



College of Science and technology

Department of Computer Science

Academic year: 2014-2015

**Module title: DISCRETE STRUCTURES
FOR COMPUTING**

Module code: COE5121

Lecturer: Dr Gahamanyi Marcel

Room: B 260 Huye Campus

Phone: 0784632389 or 0788530566

E-mail: m.gahamanyi@ur.ac.rw

COURSE OUTLINE

Title: DISCRETE STRUCTURES FOR COMPUTING

MODULE CODE: COE5121

Description of aims and content

This course provides a basic understanding of discrete mathematical topics that are fundamental for academic work in computer science. Students will develop algorithms and prove their efficacy. Topics include basic logic, basic proof techniques, set, function, relation, basic proof techniques, basics of counting and recursion .

Learning Outcomes:

1.1. LEARNING OUTCOMES :

1.1.1. KNOWLEDGE & UNDERSTANDING :

Having successfully completed the module, students should be able to demonstrate knowledge and understanding of:

A1 – The application of mathematics in computer Software and Hardware

A2 – Basic concepts and theory of computing

1.1.2. COGNITIVE/ INTELLECTUAL SKILLS/ APPLICATION OF KNOWLEDGE :

Having successfully completed the module, students should be able to:

B1 – Select appropriate structures in solving computation

B2 – Apply selective mathematical techniques relevant to discrete structures

1.1.3. GENERAL TRANSFERABLE SKILLS :

Having successfully completed the module, students should be able to:

– Demonstrate numerical techniques involving discrete structures

INDICATIVE CONTENT

Unit 1: History and overview

1.1 Knowledge themes include sets, logic, functions, and graphs

1.2 Contributors to the subject

1.3 Purpose and role of discrete structures in computer engineering

1.4 Contrasts between discrete-time models vs. continuous-time models

Unit 2: Basic logic

2.1 Propositional logic

2.2 Logical connectives

2.3 Truth tables

2.4 Use of logic to illustrate connectives

2.5 Normal forms (conjunctive and disjunctive)

2.6 Predicate logic

2.7 Universal and existential quantification

2.8 Limitations of predicate logic

2.9 Boolean algebra

2.10 Applications of logic to computer engineering

Unit 3: Sets, Functions and Relations

3.1 Sets (Venn diagrams, complements, Cartesian products, power sets)

3.2 Functions (one-to-one, onto, inverses, composition)

3.2 Relations (reflexivity, symmetry, transitivity, equivalence relations)

3.3 Discrete versus continuous functions and relations

3.5 Cardinality and countability

Unit 4: Proof techniques

4.1 Notions of implication, converse, inverse, negation, and contradiction

4.2 The structure of formal proofs

4.3 Direct proofs

4.4 Proof by counterexample, contraposition, and contradiction

4.5 Mathematical induction and strong induction

Unit 5: Basics of counting

5.1 Permutations and combinations

5.2 Counting arguments rule of products, rule of sums

5.3 The pigeonhole principle

5.4 Generating functions

5.6 Applications to computer engineering

Unit 6: Recursion

6.1 Recursive mathematical definitions

6.2 Developing recursive equations

6.3 Solving recursive equations

6.7 Applications of recursion to computer engineering

REFERENCES

Rosen, Kenneth H. : Discrete mathematics and its applications — 7th edition

ASSESSMENT:

Assignments (Two): 20%

Quiz (Two): 10%

Examinations: CATS (Two):	20%
FINAL	50%

UNIT1: INTRODUCTION

Meaning of the word “**structures**”: It means a complex composition of knowledge as *elements* and *their combinations*. Discrete Structures are the abstract mathematical structures used to represent *discrete objects* and relationships between these objects. These discrete structures include logic, sets, permutations, relations, graphs, trees, and finite-state machines.

Discrete structures are those complex compositions of knowledge that are taken from the field of discrete mathematics. Discrete mathematics is mathematics that deals with discrete objects. Discrete objects are those which are separated from (not connected to/distinct from) each other. Integers, automobiles, houses, people etc. are all discrete objects.

On the other hand, real numbers which include irrational as well as rational numbers are not discrete. As you know between any two different real numbers there is another real number different from either of them. So they are packed without any gaps and cannot be separated from their immediate neighbors. In that sense they are not discrete. In this course we will be concerned with objects such as integers, propositions, sets, relations and functions, which are all discrete. We are going to learn concepts associated with them, their properties, and relationships among them among others.

Discrete mathematics is about the mathematics of integers and of collections of objects. It underlies the operation of digital computers, and is used widely in all fields of computer science for reasoning about data structures, algorithms and complexity. Topics covered in the module include logic, proof techniques and sets, functions, relations, summations and recurrences, counting techniques and recursion.

Note: Discrete structures is the term used for discrete mathematics for computer science whereas Discrete mathematics is often referred to as finite mathematics.

Example-Exercise

State which of the following represent discrete data and which represent continuous data.

- (a) Numbers of shares sold each day in the stock market. **Ans. D**
- (b) Temperature recorded every half hour at a weather bureau. **Ans. C**
- (c) Lifetimes of television tubes produced by a company **Ans. C**
- (d) Yearly incomes of college professors. **Ans. D**
- (e) Lengths of 1000 bolts produced in a factory. **Ans. C**
- (f) Number of millimetres of rainfall in a city during various months of the year. **Ans. C**
- (g) Speed of an automobile in kilometres per hour. **Ans. C**
- (h) Number of 5000 Rwandan francs notes circulating in Rwanda at any time. **Ans. D**
- (i) Student enrolment in a university over a number of years. **Ans. D**
- (j) The exam results in(marks) in a given module. **Ans. C**

UNIT 2: BASIC LOGIC

It (logic) is covered in Chapter 1 of the reference textbook. It (logic) is a language that captures the essence of our reasoning, and correct reasoning must follow the rules of this language. It allows us to reason with statements involving variables among others.

