College of Science and technology

Department of Computer Science

Academic year: 2014-2015

Module title: DISCRETE STRUCTURES FOR COMPUTING

Module code: COE5121

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COURSE OUTLINE

Title: DISCRETE STRUCTURES FOR COMPUTING

MODULE CODE: COE5121

Description of aims and content

This course provides a basic understanding of discrete mathematical topics that are fundamental for academic work in computer science. Students will develop algorithms and prove their efficacy. Topics include basic logic, basic proof techniques, set, function, relation, basic proof techniques, basics of counting and recursion .

Learning Outcomes:

1.1. LEARNING OUTCOMES :

1.1.1. KNOWLEDGE & UNDERSTANDING :

Having successfully completed the module, students should be able to demonstrate knowledge and understanding of:

A1 – The application of mathematics in computer Software and Hardware

A2 – Basic concepts and theory of computing

1.1.2. COGNITIVE/ INTELLECTUAL SKILLS/ APPLICATION OF KNOWLEDGE :

Having successfully completed the module, students should be able to:

B1 – Select appropriate structures in solving computation

B2 – Apply selective mathematical techniques relevant to discrete structures

1.1.3. GENERAL TRANSFERABLE SKILLS :

Having successfully completed the module, students should be able to:

– Demonstrate numerical techniques involving discrete structures

INDICATIVE CONTENT

Unit 1: History and overview

1.1 Knowledge themes include sets, logic, functions, and graphs

1.2 Contributors to the subject

1.3 Purpose and role of discrete structures in computer engineering

1.4 Contrasts between discrete-time models vs. continuous-time models

Unit 2: Basic logic

2.1 Propositional logic

2.2 Logical connectives

2.3 Truth tables

2.4 Use of logic to illustrate connectives

2.5 Normal forms (conjunctive and disjunctive)

2.6 Predicate logic

2.7 Universal and existential quantification

2.8 Limitations of predicate logic

2.9 Boolean algebra

2.10 Applications of logic to computer engineering

Unit 3: Sets, Functions and Relations

3.1 Sets (Venn diagrams, complements, Cartesian products, power sets)

3.2 Functions (one-to-one, onto, inverses, composition)

3.2 Relations (reflexivity, symmetry, transitivity, equivalence relations)

3.3 Discrete versus continuous functions and relations

3.5 Cardinality and countability

Unit 4: Proof techniques

4.1 Notions of implication, converse, inverse, negation, and contradiction

4.2 The structure of formal proofs

4.3 Direct proofs

4.4 Proof by counterexample, contraposition, and contradiction

4.5 Mathematical induction and strong induction

Unit 5: Basics of counting

5.1 Permutations and combinations

5.2 Counting arguments rule of products, rule of sums

5.3 The pigeonhole principle

5.4 Generating functions

5.6 Applications to computer engineering

Unit 6: Recursion

6.1 Recursive mathematical definitions

6.2 Developing recursive equations

6.3 Solving recursive equations

6.7 Applications of recursion to computer engineering

REFERENCES

Rosen, Kenneth H. : Discrete mathematics and its applications — 7th edition

ASSESSMENT:

Assignments (Two): 20%

Quiz (Two): 10%

Examinations: CATS (Two): 20%
FINAL 50%

UNIT1: INTRODUCTION

Meaning of the word "**structures**": It means a complex composition of knowledge as ***elements*** and ***their combinations***. Discrete Structures are the abstract mathematical structures used to represent ***discrete objects*** and relationships between these objects. These discrete structures include logic, sets, permutations, relations, graphs, trees, and finite-state machines.

Discrete structures are those complex compositions of knowledge that are taken from the field of discrete mathematics. Discrete mathematics is mathematics that deals with discrete objects. Discrete objects are those which are separated from (not connected to/distinct from) each other. Integers, automobiles, houses, people etc. are all discrete objects.
On the other hand, real numbers which include irrational as well as rational numbers are not discrete. As you know between any two different real numbers there is another real number different from either of them. So they are packed without any gaps and cannot be separated from their immediate neighbors. In that sense they are not discrete. In this course we will be concerned with objects such as integers, propositions, sets, relations and functions, which are all discrete. We are going to learn concepts associated with them, their properties, and relationships among them among others.
Discrete mathematics is about the mathematics of integers and of collections of objects. It underlies the operation of digital computers, and is used widely in all fields of computer science for reasoning about data structures, algorithms and complexity. Topics covered in the module include logic , proof techniques and sets, functions, relations, summations and recurrences, counting techniques and recursion.

Note: Discrete structures is the term used for discrete mathematics for computer science whereas Discrete mathematics is often referred to as finite mathematics.

Exemple-Exercise

State which of the following represent discrete data and which represent continuous data.

(a) Numbers of shares sold each day in the stock market. **Ans. D**
(b) Temperature recorded every half hour at a weather bureau. **Ans. C**
(c) Lifetimes of television tubes produced by a company **Ans. C**
(d) Yearly incomes of college professors.   **Ans. D**
(e) Lengths of 1000 bolts produced in a factory. **Ans. C**
(f) Number of millimetres of rainfall in a city during various months of the year. **Ans. C**
(g) Speed of an automobile in kilometres per hour. **Ans. C**
(h) Number of 5000 Rwandan francs notes circulating in Rwanda at any time. **Ans. D**
(i) Student enrolment in a university over a number of years. **Ans. D**
(j) The exam results in(marks) in a given module. **Ans. C**

UNIT 2: BASIC LOGIC

It (logic) is covered in Chapter 1 of the reference textbook. It (logic) is a language that captures the essence of our reasoning, and correct reasoning must follow the rules of this language. It allows us to reason with statements involving variables among others.

2.1 PROPOSITIONAL LOGIC

Definition

A **proposition** is declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both. Propositional logic is a logic at the sentential level. Thus sentences considered in this logic are not arbitrary sentences rather are the ones that are true or false. This kind of sentences are called **propositions**.

If a proposition is true, then we say it has a **truth value** of "**true**"; if a proposition is false, its truth value is "**false**".

Example1

"Grass is green", and "2 + 5 = 5" are propositions.
The first proposition has the truth value of "**true**" and the second "**false**".

Example2

Consider the following sentences.

1. What time is it?
2. Read this carefully.
3. **x+1=2.**
4. **x+y = z**.

Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables.
We use letters to **denote propositional variables** (or statement variables), that is, variables that represent propositions, just as letters are used to denote numerical variables. The conventional letters used for propositional variables are **p,q,r,s,...** .The truth value of a proposition is true, denoted by **T**, if it is a true proposition, and the truth value of a proposition is false, denoted by **F**, if it is a false proposition.
The area of logic that deals with propositions is called the **propositional calculus** or **propositional logic.**
We now turn our attention to methods for producing new propositions from those that we already have. Many mathematical statements are constructed by combining one or more propositions. New propositions, called **compound propositions**, are formed from existing propositions using logical operators.

Elements of Propositional Logic

Simple sentences which are true or false are basic propositions. Larger and more complex sentences are constructed from basic propositions by combining them with **connectives**. Thus **propositions** and **connectives** are the basic elements of propositional logic. Though there are many connectives, we are going to use the following **five basic connectives** here:

NOT(negation) is denoted by the symbol $\neg$
AND (conjunction) is denoted by the symbol $\wedge$
OR (disjunction) is denoted by the symbol $\vee$
IF_THEN (or IMPLY) (conditional statement or implication) is denoted by the symbol $\rightarrow$
IF_AND_ONLY_IF(biconditional statement) and is denoted by the symbol $\leftrightarrow$

Truth Table

Often we want to discuss properties/relations common to all propositions. In such a case rather than stating them for each individual proposition we use variables representing an arbitrary proposition and state properties/relations in terms of those variables. Those variables are called a **propositional variable**. **Propositional variables are also considered a proposition and called a proposition** since they represent a proposition hence they behave the same way as propositions.

A proposition in general contains a number of variables. For example (**p** $\vee$ **q**) contains variables **p** and **q** each of which represents an arbitrary proposition. Thus a proposition takes different values depending on the values of the constituent variables. This relationship of the value of a proposition and those of its constituent variables can be represented by a table. It tabulates the value of a proposition for all possible values of its variables and it is called a **truth table**.

(a) Truth Table of Negation of a proposition p

p	$\neg$ p
T	**F**
F	**T**

This table shows that if **p** is true, then ($\neg$ **p**) is false, and that if **p** is false, then ($\neg$ **p**) is true.

(b) Truth Table of Conjunction of propositions p and q

p	**q**	**p$\wedge$ q**
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

This table shows that (**p** $\wedge$ **q**) is true if both **p** and **q** are true, and that it is false in any other case. Similarly for the rest of the tables.

(c) Truth Table of Disjunction of propositions p and q

p	q	p$\vee$q
T	T	T
T	F	T
F	T	T
F	F	F

This table shows that (**p** $\vee$ **q**) is false if both **p** and **q** are false and is true otherwise.

(d) Implication (conditional statement) of propositions p and q

p	q	p$\rightarrow$q
T	T	T
T	F	F
F	T	T
F	F	T

The conditional statement **p**$\rightarrow$ **q** is the proposition "if **p**, then **q**." The conditional statement **p**$\rightarrow$**q** is false when p is true and q is false, and true otherwise. In the conditional statement **p**$\rightarrow$ **q**, **p** is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).When **p** $\rightarrow$ **q** is always true, we express that by **p**$\Rightarrow$ **q**. That is **p**$\Rightarrow$ **q** is used when proposition **p** always implies proposition **q** regardless of the value of the variables in them.

