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Homework #8
Recurrences and Generating Functions

3. Consider a recurrence relation $a_n = 2a_{n-1} + 2a_{n-2}$ with $a_0 = a_1 = 2$. Solve it (*i.e.* find a closed formula) and show how it can be used to estimate the value of $\sqrt{3}$ (hint: observe $\lim_{n\to\infty} a_n/a_{n-1}$). After that, devise an algorithm for constructing a recurrence relation with integer coefficients and initial conditions that can be used to estimate the square root $\sqrt{k}$ of a given integer k.

$k^n - 2k^{n-1} - 2k^{n-2} = 0$ — хар. ур.

$k^2 - 2k - 2 = 0$

$k_{1,2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3} \iff a(1+\sqrt{3})^n + b(1-\sqrt{3})^n$

Найдём a, b:

$$\begin{cases} a_0 = a + b \\ a_1 = a(1+\sqrt{3}) + b(1-\sqrt{3}) \end{cases};$$

Подставив $a_1 = a_0 = 2$ (из усл.) получим $a = b = 1$, тогда

$a_n = (1+\sqrt{3})^n + (1-\sqrt{3})^n$

$\lim\limits_{n\to\infty} \frac{a_n}{a_{n-1}} = 1 + \sqrt{3}$

$\sqrt{3} = \frac{a_n}{a_{n-1}} - 1$

Пусть вместо „3" будет произвольное k

Найдём рекуррентное соотношение для $a_n = (1+\sqrt{k})^n + (1-\sqrt{k})^n$

Зная, что $\lambda_{1,2} = 1 \pm \sqrt{k}$, тогда

$(\lambda - 1 - \sqrt{k})(\lambda - 1 + \sqrt{k}) = 0$

Соответствует $a_n = 2a_{n-1} + (k-1)a_{n-2}$

4. Find a closed formula for the n-th term of the sequence with generating function $\frac{3x}{1-4x} + \frac{1}{1-x}$.

$$\frac{3x}{1-4x} = 3x \sum_{n=0}^{\infty} (4x)^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$3x \sum_{n=0}^{\infty} (4x)^n + \sum_{n=0}^{\infty} x^n$$

$$\begin{cases} a_n = 3 \cdot 4^{n-1} + 1, \text{ при } n > 0 \\ a_0 = 1 \end{cases}$$

5. Given the generating function $G(x) = \frac{5x^2+2x+1}{(1-x)^3}$, decompose it into partial fractions and find the sequence that it represents.

$$5x^2+2x+1 = A(1-x)^2 + B(1-x) + C$$

$$\begin{cases} A = 5 \\ -2A - B = 2 \\ A + B + C = 1 \end{cases} ; \begin{cases} A = 5 \\ B = -12 \\ C = 8 \end{cases}$$

$$G(x) = \frac{5}{1-x} - \frac{12x}{(1-x)^2} + \frac{8}{(1-x)^3}$$

$$G(x) = \sum_{n=0}^{\infty} 5x^n - \sum_{n=1}^{\infty} 12nx^n + \sum_{n=2}^{\infty} 8\frac{(n-1)n}{2} x^n$$

$$a_n = 5 - 12n + 4(n-1)n$$

6. Pell–Lucas numbers are defined by $Q_0 = Q_1 = 2$ and $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$. Derive the corresponding generating function and find a closed formula for the n-th Pell–Lucas number.

Пусть $Q(x) = \sum_{n=0}^{\infty} Q_n x^n$

Для данных Q:

$$Q(x) = Q_0 + Q_1 x + \sum_{n=2}^{\infty} Q_n x^n = 2 + 2x + 2x \sum_{n=2}^{\infty} (2Q_{n-1} + Q_{n-2}) x^n$$

$$Q(x) = 2 + 2x + 2x \sum_{n=1}^{\infty} Q_n x^n + x^2 \sum_{n=0}^{\infty} Q_n x^n = 2 + 2x + 2x(Q(x) - Q_0) + x^2 Q(x) =$$

$$= 2 + 2x + 2x Q(x) - 4x + x^2 Q(x)$$

$$Q(x)(1 - 2x - x^2) = 2 - 2x$$

$$Q(x) = \frac{2 - 2x}{1 - 2x - x^2}$$

8. Find the number of non-negative integer solutions to the Diophantine equation $3x + 5y = 100$ using generating functions.

$$(1 + x^3 + x^6 \ldots)(1 + x^5 + x^{10} \ldots)$$

$$\frac{1}{1-x^3} \cdot \frac{1}{1-x^5} = \left(\sum_{n=0}^{\infty} x^{3n}\right)\left(\sum_{n=0}^{\infty} x^{5n}\right)$$

Нужно найти коэфф. при x^{100}

Ответ: 7 (по Wolfram)

7. For each given recurrence relation, derive the corresponding generating function and find a closed formula for the n-th term of the sequence.

