The Row and Column Spaces of a Matrix

- Let A be an $m \times n$ matrix, and let S denote the set of n column vectors of A
- A subset $v_1, \ldots, v_k$ of S is *linearly independent* if the only solution to $\sum_{1 \le i \le k} c_i v_i = 0$ is $c_1 = \ldots = c_k = 0$
  - Remark: The empty set of column vectors is linearly independent
- A maximal independent set of column vectors forms a *basis* of the column space
  - Every such basis has the same cardinality, which is equal to the dimension of the column space and defined as the column rank of A
  - The row rank can be defined analogously and is equal to the column rank; we refer to these quantities as the *rank* of A

Matroids

- Introduced by Whitney in 1935, matroids are combinatorial structures that capture the abstract properties of linear independence defined for vector spaces
- A *matroid* M is a pair $(S, \mathcal{I})$ where S is a finite ground set and $\mathcal{I} \subseteq 2^S$ is a nonempty collection of *independent sets* such that the following conditions hold
  - (Hereditary Property) If $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$
  - (Exchange Property) If $X, Y \in \mathcal{I}$ and $|X| < |Y|$, then there exists $y \in Y \setminus X$ such that $X + y \in \mathcal{I}$
- A maximal independent set of M is referred to as a *basis* of M
- Observe that every basis of M has the same cardinality
  - This cardinality is referred to as the *rank* of M

Vector Matroids

- Let S denote the set of column vectors of a given matrix A, and let $\mathcal{I}$ denote the set of linearly independent subsets of S
  - The empty set belongs to $\mathcal{I}$, so $\mathcal{I}$ is nonempty
  - The hereditary property is immediate
  - The exchange property holds since $X, Y \in \mathcal{I}$ and $|X| < |Y|$ implies that at least one column vector $y \in Y$ cannot be represented as a linear combination of the column vectors in X
- Thus $M = (S, \mathcal{I})$ is a matroid
  - The rank of M is equal to the rank of A

A Simpler Example: Uniform Matroids

- Let S be a finite set, let k be a nonnegative integer, and let $\mathcal{I}$ denote the set of all subsets of S of cardinality at most k
- The empty set belongs to $\mathcal{I}$
- The hereditary property is immediate
- Suppose $X, Y \in \mathcal{I}$ and $|X| < |Y|$
  - Thus $Y \setminus X$ is nonempty
  - Let y be an element of $Y \setminus X$
  - Observe that $X + y \in \mathcal{I}$, so the exchange property holds
- We refer to $(S, \mathcal{I})$ as a uniform matroid

Partition Matroids

- Let $S_1, \ldots, S_n$ be a partition of a finite set S, let $k_1, \ldots, k_n$ be nonnegative integers, and let $\mathcal{I}$ denote the set of all subsets I of S such that $|I \cap S_i| \leq k_i$ for $1 \leq i \leq n$
- The empty set belongs to $\mathcal{I}$
- The hereditary property is immediate
- Suppose $X, Y \in \mathcal{I}$ and $|X| < |Y|$
  - Thus there exists an integer i, $1 \leq i \leq n$, such $|X \cap S_i| < |Y \cap S_i|$
  - Let y be an element of $(Y \cap S_i) \setminus (X \cap S_i)$
  - Observe that $X + y$ belongs to $\mathcal{I}$, so the exchange property holds
- Such a matroid is called a partition matroid

Laminar Matroids

- Let S be a finite set, and let $\mathcal{F}$ be a "laminar family" over S
  - That is, $\mathcal{F}$ is a subset of 2^S such that if $X, Y \in \mathcal{F}$ then $X \cap Y = \emptyset$, $X \subseteq Y$, or $Y \subseteq X$
- For each set X in $\mathcal{F}$, let k_X be a nonnegative integer
- Let $\mathcal{I}$ denote the set of all subsets I of S such that $|I \cap X| \leq k_X$ for all $X \in \mathcal{F}$
- Exercise: Prove that $(S, \mathcal{I})$ is a matroid
  - Such a matroid is called a laminar matroid
  - Laminar matroids generalize partition matroids, which generalize uniform matroids

Computing a Basis of a Matroid

- There are generic frameworks for solving various natural computational problems related to matroids
- Example: Suppose we are given a matroid $M = (S, \mathcal{I})$, and we wish to compute a basis X of M
  - Initialize X to $\emptyset$, which belongs to $\mathcal{I}$
  - For each element x of S (in arbitrary order), add x to X if the resulting set belongs to $\mathcal{I}$
  - Observe that if x cannot be added to some intermediate X (without leaving $\mathcal{I}$), then x cannot be added to the final X
  - Thus X is a basis upon termination
- The precise computational complexity of this algorithm depends on the structure of the specific matroid family for which it is implemented

Weighted Matroids

- A *weighted matroid* is a matroid $M = (S, \mathcal{I})$ where each element x of S has an associated weight $w(x)$
- The weight of a subset X of S is defined as $\sum_{x \in X} w(x)$
- Certain optimization problems can be interpreted as asking for a maximum-weight basis of a suitable matroid
- There is a generic framework for computing such a maximum-weight basis
  - The same framework can be used to compute a minimum-weight basis
- It is called the matroid greedy algorithm

