Chapter 5
Statistical Models in Simulation

Banks, Carson, Nelson & Nicol

Discrete-Event System Simulation

Purpose & Overview

- The world the model-builder sees is probabilistic rather than deterministic.
  - Some statistical model might well describe the variations.

- An appropriate model can be developed by sampling the phenomenon of interest:
  - Select a known distribution through educated guesses
  - Make estimate of the parameter(s)
  - Test for goodness of fit

- In this chapter:
  - Review several important probability distributions
  - Present some typical application of these models

Review of Terminology and Concepts

- In this section, we will review the following concepts:
  - Discrete random variables
  - Continuous random variables
  - Cumulative distribution function
  - Expectation

Discrete Random Variables

- *X* is a discrete random variable if the number of possible values of *X* is finite, or countably infinite.
- Example: Consider jobs arriving at a job shop.
  - Let *X* be the number of jobs arriving each week at a job shop.
  - R_x = possible values of *X* (range space of *X*) = $\{0,1,2,\ldots\}$
  - $p(x_i)$ = probability the random variable is $x_i = P(X = x_i)$
  - $p(x_i), i = 1,2, \ldots$ must satisfy:

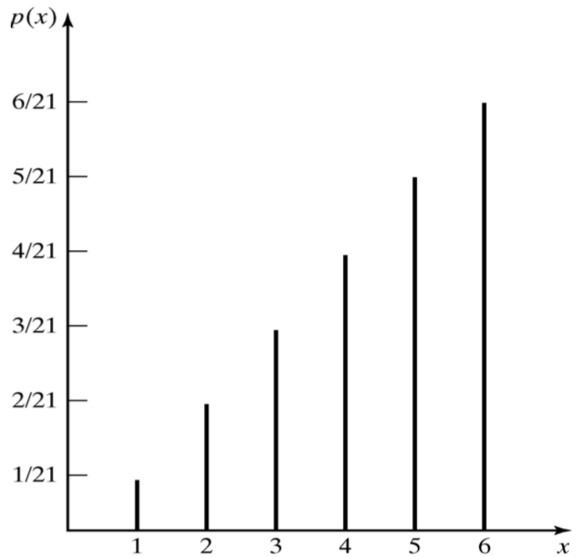
    1. $p(x_i) \geq 0, \text{ for all } i$
    2. $\sum_{i=1}^{\infty} p(x_i) = 1$

  - The collection of pairs $[x_i, p(x_i)], i = 1,2,\ldots,$ is called the probability distribution of *X*, and $p(x_i)$ is called the probability mass function (pmf) of *X*.

Discrete Random Variables

- Example: Assume the die is loaded so that the probability that a given face lands up is proportional to the number of spots showing.

x_i	1	2	3	4	5	6
$P(x_i)$	1/21	2/21	3/21	4/21	5/21	6/21



- *$p(x_i)$, $i = 1,2, \ldots$* must satisfy:

  1. $p(x_i) \geq 0$, for all i
  2. $\sum_{i=1}^{\infty} p(x_i) = 1$

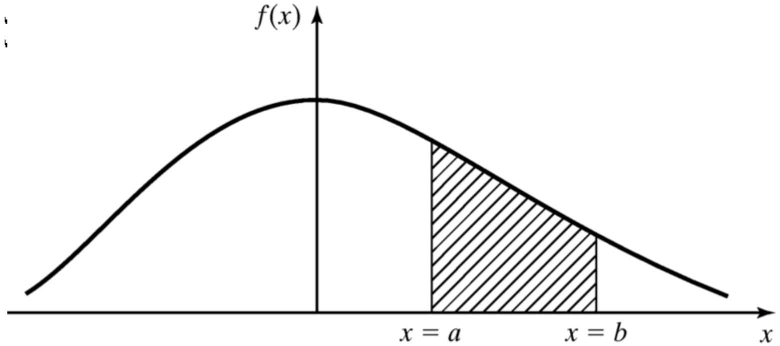
Continuous Random Variables

- X is a continuous random variable if its range space R_X is an interval or a collection of intervals.
- The probability that X lies in the interval *[a,b]* is given by:

$$P(a \le X \le b) = \int_a^b f(x)dx$$

- *f(x)*, denoted as the pdf of *X*, satisfies:

  1. $f(x) \ge 0$, for all x in R_X
  2. $\int_{R_X} f(x)dx = 1$
  3. $f(x) = 0$, if x is not in R_X



- Properties

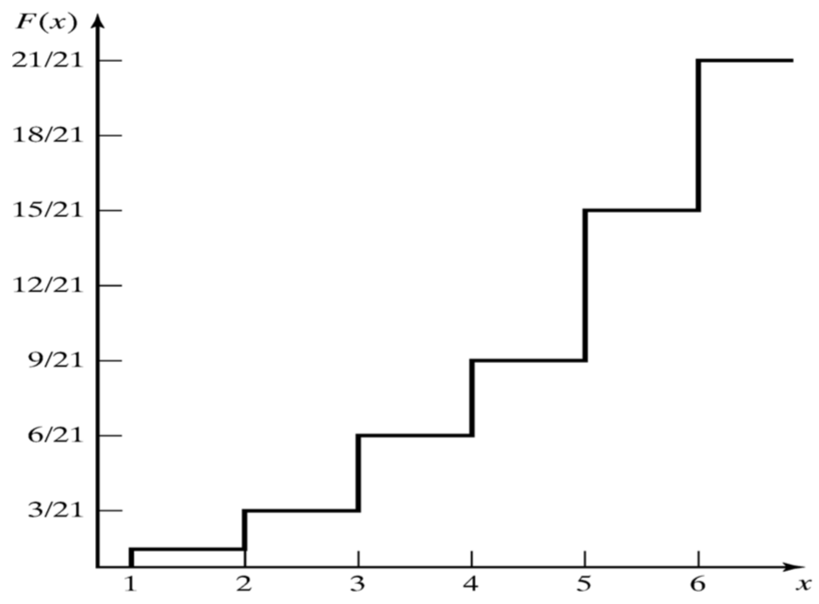
  1. $P(X = x_0) = 0$, because $\int_{x_0}^{x_0} f(x)dx = 0$
  2. $P(a \le X \le b) = P(a < X \le b) = P(a \le X < b) = P(a < X < b)$

Continuous Random Variables

- Example: The die-tossing experiment described in last example has a cdf given as follows:

x	(-∞,1)	[1,2)	[2,3)	[3,4)	[4,5)	[5,6)	[6,∞)
F(x)	0	1/21	3/21	6/21	10/21	15/21	21/21

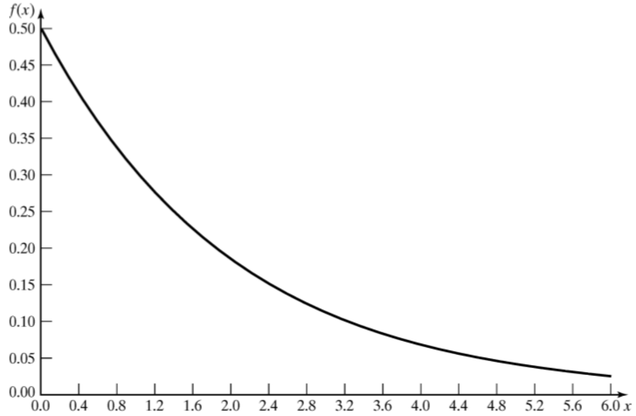
  - [a,b) = {a ≤ x ≤ b}



Continuous Random Variables

- Example: Life of an inspection device is given by *X*, a continuous random variable with pdf:

