What is probability?

Probability is simply how likely something is to happen.

Example *If a coin is randomly flipped, how likely it lands on head or tail?*

Why we should learn probability?

- *Nowadays, randomness becomes prevalent in real life.*
  - *e.g., the revenue of apple store at APM next month is random*
  - *e.g., the length a patient with cancer can survive is random*
- *Probability is the fundamental **tool** for modelling, analyzing randomness*
  - *e.g., what is the most likely revenue of apple store at APM next month?*
  - *e.g., what is the most likely length a patient with cancer can survive?*
- *Probability is the fundamental **tool** for statistics, machine learning, etc.*

Chapter 1. Probability by Combinatorial Analysis

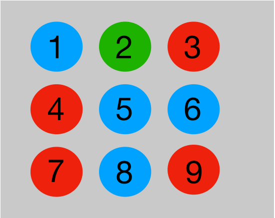
Outline

1.1 Introduction

Simple intuitive examples to understand probability

Example *If we randomly pick up a ball in the following box, what is the chance of selecting a red ball?*

we can view this chance as probability.



Solution *We see there are 9 balls in the box and 4 of them are red. Intuitively, we know the chance of selecting a red ball is 4/9.*

$$\frac{4}{9} =$$

1.1 Introduction

Simple intuitive examples to understand probability

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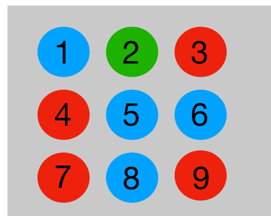


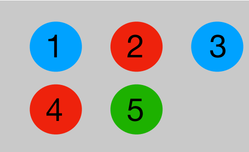
Solution *We see there are 9 balls in the box and 4 of them are red. Intuitively, we know the chance of selecting a red ball is 4/9.*

$$\frac{4}{9} = \frac{\text{4 outcomes are red balls } \{3,4,7,9\}}{\text{9 possible outcomes } \{1,2,3,4,5,6,7,8,9\}} = \frac{\#\text{ of outcomes of interest}}{\text{total } \#\text{ of possible outcomes}}$$

1.1 Introduction

Class Discussion

If we randomly pick up 2 balls simultaneously in the following box, what is the chance that the 2 selected balls are both red?



All possible outcomes:

1, 2	2, 3	3, 4	4, 5
1, 3	$\boxed{2, 4}$	3, 5	
1, 4	2, 5		
1, 5			

Total # of possible outcomes $= 4+3+2+1$

$= 10$

Choose your answer?

Choice A

$\frac{1}{4}$

Choice B

$\frac{1}{5}$

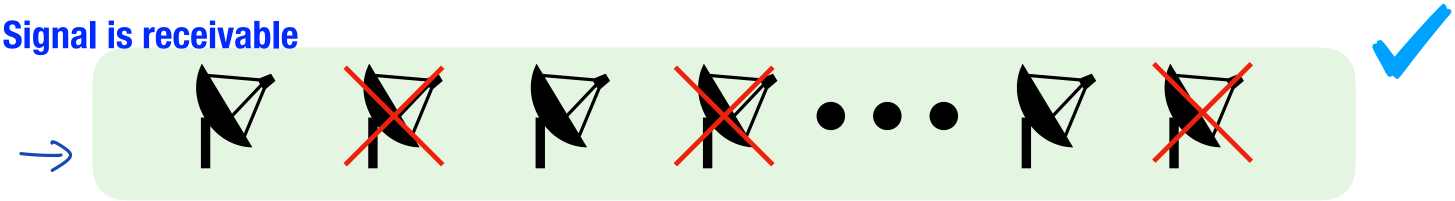
Choice C

$\frac{1}{10}$

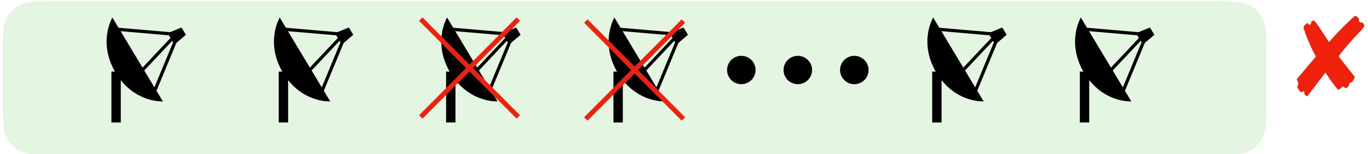
1.1 Introduction

Example A communication system is to consist of n identical antennas lined up in a linear order. The system can receive signals as long as no two consecutive antennas are defective. If it turns that exactly m of the n antennas are defective, what is the probability that the system can receive signals.

Signal is receivable



Signal is not receivable



1.1 Introduction

Example A communication system is to consist of n identical antennas lined up in a linear order. The system can receive signals as long as no two consecutive antennas are defective. If it turns that exactly m of the n antennas are defective, what is the probability that the system can receive signals.

Solution

We will be able to solve this problem in more general (values of m and n) case after this Chapter.