2.1 Propositional logic

2.1.1 Definition

A **proposition** is declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both. Propositional logic is a logic at the sentential level. Thus sentences considered in this logic are not arbitrary sentences rather are the ones that are true or false. This kind of sentences are called **propositions**.

If a proposition is true, then we say it has a **truth value** of "**true**"; if a proposition is false, its truth value is "**false**".

Example1

"Grass is green", and " $2 + 5 = 5$ " are propositions.

The first proposition has the truth value of "**true**" and the second "**false**".

Example2

Consider the following sentences.

1. What time is it?
2. Read this carefully.
3. $x+1=2$.
4. $x+y = z$.

Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables.

We use letters to **denote propositional variables** (or statement variables), that is, variables that represent propositions, just as letters are used to denote numerical variables. The conventional letters used for propositional variables are **p,q,r,s,...**. The truth value of a proposition is true, denoted by **T**, if it is a true proposition, and the truth value of a proposition is false, denoted by **F**, if it is a false proposition.

The area of logic that deals with propositions is called the **propositional calculus** or **propositional logic**.

We now turn our attention to methods for producing new propositions from those that we already have. Many mathematical statements are constructed by combining one or more propositions. New propositions, called **compound propositions**, are formed from existing propositions using logical operators.

2.1.2 Elements of Propositional Logic

Simple sentences which are true or false are basic propositions. Larger and more complex sentences are constructed from basic propositions by combining them with **connectives**. Thus **propositions** and **connectives** are the basic elements of propositional logic. Though there are many connectives, we are going to use the following **five basic connectives** here:

NOT(negation) is denoted by the symbol \neg

AND (conjunction) is denoted by the symbol \wedge

OR (disjunction) is denoted by the symbol \vee

IF_THEN (or IMPLY) (conditional statement or implication) is denoted by the symbol \rightarrow

IF_AND_ONLY_IF(biconditional statement) and is denoted by the symbol \leftrightarrow

1.1.3 Truth Table

Often we want to discuss properties/relations common to all propositions. In such a case rather than stating them for each individual proposition we use variables representing an arbitrary proposition and state properties/relations in terms of those variables. Those variables are called a **propositional variable**. **Propositional variables are also considered a proposition and called a proposition** since they represent a proposition hence they behave the same way as propositions.

A proposition in general contains a number of variables. For example $(p \vee q)$ contains variables **p** and **q** each of which represents an arbitrary proposition. Thus a proposition takes different values depending on the values of the constituent variables. This relationship of the value of a proposition and those of its constituent variables can be represented by a table. It tabulates the value of a proposition for all possible values of its variables and it is called a **truth table**.

(a) Truth Table of Negation of a proposition p

p	$\neg p$
T	F
F	T

This table shows that if **p** is true, then $(\neg p)$ is false, and that if **p** is false, then $(\neg p)$ is true.

(b) Truth Table of Conjunction of propositions p and q

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

This table shows that $(p \wedge q)$ is true if both **p** and **q** are true, and that it is false in any other case. Similarly for the rest of the tables.

(c) Truth Table of Disjunction of propositions p and q

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

This table shows that $(p \vee q)$ is false if both **p** and **q** are false and is true otherwise.

(d) Implication (conditional statement) of propositions p and q

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The conditional statement $p \rightarrow q$ is the proposition “if **p**, then **q**.” The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \rightarrow q$, **p** is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence). When $p \rightarrow q$ is always true, we express that by $p \Rightarrow q$. That is $p \Rightarrow q$ is used when proposition **p** always implies proposition **q** regardless of the value of the variables in them.

(e) Biconditional statement of propositions p and q

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

The biconditional statement $p \leftrightarrow q$ is the proposition “**p if and only if q**.” The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truthvalues, and is false otherwise. Biconditional statements are also called bi-implications.

When $p \leftrightarrow q$ is always true, we express that by $p \Leftrightarrow q$. That is \Leftrightarrow is used when two propositions always take the same value regardless of the value of the variables in them.

1.1.4 Construction of Complex Propositions

(a) Converse and Contrapositive

For the proposition $p \rightarrow q$, the proposition $q \rightarrow p$ is called its **converse**, and the proposition $\neg q \rightarrow \neg p$ is called its **contrapositive**.

For example for the proposition "If it rains, then I get wet",

Converse: If I get wet, then it rains.

Contrapositive: If I don't get wet, then it does not rain.

The converse of a proposition is not necessarily logically equivalent to it, that is they may or may not take the same truth value at the same time.

On the other hand, *the contrapositive of a proposition is always logically equivalent to the proposition*. That is, they take the same truth value regardless of the values of their constituent variables. Therefore, "If it rains, then I get wet." and "If I don't get wet, then it does not rain." are logically equivalent. If one is true then the other is also true, and vice versa.

Exercises

1. Which of the following sentences are propositions? What are the truth values of those that are propositions?

- a. Kigali is the capital of Rwanda.
- b. $2 + 3 = 7$.
- c. Open the door.
- d. $5 + 7 < 10$.
- e. The moon is a satellite of the earth.
- f. $x + 5 = 7$.
- g. $x + 5 > 9$ for every real number x .

