02 | −2.07E−02 |
| $\gamma_1$ | min | −3.61E−03 | −1.47E−01 | 5.52E−03 | 1.72E−03 | 3.52E−03 | 1.60E−02 | 1.68E−02 | 1.36E−03 |
| | max | 7.77E−03 | 1.26E−02 | 7.53E−02 | 7.99E−02 | 9.37E−02 | 2.90E−01 | 2.65E−01 | 2.17E−01 |
| | average | 1.39E−03 | −5.65E−02 | 3.36E−02 | 3.56E−02 | 4.43E−02 | 1.66E−01 | 1.42E−01 | 1.21E−01 |
| $\gamma_2$ | min | −2.25E−02 | −2.87E−02 | −3.12E−01 | −3.44E−01 | −3.13E−01 | −1.13E+00 | −8.87E−01 | −7.72E−01 |
| | max | −5.27E−15 | 3.05E−02 | −2.66E−02 | −1.80E−02 | −2.42E−02 | −8.06E−02 | −8.47E−02 | −1.49E−02 |
| | average | −7.50E−03 | 9.64E−04 | −1.28E−01 | −1.25E−01 | −1.38E−01 | −6.43E−01 | −5.58E−01 | −4.77E−01 |

Notes: Correlation error: Root-mean-square error (RMSE) of the correlation matrix fit to given data. Moment matching error: difference between the exact moments provided in Appendix 1 (calculated using the parameters of the marginal process) and the moments reconstructed using Equation (2).

Table 8. Calibration of the multivariate VG-CIR model.

| VG-CIR model | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Margins | | | | | Moment matching error | | | |
| | | F | ABT | BAX | | F | ABT | BAX |
| | $\theta$ | −3.1330 | −0.7165 | −0.7366 | $\mathbb{E}X(1)$ | 0.00E+00 | 0.00E+00 | 0.00E+00 |
| VG | $\sigma$ | 1.0542 | 0.3296 | 0.3513 | $stdX(1)$ | −4.31E−02 | −2.94E−02 | −1.58E−02 |
| | $k$ | 0.0314 | 0.1836 | 0.0927 | $\gamma_1(1)$ | 7.15E−03 | −7.07E−01 | −9.79E−03 |
| | $b$ | 1.0000 | 0.2351 | 0.2220 | $\gamma_2(1)$ | −1.98E−02 | −7.03E−01 | −2.80E−01 |
| CIR | $\lambda$ | 0.8333 | 0.4040 | 0.3926 | | | | |
| | $\kappa$ | 1.0993 | 1.0993 | 1.0993 | | | | |
| | $\eta$ | 1.1275 | 0.2651 | 0.2503 | Calibration error | | | |
| | RMSE | 7.63E−03 | 1.33E−01 | 7.39E−02 | RMSE | −8.92E−11 | 8.22E−13 | 5.15E−11 |
| | (w)RMSE | 8.53E−04 | 8.18E−03 | 3.66E−03 | (w)RMSE | −7.55E−10 | 4.93E−09 | 2.28E−09 |
| Idiosyncratic part | | | | | Correlation error | | | |
| | $\beta$ | −1.8899 | 0.0962 | −0.1144 | F | – | | |
| | $\gamma$ | 0.9680 | 0.1849 | 0.2825 | ABT | −2.73E−07 | – | |
| | $\nu$ | 0.0372 | 2.2361 | 0.1727 | BAX | −2.18E−07 | −9.97E−07 | – |
| | $a$ | 1.1932 | 0.7801 | 0.5972 | | | | |
| Systemic part | | | | | | | | |
| | $\beta_Z$ | | −1.0418 | | | | | |
| | $\gamma_Z$ | | 0.3498 | | | | | |
| | $\nu_Z$ | | 0.2000 | | | | | |

