Linear Programming

- Maximize (or minimize) a linear function subject to linear constraints
- One canonical form asks us to maximize $c^\intercal x$ subject to $Ax \leq b$ and $x \geq 0$ where
  - A is a given $m \times n$ matrix
  - b is a given $m \times 1$ column vector
  - c is a given $n \times 1$ column vector
  - x is an $n \times 1$ column vector of variables

An Example of a Linear Program

- Maximize $3x_1 + 2x_2$ subject to $x_1 + x_2 \leq 10$, $0 \leq x_1 \leq 8$, and $0 \leq x_2 \leq 5$
  - What are the corresponding m, n, A, b, and c?
  - Using a geometric interpretation, we can easily determine that an optimal solution is $(x_1, x_2) = (8, 2)$
  - The associated optimal value of the objective function is 28
- This linear program is feasible and has a finite optimal objective function value
- Some linear program are infeasible, for example, maximize x_1 subject to $x_1 \leq -5$ and $x_1 \geq 0$
- Some linear programs are feasible and unbounded, for example, maximize x_1 subject to $x_1 \geq 0$

Another Canonical Form

- Other canonical forms are possible, for example "maximize $c^\intercal x$ subject to $Ax = b$ and $x \geq 0$"
- It is easy to take an LP in the above form and rewrite it in the form of the previous slide
  - We can write each equality constraint as a pair of inequalities
- Given an LP in the form of the previous slide, we can rewrite it in the above form by introducing so-called "slack" variables
  - For example, the constraint $x_1 + x_2 \leq 7$ where x_1 and x_2 are nonnegative can be rewritten as $x_1 + x_2 + x_3 = 7$ where x_1, x_2, and x_3 are nonnegative

Example: The Maximum Flow Problem

- We can formulate the maximum flow problem as a linear program
- Let $G = (V, E)$ be a given flow network
- We wish to maximize the net flow out of s subject to the capacity and conservation constraints
  - Our LP will have a nonnegative variable x_e for each edge e in E that represents the flow assigned to edge e
  - It is easy to express the capacity and conservation constraints as linear constraints
  - The objective function can be expressed as $(\sum_{(s,v)\in E} x_{(s,v)}) - (\sum_{(v,s)\in E} x_{(v,s)})$

Other Network Flow Problems

- A number of generalizations of the maximum flow problem have been studied in the literature
- The min-cost flow problem introduces edge-specific costs (per unit of flow)
  - In this setting, we may seek a minimum cost flow routing a specified number of units from the source to the sink
- The multicommodity flow problem introduces multiple source-sink pairs s_i-t_i, and we seek to route a specified amount d_i of flow of commodity i from s_i to t_i
  - In this setting, the capacity of an edge bounds the total flow (over all commodities) that can traverse the edge
- The min-cost flow and multicommodity flow problems are easy to express as linear programs

The Computational Complexity of Linear Programming

- Dantzig developed the simplex algorithm in the 1940s
  - Tends to run rapidly on practical instances, but the worst-case time complexity is exponential
  - Certain variants are known to have polynomial "smoothed complexity"
- Khachiyan's ellipsoid algorithm (1979) is the first polynomial-time algorithm for linear programming
  - Prior to this work, it was a major open problem whether linear programming could be solved in polynomial time
- Karmarkar's algorithm (1984) also runs in polynomial time
  - Falls within the class of interior point methods

The Dual of a Linear Program

- Every linear program has a dual, which is also a linear program
- If a linear program is given in the canonical form "maximize $c^\intercal x$ subject to $Ax \leq b$ and $x \geq 0$", then its dual can be written as "minimize $y^\intercal b$ subject to $A^\intercal y \geq c$ and $y \geq 0$"
  - Assume as before that A has m rows and n columns
  - Then y is an $m \times 1$ column vector of dual variables
- Often we formulate an initial linear program, and then form its dual
  - When discussing such a pair of linear programs, we sometimes refer to the initial linear program as the "primal"

A Primal-Dual Pair of Linear Programs

- Assume that the primal is given in the canonical form "maximize $c^\intercal x$ subject to $Ax \leq b$ and $x \geq 0$"
- Thus the dual can be written as "minimize $y^\intercal b$ subject to $A^\intercal y \geq c$ and $y \geq 0$"
  - Ignoring the nonnegativity constraints $x \geq 0$, the primal has m constraints: $\sum_{1 \leq j \leq n} a_{i,j} x_j \leq b_i$ for $1 \leq i \leq m$
  - The dual has one variable y_i for each primal constraint
  - Likewise, for $1 \leq j \leq n$, the primal has a variable x_j for each dual constraint $\sum_{1 \leq i \leq m} a_{i,j} y_i \geq c_j$

What is the Dual of the Dual?

