POLITECNICO

MILANO 1863

Financial Engineering

Assignment 5, Group 16

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Contents

1. Introduction and data

In this Report our aim is to explain and price a certificate made by a basket of different stocks: AXA and ENEL. We also priced a digital option with different method's, analyzing the different approaches and the risks embedded in them. Moreover we price a call option with the Lewis Formula computing the integral with multiple methods confronting and observing the key differences and limits.

Data for ENEL and AXA

Here we summarize the Data for ENEL and AXA Underlying assets.

Ticker	ENEL	CS
Correlation	16.1%	20.0%
Dividend Yileds	2.5%	2.7%
Stock Prices	100 €	200€

Table 1: Data for ENEL and AXA

Data for Certificate Pricing

Here we summarize the data one can find in the certificate's annex (Swap Termsheet).

Principal Amount	100 mln €
Start date	18th February 2008
Maturity date	20th February 2012

The data of Party A (Bank XX in our exercise) can be summarized as follows:

s_{spol}	100 bps
Quarterly payment	Euribor 3m + s_{spol}
Daycount convention	ACT/360
Protection (P)	95%
Maturity Payment	(1-P) of the principal amount

Table 2: Party A

The data of Party B (IB in our exercise) can be summarized as follows:

Upfront payment	2% of principal amount
Participation coefficient	α
Maturity Payment	$\alpha(S(t) - P)^+$
Basket	$S(t) = \frac{1}{4}\sum_{s=1}^{4}\sum_{n=1}^{2} W_n \frac{E_s^n}{E_{s-1}^n}$
Monitoring dates	yearly

Table 3: Party B

Where:
E_s^n : Value of the nth element of the basket at monitoring date s
W_n : Weight of the nth element of the basket, each one equal to one half

Data for Digital Option

Here we summarize the data for the digital option.

Notional	10 mln €
S_0	2973.8740
Strike	ATM
Digital Payoff	5% of the notional
Settlement date	18th February 2008
Maturity	1 year
Day count convention	ACT/365

Table 4: Digital Option

Data for Call Option

Here we summarize the data for the pricing of the Call Option.

F_0	2972.153
Maturity	1 year
Discount	0.9613
α	1/2
σ	20%
κ	1
μ	3
t	1
x_{min}	-25%
x_{max}	25%
Δx	1%
Settlement date	18th February 2008
Maturity	1 year

Table 5: Call Option

In this case we have taken F_0, the Time To Maturity and the discount factor from the previous exercise.

Data for Volatility Surface Calibration

In this case we have to calibrate considering constant weights, $\alpha = \frac{1}{3}$ and F_0, the Time To Maturity and the discount factor from the Digital pricing exercise.

2. Certificate pricing

In this first part we aim to determine the participation coefficient α. Indeed we can see that the correct value for this coefficient is the value such that the expected values of the two parties' cashflows are equal. In other words we must find the α such that party A's net present value is the same as the NPV of party B.
The certificate is written on two underlyings (ENEL and AXA), for which we assumed Black dynamics. The coupon is computed as follows:

$$\alpha(S(t)-P)^+$$
$$\text{where } S(t)=\frac{1}{4}\sum_{s=1}^{4}\sum_{n=1}^{2}\frac{1}{2}\frac{E_s^n}{E_{s-1}^n}$$

where E_s^n is the value of the nth element at date s.
Thanks to the fact that an ISDA CSA agreement has been signed, we can safely ignore the problem of counterparty risk in our pricing operation. Furthermore, for the discount factors and zero rates we assumed a single curve model and used the values we have found via bootstrap and neglected the dynamics of interest rates.
In order to obtain our result we ran a Monte Carlo simulation with $N_{sim}=10^7$. The only real simulation we have to run is on the underlyings, since everything other than the final coupon can be computed in a deterministic fashion under our assumptions.
Indeed the payments of party A are the following:

$$NPV_A(t_0)=(1-B(t_0,T_n))+s_{spol}\cdot BPV(t_0,T_n)+(1-P)\cdot B(t_0,T_n)$$

And the payments of party B are:

$$NPV_B(t_0)=X+B(t_0,T_n)\cdot\alpha\cdot\mathbb{E}[(S(t)-P)^+]$$

Hence, after running the Monte Carlo simulation and estimating the expected value of the basket, we can write:

$$\alpha=\frac{NPV_A(t_0)-X}{B(t_0,T_n)\cdot\mathbb{E}[(S(t)-P)^+]}$$

In the NPVs we have omitted the Principal amount since in the end it simplifies in the computations.
We then computed the confidence interval for α by simply estimating the variance of the above quantity since we have only one random variable and it is guaranteed to be different than zero.
The results are the following:

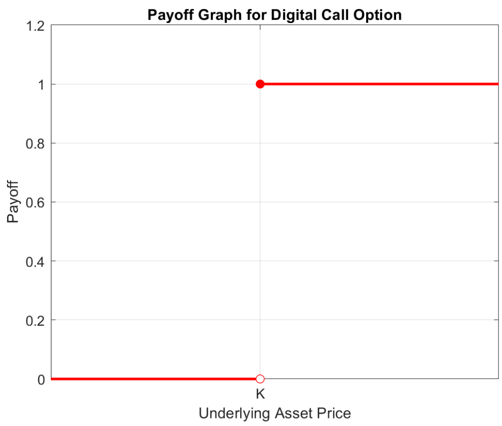
α	3.351
IC at 95%	[3.3504, 3.3515]

3. Pricing digital option

In this section, we delve into the analysis of two different methods for pricing a digital option, highlighting the distinction between pricing based on the Black model and considering the volatility smile in the implied volatility curve.
To begin, we recall the pricing formula for a European Call Option under Black's model:

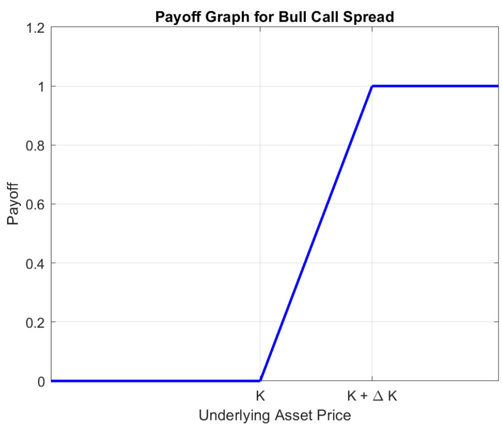
$$C(K,T)=B(t_0,t)(F_0N(d_1)-KN(d_2))$$
$$d_{1,2}=\frac{\ln(F_0/K)}{\sqrt{(t-t_0)\,\sigma^2}}\pm\frac{1}{2}\sqrt{(t-t_0)\,\sigma^2}$$

where F_0 represents the forward price at time t_0 of the underlying asset at time t.
Now, let's introduce the payoff of a digital option, also known as a cash-or-nothing option. This exotic option pays a predetermined coupon if, at maturity, the underlying asset price exceeds the strike price K.



$$\begin{cases} 1 & \text{if } S_t \geq K \\ 0 & \text{if } S_t < K \end{cases}$$

Combining simple European Call options, we can attempt to replicate the payoff of a digital option. Utilizing a "bull-spread" strategy, the following payoff emerges:



$$\begin{cases} \text{long call K} \\ \text{short call K} + \Delta\text{K} \end{cases}$$

By taking the limit as ΔK tends to zero, we approach a payoff that resembles that of a digital option. This leads to the following computations:

$$\begin{aligned} dc_B(K,T) &= \lim_{\Delta K \to 0} \frac{C(K,T) - C(K+\Delta K,T)}{\Delta K} \\ &= -\lim_{\Delta K \to 0} \frac{C(K+\Delta K,T) - C(K,T)}{\Delta K} \\ &= -\frac{\partial C(K,T)}{\partial K} = B(t_0,t)N(d_2) \end{aligned}$$

However, under the Black model dynamics, the assumptions are too strong to accurately price digital options. This is because the model assumes a constant volatility across all strikes, whereas real volatilities exhibit dependency on strikes and maturities, resulting in a volatility surface with a range of volatility values. Now, let us fix the time to maturity to 1 year and observe the volatility smile as depicted in the provided dataset. Plotting the smile reveals a decrease in volatility as strikes increase.