(a) $a_n = 2a_{n-1} - a_{n-2}$ with $a_0 = 3$, $a_1 = 5$

(b) $a_n = a_{n-1} + a_{n-2} - a_{n-3}$ with $a_0 = 1$, $a_1 = 1$, $a_2 = 5$

(c) $a_n = a_{n-1} + n$ with $a_0 = 0$

(d) $a_n = a_{n-1} + 2a_{n-2} + 2^n$ with $a_0 = 2$, $a_1 = 1$

a) $A = 3 + 5x + 7x^2 + 9x^3 \ldots$

$-xA = -3x - 5x^2 - 7x^3 - 9x^4 \ldots$

$$(1-x)A = 3 + 2x + 2x^2 + 2x^3 \ldots = 1 + \frac{2}{1-x} = \frac{3-x}{1-x}$$

$$A = \frac{3-x}{(1-x)^2} = 1 + 2(n+1) = 2n + 3$$

$a_n = 2n + 3$

b) Аналогичные действия

$$A = \frac{1 + 3x^2}{(1-x)(1-x^2)}$$

$a_n = (-1)^n + 2n$

c) Аналогичные действия

$$A = \frac{x}{(1-x)^3}$$

$$a_n = \frac{(n+1)(n+2)}{2}$$

d) Аналогичные действия

$$A = \frac{2 - 5x + 6x^2}{(1-2x)^2 (1+x)}$$

$$a_n = \frac{1}{9}\left(3n \cdot 2^{n+1} + 5 \cdot 2^n + 13(-1)^n\right)$$

9. Consider a $2n$-digit ticket number to be "lucky" if the sum of its first n digits is equal to the sum of its last n digits. Each digit (including the first one!) in a number can take value from 0 to 9. For example, a 6-digit ticket 345 264 is lucky since $3+4+5=2+6+4$.

(a) Find the number of lucky 6-digit and 8-digit tickets.

(b) Find the generating function for the number of $2n$-digit lucky tickets.

(c) Find a closed formula for the number of $2n$-digit lucky tickets.

b) $(1+x+x^2 \dots +x^9)$ – для выбора одной цифры

$(1+x+x^2 \dots +x^9)^n$ – для выбора n цифр

$$G(x) = (1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9)^{2n}$$

a) 6-digit:

Максимальная сумма половины – 27, тогда ответ – коэфф при x^{27}

$$G(x) = x^{54} + 6x^{53} + 21x^{52} + 56x^{51} + 126x^{50} + 252x^{49} + 462x^{48} + 792x^{47} + 1287x^{46} + 2002x^{45} + 2997x^{44} + 4332x^{43} + 6062x^{42} + 8232x^{41} + 10872x^{40} + 13992x^{39} + 17577x^{38} + 21582x^{37} + 25927x^{36} + 30492x^{35} + 35127x^{34} + 39662x^{33} + 43917x^{32} + 47712x^{31} + 50877x^{30} + 53262x^{29} + 54747x^{28} + 55252x^{27} + 54747x^{26} + 53262x^{25} + 50877x^{24} + 47712x^{23} + 43917x^{22} + 39662x^{21} + 35127x^{20} + 30492x^{19} + 25927x^{18} + 21582x^{17} + 17577x^{16} + 13992x^{15} + 10872x^{14} + 8232x^{13} + 6062x^{12} + 4332x^{11} + 2997x^{10} + 2002x^9 + 1287x^8 + 792x^7 + 462x^6 + 252x^5 + 126x^4 + 56x^3 + 21x^2 + 6x + 1$$

Ответ: 55252

8-digit:

Максимальная сумма половины – 36, тогда ответ – коэфф при x^{36}

$$G(x) = x^{72} + 8x^{71} + 36x^{70} + 120x^{69} + 330x^{68} + 792x^{67} + 1716x^{66} + 3432x^{65} + 6435x^{64} + 11440x^{63} + 19440x^{62} + 31760x^{61} + 50100x^{60} + 76560x^{59} + 113640x^{58} + 164208x^{57} + 231429x^{56} + 318648x^{55} + 429220x^{54} + 566280x^{53} + 732474x^{52} + 929672x^{51} + 1158684x^{50} + 1419000x^{49} + 1708575x^{48} + 2023680x^{47} + 2358840x^{46} + 2706880x^{45} + 3059100x^{44} + 3405600x^{43} + 3735720x^{42} + 4038560x^{41} + 4303545x^{40} + 4521000x^{39} + 4682700x^{38} + 4782360x^{37} + 4816030x^{36} + 4782360x^{35} + 4682700x^{34} + 4521000x^{33} + 4303545x^{32} + 4038560x^{31} + 3735720x^{30} + 3405600x^{29} + 3059100x^{28} + 2706880x^{27} + 2358840x^{26} + 2023680x^{25} + 1708575x^{24} + 1419000x^{23} + 1158684x^{22} + 929672x^{21} + 732474x^{20} + 566280x^{19} + 429220x^{18} + 318648x^{17} + 231429x^{16} + 164208x^{15} + 113640x^{14} + 76560x^{13} + 50100x^{12} + 31760x^{11} + 19440x^{10} + 11440x^9 + 6435x^8 + 3432x^7 + 1716x^6 + 792x^5 + 330x^4 + 120x^3 + 36x^2 + 8x + 1$$

