Module 3
Partial Differentiation and Application

Unit 3.3

Maxima and Minima of Function of Two Independent Variables

❖ **Maxima:**

A function of two variable, $f(x, y)$ is said to be maximum at point (a, b) if $f(a, b) > f(a + h, b + k)$ for some h, k.

❖ **Maxima:**

A function of two variable, $f(x, y)$ is said to be minimum at point (a, b) if

$f(a, b) < f(a + h, b + k)$ for some h, k.

❖ **Method To find maxima or minima of a function of two variables**

1) Given $f(x, y)$, solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ for x and y simultaneously.
2) Suppose (a, b) is solution of above equations then it is called as **Stationary Point**.
3) Calculate the values of $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$ at (a, b).
4) DECISION:

   For point (a, b),

   a. If $rt - s^2 > 0$ and $r < 0$ or $t < 0$, $f(x, y)$ is **maximum** at (a, b).
   b. If $rt - s^2 > 0$ and $r > 0$ or $t > 0$, $f(x, y)$ is **minimum** at (a, b).
   c. If $rt - s^2 < 0$ then $f(x, y)$ is **neither maximum nor minimum** at (a, b).

      Such point is known as a saddle point.
   d. If $rt - s^2 = 0$, **test fails**.

SOME SOLVED EXAMPLES

1. Discuss the maxima and minima $x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10$.

Solution:

We have $f(x, y) = x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10$.

$$f_x = 3x^2 + y^2 - 24x + 21$$
$$f_y = 2xy - 4y$$

We now solve the equations $f_x = 0, f_y = 0$

$$3x^2 + y^2 - 24x + 21 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

and $2xy - 4y = 0$

$$\Rightarrow 2y(x-2) \Rightarrow x = 2 \text{ or } y = 0.$$

When x = 2, (i)gives

$$12 + y^2 - 48 + 21 = 0$$

$$\therefore \quad y^2 - 15 = 0 \qquad \therefore y^2 = 15 \qquad \therefore \quad y = \pm\sqrt{15}.$$

$\therefore$ The possible stationary points are $(2, \sqrt{15}), (2, -\sqrt{15})$

When y = 0, (1) gives

$$3x^2 - 24x + 21 = 0 \Rightarrow x^2 - 8x + 7 = 0$$

$$\therefore \quad (x-7)(x-1) = 0 \quad \therefore \quad x = 7, 1.$$

The other possible stationary points are (7, 0), (1, 0).

Now,

$$r = f_{xx} = 6x - 24,$$
$$s = f_{xy} = 2y,$$
$$t = f_{yy} = 2x - 4$$

Hence, all stationary points of $f(x, y)$ are $(7,0), (1,0), (2, \sqrt{15}), (2, -\sqrt{15})$

We write a tabular form and find Maxima and Minima

Sr. No.	Point	r	t	s	$rt - s^2$	Conclusion
1	(7,0)	$18 > 0$	10	0	$180 > 0$	(7,0) is minima.
2	(1,0)	$-18 < 0$	–2	0	$36 > 0$	(1,0) is maxima.
3	$(2, \sqrt{15})$	$-12 < 0$	0	$2\sqrt{15}$	$-60 < 0$	Neither maxima nor minima
4	$(2, -\sqrt{15})$	$-12 < 0$	0	$-2\sqrt{15}$	$-60 < 0$	Neither maxima nor minima

Maximum Value at (1,0) is $f(1,0) = 1 + 0 - 12 - 0 + 21 + 10 = 20$

Minimum Value at (7,0) is $f(7,0)$ =343 + 0 – 588 – 0 + 147 + 10 = - 88.