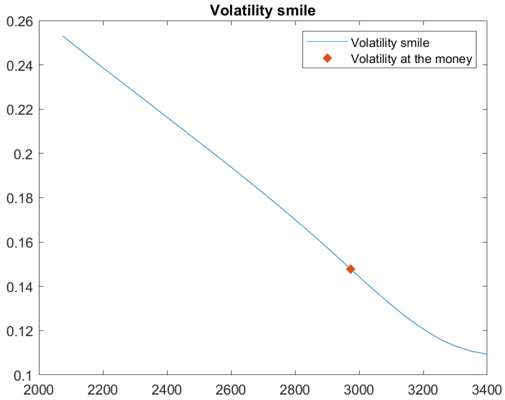


Figure 1: volatility smile

Using an implied volatility approach provides a more accurate pricing method for digital options. The concept involves leveraging the skew of volatility and the vega, which denotes the sensitivity of the call price to changes in volatility σ. Similar to our previous approach, we take the limit as ΔK tends to zero, but this time, the call price depends on both K and σ, which itself depends on K.

$$\begin{aligned} dc_{vol}(K,T) &= \lim_{\Delta K \to 0} \frac{C(K,\sigma(K)) - C(K+\Delta K, \sigma(K+\Delta K))}{\Delta K} \\ &= -\frac{d}{dK} C(K,\sigma(K)) \\ &= \underbrace{-\frac{\partial}{\partial K} C(K,\sigma(K))}_{Black\ term} - \underbrace{\frac{\partial \sigma(K)}{\partial K}}_{skew} \cdot \underbrace{\frac{\partial C(K,\sigma(K))}{\partial \sigma(K)}}_{vega} \end{aligned}$$

Subsequently, we present the graphs of the skew and vega relative to the strikes K. The vega typically exhibits a bell-shaped curve. For any fixed maturity T, the implied volatility decreases with strike K, so the skew has a negative value and it approaches zero as the option is significantly out of the money.

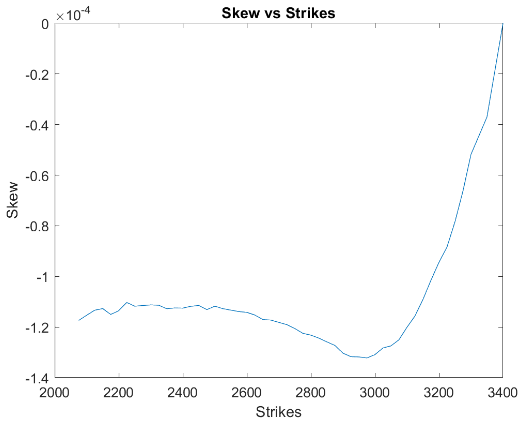


Figure 2: Skew

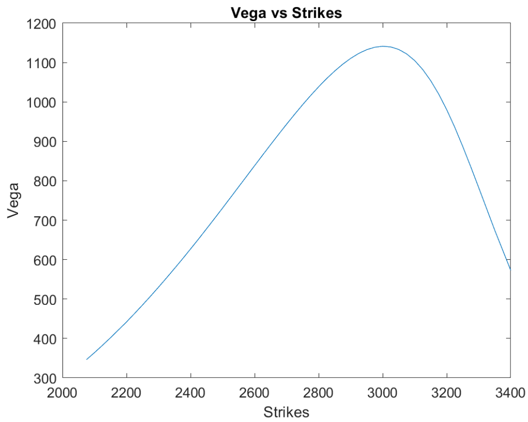


Figure 3: Vega

Two principal explanations for the skew are discussed in the literature: risk aversion and the leverage effect. Risk aversion prompts investors to protect their portfolios by purchasing out-of-the-money puts, leading to higher demand for options with lower strikes. Meanwhile, the leverage effect suggests that equity volatility increases as equity value decreases; indeed the total value of company assets, i.e. debt + equity, is a more natural candidate to follow GBM leading to:

$$\sigma_E \approx \frac{V}{E} \sigma_V$$

Then, by interpolating the skew and vega in a neighborhood of strike K corresponding to the option contract, we determine the implied volatility approach's price for the digital option:

$$dc_{vol} = dc_B - skew \cdot \nu$$

This yields the following results:

Black Model	225274.3393 €
Implied Volatility	300521.2646 €
Discrepancy	75246.93 €
Discrepancy (%)	33.40%

