Module 3
Partial Differentiation and Application

Unit 3.4

Jacobian of Two and Three Independent Variables

- If u & v are functions of two independent variables x & y, then the Jacobian of u, v with respect to x, y is denoted and defined by

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,\ v)}{\partial(x,\ y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

- Similarly, If u, v &w are functions of three independent variables x, y & z, then the Jacobian of u, v, w with respect to x, y, z is denoted and defined by

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

- If $\frac{\partial(u,v,w)}{\partial(x,y,z)} = J$ & $\frac{\partial(x,y,z)}{\partial(u,v,w)} = J'$ then $JJ' = 1.$

SOME SOLVED EXAMPLES

1. If $u = \frac{x+y}{1-xy}$, $v = tan^{-1}x + tan^{-1}y$, $find\ \frac{\partial(u,v)}{\partial(x,y)}$

Solution:

$\frac{\partial u}{\partial x} = \frac{(1-xy)(1)-(x+y)(-y)}{(1-xy)^2} = \frac{1-xy+xy+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$

Similarly, $\frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}$

$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}$

$\frac{\partial v}{\partial y} = \frac{1}{1+y^2}$

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

2. If $u = x(1-y)$, $v = xy\,(1-z)$, $w = xyz$, $find\ \frac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution:

$u = x - xy, \quad V = xy - xyz, \quad w = xyz$

$$\frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1-y & -x & 0 \\ y(1-z) & x(1-z) & -xy \\ yz & zx & xy \end{vmatrix}$$

$$= (1-y)[(x - xz)xy + xy\,zx] + x\,[(y-yz)xy + xy\,yz]$$

$$= (1-y)[x^2y - x^2yz + x^2yz] + x[xy^2 - xy^2z + xy^2]$$

$$= (1-y)(x^2\,y) + x\,(xy^2)$$

$$= x^2y - x^2y^2 + x^2y^2$$

$$= x^2y$$

3. If $x = r\sin\theta\cos\emptyset$, $y = r\sin\theta\sin\emptyset$ and $z = r\cos\theta$ then evaluate $\frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)}$ and $\frac{\partial(r,\theta,\emptyset)}{\partial(x,y,z)}$.

Solution:

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \emptyset} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \emptyset} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \emptyset} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\emptyset & r\cos\theta\cos\emptyset & -r\sin\theta\sin\emptyset \\ \sin\theta\sin\emptyset & r\cos\theta\sin\emptyset & r\sin\theta\cos\emptyset \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

Since $JJ' = 1 \therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)} \cdot \frac{\partial(r,\theta,\emptyset)}{\partial(x,y,z)} = 1$

$$\therefore \frac{\partial(r,\theta,\emptyset)}{\partial(x,y,z)} = \frac{1}{J} = \frac{1}{r^2\sin\theta}$$

4. If $x = \frac{u^2-v^2}{2}, y = uv, z = w$, Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

Solution:

Given, $x = \frac{u^2-v^2}{2}$, $y = uv, z = w$

Lets calculate $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$

$$= \begin{vmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= u^2 + v^2$$

Then, $J' = \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \dfrac{1}{J} = \dfrac{1}{u^2+v^2}$

5. If $x = a\cosh u\cos v, y = a\sinh u\sin v$, Show that $\frac{\partial(x,y)}{\partial(u,v)} = \frac{a^2(\cosh 2u - \cos 2v)}{2}$.

Solution:

Given, $x = a\cosh u\cos v, y = a\sinh u\sin v$,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} asinhu.cosv & -acoshu.sinv \\ acoshu.sinv & asinhu.cosv \end{vmatrix}$$

$$= a^2 sinh^2 u cos^2 v + a^2 cosh^2 u sin^2 v$$

$$= a^2\left(\frac{cosh2u-1}{2}\right)\left(\frac{1+cos2v}{2}\right) + a^2\left(\frac{1+cosh2u}{2}\right)\left(\frac{1-cos2v}{2}\right)$$

$$= \frac{a^2[cosh2u+cosh2ucos2v-1-cos2v+1-cos2v+cosh2u-cosh2ucos2v]}{4}$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \frac{a^2}{2}(cosh2u - cos2v)$$

6. If $x = e^u \cos v$, $y = e^u sinv$, prove that $JJ' = 1$

Solution:

$$\mathrm{J} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix}$$

$$= e^{2u} cos^2 v + e^{2u} sin^2 v = e^{2u}$$

$$Now, x^2 + y^2 = e^{2u}\ and\ \frac{x}{y} = \tan v \quad \therefore \quad 2u = \log(x^2+y^2)$$

$$\therefore \quad u = \frac{1}{2}\log(x^2+y^2) \quad \text{and} \quad v = tan^{-1}\frac{y}{x}$$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}\cdot\frac{2x}{x^2+y^2} & \frac{1}{2}\cdot\frac{2y}{x^2+y^2} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2} = \frac{1}{e^{2u}}$$

$$\therefore \quad JJ' = \frac{\partial(x,y)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(x,y)} = e^{2u}\cdot\frac{1}{e^{2u}} = 1$$

