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Homework 2 Binary Relations

1. For each given relation $R_i \subseteq M_i^2$, determine whether it is *reflexive, irreflexive, coreflexive, symmetric, antisymmetric, asymmetric, transitive, antitransitive, semiconnex, connex, left/right Euclidean.* Provide a counterexample for each non-complying property (*e.g.*, "transitivity does not hold for $x, y, z = (3, 1, 2)$"). Organize your answer in a table (*e.g.*, columns — relations, rows — properties).
   (a) $M_1 = \mathbb{R}$
   $x \, R_1 \, y \leftrightarrow |x - y| \leq 1$
   (b) $M_2 = \mathcal{P}(\{a, b, c\})$
   $R_2 = \text{“}\subseteq\text{”}$
   (c) $M_3 = \{a, b, c, d\} \quad \|R_3\| = \begin{bmatrix} 0&1&0&1 \\ 0&0&0&1 \\ 1&1&0&0 \\ 0&0&1&0 \end{bmatrix}$
   (d) $M_4 = \{\text{“rock”}, \text{“scissors”}, \text{“paper”}\}$
   $R_4 = \{\langle x, y\rangle \mid x \text{ beats } y\}$

● – rock
✂ – scissors
📄 – paper

	a	b	c	d
reflexive	true	true	$x=a$	$x=$ ●
irreflexive	$x=2$	$x=\{a\}$	true	true
coreflexive	$x=1$ $y=2$	$x=\varnothing$ $y=\{a\}$	$x=a$ $y=b$	$x=$ ● $y=$ ✂
symmetric	true	$x=\varnothing$ $y=\{a\}$	$x=a$ $y=b$	$x=$ 📄 $y=$ ●
antisymmetric	$x=2$ $y=1$	true	true	true
asymmetric	$x=2$ $y=1$	$x=\{a\}$ $y=\{a\}$	true	true
transitive	$x=0; y=1$ $z=2$	true	$x=c; y=a$ $z=b$	$x=$ ● $y=$ ✂ ; $z=$ 📄
antitransitive	$x=0; y=0{,}1$ $z=0{,}2$	$x=\{a\}; y=\{a,b\}$ $z=\{a,b,c\}$	$x=a; y=b$ $z=d$	true
semiconnex	$x=0$ $y=4$	$x=\{a\}$ $y=\{b\}$	true	true
connex	$x=0$ $y=3$	$x=\{a\}$ $y=\{b\}$	$x=a$ $y=a$	$x=$ 📄 $y=$ 📄
left Euclidean	$x=1; y=0$ $z=2$	$x=\{a,b\}$ $y=\{a\}; z=\{b\}$	$x=b; y=a$ $z=c$	true
right Euclidean	$x=1; y=0$ $z=2$	$x=\{a\}; y=\{a,b\}$ $z=\{a,c\}$	$x=c; y=b$ $z=a$	true

2. Prove (rigorously) or disprove (by providing a counterexample) the following statements about arbitrary homogeneous relations $R \subseteq M^2$ and $S \subseteq M^2$:

(a) If R and S are *reflexive*, then $R \cap S$ is so.
(b) If R and S are *symmetric*, then $R \cap S$ is so.
(c) If R and S are *transitive*, then $R \cap S$ is so.
(d) If R and S are *reflexive*, then $R \cup S$ is so.
(e) If R and S are *symmetric*, then $R \cup S$ is so.
(f) If R and S are *transitive*, then $R \cup S$ is so.

a)

$\forall a_i \in M : (a_i R a_i) \wedge (a_i S a_i)$

$R \cap S = \{(a, b) \mid (aRb) \wedge (aSb)\}$

Т.к. $\forall a_i \in M : (a_i R a_i) \wedge (a_i S a_i)$, то $\forall a_i \in M : (a_i, a_i) \in R \cap S$

Значит $R \cap S$ - reflexive

b) $aRb \Rightarrow bRa$
$aSb \Rightarrow bRa$ (т.к. symmetric)