Solution

- a. Yes, True
- b. Yes, False
- c. No
- d. Yes, False
- e. Yes, True
- f. No
- g. Yes, False

2. What is the negation of each of the following propositions?

- a. Kigali is the capital of Rwanda.
- b. Food is not expensive in Huye.
- c. $3 + 5 = 7$.
- d. The summer in France is hot and sunny.

Solution

- a. Kigali is not the capital of Rwanda.
- b. Food is expensive in Huye.
- c. $3 + 5 \neq 7$.
- d. The summer in France is not hot or (the summer in France is) not sunny.

3. Let p and q be the propositions

p : Your car is out of gas.

q : You can't drive your car.

Write the following propositions using p and q and logical connectives.

- a) Your car is not out of gas.
- b) You can't drive your car if it is out of gas.
- c) Your car is not out of gas if you can drive it.
- d) If you can't drive your car then it is out of gas.

Solution

- a) $\neg p$
- b) $p \rightarrow q$
- c) $\neg q \rightarrow \neg p$
- d) $q \rightarrow p$

4. Determine whether each of the following implications is true or false.

- a) If 0.5 is an integer, then $1 + 0.5 = 3$.
- b) If cars can fly, then $1 + 1 = 3$.
- c) If $5 > 2$ then pigs can fly.
- d) If $3 \cdot 5 = 15$ then $1 + 2 = 3$.

Solution

- a) True (because the condition is false)
- b) True (because the condition is false)
- c) False
- d) True

5. State the converse and contrapositive of each of the following implications.

- a. If it snows today, I will stay home.
- b. We play the game if it is sunny.
- c. If a positive integer is a prime then it has no divisors other than 1 and itself.

Solution

- a. Converse: "If I stay home, then it snows today. "
Contrapositive: "If I do not stay home, then it does not snow today. quot;
- b. Converse: "If we play the game, then it is sunny."
Contrapositive: "If we don't play the game, then it is not sunny."
- c. Converse: "If a positive integer has no divisors other than 1 and itself then it is a prime."
Contrapositive: "If a positive integer has a divisors other than 1 and itself then it is not a prime."

6. Construct a truth table for each of the following compound propositions.

- a) $p \wedge \neg p$
- b) $(p \vee \neg q) \rightarrow q$
- c) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$

Solutions

a)

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

b)

p	q	$\neg q$	$p \vee \neg q$	$(p \vee \neg q) \rightarrow q$
T	T	F	T	T
T	F	T	T	F
F		F	F	
F	F	T	T	F

p	q	$p \rightarrow q$	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

1.1.4 Use of logic to illustrate connectives

1. If_Then Variations

- different ways of saying if_then: only if, necessary, sufficient

If-then statements appear in various forms in practice. The following list presents some of the variations. **These are all logically equivalent**, that is as far as true or false of statement is concerned there is no difference between them. Thus if one is true then all the others are also true, and if one is false all the others are false.

- If p , then q .
- p implies q .
- If p , q .
- p only if q .
- p is sufficient for q .
- q if p .
- q whenever p .
- q is necessary for p .
- It is necessary for p that q .

For instance, instead of saying "**If** she smiles **then** she is happy", we can say "If she smiles, she is happy", "She is happy whenever she smiles", "She smiles only if she is happy" etc. without changing their truth values.

"Only if" can be translated as "then". For example, "She smiles only if she is happy" is equivalent to "If she smiles, then she is happy".

Note that "She smiles only if she is happy" means "If she is not happy, she does not smile", which is the contrapositive of "If she smiles, she is happy".

You can also look at it this way: "She smiles only if she is happy" means "She smiles only when she is happy". So any time you see her smile you know she is happy. Hence "If she smiles, then she is happy". Thus they are logically equivalent.

Also **"If she smiles, she is happy" is equivalent to "It is necessary for her to smile that she is happy"**. For "If she smiles, she is happy" means "If she smiles, she is *always* happy". That is, she never fails to be happy when she smiles. "Being happy" is inevitable consequence/**necessity** of "smile". Thus if "being happy" is missing, then "smile" can not be there either. "Being happy" is **necessary** "for her to smile" or equivalently "It is **necessary** for her to smile that she is happy".

2.From English to Proposition

- translation of English sentences to propositions

As we are going to see in the next section, reasoning is done on propositions using inference rules. For example, if the two propositions "if it snows, then the school is closed", and "it snows" are true, then we can conclude that "the school is closed" is true. In everyday life, that is how we reason.

To check the correctness of reasoning, we must check whether or not rules of inference have been followed to draw the conclusion from the premises. However, for reasoning in English or in general for reasoning in a natural language, that is not necessarily straightforward and it often encounters some difficulties. Firstly, connectives are not necessarily easily identified as we can get a flavor of that from the previous topic on variations of if_then statements. Secondly, if the argument becomes complicated involving many statements in a number of different forms twisted and tangled up, it can easily get out of hand unless it is simplified in some way.

One solution for that is to use symbols (and mechanize it). Each sentence is represented by symbols representing building block sentences, and connectives. For example, if P represents "it snows" and Q represents "the school is closed", then the previous argument can be expressed as

$$[[P \rightarrow Q] \wedge P] \rightarrow Q,$$

This representation is concise, much simpler and much easier to deal with. In addition today there are a number of automatic reasoning systems and we can verify our arguments in symbolic form using them.

To convert English statements into a symbolic form, we restate the given statements using the building block sentences, those for which symbols are given, and the connectives of propositional logic (not, and, or, if_then, if_and_only_if), and then substitute the symbols for the building blocks and the connectives.