The Matroid Greedy Algorithm

- Let $M = (S, \mathcal{I})$ be a weighted matroid with weight function w
- Initialize A to the empty set
- For each element x of S in nonincreasing order of weight (breaking ties arbitrarily), add x to A if $A + x \in \mathcal{I}$
- Upon termination, we claim that A is a maximum-weight basis of M
- Let $S = \{x_1, \ldots, x_n\}$ where x_i is the element processed in iteration i; thus $w(x_1) \geq \ldots \geq w(x_n)$

Proof of Correctness

- Let B be a maximum-weight basis of M, and let i denote the least index such that $x_i \in A \oplus B$
  - Thus $x_i \in A \setminus B$
  - Let A' denote $A \cap \{x_1, \ldots, x_i\}$
  - By repeated application of the exchange property, we can add elements of B to A' until we get a basis $B' = B + x_i - x_j$ for some $j > i$ (so $w(x_j) \leq w(x_i)$)
  - Since B' is a basis and B is a maximum-weight basis, we have $w(x_i) \leq w(x_j)$; thus $w(x_i) = w(x_j)$
  - Thus B' is a maximum-weight basis that agrees with A on $\{x_1, \ldots, x_i\}$, i.e., $(A \oplus B') \cap \{x_1, \ldots, x_i\} = \emptyset$
  - Repeating the above argument, we eventually obtain a maximum-weight basis that is equal to A

Relationship to Kruskal's MST Algorithm

- Note the similarity between (the weight-minimizing version of) the matroid greedy algorithm and Kruskal's MST algorithm
- This suggests that it may be possible to interpret the MST problem as the problem of computing a minimum-weight basis of a suitable matroid
- Let $G = (V, E)$ be a connected, undirected graph where each each e in E has an associated weight $w(e)$
- How can we interpret G as a weighted matroid?
- Since w assigns weights to the edges of G, we should choose the ground set of our matroid to be E
- Which subsets of the edges should we consider to be independent?

Graphic Matroids

- Let $G = (V, E)$ be a connected, undirected graph where each each e in E has an associated weight $w(e)$
- We claim that $(E, \mathcal{I})$ is a matroid, where $\mathcal{I}$ denotes the set of all acyclic subsets of E
  - Note the connection to Kruskal's algorithm, which grows an MST T by starting with the empty set, and adds an edge e to T in a given iteration if $T + e$ remains acyclic
- Let's prove that $(E, \mathcal{I})$ is in fact a matroid
  - The empty set of edges belongs to $\mathcal{I}$
  - The hereditary property is immediate
  - It remains to establish the exchange property

Proof of the Exchange Property

- Let X and Y be two acyclic subsets of E with $|X| < |Y|$
- The number of connected components in the graph $G_X = (V, X)$ (resp., $G_Y = (V, Y)$) is $|V| - |X|$ (resp., $|V| - |Y|$)
- Thus the number of connected components in G_X exceeds the number in G_Y
- Thus some connected component $C = (V', E')$ of G_Y includes vertices drawn from more than one connected component of G_X
- Thus C includes an edge (u, v) such that u and v belong to distinct connected components of G_X
- Edge (u, v) belongs to $Y \setminus X$ and $X + (u, v)$ is acyclic

Other MST Properties that Generalize to Matroids

- Many properties of MSTs generalize to arbtrary weighted matroids
- For example, consider the matroid analogues of some of the properties discussed in the previous lecture
  - Every maximum-weight basis has the same distribution of element weights
  - If all element weights are distinct, there is a unique maximum-weight basis
  - The matroid greedy algorithm can generate every maximum-weight basis
  - The set of maximum-weight bases does not change if we replace each weight x with $f(x)$ for some increasing function f

A Scheduling Problem

- Suppose we are given n unit-time tasks, indexed from 1 to n, to be scheduled on a single resource starting at time zero
- Task i has a nonnegative integer deadline d_i and value v_i
- We say that a schedule for a set of tasks X is *feasible* if no task in X misses its deadline
- We say that a set of tasks is feasible if it admits a feasible schedule
  - It is easy to argue that a set of tasks X is feasible if and only if any earliest deadline (ED) schedule of X is feasible for X
- The value of a set of tasks X is defined as $\sum_{i \in X} v_i$
- We wish to compute a maximum-value feasible set of tasks

A Matroid Interpretation

- Let S denote the set of tasks, and let $\mathcal{I}$ denote the set of all feasible subsets S
- We claim that $(S, \mathcal{I})$ is a matroid
  - The empty set of tasks belongs to $\mathcal{I}$
  - The hereditary property is immediate
  - It remains to prove the exchange property