Notes: Valuation date: 27/02/2009. Parameters of marginal distributions $(\theta_j, \sigma_j, k_j, \lambda_j, \kappa, \eta_j)$ are obtained by direct calibration to market prices (note: $\lambda_j = \lambda\sqrt{b_j}$, $\eta_j = b_j\eta$, where $\lambda$ and $\eta$ are the parameters of the common time change). Remaining parameters (idiosyncratic and systematic components of the VG process) are obtained by fitting the correlation matrix subject to relevant convolution conditions. Moment matching error: difference between the exact moments provided in Appendix 1 (calculated using the parameters of the marginal process) and the moments reconstructed using Equations (A2)–(A6) in conjunction with Equation (2). Correlation error: difference between the model and the sample correlation. Calibration error: difference between the errors of the calibration of the margin process and the linear combination to market prices.

For a simple construction, let $\mathbf{V}(t)$ be a $n$-dimensional absolutely continuous time change with components of the form

$$V_j(t) = b_j V(t) = b_j \int_0^t v(s)\,\mathrm{d}s, \quad j = 1, \ldots, n,$$

for positive constants $b_j$, $j = 1, \ldots, n$, and a positive integrable process $v(t)$ representing the instantaneous (common) business activity rate. A multivariate time-changed Lévy process $\mathbf{B}(t)$ can be then obtained by evaluating each component of a $n$-dimensional Lévy process $\mathbf{X}(t)$ as given in Proposition 1 on a time scale governed by $\mathbf{V}(t)$ so that

$$B_j(t) = X_j(V_j(t)), \quad j = 1, \ldots, n.$$

The corresponding characteristic function of the margin process is given by

$$\phi_{Bj}(u; t) = \mathbb{E}[\mathbb{E}(\mathrm{e}^{\mathrm{i}uX_j(V_j(t))} \mid V_j(t))] \quad u \in \mathbb{R}, \ j = 1, \ldots, n; \tag{18}$$

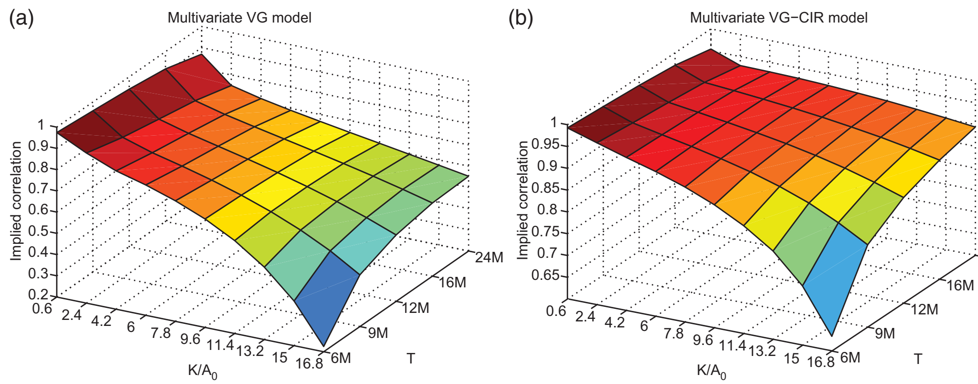


Figure 3. Spread call options: recovering implied correlation with multivariate VG and VG-CIR processes. Notes: Implied correlation recovered from the standard model in correspondence with input spread option prices obtained using the given multivariate process. Multivariate model parameters: Tables 2 and 8. Standard (log-normal) model parameters: implied volatility recovered from the vanilla option prices computed under the given multivariate process. Valuation date: 27/02/2009. $A(0) = S_1(0) - S_2(0)$. $S_1$: Baxter; $S_2$: Abbott Lab.

if $V_j(t)$ is independent of $X_j(t)$ for $j = 1, \dots, n$, Equation (18) reduces to

$$\phi_{Bj}(u;t) = \phi_V(-\mathrm{i}b_j\varphi_{Xj}(u);t), \quad j = 1,\dots,n. \tag{19}$$