Table 6: Prices for a digital option

Analyzing the price difference of the digital option using the two methodologies reveals a significant discrepancy. This variation can be interpreted as a mispricing of exotic payoffs within the Black model framework. The constant volatility approach fails to adequately incorporate digital risk, where the binary payoff exhibits a jump discontinuity at the strike value. Consequently, relying on a constant volatility may create arbitrage opportunities for other market participants. This underscores the need for more sophisticated methodologies that leverage information from the volatility surface. In our approach, we utilize the skew and vega as corrective factors to adjust the Black price, resulting in a more reliable valuation that accounts for digital risk. As this risk pertains to product selling, the correct option price is higher than one that neglects this risk. As a final note, let us remark that the discrepancy is quite significant (more than 30%), this is due to the fact that the vega has it maximum value around the ATM-spot strike price.

4. Pricing

In this section we pass to pricing options using characteristic functions and the lewis formula.
In our specific case we will price Call options for each strike corresponding to a log-moneyness that goes from $x = -25\%$ to $x = 25\%$ with a 1% step.
Initially we chose to approach the problem with a normal mean-variance mixture model with parameters given in the data section. Let us recall that in such a model the logarithm of the forward and the LaPlace exponent can be written as:

$$f_t = \sigma\,\sqrt{t\,G}\,g - \left(\frac{1}{2}+\eta\right)\,\sigma^2 t\,G - \ln\mathcal{L}(\eta) \quad f_t = \ln\left(\frac{F_t}{F_0}\right)$$
$$\ln\left(\mathcal{L}(\omega)\right) = \frac{\Delta t}{\kappa}\frac{1-\alpha}{\alpha}\left[1-\left(1+\frac{\omega\kappa\sigma^2}{1-\alpha}\right)^{\alpha}\right]$$

Let us also recall that while g is a standard normal and in general G is a random variable with unitary mean and variance $\kappa/\Delta t$ whose distribution depends on the value of α, for our particular case of $\alpha = 1/2$ G follows an inverse Gaussian distribution $G \sim IG(1, \Delta t/\kappa)$.
For the initial price of the forward F_0, the time to maturity and the discount we used the same values as of the previous point.
Furthermore, let us recall that the Lewis formula states that:

$$\frac{c(x)}{B(t_0,t)F_0} = 1 - e^{-x/2}\underbrace{\int_{-\infty}^{\infty}\frac{e^{-i\xi x}}{2\pi}\phi\left(-\xi-\frac{i}{2}\right)\frac{1}{\xi^2+\frac{1}{4}}\,d\xi}_{I(x)}$$

where $\phi(\xi)$ is the characteristic function of our chosen model. In particular in the case of a normal mean-variance mixture we can write:

$$\phi(\xi) = e^{-i\xi\ln\mathcal{L}(\eta)}\cdot\mathcal{L}\left(\frac{\xi^2+i(1+2\eta)\xi}{2}\right)$$

As a final remark let us note that since in our case $\eta = 3 > -\bar{\omega} = \frac{1-\alpha}{\kappa\sigma^2} = -12.5$ we can apply the Lewis Formula thanks to Lukiec's theorem that states that a function that is analytical near the origin is also analytical in an horizontal strip. Indeed for us the condition $\eta > -\bar{\omega}$ is equivalent to the LaPlace exponent (and hence the whole characteristic function) being well-behaved in a neighborhood of the origin.

4.1. Quadrature

In order to evaluate the integral we applied the quadrature method. Specifically, we employed the *quadgk* function already integrated into MATLAB, which employs high-order global adaptive quadrature. In our case we checked that the integration interval was wide enough for the integrand to decay (i.e. we checked that the value at ξ_1 was less than 10^{-10}). Moreover we provided *quadgk* with a maximum number of points to use, but left it up to the function to determine the optimal number of points to use. Taking $M = 20$ (i.e. $maxN = 2^{20}$), we took this more robust method as a frame of reference for the FFT algorithm.

