Module 3
Partial Differentiation and Application

Unit 3.2
Composite Functions

❖ **$z = f(x, y)$ as composite function of one variable :**

Let $z = f(x, y)$ and $x = \Phi(t)$, $y = \Psi(t)$ so that z is function of x , y and x , y are function of third variable t.

The three relations define z as a function of t. In such cases z is called a **composite function of t**.

e.g. (i) $z = x^2 + y^2$, $x = at^2$, $y = 2at$

(ii) $z = x^2y + xy^2$, $x = a\cos t$, $y = b\sin t$

In above examples z is a composite function of one variable t.

Differentiation: Let $z = f(x, y)$ posses continuous first order partial derivatives and $x = \Phi(t)$, $y = \Psi(t)$ posses continuous first order derivatives then,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$$

❖ **$z = f(x, y)$ as composite function of two variables :**

Let $z = f(x, y)$ and $x = \Phi(u, v)$, $y = \Psi(u, v)$ so that z is function of x , y and x , y are function of u , v.

The three relations define z as a function of u , v. In such cases z is called a **composite function of u , v**.

e.g. (i) $z = xy$, $x = e^u + e^{-v}$, $y = e^{-u} + e^v$

(ii) $z = x^2 - y^2$, $x = 2u - 3v$, $y = 3u + 2v$

In above examples z is a composite function of two variables u and v

Differentiation: Let $z = f(x, y)$ possess continuous first order partial derivatives and $x = \Phi(u, v)$, $y = \Psi(u, v)$ possess continuous first order partial derivatives then,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$$

SOME SOLVED EXAMPLES

1. If $u = x^2y^3$, $x = \log t$, $y = e^t$, find $\frac{du}{dt}$

Solution:

$u = x^2y^3$, $x = \log t$, $y = e^t$

$\therefore u$ is Composite Function of one variable t.

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$
$$= (2xy^3)\frac{1}{t} + (3x^2y^2)e^t$$

Substituting x and y,

$$\frac{du}{dt} = 2(\log t)e^{3t} \cdot \frac{1}{t} + 3(\log t)^2 e^{2t} \cdot e^t$$
$$= \frac{2}{t}\log t\, e^{3t} + 3(\log t)^2 e^{3t}$$

2. If $u = xy + yz + zx$ where $x = \frac{1}{t}$, $y = e^t$, $z = e^{-t}$, find $\frac{du}{dt}$

Solution:

$u = xy + yz + zx$, $x = \dfrac{1}{t}$, $y = e^t$, $z = e^{-t}$

$\therefore u$ is Composite Function of one variable t.

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$
$$= (y+z)\left(-\frac{1}{t^2}\right) + (x+z)e^t + (y+x)(-e^{-t})$$

Substituting x, y and z,

$$\frac{du}{dt} = -\frac{1}{t^2}(e^t + e^{-t}) + \left(\frac{1}{t} + e^{-t}\right)e^t - \left(e^t + \frac{1}{t}\right)e^{-t}$$
$$= -\frac{1}{t^2}(e^t + e^{-t}) + \frac{1}{t}(e^t - e^{-t})$$

3. If $z = x^2y + y^2x$, $x = at^2$, $y = 2at$ find $\frac{dz}{dt}$

Solution:

$z = x^2y + y^2x$, $x = at^2$, $y = 2at$

$\therefore z$ is Composite Function of one variable t.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{dz}{dt} = (2xy + y^2)(2at) + (x^2 + 2yx)(2a)$$

Substituting x and y,

$$\frac{dz}{dt} = (2at^2.2at + (2at)^2)2at + ((at^2)^2 + 2(2at)at^2)2a$$
$$= (4a^2t^3 + 4a^2t^2)2at + (a^2t^4 + 4a^2t^3)2a$$
$$= 8a^3t^4 + 8a^3t^3 + 2a^3t^4 + 8a^3t^3$$
$$= 10a^3t^4 + 16a^3t^3$$

4. If $z = e^{xy}$, $x = t\cos t$, $y = t\sin t$, find $\frac{dz}{dt}$ at $t = \frac{\pi}{2}$

Solution:
$z = e^{xy}$, $x = t\cos t$, $y = t\sin t$
$\therefore$ z is Composite Function of one variable t.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$
$$= e^{xy}y(\cos t - t\sin t) + e^{xy}x(\sin t + t\cos t)$$