We note that the generalization of the previous result to the case in which $X_j(t)$ and $V_j(t)$ are correlated (to capture the so-called leverage effect) can be obtained using the leverage-neutral measure of Carr and Wu (2004). The corresponding multivariate characteristic function is therefore

$$\phi_{\mathbf{B}}(\mathbf{u};t) = \phi_V(-\mathrm{i}g(\mathbf{u};\mathbf{a},\mathbf{b});t), \tag{20}$$

$$g(\mathbf{u};\mathbf{a},\mathbf{b}) = \sum_{j=1}^{n} b_j\varphi_{Yj}(u_j)$$
$$+ b_{(1)}\varphi_Z\left(\sum_{l=(1)}^{(n)} u_l a_l\right) + \sum_{l=(2)}^{(n)} (b_l - b_{l-1})\varphi_Z\left(\sum_{k=(l)}^{(n)} u_k a_k\right), \tag{21}$$

where $b_{(j)}$ is the $j$th element of the sequence $(b_{(1)}, b_{(2)}, \dots, b_{(n)})$ obtained by rearranging in increasing order the sequence of parameters $(b_1, b_2, \dots, b_n)$ (and $(1), (2), \dots, (n)$ is a permutation of $1, 2, \dots, n$). Further, for any $j \neq l$, the covariance between the $j$th and $l$th components of $\mathbf{B}(t)$ is

$$\mathbb{C}\mathrm{ov}(B_j(t), B_l(t)) = a_j a_l \min(b_j, b_l)\mathbb{E}(V(t))\mathbb{V}\mathrm{ar}(Z(1))$$
$$+ b_j b_l \mathbb{E}(X_j(1))\mathbb{E}(X_l(1))\mathbb{V}\mathrm{ar}(V(t)), \tag{22}$$

from which the correlation coefficient follows (the proof of Equations (20)–(22) is provided in Appendix A.2). We note the limited dependence structure offered by the proposed construction due to the common time change applied to the base Lévy process; a full, richer construction of multivariate time-changed Lévy processes is left to future research.

For an illustration, we consider the case of a multivariate VG process (as given in Section 2.2) time changed by an independent integrated Cox–Ingersoll–Ross (CIR) process (as in Carr et al.

2003), so that

$$\mathrm{d}v(t) = \kappa(\eta - v(t))\,\mathrm{d}t + \lambda\sqrt{v(t)}\,\mathrm{d}\bar{W}(t),$$

where $\bar{W}(t)$ is a standard Brownian motion independent of the base process $\mathbf{X}(t)$. The characteristic function of $V(t)$ is well known from standard results on affine processes (see, e.g. Filipović 2009); therefore, Equation (19) reads

$$\phi_{Bj}(u;t) = \mathrm{e}^{\Phi_j(u,t)+\Psi_j(u,t)v(0)} \quad j = 1,\dots,n,$$

with

$$\begin{aligned}
\Phi_j(u,t) &= \frac{2\kappa\eta}{\lambda^2}\ln\left(\frac{2\zeta_j(u)\mathrm{e}^{((\zeta_j(u)+\kappa)/2)t}}{(\zeta_j(u)+\kappa)(\mathrm{e}^{\zeta_j(u)t}-1)+2\zeta_j(u)}\right),\\
\Psi_j(u,t) &= \frac{2\xi_j(u)(\mathrm{e}^{\zeta_j(u)t}-1)}{(\zeta_j(u)+\kappa)(\mathrm{e}^{\zeta_j(u)t}-1)+2\zeta_j(u)},\\
\zeta_j(u) &= \sqrt{\kappa_j^2 - 2\lambda^2\xi_j(u)},\\
\xi_j(u) &= -\frac{b_j}{k_j}\ln\left(1 - \mathrm{i}u\theta k_j + u^2\frac{\sigma^2}{2}k_j\right).
\end{aligned}$$