Построим табл. истинности:

aRb	bRa	aSb	bSa	$a(R\cap S)b$	$b(R\cap S)a$
0	0	0	0	0	0
0	0	1	1	0	0
1	1	0	0	0	0
1	1	1	1	1	1

Видно, что:
$a(R\cap S)b \rightarrow b(R\cap S)a$
$b(R\cap S)a \rightarrow a(R\cap S)b$

Значит $R \cap S$ - symmetric

c) $(aRb) \wedge (bRc) \Rightarrow (aRc)$
$(aSb) \wedge (bSc) \Rightarrow (aSc)$

Построим табл. истинности:

aRb	bRc	aRc	aSb	bSc	aSc	$a(R\cap S)b$	$b(R\cap S)c$	$a(R\cap S)c$
0	0		0	0		0	0	
0	0		0	1		0	0	
0	0		1	0		0	0	
0	0		1	1	1	0	0	
0	1		0	0		0	0	
0	1		0	1		0	1	
0	1		1	0		0	0	
0	1		1	1	1	0	1	
1	0		0	0		0	0	
1	0		0	1		0	0	
1	0		1	0		1	0	
1	0		1	1	1	1	0	
1	1	1	0	0		0	0	
1	1	1	0	1		0	1	
1	1	1	1	0		1	0	
1	1	1	1	1	1	1	1	1

$(a(R\cap S)b) \wedge (b(R\cap S)c) \Rightarrow (a(R\cap S)c) = 1$

Значит $R \cap S$ - transitive

f)

$\|R\| = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}$

$\|R \cup S\| = \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}$

$\|S\| = \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$

Значит $R \cup S$ – !transitive

d)

$\forall a_i \in M: (a_i R a_i) \wedge (a_i S a_i)$

$R \cup S = \{(a, b) \mid (aRb) \wedge (aSb)\}$

т.к. $\forall a_i \in M: (a_i R a_i) \wedge (a_i S a_i)$, то $\forall a_i \in M: (a_i, a_i) \in R \cup S$

Значит $R \cup S$ - reflexive

e) $aRb \Rightarrow bRa$
$aSb \Rightarrow bRa$ (т.к. symmetric)

Построим табл. истинности:

aRb	bRa	aSb	bSa	$a(R\cup S)b$	$b(R\cup S)a$
0	0	0	0	0	0
0	0	1	1	0	0
1	1	0	0	0	0
1	1	1	1	1	1

Видно, что:

$a(R\cup S)b \rightarrow b(R\cup S)a$

$b(R\cup S)a \rightarrow a(R\cup S)b$

Значит $R \cup S$ - symmetric

3. An equinumerosity relation $\sim$ over sets is defined as follows: $A \sim B \leftrightarrow |A| = |B|$.

(a) Show that $\sim$ is an equivalence relation over finite sets.

(b) Show that $\sim$ is an equivalence relation over infinite sets[1].

(c) Find the quotient set of $\mathcal{P}(\{a, b, c, d\})$ by $\sim$.

relation in $\boxed{}$

$\boxed{r}$ – reflexive

$\boxed{s}$ – symmetric

$\boxed{t}$ – transitive

a) $|A| = n \in \mathbb{N}_0$

$\boxed{r}$ $|A| = |A| \Leftrightarrow n = n$

$\boxed{s}$ $|A| = |B| \Rightarrow |B| = |A|$

$\boxed{t}$ $|A| = |B| \wedge |B| = |C| \Rightarrow |A| = |C|$

$\Rightarrow$ $\sim$ equivalence relation

b) Infinite sets равномощные, если между ними можно построить биекцию

$\boxed{r}$ Every element of set A соотнесён с самим собой

Т.к. $|A| = |B|$, существует биекция между элементами A и B

$a_1 \leftrightarrow b_2$

$a_2 \leftrightarrow b_1$

...