Let us solve the problem in the special case where $n = 4$ and $m = 2$. In this case, there are 6 possible system configurations, namely,

$$\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{array}$$

where 1 means that the antenna is working and 0 that it is defective. Note that the system can receive signals in the first 3 arrangements and fails in the remaining 3, it seems reasonable to take $\frac{3}{6} = \frac{1}{2}$ as the desired probability.

1.1 Introduction

Continue from last page…

For general n and m, we could regard the probability as

$$\textbf{Probability that system works} = \frac{\text{number of configurations that the system works well}}{\text{total number of all possible configurations}}$$

Why counting matters

From the above example, we see that it would be useful to have an effective method for counting the number of ways that things can occur. The mathematical theory of counting is formally known as *combinatorial analysis*

1.2 Principle of Counting

Concepts

Experiment

We use the concept "experiment" to denote a process whose outcome is random

Example of experiment

e.g. randomly pick a number from {10,100,1000,10000}

e.g. randomly toss a coin

e.g. randomly roll a die

1.2 Principle of Counting

Theorem (The basic principle of counting). *Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if for each outcome of experiment* 1 *there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.*

1.2 Principle of Counting

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Proof We prove it by enumerating all the possible outcomes of the two experiments as follows:

$$\begin{array}{cccc} (1,1) & (1,2) & \cdots & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n) \\ \vdots & \vdots & \vdots & \vdots \\ (m,1) & (m,2) & \cdots & (m,n) \end{array}$$

where we say that the outcome is (i,j) if experiment 1 results in its ith possible outcome and experiment 2 then results in the jth of its possible outcomes. Hence, the set of possible outcomes consists of m rows, each row containing n elements, which proves the result.

1.2 Principle of Counting

Example A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

Solution

$10 \times 3 = 30$ possible choices.

Example In a class of 40 students, we choose a president and a vice president. There are

$$40 \times 39 = 1560$$

possible choices.

1.2 Principle of Counting

Theorem (The generalized basic principle of counting). *If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes, and if for each of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment, and if for each of the possible outcomes of the first two experiments there are n_3 possible outcomes of the third experiment, and if ... then there is a total of $n_1 n_2 \cdots n_r$ possible outcomes of the r experiments.*

Proof is simple and skipped.

Example How many different 7-place license plates are possible if the first 3 places for letters and the final 4 places by numbers?

Solution:

(26) (10)

L	L	L	N	N	N	N
26	26	26	10	10	10	10

$26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 175,760,000$

1.2 Principle of Counting

Example How many different 7-place license plates are possible if the first 3 places for letters and the final 4 places by numbers and repetition among letters or numbers were prohibited?

L	L	L	N	N	N	N
26	25	24	10	9	8	7

$26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 = 78,624,000$

1.3 Permutations

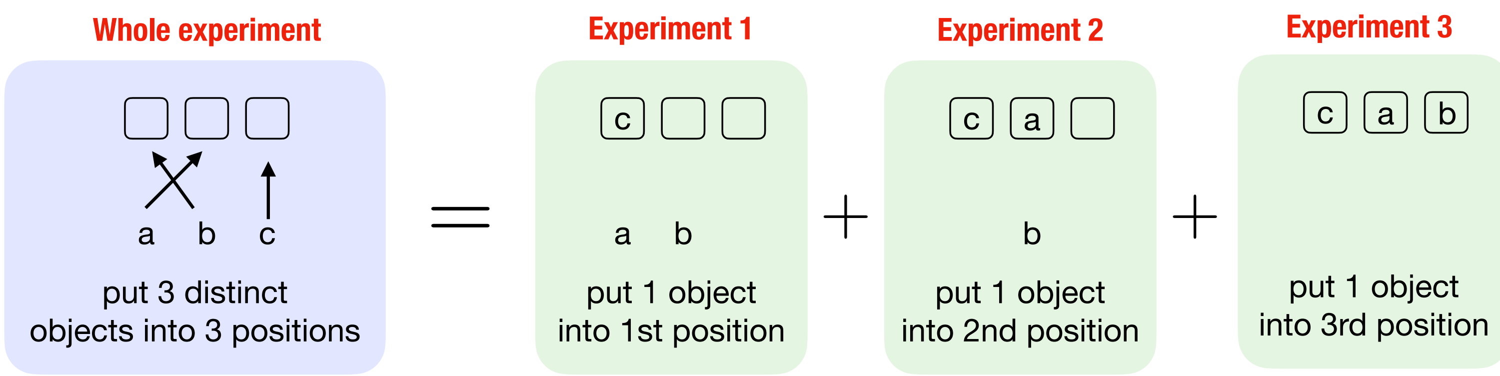
Definition

How many different *ordered arrangements* of the letters a, b, c are possible. By direct enumeration we see that there are 6:

$$abc \qquad acb \qquad bac \qquad bca \qquad cab \qquad cba$$

Each arrangement is known as a *permutation*.

For 3 objects, there are 6 possible permutations. This can be explained by the basic principle, since the first object in the permutation can be any of the 3, the second object in the permutation can then be chosen from any of the remaining 2, and the third object in the permutation is then chosen from the remaining one. Thus there are $3 \cdot 2 \cdot 1 = 6$ permutations.