2. Discuss the maxima & minima of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

Solution:

Let $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$$f_x = \frac{\partial f}{\partial x} = 4x^3 - 4x + 4y$$

$$f_y = \frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$$

We now solve the equations $f_x = 0, f_y = 0$ simultaneously,

$$4x^3 - 4x + 4y = 0 \Rightarrow x^3 - x + y = 0 \dots\dots\dots \; (i)$$

$$4y^3 + 4x - 4y = 0 \Rightarrow y^3 + x - y \dots\dots\dots \; (ii)$$

On adding (i) & (ii), we get

$$x^3 + y^3 = 0$$

$$\therefore (x + y)(x^2 - xy + y^2) = 0$$

$$\therefore x = -y$$

Substituting $x = -y$ in eq (i),

$$x^3 - x - x = 0$$

$$\therefore x^3 - 2x = 0$$

$$\therefore x(x^2 - 2) = 0$$

$$\therefore x = 0 \quad \text{or} \quad x = \pm\sqrt{2}$$

Since $x = -y$,

For $x = 0 \Rightarrow y = 0$

For $x = \sqrt{2} \Rightarrow y = -\sqrt{2}$

For $x = -\sqrt{2} \Rightarrow y = \sqrt{2}$

$\therefore$ Stationary points are $(0,0)$, $(\sqrt{2}, -\sqrt{2})$&$(-\sqrt{2}, \sqrt{2})$.

$$r = f_{xx} = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$$

$$s = f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 4$$

$$t = f_{yy} = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$$

We write a tabular form and find Maxima and Minima

Sr. No.	Point	r	t	s	$rt - s^2$	Conclusion
1	$(0,0)$	$-4 < 0$	-4	4	0	Test fails.
2	$(\sqrt{2}, -\sqrt{2})$	$20 > 0$	20	4	$384 > 0$	$(\sqrt{2}, -\sqrt{2})$is minima.
3	$(-\sqrt{2}, \sqrt{2})$	$20 > 0$	20	4	$384 > 0$	$(-\sqrt{2}, \sqrt{2})$is minima.

Minimum Value at $(\sqrt{2}, -\sqrt{2})$ *and* $(-\sqrt{2}, \sqrt{2})$ is

$$f(\sqrt{2}, -\sqrt{2}) = f(-\sqrt{2}, \sqrt{2}) = 4 + 4 - 4 - 8 - 4 = -8$$

3. Find the stationary values of $x^3 + y^3 - 3a\,xy,\ a > 0$

Solution:

We have $f\,(x,\,y) = x^3 + y^3 - 3a\,xy$

$f_x = 3x^2 - 3ay,$

$f_y = 3y^2 - 3ax$

We now solve, $f_x = 0,\ \&\ f_y = 0.$

$$x^2 - ay = 0 \Rightarrow y = x^2/a$$

and $y^2 - ax = 0$

To eliminate y, we put $y = x^2/a$ in this equation.

$\therefore \quad x^4 - a^3x = 0 \qquad \therefore \quad x(x^3 - a^3) = 0$

Hence, $x = 0$ or $x = a$.

When $x = 0 \Rightarrow y = 0$ and when $x = a \Rightarrow y = a$.

$\therefore$ $(0,\,0)$ and $(a,\,a)$ are stationary points.

Now,

$r = f_{xx} = 6x,$

$s = f_{xy} = -3a$

$t = f_{yy} = 6y$

We write a tabular form and find Maxima and Minima

Sr. No.	Point	r	t	s	$rt - s^2$	Conclusion
1	(0,0)	0	$-3a$	0	$-9a^2 < 0$	Neither maxima nor minima
2	(a,a)	$6a > 0$, as $a > 0$	$-3a$	$6a$	$27a^2 > 0$	(a,a) is minima.

Minimum Value at (a,a) is $f(a,a) = a^3 + a^3 - 3a^3 = -a^3$

4. Find the stationary values of $sin\,x \cdot siny \cdot sin\,(x + y)$.

Solution:

We have $f\,(x,\,y) = sin\,x \cdot \sin y \cdot \sin\,(x + y)$

$f_x = sin\,y\,[cos\,x \cdot sin\,(x + y) + sin\,x \cdot cos\,(x + y)] = \sin y \cdot \sin\,(2x + y)$

Similarly, $f_y = sin\,x \cdot sin\,(x + 2y)$

Now, we solve $f_x = 0 \; and \; f_y = 0$.

