Basic Counting Principles

Counting problems are of the following kind:

"**How many** different 8-letter passwords are there?"

"**How many** possible ways are there to pick 11 soccer players out of a 20-player team?"

Most importantly, counting is the basis for computing **probabilities of discrete events**.

("What is the probability of winning the lottery?")

Basic Counting Principles

The sum rule:
If a task can be done in n_1 ways and a second task in n_2 ways, and if these two tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to do either task.

Example:
The department will award a free computer to either a CS student or a CS professor.
How many different choices are there, if there are 530 students and 15 professors?

There are 530 + 15 = 545 choices.

Basic Counting Principles

Generalized sum rule:

If we have tasks T_1, T_2, …, T_m that can be done in n_1, n_2, …, n_m ways, respectively, and no two of these tasks can be done at the same time, then there are $n_1 + n_2 + \ldots + n_m$ ways to do one of these tasks.

Basic Counting Principles

The product rule:

Suppose that a procedure can be broken down into two successive tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $n_1 n_2$ ways to do the procedure.

Basic Counting Principles

Example:

How many different license plates are there that containing exactly three English letters ?

Solution:

There are 26 possibilities to pick the first letter, then 26 possibilities for the second one, and 26 for the last one.

So there are $26 \cdot 26 \cdot 26 = 17576$ different license plates.

Basic Counting Principles

Generalized product rule:

If we have a procedure consisting of sequential tasks $T_1, T_2, \ldots, T_m$ that can be done in $n_1, n_2, \ldots, n_m$ ways, respectively, then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_m$ ways to carry out the procedure.

Basic Counting Principles

The sum and product rules can also be phrased in terms of **set theory**.

Sum rule: Let $A_1, A_2, \ldots, A_m$ be disjoint sets. Then the number of ways to choose any element from one of these sets is $|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|$.

Product rule: Let $A_1, A_2, \ldots, A_m$ be finite sets. Then the number of ways to choose one element from each set in the order $A_1, A_2, \ldots, A_m$ is $|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_m|$.

The Pigeonhole Principle

The pigeonhole principle: If (k + 1) or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Example 1: If there are 11 players in a soccer team that wins 12-0, there must be at least one player in the team who scored at least twice.

Example 2: If you have 6 classes from Monday to Friday, there must be at least one day on which you have at least two classes.

The Pigeonhole Principle

The generalized pigeonhole principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ of the objects.

Example 1: In our 60-student class, at least 12 students will get the same letter grade (A, B, C, D, or F).

Permutations and Combinations

A **permutation** of a set of distinct objects is an ordered arrangement of these objects.

An ordered arrangement of r elements of a set is called an **r-permutation**.

Permutations and Combinations

Example: Let $S = \{1, 2, 3\}$.
The arrangement 3, 1, 2 is a permutation of S.
The arrangement 3, 2 is a 2-permutation of S.

The number of r-permutations of a set with n distinct elements is denoted by **$P(n, r)$**.

We can calculate $P(n, r)$ with the product rule:

$P(n, r) = n\cdot(n - 1)\cdot(n - 2) \cdot\ldots\cdot(n - r + 1)$.

(n choices for the first element, (n – 1) for the second one, (n – 2) for the third one…)

Permutations and Combinations

Example:

$P(8, 3) = 8 \cdot 7 \cdot 6 = 336$
$= (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)/(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$

General formula:

$P(n, r) = n!/(n - r)!$

Permutations and Combinations

An **r-combination** of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

Example: Let $S = \{1, 2, 3, 4\}$.
Then $\{1, 3, 4\}$ is a 3-combination from S.

The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$.

Example: $C(4, 2) = 6$, since, for example, the 2-combinations of a set $\{1, 2, 3, 4\}$ are $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$.

Permutations and Combinations

How can we calculate $C(n, r)$?

Consider that we can obtain the r-permutation of a set in the following way:

First, we form all the r-combinations of the set (there are $C(n, r)$ such r-combinations).

Then, we generate all possible orderings in each of these r-combinations (there are $P(r, r)$ such orderings in each case).

Therefore, we have:

$$P(n, r) = C(n, r) \cdot P(r, r)$$

Permutations and Combinations

$C(n, r) = P(n, r)/P(r, r)$
$= n!/(n - r)!/(r!/(r - r)!)$
$= n!/(r!(n - r)!)$

Question

How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?