1.3 Permutations

Theorem

Suppose there are n (distinct) objects, then the total number of different arrangements is **(Number of different permutations)**

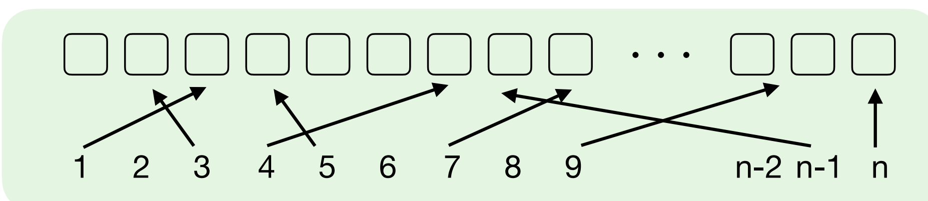
$$n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!$$

"n factorial"

with the convention that

$$0! = 1.$$

Proof *The experiment is equivalent to arranging n distinct objects into n positions.*



To fill the 1st position, there are n choices.

To fill the 2nd position, there are n-1 choices.

To fill the 3rd position, there are n-2 choices.

$\cdots$

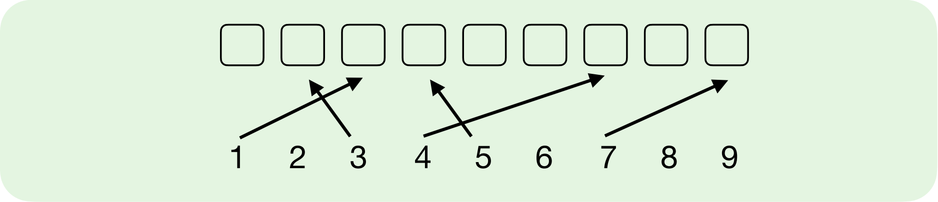
To fill the nth position, there are 1 choices.

In total

$$n \times (n-1) \times (n-2) \times \cdots \times 1 = n!$$

1.3 Permutations

Example Seating arrangement in a row: 9 people sitting in a row. There are $9! = 362,880$ ways.



1.3 Permutations

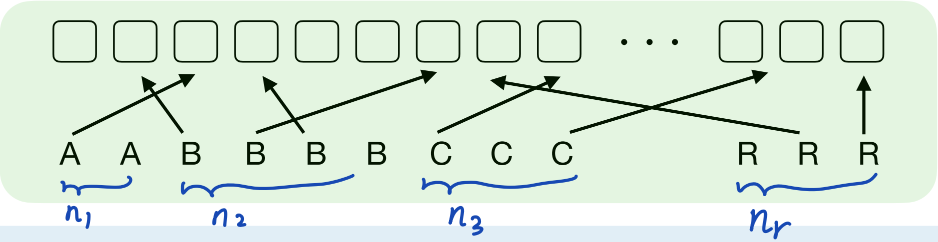
Theorem

For n objects of which n_1 are alike, n_2 are alike, $\ldots$, n_r are alike, there are

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

different permutations of the n objects.

Proof



- *First, we view these n objects as all distinct objects and permute them, there are n! permutations*
- *However, there are many objects are alike, so some permutations are essentially the same but counted more than once.*
- *For instance, consider the permutation ABBABBCCC……RRR*
- *When we permute all As, or all Bs, or all Cs, … or all Rs, essentially they are the same permutation. In other words, each permutation is counted* $n_1!n_2!\cdots n_r!$ *times.*

1.3 Permutations

Example How many ways to rearrange Mississippi?

$n = 11$

4 distinct letters:

$M \quad n_1 = 1$

$I \quad n_2 = 4$

$S \quad n_3 = 4$

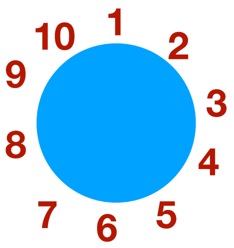
$P \quad n_4 = 2$

$$\text{Total} = \frac{n!}{n_1!\,n_2!\,n_3!\,n_4!} = \frac{11!}{1!\,4!\,4!\,2!} = 34{,}650.$$

1.3 Permutations

Seating in circle

Example (Seating in circle). 10 people sitting around a round dining table. It is the relative positions that really matters – who is on your left, on your right. No. of seating arrangements is



Choose your answer?