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



  - *X* has an exponential distribution with mean 2 years
  - Probability that the device's life is between 2 and 3 years is:

$$P(2 \leq x \leq 3) = \frac{1}{2} \int_2^3 e^{-x/2} dx = 0.14$$

Continuous Distributions [Probability Review]

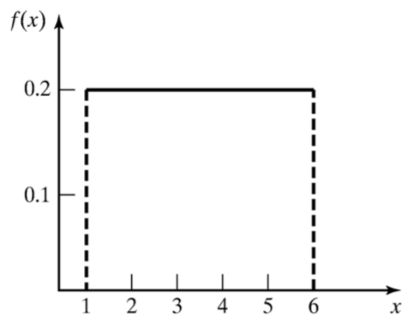
- A random variable *X is* uniformly distributed on the interval *(a, b)* if its PDF is given by

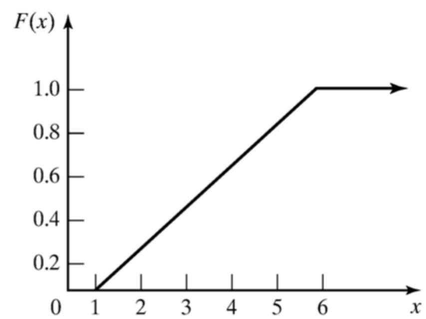
$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

- The CDF is given by

$$F(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x < b \\ 1, & x \ge b \end{cases}$$

- The PDF and CDF when *a=1* and *b=6:*

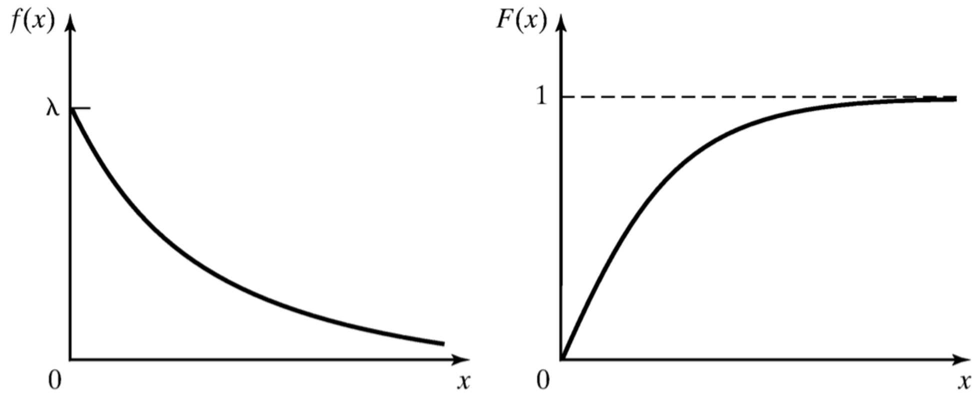




Exponential Distribution [Probability Review]

- A random variable X is said to be exponentially distributed with parameter $\lambda > 0$ if its PDF is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



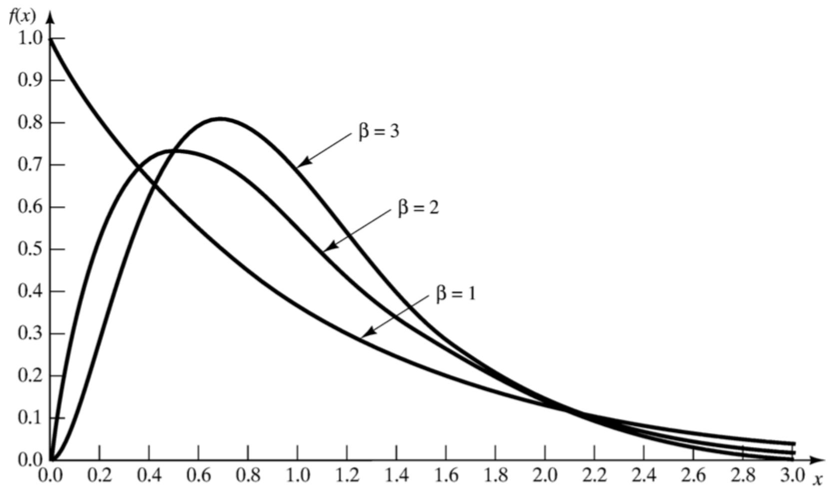
Gamma Distribution [Probability Review]

- A function used in defining the gamma distribution is the gamma function, which is defined for all $\beta > 0$ as

$$\Gamma(\beta) = \int_0^{\infty} x^{\beta-1} e^{-x} dx$$

- A random variable X is gamma distributed with parameters β and θ if its PDF is given by

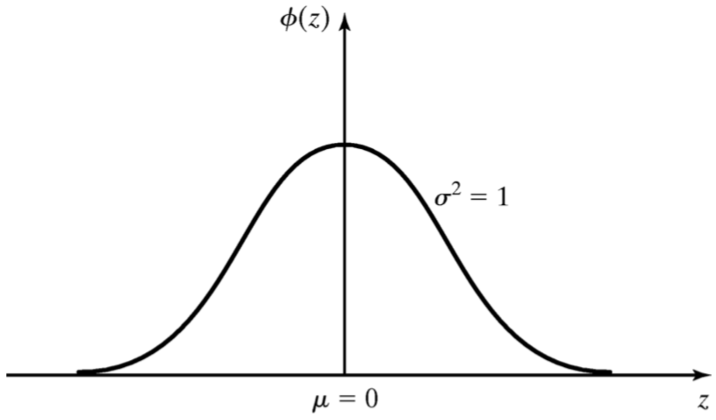
$$f(x) = \begin{cases} \dfrac{\beta\theta}{\Gamma(\beta)} (\beta\theta x)^{\beta-1} e^{-\beta\theta x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$



Normal Distribution [Probability Review]

- A random variable *X* with mean $-\infty < x < \infty$ and variance $\sigma^2 > 0$ has a normal distribution if it has the PDF

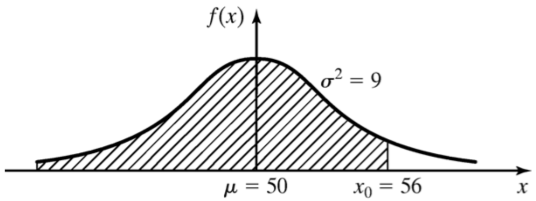
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], -\infty < x < \infty$$



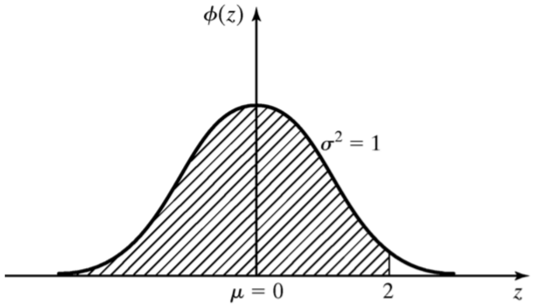
Normal Distribution [Probability Review]