- The dual can be rewritten in our canonical form as "maximize $(-b)^{\intercal}y$ subject to $-A^{\intercal}y \leq (-c)$ and $y \geq 0$"
- Thus the dual of the dual can be written as "minimize $(-c)^{\intercal}x$ subject to $(-A)x \geq -b$ and $x \geq 0$"
- The latter LP is equivalent to "maximize $c^{\intercal}x$ subject to $Ax \leq b$ and $x \geq 0$", i.e., the primal
- Thus the dual of the dual is the primal
- When working with a primal-dual pair of linear programs, it doesn't really matter which one we designate as the primal

Weak Duality Theorem

- Primal: Maximize $c^\intercal x$ subject to $Ax \leq b$ and $x \geq 0$
- Dual: Minimize $y^\intercal b$ subject to $A^\intercal y \geq c$ and $y \geq 0$
- Theorem: If x is feasible for the primal and y is feasible for the dual, then $c^\intercal x \leq y^\intercal b$
    - $A^\intercal y \geq c$ implies $y^\intercal A \geq c^\intercal$
    - Since $x \geq 0$, we have $y^\intercal Ax \geq c^\intercal x$
    - Since $Ax \leq b$, we have $y^\intercal Ax \leq y^\intercal b$

Some Consequences of Weak Duality

- Primal: Maximize $c^\intercal x$ subject to $Ax \leq b$ and $x \geq 0$
- Dual: Minimize $y^\intercal b$ subject to $A^\intercal y \geq c$ and $y \geq 0$
- Weak Duality Theorem: If x is feasible for the primal and y is feasible for the dual, then $c^\intercal x \leq y^\intercal b$
- Corollary: If x is feasible for the primal, y is feasible for the dual, and $c^\intercal x = y^\intercal b$, then x is optimal for the primal and y is optimal for the dual
- Corollary: If the primal is feasible and unbounded, then the dual is infeasible
- Remark: It is possible for both the primal and dual to be infeasible

Strong Duality Theorem

- Primal: Maximize $c^\intercal x$ subject to $Ax \leq b$ and $x \geq 0$
- Dual: Minimize $y^\intercal b$ subject to $A^\intercal y \geq c$ and $y \geq 0$
- Theorem: If either (1) the primal and dual are feasible, or (2) the primal is feasible and bounded, then the primal and dual are feasible and bounded and have the same optimal objective function value
- The proof of the strong duality theorem is beyond the scope of this course

Possible Combinations of Primal-Dual Pairs

- Any linear program is feasible and bounded, feasible and unbounded, or infeasible
- Which of the $3^2 = 9$ combinations of categories are possible for a primal-dual pair of linear programs?
  - If the primal is feasible and bounded, then the dual is feasible and bounded
  - If the primal is feasible and unbounded, then the dual is infeasible
  - If the primal is infeasible, then the dual is either feasible and unbounded or infeasible

Example: Dual of the Maximum Flow LP

- The maximum flow LP can be expressed as "maximize $(\sum_{(s,v)\in E} x_{(s,v)}) - (\sum_{(v,s)\in E} x_{(v,s)})$ subject to $0 \leq x_e \leq c(e)$ for all e in E, and $\sum_{(u,v)\in E} x_{(u,v)} = \sum_{(v,u)\in E} x_{(v,u)}$ for all v in $V \setminus \{s,t\}$"
- We can rewrite this in our canonical form by expressing each of the inequality constraints using two inequality constraints
- We can then mechanically form the dual, which has a dual variable α_e for each edge e in E (corresponding to the capacity constraint on edge e) and dual variables β_v and γ_v for each vertex v in $V \setminus \{s,t\}$ (corresponding to the two inequalities expressing flow conservation at v)

Dual of the Maximum Flow LP (cont'd)

