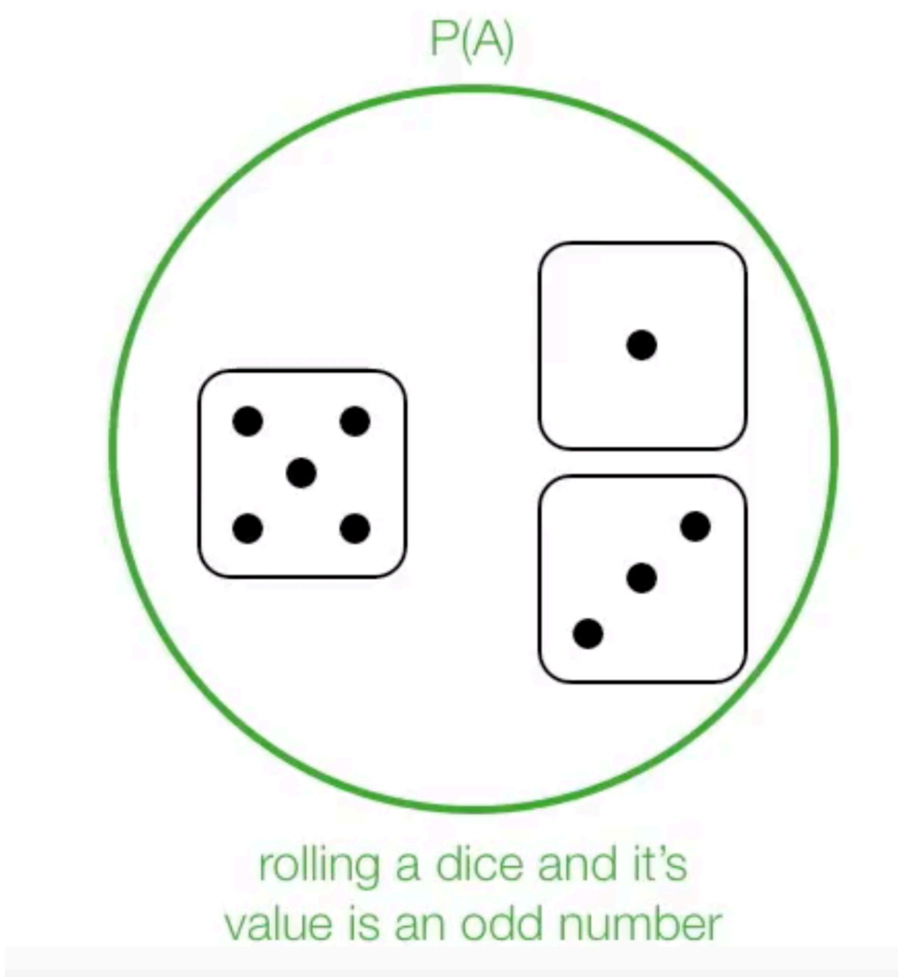
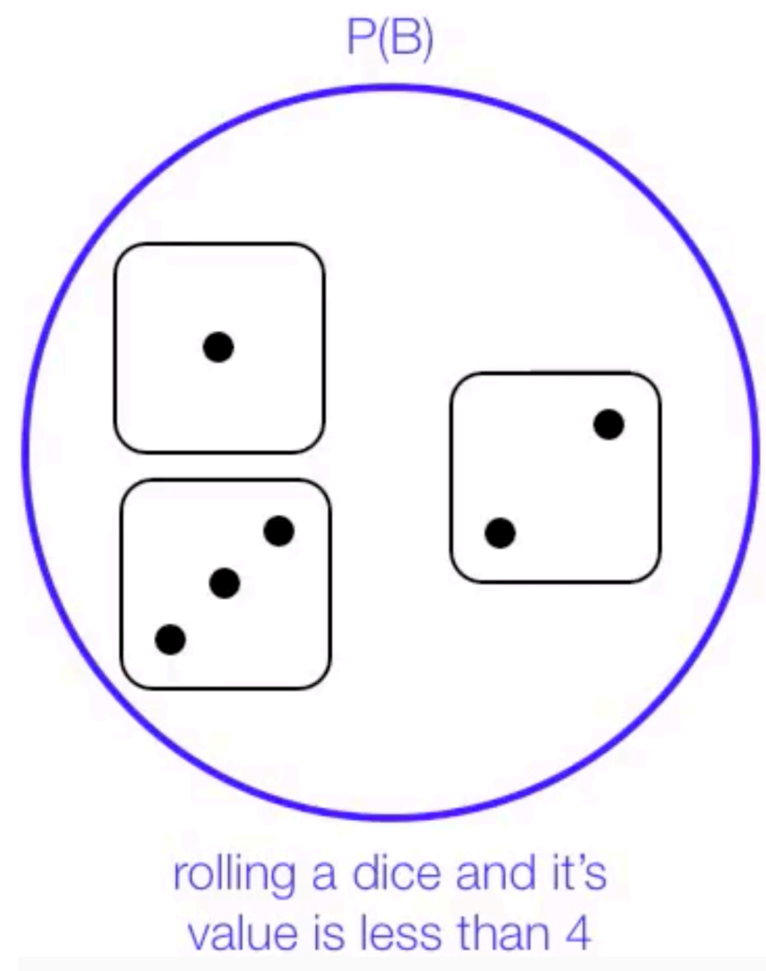
Chapter 3 Conditional Probability and Independence





