COST FUNCTIONS

Saba Bukhari

Cost Functions

Short-Run Variations in Cost

Dr. Manuel Salas-Velasco

Cost Concepts Defined

- Cost is the value of the inputs used to produce its output; e.g. the firm hires labor, and the cost is the wage rate that must be paid for the labor services
- **Total cost** (TC) is the full cost of producing any given level of output, and it is divided into two parts:
  - **Total fixed cost** (TFC): it is the part of the TC that doesn't vary with the level of output
  - **Total variable cost** (TVC): it is the part of the TC that changes directly with the output

Average Costs

- **Average total cost** (ATC) is the total cost per unit of output

$$ATC = \frac{TC}{units\,of\ output}$$

- **Average fixed cost** (AFC) is the fixed cost per unit of output

$$AFC = \frac{TFC}{units\ of\ output}$$

- **Average variable cost** (AVC) is the variable cost per unit of output

$$AVC = \frac{TVC}{units\ of\ output}$$

Marginal costs

- **Marginal cost** (incremental cost) is the increase in total cost resulting from increasing the level of output by one unit

$$MC = \frac{\text{Change in total cost}}{\text{Change in output}} = \frac{\Delta TC}{\Delta Q}$$

- Since some of total costs are fixed costs, which do not change as the level of output changes, marginal cost is also equal to the increase in variable cost, that results when output is increased by one unit

Cost Functions

Short-Run Cost Curves

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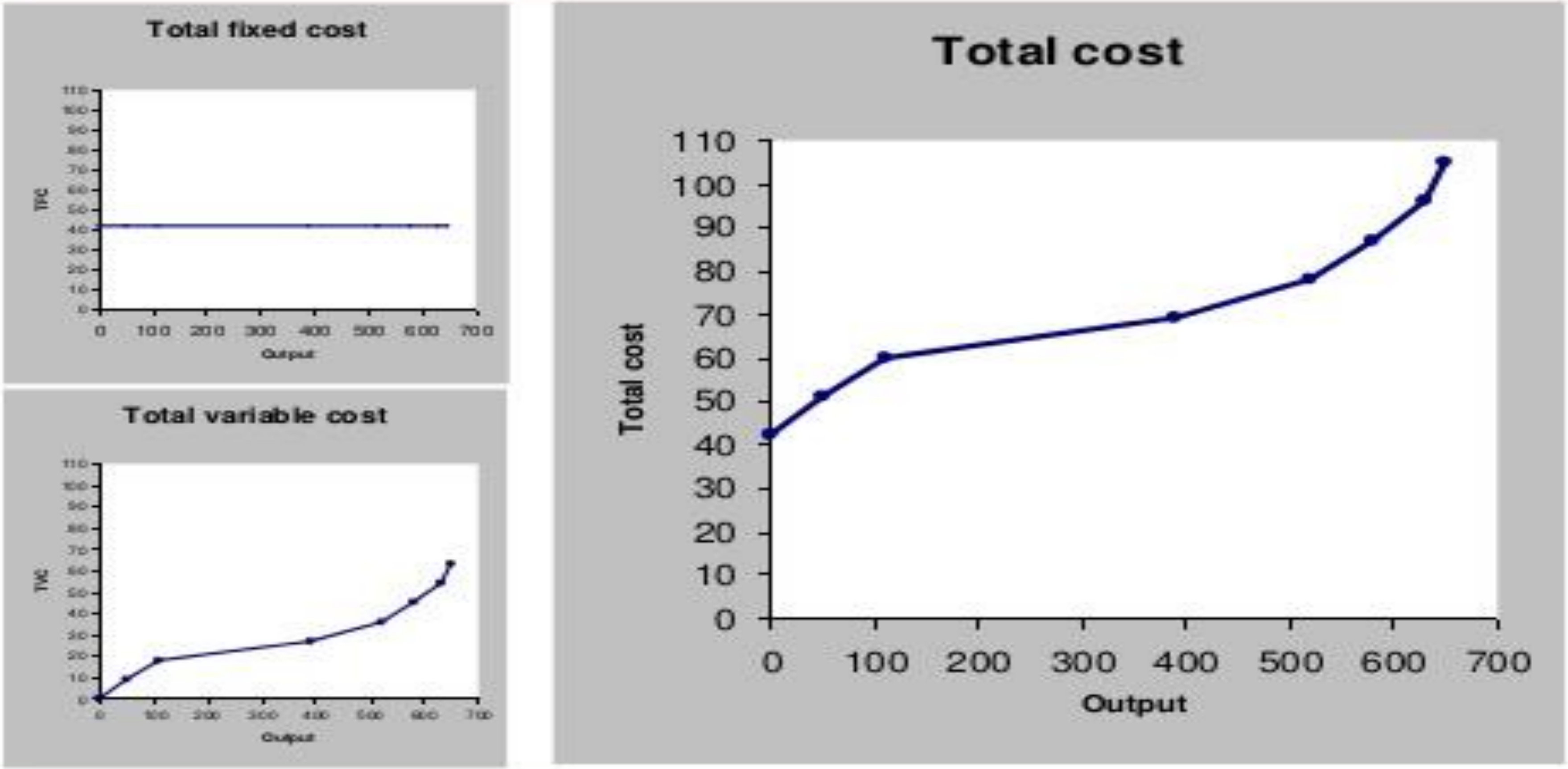
Short Run versus Long Run

- Short run
  - At least one input is *fixed*
  - Cost curves are *operating curves*
- Long run
  - All inputs are variable
  - Cost curves are *planning curves*
- Fixed costs--incurred even if firm produces nothing
- Variable costs--change with the level of output

Costs with Capital Fixed and Labor Variable

L	$TP_L = Q$	MP_L	TFC	TVC	TC
0	0		42	0	42
1	50	50	42	9	51
2	110	60	42	18	60
3	390	280	42	27	69
4	520	130	42	36	78
5	580	60	42	45	87
6	630	50	42	54	96
7	650	20	42	63	105

