Relations

If we want to describe a relationship between elements of two sets A and B, we can use **ordered pairs** with their first element taken from A and their second element taken from B.

Since this is a relation between **two sets**, it is called a **binary relation**.

Definition: Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

In other words, for a binary relation R we have $R \subseteq A \times B$. We use the notation aRb to denote that $(a, b) \in R$ and $a\underline{R}b$ to denote that $(a, b) \notin R$.

Relations

When (a, b) belongs to R, a is said to be **related** to b by R.

Example: Let P be a set of people, C be a set of cars, and D be the relation describing which person drives which car(s).

P = {Carl, Suzanne, Peter, Carla},

C = {Mercedes, BMW, tricycle}

D = {(Carl, Mercedes), (Suzanne, Mercedes), (Suzanne, BMW), (Peter, tricycle)}

This means that Carl drives a Mercedes, Suzanne drives a Mercedes and a BMW, Peter drives a tricycle, and Carla does not drive any of these vehicles.

Functions as Relations

You might remember that a **function** f from a set A to a set B assigns a unique element of B to each element of A.

The **graph** of f is the set of ordered pairs (a, b) such that b = f(a).

Since the graph of f is a subset of $A \times B$, it is a **relation** from A to B.

Moreover, for each element **a** of A, there is exactly one ordered pair in the graph that has **a** as its first element.

Functions as Relations

Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph.

This is done by assigning to an element $a \in A$ the unique element $b \in B$ such that $(a, b) \in R$.

Relations on a Set

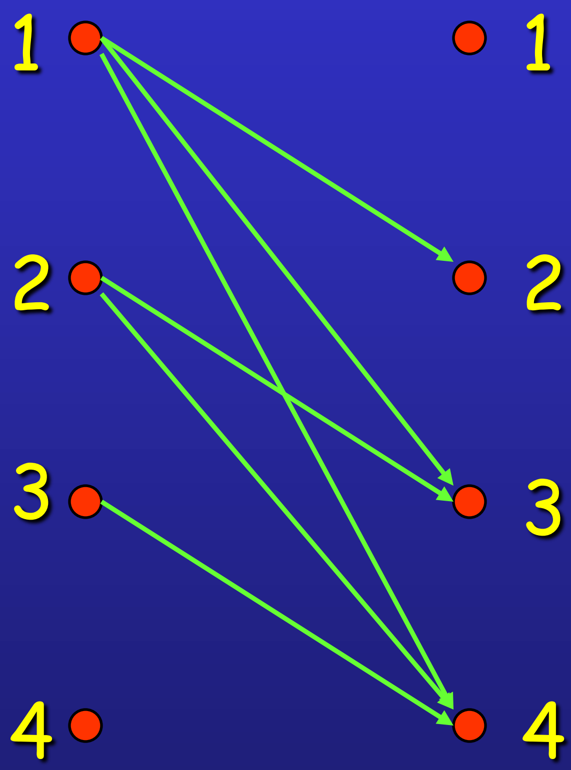
Definition: A relation on the set A is a relation from A to A.

In other words, a relation on the set A is a subset of $A \times A$.

Example: Let $A = \{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a < b\}$?

Relations on a Set

Solution: R = {(1, 2),(1, 3),(1, 4),(2, 3),(2, 4),(3, 4)}



R	1	2	3	4
1		X	X	X
2			X	X
3				X
4				

Relations on a Set

How many different relations can we define on a set A with n elements?

A relation on a set A is a subset of $A \times A$.

How many elements are in $A \times A$?

There are n^2 elements in $A \times A$, so how many subsets (= relations on A) does $A \times A$ have?

The number of subsets that we can form out of a set with m elements is 2^m. Therefore, 2^{n^2} subsets can be formed out of $A \times A$.

Answer: We can define 2^{n^2} different relations on A.

Properties of Relations

We will now look at some useful ways to classify relations.

Definition: A relation R on a set A is called **reflexive** if $(a, a) \in R$ for every element $a \in A$.

Are the following relations on $\{1, 2, 3, 4\}$ reflexive?

$R = \{(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)\}$ No.

$R = \{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$ Yes.

$R = \{(1, 1), (2, 2), (3, 3)\}$ No.

Definition: A relation on a set A is called **irreflexive** if $(a, a) \notin R$ for every element $a \in A$.

Properties of Relations

Definitions:

A relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

A relation R on a set A is called **antisymmetric** if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$.

A relation R on a set A is called **asymmetric** if $(a, b) \in R$ implies that $(b, a) \notin R$ for all $a, b \in A$.

Properties of Relations

Are the following relations on {1, 2, 3, 4} symmetric, antisymmetric, or asymmetric?

R = {(1, 1), (1, 2), (2, 1), (3, 3), (4, 4)} symmetric

R = {(1, 1)} sym. and antisym.

R = {(1, 3), (3, 2), (2, 1)} asym.

R = {(4, 4), (3, 3), (1, 4)}

Properties of Relations

Definition: A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

Are the following relations on $\{1, 2, 3, 4\}$ transitive?