(e) Biconditional statement of propositions p and q

p	q	p$\leftrightarrow$q
T	T	T
T	F	F
F	T	F
F	F	T

The biconditional statement **p**$\leftrightarrow$ **q** is the proposition "**p if and only if q**." The biconditional statement **p**$\leftrightarrow$ **q** is true when p and q have the same truthvalues, and is false otherwise. Biconditional statements are also called bi-implications.

When **p** $\leftrightarrow$ **q** is always true, we express that by **p** $\Leftrightarrow$ **q**. That is $\Leftrightarrow$ is used when two propositions always take the same value regardless of the value of the variables in them.

Construction of Complex Propositions

(a) Converse and Contrapositive

For the proposition **p** $\rightarrow$ **q**, the proposition **q** $\rightarrow$ **p** is called its **converse**, and the proposition $\neg$ **q** $\rightarrow$ $\neg$ **p** is called its **contrapositive**.

For example for the proposition "If it rains, then I get wet",

Converse: If I get wet, then it rains.
Contrapositive: If I don't get wet, then it does not rain.

The converse of a proposition is not necessarily logically equivalent to it, that is they may or may not take the same truth value at the same time.
On the other hand, ***the contrapositive of a proposition is always logically equivalent to the proposition***. That is, they take the same truth value regardless of the values of their constituent variables. Therefore, "If it rains, then I get wet." and "If I don't get wet, then it does not rain." are logically equivalent. If one is true then the other is also true, and vice versa.

Exercises

1. Which of the following sentences are propositions? What are the truth values of those that are propositions?

a. Kigali is the capital of Rwanda.
b. $2 + 3 = 7$.
c. Open the door.
d. $5 + 7 < 10$.
e. The moon is a satellite of the earth.
f. $x + 5 = 7$.
g. $x + 5 > 9$ for every real number x.

Solution
a. Yes, True
b. Yes, False
c. No
d. Yes, False
e. Yes, True
f. No
g. Yes, False

2. What is the negation of each of the following propositions?

a. Kigali is the capital of Rwanda.
b. Food is not expensive in Huye.
c. $3 + 5 = 7$.
d. The summer in France is hot and sunny.

Solution
a. Kigali is not the capital of Rwanda.
b. Food is expensive in Huye.
c. $3 + 5 \neq 7$.
d. The summer in France is not hot or (the summer in France is) not sunny.

3. Let p and q be the propositions

p: Your car is out of gas.
q: You can't drive your car.

Write the following propositions using **p** and **q** and logical connectives.
a) Your car is not out of gas.
b) You can't drive your car if it is out of gas.
c) Your car is not out of gas if you can drive it.
d) If you can't drive your car then it is out of gas.

Solution

a) $\neg p$
b) $p \rightarrow q$
c) $\neg q \rightarrow \neg p$
d) $q \rightarrow p$

4. Determine whether each of the following implications is true or false.
a) If 0.5 is an integer, then 1 + 0.5 = 3.
b) If cars can fly, then 1 + 1 = 3.
c) If 5 > 2 then pigs can fly.
d) If 3*5 = 15 then 1 + 2 = 3.

Solution

a) True (because the condition is false)
b) True (because the condition is false)
c) False
d) True

5. State the converse and contrapositive of each of the following implications.

a. If it snows today, I will stay home.
b. We play the game if it is sunny.
c. If a positive integer is a prime then it has no divisors other than 1 and itself.

Solution

a. Converse: "If I stay home, then it snows today. "
   Contrapositive: "If I do not stay home, then it does not snow today. quot;
b. Converse: "If we play the game, then it is sunny."
   Contrapositive: "If we don't play the game, then it is not sunny."
c. Converse: "If a positive integer has no divisors other than 1 and itself then it is a prime."
   Contrapositive: "If a positive integer has a divisors other than 1 and itself then it is not a prime.

6. Construct a truth table for each of the following compound propositions.

a) $p \land \neg p$
b) $(p \lor \neg q) \to q$
c) $(p \to q) \leftrightarrow (\neg q \to \neg p)$

Solutions

a)

p	**$\neg$p**	**p $\land \neg$p**
T	F	F
F	T	F

b)

p	**q**	**$\neg$q**	**p $\lor \neg$q**	**(p $\lor \neg$q) $\to$ q**
T	T	F	T	T
T	F	T	T	F
F		F	F	
F	F	T	T	F

p	**q**	**p $\to$q**	**$\neg$q**	**$\neg$p**	**$\neg$q $\to \neg$p**	**(p $\to$q) $\leftrightarrow$($\neg$q $\to \neg$p)**
T	**T**	**T**	**F**	**F**	**T**	**T**
T	**F**	**F**	**T**	**F**	**F**	**T**
F	**T**	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

Use of logic to illustrate connectives

1. If_Then Variations

- different ways of saying if_then: only if, necessary, sufficient

If-then statements appear in various forms in practice. The following list presents some of the variations. **These are all *logically* equivalent,** that is as far as true or false of statement is concerned there is no difference between them. Thus if one is true then all the others are also true, and if one is false all the others are false.

- **If p , then q.**
- **p implies q.**
- **If p,   q.**
- **p only if q.**
- **p is sufficient for q.**
- **q if p.**
- **q whenever p.**
- **q is necessary for p.**
- **It is necessary for p that q.**

For instance, instead of saying "**If** she smiles **then** she is happy", we can say "If she smiles, she is happy", "She is happy whenever she smiles", "She smiles only if she is happy" etc. without changing their truth values.

"Only if" can be translated as "then". For example, "She smiles only if she is happy" is equivalent to "If she smiles, then she is happy".
Note that "She smiles only if she is happy" means "If she is not happy, she does not smile", which is the contrapositive of "If she smiles, she is happy".
You can also look at it this way: "She smiles only if she is happy" means "She smiles only when she is happy". So any time you see her smile you know she is happy. Hence "If she smiles, then she is happy". Thus they are logically equivalent.

Also **"If she smiles, she is happy" is equivalent to "It is necessary for her to smile that she is happy".** For "If she smiles, she is happy" means "If she smiles, she is *always* happy". That is, she never fails to be happy when she smiles. "Being happy" is inevitable consequence/**necessity of** "smile". Thus if "being happy" is missing, then "smile" can not be there either. "Being happy" is **necessary** "for her to smile" or equivalently "It is **necessary** for her to smile that she is happy".

2.From English to Proposition

- translation of English sentences to propositions

As we are going to see in the next section, reasoning is done on propositions using inference rules. For example, if the two propositions "if it snows, then the school is closed", and "it snows" are true, then we can conclude that "the school is closed" is true. In everyday life, that is how we reason.

To check the correctness of reasoning, we must check whether or not rules of inference have been followed to draw the conclusion from the premises. However, for reasoning in English or in general for reasoning in a natural language, that is not necessarily straightforward and it often encounters some difficulties. Firstly, connectives are not necessarily easily identified as we can get a flavor of that from the previous topic on variations of if_then statements. Secondly, if the argument becomes complicated involving many statements in a number of different forms twisted and tangled up, it can easily get out of hand unless it is simplified in some way.

One solution for that is to use symbols (and mechanize it). Each sentence is represented by symbols representing building block sentences, and connectives. For example, if P represents "it snows" and Q represents "the school is closed", then the previous argument can be expressed as

$[\,[\,P \rightarrow Q\,] \wedge P\,] \rightarrow Q$,

This representation is concise, much simpler and much easier to deal with. In addition today there are a number of automatic reasoning systems and we can verify our arguments in symbolic form using them.

To convert English statements into a symbolic form, we **restate the given statements using the building block sentences, those for which symbols are given, and the connectives of propositional logic** (**not, and, or, if_then, if_and_only_if**), and **then substitute the symbols for the building blocks and the connectives**.

For example, let P be the proposition "It is snowing", Q be the proposition "I will go the beach", and R be the proposition "I have time".

Then first "I will go to the beach if it is not snowing" is restated as "If it is not snowing, I will go to the beach". Then symbols P and Q are substituted for the respective sentences to obtain $\neg P \rightarrow Q$.

Similarly, "It is not snowing and I have time only if I will go to the beach" is restated as "If it is not snowing and I have time, then I will go to the beach", and it is translated as

$(\neg P \wedge R) \rightarrow Q$.

Exercise

1. Write each of the following statements in the form "if *p*, then *q*" in English. (Hint: Refer to the list of common ways to express implications listed in this section.)

a. The newspaper will not come if there is an inch of snow on the street.
b. It snows whenever the wind blows from the northeast.
c. That prices go up implies that supply will be plentiful.
d. It is necessary to read the textbook to understand the materials of this course.
e. For a number to be divisible by 3, it is sufficient that it is the sum of three consecutive integers.
f. Your guarantee is good only if you bought your TV less than 90 days ago.

Solution

a. If there is an inch of snow on the street, the newspaper will not come.
b. If the wind blows from the northeast, then it snows.
c. If prices go up, then supply will be plentiful.
d. If you are going to understand the materials of this course, you must read the textbook.
e. If a number is the sum of three consecutive integers, then it is divisible by 3.
f. If you guarantee is good, then you must have bought your TV less than 90 days ago.

