SOME PRACTICE PROBLEMS

1. Simplify

(i) $\frac{(\cos 2\theta - i\sin 2\theta)^5(\cos 3\theta + i\sin 3\theta)^6}{(\cos 4\theta + i\sin 4\theta)^7(\cos\theta - i\sin\ \ )^8}$ **(ii)** $\frac{(\cos 2\theta + i\sin 2\theta)^3(\cos 3\theta - i\sin 3\theta)^2}{(\cos 4\theta + i\sin 4\theta)^5(\cos 5\theta - i\sin 5\theta)^4}$

2. Prove that

(i) $\frac{(1+i)^8(1-i\sqrt{3})^3}{(1-i)^6(1+i\sqrt{3})^9} = \frac{i}{32}$ **(ii)** $\frac{(1+i\sqrt{3})^9(1-i)^4}{(\sqrt{3}+i)^{12}(1+i)^4} = -\frac{1}{8}$

3. Find the modulus and the principal value of the argument of $\frac{(1+i\sqrt{3})^{17}}{(\sqrt{3}-i)^{15}}$

4. Express $(1+7i)(2-i)^{-2}$ in the form of $r(\cos\theta + i\sin\theta)$ and prove that the second power is a negative imaginary number and the fourth power is a negative real number.

5. If $x_n + iy_n = (1+i\sqrt{3})^n$, prove that $x_{n-1}y_n - x_n y_{n-1} = 4^{n-1}\sqrt{3}$.

6. Simplify $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n$

7. Prove that $\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta} = \sin\theta + i\cos\theta$ Hence deduct that

$\left(1+\sin\frac{\pi}{5} + i\cos\frac{\pi}{5}\right)^5 + i\left(1+\sin\frac{\pi}{5} - i\cos\frac{\pi}{5}\right)^5 = 0.$

8. If $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ *and* $\overline{z}$ is the conjugate of z find the value of $(z)^{15} + (\overline{z})^{15}$.

9. Prove that, if n is a positive integer, then

(i) $(a+ib)^{m/n} + (a-ib)^{m/n} = 2\left(\sqrt{a^2+b^2}\right)^{m/n}\cos\left(\frac{m}{n}\tan^{-1}\frac{b}{a}\right)$

(ii) $\left(\sqrt{3}+i\right)^{120} + \left(\sqrt{3}-i\right)^{120} = 2^{121}$

10. If n is a positive integer, prove that $(1+i)^n + (1-i)^n = 2\ 2^{n/2}\cos n\ \pi/4$

Hence, deduce that $(1+i)^{10} + (1-i)^{10} = 0$

11. Prove that $\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n$ is equal to -1 if $n = 3k \pm 1$

and 2 if $n = 3k$ where k is an integer.

12. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$,

prove that $\alpha^n + \beta^n = 2^{n+1}\cos(n\pi/3)$.

(i) Deduce that $\alpha^{15} + \beta^{15} = -2^{16}$ **(ii)** Deduce that $\alpha^6 + \beta^6 = 128$

13. If α, β are the roots of the equation $z^2\sin^2\theta - z.\sin 2\theta + 1 = 0$, prove that

$\alpha^n + \beta^n = 2\cos n\,\theta\, cosec^n\theta$

14. If $a = \cos 3\alpha + i\sin 3\alpha, b = \cos 3\beta + i\sin 3\beta, c = \cos 3\gamma + i\sin 3\gamma$, prove that

$\sqrt[3]{\frac{ab}{c}} + \sqrt[3]{\frac{c}{ab}} = 2\cos(\alpha+\beta-\gamma)$

15. If $x + \frac{1}{x} = 2\cos\theta$, $y + \frac{1}{y} = 2\cos\emptyset$, $z + \frac{1}{z} = 2\cos\psi$, prove that

(i) $xyz + \frac{1}{xyz} = 2\cos(\theta + \Phi + \psi)$ **(ii)** $\sqrt{xyz} + \frac{1}{\sqrt{xyz}} = 2\,cos\left(\frac{\theta+\Phi+\psi}{2}\right)$

(iii) $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\Phi)$ **(iv)** $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2\,cos\left(\frac{\theta}{m} - \frac{\emptyset}{n}\right)$

16. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$, prove that $\frac{(b+c)(c+a)(a+b)}{abc} = 8\,cos\frac{(\alpha-\beta)}{2}cos\frac{(\beta-\gamma)}{2}cos\frac{(\gamma-\alpha)}{2}$.