- The resulting dual LP seeks to minimize $\sum_{e \in E} \alpha_e c(e)$ subject to the following constraints:
  - $\alpha_{(u,v)} + \beta_u - \gamma_u - (\beta_v - \gamma_v) \geq 0$ for all (u, v) in E such that $\{u, v\} \cap \{s, t\} = \emptyset$
  - $\alpha_{(s,v)} - (\beta_v - \gamma_v) \geq 1$ for all (s, v) in $E \setminus \{(s, t)\}$
  - $\alpha_{(u,s)} + \beta_u - \gamma_u \geq -1$ for all (u, s) in $E \setminus \{(t, s)\}$
  - $\alpha_{(t,v)} - (\beta_v - \gamma_v) \geq 0$ for all (t, v) in $E \setminus \{(t, s)\}$
  - $\alpha_{(u,t)} + \beta_u - \gamma_u \geq 0$ for all (u, t) in $E \setminus \{(s, t)\}$
  - $\alpha_e \geq 1$ for all e in $E \cap \{(s, t)\}$
  - $\alpha_e \geq -1$ for all e in $E \cap \{(t, s)\}$
  - $\alpha \geq 0$
  - $\beta \geq 0$
  - $\gamma \geq 0$

Dual of the Maximum Flow LP (cont'd)

- We can get a simpler, equivalent formulation of the dual by introducing the unrestricted variable $\delta_v = \gamma_v - \beta_v$ for each v in $V \setminus \{s, t\}$, along with variables $\delta_s = 1$ and $\delta_t = 0$
- The dual constraints simplify to $\alpha_{(u,v)} + \delta_v - \delta_u \geq 0$ for all (u, v) in E, $\alpha \geq 0$, $\delta_s = 1$, and $\delta_t = 0$
- Claim 1: There is an optimal solution where $\delta_v \leq 1$ for all v in V
  - Fix an optimal solution α, δ
  - Let δ' be defined by $\delta'_v = \min(1, \delta_v)$ for all v in V
  - For any edge (u, v), if $\delta'_v < \delta_v$ then $\delta'_v = 1 \geq \delta'_u$ and hence $\alpha_{(u,v)} + \delta'_v - \delta'_u \geq 0$
  - It follows that α, δ' is an optimal solution

Dual of the Maximum Flow LP (cont'd)

- Claim 2: There is an optimal solution where $0 \leq \delta_v \leq 1$ for all v in V
  - Fix an optimal solution α, δ such that $\delta_v \leq 1$ for all v in V
  - Let δ' be defined by $\delta'_v = \max(0, \delta_v)$ for all v in V
  - For any edge (u, v), if $\delta'_u > \delta_u$ then $\delta'_u = 0 \leq \delta'_v$ and hence $\alpha_{(u,v)} + \delta'_v - \delta'_u \geq 0$
  - It follows that α, δ' is an optimal solution
- On the next problem set, we will establish that there is an optimal solution in which all of the $\alpha_{(u,v)}$'s and all of the δ_v's belong to $\{0, 1\}$

Dual of the Maximum Flow LP (cont'd)

- Thus the dual of the maximum flow LP is equivalent to "minimize $\sum_{e \in E} \alpha_e c(e)$ subject to $\alpha_{(u,v)} + \delta_v - \delta_u \geq 0$ for all (u, v) in E, $\alpha_e \in \{0, 1\}$ for all e in E, $\delta_v \in \{0, 1\}$ for all v in V, $\delta_s = 1$, and $\delta_t = 0$"
- Since the edge capacities are nonnegative, for any optimal solution α, δ we can get another optimal solution α', δ by setting $\alpha'_{(u,v)}$ to $\max(0, \delta_u - \delta_v)$ for all (u, v) in E
- Thus the dual of the maximum flow LP is equivalent to "minimize $\sum_{e=(u,v) \in E} \max(0, \delta_u - \delta_v) \cdot c(e)$ subject to $\delta_v \in \{0, 1\}$ for all v in V, $\delta_s = 1$, and $\delta_t = 0$"

Dual of the Maximum Flow LP (cont'd)

- The dual of the maximum flow LP is equivalent to "minimize $\sum_{(u,v)\in E} \max(0, \delta_u - \delta_v) \cdot c(e)$ subject to $\delta_v \in \{0,1\}$ for all v in V, $\delta_s = 1$, and $\delta_t = 0$"
- We can think of a feasible δ as encoding the cut (S, T) where $S = \{v \in V \mid \delta_v = 1\}$ and $T = \{v \in V \mid \delta_v = 0\}$
  - The value of the objective function is equal to the total capacity of all edges from S to T, i.e., the capacity of cut (S, T)
- Thus the dual of the maximum flow LP is equivalent to the minimum-capacity cut problem
- Remark: The strong duality theorem of linear programming provides an alternate proof of the max-flow min-cut theorem