2. Write each of the following propositions in the form "*p* if and only if *q*" in English.

a. If it is hot outside you drink a lot of water, and if you drink a lot of water it is hot outside.
b. For a program to be readable it is necessary and sufficient that it is well structured.
c. I like fruits only if they are fresh, and fruits are fresh only if I like them.
d. If you eat too much sweets your teeth will decay, and conversely.
e. The store is closed on exactly those days when I want to shop there.

Solution

a. You drink a lot of water if and only if it is hot outside.
b. A program is readable if and only if it is well structured.
c. I like fruits if and only if they are fresh.
d. Your teeth will decay if and only if you eat too much sweets.
e. The store is closed if and only if it is the day when I want to shop there.

Introduction to Reasoning

Introduction

Logical reasoning is the process of drawing conclusions from premises using rules of inference. Here we are going to study reasoning with propositions. Later we are going to see reasoning with predicate logic, which allows us to reason about individual objects. However, ***inference rules of propositional logic are also applicable to predicate logic*** and reasoning with propositions is fundamental to reasoning with predicate logic.
These inference rules are results of observations of human reasoning over centuries. Though there is nothing absolute about them, they have contributed significantly in the scientific and engineering progress the mankind have made. Today they are universally accepted as the rules of logical reasoning and they should be followed in our reasoning.

Since inference rules are based on identities and implications, we are going to study them first. We start with three types of proposition which are used to define the meaning of "identity" and "implication".

Types of Proposition

Some propositions are always true regardless of the truth value of its component propositions.
For example (**p** $\vee\neg$**p**) is always true regardless of the value of the proposition **p**.
a proposition that is always true called a **tautology**.
there are also propositions that are always false such as (**p** $\wedge\neg$**p**).
such a proposition is called a **contradiction**.
a proposition that is neither a tautology nor a contradiction is called a **contingency**.
for example (**p** $\vee$**q**) is a contingency.

These types of propositions play a **crucial role in reasoning**. In particular every inference rule is a tautology as we see in *identities* and *implications*.

Identities of propositional logic and Dual propositions

Identities of propositional logic

From the definitions of connectives, a number of relations between propositions which are useful in reasoning can be derived. Below some of the often encountered pairs of **logically equivalent** propositions, also called *identities*, are listed.
These identities are used in logical reasoning. In fact we use them in our daily life, often more than one at a time, without realizing it.

If two propositions are logically equivalent, one can be substituted for the other in any proposition in which they occur without changing the logical value of the proposition.

Below $\Leftrightarrow$ corresponds to $\leftrightarrow$ and it means that the equivalence is always true (a tautology), while $\leftrightarrow$ means the equivalence may be false in some cases, that is in general a contingency.

That these equivalences hold can be verified by constructing truth tables for them.
First the identities are listed, then examples are given to illustrate them.

list of identities:

1. **P** $\Leftrightarrow$ **(P** $\vee$ **P**) ----- idempotence of $\vee$
2. **P** $\Leftrightarrow$ **(P** $\wedge$ **P**) ----- idempotence of $\wedge$
3. (**p** $\vee$ **q)** $\Leftrightarrow$ **(q** $\vee$ **p**) ----- commutativity of $\vee$
4. (**p** $\wedge$ **q)** $\Leftrightarrow$ **(q** $\wedge$ **p**) ----- commutativity of $\wedge$
5. [(**p** $\vee$ **q)** $\vee$ **r]** $\Leftrightarrow$ **[p** $\vee$ **(q** $\vee$ **r**)] ----- associativity of $\vee$
6. [(**p** $\wedge$ **q)** $\wedge$ **r]** $\Leftrightarrow$ **[p** $\wedge$ **(q** $\wedge$ **r**)] ----- associativity of $\wedge$
7. $\neg$ (**p** $\vee$ **q)** $\Leftrightarrow$ **(**$\neg$ **p** $\wedge$ $\neg$ **q**) ----- de Morgan's law
8. $\neg$ (**p** $\wedge$ **q)** $\Leftrightarrow$ **(**$\neg$ **p** $\vee$ $\neg$ **q**) ----- de Morgan's law
9. [p $\wedge$ (q $\vee$ r] $\Leftrightarrow$ [(p $\wedge$ q) $\vee$ (p $\wedge$ r)] ----- distributivity of $\wedge$ over $\vee$
10. [**p** $\vee$ **(q** $\wedge$ **r]** $\Leftrightarrow$ **[(p** $\vee$ **q)** $\wedge$ **(p** $\vee$ **r**)] ----- distributivity of $\vee$ over $\wedge$
11. (**p** $\vee$ true) $\Leftrightarrow$ true
12. (**p** $\wedge$ false) $\Leftrightarrow$ false
13. (**p** $\vee$ false) $\Leftrightarrow$ **p**
14. (**p** $\wedge$ true) $\Leftrightarrow$ **p**
15. (**p** $\vee$ $\neg$ **p**) $\Leftrightarrow$ true
16. (**p** $\wedge$ $\neg$ **p**) $\Leftrightarrow$ false
17. **p** $\Leftrightarrow$ $\neg$ **(**$\neg$ **p**) ----- double negation
18. (**p** $\rightarrow$ **q)** $\Leftrightarrow$ **(**$\neg$ **p** $\vee$ **q**) ----- implication
19. (**p** $\leftrightarrow$ **q)** $\Leftrightarrow$ **[(p** $\rightarrow$ **q)** $\wedge$ **(q** $\rightarrow$ **p**)]----- equivalence
20. [(**p** $\wedge$ **q)** $\rightarrow$ **r]** $\Leftrightarrow$ **[p** $\rightarrow$ **(q** $\rightarrow$ **r**)] ----- exportation
21. [(**p** $\rightarrow$ **q)** $\wedge$ **(p** $\rightarrow$ $\neg$ **q)]** $\Leftrightarrow$ $\neg$ **p** ----- absurdity
22. (**p** $\rightarrow$ **q)** $\Leftrightarrow$ **(**$\neg$ **q** $\rightarrow$ $\neg$ **p**) ----- contrapositive

1. p $\Leftrightarrow$ **(p** $\vee$ **p)** ----- idempotence of $\vee$
What this says is, for example, that "Tom is happy." is equivalent to "Tom is happy or Tom is happy". This and the next identity are rarely used, if ever, in everyday life. However, these are useful when manipulating propositions in reasoning in symbolic form.

2**. p** $\Leftrightarrow$ **(p** $\wedge$ **p**) ----- idempotence of $\wedge$
Similar to 1. above.

3. (**p** $\vee$ **q)** $\Leftrightarrow$ **(q** $\vee$ **p**) ----- commutativity of $\vee$
What this says is, for example, that "Tom is rich or (Tom is) famous." is equivalent to "Tom is famous or (Tom is) rich".

4. $(\mathbf{p} \wedge \mathbf{q}) \Leftrightarrow (\mathbf{q} \wedge \mathbf{p})$ ----- commutativity of $\wedge$
What this says is, for example, that "Tom is rich and (Tom is) famous." is equivalent to "Tom is famous and (Tom is) rich".

5. $[(\mathbf{p} \vee \mathbf{q}) \vee \mathbf{r}] \Leftrightarrow [\mathbf{p} \vee (\mathbf{q} \vee \mathbf{r})]$ ----- associativity of $\vee$
What this says is, for example, that "Tom is rich or (Tom is) famous, or he is also happy." is equivalent to "Tom is rich, or he is also famous or (he is) happy".

6. $[(\mathbf{p} \wedge \mathbf{q}) \wedge \mathbf{r}] \Leftrightarrow [\mathbf{p} \wedge (\mathbf{q} \wedge \mathbf{r})]$ ----- associativity of $\wedge$
Similar to 5. above.

7. $\neg (\mathbf{p} \vee \mathbf{q}) \Leftrightarrow (\neg \mathbf{p} \wedge \neg \mathbf{q})$ ----- DeMorgan's Law
For example, "It is not the case that Tom is rich or famous." is true if and only if "Tom is not rich and he is not famous."

8. $\neg (\mathbf{p} \wedge \mathbf{q}) \Leftrightarrow (\neg \mathbf{p} \vee \neg \mathbf{q})$ ----- DeMorgan's Law
For example, "It is not the case that Tom is rich and famous." is true if and only if "Tom is not rich or he is not famous."

9. $[\mathbf{p} \wedge (\mathbf{q} \vee \mathbf{r})] \Leftrightarrow [(\mathbf{p} \wedge \mathbf{q}) \vee (\mathbf{p} \wedge \mathbf{r})]$ ----- distributivity of $\wedge$ over $\vee$
What this says is, for example, that "Tom is rich, and he is famous or (he is) happy." is equivalent to "Tom is rich and (he is) famous, or Tom is rich and (he is) happy".

10. $[\mathbf{p} \vee (\mathbf{q} \wedge \mathbf{r})] \Leftrightarrow [(\mathbf{p} \vee \mathbf{q}) \wedge (\mathbf{p} \vee \mathbf{r})]$ ----- distributivity of $\vee$ over $\wedge$
Similarly to 9. above, what this says is, for example, that "Tom is rich, or he is famous and (he is) happy." is equivalent to "Tom is rich or (he is) famous, and Tom is rich or (he is) happy".