17. If a, b, c are three complex numbers such that $a + b + c = 0$, prove that

(i) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$ and **(ii)** $a^2 + b^2 + c^2 = 0$

18. If $\cos\alpha + \cos\beta + \cos\gamma = 0$ *and* $\sin\alpha + \sin\beta + \sin\gamma = 0$, Prove that

(i) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$, $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$.

(ii) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$

(iii) $cos(\alpha+\beta) + cos(\beta+\gamma) + cos(\gamma+\alpha) = 0$.

(iv) $\sin(\alpha+\beta) + sin(\beta+\gamma) + sin(\gamma+\alpha) = 0$.

(v) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\,cos(\alpha+\beta+\gamma)$

(vi) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

19. If $a\cos\alpha + b\cos\beta + c\cos\gamma = a\sin\alpha + b\sin\beta + c\sin\gamma = 0$, Prove that $a^3\cos 3\alpha + b^3\cos 3\beta + c^3\cos 3\gamma = 3abc\cos(\alpha+\beta+\gamma)$ and $a^3\sin 3\alpha + b^3\sin 3\beta + c^3\sin 3\gamma = 3\,abc\,sin(\alpha+\beta+\gamma)$

20. If $x_r = cos\left(\frac{2}{3}\right)^r\pi + i\,sin\left(\frac{2}{3}\right)^r\pi$, prove that

(i) $x_1x_2x_3 \ldots \infty = 1$, **(ii)** $x_0x_1x_2 \ldots \infty = -1$

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SOME PRACTICE PROBLEMS

1. Find the cube roots of unity. If ω is a complex cube root of unity prove that

(i) $1+\omega+\omega^2=0$ **(ii)** $\frac{1}{1+2\omega}+\frac{1}{2+\omega}-\frac{1}{1+\omega}=0$

2. Prove that the n nth roots of unity are in geometric progression.

3. Show that the sum of the n nth roots of unity is zero.

4. Prove that the product of n nth roots of unity is $(-1)^{n-1}$

5. Find all the values of the following :

(i) $(-1)^{1/5}$ **(ii)** $(-i)^{1/3}$ **(ix)** $\left(1-i\sqrt{3}\right)^{1/4}$

6. Find the continued product of all the values of $\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{3/4}$

7. Find all the value of $(1+i)^{2/3}$ and find the continued product of these values.

8. Solve the equations

(i) $x^9+8x^6+x^3+8=0$ **(ii)** $x^4-x^3+x^2-x+1=0$

(iii) $(x+1)^8+x^8=0$

9. If $(x+1)^6=x^6, show\ that\ x=-\frac{1}{2}-i\cot\frac{\theta}{2}\ where\ \theta=\frac{2k\pi}{6}, k=0,1,2,3,4,5.$

10. Show that the roots of $(x+1)^7=(x-1)^7$ are given by $\pm i\cot\frac{r\pi}{7}, r=1,2,3.$

11. If $\alpha,\alpha^2,\alpha^3,\ldots\alpha^6$ are the roots of $x^7-1=0$, find them and prove that $(1-\alpha)(1-\alpha^2)\ldots\ldots\ldots\ldots(1-\alpha^6)=7.$

12. Prove that $x^5-1=(x-1)\left(x^2+2x\cos\frac{\pi}{5}+1\right)\left(x^2+2x\cos\frac{3\pi}{5}+1\right)=0.$

13. Solve the equation $z^n=(z+1)^n$ and show that the real part of all the roots is $-1/2$.

14. If $a=e^{i\,2\pi/7}$ and $b=a+a^2+a^4$, $c=a^3+a^5+a^6$. then prove that b & c are roots of quadratic equation $x^2+x+2=0$.

15. Prove that **(i)** $\sqrt{1-cosce(\theta/2)}=\left(1-e^{i\theta}\right)^{-1/2}-\left(1-e^{-i\theta}\right)^{-1/2}$

(iv) $\sqrt{1-sce(\theta/2)}=\left(1+e^{i\theta}\right)^{-1/2}-\left(1+e^{-i\theta}\right)^{-1/2}$

16. If $1+2i$ is a root of the equation $x^4-3x^3+8x^2-7x+5=0$, find all the other roots.