- Example: Suppose that *X ~ N (50, 9).*

F(56) = $\Phi(\frac{56-50}{3}) = \Phi(2) = 0.9772$



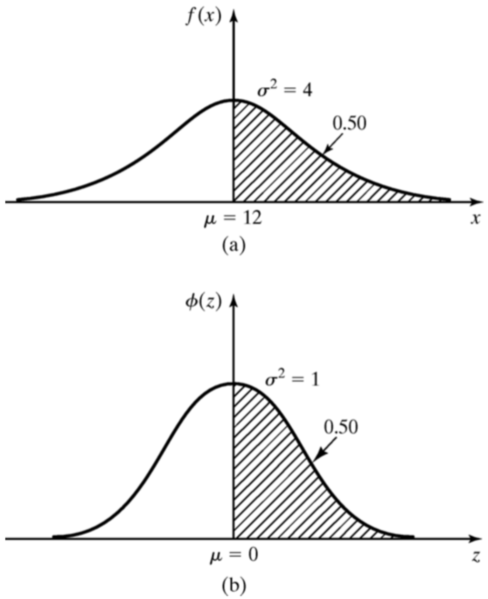
(a)



(b)

Normal Distribution [Probability Review]

- Example: The time in hours required to load a ship, *X*, is distributed as *N(12, 4)*. The probability that 12 or more hours will be required to load the ship is:



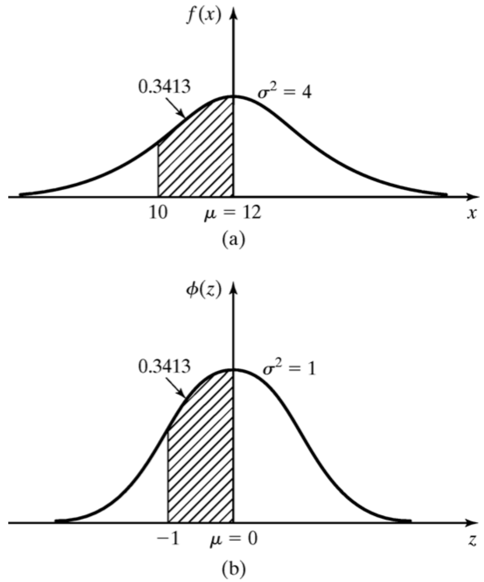
$$P(X > 12) = 1 - F(12) = 1 - 0.50 = 0.50$$

(The shaded portions in both figures)

Normal Distribution [Probability Review]

- Example (cont.):

  The probability that between 10 and 12 hours will be required to load a ship is given by

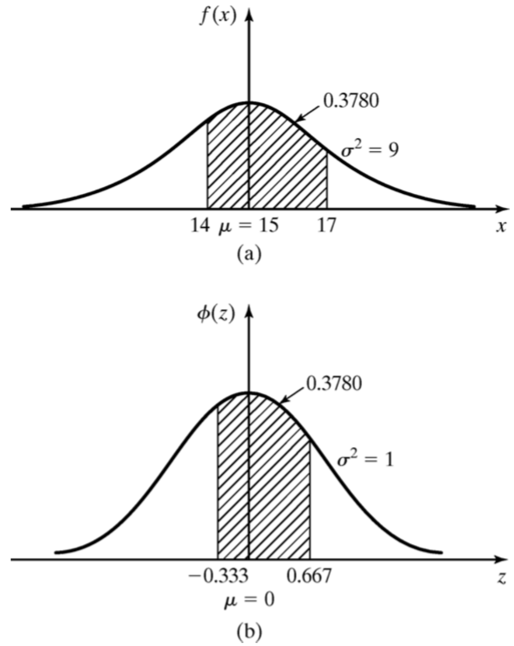


$$P(10 \leq X \leq 12) = F(12) - F(10) = 0.5000 - 0.1587 = 0.3413$$

The area is shown in shaded portions of the figure

Normal Distribution [Probability Review]

- Example: The time to pass through a queue is *N(15, 9)*. The probability that an arriving customer waits between 14 and 17 minutes is:



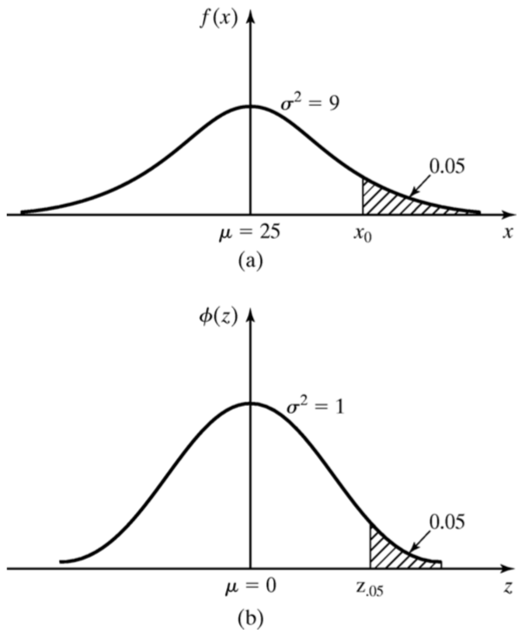
$$P(14 \le X \le 17) = F(17) - F(14) =$$

$$\Phi\left(\frac{17-15}{3}\right) - \Phi\left(\frac{14-15}{3}\right) = \Phi(0.667) - \Phi(-0.333) = 0.7476 - 0.3696 = 0.3780$$

Normal Distribution [Probability Review]

- Example: Lead-time demand, *X*, for an item is *N(25, 9)*.

  Compute the value for lead-time that will be exceeded only *5%* of time.



$$P(X > x_0) = P(Z > \frac{x_0 - 25}{3}) = 1 - \Phi(\frac{x_0 - 25}{3}) = 0.05$$

$$\frac{x_0 - 25}{3} = 1.645$$

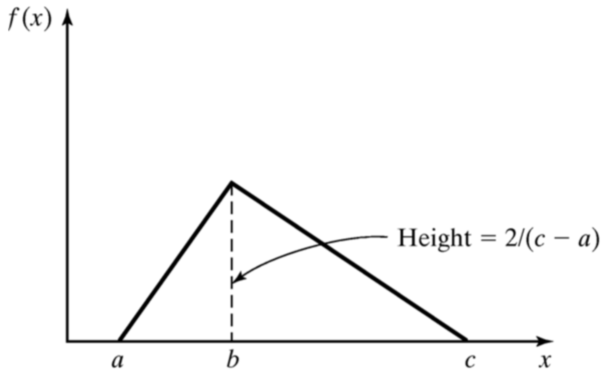
$$x_0 = 29.935$$

Triangular Distribution

- A random variable *X* has a triangular distribution if its PDF is given by

$$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(c-a)}, & a \le x \le b \\[2ex] \dfrac{2(c-x)}{(c-b)(c-a)}, & b < x \le c \\[2ex] 0, & \text{elsewhere} \end{cases}$$

where $a \le b \le c$.

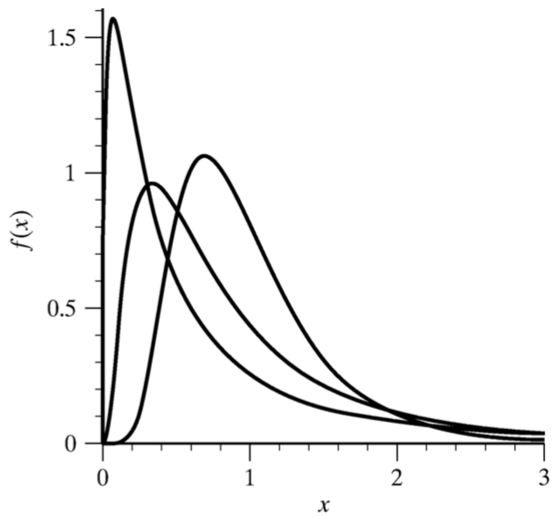


Lognormal Distribution

- A random variable *X* has a lognormal distribution if its PDF is given by

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\sigma x} \exp[-\dfrac{(\ln x - \mu)^2}{2\sigma^2}], & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\sigma^2 > 0$.