$\therefore \quad \sin y \sin(2x + y) = 0 \Rightarrow \frac{1}{2}[\cos 2x - \cos(2x + 2y)] = 0 \; \ldots\ldots\ldots.. \; (i)$

and $\sin x \sin(x + 2y) = 0 \Rightarrow \frac{1}{2}[\cos 2y - \cos(2x + 2y)] = 0 \; \ldots\ldots\ldots.. \; (ii)$

Equating (i) & (ii), we get

$[\cos 2x - \cos(2x + 2y)] = [\cos 2y - \cos(2x + 2y)] \Rightarrow \cos 2x = \cos 2y$

$\Rightarrow x = y$

From (i) we get,

$\frac{1}{2}[\cos 2x - \cos 4x] = 0 \Rightarrow \cos 2x - (2\cos^2 2x - 1) = 0$

$\Rightarrow 2\cos^2 2x - \cos 2x - 1 = 0$

$\Rightarrow \cos 2x = 1$ or $\cos 2x = -\frac{1}{2}$

$\Rightarrow 2x = 0$ or $2x = \frac{2\pi}{3} \Rightarrow x = 0$ or $x = \frac{\pi}{3}$

As $x = y$,

$y = 0$ or $y = \frac{\pi}{3}$

$\therefore$ (0, 0) and $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ are possible stationary points.

$r = f_{xx} = 2\,\sin y \cdot \cos(2x + y)$

$s = f_{xy} = \cos y \cdot \sin(2x + y) + \sin y \cdot \cos(2x + y) = \sin(2x + 2y)$

$t = f_{yy} = 2\sin x \cdot \cos(x + 2y)$

We write a tabular form and find Maxima and Minima

Sr. No.	Point	r	t	s	$rt - s^2$	Conclusion
1	(0,0)	0	0	0	0	Test Fails
2	$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$	$-\sqrt{3} < 0$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{9}{4} > 0$	$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is maxima.

Maximum Value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

5. Divide 90 into three parts such that the sum of their products taken two at a time is maximum.

Solution:

Let three parts of the 90 are x, y & z.

$\therefore x + y + z = 90 \dots\dots\dots (i)$

Function to be maximized $f\left(x,\ y\right) = xy + yz + zx$

$$= xy + y\,(90 - x - y) + x\,(90 - x - y) \quad from(i)$$

$$= 90x + 90y - xy - x^2 - y^2$$

$f_x = 90 - y - 2x,$

$f_y = 90 - x - 2y$

Solving $f_x = 0$ & $f_y = 0$

$\therefore \quad 2x + y = 90 \quad \& \quad x + 2y = 90$

Solving above equations we get,

$x = 30$ & $y = 30$

Hence (30,30) is stationary point.

$r = f_{xx} = -2,$

$s = f_{xy} = -1,$

$t = f_{yy} = -2$

At (30,30)

$r = -2 < 0,\ S = -1,\ t = -2$

$rt - s^2 = 4 > 0$

Function has maxima at (30,30).

From (i)

$z = 90 - x - y = 30$

$\therefore$ Required three parts of the 90 are 30,30 &30.

6. A rectangular box with open top has capacity of 32 cubiccms. Find the dimensions of the box such that the material required is minimum.

Solution:

Let the dimensions of the box be x, y, z.

$\therefore Volume = V = xyz = 32 \quad \ldots\ldots\ldots..(i)$

Minimum material required if surface area is minimum.

Considering that the box is rectangular with open top,

$Surface\ Area = xy + 2yz + 2zx$

$$f(x,\ y) = xy + 2yz + 2zx = xy + \frac{64}{x} + \frac{64}{y} \qquad from(i)$$

$f_x = y - \frac{64}{x^2}$,

$f_y = x - \frac{64}{y^2}$

Solving $f_x = 0$ & $f_y = 0$

$y - \frac{64}{x^2} = 0 \quad \therefore 64 = x^2 y \quad \therefore y = \frac{64}{x^2} \quad \ldots\ldots\ldots..(ii)$

$x - \frac{64}{y^2} = 0 \quad \therefore \quad 64 = xy^2 \quad \ldots\ldots\ldots..(iii)$

$\therefore \quad 64 = x \cdot \frac{(64)^2}{x^4}$

$\therefore x^3 = 64 \quad \therefore \quad x = 4$

For $x = 4$, $\quad y = \frac{64}{x^2} = 4$

Hence, (4,4) is stationary point

$r = f_{xx} = \frac{64(2)}{x^3}$,

$s = f_{xy} = 1$,

$t = f_{yy} = \frac{64(2)}{y^3}$

At (4,4), $r = 2$, $s = 1$ and $t = 2$

$rt - s^2 = 4 - 1 = 3 > 0$, & $r = 2 > 0$, So, function has minima at (4,4)

From (i)

$z = \frac{32}{xy} = 2$

Hence, surface area is minimum if dimensions of box are $x = 4, y = 4$ and $z = 2$.