For example, let P be the proposition "It is snowing", Q be the proposition "I will go the beach", and R be the proposition "I have time".

Then first "I will go to the beach if it is not snowing" is restated as "If it is not snowing, I will go to the beach". Then symbols P and Q are substituted for the respective sentences to obtain $\neg P \rightarrow Q$.

Similarly, "It is not snowing and I have time only if I will go to the beach" is restated as "If it is not snowing and I have time, then I will go to the beach", and it is translated as

$$(\neg P \wedge R) \rightarrow Q.$$

Exercise

1. Write each of the following statements in the form "if p , then q " in English. (Hint: Refer to the list of common ways to express implications listed in this section.)

- a. The newspaper will not come if there is an inch of snow on the street.
- b. It snows whenever the wind blows from the northeast.
- c. That prices go up implies that supply will be plentiful.
- d. It is necessary to read the textbook to understand the materials of this course.
- e. For a number to be divisible by 3, it is sufficient that it is the sum of three consecutive integers.
- f. Your guarantee is good only if you bought your TV less than 90 days ago.

Solution

- a. If there is an inch of snow on the street, the newspaper will not come.
- b. If the wind blows from the northeast, then it snows.
- c. If prices go up, then supply will be plentiful.
- d. If you are going to understand the materials of this course, you must read the textbook.
- e. If a number is the sum of three consecutive integers, then it is divisible by 3.
- f. If your guarantee is good, then you must have bought your TV less than 90 days ago.

2. Write each of the following propositions in the form " p if and only if q " in English.

- a. If it is hot outside you drink a lot of water, and if you drink a lot of water it is hot outside.
- b. For a program to be readable it is necessary and sufficient that it is well structured.
- c. I like fruits only if they are fresh, and fruits are fresh only if I like them.
- d. If you eat too much sweets your teeth will decay, and conversely.

- e. The store is closed on exactly those days when I want to shop there.

Solution

- You drink a lot of water if and only if it is hot outside.
- A program is readable if and only if it is well structured.
- I like fruits if and only if they are fresh.
- Your teeth will decay if and only if you eat too much sweets.
- The store is closed if and only if it is the day when I want to shop there.

1.1.5 Introduction to Reasoning

- tautology
- contradiction
- contingency

Logical reasoning is the process of drawing conclusions from premises using rules of inference. Here we are going to study reasoning with propositions. Later we are going to see reasoning with predicate logic, which allows us to reason about individual objects. However, *inference rules of propositional logic are also applicable to predicate logic* and reasoning with propositions is fundamental to reasoning with predicate logic.

These inference rules are results of observations of human reasoning over centuries. Though there is nothing absolute about them, they have contributed significantly in the scientific and engineering progress the mankind have made. Today they are universally accepted as the rules of logical reasoning and they should be followed in our reasoning.

Since inference rules are based on identities and implications, we are going to study them first. We start with three types of proposition which are used to define the meaning of "identity" and "implication".

Types of Proposition

Some propositions are always true regardless of the truth value of its component propositions.

For example $(p \vee \neg p)$ is always true regardless of the value of the proposition p .

a proposition that is always true called a **tautology**.

there are also propositions that are always false such as $(p \wedge \neg p)$.

such a proposition is called a **contradiction**.

a proposition that is neither a tautology nor a contradiction is called a **contingency**.

for example $(p \vee q)$ is a contingency.

These types of propositions play a **crucial role in reasoning**. In particular every inference rule is a tautology as we see in *identities* and *implications*.

Identities

- Identities (tautologies) of propositional logic
- Dual of proposition

From the definitions(meaning) of connectives, a number of relations between propositions which are useful in reasoning can be derived. Below some of the often encountered pairs of **logically equivalent** propositions, also called *identities*, are listed.

These identities are used in logical reasoning. In fact we use them in our daily life, often more than one at a time, without realizing it.

If two propositions are logically equivalent, one can be substituted for the other in any proposition in which they occur without changing the logical value of the proposition.

Below \Leftrightarrow corresponds to \leftrightarrow and it means that the equivalence is always true (a tautology), while \leftrightarrow means the equivalence may be false in some cases, that is in general a contingency.

That these equivalences hold can be verified by constructing truth tables for them. First the identities are listed, then examples are given to illustrate them.

list of identities:

1. $p \Leftrightarrow (p \vee p)$ ----- idempotence of \vee
2. $p \Leftrightarrow (p \wedge p)$ ----- idempotence of \wedge
3. $(p \vee q) \Leftrightarrow (q \vee p)$ ----- commutativity of \vee
4. $(p \wedge q) \Leftrightarrow (q \wedge p)$ ----- commutativity of \wedge
5. $[(p \vee q) \vee r] \Leftrightarrow [p \vee (q \vee r)]$ ----- associativity of \vee
6. $[(p \wedge q) \wedge r] \Leftrightarrow [p \wedge (q \wedge r)]$ ----- associativity of \wedge
7. $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$ ----- de Morgan's law
8. $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$ ----- de Morgan's law
9. $[p \wedge (q \vee r)] \Leftrightarrow [(p \wedge q) \vee (p \wedge r)]$ ----- distributivity of \wedge over \vee
10. $[p \vee (q \wedge r)] \Leftrightarrow [(p \vee q) \wedge (p \vee r)]$ ----- distributivity of \vee over \wedge
11. $(p \vee \text{true}) \Leftrightarrow \text{true}$
12. $(p \wedge \text{false}) \Leftrightarrow \text{false}$
13. $(p \vee \text{false}) \Leftrightarrow p$
14. $(p \wedge \text{true}) \Leftrightarrow p$
15. $(p \vee \neg p) \Leftrightarrow \text{true}$
16. $(p \wedge \neg p) \Leftrightarrow \text{false}$
17. $p \Leftrightarrow \neg(\neg p)$ ----- double negation
18. $(p \rightarrow q) \Leftrightarrow (\neg p \vee q)$ ----- implication
19. $(p \leftrightarrow q) \Leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow p)]$ ----- equivalence
20. $[(p \wedge q) \rightarrow r] \Leftrightarrow [p \rightarrow (q \rightarrow r)]$ ----- exportation
21. $[(p \rightarrow q) \wedge (p \rightarrow \neg q)] \Leftrightarrow \neg p$ ----- absurdity
22. $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$ ----- contrapositive