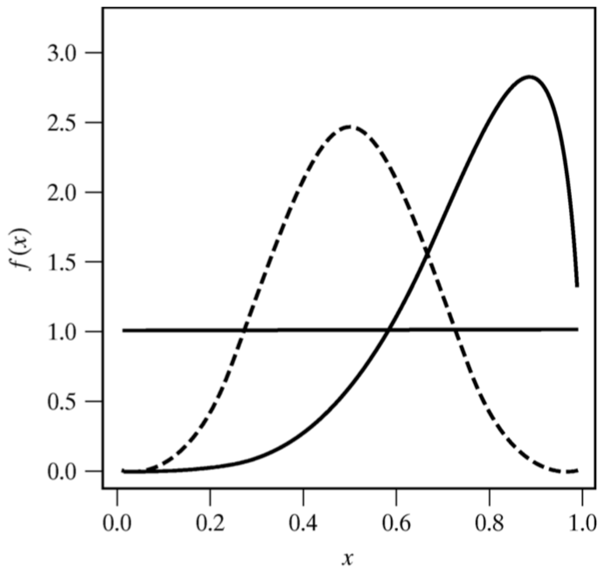
Beta Distribution

- A random variable X is beta-distributed with parameters $\beta_1 > 0$ and $\beta_2 > 0$ if its PDF is given by

$$f(x) = \begin{cases} \dfrac{x^{\beta_1 - 1}(1-x)^{\beta_2 - 1}}{B(\beta_1, \beta_2)}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

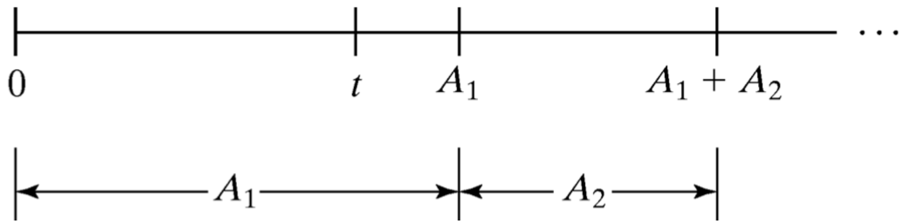
where

$$B(\beta_1, \beta_2) = \Gamma(\beta_1)\Gamma(\beta_2) / \Gamma(\beta_1 + \beta_2)$$



Poisson Process

- Consider the time at which arrivals occur.
- Let the first arrival occur at time A_1, the second occur at time A_1+A_2, and so on.



- The probability that the first arrival will occur in *[0, t]* is given by

$$P(A_1 \le t) = 1 - e^{-\lambda t}$$

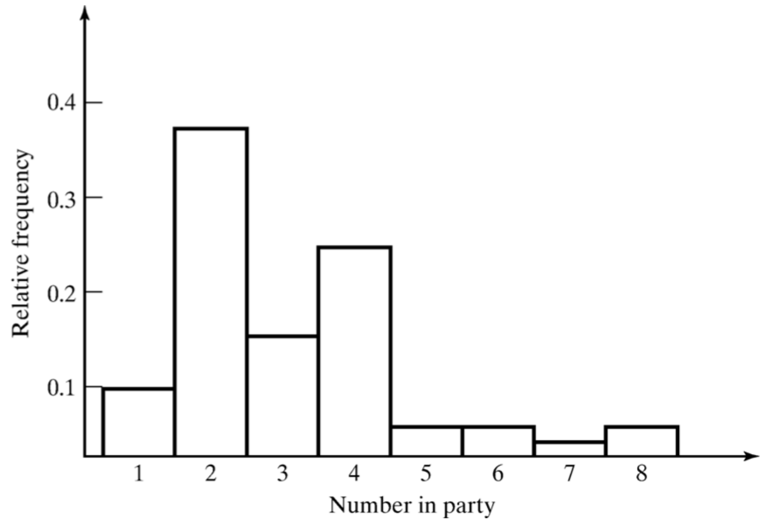
Empirical Distributions

- Example:
- Customers arrive at lunchtime in groups of from one to eight persons.
- The number of persons per party in the last 300 groups has been observed.
- The results are summarized in a table.
- The histogram of the data is also included.

Empirical Distributions (cont.)

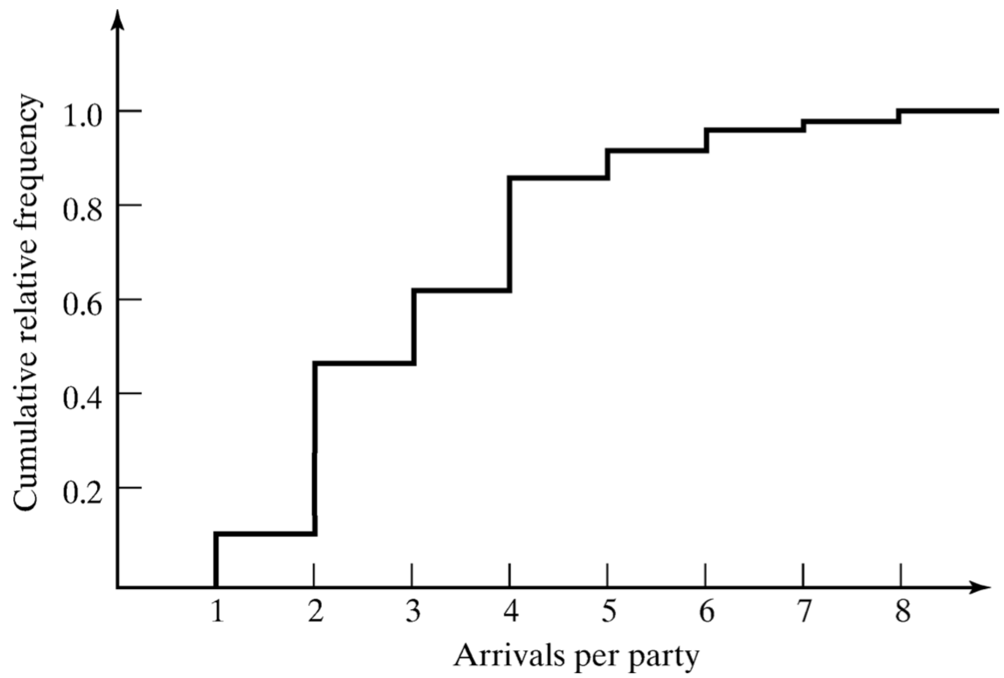
[Probability Review]

Arrivals per Party	*Frequency*	*Relative Frequency*	*Cumulative Relative Frequency*
1	30	0.10	0.10
2	110	0.37	0.47
3	45	0.15	0.62
4	71	0.24	0.86
5	12	0.04	0.90
6	13	0.04	0.94
7	7	0.02	0.96
8	12	0.04	1.00



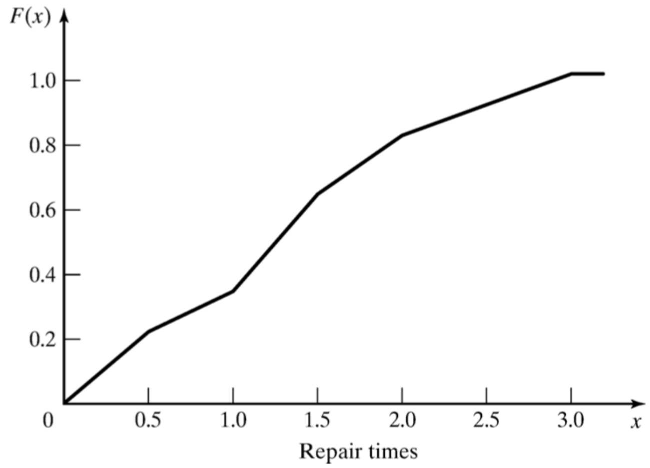
Empirical Distributions (cont.)