11. $(\mathbf{p} \vee \text{true}) \Leftrightarrow \text{True}$. Here True is a proposition that is always true. Thus the proposition $(P \vee \text{True})$ is always true regardless of what P is.
This and the next three identities, like identities 1 and 2, are rarely used, if ever, in everyday life. However, these are useful when manipulating propositions in reasoning in symbolic form.

12. $(\mathbf{p} \wedge \text{False}) \Leftrightarrow \text{False}$

13. $(\mathbf{p} \vee \text{False}) \Leftrightarrow \mathbf{p}$

14. $(\mathbf{p} \wedge \text{True}) \Leftrightarrow \mathbf{p}$

15. $(\mathbf{p} \vee \neg \mathbf{p}) \Leftrightarrow \text{True}$

What this says is that a statement such as "Tom is 6 foot tall or he is not 6 foot tall." is always true.

16. $(\mathbf{p} \wedge \neg \mathbf{p}) \Leftrightarrow$ False
What this says is that a statement such as "Tom is 6 foot tall and he is not 6 foot tall." is always false.

17. $\mathbf{p} \Leftrightarrow \neg(\neg \mathbf{p})$ ----- double negation
What this says is, for example, that "It is not the case that Tom is not 6 foot tall." is equivalent to "Tom is 6 foot tall."

18. $(\mathbf{p} \rightarrow \mathbf{q}) \Leftrightarrow (\neg \mathbf{p} \vee \mathbf{q})$ ----- implication
For example, the statement "If I win the lottery, I will give you a million dollars." is not true, that is, I am lying, if I win the lottery and don't give you a million dollars. It is true in all the other cases. Similarly, the statement "I don't win the lottery or I give you a million dollars." is false, if I win the lottery and don't give you a million dollars. It is true in all the other cases. Thus these two statements are logically equivalent.

19. $(\mathbf{p} \leftrightarrow \mathbf{q}) \Leftrightarrow [(\mathbf{p} \rightarrow \mathbf{q}) \wedge (\mathbf{q} \rightarrow \mathbf{p})]$----- equivalence
What this says is, for example, that "Tom is happy if and only if he is healthy." is logically equivalent to ""if Tom is happy then he is healthy, and if Tom is healthy he is happy."

20. $[(\mathbf{p} \wedge \mathbf{q}) \rightarrow \mathbf{r}] \Leftrightarrow [\mathbf{p} \rightarrow (\mathbf{q} \rightarrow \mathbf{r})]$ ----- exportation
For example, "If Tom is healthy, then if he is rich, then he is happy." is logically equivalent to "If Tom is healthy and rich, then he is happy."

21. $[(\mathbf{p} \rightarrow \mathbf{q}) \wedge (\mathbf{p} \rightarrow \neg \mathbf{q})] \Leftrightarrow \neg \mathbf{p}$ ----- absurdity
For example, if "If Tom is guilty then he must have been in that room." and "If Tom is guilty then he could not have been in that room." are both true, then there must be something wrong about the assumption that Tom is guilty.

22. $(\mathbf{p} \rightarrow \mathbf{q}) \Leftrightarrow (\neg \mathbf{q} \rightarrow \neg \mathbf{p})$ ----- contrapositive
For example, "If Tom is healthy, then he is happy." is logically equivalent to "If Tom is not happy, he is not healthy."

The identities 1 ~ 16 listed above can be paired by duality relation, which is defined below, as 1 and 2, 3 and 4, ..., 15 and 16. That is 1 and 2 are dual to each other, 3 and 4 are dual to each other, ... Thus if you know one of a pair, you can obtain the other of the pair by using the duality.

Dual of Proposition

Let ***X*** be a proposition involving only $\neg$, $\wedge$, and $\vee$ as a connective. Let X^* be the proposition obtained from ***X*** by replacing $\wedge$ with $\vee$, $\vee$ with $\wedge$, ***T*** with ***F***, and ***F*** with ***T***. Then X^* is called the **dual** of ***X***.

For example, the dual of $[\mathbf{p} \wedge \mathbf{q}] \vee \mathbf{p}$ is $[\mathbf{p} \vee \mathbf{q}] \wedge \mathbf{p}$, and the dual of $[\neg \mathbf{p} \wedge \mathbf{q}] \vee \neg [\mathbf{t} \wedge \neg \mathbf{r}]$ is $[\neg \mathbf{p} \vee \mathbf{q}] \wedge \neg [\mathbf{f} \vee \neg \mathbf{r}]$.

Property of Dual: If two propositions ***p*** and ***q*** involving only $\neg$, $\wedge$, and $\vee$ as connectives are equivalent, then their duals p^* and q^* are also equivalent.

Examples of Use of Identities

Here a few examples are presented to show how the identities in Identities can be used to prove some useful results.

1. $\neg (p \rightarrow q) \Leftrightarrow (p \wedge \neg q)$

What this means is that the **negation of "if p then q"** is **"p** but **not q"**. For example, if you said to someone "If I win a lottery, I will give you $100,000." and later that person says "You lied to me." Then what that person means is that you won the lottery but you did not give that person $100,000 you promised.

To prove this, first let us get rid of $\rightarrow$ using one of the identities**:** $(p \rightarrow q) \Leftrightarrow (\neg p \vee q)$.
That is, $\neg (p \rightarrow q) \Leftrightarrow \neg (\neg p \vee q)$.
Then by De Morgan, it is equivalent to $\neg\neg p \wedge \neg q$, which is equivalent to $P \wedge \neg Q$, since the double negation of a proposition is equivalent to the original proposition as seen in the identities.

2. $p \vee (p \wedge q) \Leftrightarrow p$ --- Absorption
What this tells us is that $p \vee (p \wedge q)$ can be simplified to **p,** or if necessary **p** can be expanded into $p \vee (p \wedge q)$ **.**

To prove this, first note that $P \Leftrightarrow (P \wedge T)$.
Hence
$P \vee (P \wedge Q)$
$\Leftrightarrow (P \wedge T) \vee (P \wedge Q)$
$\Leftrightarrow P \wedge (T \vee Q)$,   by the distributive law.
$\Leftrightarrow (P \wedge T)$,   since $(T \vee Q) \Leftrightarrow T$.
$\Leftrightarrow P$,   since $(P \wedge T) \Leftrightarrow P$.

Note that by the duality
$P \wedge (P \vee Q) \Leftrightarrow P$ also holds.

Exercises

1. Use truth table to verify the following equivalences.

a) p $\wedge$ **False** $\Leftrightarrow$False

b) p $\vee$ **True** $\Leftrightarrow$**Ttrue**

c) p $\vee$ p $\Leftrightarrow$ p

Solution

p	p$\wedge$ **False**	p $\vee$ **True**	p $\vee$ p
T	**F**	**T**	**T**
F	**F**	**F**	**F**

2. Use truth tables to verify the distributive law p $\wedge$(q $\vee$ r) $\Leftrightarrow$(p $\wedge$ q) $\vee$(p $\wedge$ r).

p	*q*	*r*	*q* $\vee$*r*	p $\wedge$(q $\vee$ r)	p $\wedge$ q	p $\wedge$ r	(p $\wedge$ q) $\vee$ (p $\wedge$ r)
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

3. Show that each of the following implications is a tautology without using truth tables.

a) **p** $\rightarrow$ **(p** $\vee$ **q**)
b) (**p** $\wedge$**q)** $\rightarrow$ **(p** $\rightarrow$ **q**)
c) $\neg$ **(p** $\rightarrow$ **q)** $\rightarrow$$\neg$ **q**

Solution

a) If the hypothesis **p** is true, by the definition of disjunction, the conclusion **p** $\vee$**q** is also true.

If **p** is false on the other hand, then by the definition of implication **p** $\rightarrow$ (**p** $\vee$**q***)*is true.

Altenatively, ***p*** $\rightarrow$ ***(p*** $\vee$***q)*** $\Leftrightarrow$***(***$\neg$***p*** $\vee$***(p*** $\vee$***q))*** $\Leftrightarrow$***((***$\neg$***p*** $\vee$***p)*** $\vee$***q)*** $\Leftrightarrow$***(T*** $\vee$***q)*** $\Leftrightarrow$***T***

b) If the hypothesis *p* $\wedge$ *q* is true, then both **p** and **q** are true so that the conclusion **p** $\rightarrow$ **q** is also true. If the hypothesis is false, then "if-then" statement is always true.

This can also be proven similarly to the alternative proof for a).

c) If the hypothesis $\neg$ (*p*$\rightarrow$ *q*) is true, then *p* $\rightarrow$ *q* is false, so that *p* is true and *q* is false. Hence, the conclusion $\neg$ *q* is true.If the hypothesis is false, then "if-then" statement is always true.

This can also be proven similarly to the alternative proof for a).

4. Verify the following equivalences, which are known as the absorption laws.

a) $\mathbf{[\, p \lor (p \land q)\,] \Leftrightarrow p}$

b) [$\mathbf{p \land (p \lor q)\,] \Leftrightarrow p}$

Solution

a) If p is true, then $p \lor (p \land q)$ is true since the first proposition in the disjunction is true. On the other hand, if p is false, then $p \land q$ is also false, so $p \lor (p \land q)$ is false. Since p and $p \lor (p \land q)$ always have the same truth value, they are equivalent.

This can also be proven similarly to b).

b) $[\, p \land (p \lor q)\,]$

$\Leftrightarrow [\, (p \lor F) \land (p \lor q)\,]$

$\Leftrightarrow [\, (p \lor (F \land q)\,] \Leftrightarrow [\, p \lor F\,] \Leftrightarrow p$

This can also be proven similarly to a).