The parameters of the multivariate VG-CIR model calibrated to the market data described in Section 3.1, under the assumption of a risk-neutral dynamic of the stock price,

$$S_j(t) = S_j(0)\mathrm{e}^{(r-q_j)t-\Phi_j(-\mathrm{i},t)-\Psi_j(-\mathrm{i},t)v(0)+B_j(t)},$$

are reported in Table 8. For illustration purposes, we only consider the valuation date as 27/02/2009. The table reports the error in fitting the correlation matrix as well as the error in reproducing the original option prices by the multivariate VG-CIR model. Comparison with Table 2 shows the improved performance of the time-changed VG construction due to the additional stochastic volatility features. Further evidence is provided in Figure 2, where we plot the implied volatilities generated by the multidimensional VG-CIR process and compare them with the ones obtained previously from the multivariate VG model. In particular, we note that the implied volatility induced by the VG-CIR construction provides a better fit especially for the more liquid contracts, as expected (see, e.g. Carr et al. 2003; Huang and Wu 2004).

In Figure 3 – panel (b), we show the implied correlation extracted from the prices of the spread option introduced in Section 3.2. Similar to the case of the multivariate VG model, we observe high values of the implied correlation for in-the-money options, which decreases as the contract moves out-of-the-money. Further, the calibrated multivariate VG-CIR process generates implied correlation values that are consistently higher than the ones generated by the multivariate VG model calibrated to the same data set. This is due to the higher variance of the Gamma clock when compared with the integrated CIR process, which in turn generates a distribution of the underlying spread with higher variance than under the VG-CIR framework. As the spread call option price is decreasing in the correlation level, the multivariate VG model is expected to give a relatively higher price for this contract.

The multivariate construction for time-changed Lévy processes introduced in this section can be further improved to a setting in which stochastic volatility can be generated separately from the diffusion and the jump component of $\mathbf{X}(t)$, by applying individual time changes as in Huang and Wu (2004). Hence, the multidimensional model proposed in this section could be considered as an alternative to the Wishart processes approach introduced, for example, by Gourieroux (2006), Da Fonseca, Grasselli, and Tebaldi (2007) and further extended by Leippold and Trojani (2010).

## 5. Conclusions

In this note, we present an alternative construction of multivariate Lévy processes which keeps the appealing properties of the approaches existing in the literature and, at the same time, addresses their limitations. The proposed model could also be used as a platform to construct multivariate time-changed Lévy processes, allowing for a richer stochastic volatility structure.

The empirical analysis presented in this paper shows that our approach is flexible enough to accommodate the full range of possible linear dependence, from negative to positive correlation, from complete linear dependence to independence, but, at the same time, it is relatively parsimonious in terms of the number of parameters involved, as this grows linearly with the number of names in the basket. The presence of restrictions on the parameters due to convolution conditions implies some accuracy error in reproducing the margin distribution when the number of assets grows. Further, model calibration requires access to the log-returns (risk-neutral) correlation matrix, which is, however, not directly observable due to lack of actively traded multi-assets securities. Hence, current research is focussed on investigating alternative estimation methods of the parameters of the systematic process based on index options and asymptotic properties of the Lévy processes considered, in order to both relax the convolution requirement and gain information on the correlation matrix as to improve the tractability of the model, especially its calibration to market data.

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## Appendix 1. Proof of Corollary 3

To prove the result, we use the conditional covariance formula for any three random variables $\xi$, $\eta$, and $\zeta$

$$\mathbb{C}\text{ov}(\xi, \eta) = \mathbb{E}(\mathbb{C}\text{ov}(\xi, \eta \mid \zeta)) + \mathbb{C}\text{ov}(\mathbb{E}(\xi \mid \zeta), \mathbb{E}(\eta \mid \zeta)) \tag{A1}$$