At $t = \frac{\pi}{2}$, $x = 0$, $y = \frac{\pi}{2}$

Hence, $\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{2}} = e^0\left[\frac{\pi}{2}\left(0 - \frac{\pi}{2}\right) + 0\right] = -\frac{\pi^2}{4}$

5. If $z = sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$ find $\frac{dz}{dt}$

Solution:
$z = sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$

$\therefore$ z is Composite Function of one variable t.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$
$$= \frac{1}{\sqrt{1-(x-y)^2}}3 + \frac{(-1)}{\sqrt{1-(x-y)^2}}12t^2$$
$$= \frac{3-12t^2}{\sqrt{1-(x-y)^2}}$$

Substituting x and y,

$$\frac{dz}{dt} = \frac{3 - 12t^2}{\sqrt{1 - (3t - 4t^3)^2}}$$

6. If $x^2 = au + bv,\ \ y^2 = au - bv \ and \ z = f(x, y)$,

Prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$.

Solution:

$z = f(x, y),\ \ x^2 = au + bv,\ \ y^2 = au - bv$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{a}{2x} + \frac{\partial z}{\partial y}\cdot\frac{a}{2y}$$

$$u\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{au}{2x} + \frac{\partial z}{\partial y}\cdot\frac{au}{2y} \qquad \text{............ (i)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{b}{2x} + \frac{\partial z}{\partial y}\left(-\frac{b}{2y}\right)$$

$$v\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{bv}{2x} - \frac{\partial z}{\partial y}\cdot\frac{bv}{2y} \qquad \text{............ (ii)}$$

Adding Equations (i) and (ii),

$$\begin{aligned} u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\cdot\frac{au}{2x} + \frac{\partial z}{\partial y}\cdot\frac{au}{2y} + \frac{\partial z}{\partial x}\cdot\frac{bv}{2x} - \frac{\partial z}{\partial y}\cdot\frac{bv}{2y} \\ &= \frac{\partial z}{\partial x}\left(\frac{au+bv}{2x}\right) + \frac{\partial z}{\partial y}\left(\frac{au-bv}{2y}\right) \\ &= \frac{\partial z}{\partial x}\left(\frac{x^2}{2x}\right) + \frac{\partial z}{\partial y}\left(\frac{y^2}{2y}\right) \\ &= \frac{1}{2}\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) \end{aligned}$$

7. If $z = f(u,\ v)\ and\ u = \log(x^2 + y^2),\ v = \frac{y}{x}$, prove that $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = (1 + v^2)\frac{\partial z}{\partial v}$

Solution:

$z = f(u,\ v),\ u = \log(x^2 + y^2),\ v = \frac{y}{x}$

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \\ &= \frac{\partial z}{\partial u}\cdot\frac{1}{x^2+y^2}\cdot 2x + \frac{\partial z}{\partial v}\left(-\frac{y}{x^2}\right) \end{aligned}$$

$$y\frac{\partial z}{\partial x} = \frac{2xy}{x^2+y^2}\cdot\frac{\partial z}{\partial u} - \frac{y^2}{x^2}\cdot\frac{\partial z}{\partial v} \qquad \text{................. (1)}$$

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} \\ &= \frac{\partial z}{\partial u}\cdot\frac{2y}{x^2+y^2} + \frac{\partial z}{\partial v}\cdot\frac{1}{x} \end{aligned}$$

$$x\frac{\partial z}{\partial y} = \frac{2xy}{x^2+y^2}\cdot\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \qquad \text{................. (2)}$$

Subtracting Eq. (1) from Eq. (2),

Hence, $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = \frac{\partial z}{\partial v} + \frac{y^2}{x^2}\frac{\partial z}{\partial v} = (1 + v^2)\frac{\partial z}{\partial v}$

8. If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$, prove that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$.