5. Find the dual of each of the following propositions.

a. $p \lor \neg q \lor \neg r$
b. $(p \lor q \lor r) \land s$
c. $(p \land \mathbf{F}) \lor (q \land \mathbf{T})$

Solution

a) $\boldsymbol{(p \land \neg q \land \neg r)}$

b) $\boldsymbol{(p \land q \land r) \lor s}$

c) $\boldsymbol{(p \lor T) \land (q \lor F)}$

6. Find a compound proposition involving the propositions *p*,*q*, and *r* that is true when exactly one of *p*, *q*, and *r* is true and is false otherwise. (*Hint*: Form a disjunction of conjunctions. Include a conjunction for each combination of values for which the propostion istrue. Each conjunction should include each of the three propositions or theirnegations.)

Solution

$(p \land \neg q \land \neg r) \lor (\neg p \land q \land \neg r) \lor (\neg p \land \neg q \land r)$

Implications of propositional logic

The following implications are some of the relationships between propositions that can be derived from the definitions of connectives.
$\Rightarrow$ below corresponds to $\rightarrow$ and it means that the implication always holds. That is it is a tautology. These implications are used in logical reasoning. When the right hand side of these implications is substituted for the left hand side appearing in a proposition, the resulting proposition is implied by the original proposition, that is, one can deduce the new proposition from the original one.

List of Implications:

1. $\mathbf{P \Rightarrow (P \vee Q)}$ ----- addition
2. $(P \wedge Q) \Rightarrow P$ ----- simplification
3. $\mathbf{[P \wedge (P \rightarrow Q] \Rightarrow Q}$ ----- modus ponens
4. $\mathbf{[(P \rightarrow Q) \wedge \neg Q] \Rightarrow \neg P}$ ----- modus tollens
5. $[\neg P \wedge (P \vee Q] \Rightarrow Q$ ----- disjunctive syllogism
6. $\mathbf{[(P \rightarrow Q) \wedge (Q \rightarrow R)] \Rightarrow (P \rightarrow R)}$ ----- hypothetical syllogism
7. $(P \rightarrow Q) \Rightarrow [(Q \rightarrow R) \rightarrow (P \rightarrow R)]$
8. $[(P \rightarrow Q) \wedge (R \rightarrow S)] \Rightarrow [(P \wedge R) \rightarrow (Q \wedge S)]$
9. $[(P \leftrightarrow Q) \wedge (Q \leftrightarrow R)] \Rightarrow (P \leftrightarrow R)$

Examples:

1. $P \Rightarrow (P \vee Q)$ ----- addition
For example, if the sun is shining, then certainly the sun is shining or it is snowing. Thus "if the sun is shining, then the sun is shining or it is snowing." "If $0 < 1$, then $0 \leq 1$ or a similar statement is also often seen.

2. $(P \wedge Q) \Rightarrow P$ ----- simplification
For example, if it is freezing and (it is) snowing, then certainly it is freezing. Thus "If it is freezing and (it is) snowing, then it is freezing."

3. $[P \wedge (P \rightarrow Q] \Rightarrow Q$ ----- modus ponens
For example, if the statement "If it snows, the schools are closed" is true and it actually snows, then the schools are closed.
This implication is the basis of all reasoning. Theoretically, this is all that is necessary for reasoning.
4. $[(P \rightarrow Q) \wedge \neg Q] \Rightarrow \neg P$ ----- modus tollens
For example, if the statement "If it snows, the schools are closed" is true and the schools are not closed, then one can conclude that it is not snowing.
Note that this can also be looked at as the application of the contrapositive and modus ponens. That is, $(P \rightarrow Q)$ is equivalent to $(\neg Q) \rightarrow (\neg P)$. Thus if in addition $\neg Q$ holds, then by the modus ponens, $\neg P$ is concluded.

5. $[\neg P \wedge (P \vee Q] \Rightarrow Q$ ----- disjunctive syllogism

For example, if the statement "It snows or (it) rains." is true and it does not snow, then one can conclude that it rains.

6. $[(P \rightarrow Q) \wedge (Q \rightarrow R)] \Rightarrow (P \rightarrow R)$ ----- hypothetical syllogism
For example, if the statements "If the streets are slippery, the school buses can not be operated." and "If the school buses can not be operated, the schools are closed." are true, then the statement "If the streets are slippery, the schools are closed." is also true.

7. $(P \rightarrow Q) \Rightarrow [(Q \rightarrow R) \rightarrow (P \rightarrow R)]$
This is actually the hypothetical syllogism in another form. For by considering $(P \rightarrow Q)$ as a proposition S, $(Q \rightarrow R)$ as a proposition T, and $(P \rightarrow R)$ as a proposition U in the hypothetical syllogism above, and then by applying the "exportation" from the identities, this is obtained.

8. $[(P \rightarrow Q) \wedge (R \rightarrow S)] \Rightarrow [(P \wedge R) \rightarrow (Q \wedge S)]$
For example, if the statements "If the wind blows hard, the beach erodes." and "If it rains heavily, the streets get flooded." are true, then the statement "If the wind blows hard and it rains heavily, then the beach erodes and the streets get flooded." is also true.

9. $[(P \leftrightarrow Q) \wedge (Q \leftrightarrow R)] \Rightarrow (P \leftrightarrow R)$
This just says that the logical equivalence is transitive, that is, if P and Q are equivalent, and if Q and R are also equivalent, then P and R are equivalent.

Reasoning with Propositions

Logical reasoning is the process of drawing conclusions from premises using rules of inference. The basic inference rule is **modus ponens**. It states that if both $\mathbf{P \rightarrow Q}$ and **P** hold, then **Q** can be concluded, and it is written as

P
$\mathbf{P \rightarrow Q}$
....
Q

Here the lines above the dotted line are **premises** and the line below it is the **conclusion** drawn from the premises.

For example if "if it rains, then the game is not played" and "it rains" are both true, then we can conclude that the game is not played.

In addition to modus ponens, one can also reason by using **identities** and implications.

If the left(right) hand side of an identity appearing in a proposition is replaced by the right(left) hand side of the identity, then the resulting proposition is logically equivalent to the original proposition. Thus the new proposition is deduced from the original proposition. For example in the proposition $\mathbf{P \wedge (Q \rightarrow R)}$**,** $\mathbf{(Q \rightarrow R)}$ can be replaced with $(\mathbf{\neg Q \vee R})$ to conclude

$P \wedge (\neg Q \vee R)$, since $(\mathbf{Q \to R}) \Leftrightarrow (\neg\ \mathbf{Q} \vee\ \mathbf{R})$

Similarly if the left(right) hand side of an implication appearing in a proposition is replaced by the right(left) hand side of the implication, then the resulting proposition is logically implied by the original proposition. Thus the new proposition is deduced from the original proposition.

The tautologies listed as "implications" can also be considered **inference rules** as shown below.

Rules of Inference	**Tautological Form**	**Name**
P ………… $P \vee Q$	$P \Rightarrow (P \vee Q)$	addition
$P \wedge Q$ ………… P	$(P \wedge Q) \Rightarrow P$	simplification
P $P \to Q$ ………… Q	$[P \wedge (P \to Q)] \Rightarrow Q$	modus ponens
$\neg Q$ $P \to Q$ ……….. $\neg P$	$[\neg Q \wedge (P \to Q)] \Rightarrow \neg P$	modus tollens
$P \vee Q$ $\neg P$ ………… Q	$[(P \vee Q) \wedge \neg P] \Rightarrow Q$	disjunctive syllogism
$P \to Q$ $Q \to R$ ………… $P \to R$	$[(P \to Q) \wedge (Q \to R)] \Rightarrow [P \to R]$	hypothetical syllogism
P Q		conjunction

....
$P \wedge Q$

$(P \to Q) \wedge (R \to S)$ $\quad$ $[(P \to Q) \wedge (R \to S) \wedge (P \vee R)] \Rightarrow [Q \vee S]$ $\quad$ constructive dilemma
$P \vee R$
....
$Q \vee S$

$(P \to Q) \wedge (R \to S)$ $\quad$ $[(P \to Q) \wedge (R \to S) \wedge (\neg Q \vee \neg S)] \Rightarrow [\neg P \vee \neg R]$ $\quad$ destructive dilemma
$\neg Q \vee \neg S$
....
$\neg P \vee \neg R$

Example of Inferencing

Consider the following argument**:**

1. Today is Tuesday or Wednesday.

2. But it can't be Wednesday, since the doctor's office is open today, and that office is always closed on Wednesdays.

3. Therefore today must be Tuesday.

This sequence of reasoning (inferencing) can be represented as a series of application of modus ponens to the corresponding propositions as follows.

The modus ponens is an inference rule which deduces Q from $P \to Q$ and P.

T: Today is Tuesday.

W: Today is Wednesday.

D: The doctor's office is open today.

C: The doctor's office is always closed on Wednesdays.

The above reasoning can be represented by propositions as follows.

1. $T \vee W$

2. ***D***
C
……….