Chapter 3 Conditional Probability and Independence

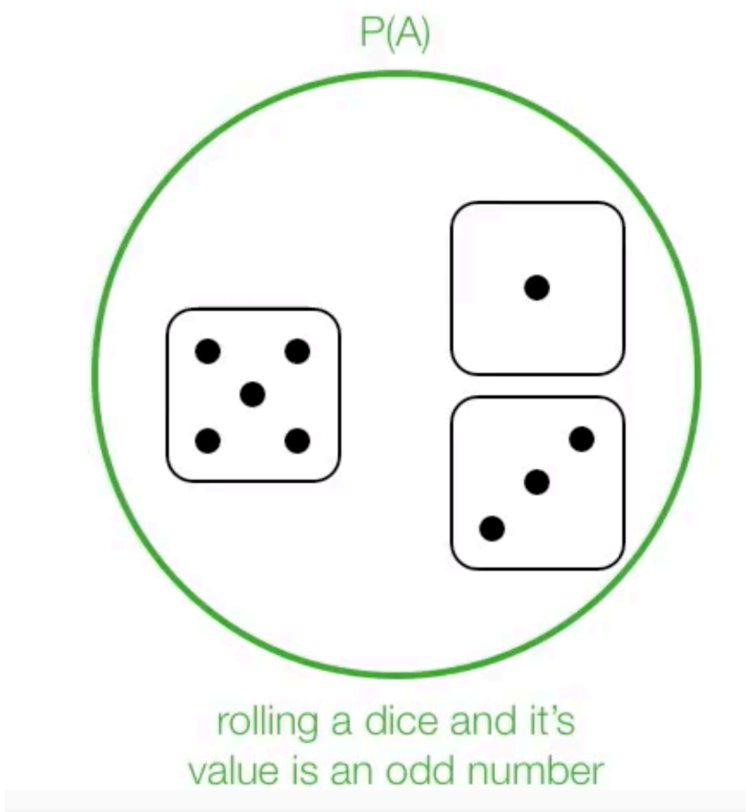
Outline

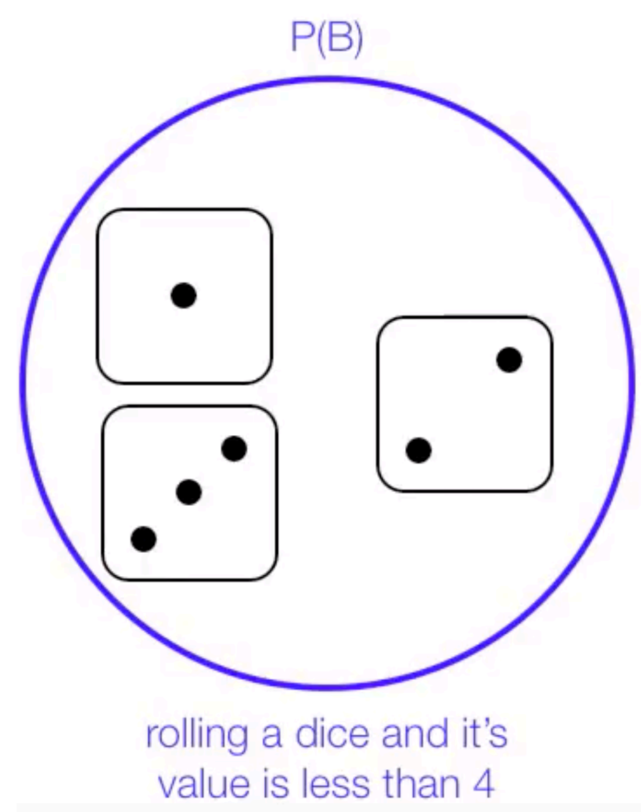
3.1 Introduction

Motivation 1

In many problems, we are interested in event B. However, we have some partial information, namely, that an event A has occurred. How to make use of this information?