- The CDF in the figure is called the empirical distribution of the given data.



Empirical Distributions

- Example:
- He time required to repair a system that has suffered a failure has been collected for the last 100 instances.
- The empirical CDF is shown in the figure



Empirical Distributions (cont.)

Intervals (Hours)	Frequency	Relative Frequency	Cumulative Frequency
0<x<0.5	21	0.21	0.21
0.5<x<1.0	12	0.12	0.33
1.0<x<1.5	29	0.29	0.62
1.5<x<2.0	19	0.19	0.81
2.0<x<2.5	8	0.08	0.89
2.5<x<3.0	11	0.11	1.00

Cumulative Distribution Function [Probability Review]

- Cumulative Distribution Function (cdf) is denoted by *F(x)*, where *F(x)* = *P(X <= x)*
  - If *X* is discrete, then $$F(x) = \sum_{\substack{\text{all} \\ x_i \le x}} p(x_i)$$
  - If *X* is continuous, then $$F(x) = \int_{-\infty}^{x} f(t)dt$$
- Properties
  1. F is nondecreasing function. If $a \prec b$, then $F(a) \le F(b)$
  2. $\lim_{x\to\infty} F(x) = 1$
  3. $\lim_{x\to-\infty} F(x) = 0$
- All probability question about *X* can be answered in terms of the cdf, e.g.:

$$P(a \prec X \le b) = F(b) - F(a), \text{ for all } a \prec b$$

Cumulative Distribution Function [Probability Review]

- Example: An inspection device has cdf:

$$F(x) = \frac{1}{2}\int_0^x e^{-t/2}dt = 1 - e^{-x/2}$$

  - The probability that the device lasts for less than 2 years:

$$P(0 \le X \le 2) = F(2) - F(0) = F(2) = 1 - e^{-1} = 0.632$$

  - The probability that it lasts between 2 and 3 years:

$$P(2 \le X \le 3) = F(3) - F(2) = (1 - e^{-(3/2)}) - (1 - e^{-1}) = 0.145$$

Expectation

- The expected value of *X* is denoted by *E(X)*
  - If *X* is discrete $$E(x) = \sum_{\text{all } i} x_i p(x_i)$$
  - If *X* is continuous $$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$
  - a.k.a the mean, m, or the 1st moment of *X*
  - A measure of the central tendency
- The variance of *X* is denoted by *V(X)* or *var(X)* or σ^2
  - Definition: $V(X) = E[(X - E[X]^2]$
  - Also, $V(X) = E(X^2) - [E(x)]^2$
  - A measure of the spread or variation of the possible values of X around the mean
- The standard deviation of *X* is denoted by σ
  - Definitio B10 square root of *V(X)*
  - Expressed in the same units as the mean

B10 after :, two spaces, then next word starts with a capital letter
Brian, 1/7/2005

Expectations

- Example: The mean of life of the previous inspection device is:

$$E(X) = \frac{1}{2}\int_0^\infty xe^{-x/2}dx = -xe^{-x/2}\Big|_0^\infty + \int_0^\infty e^{-x/2}dx = 2$$

- To compute variance of *X*, we first compute *E(X²)*:

$$E(X^2) = \frac{1}{2}\int_0^\infty x^2e^{-x/2}dx = -x^2e^{-x/2}\Big|_0^\infty + \int_0^\infty e^{-x/2}dx = 8$$

- Hence, the variance and standard deviation of the device's life are:

$$V(X) = 8 - 2^2 = 4$$

$$\sigma = \sqrt{V(X)} = 2$$

Useful Statistical Models

- In this section, statistical models appropriate to some application areas are presented. The areas include:
  - Queueing systems
  - Inventory and supply-chain systems
  - Reliability and maintainability
  - Limited data

Queueing Systems

[Useful Models]

- In a queueing system, interarrival and service-time patterns can be probablistic (for more queueing examples, see Chapter 2).
- Sample statistical models for interarrival or service time distribution:
  - Exponential distribution: if service times are completely random
  - Normal distribution: fairly constant but with some random variability (either positive or negative)
  - Truncated normal distribution: similar to normal distribution but with restricted value.
  - Gamma and Weibull distribution: more general than exponential (involving location of the modes of pdf's and the shapes of tails.)

Inventory and supply chain [Useful Models]

- In realistic inventory and supply-chain systems, there are at least three random variables:
  - The number of units demanded per order or per time period
  - The time between demands
  - The lead time
- Sample statistical models for lead time distribution:
  - Gamma
- Sample statistical models for demand distribution:
  - Poisson: simple and extensively tabulated.
  - Negative binomial distribution: longer tail than Poisson (more large demands).
  - Geometric: special case of negative binomial given at least one demand has occurred.

Reliability and maintainability [Useful Models]

- Time to failure (TTF)
  - Exponential: failures are random
  - Gamma: for standby redundancy where each component has an exponential TTF
  - Weibull: failure is due to the most serious of a large number of defects in a system of components
  - Normal: failures are due to wear

Other areas

- For cases with limited data, some useful distributions are:
  - Uniform, triangular and beta
- Other distribution: Bernoulli, binomial and hyperexponential.

Discrete Distributions

- Discrete random variables are used to describe random phenomena in which only integer values can occur.
- In this section, we will learn about:
  - Bernoulli trials and Bernoulli distribution
  - Binomial distribution
  - Geometric and negative binomial distribution
  - Poisson distribution

Bernoulli Trials and Bernoulli Distribution

[Discrete Dist'n]

- Bernoulli Trials:
  - Consider an experiment consisting of n trials, each can be a success or a failure.
    - Let $X_j = 1$ if the jth experiment is a success
    - and $X_j = 0$ if the jth experiment is a failure
  - The Bernoulli distribution (one trial):

$$p_j(x_j) = p(x_j) = \begin{cases} p, & x_j = 1, j = 1,2,\ldots,n \\ 1 - p = q, & x_j = 0, j = 1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

  - where $E(X_j) = p$ and $V(X_j) = p(1-p) = pq$
- Bernoulli process:
  - The n Bernoulli trials where trails are independent:

$$p(x_1, x_2, \ldots, x_n) = p_1(x_1)p_2(x_2) \ldots p_n(x_n)$$

Binomial Distribution

[Discrete Dist'n]

- The number of successes in *n* Bernoulli trials, *X*, has a binomial distribution.

$$p(x)=\begin{cases}\binom{n}{x}\underbrace{p^{x}q^{n-x}}, & x=0,1,2,\ldots,n\\ 0, & \text{otherwise}\end{cases}$$

The number of outcomes having the required number of successes and failures

Probability that there are x successes and (n-x) failures

- The mean, $E(x) = p + p + \ldots + p = n*p$
- The variance, $V(X) = pq + pq + \ldots + pq = n*pq$

Geometric & Negative Binomial Distribution

[Discrete Dist'n]

- Geometric distribution
  - The number of Bernoulli trials, *X*, to achieve the 1st success:

$$p(x) = \begin{cases} q^{x-1}p, & x = 0,1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