Solution:

Let $l = x^2 - y^2$, $m = y^2 - z^2$, $n = z^2 - x^2$

$$\frac{\partial l}{\partial x} = 2x, \quad \frac{\partial m}{\partial x} = 0, \quad \frac{\partial n}{\partial x} = -2x$$

$$\frac{\partial l}{\partial y} = -2y, \quad \frac{\partial m}{\partial y} = 2y, \quad \frac{\partial n}{\partial y} = 0$$

$$\frac{\partial l}{\partial z} = 0, \quad \frac{\partial m}{\partial z} = -2z, \quad \frac{\partial n}{\partial z} = 2z$$

Consider, $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2) \ = f(l, m, n)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x} = \frac{\partial u}{\partial l}\cdot 2x + \frac{\partial u}{\partial m}\cdot 0 + \frac{\partial u}{\partial n}\cdot(-2x)$$

$$\therefore \frac{1}{x}\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial l} - 2\frac{\partial u}{\partial n} \qquad \text{………..} (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y} = \frac{\partial u}{\partial l}(-2y) + \frac{\partial u}{\partial m}(2y) + \frac{\partial u}{\partial n}(0)$$

$$\therefore \frac{1}{y}\frac{\partial u}{\partial y} = -2\frac{\partial u}{\partial l} + 2\frac{\partial u}{\partial m} \qquad \text{………..} (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z} = \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}(-2z) + \frac{\partial u}{\partial n}(2z)$$

$$\therefore \frac{1}{z}\frac{\partial u}{\partial z} = -2\frac{\partial u}{\partial m} + 2\frac{\partial u}{\partial n} \qquad \text{………..} (3)$$

Adding Eqs (1), (2) and (3),

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$$

9. If $x = \sqrt{vw},\ y = \sqrt{wu},\ z = \sqrt{uv}$ and φ is function of x, y, z then prove that

$$x\frac{\partial \varphi}{\partial x} + y\frac{\partial \varphi}{\partial y} + z\frac{\partial \varphi}{\partial z} = u\frac{\partial \varphi}{\partial u} + v\frac{\partial \varphi}{\partial v} + w\frac{\partial \varphi}{\partial w}$$

Solution:

φ is function of x, y, z and $\quad x = \sqrt{vw},\ y = \sqrt{wu},\ z = \sqrt{uv}$

Hence φ is composite function of three variables u, v, w

$$\frac{\partial \varphi}{\partial u} = \frac{\partial \varphi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \varphi}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \varphi}{\partial z}\frac{\partial z}{\partial u}$$

$$= \frac{\partial \varphi}{\partial x}(0) + \frac{\partial \varphi}{\partial y}\frac{\sqrt{w}}{2\sqrt{u}} + \frac{\partial \varphi}{\partial z}\frac{\sqrt{v}}{2\sqrt{u}}$$

$$u\frac{\partial\varphi}{\partial u}=\frac{u\sqrt{w}}{2\sqrt{u}}\frac{\partial\varphi}{\partial y}+\frac{u\sqrt{v}}{2\sqrt{u}}\frac{\partial\varphi}{\partial z}=\frac{\sqrt{uw}}{2}\frac{\partial\varphi}{\partial y}+\frac{\sqrt{uv}}{2}\frac{\partial\varphi}{\partial z}$$

As, $y=\sqrt{wu},\ z=\sqrt{uv}$

$$u\frac{\partial\varphi}{\partial u}=\frac{1}{2}\left(y\frac{\partial\varphi}{\partial y}+z\frac{\partial\varphi}{\partial z}\right) \qquad \ldots\ldots\ldots..(i)$$

Similarly,

$$v\frac{\partial\varphi}{\partial v}=\frac{1}{2}\left(x\frac{\partial\varphi}{\partial x}+z\frac{\partial\varphi}{\partial z}\right) \qquad \ldots\ldots\ldots..(ii)$$

$$w\frac{\partial\varphi}{\partial u}=\frac{1}{2}\left(x\frac{\partial\varphi}{\partial x}+\frac{\partial\varphi}{\partial y}y\right) \qquad \ldots\ldots\ldots..(iii)$$