Example





3.1 Introduction

Motivation 2

In calculating $P(B)$, there are occasions that we can consider it under different cases. How do we compute the probability of B under these cases? How do we combine them to give $P(B)$?

Example

Randomly choose a dice and roll it

event $B = \{4 \text{ is observed}\}$

Case 1: 6-faced die is chosen

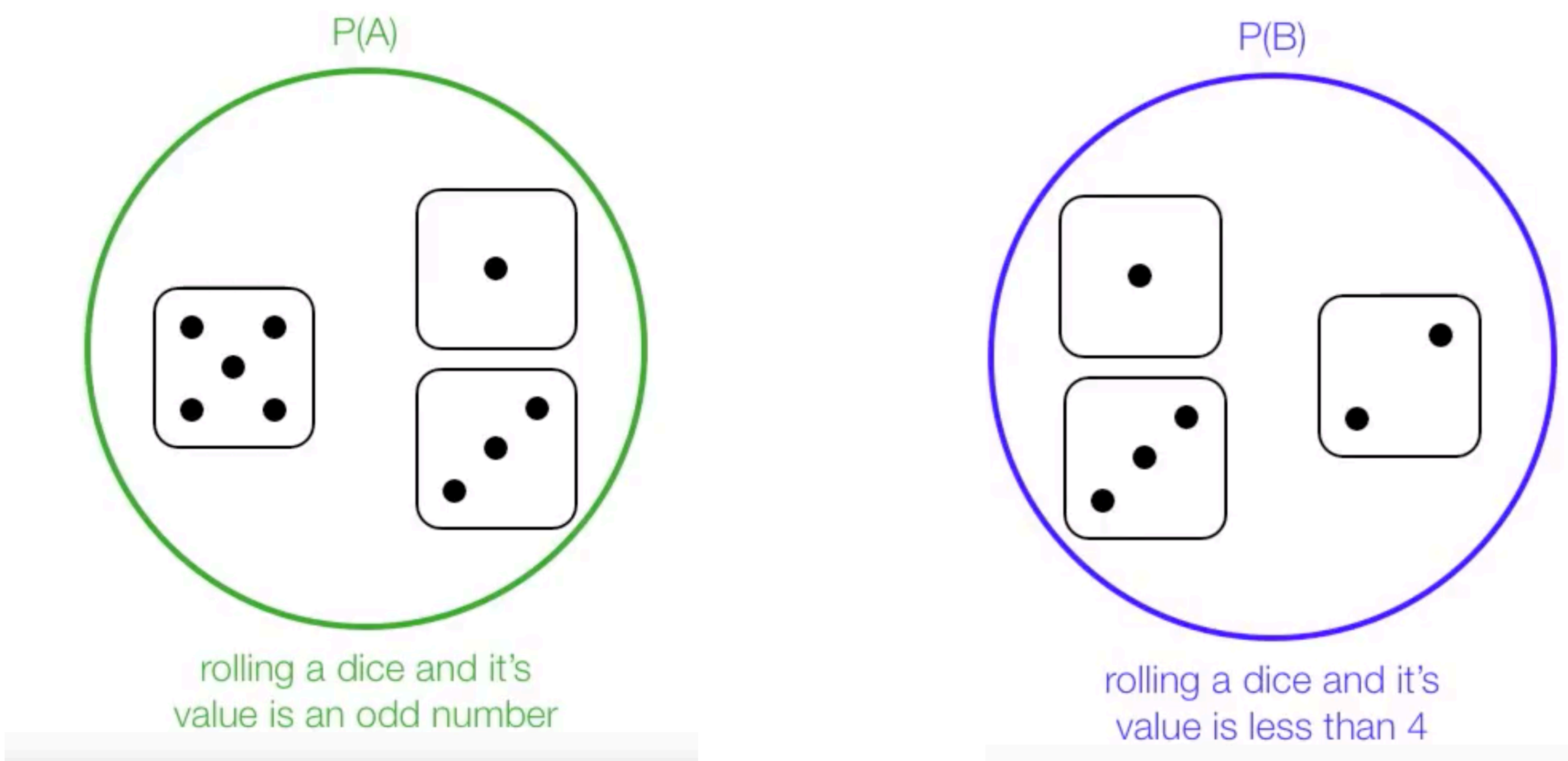
Case 2: 4-faced die is chosen

Case 3: 10-faced die is chosen

Case 4: 12-faced die is chosen

3.2 Conditional Probability

Example



The probability of B when we know that event A occurs?