Можно поменять a_i и b_i местами

$b_1 \leftrightarrow a_2$

$b_2 \leftrightarrow a_1$

...

$\boxed{s}$ Получается каждому элементу из B соответствует элемент A и наоборот

Существует биекция между (A и B) и (B и C)

$a_1 \leftrightarrow b_3$ $\quad$ $b_1 \leftrightarrow c_1$

$a_2 \leftrightarrow b_1$ $\quad$ $b_2 \leftrightarrow c_3$

$a_3 \leftrightarrow b_2$ $\quad$ $b_3 \leftrightarrow c_2$

... $\quad$ ...

Каждому элементу множества B соответствует элемент множества A и C

$a_1 \leftrightarrow c_2$

$a_2 \leftrightarrow c_1$

$a_3 \leftrightarrow c_3$

$\boxed{t}$ Теперь каждому элементу множества A соответствует элемент множества C

$\Rightarrow$ $\sim$ equivalence relation

c) $P(\{a,b,c,d\}) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\},$

$\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}\}$

$[\varnothing]_\sim = \{\varnothing\}$

$[\{a\}]_\sim = \{\{a\}, \{b\}, \{c\}, \{d\}\}$

$[\{a,b\}]_\sim = \{\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}\}$

$[\{a,b,c\}]_\sim = \{\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$

$[\{a,b,c,d\}]_\sim = \{\{a,b,c,d\}\}$

$P(\{a,b,c,d\})_\sim = \{[\varnothing]_\sim, [\{a\}]_\sim, [\{a,b\}]_\sim, [\{a,b,c\}]_\sim, [\{a,b,c,d\}]_\sim\}$

5. The characteristic function f_S of a set S is defined as follows:

$$f_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

Let A and B be finite sets. Show that for all $x \in \mathfrak{U}$:

(a) $f_{\overline{A}}(x) = 1 - f_A(x)$
(b) $f_{A \cap B}(x) = f_A(x) \cdot f_B(x)$
(c) $f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$
(d) $f_{A \oplus B}(x) = f_A(x) + f_B(x) - 2 f_A(x) \cdot f_B(x)$

d)

$$\left.\begin{array}{ll} x \in A & \begin{cases} x \notin \overline{A} \\ f_A(x) = 1 \\ f_{\overline{A}}(x) = 0 \end{cases} \\ x \notin A & \begin{cases} x \in A \\ f_{\overline{A}}(x) = 1 \\ f_A(x) = 0 \end{cases} \end{array}\right\} \quad f_{\overline{A}}(x) = 1 - f_A(x)$$

$x \in A$	$f_A(x)$	$f_{\overline{A}}(x)$	$1 - f_A(x)$
0	0	1	1
1	1	0	0

b)

$$x \in A \wedge x \in B \; \begin{cases} f_A(x) = 1 \\ f_B(x) = 1 \\ f_{A \cap B}(x) = 1 \end{cases}$$

$$x \in A \wedge x \notin B \; \begin{cases} f_A(x) = 1 \\ f_B(x) = 0 \\ f_{A \cap B}(x) = 0 \end{cases}$$

$$x \notin A \wedge x \in B \; \begin{cases} f_A(x) = 0 \\ f_B(x) = 1 \\ f_{A \cap B}(x) = 0 \end{cases}$$

$$x \notin A \wedge x \notin B \; \begin{cases} f_A(x) = 0 \\ f_B(x) = 0 \\ f_{A \cap B}(x) = 0 \end{cases}$$

$x \in A$	$x \in B$	$f_A(x)$	$f_B(x)$	$f_{A \wedge B}(x)$	$f_A(x) \cdot f_B(x)$
0	0	0	0	0	0
0	1	0	1	0	0
1	0	1	0	0	0
1	1	1	1	1	1

c) $f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x) = f_A(x) + f_B(x) - f_{A \cap B}(x)$