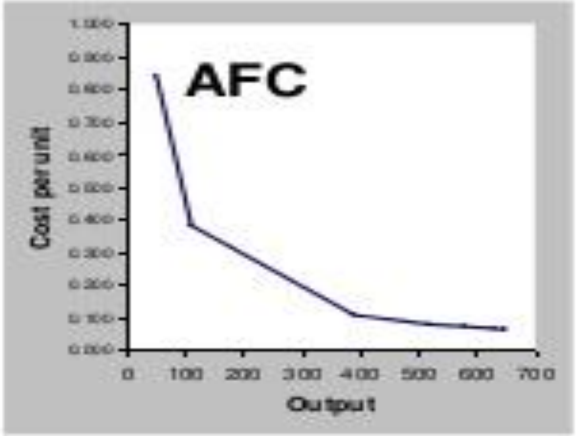
Short-Run Total Cost Curves

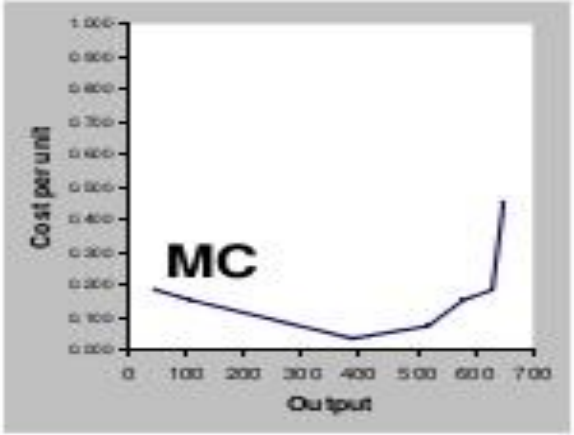


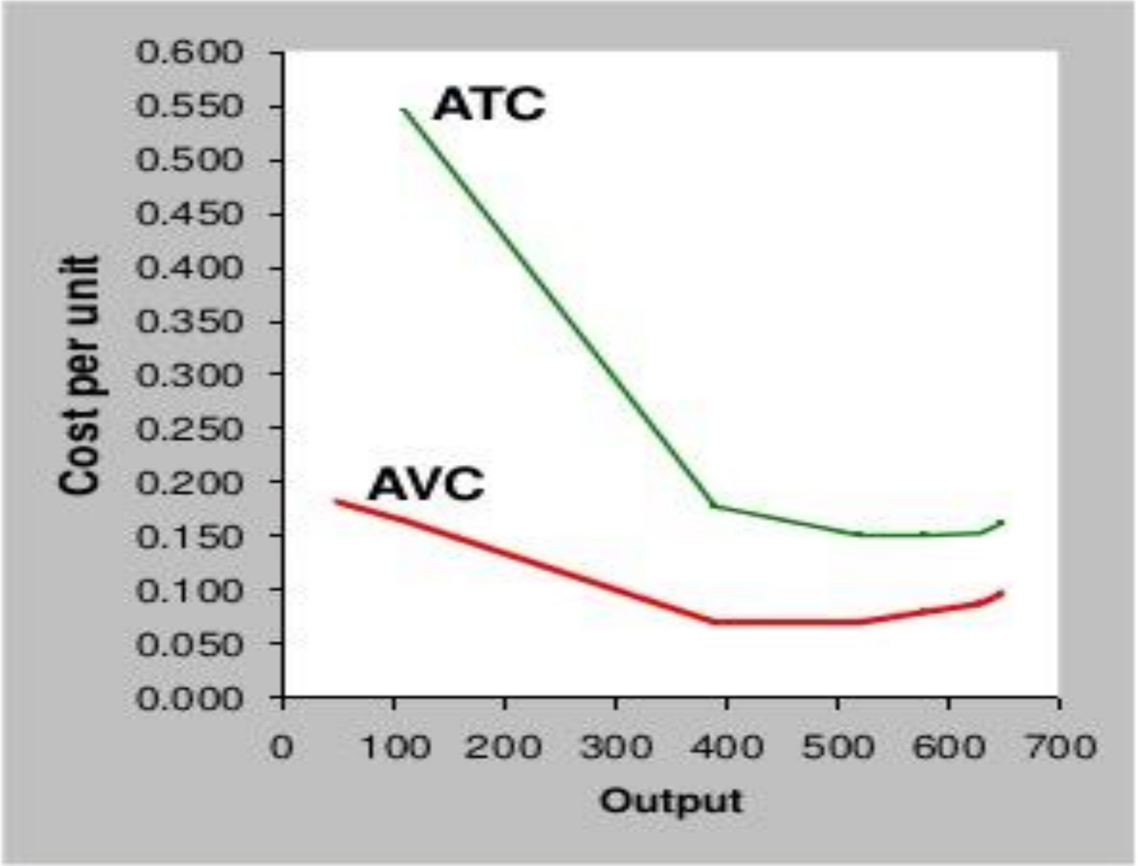
Costs with Capital Fixed and Labor Variable

L	$TP_L=Q$	MP_L	AFC	AVC	ATC	MC
0	0					
1	50	50	0.840	0.180	1.020	0.180
2	110	60	0.382	0.164	0.545	0.150
3	390	**280**	0.108	0.069	0.177	**0.032**
4	520	130	0.081	**0.069**	0.150	**0.069**
5	580	60	0.072	0.078	**0.150**	**0.150**
6	630	50	0.067	0.086	0.152	0.180
7	650	20	0.065	0.097	0.162	0.450