Solution

3.2 Conditional Probability

Example Suppose that 2 dice are tossed, and all 36 outcomes are equally likely to occur. Suppose further that we observe that the first die is a 3. Then, given this information, what is the probability that the sum of the 2 dice equals 8?

Solution

3.2 Conditional Probability

Definition

Let E be the event that the sum of the dice is 8, and let F be the event that the first die is 3. Then the probability just obtained is called *conditional probability that E occurs given that F has occurred*. It is usually denoted by

$$P(E|F).$$

Generally, we can talk about $P(E|F)$ for all events E and F. It can be derived in the same manner: If F occurs, then in order for E to occur as well, it is necessary that the outcome is in both EF. Now, as we know, F has occurred, it follows that F becomes our new or reduced sample space since anything in F^c will not occur. Hence the probability of EF given F shall equal the probability of EF *relative to* the probability of F.

We sometimes write $E \cap F$ as EF

3.2 Conditional Probability

Definition (Conditional probability). *If* $P(F) > 0$, *then*

$$P(E|F) = \frac{P(EF)}{P(F)}.$$

Remark Especially, when the sample space S has equally likely outcomes, $P(E|F)$ is just the proportion of EF divided by the proportion of F.

$$P(E|F) = \frac{\text{number of outcomes in } E \cap F}{\text{number of outcomes in } F}$$

3.2 Conditional Probability

Class Discussion

Let F be an event, and we define the conditional probability $P(.|F)$ so that for any event E, we assign the number $P(E|F)=P(EF)/P(F)$ to event E at its probability.

By Kolmogorov's axioms, is $P(.|F)$ a valid probability?

3.2 Conditional Probability

Example A student is taking a one-hour-time exam. Suppose the probability that the student will finish the exam in less than x hours is $\frac{x}{2}$, for all $0 \leq x \leq 1$. Given that the student is still working after 0.75 hours, what is the conditional probability that the full hour is used?

It is important to properly define the events at the start of the solutions.

Solution

3.2 Conditional Probability

Example A coin is flipped twice. Assume that all 4 possible outcomes in the sample space $S = \{(H,H),(H,T),(T,H),(T,T)\}$ are equally likely. What is the conditional probability both flips give heads, given that (a) the first flip give heads; (b) at least one flip gives heads?

Solution Let $B = \{(H,H)\}$ be the event that both flips give heads; let $F = \{(H,H),(H,T)\}$ be the event that the first flip gives the heads; and let $A = \{(H,H),(H,T),(T,H)\}$ be the event that at least one flip gives the heads. The probability for (a) can be obtained from

$$P(B|F) = \frac{P(BF)}{P(F)} = \frac{P(B)}{P(F)} = \frac{1/4}{2/4} = \frac{1}{2}.$$

For (b), we have

$$P(B|A) = \frac{P(BA)}{P(A)} = \frac{P(B)}{P(A)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

3.2 Conditional Probability

Example A bin contains 25 light bulbs, of which 5 are good and function at least 30 days, 10 are partially defective and will fail in the second day of use, while the rest are totally defective and won't light up at all. Given that a randomly chosen bulb initially lights up, what is the probability that it will still be working after one week?

It is important to properly define the events at the start of the solutions.

Solution

3.2 Conditional Probability

(Multiplication Rule)
Suppose that $P(F) > 0$, then

$$\underbrace{P(EF)}_{E \cap F} = P(E|F)P(F).$$

In words, the above equation states that the probability that both E and F occur is equal to the probability that F occurs multiplied by the conditional probability of E given that F occurred. This equation is very useful in computing the probability of the intersection of events.

3.2 Conditional Probability

Example Celine is undecided as to whether to take a French course or a chemistry course. She estimates that her probability of receiving an $\boldsymbol{A}$ grade would be $\frac{1}{2}$ in a French course, and $\frac{2}{3}$ in a chemistry course. If Celine decides to base her decision on the flip of a fair coin, what is the probability that she gets an $\boldsymbol{A}$ in chemistry.