Choice A	Choice B	Choice C
$9!$	$10!$	$10! - 9!$

1.3 Permutations

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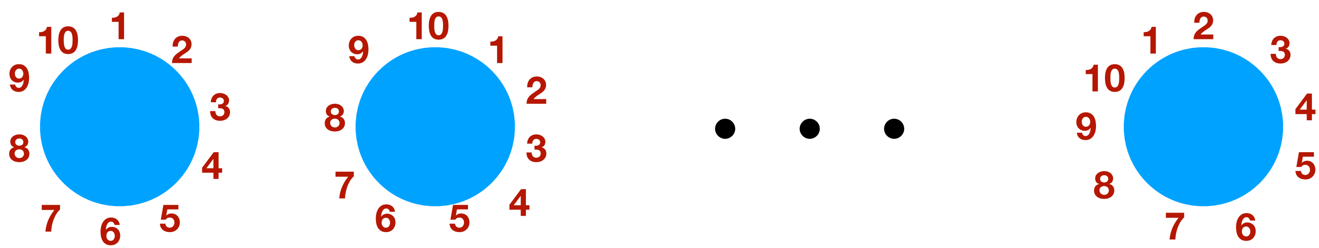


Rotating it clock wisely gives the same arrangements

$$\frac{10!}{10} = 9!$$

1.3 Permutations

Seating in circle

Theorem

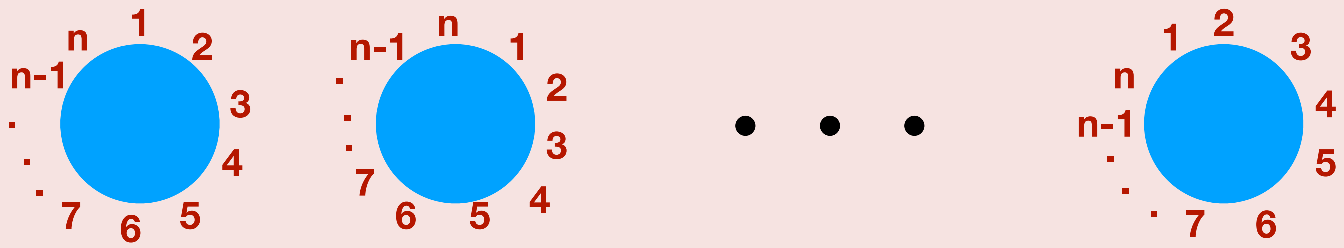
Generally, for n people sitting in a circle, there are

$$\frac{n!}{n} = (n-1)!$$

$n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

possible arrangements.

Proof

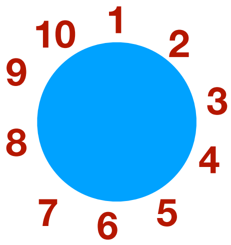


Rotating it clock wisely
gives the same arrangements

1.3 Permutations

Seating in circle

Example (Making necklaces). n different pearls string in a necklace. Number of ways of stringing the pearls is



1.3 Permutations

Seating in circle

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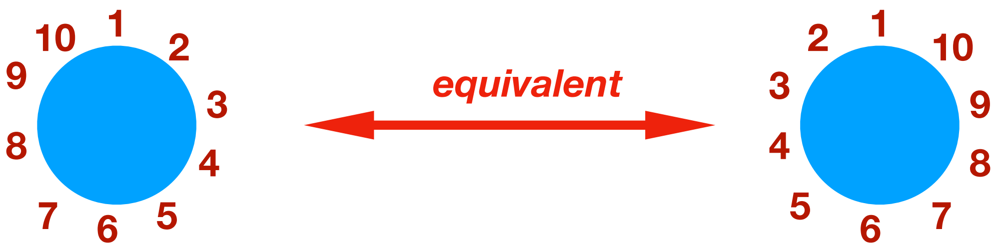

First, like arranging n objects in a round table, there are (n-1)! arrangements

In addition, for a necklace, we can flip it reversely, e.g., like a mirror. They are actually the same necklace

Mirroring the necklaces gives the same one

Therefore, the answer is $\boxed{\dfrac{(n-1)!}{2}}$

1.4 Combinations

Example

In how many ways can we choose 3 items simultaneously from A, B, C, D, E?

1.4 Combinations

Example

In how many ways can we choose 3 items simultaneously from A, B, C, D, E?

5 ways to choose first item,
4 ways to choose second item, and
3 ways to choose third item
So the number of ways (in this order) is $5 \cdot 4 \cdot 3$.
However,

$$ABC, ACB, BAC, BCA, CAB \text{ and } CBA$$

will be considered as the same group. $\{A, B, C\}$.
So the number of different groups (order not important) is $\dfrac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1}$.

Basically, in combinations, we do not care about the order the items are chosen or arranged.

1.4 Combinations

Theorem

Generally, if there are n distinct objects, of which we choose a group of r items,

$$\text{Number of possible groups} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

1.4 Combinations

Theorem

Generally, if there are n distinct objects, of which we choose a group of r items,

$$\text{Number of possible groups} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \times \frac{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}$$

$$\boxed{= \frac{n!}{r!(n-r)!}.} \xrightarrow{\text{denoted by}} \boxed{{}_nC_r \text{ or } \binom{n}{r}}$$

called "n choose r"

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

1.4 Combinations

Theorem

Generally, if there are n distinct objects, of which we choose a group of r items,

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$$= \frac{n!}{r!(n-r)!}.$$

denoted by $\; {}_nC_r$ or $\binom{n}{r}$

called "n choose r"

Proof: Method 1. $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-(n-r))!} = \binom{n}{n-r}$. (Algebraic proof)

Method 2. Taking a group of r items out of n objects $\Longleftrightarrow$ $(n-r)$ (Combinatorial proof).