Complementary Slackness

- Theorem: Feasible solutions x and y for the primal and dual are optimal if and only if (1) for each non-tight primal constraint, the corresponding dual variable is zero, and (2) for each non-tight dual constraint, the corresponding primal variable is zero
- "If" direction
  - Assume x and y are feasible solutions for the primal and dual satisfying conditions (1) and (2)
  - Condition (1) implies $y^\intercal(b - Ax) = 0$ since $y \geq 0$ and $Ax \leq b$; hence $y^\intercal Ax = y^\intercal b$
  - Condition (2) implies $(y^\intercal A - c^\intercal)x = 0$ since $y^\intercal A - c^\intercal \geq 0$ and $x \geq 0$; hence $y^\intercal Ax = c^\intercal x$
  - Hence $c^\intercal x = y^\intercal b$, so x and y are optimal for the primal and dual

Complementary Slackness (cont'd)

- Theorem: Feasible solutions x and y for the primal and dual are optimal if and only if (1) for each non-tight primal constraint, the corresponding dual variable is zero, and (2) for each non-tight dual constraint, the corresponding primal variable is zero
- "Only if" direction
  - Assume x and y are optimal solutions for the primal and dual
  - We have $c^\intercal x \leq y^\intercal Ax \leq y^\intercal b$ and so by strong duality $c^\intercal x = y^\intercal Ax = y^\intercal b$
  - Hence $y^\intercal(b - Ax) = 0$ and (1) follows since $y \geq 0$ and $b - Ax \geq 0$
  - Similarly, $(y^\intercal A - c^\intercal)x = 0$ and (2) follows since $y^\intercal A - c^\intercal \geq 0$ and $x \geq 0$

Example: Minimum-Cost Perfect Matching

- In the previous lecture, we studied the minimum-cost perfect matching problem on a bipartite graph $G = (V, E)$ where V has associated bipartition (X, Y) where $|X| = |Y| = n$
- The iterative algorithm that we presented maintains a matching M together with compatible prices p such that the following conditions hold
  - For each edge $e = (i, j)$ in $E \setminus M$, the reduced cost $p(i) + c_e - p(j)$ is nonnegative
  - For each edge $e = (i, j)$ in M, the reduced cost $p(j) - c_e - p(i)$ is zero
  - The vertex prices are nonnegative, and are zero for any unmatched vertices in X
- How might one come to consider such prices?

Minimum-Cost Perfect Matching: LP Formulation

- Today we will show how to use LP duality to deduce the existence of compatible prices with respect to a minimum-cost matching M
- Consider the LP that minimizes $\sum_{e \in E} c_e x_e$ subject to the following constraints
  - For any vertex i in X, we have $\sum_{j:(i,j)\in E} x_{(i,j)} = 1$
  - For any vertex j in Y, we have $\sum_{i:(i,j)\in E} x_{(i,j)} = 1$
  - For any edge e in E, we have $x_e \geq 0$
- On the next assignment, you will prove that if the above LP is feasible, then it has an integral (and hence 0-1) optimal solution

Dual of the Minimum-Cost Perfect Matching LP

- By similar reasoning as we used to form the dual of the network flow LP, we can deduce that the dual of the minimum-cost perfect matching LP is equivalent to maximizing $\sum_{i \in X} \alpha_i + \sum_{j \in Y} \alpha_j$ subject to the following constraints
  - For any edge $e = (i, j)$ in E, we have $\alpha_i + \alpha_j \leq c_e$
  - The α_i's and α_j's are unrestricted
- Let α^* be an optimal solution for the dual
- For any real Δ, we can obtain another optimal solution β^* as follows
  - For any vertex i in X, set β_i^* to $\alpha_i^* - \Delta$
  - For any vertex j in Y, set β_j^* to $\alpha_j^* + \Delta$

Reformulating the Dual

- The dual of the minimum-cost perfect matching LP is equivalent to maximizing $\sum_{i \in X} \beta_i + \sum_{j \in Y} \beta_j$ subject to the following constraints
  - For any edge $e = (i, j)$ in E, we have $\beta_i + \beta_j \leq c_e$
  - For any vertex i in X, we have $\beta_i \leq 0$
  - For any vertex j in Y, we have $\beta_j \geq 0$
- This is equivalent to maximizing $-\sum_{i \in X} \gamma_i + \sum_{j \in Y} \gamma_j$ subject to the following constraints
  - For any edge $e = (i, j)$ in E, we have $\gamma_i + c_e - \gamma_j \geq 0$
  - The γ_i's and γ_j's are nonnegative