Solution

3.2 Conditional Probability

Example Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls, one at a time, from the urn without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red?

meaning that we do not put the selected ball back to the urn

Solution 1 (by conditional probability)

3.2 Conditional Probability

Example Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls, one at a time, from the urn without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red?

view every ball as a distinct ball, then the sample space has equally likely outcomes

Solution 2 (by combinatorial analysis)

3.2 Conditional Probability

(General Multiplication Rule)
Let $A_1, A_2, \ldots, A_n$ be n events, then

$A_1 \cap A_2 \cap \cdots \cap A_n$

$$\begin{aligned} & P(A_1 A_2 \cdots A_n) \\ = \; & P(A_1)P(A_2|A_1)P(A_3|A_1A_2)\cdots P(A_n|A_1A_2\cdots A_{n-1}). \end{aligned}$$

Proof. (Sketch)

$$RHS = P(A_1)\frac{P(A_1A_2)}{P(A_1)}\frac{P(A_1A_2A_3)}{P(A_1A_2)}\cdots\frac{P(A_1\cdots A_n)}{P(A_1\cdots A_{n-1})}$$

3.2 Conditional Probability

Example

Three cards are selected successively at random and removed without replacement from a standard deck of 52 playing cards. Calculate the probability of receiving, in order, a king, a queen, a jack.

Solution 1 (by conditional probability)

3.2 Conditional Probability

Example

Three cards are selected successively at random and removed without replacement from a standard deck of 52 playing cards. Calculate the probability of receiving, in order, a king, a queen, a jacket.

order matters here

Solution 2 (by combinatorial analysis)

3.2 Conditional Probability

Example

A box of fuses contains 20 fuses, of which 5 are defective. If three of the fuses are selected randomly and removed from the box in succession without replacement, calculate the probability that all three fuses are defective.

Solution 1 (by conditional probability)

3.2 Conditional Probability

Example

A box of fuses contains 20 fuses, of which 5 are defective. If three of the fuses are selected randomly and removed from the box in succession without replacement, calculate the probability that all three fuses are defective.

Solution 2 (by combinatorial analysis)

regard all fuses are different, and the order does not matter

3.2 Conditional Probability

Class Discussion

If six cards are selected at random (without replacement) from a standard deck of 52 cards, what is the probability there will be no pairs?

Solution 1 (by conditional probability)

3.2 Conditional Probability

Class Discussion

If six cards are selected at random (without replacement) from a standard deck of 52 cards, what is the probability there will be no pairs?

Solution 2 (by combinatorial analysis)

3.3 Total Probability

In calculating $P(B)$, there are occasions that we can consider it under different cases. How do we compute the probability of B under these cases? How do we combine them to give $P(B)$?

Example

Randomly choose a dice and roll it

event $B = \{3 \text{ is observed}\}$

Case 1: 6-faced die is chosen

Case 2: 4-faced die is chosen

Compute $P(B)$?

3.3 Total Probability

Class Discussion

We have 2 fair dice. The 1st die has 4 faces, with numbers 1, 2, 3, 4. The 2nd die has 6 faces, with numbers 1, 2, 3, 4, 5, 6.
Now, we randomly choose a die and roll it. Let X be the number we get.

What is the probability that $X = 3$?

Case 1: the 4-faced die is chosen

$$P(X = 3) =$$

Case 2: the 6-faced die is chosen

$$P(X = 3) =$$

In total

$$P(X = 3) =$$

3.3 Total Probability

Theorem

Let A and B be any two events, then

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c).$$

Proof

3.3 Total Probability

Example In answering a question on a multiple-choice test, a student either knows the answer or guesses the answer at random. Let p be the probability that the student knows the answer, and $1 - p$ the probability that he doesn't. Suppose there are m alternatives in the question.

(a) What is the probability that he answered it correctly?

(b) What is the probability that the student knew the answer given that he answered it correctly?

Solution

3.3 Total Probability

Example An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. Their statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas the probability is only 0.2 for a non-accident-prone person. If we assume that 30 percent of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy.