$C(6, 3) = 6!/(3!\cdot 3!) = 720/(6\cdot 6) = 720/36 = 20$

There are 20 different ways, that is, 20 different groups to be picked.

Permutations and Combinations

Corollary:
Let n and r be nonnegative integers with $r \leq n$.
Then $C(n, r) = C(n, n - r)$.

Note that "**picking a group of r people from a group of n people**" is the same as "**splitting a group of n people into a group of r people and another group of (n – r) people**".

Permutations and Combinations

Example:

A soccer club has 8 female and 7 male members. For today's match, the coach wants to have 6 female and 5 male players on the grass. How many possible configurations are there?

$$C(8, 6) \cdot C(7, 5) = 8!/(6!\cdot 2!) \cdot 7!/(5!\cdot 2!)$$
$$= 28\cdot 21$$
$$= 588$$

Combinations

We also saw the following:

$$C(n, n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} = C(n,r)$$

This symmetry is intuitively plausible. For example, let us consider a set containing six elements (n = 6).

Picking two elements and **leaving four** is essentially the same as **picking four** elements and **leaving two**.

In either case, our number of choices is the number of possibilities to **divide** the set into one set containing two elements and another set containing four elements.

Combinations

Pascal's Identity:

Let n and k be positive integers with $n \geq k$.
Then $C(n + 1, k) = C(n, k - 1) + C(n, k)$.

How can this be explained?

What is it good for?

Combinations

Imagine a set S containing n elements and a set T containing (n + 1) elements, namely all elements in S plus a new element **a**.

Calculating C(n + 1, k) is equivalent to answering the question: How many subsets of T containing k items are there?

Case I: The subset contains (k – 1) elements of S plus the element **a**: C(n, k – 1) choices.

Case II: The subset contains k elements of S and does not contain **a**: C(n, k) choices.

Sum Rule: C(n + 1, k) = C(n, k – 1) + C(n, k).

Pascal's Triangle

In Pascal's triangle, each number is the sum of the numbers to its upper left and upper right:

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
…		…		…		…		…		…

Pascal's Triangle

Since we have $C(n + 1, k) = C(n, k - 1) + C(n, k)$ and $C(0, 0) = 1$, we can use Pascal's triangle to simplify the computation of $C(n, k)$:

k

n

$C(0, 0) = \mathbf{1}$

$C(1, 0) = \mathbf{1}$ $C(1, 1) = \mathbf{1}$

$C(2, 0) = \mathbf{1}$ $C(2, 1) = \mathbf{2}$ $C(2, 2) = \mathbf{1}$

$C(3, 0) = \mathbf{1}$ $C(3, 1) = \mathbf{3}$ $C(3, 2) = \mathbf{3}$ $C(3, 3) = \mathbf{1}$

$C(4, 0) = \mathbf{1}$ $C(4, 1) = \mathbf{4}$ $C(4, 2) = \mathbf{6}$ $C(4, 3) = \mathbf{4}$ $C(4, 4) = \mathbf{1}$

Binomial Coefficients

Expressions of the form $C(n, k)$ are also called **binomial coefficients**.

A **binomial expression** is the sum of two terms, such as $(a + b)$.

Now consider $(a + b)^2 = (a + b)(a + b)$.

When expanding such expressions, we have to form all possible products of a term in the first factor and a term in the second factor:

$(a + b)^2 = a \cdot a + a \cdot b + b \cdot a + b \cdot b$

Then we can sum identical terms:

$(a + b)^2 = a^2 + 2ab + b^2$

Binomial Coefficients

For $(a + b)^3 = (a + b)(a + b)(a + b)$ we have

$(a + b)^3 = aaa + aab + aba + abb + baa + bab + bba + bbb$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

There is only one term a^3, because there is only one possibility to form it: Choose **a** from all three factors: $C(3, 3) = 1$.

There is three times the term a^2b, because there are three possibilities to choose **a** from two out of the three factors: $C(3, 2) = 3$.

Similarly, there is three times the term ab^2 ($C(3, 1) = 3$) and once the term b^3 ($C(3, 0) = 1$).

Binomial Coefficients

This leads us to the following formula:

$$(a+b)^n = \sum_{j=0}^{n} C(n, j) \cdot a^{n-j} b^j \qquad \text{(Binomial Theorem)}$$

With the help of Pascal's triangle, this formula can considerably simplify the process of expanding powers of binomial expressions.

For example, the fifth row of Pascal's triangle (1 – 4 – 6 – 4 – 1) helps us to compute $(a + b)^4$:

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$