1. $p \Leftrightarrow (p \vee p)$ ----- idempotence of \vee

What this says is, for example, that "Tom is happy." is equivalent to "Tom is happy or Tom is

happy". This and the next identity are rarely used, if ever, in everyday life. However, these are useful when manipulating propositions in reasoning in symbolic form.

2. $p \Leftrightarrow (p \wedge p)$ ----- idempotence of \wedge

Similar to 1. above.

3. $(p \vee q) \Leftrightarrow (q \vee p)$ ----- commutativity of \vee

What this says is, for example, that "Tom is rich or (Tom is) famous." is equivalent to "Tom is famous or (Tom is) rich".

4. $(p \wedge q) \Leftrightarrow (q \wedge p)$ ----- commutativity of \wedge

What this says is, for example, that "Tom is rich and (Tom is) famous." is equivalent to "Tom is famous and (Tom is) rich".

5. $[(p \vee q) \vee r] \Leftrightarrow [p \vee (q \vee r)]$ ----- associativity of \vee

What this says is, for example, that "Tom is rich or (Tom is) famous, or he is also happy." is equivalent to "Tom is rich, or he is also famous or (he is) happy".

6. $[(p \wedge q) \wedge r] \Leftrightarrow [p \wedge (q \wedge r)]$ ----- associativity of \wedge

Similar to 5. above.

7. $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$ ----- DeMorgan's Law

For example, "It is not the case that Tom is rich or famous." is true if and only if "Tom is not rich and he is not famous."

8. $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$ ----- DeMorgan's Law

For example, "It is not the case that Tom is rich and famous." is true if and only if "Tom is not rich or he is not famous."

9. $[p \wedge (q \vee r)] \Leftrightarrow [(p \wedge q) \vee (p \wedge r)]$ ----- distributivity of \wedge over \vee

What this says is, for example, that "Tom is rich, and he is famous or (he is) happy." is equivalent to "Tom is rich and (he is) famous, or Tom is rich and (he is) happy".

10. $[p \vee (q \wedge r)] \Leftrightarrow [(p \vee q) \wedge (p \vee r)]$ ----- distributivity of \vee over \wedge

Similarly to 9. above, what this says is, for example, that "Tom is rich, or he is famous and (he is) happy." is equivalent to "Tom is rich or (he is) famous, and Tom is rich or (he is) happy".

11. $(p \vee \text{true}) \Leftrightarrow \text{True}$. Here True is a proposition that is always true. Thus the proposition $(p \vee \text{True})$ is always true regardless of what P is.

This and the next three identities, like identities 1 and 2, are rarely used, if ever, in everyday life.

However, these are useful when manipulating propositions in reasoning in symbolic form.

$$12. (p \wedge \text{False}) \Leftrightarrow \text{False}$$

$$13. (p \vee \text{False}) \Leftrightarrow p$$

$$14. (p \wedge \text{True}) \Leftrightarrow p$$

$$15. (p \vee \neg p) \Leftrightarrow \text{True}$$

What this says is that a statement such as "Tom is 6 foot tall or he is not 6 foot tall." is always true.

$$16. (p \wedge \neg p) \Leftrightarrow \text{False}$$

What this says is that a statement such as "Tom is 6 foot tall and he is not 6 foot tall." is always false.

$$17. p \Leftrightarrow \neg(\neg p) \text{ ----- double negation}$$

What this says is, for example, that "It is not the case that Tom is not 6 foot tall." is equivalent to "Tom is 6 foot tall."

$$18. (p \rightarrow q) \Leftrightarrow (\neg p \vee q) \text{ ----- implication}$$

For example, the statement "If I win the lottery, I will give you a million dollars." is not true, that is, I am lying, if I win the lottery and don't give you a million dollars. It is true in all the other cases. Similarly, the statement "I don't win the lottery or I give you a million dollars." is false, if I win the lottery and don't give you a million dollars. It is true in all the other cases. Thus these two statements are logically equivalent.

$$19. (p \leftrightarrow q) \Leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow p)] \text{ ----- equivalence}$$

What this says is, for example, that "Tom is happy if and only if he is healthy." is logically equivalent to "if Tom is happy then he is healthy, and if Tom is healthy he is happy."

$$20. [(p \wedge q) \rightarrow r] \Leftrightarrow [p \rightarrow (q \rightarrow r)] \text{ ----- exportation}$$

For example, "If Tom is healthy, then if he is rich, then he is happy." is logically equivalent to "If Tom is healthy and rich, then he is happy."

$$21. [(p \rightarrow q) \wedge (p \rightarrow \neg q)] \Leftrightarrow \neg p \text{ ----- absurdity}$$

For example, if "If Tom is guilty then he must have been in that room." and "If Tom is guilty then he could not have been in that room." are both true, then there must be something wrong about the assumption that Tom is guilty.