$x \in A$	$x \in B$	$f_A(x)$	$f_B(x)$	$f_{A \wedge B}(x)$	$f_{A \cup B}(x)$	$f_A(x) + f_B(x) - f_{A \cap B}(x)$
0	0	0	0	0	0	0
0	1	0	1	0	1	1
1	0	1	0	0	1	1
1	1	1	1	1	1	1

d) $f_{A \oplus B} = f_A(x) + f_B(x) - 2 f_A(x) \cdot f_B(x) = (f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)) - f_A(x) \cdot f_B(x) = f_{A \cup B}(x) - f_{A \cap B}(x)$

$x \in A$	$x \in B$	$f_{A \wedge B}(x)$	$f_{A \cup B}(x)$	$f_{A \cup B}(x) - f_{A \cap B}(x)$
0	0	0	0	0
0	1	0	1	1
1	0	0	1	1
1	1	1	1	0

$A \oplus B = \{ x \mid (x \in A \cup x \in B) \wedge \overline{(x \in A \wedge x \in B)} \}$

6. Find the error in the "proof" of the following "theorem".

"*Theorem*": Let R be a relation on a set A that is symmetric and transitive. Then R is reflexive.

"*Proof*": Let $a \in A$. Take an element $b \in A$ such that $\langle a, b \rangle \in R$. Because R is symmetric, we also have $\langle b, a \rangle \in R$. Now using the transitive property, we can conclude that $\langle a, a \rangle \in R$ because $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$.

Доказательство предполагает, что для $a \in A$ $\exists\, b \in A : aRb$

Если такое $b \in A$ не сущ., то нельзя применить свойства symmetric и transitive

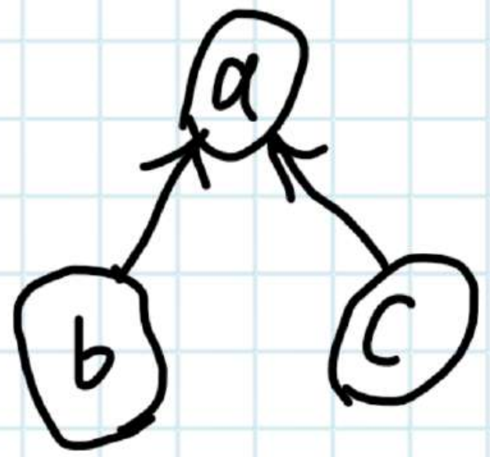
Counter-Example:

Пусть R – empty relation

$$\|R\| = \begin{matrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{matrix}$$

R is symmetric и transitive, но !reflexive

7. Give an example of a relation R on the set $\{a, b, c\}$ such that the symmetric closure of the reflexive closure of the transitive closure of R is not transitive.



$$\|R\| = \begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{matrix}$$

$$\|R^{+}\| = \begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{matrix}$$ – transitive замыкание

$$\|R^{+=}\| = \begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}$$ – reflexive замыкание of transitive замыкание

$$\|R^{+=\sim}\| = \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}$$ – symmetric замыкание of reflexive замыкание of transitive замыкание

$$\begin{array}{c|ccc} & a & b & c \\ a & 1 & 1 & 1 \\ b & 1 & 1 & 0 \\ c & 1 & 0 & 1 \end{array}$$

$cRa \wedge aRb$, но $c \not R b$, so relation is !transitive

4. Let R_θ be a relation of θ-similarity (clearly, $\theta \in [0;1] \subseteq \mathbb{R}$) of finite non-empty sets defined as follows: a set A is said to be θ-*similar* to B *iff* the Jaccard index $\mathrm{Jac}(A,B) = \frac{|A\cap B|}{|A\cup B|}$ for these sets is at least θ, *i.e.* $\langle A, B\rangle \in R_\theta \leftrightarrow \mathrm{Jac}(A,B) \geq \theta$.