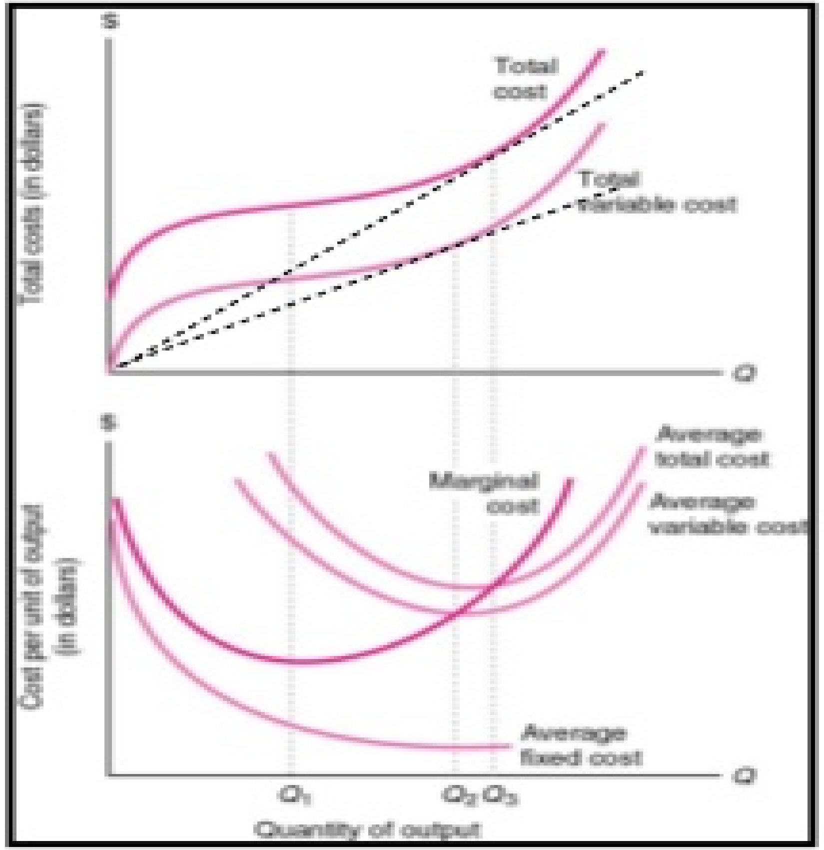
Short-Run Average and Marginal Cost Curves







Short-Run Cost Curves



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Cost Functions

Mathematical Derivation of
Short-Run Costs

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Short-Run Costs with One Variable Input

- Let's consider the production function containing two inputs:

$$Q = f(L, K)$$

- Since the amount of capital is fixed, the short-run production function would be:

$Q = f(L, \overline{K})$ or $PT_L = f(L, \overline{K})$ Total product of labor function

- The inverse of the total product of labor is:

$L = f(Q, \overline{K})$ or $L = L(Q, \overline{K})$ The amount of labor, L, that must be used to produce various levels of output, Q

Short-Run Costs with One Variable Input

- Given the prices of labor (P_L) and capital (P_K), assuming that these prices are determined in competitive input markets, the short-run total cost would be:

$$\underbrace{\overline{P}_K \, \overline{K}}_{\text{fixed cost}} + \underbrace{\overline{P}_L \, L}_{\text{variable cost}} = TC$$

Let's consider the production function: $Q = 10\, K^{1/2}\, L^{1/2}$

Assuming capital is fixed at 4 units, and the prices of capital and labor have been specified at \$40 and \$10:

$$40\,(4) + 10\,L = TC$$

$$Q = 10 \bullet 4^{1/2}\, L^{1/2} \;\Rightarrow\; Q = 20\,L^{1/2} \;\Rightarrow\; \frac{Q}{20} = L^{1/2} \;\Rightarrow\; L = \frac{Q^2}{400}$$

$$160 + 10\,\frac{Q^2}{400} = TC \;\Rightarrow\; TC = 160 + 0.025\,Q^2$$

Short-Run Costs with One Variable Input

Short-run total cost:

$$TC = 160 + 0.025Q^2$$

Short-run marginal cost:

$$MC = \frac{dTC}{dQ} = 0.05Q$$

Short-run average total cost:

$$ATC = \frac{TC}{Q} = \frac{160}{Q} + 0.025Q$$

We can use the production function to show that marginal cost is equal to minimum short-run average total cost

Capacity of the Firm

Output level that minimizes average total cost (Q*):

$$ATC = \frac{TC}{Q} = \frac{160}{Q} + 0.025Q$$

$$\frac{d\,ATC}{dQ} = -\frac{160}{Q^2} + 0.025 = 0 \quad \Rightarrow \quad 0.025 = \frac{160}{Q^2} \quad \Rightarrow \quad Q^2 = 6400 \quad \Rightarrow \quad Q^* = 80$$

If the firm produces 80 units of output, then ATC = $4

If the firm produces 80 units of output, then MC = $4

If the firm produces 80 units of output, then TC = $320

The level of output that corresponds to the minimum short-run average total cost is often called the **capacity of the firm**

A firm that is producing at an output less than the point of minimum average total cost is said to have excess capacity