Properties

For $r = 0, 1, \ldots, n$,
$$\binom{n}{r} = \binom{n}{n-r}.$$

Convention:
When n is a nonnegative integer, and $r < 0$ or $r > n$, take
$$\binom{n}{r} = 0.$$

$$\binom{n}{0} = \binom{n}{n} = 1.$$

1.4 Combinations

Example A committee of 3 is to be formed from a group of 20 people.

1. How many possible committees can be formed?

No. of possible committees $= \binom{20}{3} = \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 1140$

2. Suppose further that, two guys: Peter and Paul refuse to serve in the same committee. How many possible committees can be formed with the restriction that these two guys don't serve together?

Solution 1:

Total No. of possible committees
− No. of possible committees that Peter and Paul serve together

$= \binom{20}{3} - \binom{18}{1} = 1140 - 18 = 1122$

Solution 2:

Case 1. None of Peter and Paul is in the committee:

$\binom{18}{3} = \frac{18 \times 17 \times 16}{6} = 816$

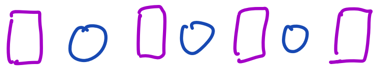
Case 2. One of Peter and Paul is in the committee:

$\binom{2}{1}\binom{18}{2} = 2\,\frac{18 \times 17}{2} = 306$

Total $= 816 + 306 = 1122$.

$n-m=3$

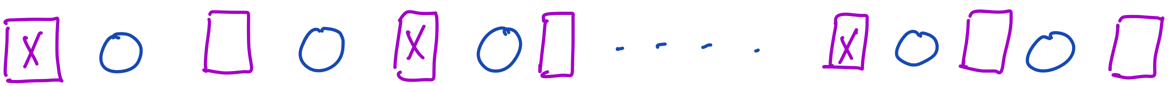
1.4 Combinations



$n-m+1$ slots

Example Consider a set of n antennas of which m are defective and $n-m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

First, place $(n-m)$ functional antennas in line (only 1 way since they are indistinguishable).



Next, place m defective antennas between functional antennas. (So that no two defectives are next two each other).

Total number of slots $\square$ is $n-m+1$.

The number of arrangements of working systems is $\binom{n-m+1}{m}$.

1.4 Combinations

Useful Combinatorial Identities

Theorem

For $1 \le r \le n$,

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Algebraic Proof

$$RHS = \frac{(n-1)!}{(r-1)!\,(n-r)!} + \frac{(n-1)!}{r!\,(n-r-1)!} = \frac{(n-1)!}{r!\,(n-r)!}\left[r + (n-r)\right] = \frac{n!}{r!\,(n-r)!}$$

$$= \binom{n}{r} = LHS.$$

Combinatorial Proof

$\binom{n}{r}$ = # of possible ways of taking r balls out of n balls.

case 1. Ball ① is chosen. $\binom{n-1}{r-1}$

case 2. Ball ① is not chosen. $\binom{n-1}{}$

1.4 Combinations

Useful Combinatorial Identities

Binomial Theorem

Let n be a nonnegative integer, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

In view of the binomial theorem, $\binom{n}{k}$ is often referred to as the binomial coefficient.

$(x+y)^2 = (x+y)(x+y) = x^2+xy+yx+y^2 = x^2+2xy+y^2$

Combinatorial Proof

is composed of sum of terms $x^k y^{n-k}$, $k = 0, 1, 2, \cdots, n$

$$(x+y)^n = (x+\underbrace{y)(x+y}_{x^2+xy+yx+y^2})(x+y)\cdots(x+y)$$

$$x^3+3x^2y+3xy^2+y^3$$

To get $x^k y^{n-k}$, it means there are k brackets contribute x,

and n-k brackets contribute y

1.4 Combinations

$n=4$, $(0,1,1,0) \updownarrow \{2,3\}$

Useful Combinatorial Identities

Method 2. Take subsets of $\{1,2,\cdots,n\}$. We represent a subset in terms of a vector $(x_1, x_2, \cdots, x_n)$. $x_i = \begin{cases} 1, & \text{if } i \text{ is in the subset} \\ 0, & \text{if } i \text{ is not in the subset.} \end{cases}$

Total # of such vectors $= 2 \cdot 2 \cdots 2 = 2^n$.

Example How many subsets are there of a set consisting of n elements?

Method 1. (Apply Binomial Theorem).

# of subsets of size $k = \binom{n}{k}$, $k = 0, 1, 2, \cdots, n$

Total # of subsets $= \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ with $x=1$, $y=1$

$= (x+y)^n = (1+1)^n = 2^n$.

Example

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Proof $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ with $x=-1$, $y=1$

$= (x+y)^n = (1-1)^n = 0$.

1.4 Combinations

Useful Combinatorial Identities

Example $$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots.$$

Proof Method 1. From previous example, move the negative terms to the RHS of the equation.

Method 2. Combinatorial proof (sketch).

Let us look at vectors of length n :

$(x_1, x_2, \cdots, x_n)$, x_i is either 0 or 1.