  - *E(x) = 1/p, and V(X) = q/p²*

- Negative binomial distribution
  - The number of Bernoulli trials, X, until the *k*th success
  - If Y is a negative binomial distribution with parameters p and k, then:

$$p(x) = \begin{cases} \binom{y-1}{k-1} q^{y-k} p^{k}, & y = k, k+1, k+2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

  - *E(Y) = k/p, and V(X) = kq/p²*

Poisson Distribution

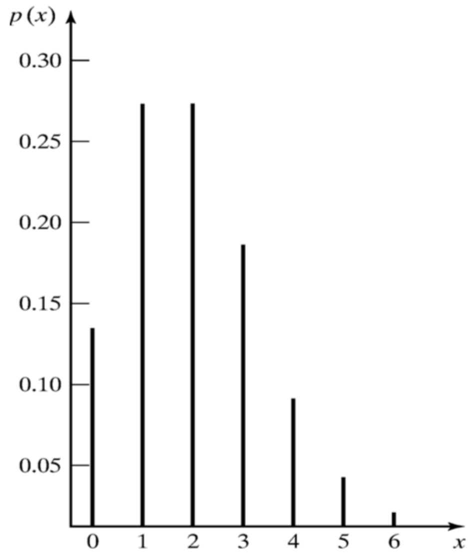
[Discrete Dist'n]

- Poisson distribution describes many random processes quite well and is mathematically quite simple.
  - where $\alpha > 0$, pdf and cdf are:

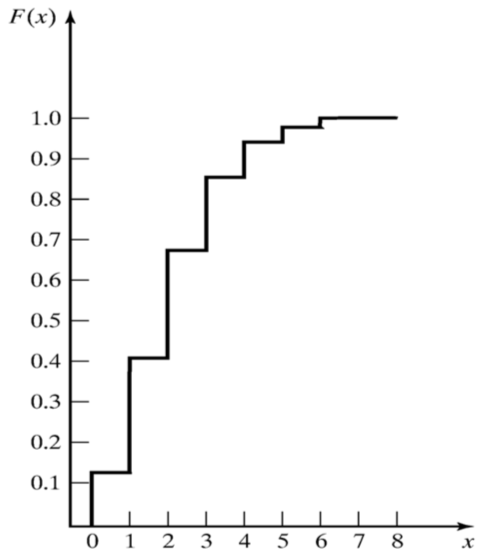
$$p(x) = \begin{cases} \dfrac{e^{-\alpha}\alpha^{x}}{x!}, & x = 0,1,\ldots \\ 0, & \text{otherwise} \end{cases}$$

$$F(x) = \sum_{i=0}^{x} \frac{e^{-\alpha}\alpha^{i}}{i!}$$

  - $E(X) = \alpha = V(X)$



(a)



(b)

Poisson Distribution

- Example: A computer repair person is "beeped" each time there is a call for service. The number of beeps per hour ~ Poisson(α = 2 per hour).

  - The probability of three beeps in the next hour:

    $p(3) = e^{-2}2^3/3! = 0.18$

    also, $p(3) = F(3) – F(2) = 0.857\text{-}0.677\text{=}0.18$

  - The probability of two or more beeps in a 1-hour period:

    $$\begin{aligned} p(2 \text{ or more}) &= 1 – p(0) – p(1) \\ &= 1 – F(1) \\ &= 0.594 \end{aligned}$$

Continuous Distributions

- Continuous random variables can be used to describe random phenomena in which the variable can take on any value in some interval.
- In this section, the distributions studied are:
  - Uniform
  - Exponential
  - Normal
  - Weibull
  - Lognormal

Uniform Distribution

[Continuous Dist'n]

- A random variable X is uniformly distributed on the interval (a,b), $U(a,b)$, if its pdf and cdf are:

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases} \qquad F(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x < b \\ 1, & x \ge b \end{cases}$$

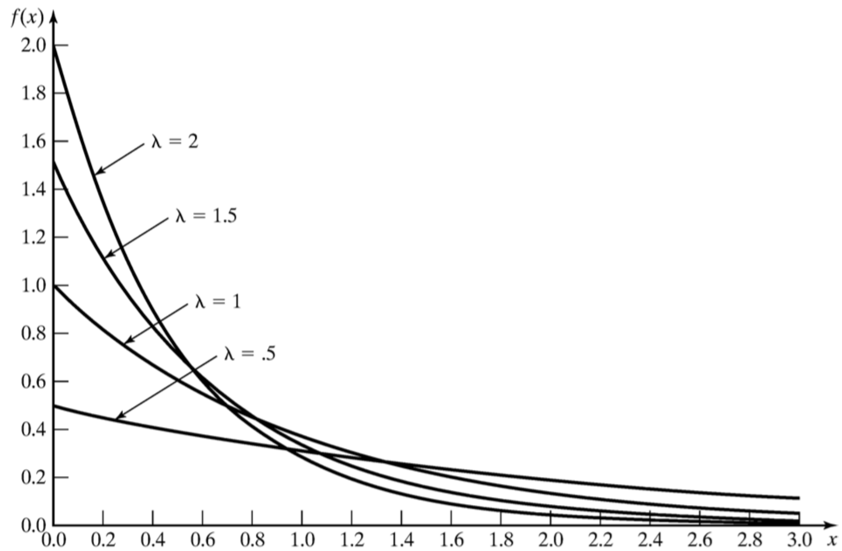
- Properties
  - $P(x_1 < X < x_2)$ is proportional to the length of the interval $[F(x_2) - F(x_1) = (x_2 - x_1)/(b-a)]$
  - $E(X) = (a+b)/2$ $\qquad V(X) = (b-a)^2/12$
- U(0,1) provides the means to generate random numbers, from which random variates can be generated.

Exponential Distribution [Continuous Dist'n]

- A random variable X is exponentially distributed with parameter $\lambda > 0$ if its pdf and cdf are:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases} \qquad F(x) = \begin{cases} 0, & x < 0 \\ \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

  - $E(X) = 1/\lambda$ $V(X) = 1/\lambda^2$
  - Used to model interarrival times when arrivals are completely random, and to model service times that are highly variable
  - For several different exponential pdf's (see figure), the value of intercept on the vertical axis is λ, and all pdf's eventually intersect.



Exponential Distribution [Continuous Dist'n]

- Memoryless property
  - For all s and t greater or equal to 0:

    $$P(X > s+t \mid X > s) = P(X > t)$$

  - Example: A lamp ~ exp(λ = 1/3 per hour), hence, on average, 1 failure per 3 hours.
    - The probability that the lamp lasts longer than its mean life is:
      $P(X > 3) = 1-(1-e^{-3/3}) = e^{-1} = 0.368$
    - The probability that the lamp lasts between 2 to 3 hours is:
      $P(2 <= X <= 3) = F(3) – F(2) = 0.145$
    - The probability that it lasts for another hour given it is operating for 2.5 hours:
      $P(X > 3.5 \mid X > 2.5) = P(X > 1) = e^{-1/3} = 0.717$

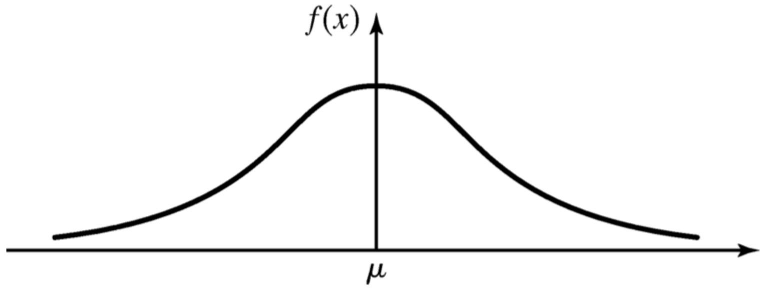
Normal Distribution

[Continuous Dist'n]

- A random variable X is normally distributed has the pdf:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], \quad -\infty < x < \infty$$

  - Mean: $-\infty < \mu < \infty$
  - Variance: $\sigma^2 > 0$
  - Denoted as $X \sim N(\mu, \sigma^2)$



- Special properties:
  - $\lim_{x \to -\infty} f(x) = 0$, and $\lim_{x \to \infty} f(x) = 0$
  - $f(\mu-x)=f(\mu+x)$; the pdf is symmetric about μ.
  - The maximum value of the pdf occurs at $x = \mu$; the mean and mode are equal.