(a) Draw the graph of a relation $R_\theta \subseteq \{A_i\}^2$, where $\theta = 0.25$, $A_1 = \{1,2,5,6\}$, $A_2 = \{2,3,4,5,7,9\}$, $A_3 = \{1,4,5,6\}$, $A_4 = \{3,7,9\}$, $A_5 = \{1,5,6,8,9\}$.

(b) Determine whether θ-similarity is a tolerance relation.

(c) Determine whether θ-similarity is an equivalence relation.

a) $Jac(A_1, A_1) = 1$

$Jac(A_1, A_2) = \frac{1}{4}$

$Jac(A_1, A_3) = \frac{3}{5}$

$Jac(A_1, A_4) = 0$

$Jac(A_1, A_5) = \frac{1}{2}$

$Jac(A_2, A_1) = \frac{1}{4}$

$Jac(A_2, A_2) = 1$

$Jac(A_2, A_3) = \frac{1}{4}$

$Jac(A_2, A_4) = \frac{1}{2}$

$Jac(A_2, A_5) = \frac{2}{9}$

$Jac(A_3, A_1) = \frac{3}{5}$

$Jac(A_3, A_2) = \frac{1}{4}$

$Jac(A_3, A_3) = 1$

$Jac(A_3, A_4) = 0$

$Jac(A_3, A_5) = \frac{1}{2}$

$Jac(A_4, A_1) = 0$

$Jac(A_4, A_2) = \frac{1}{2}$

$Jac(A_4, A_3) = 0$

$Jac(A_4, A_4) = 1$

$Jac(A_4, A_5) = \frac{1}{7}$

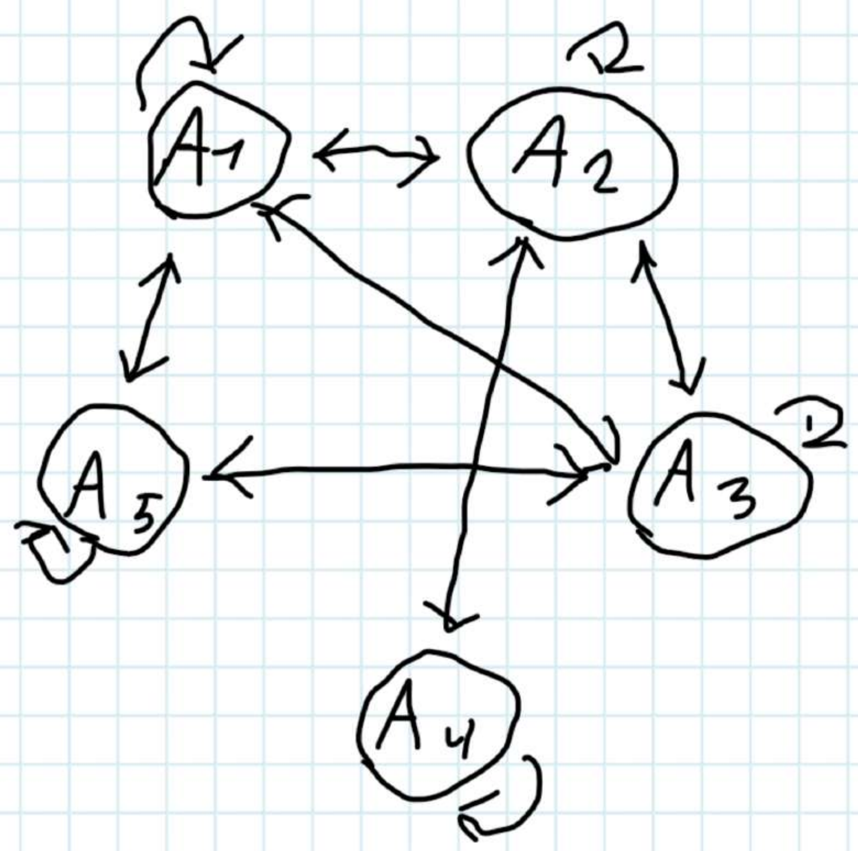
$Jac(A_5, A_1) = \frac{1}{2}$

$Jac(A_5, A_2) = \frac{2}{9}$

$Jac(A_5, A_3) = \frac{1}{2}$

$Jac(A_5, A_4) = \frac{1}{7}$

$Jac(A_5, A_5) = 1$



R_θ	A_1	A_2	A_3	A_4	A_5
A_1	1	1	1	0	1
A_2	1	1	1	1	0
A_3	1	1	1	0	1
A_4	0	1	0	1	0
A_5	1	0	1	0	1

b)

$Jac(A, A) \geq \theta \in [0; 1] \subseteq \mathbb{R}$

$Jac(A, A) = \frac{|A \cap A|}{|A \cup A|} = \frac{|A|}{|A|} = 1$

$Jac(A, A) = 1 \geq \theta$

Тогда $A R_\theta A$ и R_θ – reflexive relation

$A R_\theta B \Rightarrow B R_\theta A$

$Jac(A, B) \geq \theta$

$Jac(A, B) = \frac{|A \cap B|}{|A \cup B|} = \frac{|B \cap A|}{|B \cup A|} = Jac(B, A)$

$Jac(B, A) = Jac(A, B) \geq \theta$

Тогда $B R_\theta A$ и R_θ – symmetric relation