LHS of this identity = # of these vectors with even number of 1's.

bijection $\updownarrow$

RHS of this identity = # of these vectors with odd number of 1's.

say, $n=3$, $2^n=8$

$(1,1,1) \longleftrightarrow (0,0,0)$

$(0,0,1) \longleftrightarrow (1,1,0)$

$(0,1,0) \longleftrightarrow (1,0,1)$

$n=4$

$(1,1,1,\underline{1}) \longleftrightarrow (0,0,0,\underline{1})$

$(1,1,0,\underline{0}) \longleftrightarrow (0,0,1,\underline{0})$

$(1,0,1,0)$: $\longleftrightarrow (0,1,0,0)$

$(1, 0, 0) \longleftrightarrow (0, 1, 1)$

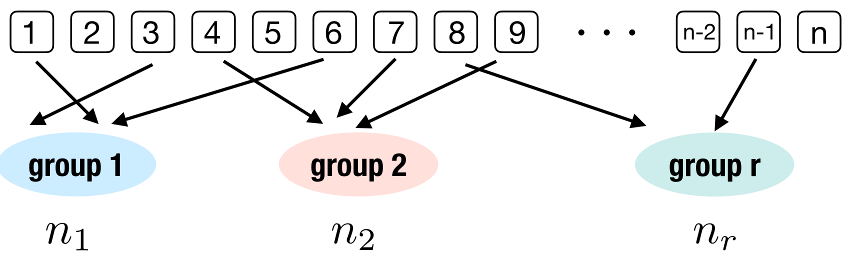
1.5 Multinomial Coefficients

Binomial coefficient $\binom{n}{s}$: a set of n distinct items is divided into 2 distinct groups; one group has s elements, the other has $(n-s)$ elements.

Example

A set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$, where $\sum_{i=1}^{r} n_i = n$. How many different divisions are possible?

Proof



We use the generalized basic principle of counting.

$$\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \binom{n-n_1-n_2}{n_3} \cdots \binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$$

1.5 Multinomial Coefficients

Example

A set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$, where $\sum_{i=1}^{r} n_i = n$. How many different divisions are possible?

Proof

$$\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \binom{n-n_1-n_2}{n_3} \cdots \binom{n-n_1-n_2 \cdots -n_{r-1}}{n_r}$$

$$= \frac{n!}{n_1! \cancel{(n-n_1)!}} \frac{\cancel{(n-n_1)!}}{n_2! \cancel{(n-n_1-n_2)!}} \frac{\cancel{(n-n_1-n_2)!}}{n_3! \cancel{(n-n_1-n_2-n_3)!}} \cdots \frac{\cancel{(n-n_1-n_2-\cdots-n_{r-1})!}}{n_r! \underbrace{(n-n_1-n_2-\cdots-n_r)!}_{0!=1}}$$

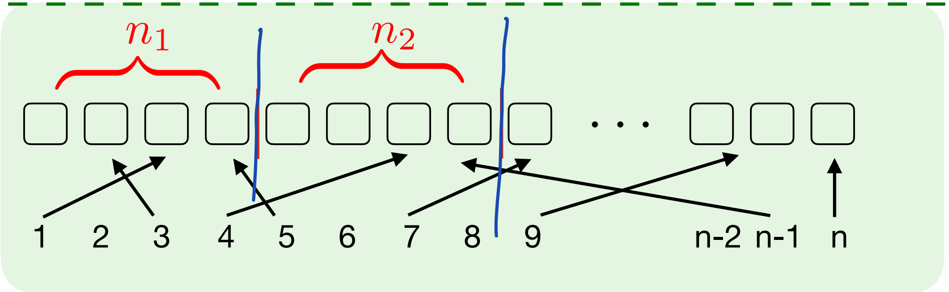
$$= \frac{n!}{n_1! n_2! \cdots n_r!}$$

1.5 Multinomial Coefficients

Example

A set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$, where $\sum_{i=1}^{r} n_i = n$. How many different divisions are possible?

Another Proof



The problem can be cast to arranging n objects in a line and then the first n_1 *objects belong to 1st group; the second* n_2 *objects belong to the 2nd group; .....*

There are n! different arrangements of n objects in a line.
However, for each arrangement, no matter how you permute the first n_1 *objects, they are the same divisions; similarly, you can permute the second* n_2 *objects, they are also the same divisions;....   So we get* $n!/(n_1!n_2!\cdots n_r!)$

1.5 Multinomial Coefficients

Notations

If $n_1 + n_2 + \cdots + n_r = n$, we define $\binom{n}{n_1, n_2, \ldots, n_r}$ by multinomial coefficient

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}.$$

= # of ways of dividing n distinct items into r distinct group with group sizes $n_1, n_2, \cdots, n_r$ such that $n_1+n_2+\cdots+n_r = n$. $(0 \le n_i \le n)$.

$$\binom{n}{s} = \binom{n}{s, n-s}$$

Example A police department in a small city consists of 10 officers. If the department policy is to have 5 of the officers patrolling the streets, 2 of the officers working full time at the station, and 3 of the officers on reserve at the station, how many different divisions of the 10 officers into the 3 groups are possible?