Adding (i), (ii) and (iii)

$$u\frac{\partial\varphi}{\partial u}+v\frac{\partial\varphi}{\partial v}+w\frac{\partial\varphi}{\partial w}=\frac{1}{2}\left(y\frac{\partial\varphi}{\partial y}+z\frac{\partial\varphi}{\partial z}+x\frac{\partial\varphi}{\partial x}+z\frac{\partial\varphi}{\partial z}+x\frac{\partial\varphi}{\partial x}+y\frac{\partial\varphi}{\partial y}\right)$$

$$=x\frac{\partial\varphi}{\partial x}+y\frac{\partial\varphi}{\partial y}+z\frac{\partial\varphi}{\partial z}$$

10. If $x=e^{u}cosec\ v,\ y=e^{u}\cot v\ and\ z$ is a function of x and y, prove that

$$\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2=e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2-sin^2v\left(\frac{\partial z}{\partial v}\right)^2\right]$$

Solution:

$$z=f(x,\ y),\ x=e^{u}cosec\ v,\ y=e^{u}\cot v$$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}=\frac{\partial z}{\partial x}e^{u}cosec\ v+\frac{\partial z}{\partial y}e^{u}\cot v$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}=\frac{\partial z}{\partial x}(-e^{u}cosec\ v\cot v)+\frac{\partial z}{\partial y}(-e^{u}cosec\ v)$$

$$\text{R.H.S}=e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2-\sin^2 v\left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$=e^{-2u}\Bigg[\left(\frac{\partial z}{\partial x}\right)^2e^{u}cosec^2\ v+\left(\frac{\partial z}{\partial y}\right)^2e^{2u}\cot^2 v+2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}e^{2u}cosec\ v\cot v+$$
$$(-\sin^2 v)\left(\frac{\partial z}{\partial x}\right)^2(e^{2u}cosec^2\ v\cot^2 v)+(-\sin^2 v)\left(\frac{\partial z}{\partial y}\right)^2e^{2u}\,\mathrm{cosec}^4\ v+$$
$$(-\sin^2 v)2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}e^{2u}cosec^3\ v\cot v\Bigg]$$

$$=\left(\frac{\partial z}{\partial x}\right)^2(cosec^2-\ \cot^2 v)+\left(\frac{\partial z}{\partial y}\right)^2(\cot^2 v-cosec^2\ v)$$

$$=\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2$$

$=$ L.H.S.

11. If $x+y=2e^{\theta}\cos\Phi$, $x-y=2\,i\,e^{\theta}\sin\Phi$, show that $\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial\Phi^2}=4xy\frac{\partial^2 u}{\partial x\partial y}$ where u is a $f(x,y)$.

Solution:

Adding $x+y=2e^{\theta}\cos\Phi$, $x-y=2\,i\,e^{\theta}\sin\Phi$,

$$2x=2e^{\theta}(\cos\Phi+i\sin\Phi)$$

$$\therefore x=e^{\theta}\cdot e^{i\Phi}=e^{\theta+i\Phi}$$

Subtracting results, $x+y=2e^{\theta}\cos\Phi$, $x-y=2\,i\,e^{\theta}\sin\Phi$

$$2y=2e^{\theta}(\cos\Phi-i\sin\Phi)$$

$$\therefore y=e^{\theta-i\Phi}$$

Now, u is a function of $x,\ y$ and $x,\ y$ are functions of θ and Φ

$$\therefore \frac{\partial u}{\partial\theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\theta}$$

$$=\frac{\partial u}{\partial x}\cdot e^{\theta+i\Phi}+\frac{\partial u}{\partial y}\cdot e^{\theta-i\Phi}=\frac{\partial u}{\partial x}\cdot x+\frac{\partial u}{\partial y}\cdot y \quad \ldots\ldots\ldots\ldots \text{(i)}$$

$$\therefore \frac{\partial}{\partial\theta}\equiv x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y} \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

$$\therefore \frac{\partial^2 u}{\partial\theta^2}=\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial\theta}\right)=\frac{\partial}{\partial\theta}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{[From (i)]}$$

$$=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{[From (ii)]}$$

$$=x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial y}+y\frac{\partial u}{\partial y} \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