$\Rightarrow R_\theta$ – tolerance relation

c) $A R_\theta B \wedge B R_\theta C \Rightarrow A R_\theta C$

$Jac(A, B) \geq \theta$

$Jac(B, C) \geq \theta$

$\frac{|A \cap B|}{|A \cup B|} \geq \theta$

$\frac{|B \cap C|}{|B \cup C|} \geq \theta$

$A = \{1, 2\}$

$B = \{2, 3\}$

$C = \{3, 4\}$

$\theta = \frac{1}{3}$

$\frac{|A \cap B|}{|A \cup B|} = \frac{1}{3}$

$\frac{|B \cap C|}{|B \cup C|} = \frac{1}{3}$

$\frac{|A \cap C|}{|A \cup C|} = 0$

$\Rightarrow R_\theta$ – !transitive

Тогда R_θ – !equivalence relation

8. Prove or disprove the following statements about the functions f and g:

(a) If f and g are injections, then $g \circ f$ is also an injection.
(b) If f and g are surjections, then $g \circ f$ is also a surjection.
(c) If f and $f \circ g$ are injections, then g is also an injection.
(d) If f and $f \circ g$ are surjections, then g is also a surjection.

a) $f: X \to Y$
$g: Y \to Z$
$g \circ f: X \to Z$

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \quad (\forall\, x_1, x_2 \in X)$

$g(y_1) = g(y_2) \Rightarrow y_1 = y_2 \quad (\forall\, y_1, y_2 \in Y)$

$g(f(x_1)) = g(f(x_2))$

Т.к g – инъекция, то $f(x_1) = f(x_2)$
Т.к. f – инъекция, то $x_1 = x_2$

Тогда $g(f(x_1)) = g(f(x_2)) \Rightarrow x_1 = x_2$

Тогда $g \circ f$ – инъекция

b) $f: X \to Y$ $\quad \forall y \in Y\ \exists x \in X: y = f(x)$
$g: Y \to Z$ $\quad \forall z \in Z\ \exists y \in Y: z\, g(y)$
$g \circ f: X \to Z$

Т.к. g – сюрьекция, то $\forall z \in Z\ \exists y \in Y: z = g(y)$
Т.к. f – сюрьекция, то $\forall y \in Y\ \exists x \in X: y = f(x)$