# of possible division $= \binom{10}{5, 2, 3} = \frac{10!}{5!\,2!\,3!} = \frac{10\times 9\times 8\times 7\times 6}{2\times 6}$

$= 2520.$

1.5 Multinomial Coefficients

Example Ten children are to be divided into an A team and a B team of 5 each. The A team will play in one league and the B team in another. How many different divisions are possible?

# of different divisions $= \binom{10}{5,5} = \frac{10!}{5!\,5!} = 252.$

Example In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

# of possible divisions $= \frac{\binom{10}{5,5}}{2} = \frac{252}{2} = 126.$

1.5 Multinomial Coefficients

Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Multinomial Theorem

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{(n_1,\ldots,n_r):n_1+\cdots+n_r=n} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

Note that the sum is over all nonnegative integer-valued vectors $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \cdots + n_r = n$.

$x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$

$$\begin{aligned}(x_1 + \cdots + x_r)^n \\ &= (x_1 + \cdots + x_r)(x_1 + \cdots + x_r)(x_1 + \cdots + x_r)\cdots(x_1 + \cdots + x_r)\end{aligned}$$

1.5 Multinomial Coefficients

Multinomial Theorem

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{(n_1,\ldots,n_r):n_1+\cdots+n_r=n} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

Note that the sum is over all nonnegative integer-valued vectors $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \cdots + n_r = n$.

$$(x_1 + \cdots + x_r)^n$$
$$= (x_1 + \cdots + x_r)(x_1 + \cdots + x_r)(x_1 + \cdots + x_r) \cdots (x_1 + \cdots + x_r)$$

Say, choose x_1 $\quad x_3 \quad x_1 \quad x_2$

Then, multiply them together $\quad x_1 x_3 x_1 \cdots x_2$

1.6 Number of Integer Solutions

Theorem *There are* $\binom{n-1}{r-1}$ *distinct positive integer-valued vectors* $(x_1, x_2, \ldots, x_r)$ *that satisfies the equation*

$$x_1 + x_2 + \cdots + x_r = n,$$

where $x_i > 0$ *for* $i = 1, \ldots, r$.

Proof (Stars and Bars method).

The problem is equivalent to the following:

we put n stars (indistinguishable) into r bins (labeled from 1 to r) such that no bin is empty.

(x_i = # of stars in the ith bin ; $x_i > 0$).

Say $n = 7$, $r = 4$ $\longleftrightarrow$ $(r-1) = 3$ bars $(n-1) = 6$ slots

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Place n stars in a line first. Then put $(r-1)$ bars to seperate them into r (non-empty) bins. (there are $(n-1)$ slots to place the bars)

$\binom{n-1}{\ }$

 # of possible ways = (r-1).

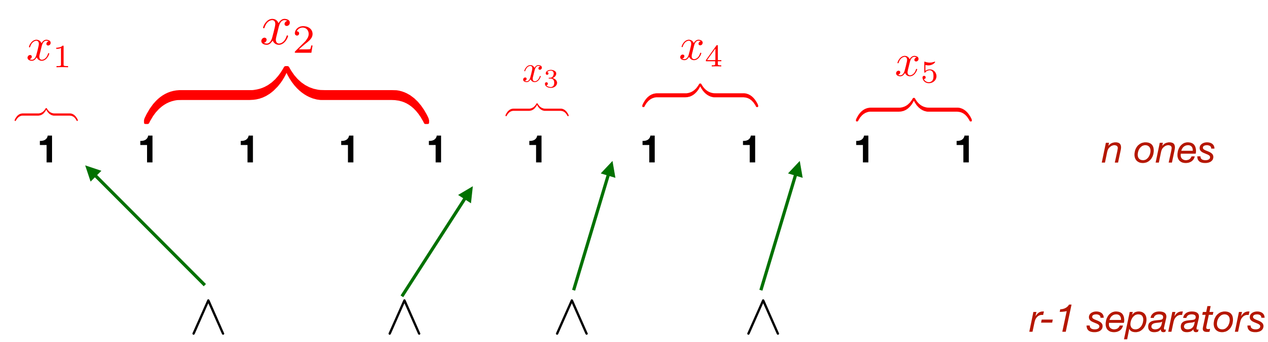
1.6 Number of Integer Solutions

(1) **Theorem** *There are $\binom{n-1}{r-1}$ distinct positive integer-valued vectors $(x_1, x_2, \ldots, x_r)$ that satisfies the equation*

$$x_1 + x_2 + \cdots + x_r = n,$$

where $x_i > 0$ for $i = 1, \ldots, r$.

Proof



Choose r-1 from n-1 places to put the separators

1.6 Number of Integer Solutions

2

Theorem *There are* $\binom{n+r-1}{r-1}$ *distinct non-negative integer-valued vectors* $(x_1, x_2, \ldots, x_r)$ *that satisfies the equation*

$$x_1 + x_2 + \cdots + x_r = n,$$

where $x_i \geq 0$ *for* $i = 1, \ldots, r$.

Proof (Combinatorial proof: use Stars and Bars).

Put n stars (indistinguishable) into r bins ; the bins can be empty.