Ответ: 4816 030

c) Можно было использовать binomial theorem для расширения G(x), но там сложно считать

Пусть w – primitive 10th root of unity, тогда $1+w+w^2 \dots w^9 = 0$ (по определению)

Теперь заменим x на w^i

$$a_k = \frac{1}{10} \sum_{i=0}^{9} (w^i)^k \cdot (1+w^i+w^{2i}+w^{3i}+w^{4i}+w^{5i}+w^{6i}+w^{7i}+w^{8i}+w^{9i})^{2n}$$

1. For each given recurrence relation, find the first five terms, derive the closed-form solution, and check it by substituting it back to the recurrence relation.
   (a) $a_n = a_{n-1} + n$ with $a_0 = 2$
   (b) $a_n = 2a_{n-1} + 2$ with $a_0 = 1$
   (c) $a_n = 3a_{n-1} + 2^n$ with $a_0 = 5$
   (d) $a_n = 4a_{n-1} + 5a_{n-2}$ with $a_0 = 1$, $a_1 = 17$
   (e) $a_n = 4a_{n-1} - 4a_{n-2}$ with $a_0 = 3$, $a_1 = 11$
   (f) $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ with $a_{0,1,2} = 3, 2, 6$

a) $a_0 = 2$

$a_1 = 3$

$a_2 = 5$

$a_3 = 8$

$a_4 = 12$

$$a_n = a_0 + \sum_{k=1}^{n} k \Rightarrow$$

$$a_n = 2 + \frac{n(n+1)}{2} = a_0 + \frac{n(n+1)}{2}$$

c) $a_0 = 5$

$a_1 = 17$

$a_2 = 55$

$a_3 = 173$

$a_4 = 535$

$a_n = A \cdot 3^n - 2^{n+1}$

$A = 7$

$a_n = 7 \cdot 3^n - 2^{n+1}$

e) $a_0 = 3$

$a_1 = 11$

$a_2 = 32$

$a_3 = 84$

$a_4 = 208$

$r^2 = 4r - 4$

$r_{1,2} = 2$

$a_n = (A + Bn)2^n$

$$\begin{cases} A = 3 \\ B = 2{,}5 \end{cases}$$

$a_n = (3 + 2{,}5n)2^n$

b) $a_0 = 1$

$a_1 = 4$

$a_2 = 10$

$a_3 = 22$

$a_4 = 46$

$a_n = A \cdot 2^n + B$

$A \cdot 2^n + B = 2(A \cdot 2^{n-1} + B) + 2$

$$\begin{cases} A = 3 \\ B = -2 \end{cases} \Rightarrow$$

$a_n = 3 \cdot 2^n - 2$

d) $a_0 = 1$

$a_1 = 17$

$a_2 = 73$

$a_3 = 377$

$a_4 = 1873$

Решим $r^2 - 4r - 5 = 0$

$r_1 = 5$

$r_2 = -1$

$a_n = A \cdot 5^n + B \cdot (-1)^n$

$$\begin{cases} A = 3 \\ B = -2 \end{cases}$$

$a_n = 3 \cdot 5^n - 2 \cdot (-1)^n$

f) $a_0 = 3$

$a_1 = 2$

$a_2 = 6$

$a_3 = 8$

$a_4 = 18$

$r^3 = 2r^2 + r - 2$

$r_1 = 1;\ r_2 = 2;\ r_3 = -1$

$a_n = A \cdot 1^n + B \cdot 2^n + C \cdot (-1)^n$

$A = 1;\ B = 1;\ C = 1$

$a_n = 1 + 1 \cdot 2^n + (-1)^n$

2. Solve the following recurrences by applying the Master theorem. For the cases where the Master theorem does not apply, use the Akra–Bazzi method. In cases where neither of these two theorems apply, explain why and solve the recurrence relation by closely examining the recursion tree. Solutions must be in the form $T(n) \in \Theta(\ldots)$.