$\neg \boldsymbol{W}$
………
3. $\boldsymbol{T}$

To see if this conclusion $\boldsymbol{T}$ is correct, let us first find the relationship among $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{W}$:

$\boldsymbol{C}$ can be expressed using $\boldsymbol{D}$ and $\boldsymbol{W}$. That is, restate $\boldsymbol{C}$ first as the doctor's office is always closed if it is Wednesday. Then $\boldsymbol{C} \Leftrightarrow (\boldsymbol{W} \rightarrow \neg \boldsymbol{D})$ Thus substituting $(\boldsymbol{W} \rightarrow \neg \boldsymbol{D})$ for $\boldsymbol{C}$, we can proceed as follows.

$\boldsymbol{D}$
$\boldsymbol{W} \rightarrow \neg \boldsymbol{D}$
…………
$\rightarrow \boldsymbol{W}$

which is correct by modus tollens.

From this $\neg \boldsymbol{W}$ combined with $\boldsymbol{T} \vee \boldsymbol{W}$ of **1.** above,

$\neg \boldsymbol{W}$
$\boldsymbol{T} \vee \boldsymbol{W}$
………
$\boldsymbol{T}$

which is correct by disjunctive syllogism.
Thus we can conclude that the given argument is correct.

To save space we also write this process as follows eliminating one of the **~*W***'s:

$\boldsymbol{D}$
$\boldsymbol{W} \rightarrow \neg \boldsymbol{D}$
…………
$\neg \boldsymbol{W}$
$\boldsymbol{T} \vee \boldsymbol{W}$
…………
$\boldsymbol{T}$

Proof of Identities using truth tables

All the identities in **Identities** can be proven to hold using truth tables as follows. In general two propositions are logically equivalent if they take the same value for each set of values of their variables. Thus to see whether or not two propositions are equivalent, we construct truth tables for them and compare to see whether or not they take the same value for each set of values of their variables.

For example consider the **commutativity of** $\vee$:
$(P \vee Q) \Leftrightarrow (Q \vee P)$.

To prove that this equivalence holds, let us construct a truth table for each of the proposition $(P \vee Q)$ and $(Q \vee P)$.

A truth table for $(P \vee Q)$ is, by the definition of $\vee$,

P	Q	$P\vee Q$
F	F	F
F	T	T
T	F	T
T	T	T

A truth table for $(Q \vee P)$ is, by the definition of $\vee$,

P	Q	$Q\vee P$
F	F	F
F	T	T
T	F	T
T	T	T

As we can see from these tables $(P \vee Q)$ and $(Q \vee P)$ take the same value for the same set of value of P and Q. Thus they are (logically) equivalent.

We can also put these two tables into one as follows:

P	Q	$P\vee Q$	$Q\vee P$
F	F	F	F
F	T	T	T
T	F	T	T
T	T	T	T

Using this convention for truth table we can show that the first of **De Morgan's** Laws also holds.

P	Q	$\neg(P\vee Q)$	$\neg P\wedge\neg Q$
F	F	T	T
F	T	F	F
T	F	F	F
T	T	F	F

By comparing the two right columns we can see that $\neg(P\vee Q)$ and $\neg P\wedge\neg Q$ are equivalent.

Proof of Implications using truth table and tautologies

1. All the implications in **Implications** can be proven to hold by constructing truth tables and showing that they are always true.

For example consider the first implication "addition": $P \Rightarrow (P \vee Q)$.
To prove that this implication holds, let us first construct a truth table for the proposition $P \vee Q$.

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	T

Then by the definition of $\rightarrow$, we can add a column for $P \rightarrow (P \vee Q)$ to obtain the following truth table.

P	Q	$P \vee Q$	$P \rightarrow (P \vee Q)$
F	F	F	T
F	T	T	T
T	F	T	T
T	T	T	T

The first row in the rightmost column results since P is false, and the others in that column follow since $(P \vee Q)$ is true.

The rightmost column shows that $P \rightarrow (P \vee Q)$ is always true.

2. Some of the implications can also be proven by using identities and implications that have already been proven.

For example suppose that the identity "exportation":
$[(X \wedge Y) \rightarrow Z] \Leftrightarrow [X \rightarrow (Y \rightarrow Z)]$ **,**
and the implication "hypothetical syllogism":
$[(P \rightarrow Q) \wedge (Q \rightarrow R)] \Rightarrow (P \rightarrow R)$
have been proven. Then the implication No. 7:
$(P \rightarrow Q) \Rightarrow [(Q \rightarrow R) \rightarrow (P \rightarrow R)]$
can be proven by applying the "exportation" to the "hypothetical syllogism" as follows**:**

Consider $(P \rightarrow Q)$, $(Q \rightarrow R)$ **,** and $(P \rightarrow R)$ in the "hypothetical syllogism" as X, Y and Z of the "exportation", respectively.
Then since $[(X \wedge Y) \rightarrow Z] \Leftrightarrow [X \rightarrow (Y \rightarrow Z)]$ implies $[(X \wedge Y) \rightarrow Z] \Rightarrow [X \rightarrow (Y \rightarrow Z)]$ **,** the implication of No. 7 follows.

Similarly the **modus ponens** (implication No. 3) can be proven as follows**:**
Noting that $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$,

$P \wedge (P \rightarrow Q)$
$\Leftrightarrow P \wedge (\neg P \vee Q)$
$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$ --- by the distributive law
$\Leftrightarrow F \vee (P \wedge Q)$
$\Leftrightarrow (P \wedge Q)$
$\Rightarrow Q$

Also the **exportation** (identity No. 20), **(** $P \rightarrow (Q \rightarrow R)$) $\Leftrightarrow$ $(P \wedge Q) \rightarrow R$ **)** can be proven using identities as follows**:**

$(P \rightarrow (Q \rightarrow R)) \Leftrightarrow \neg P \vee (Q \rightarrow R)$
$\Leftrightarrow \neg P \vee (\neg Q \vee R)$
$\Leftrightarrow (\neg P \vee \neg Q) \vee R$
$\Leftrightarrow \neg (P \wedge Q) \vee R$
$\Leftrightarrow (P \wedge Q) \rightarrow R$

3. Some of them can be proven by noting that a proposition in an implication can be replaced by an equivalent proposition without affecting its value.

For example by substituting $(\neg Q \rightarrow \neg P)$ for $(P \rightarrow Q)$ **,** since they are equivalent being contrapositive to each other, **modus tollens** (the implication No. 4)**:** $[(P \rightarrow Q) \wedge \neg Q] \Rightarrow \neg P$, reduces to the modus ponens: $[X \wedge (X \rightarrow Y)] \Rightarrow Y$. Hence if the modus ponens and the "contrapositive" in the "Identities" have been proven, then the modus tollens follows from them.

Exercises

1. What rule of inference is used in each of the following arguments?

a. John likes apple pies. Therefore, John likes apple pies or icecream.
b. Mary likes chocolate and icecream. Therefore, Mary likes chocolate.
c. If it snows, then the roads are closed; it snows . Therefore, the roads are closed.
d. If it snows, then the roads are closed; the roads are not closed. Therefore, it does not snow.
e. To go to Tahiti, one must fly or take a boat; there is no seat on any flight to Tahiti this year. Therefore, one must take a boat to go to Tahiti this year.

Solution

a. Addition
b. Simplification
c. Modus ponens
d. Modus tollens
e. Disjunctive syllogism

2. Express the following arguments using the symbols indicated. What rules of inference are used in each of them ?

a. If the teens like it, then the sales volume will go up; Either the teens like it or the store will close; The sales volume will not go up. Therefore, the store will close.
Symbols to be used: The teens like it (T). The sales volume will go up (S). The store will close (C).
b. It is not the case that if there is not a lot of sun, then there is enough water, nor is it true that either there is a lot of rain or the crop is good. Therefore, there is not enough water and the crop is not good.
Symbols to be used: There is not a lot of sun (S). There is enough water (W). There is a lot of rain (R). The crop is good (C).

c. If flowers are colored, they are always scented; I don't like flowers that are not grown in the open air; All flowers grown in the open air are colored . Therefore, I don't like any flowers that are scentless.
Symbols to be used: Flowers are colored (C). Flowers are scented (S). I like flowers (L). Flowers are grown in the open air (O).

d. No animals, except giraffes, are 15 feet or higher; There are no animals in this zoo that belong to anyone but me; I have no animals less than 15 feet high. Therefore, all animals in this zoo are giraffes.
Symbols to be used: Animals are giraffes (G). Animals are 15 feet or higher (F). Animals are in the zoo (Z). Animals belong to me (M).

e. Bees like red flowers, or my hat is red and bees like hats; However, my hat is not red, or bees don't like hats but they like red flowers. Therefore bees like red flowers.
Symbols to be used: Bees like red flowers (R). My hat is red (H). Bees like hats (L).