22. $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$ ----- contrapositive

For example, "If Tom is healthy, then he is happy." is logically equivalent to "If Tom is not happy, he is not healthy."

The identities 1 ~ 16 listed above can be paired by duality relation, which is defined below, as 1 and 2, 3 and 4, ..., 15 and 16. That is 1 and 2 are dual to each other, 3 and 4 are dual to each other, Thus if you know one of a pair, you can obtain the other of the pair by using the duality.

Dual of Proposition

Let X be a proposition involving only \neg , \wedge , and \vee as a connective. Let X^* be the proposition obtained from X by replacing \wedge with \vee , \vee with \wedge , T with F , and F with T . Then X^* is called the **dual** of X .

For example, the dual of $[p \wedge q] \vee p$ is $[p \vee q] \wedge p$, and the dual of $[\neg p \wedge q] \vee \neg [t \wedge \neg r]$ is $[\neg p \vee q] \wedge \neg [f \vee \neg r]$.

Property of Dual: If two propositions p and q involving only \neg , \wedge , and \vee as connectives are equivalent, then their duals p^* and q^* are also equivalent.

Examples of Use of Identities

Here a few examples are presented to show how the identities in [Identities](#) can be used to prove some useful results.

$$1. \neg(p \rightarrow q) \Leftrightarrow (p \wedge \neg q)$$

What this means is that the **negation of "if p then q"** is "**p but not q**". For example, if you said to someone "If I win a lottery, I will give you \$100,000." and later that person says "You lied to me." Then what that person means is that you won the lottery but you did not give that person \$100,000 you promised.

To prove this, first let us get rid of \rightarrow using one of the identities: $(p \rightarrow q) \Leftrightarrow (\neg p \vee q)$.

That is, $\neg(p \rightarrow q) \Leftrightarrow \neg(\neg p \vee q)$.

Then by De Morgan, it is equivalent to $\neg\neg p \wedge \neg q$, which is equivalent to $P \wedge \neg Q$, since the double negation of a proposition is equivalent to the original proposition as seen in the identities.

$$2. p \vee (p \wedge q) \Leftrightarrow p \text{ --- Absorption}$$

What this tells us is that $p \vee (p \wedge q)$ can be simplified to p , or if necessary p can be expanded into $p \vee (p \wedge q)$.

To prove this, first note that $P \Leftrightarrow (P \wedge T)$.

Hence

$$P \vee (P \wedge Q)$$

$$\Leftrightarrow (P \wedge T) \vee (P \wedge Q)$$

$$\Leftrightarrow P \wedge (T \vee Q), \text{ by the distributive law.}$$

$$\Leftrightarrow (P \wedge T), \text{ since } (T \vee Q) \Leftrightarrow T.$$

$$\Leftrightarrow P, \text{ since } (P \wedge T) \Leftrightarrow P.$$

Note that by the duality

$$P \wedge (P \vee Q) \Leftrightarrow P \text{ also holds.}$$

Exercises

1. Use truth table to verify the following equivalences.

a) $p \wedge \text{False} \Leftrightarrow \text{False}$

b) $p \vee \text{True} \Leftrightarrow \text{True}$

c) $p \vee p \Leftrightarrow p$

Solution

p	$p \wedge \text{False}$	$p \vee \text{True}$	$p \vee p$
T	F	T	T
F	F	F	F

2. Use truth tables to verify the distributive law $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

3. Show that each of the following implications is a tautology without using truth tables.

a) $p \rightarrow (p \vee q)$

b) $(p \wedge q) \rightarrow (p \rightarrow q)$

c) $\neg(p \rightarrow q) \rightarrow \neg q$

Solution

a) If the hypothesis p is true, by the definition of disjunction, the conclusion $p \vee q$ is also true.

If \mathbf{p} is false on the other hand, then by the definition of implication $\mathbf{p} \rightarrow (\mathbf{p} \vee \mathbf{q})$ is true.

Alternatively, $\mathbf{p} \rightarrow (\mathbf{p} \vee \mathbf{q}) \Leftrightarrow (\neg \mathbf{p} \vee (\mathbf{p} \vee \mathbf{q})) \Leftrightarrow ((\neg \mathbf{p} \vee \mathbf{p}) \vee \mathbf{q}) \Leftrightarrow (\mathbf{T} \vee \mathbf{q}) \Leftrightarrow \mathbf{T}$

b) If the hypothesis $\mathbf{p} \wedge \mathbf{q}$ is true, then both \mathbf{p} and \mathbf{q} are true so that the conclusion $\mathbf{p} \rightarrow \mathbf{q}$ is also true. If the hypothesis is false, then "if-then" statement is always true.

This can also be proven similarly to the alternative proof for a).

c) If the hypothesis $\neg(\mathbf{p} \rightarrow \mathbf{q})$ is true, then $\mathbf{p} \rightarrow \mathbf{q}$ is false, so that \mathbf{p} is true and \mathbf{q} is false. Hence, the conclusion $\neg \mathbf{q}$ is true. If the hypothesis is false, then "if-then" statement is always true.

This can also be proven similarly to the alternative proof for a).