$R = \{(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)\}$ No.

$R = \{(1, 3), (3, 2), (2, 1)\}$ No.

$R = \{(2, 4), (4, 3), (2, 3), (4, 1)\}$ Yes.

Counting Relations

Example: How many different reflexive relations can be defined on a set A containing n elements?

Solution: Relations on R are subsets of $A \times A$, which contains n^2 elements.

Therefore, different relations on A can be generated by choosing different subsets out of these n^2 elements, so there are 2^{n^2} relations.

A **reflexive** relation, however, **must** contain the n elements (a, a) for every $a \in A$.

Consequently, we can only choose among $n^2 - n = n(n - 1)$ elements to generate reflexive relations, so there are $2^{n(n-1)}$ of them.

Equivalence Relations

Equivalence relations are used to relate objects that are similar in some way.

Definition: A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Two elements that are related by an equivalence relation R are called **equivalent**.

Equivalence Relations

Since R is **symmetric**, a is equivalent to b whenever b is equivalent to a.

Since R is **reflexive**, every element is equivalent to itself.

Since R is **transitive**, if a and b are equivalent and b and c are equivalent, then a and c are equivalent.

Obviously, these three properties are necessary for a reasonable definition of equivalence.

Equivalence Relations

Example: Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if l(a) = l(b), where l(x) is the length of the string x. Is R an equivalence relation?

Solution:

- R is reflexive, because l(a) = l(a) and therefore aRa for any string a.
- R is symmetric, because if l(a) = l(b) then l(b) = l(a), so if aRb then bRa.
- R is transitive, because if l(a) = l(b) and l(b) = l(c), then l(a) = l(c), so aRb and bRc implies aRc.

R is an equivalence relation.

Equivalence Classes

Definition: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class** of a.

The equivalence class of a with respect to R is denoted by $[a]_R$.

When only one relation is under consideration, we will delete the subscript R and write **[a]** for this equivalence class.

If $b \in [a]_R$, b is called a **representative** of this equivalence class.

Equivalence Classes

Example: In the previous example (strings of identical length), what is the equivalence class of the word mouse, denoted by [mouse] ?

Solution: [mouse] is the set of all English words containing five letters.

For example, ‘horse’ would be a representative of this equivalence class.

Equivalence Classes

Theorem: Let R be an equivalence relation on a set A. The following statements are equivalent:

- aRb
- $[a] = [b]$
- $[a] \cap [b] \neq \emptyset$

Definition: A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, $i \in I$, forms a partition of S if and only if

(i) $A_i \neq \emptyset$ for $i \in I$

- $A_i \cap A_j = \emptyset$, if $i \neq j$
- $\cup_{i \in I} A_i = S$

Equivalence Classes

Examples: Let S be the set {u, m, b, r, o, c, k, s}.
Do the following collections of sets partition S ?

{{m, o, c, k}, {r, u, b, s}}	yes.
{{c, o, m, b}, {u, s}, {r}}	no (k is missing).
{{b, r, o, c, k}, {m, u, s, t}}	no (t is not in S).
{{u, m, b, r, o, c, k, s}}	yes.
{{b, o, o, k}, {r, u, m}, {c, s}}	yes ({b,o,o,k} = {b,o,k}).
{{u, m, b}, {r, o, c, k, s}, ∅}	no (∅ not allowed).

Equivalence Classes

Theorem: Let R be an equivalence relation on a set S. Then the **equivalence classes** of R form a **partition** of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Equivalence Classes

Example: Let us assume that Frank, Suzanne and George live in Boston, Stephanie and Max live in Lübeck, and Jennifer lives in Sydney.

Let R be the **equivalence relation** {(a, b) | a and b live in the same city} on the set P = {Frank, Suzanne, George, Stephanie, Max, Jennifer}.

Then R = {(Frank, Frank), (Frank, Suzanne), (Frank, George), (Suzanne, Frank), (Suzanne, Suzanne), (Suzanne, George), (George, Frank), (George, Suzanne), (George, George), (Stephanie, Stephanie), (Stephanie, Max), (Max, Stephanie), (Max, Max), (Jennifer, Jennifer)}.

Equivalence Classes

Then the **equivalence classes** of R are:

{{Frank, Suzanne, George}, {Stephanie, Max}, {Jennifer}}.

This is a **partition** of P.

The equivalence classes of any equivalence relation R defined on a set S constitute a partition of S, because every element in S is assigned to **exactly one** of the equivalence classes.

Equivalence Classes

Another example: Let R be the relation $\{(a, b) \mid a \equiv b \pmod 3\}$ on the set of integers.

Is R an equivalence relation?

Yes, R is reflexive, symmetric, and transitive.

What are the equivalence classes of R ?

$\{\{\ldots, -6, -3, 0, 3, 6, \ldots\},$
$\{\ldots, -5, -2, 1, 4, 7, \ldots\},$
$\{\ldots, -4, -1, 2, 5, 8, \ldots\}\}$