SVD Sample Problems

Problem 1. Find the singular values of the matrix $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$.

Solution. We compute AA^T. (This is the smaller of the two symmetric matrices associated with A.) We get $AA^T = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$. We next find the eigenvalues of this matrix. The characteristic polynomial is $\lambda^3 - 6\lambda^2 + 6\lambda = \lambda(\lambda^2 - 6\lambda + 6)$. This gives three eigenvalues: $\lambda = 3 + \sqrt{3}$, $\lambda = 3 - \sqrt{3}$ and $\lambda = 0$. Note that all are positive, and that there are two nonzero eigenvalues, corresponding to the fact that A has rank 2.

For the singular values of A, we now take the square roots of the eigenvalues of AA^T, so $\sigma_1 = \sqrt{3+\sqrt{3}}$ and $\sigma_2 = \sqrt{3-\sqrt{3}}$. (We don't have to mention the singular values which are zero.)

Problem 2. Find the singular values of the matrix $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Solution. We use the same approach: $AA^T = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$. This has characteristic polynomial $\lambda^2 - 10\lambda + 9$, so $\lambda = 9$ and $\lambda = 1$ are the eigenvalues. Hence the singular values are 3 and 1.

Problem 3. Find the singular values of $A = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and find the SDV decomposition of A.

Solution. We compute AA^T and find $AA^T = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. The characteristic polynomial is

$$\begin{aligned} -\lambda^3 + 10\lambda^2 - 16\lambda &= -\lambda(\lambda^2 - 10\lambda + 16) \\ &= -\lambda(\lambda - 8)(\lambda - 2) \end{aligned}$$

So the eigenvalues of AA^T are $\lambda = 8, \lambda = 2, \lambda = 0$. Thus the singular values are $\sigma_1 = 2\sqrt{2}, \sigma_2 = \sqrt{2}$ (and $\sigma_3 = 0$).

To give the decomposition, we consider the diagonal matrix of singular values $\Sigma = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Next, we find an orthonormal set of eigenvectors for AA^T. For $\lambda = 8$, we find an eigenvector $(1, 2, 1)$ - normalizing gives $p_1 = (\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})$. For $\lambda = 2$ we find $p_2 = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, and finally for $\lambda = 0$ we get $p_3 = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$.

This gives the matrix $P = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

Finally, we have to find an orthogonal set of eigenvectors for $A^TA == \begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix}$.

This can be done in two ways. We show both ways, starting with orthogonal diagonalization. We already know that the eigenvalues will be $\lambda = 8, \lambda = 2, \lambda = 0$. This gives eigenvectors $q_1 = (\frac{1}{\sqrt{6}}, \frac{3}{\sqrt{12}}, \frac{1}{\sqrt{12}})$, $q_2 = (\frac{1}{\sqrt{3}}, 0, -\frac{2}{\sqrt{6}})$ and $q_3 = (\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2})$. Put these together to get

$$Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{12}} & 0 & -\frac{1}{2} \\ \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{6}} & \frac{1}{2} \end{bmatrix}$$

For a quicker method, we calculate the columns of Q using those of P using the formula

$$p_i = \frac{1}{\sigma_i} A^T p_i.$$

Thus we calculate

$$p_1 = \frac{1}{\sigma_1} A^T p_1 = \frac{1}{\sqrt{8}} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} = q_1$$

and similarly for the other two columns.

Either way we can now we verify that we have $A = P\Sigma Q^T$.

Problem 4. Find the SDV of the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

Solution. We first compute

$$AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad A^TA = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

We see immediately that the eigenvalues of AA^T are $\lambda_1 = \lambda_2 = 2$ (and hence that the eigenvalues of A^TA are 2 and 0, both with multiplicity 2), and thus the matrix A has singular value $\sigma_1 = \sigma_2 = \sqrt{2}$.

Next, an orthonormal basis of eigenvectors of AA^T is $p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $p_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. (You can choose any orthonormal basis for $\mathbb{R}^2$ here, but this one makes computation easiest.) Thus we set

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Lastly we have to find Q. We use the formula

$$q_1 = \frac{1}{\sigma_1} A^T p_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and

$$q_2 = \frac{1}{\sigma_2} A^T p_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

We also need q_3 and q_4 but we can't compute those using the same formula, since we just ran our of p_i's. However, we know that the q_1, q_2, q_3, q_4 should be an orthonormal basis for $\mathbb{R}^4$, so we need to choose q_3 and q_4 in such a way that this indeed works out. We choose

$$q_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \qquad q_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

giving

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

It is now easy to check that $A = P\Sigma Q^T$, where $\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}$.

Note: we could also have diagonalized $A^T A$ to obtain Q, but we need to be careful, because if we choose the eigenvectors in the wrong way, we don't get $A = P\Sigma Q^T$; however, this can always be fixed by multiplying the eigenvectors by -1 as needed.