4.2. FFT

An alternative, and potentially faster approach is the FFT algorithm. Indeed, we can easily see that the integral can be written as a Fourier transform. This is true if we perform the following:

$$I(x) = \mathcal{F}[f(\xi)](x) = \int_{-\infty}^{\infty} f(\xi)e^{-i\xi x}d\xi \text{ if we choose } f(\xi) = \frac{1}{2\pi}\phi\left(-\xi - \frac{i}{2}\right)\frac{1}{\xi^2 + \frac{1}{4}}$$

In order to perform the Fourier transform we can leverage the FFT algorithm that also allows us to compute the Fourier Transform for multiple choices of the log-moneyness at the same time. We also need to make a choice of parameters, in order for the FFT to work the following constraints must be satisfied:

$$\begin{cases} x_1 & = -x_N \\ \xi_1 & = -\xi_N \\ dx & = \frac{x_N - x_1}{N} \\ d\xi & = \frac{\xi_N - \xi_1}{N} \\ d\xi \cdot dx & = \frac{2\pi}{N} \end{cases}$$

Hence of the original 7 parameters we need only specify two. As a rule of thumb the first one will always be the number of points to use in the grid to perform the discretization step of the algorithm (we chose $N = 2^M$) the other can be chosen between either ξ_1 or dx. In our case we chose to model our FFT using M and dx. Indeed modeling the FFT using the log-moneyness step lets us anchor our modelization to real financial quantities.

The FFT algorithm was performed on the grid obtained from these parameters, then we interpolated these values to determine the value of the integral at the log-moneynesses of interest. For further details on how the FFT is performed see the appendix.

Initially, since we want a fine mesh that encompasses our required log-moneynesses, we chose a step of $dx = 0.0025$. Empirically, having fixed the value of dx as above, we observed that for $M = 12$ there was a significant MSE between the two methods, so we rejected this choice. Comparing the MSE for different choices of M, we detected that the FFT algorithm reaches the same values as the quadrature for $M = 15$ (with a tolerance of 1bp). Of course taking a larger value of dx lets the algorithm converge faster with a smaller number of points, (indeed if we take $dx = 0.01$ the two methods converge for about $M = 12$) but this gives less points to interpolate between when obtaining our results.

The prices ranges from around 20 € to about 700 €. These results are quite reasonable. We observe a steep increase in price for in-the-money options (indeed it is almost linear) while out-of-the-money options show a more gradual behaviour that gets close to zero for very low value of the log-moneyness.

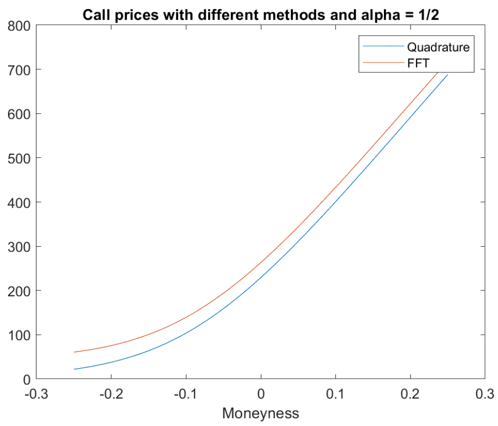


Figure 4: Quadrature vs FFT (M=12)

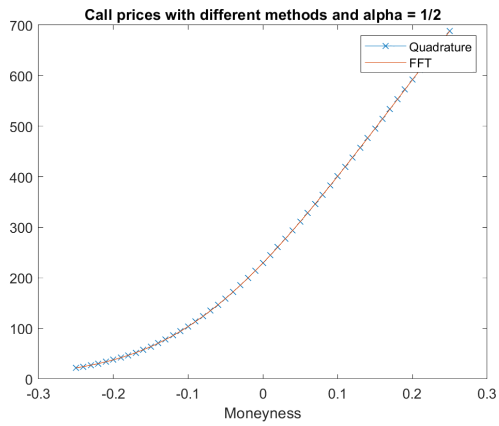


Figure 5: Quadrature vs FFT (M=15)

4.3. MonteCarlo method

To check our results in a non-integral fashion, we also applied a Monte Carlo simulation approach.
In order to perform this, we must draw two random variables: the standard gaussian $g \sim \mathcal{N}(0,1)$ and an inverse Gaussian $G \sim IG(1, \Delta t/\kappa)$. Then we simply write the f_t as above and apply the exponential, the price can thus be obtained as the mean of the discounted payoff. We can easily check that the values overlap with the quadrature and FFT methods:

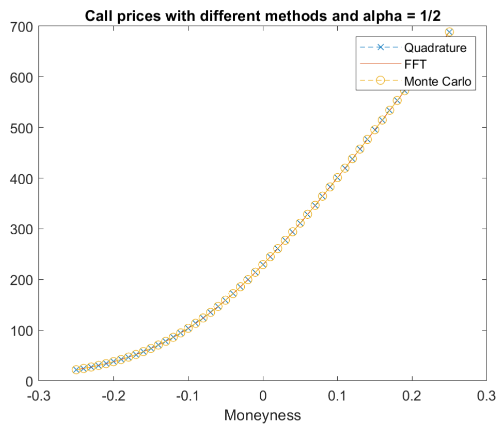


Figure 6: Monte Carlo ($N_{sim} = 10^7$) vs Quadrature vs FFT

In order to be sure that the drawn Inverse Gaussian Variable is correctly sampled we computed the first 4 analytical moments and compared them with the numerical ones:

Numerical	Analytical
0.9998	1
1.998	2
6.984	7
36.877	37

Table 7: Analytical and Numerical moment of the IG

The analytical values were obtained as follows:

$$\mu_1 = \mu$$
$$\mu_2 = \frac{\mu^2(\lambda+\mu)}{\lambda}$$
$$\mu_3 = \frac{\mu^3(\lambda^2+3\lambda\mu+3\mu^2)}{\lambda^2}$$
$$\mu_4 = \frac{5\mu^2}{\lambda}\mu_3 + \mu^2\mu_2$$

If we plot the three methods, as well as the real observed prices, we obtain the following graph:

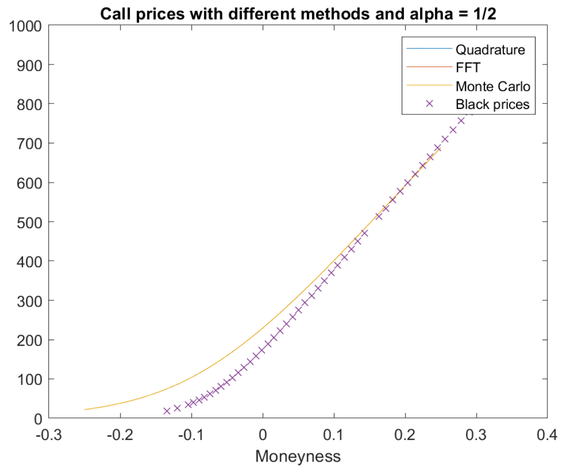


Figure 7: $\alpha = 1/2$

We can easily observe that all three methods do not capture the real observed prices. This makes sense, the financial parameters of the model are σ, κ and η while α is a tempering parameter for the distribution of G. Hence we can deduce that the given parameters do not fully capture the market prices.

4.4. Mean-Variance mixture model with $\alpha = 2/3$

To further explore possible choices for the hyper-parameters for the model we also explored a mean-variance mixture with $\alpha = \frac{2}{3}$.
If we ran both methods with the new α we obtain, as can be seen from the graph below, results which are remarkably similar to the case of $\alpha = 1/2$. Indeed the difference is quite negligible. Indeed, evaluating the difference from the two methods as

$$\textbf{Difference} = \frac{\sum_{n=1}^{N} |C_{\frac{2}{3}} - C_{\frac{1}{2}}|}{N}$$

we found an average distance between the two cases of approximately 2.702 which is around 1% of the average price found in the $\alpha = \frac{1}{2}$ case.

Both cases give results which are remarkably close. Thus, again, the model does not fully capture the market prices.
To us this makes rather sense, α is a tempering parameter for the distribution of G, it is sensible to say that changing the α when pricing simple derivatives such as a vanilla call-option does not influence the results and their convergence in a tangible way.