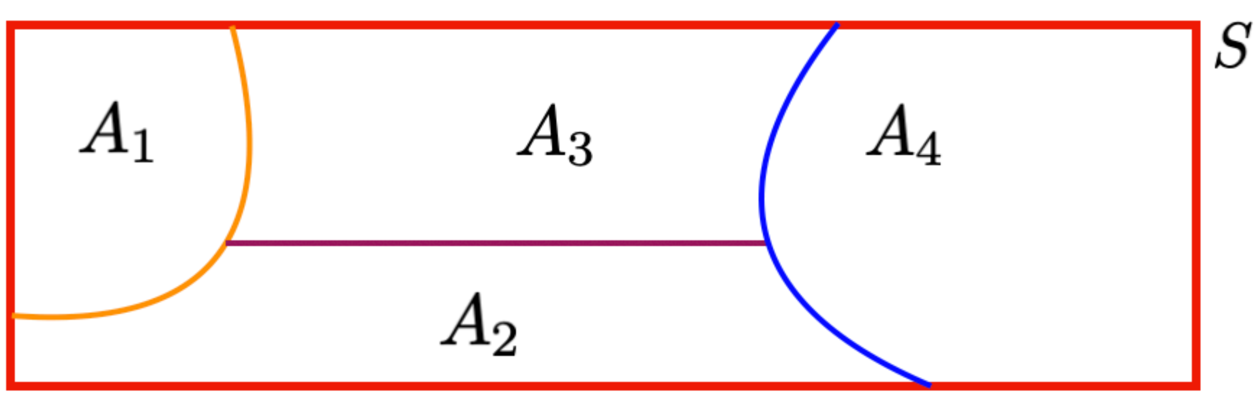
Solution

3.3 Total Probability

Definition *We say that $A_1, A_2, \ldots, A_n$ **partitions** the sample space S if:*

(a) *They are "mutually exclusive", meaning $A_i \cap A_j = \emptyset$, for all $i \neq j$.*

(b) *They are "*exhaustive*", meaning $\cup_{i=1}^{n} A_i = S$.*



3.3 Total Probability

Law of Total Probability *Suppose the events $A_1, A_2, \ldots, A_n$ partitions the sample space. Assume further that $P(A_i) > 0$ for $1 \leq i \leq n$. Let B be any event, then*

$$P(B) = P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n).$$

Proof

3.3 Total Probability

Example

Suppose that in country S, 40% of the people support party A, 30% of the people support party B, 20% support party C, and 10% support party D. Let Q be a certain policy. We're given that 50% of the supporters of party A are in favor of Q, 40% of the supporters of party B are in favor of Q, 30% of the supporters of party C are in favor of Q, and 100% of the supporters of party D are in favor of Q. If we draw a citizen from this imaginary country at random, what is the probability that the citizen supports Q? Let Q denote the event "is in favor of policy Q", A be the event "supports party A" (and so on for the rest of the parties).

Solution

3.3 Total Probability

Class Discussion

- In a certain county
  - 60% of registered voters are Republicans
  - 30% are Democrats
  - 10% are Independents.
- When those voters were asked about increasing military spending
  - 40% of Republicans opposed it
  - 65% of the Democrats opposed it
  - 55% of the Independents opposed it.
- What is the probability that a randomly selected voter in this county opposes increased military spending?

3.4 Bayes' Theorem

Bayes' formula *Suppose the events $A_1, A_2, \ldots, A_n$ partitions the sample space. Assume further that $P(A_i) > 0$ for $1 \le i \le n$. Let B be any event, then for any $1 \le i \le n$,*

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)}.$$

Proof

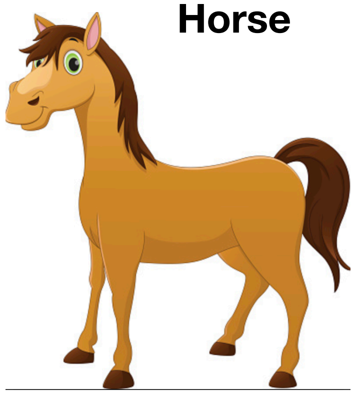
3.4 Bayes' Theorem

Mule

I 'saw' a mule in the village today.

I am pretty sure that my eyes fooled my brain. Because my brain told me that there are only two animals in my village: horse and donkey.

Horse



Donkey

So, it must be a horse or a donkey. How do I determine which one it is?

30%

70%

I know there are 30% horses and 70% donkeys in the village. Therefore, I have a ***'prior belief'*** *that I have 30% chance to encounter a horse and 70% to encounter a donkey.*

Suppose the a horse is mistaken as a mule with probability 0.4 and a donkey is mistaken as a mule with probability 0.8.