Say $n=7$, $r=4$. bins $\Longleftrightarrow$ $(r-1)=3$ bars

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We arrange a total of $7+3 = n+r-1$ objects in a line and choose $(r-1)=3$ of them to be bars.

Total possible ways $= \binom{n+r-1}{r-1}$

1.6 Number of Integer Solutions

Theorem *There are* $\binom{n+r-1}{r-1}$ *distinct non-negative integer-valued vectors* $(x_1, x_2, \ldots, x_r)$ *that satisfies the equation*

$$x_1 + x_2 + \cdots + x_r = n,$$

where $x_i \geq 0$ *for* $i = 1, \ldots, r$.

$x_i \geq 0$

Proof

Proof. Let $y_i = x_i + 1$, then $y_i > 0$ and the number of non-negative solutions of

$$x_1 + x_2 + \cdots + x_r = n$$

is the same as the number of positive solutions of

$$(y_1 - 1) + (y_2 - 1) + \cdots + (y_r - 1) = n$$

i.e.,

$$y_1 + y_2 + \cdots + y_r = n + r,$$

which is $\binom{n+r-1}{r-1}$. □

1.6 Number of Integer Solutions

Example An investor has 20 thousand dollars to invest among 4 possible investments. Each investment must be in units of a thousand dollars. If the total 20 thousand is to be invested, how many different investment strategies are possible? What if not all the money need be invested?

Let x_i, $i=1,2,3,4$ be the number of thousands spent in investment i.

If all 20 thousand is invested, then x_i's are integers satisfying

$$x_1+x_2+x_3+x_4=20 \qquad \binom{n=20}{r=4}.$$

$$x_i \geq 0, \quad i=1,2,3,4,$$

then the # of possible investment strategies $=\binom{n+r-1}{r-1}$

$$=\binom{23}{3}=\frac{23\times 22\times 21}{6}=1771.$$

If not all 20 thousand need to be invested, let x_5 be the amount spent in reserve.

then $x_1, \cdots, x_5$ are integer solutions to

$$x_1+\cdots+x_5=20 \qquad n=20,\ r=5.$$

$$x_i \geq 0, \quad i=1,2,\cdots,5$$

 $\Rightarrow$ # of possible investments $= \binom{n+r-1}{r-1} = \binom{24}{4}$

1.6 Number of Integer Solutions

$= 10,626.$

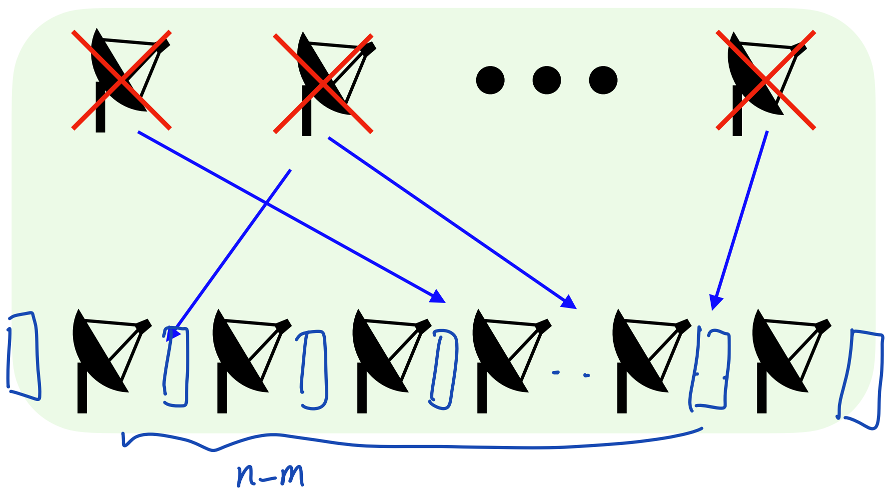
Example Consider a set of n antennas of which m are defective and $n-m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

How do we arrange them so that no two defectives are consecutive?

Previously,

Fix the functional ones ($n-m$) and insert defective ones into them

$\binom{n-m+1}{m}$



1.6 Number of Integer Solutions

Example Consider a set of n antennas of which m are defective and $n-m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

Fix the m defective ones. Then insert the functional ones.

$$x_1 \quad X \quad x_2 \quad X \quad x_3 \quad X \quad \cdots\cdots \quad x_m \quad X \quad x_{m+1}$$

$x_i = \#$ of functional ones.

Require $x_1 \geq 0$, $x_2 > 0$, $x_3 > 0$, $\cdots$, $x_m > 0$, $x_{m+1} \geq 0$.

$$x_1 + x_2 + \cdots + x_m + x_{m+1} = n-m.$$

Let $y_1 = x_1 + 1 > 0$, $y_2 = x_2$, $y_3 = x_3$, $\cdots$, $y_m = x_m$, $y_{m+1} = x_{m+1}+1$

then $y_i > 0$, $1 \leq i \leq m+1$

$$y_1 + \cdots + y_{m+1} = n-m+2$$

The # of integer solutions of y_i's is $\binom{n-m+1}{m}$.