Cost Functions

Long-Run: Optimal
Combination of Inputs

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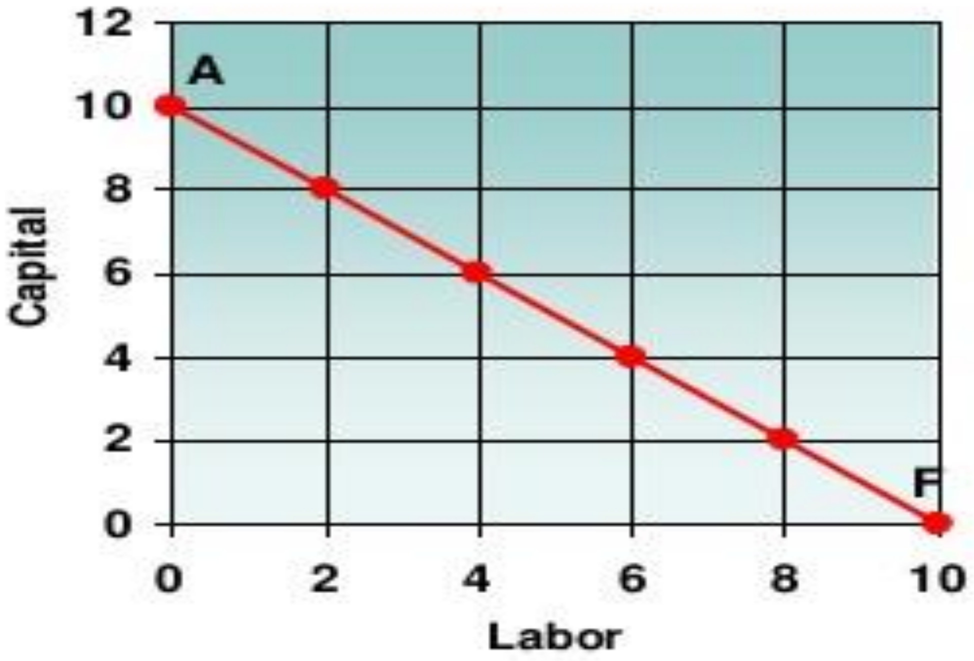
Labor and Capital in Production

	P_K	K	Expend.in capital	P_L	L	Expend. in labor	TC
A	$10	10	100	$10	0	0	$100
B	$10	8	80	$10	2	20	$100
C	$10	6	60	$10	4	40	$100
D	$10	4	40	$10	6	60	$100
E	$10	2	20	$10	8	80	$100
F	$10	0	0	$10	10	10	$100

The Isocost Line



$$\overline{P}_K\ K + \overline{P}_L\ L = TC$$

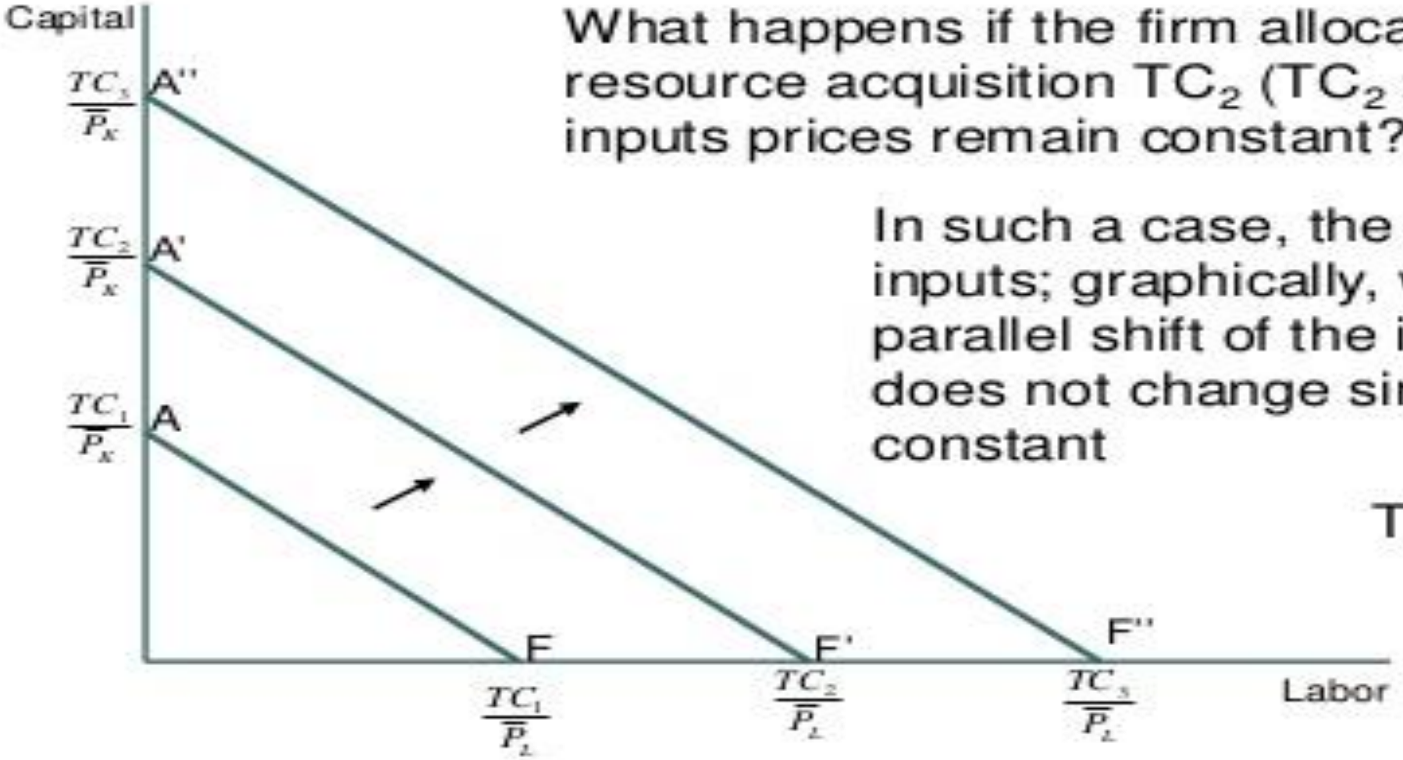
By subtracting "P_L L" from both sides and then dividing by P_L, we get the general equation of the isocost line

$$K = \underbrace{\frac{TC}{\overline{P}_K}}_{\text{vertical intercept of the isocost line}} - \underbrace{\frac{\overline{P}_L}{\overline{P}_K}}_{\text{slope of the isocost line}} L$$

- A **isocost line** shows the various combinations of inputs that a firm can purchase o hire at a given cost