Solution

a. The argument is translated as follows:
$T \rightarrow S$
$T \vee C$
$\neg S$
……………………………………
C

The inference rules used are:
From ($T \rightarrow S$) and $\neg S$
by modus tollens we deduce $\neg T$
From $\neg T$ and ($T \vee C$)
by disjunctive syllogism we conclude C.

b. The argument is translated as follows:
$T \rightarrow S$
$T \vee C$
$\neg S$
……………………………………
C

The inference rules used are:
From ($T \rightarrow S$) and $\neg S$
by modus tollens we deduce $\neg T$
From $\neg T$ and ($T \vee C$)
by disjunctive syllogism we conclude C.

c. The argument is translated as follows:
$\neg (S \rightarrow W) \wedge \neg (R \vee C)$
…………………………….
$(\neg W \wedge \neg C)$

The inference rules used are:
$\neg (S \rightarrow W)$ is equivalent to $(S \wedge \neg W)$
by implication and De Morgan
Also $\neg (R \vee C)$ is equivalent to $(\neg R \wedge \neg C)$ by De Morgan.
Hence $\neg (S \rightarrow W) \wedge \neg (R \vee C)$ is equivalent to $(S \wedge \neg W) \wedge (\neg R \wedge \neg C)$, which is equivalent to
$(S \wedge \neg R) \wedge (\neg W \wedge \neg C)$
Hence by simplification $(\neg W \wedge \neg C)$

d. The argument is translated as follows:
$C \rightarrow S$
$\neg O \rightarrow \neg L$
$O \rightarrow C$
……………….
$\neg S \rightarrow \neg L$

The inference rules used are:
$\neg S \rightarrow \neg C$
by Contrapositive of $C \rightarrow S$
$\neg C \rightarrow \neg O$
by Contrapositive of $O \rightarrow C$
By hypothetical syllogism from the last two
$\neg S \rightarrow \neg O$
By another hypothetical syllogism from this and $\neg O \rightarrow \neg L$,
$\neg S \rightarrow \neg L$
is obtained.

e. The argument is translated as follows:
$\neg G \rightarrow \neg F$
$Z \rightarrow M$
$M \rightarrow F$
…………….
$Z \rightarrow G$

The inference rules used are:
>From $Z \rightarrow M$ and
$M \rightarrow F$
by hypothetical syllogism
$Z \rightarrow F$ is obtained.
Then from $\neg G \rightarrow \neg F$
by taking contrapositive $M \rightarrow F$ is obtained.
>From this and $Z \rightarrow F$
by hypothetical syllogism $Z \rightarrow G$ is obtained.

f. The argument is translated as follows:
$R \vee (H \wedge L)$
$\neg H \vee (\neg L \wedge R)$
…………………..

R

The inference rules used are:
From $\neg H \vee (\neg L \wedge R)$ by distributive law
$(\neg H \vee \neg L) \wedge (\neg H \vee R)$
From this by simplification
$(\neg H \vee \neg L)$
From this by De Morgan $\neg (H \wedge L)$
With this and $R \vee (H \wedge L)$
by disjunctive syllogism R is concluded.

2.2 INTRODUCTION TO PREDICATE LOGIC

Introduction

The propositional logic is not powerful enough to represent all types of assertions that are used in computer science and mathematics, or to express certain types of relationship between propositions such as equivalence.

For example, the assertion **"x is greater than 1",** where **x** is a variable, is not a proposition because you can not tell whether it is true or false unless you know the value of **x.** Thus the propositional logic can not deal with such sentences. However, such assertions appear quite often in mathematics and we want to do inferencing on those assertions.

Also the pattern involved in the following logical equivalences can not be captured by the propositional logic:

"Not all birds fly" is equivalent to "Some birds don't fly".
"Not all integers are even" is equivalent to "Some integers are not even".
"Not all cars are expensive" is equivalent to "Some cars are not expensive",
... .

Each of those propositions is treated independently of the others in propositional logic. For example, if P represents "Not all birds fly" and Q represents "Some integers are not even", then there is no mechanism in propositional logic to find out whether or not P is equivalent to Q. Hence to be used in inferencing, each of these equivalences must be listed individually rather than dealing with a general formula that covers all these equivalences collectively and instantiating it as they become necessary, if only propositional logic is used.

Thus we need more powerful logic to deal with these and other problems. The predicate logic is one of such logic and it addresses these issues among others.

2.1 Predicate and quantifiers

A **predicate** is a verb phrase template that describes a property of objects, or a relationship among objects represented by the variables.

For example, the sentences "The car Tom is driving is blue", "The sky is blue", and "The cover of this book is blue" come from the template "is blue" by placing an appropriate noun/noun phrase in front of it. The phrase **"is blue"** is a predicate and it describes the property of being blue. Predicates are often given a **name**. For example any of "is blue", "Blue" or "B" can be used to represent the predicate "is blue" among others. If we adopt B as the name for the predicate "is blue", sentences that assert an object is blue can be represented as "B(x)", where x represents an arbitrary object. B(x) reads as "x is blue".

Similarly the sentences "John gives the book to Mary", "Jim gives a loaf of bread to Tom", and "Jane gives a lecture to Mary" are obtained by substituting an appropriate object for variables *x*, *y*, and *z* in the sentence "*x* gives *y* to *z*". The template "... gives ... to ..." is a predicate and it describes a relationship among three objects. This predicate can be represented by Give (x, y, z) or G(x, y, z), for example.

Note: The sentence "John gives the book to Mary" can also be represented by another predicate such as "gives a book to". Thus if we use B(x, y) to denote this predicate, "John gives the book to Mary" becomes B(John, Mary). In that case, the other sentences, "Jim gives a loaf of bread to Tom", and "Jane gives a lecture to Mary", must be expressed with other predicates.

Quantifiers Forming Propositions from Predicates

- universe
- universal quantifier
- existential quantifier
- free variable
- bound variable
- scope of quantifier
- order of quantifiers

A predicate with variables is not a proposition. For example, the statement $x > 1$ with variable x over the universe of real numbers is neither true nor false since we don't know what x is. It can be true or false depending on the value of x.
For $x > 1$ to be a proposition either we substitute a specific number for x or change it to something like "There is a number x for which $x > 1$ holds", or "For every number x, $x > 1$ holds".

More generally, a predicate with variables (called an **atomic formula**) can be made a **proposition** by applying one of the following two operations to each of its variables:

1. assign a value to the variable
2. quantify the variable using a **quantifier**.

For example, $x > 1$ becomes $3 > 1$ if 3 is assigned to x, and it becomes a true statement, hence a proposition.

In general, a quantification is performed on formulas of predicate logic (called wff), such as ***x > 1*** or ***P*(*x*)**, by using quantifiers on variables. There are two types of quantifiers: **universal quantifier** and **existential quantifier**.

The **universal quantifier** turns, for example, the statement $x > 1$ to "for every object x in the universe, $x > 1$", which is expressed as "$\forall x, x > 1$". This new statement is true or false in the universe of discourse. Hence it is a proposition once the universe is specified.

Similarly the **existential quantifier** turns, for example, the statement $x > 1$ to "for some object x in the universe, $x > 1$", which is expressed as "$\exists\, x, x > 1$." Again, it is true or false in the universe of discourse, and hence it is a proposition once the universe is specified.

Universe of Discourse

The universe of discourse, also called **universe**, is the set of objects of interest. The propositions in the predicate logic are statements on objects of a universe. The universe is thus the domain of the (individual) variables. It can be the set of real numbers, the set of integers, the set of all cars on a parking lot, the set of all students in a classroom etc. The universe is often left implicit in practice. But it should be obvious from the context.

The Universal Quantifier

The expression: $\forall x\, P(x)$, denotes the **universal quantification** of the atomic formula P(x). Translated into the English language, the expression is understood as: "*For all x, P(x)* holds", "*for each x, P(x)* holds" or "*for every x, P(x)* holds". $\forall$is called the **universal quantifier**, and $\forall x$ means all the objects x in the universe. If this is followed by $P(x)$ then the meaning is that $P(x)$ is true for every object x in the universe. For example, "All cars have wheels" could be transformed into the propositional form, $\forall x\, P(x)$, where:

- $P(x)$ is the predicate denoting: ***x* has wheels**, and
- the universe of discourse is only populated by cars.

Universal Quantifier and **Connective AND**

If all the elements in the universe of discourse can be listed then the universal quantification $\forall x\, P(x)$ is equivalent to the conjunction: $\boldsymbol{P(x_1)) \wedge P(x_2) \wedge P(x_3) \wedge \ldots \wedge P(x_n)}$.

For example, in the above example of $\forall x\, P(x)$, if we knew that there were **only** 4 cars in our universe of discourse *(c1, c2, c3* and *c4)* then we could also translate the statement as: $\boldsymbol{P(c1) \wedge}$

P(c2)* $\wedge$*P(c3)* $\wedge$*P(c4)

The Existential Quantifier

The expression: $\exists$ *x P(x)*, denotes the **existential quantification** of *P(x)*. Translated into the English language, the expression could also be understood as: "There exists an *x* such that *P(x)*" or "There is at least one *x* such that *P(x)*" $\exists$is called the **existential quantifier**, and $\exists$*x* means at least one object *x* in the universe. If this is followed by *P(x)* then the meaning is that *P(x)* is true for at least one object *x* of the universe. For example, "*Someone loves you*" could be transformed into the propositional form, $\exists$*x P(x)*, where:

- *P(x)* is the predicate meaning: ***x* loves you**,
- The universe of discourse contains (but is not limited to) all living creatures.

Existential Quantifier and **Connective OR**
If all the elements in the universe of discourse can be listed, then the existential quantification $\exists$*x P(x)* is equivalent to the disjunction: $\boldsymbol{P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots \vee P(x_n)}$.