7. If $x = u(1-v)$, $y = uv$, prove that $JJ' = 1$

Solution:

$$\mathrm{J} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u - uv + uv = u \quad \ldots\ldots\ldots\ldots(i)$$

Now, $\quad x = u - uv, \qquad y = uv$

$\therefore x + y = u$ and $\quad v = \frac{y}{u}$ ie. $v = \frac{y}{x+y}$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -\frac{y}{(x+y)^2} & \frac{(x+y)\cdot 1 - y\cdot 1}{(x+y)^2} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ -\frac{y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix} = \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2}$$

$$= \frac{x+y}{(x+y)^2} = \frac{1}{x+y}$$

As $x + y = u$,

Hence $J' = \frac{1}{u}$ $\quad\ldots\ldots\ldots\ldots (ii)$

By (i)& (ii)

$$JJ' = \frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = u \times \frac{1}{u} = 1$$

8. If $x = uv$, $y = \frac{u+v}{u-v}$, find $\frac{\partial(u,v)}{\partial(x,y)}$

Solution:

$$\mathrm{J} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{(u-v)1-(u+v)1}{(u-v)^2} & \frac{(u-v)1+(u+v)}{(u-v)^2} \end{vmatrix}$$

$$= \begin{vmatrix} v & u \\ \frac{-2v}{(u-v)^2} & \frac{2v}{(u-v)^2} \end{vmatrix} = \frac{2v}{(u-v)^2} + \frac{2uv}{(u-v)^2} = \frac{4uv}{(u-v)^2}$$

$$\therefore \quad \mathrm{J}' = \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{J} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}} = \frac{(u-v)^2}{4uv}$$

Since $(y^2 - 1) = \frac{(u+v)^2}{(u-v)^2} - 1 = \frac{4uv}{(u-v)^2}$

$$\therefore \quad \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{y^2-1}$$

9. Show that $JJ' = 1$ where $x = e^v \sec u, y = e^v \tan u$.

Solution:

Given, $x = e^v secu, y = e^v tanu$ $\quad\ldots\ldots\ldots\ldots (i)$

Let $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} e^v secu.tanu & e^v secu \\ e^v sec^2 u & e^v tanu \end{vmatrix}$

$$= e^{2v} secutan^2 u - e^{2v} sec^3 u$$

$$= e^{2v} secu(tan^2 u - sec^2 u)$$

$$= -e^{2v} secu = -xe^v$$

$$\therefore J = -xe^{v} \quad \ldots\ldots\ldots\ldots (ii)$$

From (i), $\frac{y}{x} = \frac{e^{v}tanu}{e^{v}secu} = sinu$

$$\therefore u = sin^{-1}\left(\frac{y}{x}\right) \quad \ldots\ldots\ldots\ldots (iii)$$

As $sec^2u - tan^2u = 1$,

$$\left(\frac{x}{e^{v}}\right)^2 - \left(\frac{y}{e^{v}}\right)^2 = 1 \quad \ldots\ldots\ldots\ldots from\ (i)$$

$$\therefore x^2 - y^2 = e^{2v}$$

$$\therefore v = \frac{\log(x^2-y^2)}{2} \quad \ldots\ldots\ldots\ldots (iv)$$

Now $J^{'} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$$= \begin{vmatrix} \frac{-y}{x\sqrt{x^2-y^2}} & \frac{1}{\sqrt{x^2-y^2}} \\ \frac{x}{x^2-y^2} & \frac{-y}{x^2-y^2} \end{vmatrix} \quad \ldots\ldots\ldots\ldots from\ (iii) and (iv)$$

$$= \frac{y^2}{x(x^2-y^2)^{3/2}} - \frac{x}{(x^2-y^2)^{3/2}}$$

$$= \frac{y^2-x^2}{x(x^2-y^2)^{3/2}}$$

$$= \frac{-1}{x\sqrt{x^2-y^2}} = \frac{-1}{x}e^{-v}$$

$$\therefore J^{'} = \frac{-1}{x}e^{-v} \quad \ldots\ldots\ldots\ldots from\ (v)$$

Consider $JJ^{'} = (-xe^{v})\left(\frac{-1}{x}e^{-v}\right) \quad \ldots\ldots\ldots\ldots from\ (ii) and (v)$

$$\therefore JJ^{'} = 1$$

10. If $x = v^2 + w^2, y = w^2 + u^2, z = u^2 + v^2$, Prove that $JJ^{'} = 1$.

Solution:

Given, $x = v^2 + w^2, y = w^2 + u^2, z = u^2 + v^2 \quad \ldots\ldots\ldots\ldots (i)$

Let $J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix} = 16uvw \quad \ldots\ldots\ldots\ldots (ii)$

From (i), $2u^2 = y + z - x$

On differentiating partially w.r.t. x, y, z respectively,

$$4u\frac{\partial u}{\partial x} = -1, \quad 4u\frac{\partial u}{\partial y} = 1, \quad 4u\frac{\partial u}{\partial z} = 1$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{-1}{4u}, \quad \frac{\partial u}{\partial y} = \frac{1}{4u}, \quad \frac{\partial u}{\partial z} = \frac{1}{4u}$$

From (i), $2v^2 = x - y + z$

On differentiating partially, w.r.t. x, y, z respectively,

$$4v\frac{\partial v}{\partial x} = 1, \qquad 4v\frac{\partial v}{\partial y} = -1, \qquad 4v\frac{\partial v}{\partial z} = 1$$

$$\frac{\partial v}{\partial x} = \frac{1}{4v}, \qquad \frac{\partial v}{\partial y} = \frac{-1}{4v}, \qquad \frac{\partial v}{\partial z} = \frac{1}{4v}$$

From (i), $2w^2 = x + y - z$

On differentiating partially, w.r.t. x, y, z respectively,

$$4w\frac{\partial w}{\partial x} = 1, 4w\frac{\partial w}{\partial y} = 1, 4w\frac{\partial w}{\partial z} = -1$$

$$\frac{\partial w}{\partial x} = \frac{1}{4w}, \frac{\partial w}{\partial y} = \frac{1}{4w}, \frac{\partial w}{\partial z} = \frac{-1}{4w}$$

Now, $J' = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} \frac{-1}{4u} & \frac{1}{4u} & \frac{1}{4u} \\ \frac{1}{4v} & \frac{-1}{4v} & \frac{1}{4v} \\ \frac{1}{4w} & \frac{1}{4w} & \frac{-1}{4w} \end{vmatrix}$

$\therefore J' = \left(\frac{1}{64uvw}\right)\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \left(\frac{1}{64uvw}\right)(4) = \frac{1}{16uvw}$ ……… (iii)

Consider $JJ' = (16uvw)\left(\frac{1}{16uvw}\right)$…..from (ii) & (iii)

$$\therefore JJ' = 1$$

11. $u = fc,\ w = f(x,y,z)$, Prove that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial u}{\partial x}.\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}$.

Solution:

Given, $u = f(x), v = f(x,y),\ w = f(x,y,z)$

Consider, $J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} u_x & 0 & 0 \\ v_x & v_y & 0 \\ w_x & w_y & w_z \end{vmatrix} = u_x v_y w_z$

$$\therefore J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}.$$

SOME PRACTICE PROBLEMS

1. If $x = r\cos\theta, y = r\sin\theta$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$.
2. If $x = uv$, $y = \frac{u+v}{u-v}$, find $\frac{\partial(u,v)}{\partial(x,y)}$.
3. If $u = \frac{x+y}{1-xy}, v = \tan^{-1}x + \tan^{-1}y$, find $\frac{\partial(u,v)}{\partial(x,y)}$.
4. If $x = \frac{u^2-v^2}{2}, y = uv, z = w$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.
5. If $u = 1 - x, v = x(1 - y), w = xy(1 - z)$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = -x^2y$.
6. If $u = x + y + z, v = x^2 + y^2 + z^2, w = xy + yz + zx$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.
7. If $u_1 = \frac{x_2x_3}{x_1}, u_2 = \frac{x_3x_1}{x_2}, u_3 = \frac{x_1x_2}{x_3}$, find the value of $\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)}$.
8. If $x = e^v\sec u, y = e^v\tan u$, find $\frac{\partial(u,v)}{\partial(x,y)}$
9. If $x = r^2\cos 2\theta, y = r^2\sin 2\theta$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$.
10. If $x = a\cosh u\cos v, y = a\sinh u\sin v$, show that $\frac{\partial(x,y)}{\partial(u,v)} = \frac{a^2(\cosh 2u-\cos 2v)}{2}$.
11. If $ux = yz, vy = zx, wz = xy$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.
12. Show that $JJ' = 1$ where $x = e^v\sec u, y = e^v\tan u$.
13. Show that $JJ' = 1$ where $x = uv, y = \frac{u}{v}$.
14. If $x = v^2 + w^2, y = w^2 + u^2, z = u^2 + v^2$, prove that $JJ' = 1$.
15. If $x = u\cos v, y = u\sin v$, show that $\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$.
16. $u = f(x), v = f(x,y), w = f(x,y,z)$, prove that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}$
17. Hence find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ if $u = e^x, v = e^{x+y}, w = e^{x+y+z}$.