Isocost line AF: K = 10 - L

Isocost Mapping for Different Levels of Cost

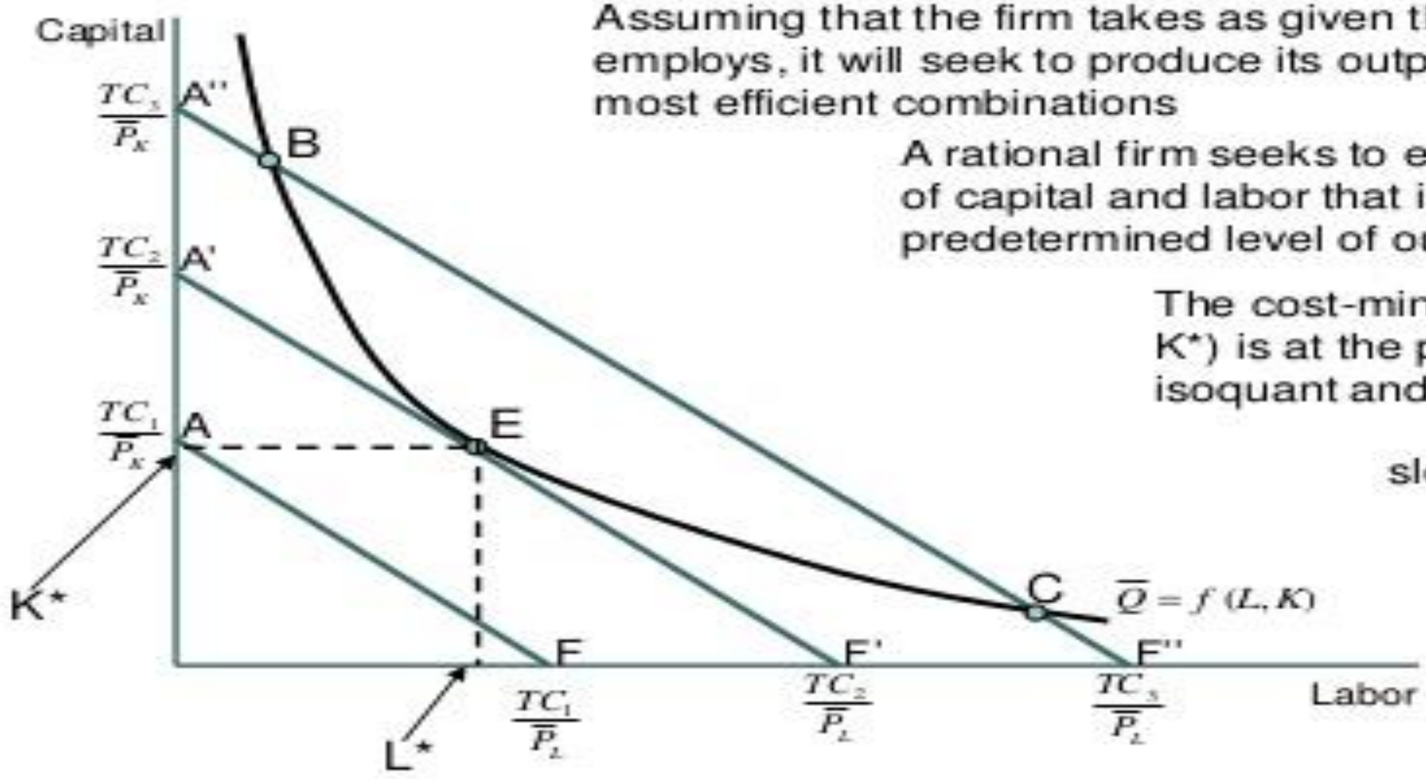


What happens if the firm allocates a larger budget to resource acquisition TC_2 ($TC_2 > TC_1$), assuming inputs prices remain constant?

In such a case, the firm can buy more inputs; graphically, we have a rightward parallel shift of the isocost line; the slope does not change since input prices are constant

$TC_3 > TC_2 > TC_1$

Cost Minimization



Assuming that the firm takes as given the prices of the two inputs it employs, it will seek to produce its output by utilizing its inputs in their most efficient combinations

A rational firm seeks to employ a unique combination of capital and labor that is capable of producing a predetermined level of output at the least possible cost

The cost-minimizing input combination (L^*, K^*) is at the point of tangency between the isoquant and the isocost line TC_2

slope of isoquant = slope of isocost

$$MRTS = \frac{MP_L}{MP_K} = \frac{\overline{P}_L}{\overline{P}_K}$$

$$\boxed{\frac{MP_L}{\overline{P}_L} = \frac{MP_K}{\overline{P}_K}}$$

The necessary condition of cost minimization

Constrained Cost Minimization: Lagrangian Multiplier Method

- We express the constrained cost minimization problem as:

$$\text{Minimize} \quad TC = \overline{P}_K\, K + \overline{P}_L\, L$$
$$\text{subject to:} \quad \overline{Q} = f(K,L)$$

Step 1. Set up the Lagrangian function:

$$\mathrm{L} = \mathrm{L}\,(K,L,\lambda) = \overline{P}_K K + \overline{P}_L L + \lambda\,(\overline{Q} - f(K,L))$$

Step 2. Determine the first-order conditions:

$$\frac{\partial \mathrm{L}}{\partial K} = \overline{P}_K - \lambda \frac{\partial (K,L)}{\partial K} = 0$$

$$\frac{\partial \mathrm{L}}{\partial L} = \overline{P}_L - \lambda \frac{\partial (K,L)}{\partial L} = 0$$

$$\frac{\partial \mathrm{L}}{\partial \lambda} = \overline{Q} - f(K,L) = 0$$

Constrained Cost Minimization: Lagrangian Multiplier Method

Step 3. Solve the first-order conditions simultaneously

$$\overline{P}_K - \lambda \frac{\partial(K,L)}{\partial K} = 0 \quad \Rightarrow \quad \overline{P}_K = \lambda \frac{\partial(K,L)}{\partial K} \quad \Rightarrow \quad \frac{\partial K}{\partial(K,L)} \overline{P}_K = \lambda$$