For example, in the above example of $\exists$*x P(x)*, if we knew that there were **only** 5 living creatures in our universe of discourse (say: me, he, she, rex and fluff), then we could also write the statement as: ***P(me)* $\vee$*P(he)* $\vee$*P(she)* $\vee$*P(rex)* $\vee$ *P(fluff)***

An appearance of a variable in a **wff** is said to be **bound** if either a specific value is assigned to it or it is quantified. If an appearance of a variable is not bound, it is called **free**. The extent of the application(effect) of a quantifier, called the **scope** of the quantifier, is indicated by square brackets **[ ]**. If there are no square brackets, then the scope is understood to be the smallest **wff** following the quantification.
For example, in $\exists$ ***x P*(*x, y*)**, the variable ***x*** is bound while ***y*** is free. In $\forall$***x* [** $\exists$ ***y P*(*x, y*)** $\vee$ ***Q*(*x, y*)] ,** ***x*** and the ***y*** in ***P*(*x, y*)** are bound, while ***y*** in ***Q*(*x, y*)** is free, because the scope of $\exists$ ***y*** is ***P*(*x, y*)**. The scope of $\forall$ ***x*** is **[**$\exists$ ***y P*(*x, y*)** $\vee$ ***Q*(*x, y*)] .**

How to read quantified formulas

When reading quantified formulas in English, **read them from left to right.** $\forall$***x*** can be read as "for every object ***x*** in the universe the following holds" and $\exists$ ***x*** can be read as "there erxists an object ***x*** in the universe which satisfies the following" or "for some object ***x*** in the universe the following holds". Those do not necessarily give us good English expressions. But they are where we can start. Get the correct reading first then polish your English without changing the truth values.

For example, let the universe be the set of airplanes and let ***F*(*x, y*)** denote "***x*** flies faster than ***y***". Then
$\forall$ ***x***$\forall$ ***y F*(*x, y*)** can be translated initially as "For every airplane ***x*** the following holds: ***x*** is faster

than every (any) airplane ***y***". In simpler English it means "Every airplane is faster than every airplane (including itself !)".

$\forall x \exists y F(x, y)$ can be read initially as "For every airplane ***x*** the following holds: for some airplane ***y***, ***x*** is faster than ***y***". In simpler English it means "Every airplane is faster than some airplane".

$\exists x \forall y F(x, y)$ represents "There exist an airplane ***x*** which satisfies the following: (or such that) for every airplane ***y***, ***x*** is faster than ***y***". In simpler English it says "There is an airplane which is faster than every airplane" or "Some airplane is faster than every airplane".

$\exists x \exists y F(x, y)$ reads "For some airplane ***x*** there exists an airplane ***y*** such that ***x*** is faster than ***y***", which means "Some airplane is faster than some airplane".

Order of Application of Quantifiers

When more than one variables are quantified in a wff such as $\exists y \forall x P(x, y)$, they are applied from the inside, that is, the one closest to the atomic formula is applied first. Thus $\exists y \forall x P(x, y)$ reads $\exists y [\forall x P(x, y)]$, and we say "there exists an ***y*** such that for every ***x***, $P(x, y)$ holds" or "for some ***y***, $P(x, y)$ holds for every ***x***".

The positions of the same type of quantifiers can be switched without affecting the truth value as long as there are no quantifiers of the other type between the ones to be interchanged.
For example $\exists x \exists y \exists z P(x, y, z)$ is equivalent to $\exists y \exists x \exists z P(x, y, z)$, $\exists z \exists y \exists x P(x, y, z)$, etc. It is the same for the universal quantifier.

However, the positions of different types of quantifiers can **not** be switched.
For example $\forall x \exists y P(x, y)$ is **not** equivalent to $\exists y \forall x P(x, y)$. For let $P(x, y)$ represent $x < y$ for the set of numbers as the universe, for example. Then $\forall x \exists y P(x, y)$ reads "for every number ***x***, there is a number ***y*** that is greater than ***x***", which is true, while $\exists y \forall x P(x, y)$ reads "there is a number that is greater than every (any) number", which is not true.

Well-Formed Formula for First Order Predicate Logic

- wff (well formed formula)
- atomic formula
- syntax of wff

Not all strings can represent propositions of the predicate logic. Those which produce a proposition when their symbols are interpreted must follow the rules given below, and they are called **wffs** (well-formed formulas) of the first order predicate logic.

Rules for constructing Wffs

A predicate name followed by a list of variables such as $P(x, y)$, where P is a predicate name, and x and y are variables, is called an **atomic formula**.

Wffs are constructed using the following rules:

1. *True* and *False* are wffs.
2. Each propositional constant (i.e. specific proposition), and each propositional variable (i.e. a variable representing propositions) are wffs.
3. Each atomic formula (i.e. a specific predicate with variables) is a wff.
4. If A, B, and C are wffs, then so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, and $(A \leftrightarrow B)$.
5. If x is a variable (representing objects of the universe of discourse), and A is a wff, then so are $\forall x\, A$ and $\exists\, x\, A$.

(Note : More generally, arguments of predicates are something called a term. Also variables representing predicate names (called predicate variables) with a list of variables can form atomic formulas. But we do not get into that here.

For example, "The capital of Virginia is Richmond." is a specific proposition. Hence it is a wff by Rule 2.

Let $\boldsymbol{B}$ be a predicate name representing "being blue" and let $\boldsymbol{x}$ be a variable. Then $\boldsymbol{B(x)}$ is an atomic formula meaning "$\boldsymbol{x}$ is blue". Thus it is a wff by Rule 3. above. By applying Rule 5. to $\boldsymbol{B(x)}$, $\boldsymbol{\forall x B(x)}$ is a wff and so is $\boldsymbol{\exists x B(x)}$. Then by applying Rule 4. to them $\boldsymbol{\forall x\, B(x) \wedge \exists x\, B(x)}$ is seen to be a wff. Similarly if $\boldsymbol{R}$ is a predicate name representing "being round". Then $\boldsymbol{R(x)}$ is an atomic formula. Hence it is a wff. By applying Rule 4 to $\boldsymbol{B(x)}$ and $\boldsymbol{R(x)}$, a wff $\boldsymbol{B(x) \wedge R(x)}$ is obtained.

In this manner, larger and more complex wffs can be constructed following the rules given above.

Note, however, that strings that can not be constructed by using those rules are not wffs. For example, $\boldsymbol{\forall x B(x) R(x)}$, and $\boldsymbol{B(\ \exists x\ )}$ are **NOT** wffs, **NOR** are $\boldsymbol{B(\ R(x)\ )}$, and $\boldsymbol{B(\ \exists x\ R(x)\ )}$ **.**

One way to check whether or not an expression is a wff is to try to state it in English. If you can translate it into a correct English sentence, then it is a wff.

More examples: To express the fact that Tom is taller than John, we can use the atomic formula ***taller*(Tom, John)**, which is a wff. This wff can also be part of some compound statement such as ***taller* (Tom, John) $\wedge \neg$*taller*(John, Tom)**, which is also a wff.

If x is a variable representing people in the world, then ***taller(x*,Tom), $\forall$*x taller(x*,Tom), $\exists$*x taller(x*,Tom), $\exists$*x* $\forall$*y taller (x,y)*** are all wffs among others.

However, *taller*($\exists$ *x*, John) and *taller* (Tom $\wedge$Mary, Jim), for example, are **NOT** wffs.

Exercises

1. Let $Q(x, y)$ denote the statement "x is greater than y." What are the truth values of the following?

a. $Q(3, 1)$
b. $Q(5, 5)$
c. $Q(6, -6)$
d. $Q(2^8, 256)$

2. Let $P(x)$ be the statement "x is happy," where the universe of discourse for x is the set of students. Express each of the following quantifications in English.

a. $\exists x\, P(x)$
b. $\forall x \neg P(x)$
c. $\exists x \neg P(x)$
d. $\neg\forall x \neg P(x)$

3. Let $P(x)$ be the statement " $x > x^2$." If the universe of discourse is the set of real numbers, what are the truth values of the following?

a. $P(0)$
b. $P(1/2)$
c. $P(2)$
d. $P(-1)$
e. $\exists x\, P(x)$
f. $\forall x\, P(x)$

4. Suppose that the universe of discourse of the atomic formula $P(x,y)$ is {1, 2, 3}. Write out the following propositions using disjunctions and conjunctions.

a. $\exists x\, P(x, 2)$
b. $\forall y\, P(3, y)$
c. $\forall x\, \forall y\, P(x, y)$
d. $\exists x \exists y\, P(x, y)$
e. $\exists x\, \forall y\, P(x, y)$
f. $\forall y\, \exists x\, P(x, y)$

Solution

1. a) T b) F c) T d) F

2. There is a student who is happy.

a. Every student is not happy.
b. There is a student who is not happy.
c. Not all students are unhappy.

3. a) F b) T c) F d)F e) T f) F

4. a) $P(1, 2) \vee P(2, 2) \vee P(3, 2)$

b) $P(3, 1) \wedge P(3, 2) \wedge P(3, 3)$

c) $P(1, 1) \wedge P(1, 2) \wedge P(1, 3) \wedge P(2, 1) \wedge P(2, 2) \wedge P(2, 3) \wedge P(3, 1) \wedge P(3, 2) \wedge P(3, 3)$

e) $(P(1, 1) \wedge P(1, 2) \wedge P(1, 3)) \vee (P(2, 1) \wedge P(2, 2) \wedge P(2, 3)) \vee (P(3, 1) \wedge P(3, 2) \wedge P(3, 3))$

f) $(P(1, 1) \vee P(2, 1) \vee P(3, 1)) \wedge (P(1, 2) \vee P(2, 2) \vee P(3, 2)) \wedge (P(1, 3) \vee P(2, 3) \vee P(3, 3))$