Also, $$\frac{\partial u}{\partial\Phi}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\Phi}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\Phi}$$

$$=\frac{\partial u}{\partial x}\cdot e^{\theta+i\Phi}\cdot i+\frac{\partial u}{\partial y}\cdot e^{\theta-i\Phi}\cdot(-i)$$

$$=\frac{\partial u}{\partial x}\cdot ix-i\frac{\partial u}{\partial y}\cdot y \quad \ldots\ldots\ldots\ldots \text{(iv)}$$

$$\therefore \frac{\partial}{\partial\Phi}\equiv i\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right) \quad \ldots\ldots\ldots\ldots \text{(v)}$$

$$\therefore \frac{\partial^2 u}{\partial\Phi^2}=\frac{\partial}{\partial\Phi}\left(\frac{\partial u}{\partial\Phi}\right)=\frac{\partial}{\partial\Phi}\left[i\left(x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}\right)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{[From (iv)]}$$

$$=i\left[i\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{[From (v)]}$$

$$=-\left[x^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

$$=-x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}-y^2\frac{\partial^2 u}{\partial y^2}-x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y} \quad \ldots\ldots\ldots\ldots \text{(vi)}$$

$\therefore$ Adding the two results, (v) and (vi) we get,

$$\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial\Phi^2}=4xy\frac{\partial^2 u}{\partial x\partial y}$$

SOME PRACTICE PROBLEMS

1. If $u = x^2 + y^2 + z^2$, where, $x = e^t, y = e^t \sin t, z = e^t \cos t$
   prove that $\frac{du}{dt} = 4e^{2t}$.
2. If $z = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$, prove that $\frac{dz}{dt} = \frac{3}{\sqrt{1-t^2}}$.
3. If $z = \tan^{-1}\left(\frac{x}{y}\right)$, $x = 2t$, $y = 1 - t^2$, prove that $\frac{dz}{dt} = \frac{2}{1+t^2}$.
4. If $u = f\,[e^{y-z}, e^{z-x}, e^{x-y}]$, then show that $u_x + u_y + u_z = 0$.
5. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.
6. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.
7. If $u = f(x^n - y^n, y^n - z^n, z^n - x^n)$,
   prove that $\frac{1}{x^{n-1}}\frac{\partial u}{\partial x} + \frac{1}{y^{n-1}}\frac{\partial u}{\partial y} + \frac{1}{z^{n-1}}\frac{\partial u}{\partial z} = 0$.
8. If $u = f(x - y, y - z, z - x)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.
9. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0$
10. If $x = u + v + w, y = uv + vw + wu, z = uvw$, *and* ϕ is a function of x, y & z,
    then prove that $x\frac{\partial \phi}{\partial x} + 2y\frac{\partial \phi}{\partial y} + 3z\frac{\partial \phi}{\partial z} = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w}$.
11. If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ *and* ϕ is a function of x, y & z then prove that,
    $x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z} = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w}$.
12. If $z = f(x, y)$, $x = r\cos\theta, y = r\sin\theta$, prove that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$.
13. If $z = f(x, y)$, $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, then show that
    $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.
14. If $w = \phi(u, v)$, $u = x^2 - y^2 - 2xy$, $v = y$, prove that $\frac{\partial w}{\partial v} = 0$ is equivalent
    to $(x + y)\frac{\partial w}{\partial x} + (x - y)\frac{\partial w}{\partial y} = 0$.
15. If $z = f(x, y), x = u\cosh v, y = u\sinh v$, prove that, $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2$.

16. If $z = f(x, y)$, $x = e^u \cos v$, $y = e^u \sin v$, prove that

$$(i)\ x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y} \qquad (ii)\ \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

17. If $z = f(x, y)$, $x = e^u \sec v$, $y = e^u \tan v$,

prove that $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - cos^2 v\left(\frac{\partial z}{\partial v}\right)^2\right]$

18. If $z = f(u, v)$, $u = e^x, v = e^y$, prove that $\frac{\partial^2 z}{\partial x \partial y} = uv\frac{\partial^2 z}{\partial u \partial v}$.

19. If $z = f(u, v)$, $u = lx + my, v = ly - mx$,

prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$.

20. If $z = f(u, v)$, $u = x^2 - y^2 - 2xy$, $v = y$,

prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 4\sqrt{u^2 + v^2}\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$