$$\overline{P}_L - \lambda \frac{\partial(K,L)}{\partial L} = 0 \quad \Rightarrow \quad \overline{P}_L = \lambda \frac{\partial(K,L)}{\partial L} \Rightarrow \quad \frac{\partial L}{\partial(K,L)} \overline{P}_L = \lambda$$

$$\frac{\partial K\, \overline{P}_K}{\partial(K,L)} = \frac{\partial L\, \overline{P}_L}{\partial(K,L)} \quad \Rightarrow \quad \partial K\, \overline{P}_K = \partial L\, \overline{P}_L \quad \Rightarrow \quad \frac{\partial K}{\partial L} = \frac{\overline{P}_L}{\overline{P}_K}$$

$$\frac{\partial Q}{\partial Q} \frac{\partial K}{\partial L} = \frac{\overline{P}_L}{\overline{P}_K} \quad \Rightarrow \quad \frac{\dfrac{\partial Q}{\partial L}}{\dfrac{\partial Q}{\partial K}} = \frac{\overline{P}_L}{\overline{P}_K} \quad \Rightarrow \quad \frac{MP_L}{MP_K} = \frac{\overline{P}_L}{\overline{P}_K} \quad or \quad \frac{MP_L}{\overline{P}_L} = \frac{MP_K}{\overline{P}_K}$$

Lagrangian Multiplier Method: Example

$$\text{Minimize} \quad TC = 40K + 10L$$
$$\text{subject to:} \quad 80 = 10K^{0.5} L^{0.5}$$

Step 1. Set up the Lagrangian function:

$$\mathcal{L} = \mathcal{L}(K, L, \lambda) = 40K + 10L + \lambda(80 - 10K^{0.5} L^{0.5})$$

Step 2. Determine the first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial K} = 40 - \lambda\, 10\, L^{0.5}\, 0.5\, K^{-0.5} = 40 - 5\lambda\, L^{0.5} K^{-0.5} = 0$$

$$\frac{\partial \mathcal{L}}{\partial L} = 10 - \lambda\, 10\, K^{0.5}\, 0.5\, L^{-0.5} = 10 - 5\lambda\, K^{0.5} L^{-0.5} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 80 - 10K^{0.5} L^{0.5} = 0$$

Lagrangian Multiplier Method: Example

Step 3. Solve the first-order conditions simultaneously for the unique values of K, L and lambda that minimizes TC

$$\left.\begin{aligned} 40-5\lambda\, L^{0.5}\, K^{-0.5}=0 \quad &\Rightarrow \quad \lambda=\frac{40}{5L^{0.5}\, K^{-0.5}} \\ 10-5\lambda\, K^{0.5}\, L^{-0.5}=0 \quad &\Rightarrow \quad \lambda=\frac{10}{5K^{0.5}\, L^{-0.5}} \end{aligned}\right\} \quad \frac{40}{5L^{0.5}\, K^{-0.5}}=\frac{10}{5K^{0.5}\, L^{-0.5}} \quad \frac{5K^{0.5}\, L^{-0.5}}{5L^{0.5}\, K^{-0.5}}=\frac{10}{40} \qquad K=0.25\, L$$

$$80=10\, K^{0.5}\, L^{0.5} \qquad 80=10\,(0.25L)^{0.5}\, L^{0.5} \quad \Rightarrow \quad L^*=16 \text{units of labor}$$

$$K^*=4 \text{ units of capital}$$

TC = $320

We can interpret the value of lambda as the effect on the value of the objective function as the constraint changes by one unit

In this case the value is 4, so if the predetermined level of output increases by one unit, the firm's minimum total expenditure increases by $4 (lambda is the firm's long-run marginal cost)

Cost Functions

Cost Curves in the Long-Run

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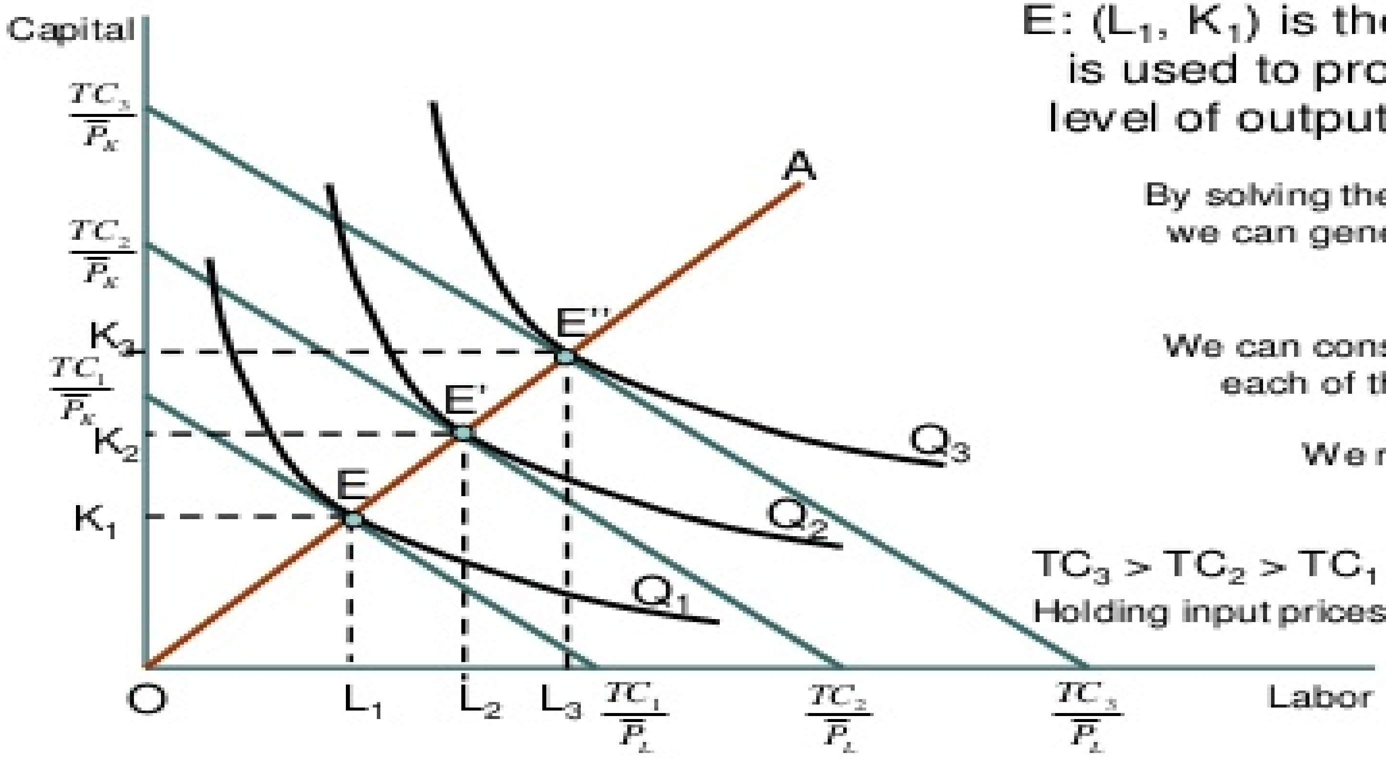
Long-Run Expansion Path