By Bayes' Theorem

'posterior belief'

P(horse | saw a mule)=0.176

P(donkey | saw a mule)=0.823

3.4 Bayes' Theorem

Example A blood test is 95 percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1 percent of the healthy persons tested. If 0.5 percent of the population actually has the disease, what is the probability a person has the disease given that the test result is positive?

Solution

3.4 Bayes' Theorem

Example A plane is missing, and it is presumed that it was equally likely to have gone down in any of 3 possible regions. Let $1 - \beta_i$ be the probability that the plane will be found upon a search of the ith region when the plane is indeed there, for $i = 1, 2, 3$. What is the conditional probability that the plane is in the ith region, given that a search of region 1 is unsuccessful, $i = 1, 2, 3$?

Solution

3.4 Bayes' Theorem

Example

In a certain assembly plant, three machines, I, II and III, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. **What is the probability that it is defective?**

Given the fact that the selected machine is defective, what is the probability that it is type I?

Solution

3.5 Independence

Definition *Two events A and B are said to be* ***independent*** *if*

$$P(AB) = P(A)P(B).$$

They are said to be **dependent** if

$$P(AB) \neq P(A)P(B).$$

Motivation: Suppose $P(B) > 0$. Then,

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A).$$

That is, A is independent of B if knowledge that B has occurred does NOT change the probability that A occurs.

3.5 Independence

Example A card is selected at random from an ordinary deck of 52 playing cards. If E is the event that the selected card is an ace and F is the event that it is spade $\spadesuit$, then E and F are independent.

Proof. This follows from the definition of independence since $P(EF) = \frac{1}{52}$, whereas $P(E) = \frac{4}{52}$ and $P(F) = \frac{13}{52}$. $\square$

Example Two coins are flipped, and all 4 outcomes are assumed to be equally likely. If E is the event that the first coin lands heads and F the event that the second lands tails, then E and F are independent.

Proof This also follows from the definition of independence easily, since

$$P(EF) = P(\{H, T\}) = \frac{1}{4};$$

$$P(E) = P(\{(H, H), (H, T)\}) = \frac{1}{2};$$

$$P(F) = P(\{(H, T), (T, T)\}) = \frac{1}{2}.$$

3.5 Independence

Example Two coins are flipped, and all 4 outcomes are assumed to be equally likely. If E is the event that the first coin lands heads and F the event that only one coin lands head then E and F are independent ?

3.5 Independence

Example Suppose that we toss 2 fair dice. Let E_1 be the event that the sum of the dice is 6 and F denote the event that first die equals 4. Then are E_1 and F independent?

$$P(E_1F) = P(\{4,2\}) = \frac{1}{36},$$

whereas

$$P(E_1)P(F) = \frac{5}{36} \times \frac{1}{6} = \frac{5}{216}.$$

Hence, E_1 and F are not independent.

Remark the reason why E_1 and F are dependent in the above example can be explained intuitively in words. Note that since we are interested in the possibility to get 6 with 2 dice we shall be happy to get any of $1, 2, 3, 4, 5$ for the first die, for then we shall still have a chance to get a total of 6. However, if the first die gives 6 already, then there is no chance to get a total 6 with 2 dice. That means, the *chance* to get a total 6 indeed depends on the outcome of the first die.

3.5 Independence

Class Discussion Suppose that we toss 2 fair dice. Let E_1 be the event that the sum of the dice is 7 and F denote the event that first die equals 4. Then are E_1 and F independent?

Solution

3.5 Independence

Theorem *If A and B are independent, then so are*

(i) A and B^c; (ii) A^c and B; (iii) A^c and B^c.

Proof

3.5 Independence

Example Two fair dice are thrown. Let A be the event that the sum of the dice is 7; B the event that first die is 4; and C the event that the second die is 3.

$A = \{(1,6),(2,6),(3,4),(4,3),(5,2),(6,1)\}$.
$B = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$.
$C = \{(1,3),(2,3),(3,3),(4,3),(5,3),(6,3)\}$.
$AB = \{(4,3)\}$
$AC = \{(4,3)\}$
$BC = \{(4,3)\}$ and
$ABC = \{(4,3)\}$
Then,

$$P(AB) = P(A)P(B)$$

and

$$P(AC) = P(A)P(C)$$

that is A and B are independent; also A and C are independent.
However,

$$P(ABC) = \frac{1}{36} \neq \frac{1}{216} = \frac{1}{6} \times \frac{1}{36} = P(A)P(BC).$$

A is dependent with $B \cap C$

This example shows that even though A, B, C are pair-wisely independent.
But A can be dependent with BC.