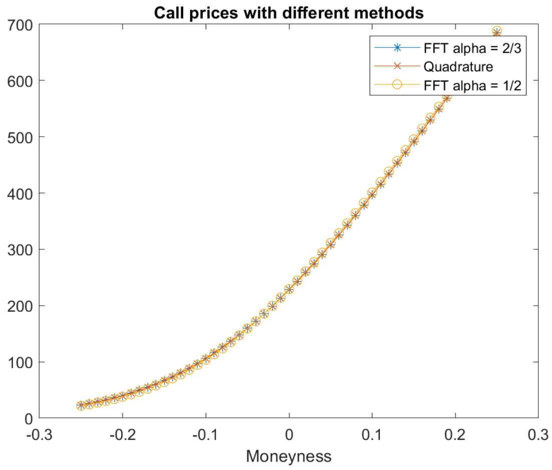


Figure 8: $\alpha = 2/3$ vs α=1/2

5. Volatility surface calibration

In this last section we set out to calibrate our mean-variance mixture model to more closely represent the market prices. In particular we chose a model with $\alpha = 1/3$ and proceeded to calibrate the parameters σ, κ and η using constant weights for all strikes. As parameters for the FFT we used $M = 15$ and $dz = 0.0025$, just like we did above.
For the calibration process, we decided to use the MATLAB function *fmincon*: this built-in function not only finds the required minimum but also allows us to set some constraints on our parameters.
In our case, we require that:

$$\eta \geq -\frac{1-\alpha}{\kappa\sigma^2} \qquad \sigma, \kappa \in \mathbb{R}^+$$

After doing this, we obtain the following results:

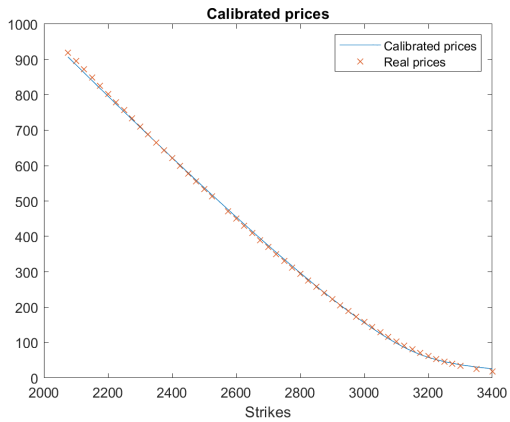


Figure 9: Calibrated prices

and parameters:

σ	0.12416
κ	1.6639
η	7.057

Observing the plot above, we can discern that the results obtained via the *fmincon* function very closely match the observed prices of the call option.

Hence now our model, almost perfectly matches the market prices. To have a better grasp of whether our model correctly reproduces the market we also investigated the implied volatilities of our model.
In order to find the implied volatility of the model, we inverted the model prices using the Black formula. Performing this operation leads to the following values:

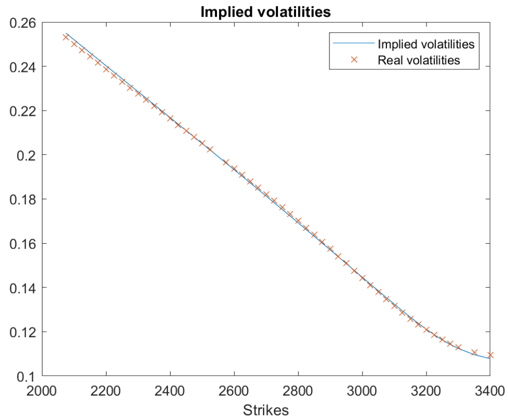


Figure 10: Implied volatilities

We also computed two metrics to judge our model:

MAPE	MRE
0.42542 %	1.4146 %

The MAPE is the mean absolute percentage error and gives an idea of how much on average we mismatch the observed volatilities. On the other hand the MRE is the maximum relative error and gives us an upper bound for the volatility error. We can observe that both metrics are rather low and give us rather good results.
This is quite reasonable as a result. The η of our model is 7, representing an asymmetric volatility curve, while our κ, which represents the vol-of-vol, has a value of 1.66 meaning we are far from the Black-Scholes model. Finally the σ represents the average value of the volatility which is around 0.12.

6. Appendix

6.1. Fast Fourier Transform

The following integral:

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(\xi) e^{-i\xi x} d\xi$$

can be approximated as follows:

$$\hat{f}(x_k) \approx \underbrace{d\xi e^{-i\xi_1 x_k}}_{prefactor} FFT(k)$$

where the FFT is computed as:

$$FFT(k) = \sum_{j=1}^{N} f_j \omega^{(j-1)(k-1)} \text{ where } \omega = e^{-\frac{2\pi i}{N}}$$

and where $f_j = e^{-1x_1(j-1)d\xi} f(\xi_j)$
Let us remark that the FFT algorithm must be run on the function $\tilde{f}$ and not directly f in order to incorporate the correct exponential values.

7. References

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