E: (L_1, K_1) is the optimal combination that is used to produce the predetermined level of output Q_1, at the least cost TC_1

By solving the optimization problem many times, we can generate an entire set of optimal input combinations

We can construct a ray OA that passes through each of these optimal input combinations

We refer to a curve such as OA as an **expansion path**

$TC_3 > TC_2 > TC_1$

Holding input prices constant

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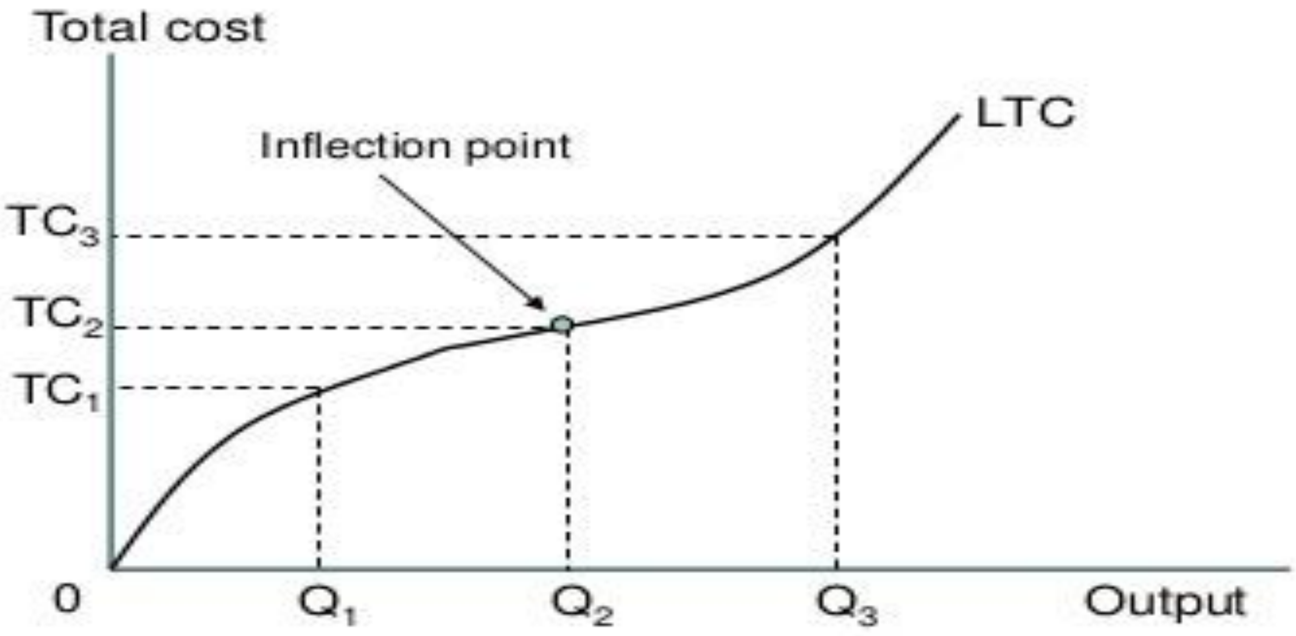
Long-Run Total Cost Function