(see Ross 2010, for example). Hence, let $f, g : \mathbb{R}^n \to \mathbb{R}$ be non-decreasing functions for which the covariances are defined. By properties of positive association (see, e.g. Müller and Stoyan 2002), $\mathbf{Y}(t)$ is positive associated because it has independent components; consequently, also $\mathbf{Y}(t) + \mathbf{a}z$ is positive associated for each fixed $z \in \mathbb{R}$. Therefore,

$$\mathbb{C}\mathrm{ov}(f(\mathbf{X}(t)), g(\mathbf{X}(t)) \mid Z(t)) = \mathbb{C}\mathrm{ov}(f(\mathbf{Y}(t) + \mathbf{a}Z(t)), g(\mathbf{Y}(t) + \mathbf{a}Z(t)) \mid Z(t)) \geq 0;$$

hence, its expectation is non-negative. Further, $\mathbb{E}(f(\mathbf{Y}(t) + \mathbf{a}z))$ and $\mathbb{E}(g(\mathbf{Y}(t) + \mathbf{a}z))$ are non-decreasing functions of $z$ if $a_j \geq 0$ for $j = 1, 2, \ldots, n$. As $Z(t)$ is positive associated, it follows from the properties of positive association (see, e.g. Müller and Stoyan 2002)) that the covariance between these two terms is non-negative as well. On the other hand, if $a_j \leq 0$ for all $j = 1, 2, \ldots, n$, then $(-\mathbb{E}(f(\mathbf{Y}(t) + \mathbf{a}z)))$ and $(-\mathbb{E}(g(\mathbf{Y}(t) + \mathbf{a}z)))$ are non-decreasing functions of $z$, and therefore,

$$\mathbb{C}\mathrm{ov}(\mathbb{E}(f(\mathbf{X}(t)) \mid Z(t)), \mathbb{E}(g(\mathbf{X}(t)) \mid Z(t))) = \mathbb{C}\mathrm{ov}(-\mathbb{E}(f(\mathbf{X}(t)) \mid Z(t)), -\mathbb{E}(g(\mathbf{X}(t)) \mid Z(t)))$$

is non-negative as well. The required result follows.

## Appendix 2. Time-changed Lévy processes

### A.1 *General facts*

Time-changed Lévy processes are obtained by observing a Lévy process $X(t)$ on a time scale governed by a non-negative, non-decreasing stochastic process $V(t)$. $X(t)$ is the base process, $V(t)$ is the time change, or stochastic clock, and the resulting process is $B(t) = X(V(t))$. Under the assumption of a stochastic clock independent of the base process, the process characteristic function is $\phi_B(u;t) = \phi_V(-\mathrm{i}(\varphi_X(u)); t)$. It follows by direct differentiation of the (logarithm of the) characteristic function of $B(t)$ that

$$\mathbb{E}B(t) = \mathbb{E}(X(1))\mathbb{E}(V(t)), \tag{A2}$$

$$\mathbb{V}\mathrm{ar}(B(t)) = \mathbb{V}\mathrm{ar}(X(1))\mathbb{E}(V(t)) + \mathbb{E}^2(X(1))\mathbb{V}\mathrm{ar}(V(t)), \tag{A3}$$

$$c_3(B(t)) = c_3(X(1))\mathbb{E}(V(t)) + 3\mathbb{E}(X(1))\mathbb{V}\mathrm{ar}(X(1))\mathbb{V}\mathrm{ar}(V(t)) \tag{A4}$$

$$+ \mathbb{E}^3(X(1))c_3(V(t)), \tag{A5}$$

$$\begin{aligned} c_4(B(t)) &= c_4(X(1))\mathbb{E}(V(t)) + 4c_3(X(1))\mathbb{E}(X(1))\mathbb{V}\mathrm{ar}(V(t)) + 3\mathbb{V}\mathrm{ar}^2(X(1))\mathbb{V}\mathrm{ar}(V(t)) \\ &\quad + 6\mathbb{E}^2(X(1))\mathbb{V}\mathrm{ar}(X(1))c_3(V(t)) + \mathbb{E}^4(X(1))c_4(V(t)), \end{aligned} \tag{A6}$$

from which the indices of skewness, $\gamma_1(t)$, and excess kurtosis, $\gamma_2(t)$, follow.