Normal Distribution

[Continuous Dist'n]

- Evaluating the distribution:
  - Use numerical methods (no closed form)
  - Independent of μ and σ, using the standard normal distribution:

    Z ~ N(0,1)
  - Transformation of variables: let Z = (X - μ) / σ,

$$F(x) = P(X \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right)$$

$$= \int_{-\infty}^{(x-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \int_{-\infty}^{(x-\mu)/\sigma} \phi(z) dz = \Phi\left(\tfrac{x-\mu}{\sigma}\right) \qquad \text{, where } \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

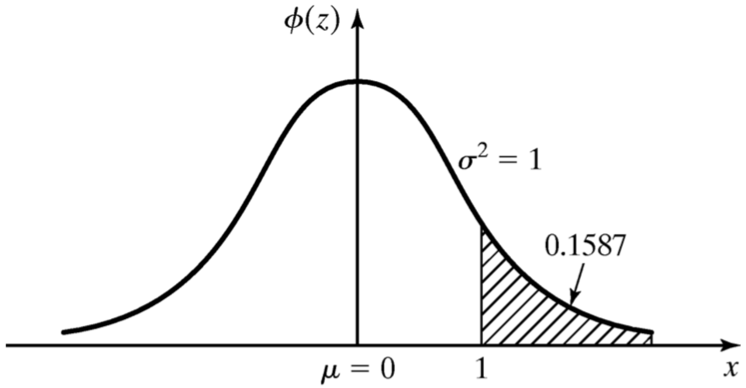
Normal Distribution

[Continuous Dist’n]

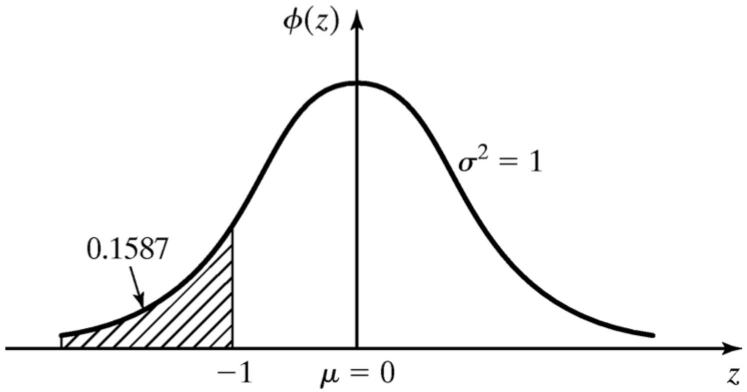
- Example: The time required to load an oceangoing vessel, *X*, is distributed as *N(12,4)*
  - The probability that the vessel is loaded in less than 10 hours:

$$F(10) = \Phi\left(\frac{10-12}{2}\right) = \Phi(-1) = 0.1587$$

    - Using the symmetry property, $\Phi(1)$ is the complement of Φ (-1)



(a)



(b)

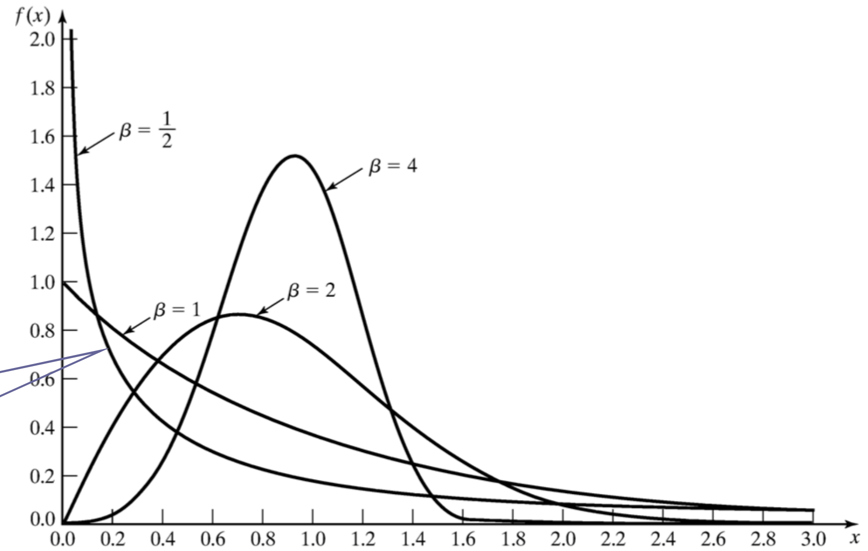
Weibull Distribution

[Continuous Dist'n]

- A random variable X has a Weibull distribution if its pdf has the form:

$$f(x)=\begin{cases}\dfrac{\beta}{\alpha}\left(\dfrac{x-\nu}{\alpha}\right)^{\beta-1}\exp\left[-\left(\dfrac{x-\nu}{\alpha}\right)^{\beta}\right], & x\geq\nu\\ 0, & \text{otherwise}\end{cases}$$

- 3 parameters:
  - Location parameter: υ, $(-\infty<\nu<\infty)$
  - Scale parameter: β, $(\beta>0)$
  - Shape parameter. α, (>0)
- Example: $\upsilon = 0$ and $\alpha = 1$:



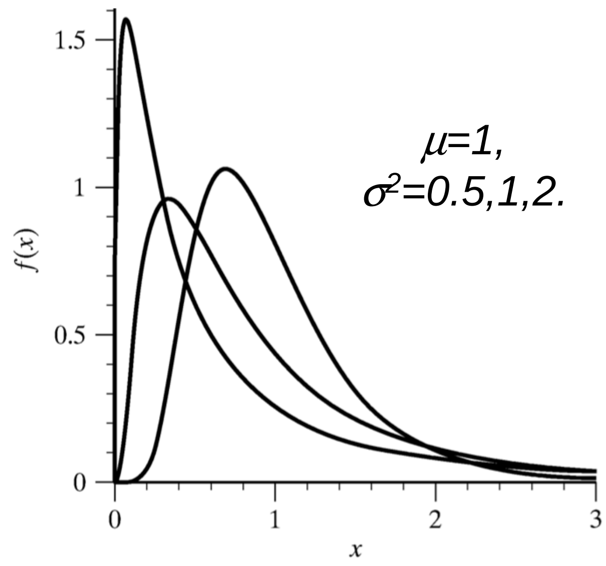
When $\beta = 1$,
$X \sim exp(\lambda = 1/\alpha)$

Lognormal Distribution

[Continuous Dist'n]

- A random variable X has a lognormal distribution if its pdf has the form:

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma x} \exp\left[-\frac{(\ln x - \mu)^2}{2\sigma^2}\right], & x > 0 \\ 0, & \text{otherwise} \end{cases}$$