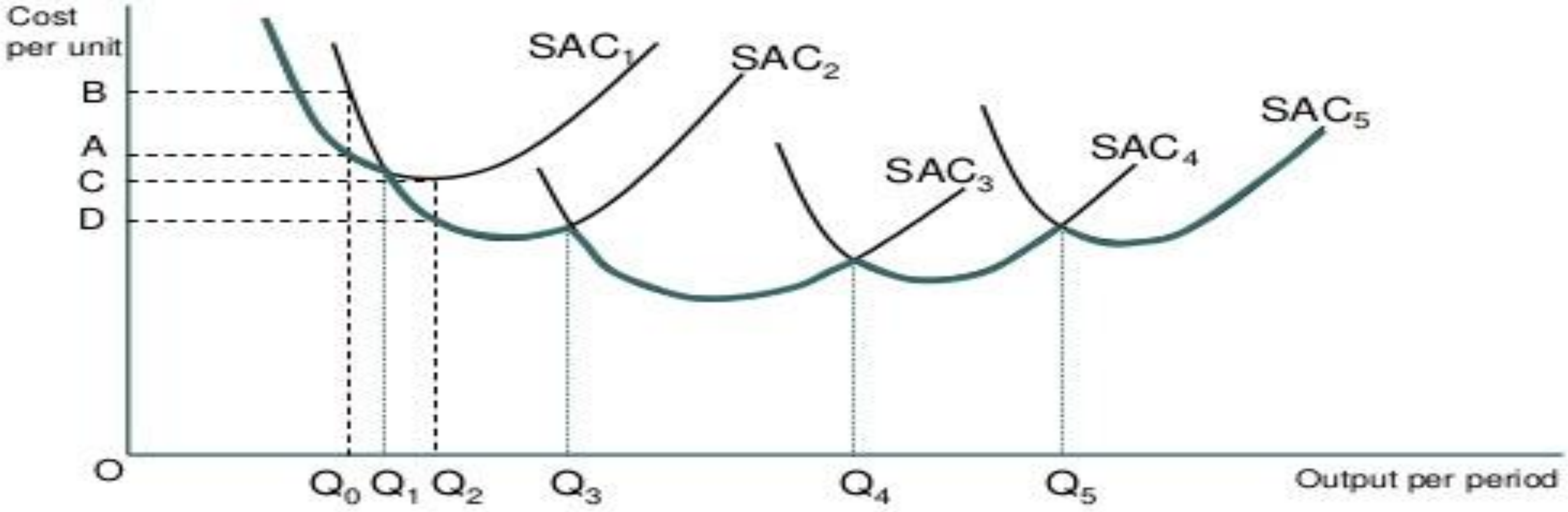


- We assume a cubic shape of the curve
  - Specifically, this LTC curve indicates that as output increases from 0 to Q_2 units of output, the corresponding costs increase from 0 to TC_2 but a decreasing rate
  - However, as output increases beyond Q_2 the costs increase at an increasing rate

The Long-Run Average Cost Curve

The long-run average cost curve (LAC) shows the lowest cost of producing each level of output when the firm can build the most appropriate plant to produce each level of output (and all factors are variable)

Assume that the firm can build only five scales of pant: SAC_1, SAC_2, SAC_3, SAC_4, SAC_5



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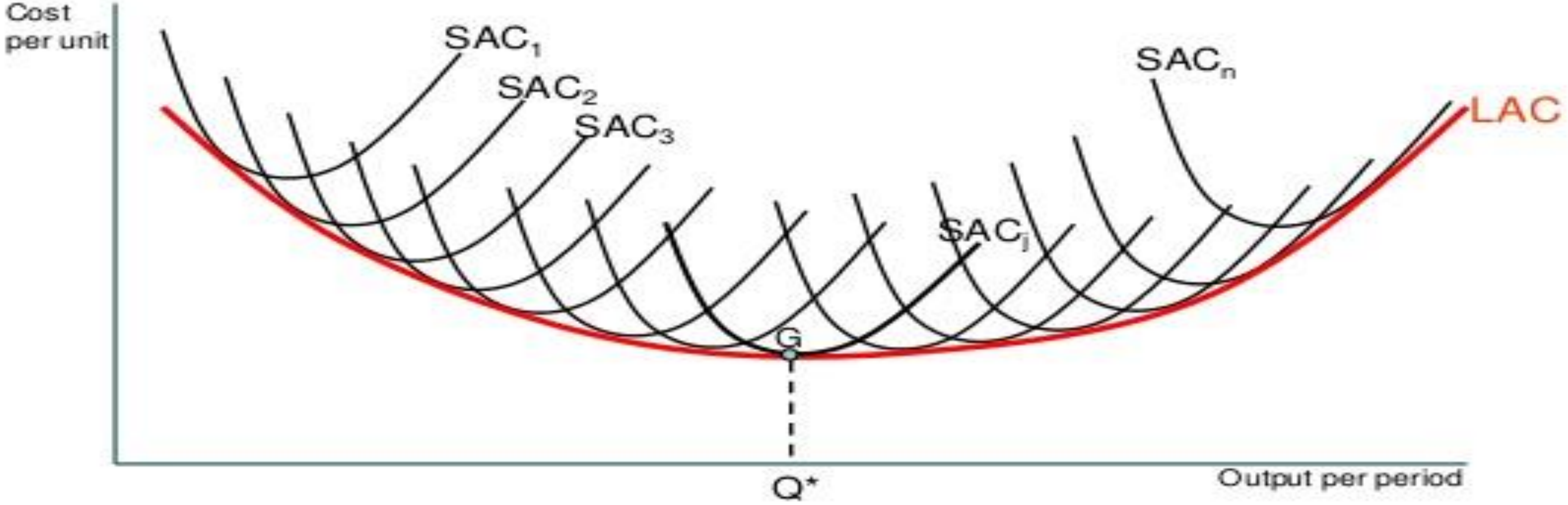
The Long-Run Average Cost Curve



If the firm could build many more scales of plant, the LAC curve would be the envelope to SAC curves

Only at point G (the lowest point on the LAC curve) the long-run average cost curve is tangent to a SAC curve in its minimum ➔ optimal scale of plant

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The U Shape of the LAC Curve

- The shape of the LAC curve depends on increasing, constant and decreasing returns to scale
- The LAC curve has been drawn as U-shaped based on the assumption that increasing returns of scale prevail at small levels of output and decreasing returns to scale prevail at larger levels of output
  - Increasing returns to scale (e.g. output more than doubles with a doubling of inputs) are reflected in a declining LAC curve. Decreasing long-run average costs are called ECONOMIES OF SCALE
  - Decreasing returns to scale (e.g. output grows at a proportionately slower rate than the use of inputs) are reflected in a LAC curve that is rising. Increasing long-run average costs are called DISECONOMIES OF SCALE

Plant Size and Dis/Economies of Scale

- Decreasing costs arise because of technological and financial reasons
  - At the technological level, economies of scale arise because as the scale of operations increases, a greater division of labor and specialization can take place and more specialized and productive machinery can be used
  - There are also financial reasons that arise as the size of the firm increases; because of bulk purchases, larger firms are more likely to receive quantity discounts in purchasing raw materials and other intermediate inputs than smaller firms
- On the other hand, diseconomies are associated with managerial coordination of a large firm
  - Many firms find that if they become too large, with many different plants or branches, it becomes difficult to coordinate the management of such a large firm, and this coordination problem may cause unit costs to rise