In this case, we say A, B, C are dependent, but they are pair-wisely independent.

3.5 Independence

Definition *Three events A, B and C are said to be independent if the following 4 conditions hold:*

$$P(ABC) = P(A)P(B)P(C) \tag{1}$$

$$P(AB) = P(A)P(B) \tag{2}$$

$$P(AC) = P(A)P(C) \tag{3}$$

$$P(BC) = P(B)P(C) \tag{4}$$

Remark Second condition implies A and B are independent;
Third condition implies A and C are independent; and
Fourth condition implies B and C are independent
That is, A, B and C are pairwise independent.

3.5 Independence

Theorem It should be noted that if A, B and C are independent, then A is independent of any event formed from B and C.

(i) A is independent of $B \cup C$.

(ii) A is independent of $B \cap C$.

Proof

3.5 Independence

Definition *Events $A_1, A_2, \ldots, A_n$ are said to be independent if, for every sub-collection of events $A_{i_1}, \ldots, A_{i_r}$, we have*

$$P(A_{i_1} \cdots A_{i_r}) = P(A_{i_1}) \cdots P(A_{i_r}). \qquad \text{for } r = 1, 2, 3, \cdots, n$$

For $n = 4$, that is, 4 events A_1, A_2, A_3 and A_4 are independent if we can verify (i), (ii) and (iii) below:

(i) $r = 4$:

$$P(A_1A_2A_3A_4) = P(A_1)P(A_2)P(A_3)P(A_4)$$

(ii) $r = 3$: there are 4 conditions,

$$\begin{aligned} P(A_1A_2A_3) &= P(A_1)P(A_2)P(A_3) \\ P(A_1A_2A_4) &= P(A_1)P(A_2)P(A_4) \\ P(A_1A_3A_4) &= P(A_1)P(A_3)P(A_4) \\ P(A_2A_3A_4) &= P(A_2)P(A_3)P(A_4) \end{aligned}$$

(iii) $r = 2$: there are 6 conditions,

$$\begin{aligned} P(A_1A_2) &= P(A_1)P(A_2) \\ P(A_1A_3) &= P(A_1)P(A_3) \\ P(A_1A_4) &= P(A_1)P(A_4) \\ P(A_2A_3) &= P(A_2)P(A_3) \\ P(A_2A_4) &= P(A_2)P(A_4) \\ P(A_3A_4) &= P(A_3)P(A_4) \end{aligned}$$

3.5 Independence

Example We are given a loaded coin with probability of getting a head $= p$; with probability of getting a tail $= 1 - p$.

This loaded coin is tossed n times independently. What is the probability of getting

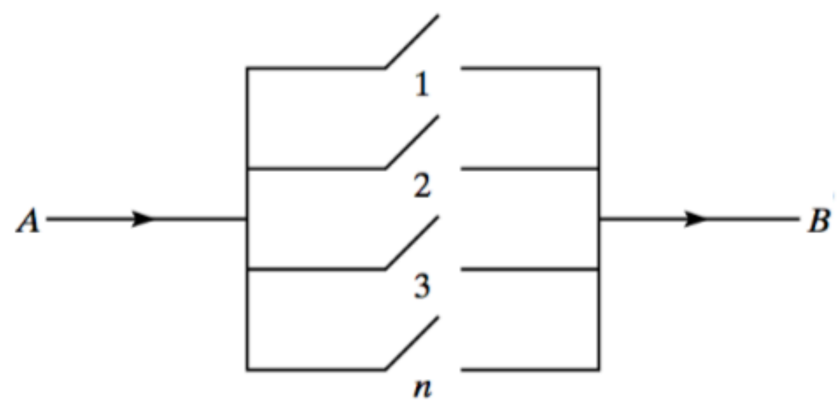
(i) at least 1 head in these n tosses?

(ii) exactly k heads in these n tosses?

Solution

3.5 Independence

Example A system composed of n separate components is said to be a parallel system if it functions when at least one of the components functions, see the figure below For



such a system, if component i, independent of other components, functions with probability $p_i, i = 1, \cdots, n$, what is the probability that the system functions?

Solution