(a) $T(n) = 2T(n/2) + n$
(b) $T(n) = T(3n/4) + T(n/4) + n$
(c) $T(n) = 3T(n/2) + n$
(d) $T(n) = 2T(n/2) + n/\log n$
(e) $T(n) = 6T(n/3) + n^2 \log n$
(f) $T(n) = T(3n/4) + n \log n$
(g) $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n$
(h) $T(n) = T(n/2) + T(n/4) + 1$
(i) $T(n) = T(n/2) + T(n/3) + T(n/6) + n$
(j) $T(n) = 2T(n/3) + 2T(2n/3) + n$
(k) $T(n) = \sqrt{2n}\,T(\sqrt{2n}) + \sqrt{n}$
(l) $T(n) = \sqrt{2n}\,T(\sqrt{2n}) + n$

a) Master theorem:

$a = 2$

$b = 2$

$f(n) = n$

$\log_2 2 = 1$

$f(n) = \Theta(n^{\log_b a})$ $\Big|$ => case 2 => $T(n) \in \Theta(n \log n)$

b) Akra-Bazzi:

$a_1 = 1$ $\quad a_2 = 1$

$b_1 = \frac{3}{4}$ $\quad b_2 = \frac{1}{4}$ $\quad p = 0$

$T(n) \in \Theta(n \log n)$

c) Master theorem:

$a = 3$

$b = 2$

$f(n) = n$

$\log_2 3 > 1$

$f(n) = \Theta(n^{\log_b a - \varepsilon})$ $\Big|$ => case 1 => $T(n) \in \Theta(n^{\log_2 3})$

d) Master theorem:

$a = 2$

$b = 2$

$f(n) = \frac{n}{\log n}$

$\log_2 2 = 1$

$f(n) = \Theta(n)$ $\Big|$ => case 2 => $T(n) \in \Theta(n \log\log n)$

e) Master theorem:

$a = 6$

$b = 3$

$f(n) = n^2 \log n$

$\log_3 6 < 2$

$f(n) = \Theta(n^{\log_b a + \varepsilon})$ $\Big|$ => case 3 => $T(n) \in \Theta(n^2 \log n)$

f) Master theorem:

$a = 1$

$b = \frac{4}{3}$

$f(n) = n \log n$

$\log_{\frac{4}{3}} 1 = 0$

$f(n) = \Theta(n^2)$ $\Big|$ => case 3 => $T(n) \in \Theta(n \log n)$

=>

2. Solve the following recurrences by applying the Master theorem. For the cases where the Master theorem does not apply, use the Akra–Bazzi method. In cases where neither of these two theorems apply, explain why and solve the recurrence relation by closely examining the recursion tree. Solutions must be in the form $T(n) \in \Theta(\dots)$.

(a) $T(n) = 2T(n/2) + n$
(b) $T(n) = T(3n/4) + T(n/4) + n$
(c) $T(n) = 3T(n/2) + n$
(d) $T(n) = 2T(n/2) + n/\log n$
(e) $T(n) = 6T(n/3) + n^2 \log n$
(f) $T(n) = T(3n/4) + n \log n$
(g) $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n$
(h) $T(n) = T(n/2) + T(n/4) + 1$
(i) $T(n) = T(n/2) + T(n/3) + T(n/6) + n$
(j) $T(n) = 2T(n/3) + 2T(2n/3) + n$
(k) $T(n) = \sqrt{2n}\,T(\sqrt{2n}) + \sqrt{n}$
(l) $T(n) = \sqrt{2n}\,T(\sqrt{2n}) + n$

g) Recursion tree:
Рекурсия разбивает на 2 равные части
Глубина $\log n$
На каждом уровне $\Theta(n)$

$T(n) \in \Theta(n \log n)$

h) Akra-Bazzi:
$a_1 = 1 \quad a_2 = 2$
$b_1 = \frac{1}{2} \quad b_2 = \frac{1}{4} \quad p = 0$

$T(n) \in \Theta(n \log n)$

i) Akra-Bazzi:
$a_1 = 1 \quad a_2 = 1 \quad a_3 = 1$
$b_1 = \frac{1}{2} \quad b_2 = \frac{1}{3} \quad b_3 = \frac{1}{6} \quad p = 0$

$T(n) \in \Theta(n \log n)$

j) Recursion tree:
Разделяем на $\frac{n}{3}$ и $\frac{2n}{3}$
На каждом уровне $\Theta(n)$
Глубина $\log \frac{3}{2}$

$T(n) \in \Theta(n \log_{\frac{3}{2}} n)$

$T(n) \in \Theta(n \log n)$

k) Akra-Bazzi:
$a_1 = \sqrt{2}$
$b_1 = \sqrt{2}$
$p = 1$

$T(n) \in \Theta(n\sqrt{n})$

l) Akra-Bazzi:
$a_1 = \sqrt{2}$
$b_1 = \sqrt{2}$
$p = 1$

$T(n) \in \Theta(n^2)$