In the special case in which the base process is a Brownian motion with drift $X(t) = \theta t + \sigma W(t)$ for $\theta \in \mathbb{R}, \sigma > 0$, then Equations (A2)–(A6) reduce to (see, e.g. Ané and Geman 2000)

$$\mathbb{E}B(t) = \theta\mathbb{E}(V(t)), \tag{A7}$$

$$\mathbb{V}\mathrm{ar}(B(t)) = \sigma^2\mathbb{E}(V(t)) + \theta^2\mathbb{V}\mathrm{ar}(V(t)), \tag{A8}$$

$$c_3(B(t)) = 3\theta\sigma^2\mathbb{V}\mathrm{ar}(V(t)) + \theta^3 c_3(V(t)), \tag{A9}$$

$$c_4(B(t)) = \theta^4 c_4(V(t)) + 6\theta^2\sigma^2 c_3(V(t)) + 3\sigma^4\mathbb{V}\mathrm{ar}(V(t)). \tag{A10}$$

### A.2 *Proof of Equations* (20)–(22)

(i) The multivariate characteristic function of the process $\mathbf{B}(t)$ can be written as

$$\phi_{\mathbf{B}}(\mathbf{u};t) = \mathbb{E}[\mathrm{e}^{V(t)\sum_{j=1}^n b_j\varphi_{Y_j}(u_j)}\mathbb{E}(\mathrm{e}^{\mathrm{i}\sum_{j=1}^n u_j a_j Z(b_j V(t))} \mid V(t))].$$

Rearrange the sequence $(b_1, b_2, \ldots, b_n)$ in increasing order to obtain $(b_{(1)}, b_{(2)}, \ldots, b_{(n)})$, where $(1), (2), \ldots, (n)$ is a permutation of $1, 2, \ldots, n$. Then, conditioned on $V(t)$,

$$\sum_{j=1}^{n} u_j a_j Z(b_j V(t)) = \sum_{l=(1)}^{(n)} u_l a_l Z(b_l V(t))$$
$$= \sum_{l=(1)}^{(n)} u_l a_l Z(b_{(1)} V(t)) + \sum_{l=(2)}^{(n)} \left( \sum_{k=l}^{(n)} u_k a_k (Z(b_l V(t)) - Z(b_{l-1} V(t))) \right).$$

Conditioned on $V(t)$, $Z(t)$ has independent and stationary increments, and Equations (20) and (21) follow.

(ii) Equation (22) follows by direct differentiation of the multivariate characteristic function. Alternatively, the covariance can be calculated using the conditional covariance formula (A1). Due to the assumptions of independence between $Y_j(t), Z(t)$, and $V(t)$, in fact,

$$\mathbb{C}\mathrm{ov}(X_j(b_j V(t)), X_l(b_l V(t)) \mid V(t)) = a_j a_l \min(b_j, b_l) V(t) \mathbb{V}\mathrm{ar}(Z(1));$$

further $\mathbb{E}(X_j(b_j V(t)) \mid V(t)) = b_j V(t) \mathbb{E}(X_j(1))$, from which Equation (22) follows.

## Appendix 3. Cumulants of a JD process

By differentiation of the characteristic exponent, it follows

$$\mathbb{E}X(t) = (\mu + \lambda \mathbb{E}(\xi))t, \tag{A11}$$

$$\mathbb{V}\mathrm{ar}(X(t)) = (\sigma^2 + \lambda \mathbb{E}(\xi^2))t, \tag{A12}$$

$$c_3(X(t)) = \lambda \mathbb{E}(\xi^3)t, \tag{A13}$$

$$c_4(X(t)) = \lambda \mathbb{E}(\xi^4)t. \tag{A14}$$