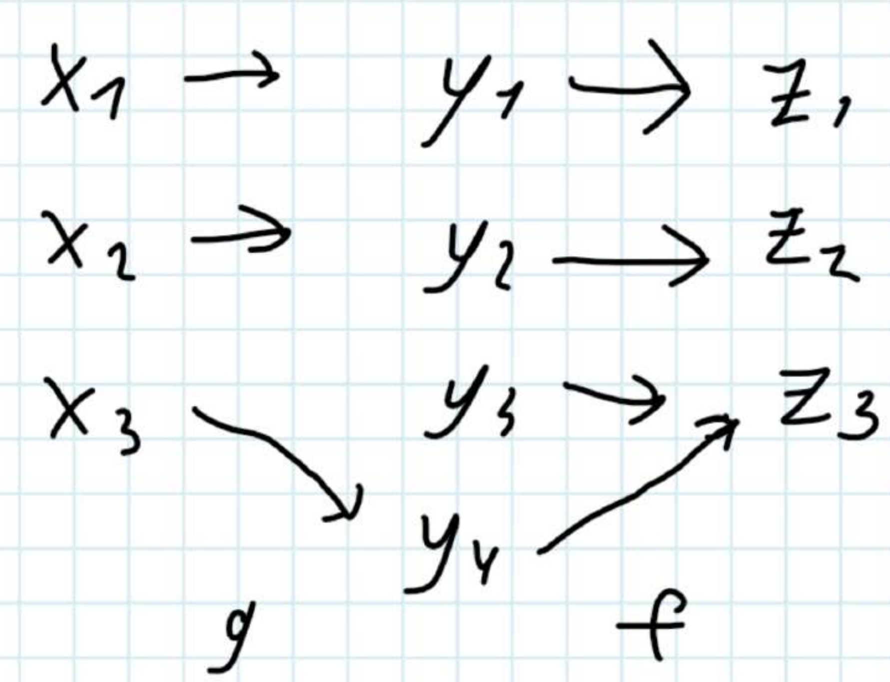
Получаем $\forall z \in Z\ \exists x \in X: z = g(f(x))$
Значит $g \circ f$ – сюрьекция

c) $f: Y \to Z$
$g: X \to Y$
$f \circ g: X \to Z$

$g(x_1) = f(x_2) \quad (\forall\, x_1, x_2 \in X)$
$f(y_1) = g(y_2) \Rightarrow y_1 = y_2 \quad (\forall\, y_1, y_2 \in Y)$
$f(g(x_1)) = f(g(x_2)) \Rightarrow x_1 = x_2 \quad (\forall\, x_1, x_2 \in X)$

Т.к. $f \circ g$ – инъекция и при $g(x_1) = g(x_2)$
$f(g(x_1)) = f(g(x_2))$, то $x_1 = x_2$

$g(x_1) = g(x_2) \Rightarrow f(g(x_1)) = f(g(x_2)) \Rightarrow x_1 = x_2$
$g(x_1) = g(x_2) \Rightarrow x_1 = x_2$
Тогда g – инъекция

d)


f – сюрьективно
$f \circ g$ – сюрьективно
но g – ! сюрьективно

11. Prove that a set S is infinite if and only if there is a proper subset $A \subset S$ such that there is a one-to-one correspondence (bijection) between A and S.

1) Пусть S – бесконечное множество $\{a_1, a_2, a_3 \dots\}$

Т.к. $A \subset S$, то в A нет хотя бы одного элемента S (пусть это будет a_1)

Попробуем провести биекцию:

S	A
a_1	a_2
a_2	a_3
a_3	a_4
$\dots$	
a_n	a_{n+1}

Биекция существует

2) Пусть S конечное множество $\{a_1, a_2, a_3 \dots a_n\}$ $n \in \mathbb{N}$

В таком случае $|S| = n \in \mathbb{N}$

Т.к. $A \subset S$, то $|A| < n \in \mathbb{N}$

Т.к. A и S конечны и неравномощны, то невозможно провести биекцию

Из 1) и 2) можно сделать вывод, что S – бесконечное множество iff, когда существует множество $A \subset S$ такое, что можно провести биекцию между A и S