Existence of (Nonnegative) Compatible Prices

- Let M be a minimum-cost perfect matching of G, and let x be a corresponding 0-1 optimal solution to the primal
- By strong duality, there exists an optimal solution γ to the latter dual formulation such that the dual objective is equal to the cost of M
  - Interpreting γ_v as the price of vertex v, we have nonnegative prices such that all of the reduced costs are nonnegative
- Complementary slackness implies that if $x_e > 0$, then the dual constraint $\gamma_i + c_e - \gamma_j \geq 0$ associated with edge $e = (i, j)$ is tight
  - Thus the reduced cost of each edge in M is zero
- Since M is a perfect matching, there are no unmatched vertices in X

Example: Two-Player Zero-Sum Games

- An $m \times n$ payoff matrix A is known to both players
- The row player choose a row index i, $1 \leq i \leq m$
- Simultaneously, the column player chooses a column index j, $1 \leq j \leq n$
- The row player pays $a_{i,j}$ to the column player
- The well-known “rock-paper-scissors” game has this form
- What is an optimal mixed strategy for the column (resp., row) player?

Maximin Strategy of the Column Player

- Under the maximin strategy, the column player seek a mixed strategy x (where x_j represents the probability of playing column j) that maximizes the minimum expected payoff to the column player
  - Maximize $\min_{1 \leq i \leq m} \sum_{1 \leq j \leq n} a_{i,j} x_j$ subject to $\sum_{1 \leq j \leq n} x_j = 1$ and $x \geq 0$
  - This can be rewritten as "maximize α where $\sum_{1 \leq j \leq n} a_{i,j} x_j \geq \alpha$ for $1 \leq i \leq m$, $\sum_{1 \leq j \leq n} x_j = 1$, $x \geq 0$, and α is unrestricted"
  - We can write this LP in our canonical form as "maximize $\alpha' - \alpha''$ where $[\sum_{1 \leq j \leq n} (-a_{i,j}) x_j] + \alpha' - \alpha'' \leq 0$ for $1 \leq i \leq m$, $\sum_{1 \leq j \leq n} x_j \leq 1$, $\sum_{1 \leq j \leq n} -x_j \leq -1$, $x \geq 0$, $\alpha' \geq 0$, and $\alpha'' \geq 0$"

The Dual of the Maximin Strategy of the Column Player

- Mechanically forming the dual of the preceding LP, we obtain "Minimize $\beta' - \beta''$ where $\sum_{1 \le i \le m}(-a_{i,j})y_i + \beta' - \beta'' \ge 0$ for $1 \le j \le n$, $\sum_{1 \le i \le m} y_i \ge 1$, $\sum_{1 \le i \le m} -y_i \ge -1$, $y \ge 0$, $\beta' \ge 0$, and $\beta'' \ge 0$"
- The dual can be rewritten as "minimize β where $\sum_{1 \le i \le m} a_{i,j} y_i \le \beta$ for $1 \le j \le n$, $\sum_{1 \le i \le m} y_i = 1$, $y \ge 0$, and β is unrestricted"
- This is in turn equivalent to "minimize $\max_{1 \le j \le n} \sum_{1 \le i \le m} a_{i,j} y_i$ subject to $\sum_{1 \le i \le m} y_i = 1$ and $y \ge 0$"
- Thus the dual of the maximin strategy of the column player corresponds to the minimax strategy of the row player

The Maximin/Minimax Strategies Yield an Equilibrium

- Assume that the column (resp., row) player chooses a mixed strategy given by an optimal solution to the maximin (resp., minimax) LP
- By strong duality, the optimal objective functions values are both equal to the same value, call it λ
- Suppose the column player publicly commits in advance to the maximin strategy
  - By the definition of the maximin strategy, the expected transfer to the column player is at least λ, no matter what strategy is employed by the row player
  - Thus the minimax strategy is a best response for the row player, since it results in an expected transfer of λ
- Similarly, the column player has no incentive to deviate from maximin if the row player commits to minimax

Properties of the Maximin/Minimax Equilibrium

- What do the complementary slackness conditions tell us about the maximin/minimax equilibrium for a two-player zero-sum game?
  - Let C be the set of columns with positive probability in the maximin strategy of the column player, and let R be the set of rows with positive probability in the minimax strategy of the row player
  - If the column player plays a column in C, and the row player plays the minimax strategy, then the expected transfer is exactly λ
  - If the row player plays a row in R, and the column player plays the maximin strategy, then the expected transfer is exactly λ