7. Divide 24 into three parts such that the product of the first, square of the second and cube of the third is maximum.

Solution:

Let three parts of the 24 are z, y & x.

$z + y + x = 24 \quad \ldots\ldots\ldots\ldots (i)$

We want product $zy^2 \cdot x^3$ to be maximum.

Hence,

$$f(x, y) = zy^2 \cdot x^3 = (24 - y - x)y^2x^3 \quad \ldots\ldots\ldots\ldots from(i) \;\; z = 24 - y - x$$

$$= 24y^2x^3 - y^3x^3 - y^2x^4$$

$f_x = 72y^2x^2 - 3y^3x^2 - 4y^2x^3$

$f_y = 48x^3y - 3x^3y^2 - 2x^4y$

Solving $f_x = 0$ & $f_y = 0$

$f_x = 0 \Rightarrow 72y^2x^2 - 3y^3x^2 - 4y^2x^3 = 0$

$\Rightarrow y^2x^2(72 - 3y - 4x) = 0$

$f_y = 0 \Rightarrow 48x^3y - 3x^3y^2 - 2x^4y = 0$

$\Rightarrow x^3y(48 - 3y - 2x) = 0$

As, we want maximum product $x, y, z \neq 0$

$\Rightarrow (72 - 3y - 4x) = 0$ *and* $(48 - 3y - 2x) = 0$

$\therefore \quad 4x + 3y = 72$ *and* $2x + 3y = 48$

$\therefore \;\; x = 12, \; y = 8$

Hence (12,8) is stationary point.

$r = f_{xx} = 144y^2x - 6y^3x - 12y^2x^2$

$s = f_{xy} = 144yx^2 - 9y^2x^2 - 8yx^3$

$t = f_{yy} = 48x^3 - 6x^3y - 2x^4$

At (12,8) ,

$r = 144 \times 64 \times 12 - 6 \times 512 \times 12 - 12 \times 64 \times 144 = -36864$

$s = 144 \times 8 \times 144 - 9 \times 64 \times 144 - 8 \times 8 \times 1728 = -27648$

$t = 48 \times 1728 - 6 \times 1728 \times 8 - 2 \times 20736 = -41472$

$rt - s^2 = 1528823808 - 764411904 > 0, \ \& \, r < 0$

So, function has minima at $(12,8)$

From (i)

$z = 24 - y - x = 24 - 8 - 12 = 4$

$\therefore \ \mathrm{z} = 4, \mathrm{y} = 8, \mathrm{x} = 12.$

Hence three parts of 24 are 4, 8 and 12 such that the product of the first, square of the second and cube of the third is maximum.

SOME PRACTICE PROBLEMS

1. Find stationary points of the following functions and discuss the maxima & minima at those points.
    1) $x^3 + 3\,xy^2 - 15x^2 - 15y^2 + 72x$
    2) $x^4 + y^4 - 2x^2 + 4xy - 2y^2$
    3) $x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10$
    4) $x^3 y^2 (1 - x - y)$
    5) $x^2 y^3 (1 - x - y)$
    6) $xy(3a - x - y)$
    7) $x^2 y - 3x^2 - 2y^2 - 4y + 3$
    8) $y^2 + 4xy + 3x^2 + x^3$
    9) $xy(3 - x - y)$
    10) $x^3 + 3x y^2 - 3x^2 - 3y^2 + 4$
    11) $2(x^2 - y^2) - x^4 + y^4$
    12) $xy + a^3 \left(\frac{1}{x} + \frac{1}{y}\right)$
2. A real number $k, k > 0$ is divided into 3 parts such that the sum of their products taken two at a time is maximum. Find the numbers.
3. A rectangular box, open at top has volume V. Find dimensions of the box requiring least material for its construction.
4. Find the maximum value of $cosA\ cosB\ cosC$, where A, B, C are angles of a triangle.
5. Find the maximum volume of a parallelepiped inscribed in a sphere $x^2 + y^2 + z^2 = a^2$.