  - Mean $E(X) = e^{\mu+\sigma^2/2}$
  - Variance $V(X) = e^{2\mu+\sigma^2/2}(e^{\sigma^2} - 1)$

- Relationship with normal distribution
  - When $Y \sim N(\mu, \sigma^2)$, then $X = e^Y \sim$ lognormal(μ, σ^2)
  - Parameters μ and σ^2 are not the mean and variance of the lognormal

Poisson Distribution

- Definition: *N(t)* is a counting function that represents the number of events occurred in *[0,t].*
- A counting process {*N(t), t>=0*} is a Poisson process with mean rate λ if:
  - Arrivals occur one at a time
  - {*N(t), t>=0*} has stationary increments
  - {*N(t), t>=0*} has independent increments
- Properties

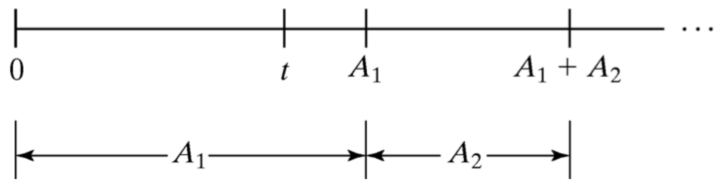
$$P[N(t)=n]=\frac{e^{-\lambda t}(\lambda t)^n}{n!}, \qquad \text{for } t \geq 0 \text{ and } n = 0,1,2,\ldots$$

  - Equal mean and variance: $E[N(t)] = V[N(t)] = \lambda t$
  - Stationary increment: The number of arrivals in time *s* to *t* is also Poisson-distributed with mean $\lambda(t-s)$

Interarrival Times

[Poisson Dist’n]

- Consider the interarrival times of a Possion process (A_1, A_2, …), where A_i is the elapsed time between arrival *i* and arrival *i+1*



  - The 1st arrival occurs after time t iff there are no arrivals in the interval [0,t], hence:

    $P\{A_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$

    $P\{A_1 <= t\} = 1 - e^{-\lambda t}$ [cdf of exp(λ)]

  - Interarrival times, A_1, A_2, …, are exponentially distributed and independent with mean $1/\lambda$

Arrival counts ~B42$i(\lambda)$ ⇔ Interarrival time ~ $Exp(1/\lambda)$

Stationary & Independent

Memoryless

Slide 52

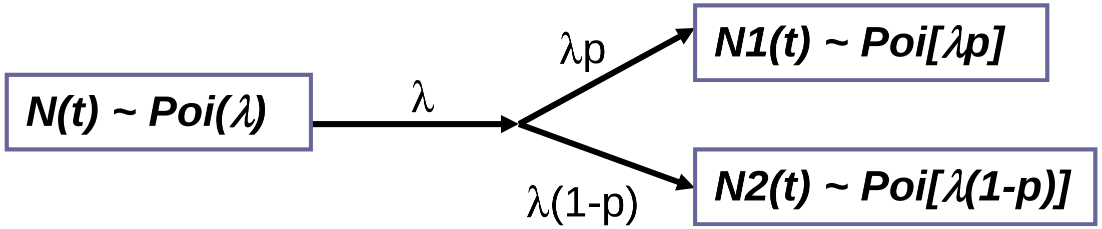
B42 Poi is not an abbreviation of Poisson that I have ever seen

Brian, 1/7/2005

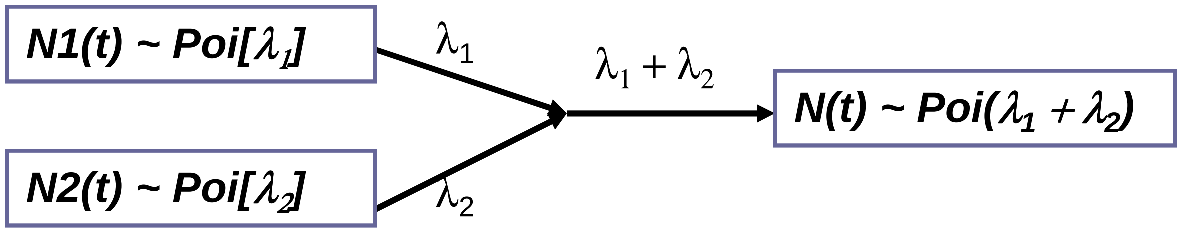
Splitting and Pooling

[Poisson Dist'n]

- Splitting:
  - Suppose each event of a Poisson process can be classified as Type I, with probability *p* and Type II, with probability *1-p.*
  - $N(t) = N1(t) + N2(t)$, where $N1(t)$ and $N2(t)$ are both Poisson processes with rates λp and $\lambda(1-p)$



- Pooling:
  - Suppose two Poisson processes are pooled together
  - $N1(t) + N2(t) = N(t)$, where $N(t)$ is a Poisson processes with rates $\lambda_1 + \lambda_2$



Nonstationary Poisson Process (NSPP)

- Poisson Process without the stationary increments, characterized by $\lambda(t)$, the arrival rate at time t.
- The expected number of arrivals by time t, $\Lambda(t)$:

$$\Lambda(t) = \int_0^t \lambda(s)ds$$

- Relating stationary Poisson process $n(t)$ *with rate* $\lambda=1$ and NSPP $N(t)$ with rate $\lambda(t)$:
  - Let arrival times of a stationary process with rate $\lambda = 1$ be $t_1, t_2, \ldots,$ and arrival times of a NSPP with rate $\lambda(t)$ be $T_1, T_2, \ldots,$ we know:

$$t_i = \Lambda(T_i)$$

$$T_i = \Lambda^{-1}(t_i)$$

Nonstationary Poisson Process (NSPP)

[Poisson Dist'n]

- Example: Suppose arrivals to a Post Office have rates 2 per minute from 8 am until 12 pm, and then 0.5 per minute until 4 pm.
- Let t = 0 correspond to 8 am, NSPP *N(t)* has rate function:

$$\lambda(t) = \begin{cases} 2, & 0 \le t < 4 \\ 0.5, & 4 \le t < 8 \end{cases}$$

Expected number of arrivals by time t:

$$\Lambda(t) = \begin{cases} 2t, & 0 \le t < 4 \\ \int_0^4 2ds + \int_4^t 0.5ds = \frac{t}{2} + 6, & 4 \le t < 8 \end{cases}$$

- Hence, the probability distribution of the number of arrivals between 11 am and 2 pm.

$$\begin{aligned} P[N(6) - N(3) = k] &= P[N(\Lambda(6)) - N(\Lambda(3)) = k] \\ &= P[N(9) - N(6) = k] \\ &= e^{(9-6)}(9-6)^k/k! \quad = e^3(3)^k/k! \end{aligned}$$

Empirical Distributions [Poisson Dist'n]

- A distribution whose parameters are the observed values in a sample of data.
  - May be used when it is impossible or unnecessary to establish that a random variable has any particular parametric distribution.
  - Advantage: no assumption beyond the observed values in the sample.
  - Disadvantage: sample might not cover the entire range of possible values.

Summary

- The world that the simulation analyst sees is probabilistic, not deterministic.
- In this chapter:
  - Reviewed several important probability distributions.
  - Showed applications of the probability distributions in a simulation context.
- Important task in simulation modeling is the collection and analysis of input data, e.g., hypothesize a distributional form for the input data. Reader should know:
  - Difference between discrete, continuous, and empirical distributions.
  - Poisson process and its properties.