4. Verify the following equivalences, which are known as the absorption laws.

$$\text{a) } [\mathbf{p} \vee (\mathbf{p} \wedge \mathbf{q})] \Leftrightarrow \mathbf{p}$$

$$\text{b) } [\mathbf{p} \wedge (\mathbf{p} \vee \mathbf{q})] \Leftrightarrow \mathbf{p}$$

Solution

a) If \mathbf{p} is true, then $\mathbf{p} \vee (\mathbf{p} \wedge \mathbf{q})$ is true since the first proposition in the disjunction is true. On the other hand, if \mathbf{p} is false, then $\mathbf{p} \wedge \mathbf{q}$ is also false, so $\mathbf{p} \vee (\mathbf{p} \wedge \mathbf{q})$ is false. Since \mathbf{p} and $\mathbf{p} \vee (\mathbf{p} \wedge \mathbf{q})$ always have the same truth value, they are equivalent.

This can also be proven similarly to b).

$$\text{b) } [\mathbf{p} \wedge (\mathbf{p} \vee \mathbf{q})]$$

$$\Leftrightarrow [(\mathbf{p} \vee \mathbf{F}) \wedge (\mathbf{p} \vee \mathbf{q})]$$

$$\Leftrightarrow [(\mathbf{p} \vee (\mathbf{F} \wedge \mathbf{q}))] \Leftrightarrow [\mathbf{p} \vee \mathbf{F}] \Leftrightarrow \mathbf{p}$$

This can also be proven similarly to a).

5. Find the dual of each of the following propositions.

$$\text{a. } \mathbf{p} \vee \neg \mathbf{q} \vee \neg \mathbf{r}$$

$$\text{b. } (\mathbf{p} \vee \mathbf{q} \vee \mathbf{r}) \wedge \mathbf{s}$$

$$\text{c. } (\mathbf{p} \wedge \mathbf{F}) \vee (\mathbf{q} \wedge \mathbf{T})$$

Solution

$$\text{a) } (p \wedge \neg q \wedge \neg r)$$

$$\text{b) } (p \wedge q \wedge r) \vee s$$

$$\text{c) } (p \vee T) \wedge (q \vee F)$$

6. Find a compound proposition involving the propositions p, q , and r that is true when exactly one of p, q , and r is true and is false otherwise. (*Hint*: Form a disjunction of conjunctions. Include a conjunction for each combination of values for which the proposition is true. Each conjunction should include each of the three propositions or their negations.)

Solution

$$(p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$$

Implications

- implications (tautologies) of propositional logic

The following implications are some of the relationships between propositions that can be derived from the definitions (meaning) of connectives.

\Rightarrow below corresponds to \rightarrow and it means that the implication always holds. That is it is a tautology. These implications are used in logical reasoning. When the right hand side of these implications is substituted for the left hand side appearing in a proposition, the resulting proposition is implied by the original proposition, that is, one can deduce the new proposition from the original one.

List of Implications:

1. $p \Rightarrow (p \vee q)$ ----- addition
2. $(p \wedge q) \Rightarrow p$ ----- simplification
3. $[p \wedge (p \rightarrow q)] \Rightarrow q$ ----- modus ponens
4. $[(p \rightarrow q) \wedge \neg q] \Rightarrow \neg p$ ----- modus tollens
5. $[\neg p \wedge (p \vee q)] \Rightarrow q$ ----- disjunctive syllogism
6. $[(p \rightarrow q) \wedge (q \rightarrow r)] \Rightarrow (p \rightarrow r)$ ----- hypothetical syllogism
7. $(p \rightarrow q) \Rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)]$
8. $[(p \rightarrow q) \wedge (r \rightarrow s)] \Rightarrow [(p \wedge r) \rightarrow (q \wedge s)]$
9. $[(p \leftrightarrow q) \wedge (q \leftrightarrow r)] \Rightarrow (p \leftrightarrow r)$

Examples:

1. $P \Rightarrow (P \vee Q)$ ----- addition

For example, if the sun is shining, then certainly the sun is shining or it is snowing. Thus "if the sun is shining, then the sun is shining or it is snowing." "If $0 < 1$, then $0 \leq 1$ or a similar statement is also often seen.

2. $(P \wedge Q) \Rightarrow P$ ----- simplification

For example, if it is freezing and (it is) snowing, then certainly it is freezing. Thus "If it is freezing and (it is) snowing, then it is freezing."

3. $[P \wedge (P \rightarrow Q)] \Rightarrow Q$ ----- modus ponens

For example, if the statement "If it snows, the schools are closed" is true and it actually snows, then the schools are closed.

This implication is the basis of all reasoning. Theoretically, this is all that is necessary for reasoning.

4. $[(P \rightarrow Q) \wedge \neg Q] \Rightarrow \neg P$ ----- modus tollens

For example, if the statement "If it snows, the schools are closed" is true and the schools are not closed, then one can conclude that it is not snowing.

Note that this can also be looked at as the application of the contrapositive and modus ponens. That is, $(P \rightarrow Q)$ is equivalent to $(\neg Q) \rightarrow (\neg P)$. Thus if in addition $\neg Q$ holds, then by the modus ponens, $\neg P$ is concluded.

5. $[\neg P \wedge (P \vee Q)] \Rightarrow Q$ ----- disjunctive syllogism

For example, if the statement "It snows or (it) rains." is true and it does not snow, then one can conclude that it rains.

6. $[(P \rightarrow Q) \wedge (Q \rightarrow R)] \Rightarrow (P \rightarrow R)$ ----- hypothetical syllogism

For example, if the statements "If the streets are slippery, the school buses can not be operated." and "If the school buses can not be operated, the schools are closed." are true, then the statement "If the streets are slippery, the schools are closed." is also true.

7. $(P \rightarrow Q) \Rightarrow [(Q \rightarrow R) \rightarrow (P \rightarrow R)]$

This is actually the hypothetical syllogism in another form. For by considering $(P \rightarrow Q)$ as a proposition S, $(Q \rightarrow R)$ as a proposition T, and $(P \rightarrow R)$ as a proposition U in the hypothetical syllogism above, and then by applying the "exportation" from the identities, this is obtained.