Proof of the Exchange Property

- Let X and Y belong to $\mathcal{I}$ with $|X| < |Y|$
- Let A be a gap-free feasible schedule for X, let B be an ED schedule for Y, and let y be the rightmost task in B that does not belong to X
- Schedule B ends with $y, x_1, \ldots, x_k$ for some $k \geq 0$ where $\{x_1, \ldots, x_k\} \subseteq X$
    - We refer to the last $k+1$ slots of B as the "source slots", which we index from 1 to $k+1$ (left to right)
    - We refer to the k slots of A containing $x_1, \ldots, x_k$ and the first empty slot of A as the "target slots", which we index from 1 to $k+1$ (left to right)
    - We modify A to obtain a feasible schedule for $X + y$ by mapping the task in source slot i to target slot i, $1 \leq i \leq k+1$

Implications for the Scheduling Problem

- We can obtain a maximum-value feasible set of tasks by implementing the matroid greedy algorithm
- How can we efficiently determine whether the set of tasks $X + x$ obtained by adding a task x to a feasible set X is itself feasible?
- It is easy to check feasibility of the ED schedule of $X + x$ in polynomial time
- Since the number of iterations is n, we can compute a maximum-value feasible set of tasks in polynomial time
- On the upcoming problem set, you will be asked to develop a faster implementation

A Further Generalization

- We can generalize the preceding scheduling problem by introducing a release time r_i for each task i
- A unit-time task with release time r_i and deadline d_i can only be scheduled in one of the unit intervals $[t, t+1]$ for t in $\{r_i, r_i + 1, \ldots, d_i - 1\}$
- It turns out that the set of feasible subsets of the tasks still corresponds to the independent sets of a matroid
- Remark: A fast implementation of the matroid greedy algorithm is known for the resulting matroid

Transversal Matroids

- Let $G = (U, V, E)$ be a bipartite graph
- A *matching* of G is a subset E' of E that induces degree at most one on any vertex
  - The endpoints of the edges in E' are said to be *matched* in E'
- Let $\mathcal{I}$ denote the set of all subsets I of U such that some matching of G matches every vertex in I
- We claim that $(U, \mathcal{I})$ is a matroid
  - The empty set of vertices belongs to $\mathcal{I}$
  - The hereditary property is immediate
  - It remains to prove the exchange property

Proof of the Exchange Property

- Let X and Y belong to $\mathcal{I}$ with $|X| < |Y|$
- There exists a matching A (resp., B) that matches X (resp., Y) and such that $|A| = |X|$ (resp., $|B| = |Y|$)
- The subgraph with edge set $A \oplus B$ consists of a collection of disjoint (simple) paths and even-length cycles
- Since $|A| < |B|$, there is at least one odd-length path P that begins and ends with an edge in B
  - Since the length of P is odd, it has one endpoint in U, call it y, this is not matched in A
  - Observe that the matching $A \oplus P$ matches all of the vertices in $X + y$

Connection to the Scheduling Matroids Considered Earlier

- We can think of U as the set of unit-time tasks to be scheduled
- We can think of V as the set of time slots
- We include an edge (u, v) in E if we are allowed to schedule task u in slot v
- We can perform the feasibility check associated with a single iteration of the matroid greedy algorithm in $O(|E|)$ time
  - The idea is to maintain a matching A for the current feasible subset X of U
  - To check whether $X + x$ remains feasible for some $x \in U \setminus X$, we can use alternating breadth-first search to search for an alternating path P from x to an unmatched node in V
  - If so, we can add x to X and update A to $A \oplus P$

Matching Matroids

- Let $G = (V, E)$ be a graph
- Let $\mathcal{I}$ denote the set of all subsets X of V such that some matching of G matches all of the vertices in X
- We claim that $(V, \mathcal{I})$ is a matroid
  - The empty set of vertices belongs to $\mathcal{I}$
  - The hereditary property is immediate
  - Exercise: Prove the exchange property
- Such a matroid is called a matching matroid

The Circuits of a Matroid

- Let $M = (S, \mathcal{I})$ be a matroid
- A *circuit* of M is a minimal dependent set
  - That is, a subset C of S such that $C \notin \mathcal{I}$ and $C - x \in \mathcal{I}$ for all x in C
- For a graphic matroid, a minimal dependent set is a (simple) cycle
- Recall the following property of spanning trees
  - If T is a spanning tree of $G = (V, E)$ and $e \in E \setminus T$, then $T + e$ contains a unique (simple) cycle
- A generalization of this claim holds for arbitrary matroids

A Property of Matroids

- Lemma: If X is a basis of matroid $M = (S, \mathcal{I})$ and $x \in S \setminus X$, then there is a unique circuit in $Y = X + x$
- Let C denote the set of all y in Y such that $Y - y \in \mathcal{I}$
- Claims 1 and 2 below imply that C is a circuit, while Claim 3 guarantees uniqueness
  - Claim 1: $C \notin \mathcal{I}$
  - Claim 2: $C - y \in \mathcal{I}$ for all y in C
  - Claim 3: If $Z \notin \mathcal{I}$ and $Z \subseteq Y$, then $C \subseteq Z$
- On the upcoming problem set, you will be asked to prove Claims 1, 2, and 3