8. $[(P \rightarrow Q) \wedge (R \rightarrow S)] \Rightarrow [(P \wedge R) \rightarrow (Q \wedge S)]$

For example, if the statements "If the wind blows hard, the beach erodes." and "If it rains heavily, the streets get flooded." are true, then the statement "If the wind blows hard and it rains

heavily, then the beach erodes and the streets get flooded." is also true.

$$9. [(P \leftrightarrow Q) \wedge (Q \leftrightarrow R)] \Rightarrow (P \leftrightarrow R)$$

This just says that the logical equivalence is transitive, that is, if P and Q are equivalent, and if Q and R are also equivalent, then P and R are equivalent.

Reasoning with Propositions

Logical reasoning is the process of drawing conclusions from premises using rules of inference. The basic inference rule is **modus ponens**. It states that if both $p \rightarrow q$ and p hold, then q can be concluded, and it is written as

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline q \end{array}$$

Here the lines above the dotted line are **premises** and the line below it is the **conclusion** drawn from the premises.

For example if "if it rains, then the game is not played" and "it rains" are both true, then we can conclude that the game is not played.

In addition to modus ponens, one can also reason by using [identities](#) and [implications](#).

If the left(right) hand side of an identity appearing in a proposition is replaced by the right(left) hand side of the identity, then the resulting proposition is logically equivalent to the original proposition. Thus the new proposition is deduced from the original proposition. For example in the proposition $p \wedge (q \rightarrow r)$, $(q \rightarrow r)$ can be replaced with $(\neg q \vee r)$ to conclude

$$p \wedge (\neg q \vee r), \text{ since } (q \rightarrow r) \Leftrightarrow (\neg q \vee r)$$

Similarly if the left(right) hand side of an implication appearing in a proposition is replaced by the right(left) hand side of the implication, then the resulting proposition is logically implied by the original proposition. Thus the new proposition is deduced from the original proposition.

The tautologies listed as "implications" can also be considered **inference rules** as shown below.

Rules of Inference	Tautological Form	Name
$\frac{P}{P \vee Q}$	$P \Rightarrow (P \vee Q)$	addition
$\frac{P \wedge Q}{P}$	$(P \wedge Q) \Rightarrow P$	simplification
$\frac{P \quad P \rightarrow Q}{Q}$	$[P \wedge (P \rightarrow Q)] \Rightarrow Q$	modus ponens
$\frac{\neg Q \quad P \rightarrow Q}{\neg P}$	$[\neg Q \wedge (P \rightarrow Q)] \Rightarrow \neg P$	modus tollens
$\frac{P \vee Q \quad \neg P}{Q}$	$[(P \vee Q) \wedge \neg P] \Rightarrow Q$	disjunctive syllogism
$\frac{P \rightarrow Q \quad Q \rightarrow R}{P \rightarrow R}$	$[(P \rightarrow Q) \wedge (Q \rightarrow R)] \Rightarrow [P \rightarrow R]$	hypothetical syllogism
$\frac{P \quad Q}{P \wedge Q}$		conjunction
$\frac{(P \rightarrow Q) \wedge (R \rightarrow S) \quad P \vee R}{Q \vee S}$	$[(P \rightarrow Q) \wedge (R \rightarrow S) \wedge (P \vee R)] \Rightarrow [Q \vee S]$	constructive dilemma

$$\begin{array}{lll}
 (P \rightarrow Q) \wedge (R \rightarrow S) & [(P \rightarrow Q) \wedge (R \rightarrow S) \wedge (\neg Q \vee \neg S)] \Rightarrow [\neg P \vee \neg R] & \text{destructive dilemma} \\
 \neg Q \vee \neg S & & \\
 \hline
 \neg P \vee \neg R & &
 \end{array}$$

Example of Inferencing

Consider the following argument:

1. Today is Tuesday or Wednesday.
2. But it can't be Wednesday, since the doctor's office is open today, and that office is always closed on Wednesdays.
3. Therefore today must be Tuesday.

This sequence of reasoning (inferencing) can be represented as a series of application of modus ponens to the corresponding propositions as follows.

The modus ponens is an inference rule which deduces Q from $P \rightarrow Q$ and P .

T : Today is Tuesday.

W : Today is Wednesday.

D : The doctor's office is open today.

C : The doctor's office is always closed on Wednesdays.

The above reasoning can be represented by propositions as follows.

1. $T \vee W$

2. D
 C

 $\sim W$

3. T

To see if this conclusion T is correct, let us first find the relationship among C , D , and W :

C can be expressed using D and W . That is, restate C first as the doctor's office is always closed if it is Wednesday. Then $C \leftrightarrow (W \rightarrow \sim D)$ Thus substituting $(W \rightarrow \sim D)$ for C , we can proceed as follows.

$$\begin{array}{l} D \\ W \rightarrow \sim D \\ \hline \sim W \end{array}$$

which is correct by modus tollens.

From this $\sim W$ combined with $T \vee W$ of 1. above,

$$\begin{array}{l} \sim W \\ T \vee W \\ \hline T \end{array}$$

which is correct by disjunctive syllogism.

Thus we can conclude that the given argument is correct.

To save space we also write this process as follows eliminating one of the $\sim W$'s:

$$\begin{array}{l} D \\ W \rightarrow \sim D \\ \hline \sim W \\ T \vee W \\ \hline T \end{array}$$

