Chapter 2. Axioms of Probability

Outline

2.1 Introduction

In Chapter 1, we used some intuitive definitions to calculate probabilities.

Basically, we translated "calculate probability" into counting the number of total outcomes, and counting the number of interesting outcomes.

However, such a counting principle can only deal with "simple" experiments. There are experiments whose # of total outcomes is ∞
and # of interesting outcomes is also ∞

e.g., the lifetime of an iPhone, the height of an adult in Hong Kong

To develop "probability" into a powerful math tool, we shall rigorously formalize it.

2.1 Introduction

Terminologies of Probability

Random experiments
Outcomes

Event
Sample space

Rigorously,
what is probability ?

概率？

2.2 Sample Space and Events

Definitions

The basic object of probability is an **experiment**: an activity or procedure that produces distinct, well-defined possibilities called **outcomes**. The **sample space** is the set of all possible outcomes of an experiment, usually denoted by S.

Example The sample space of tossing a coin:

$$S = \{ \text{ head, tail } \}.$$

Example Tossing two dice:

$$\begin{aligned} S &= \{(1,1),(1,2),\ldots,(1,6),(2,1),\ldots,(6,6)\} \\ &= \{(i,j) : 1 \leq i,j \leq 6\}. \end{aligned}$$

2.2 Sample Space and Events

Example The lifetime of a transistor:

$$S = [0, \infty).$$

Definitions

Any subset E of the sample space is an **event**.

A sample space of a random experiment is the collection of ALL possible outcomes

An event of a random experiment is the collection of SOME possible outcomes

2.2 Sample Space and Events

Example The sample space of tossing a coin:

$$S = \{\text{ head, tail }\}.$$

$$E = \{\text{ head }\} \text{ is an possible event.}$$

Example Tossing two dice:

$$\begin{aligned} S &= \{(1,1),(1,2),\ldots,(1,6),(2,1),\ldots,(6,6)\} \\ &= \{(i,j) : 1 \le i,j \le 6\}. \end{aligned}$$

$$\begin{aligned} E &= \{\text{ sum of 2 dice is 7}\} \\ &= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} \end{aligned}$$

is an possible event.

2.2 Sample Space and Events

Example The lifetime of a transistor:

$$S = [0, \infty).$$

$E = \{x : 0 \leq x \leq 5\}$ is an possible event.

The number of elements in the sample space and event can be finite or infinite.

If the random experiment produces an outcome in event E, we say that "event E occurs"

2.2 Sample Space and Events

Class Discussion

If we randomly roll 2 six-faced dice, which can be fair or unfair, let x denote the number of 1st die, and y denote the number of 2nd die.
Among the following events, which are the most likely to occur?

Choose your answer?

Choice A

$E_1 = \{(x, y) : x = 1, y = 2\}$

Choice B

$E_2 = \{(x, y) : x = 2, y = 1\}$

Choice C

$E_3 = \{(x, y) : x + y = 3\}$

2.3 Operations on Events

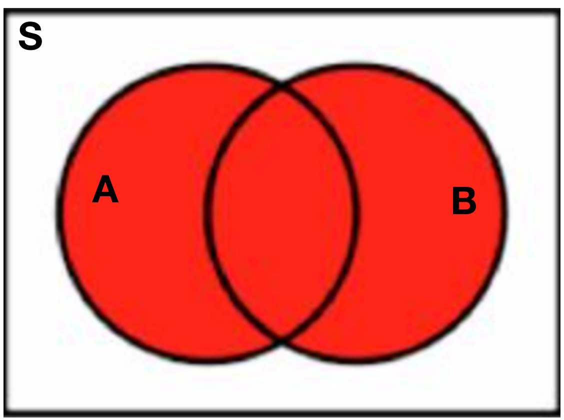
Four basic operations

Union: A U B represents the union of event A and event B, it contains the outcomes in either A or B or both. Since AUB is also a collection of outcomes, it is also an event.

Example: A={the selected card is King}
B={the selected card is heart}
, then AUB={the selected card is either King or heart}.

Either event A or event B occurs.

Venn diagram $A \cup B$

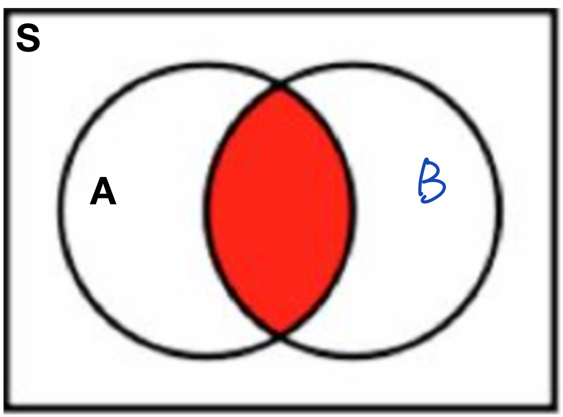


Intersection: $A \cap B$ represents the intersection of event A and event B, it contains only the outcomes in both A and B. Since $A \cap B$ is also a collection of outcomes, it is also an event.

Example: A={the selected card is King}
B={the selected card is heart}
, then $A \cap B$={the selected card is King of heart}.

Both event A and event B occur.

Venn diagram $A \cap B$ *(or AB in short)*



Remember that S is the sample space

2.3 Operations on Events

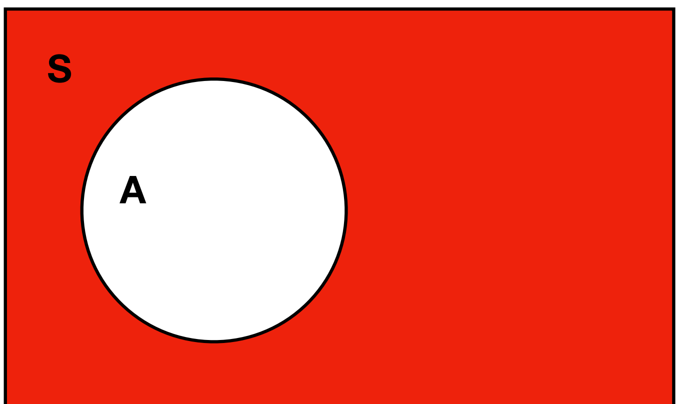
Four basic operations

Complement: A^c represents the complement of event A , it contains the outcomes NOT in A. Since A^c is also a collection of outcomes, it is also an event.
Example: A={the selected card is King}
, then A^c={the selected card is not King}.

Event A does NOT occur.

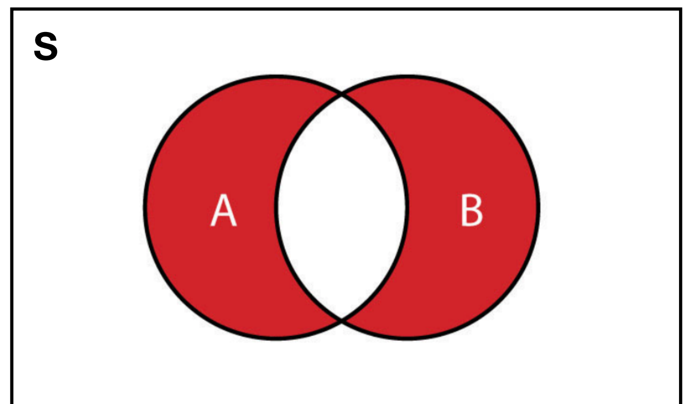
"\": minus
or substract

Venn diagram $A^c = S \setminus A$



Symmetric difference: $A \Delta B$ represents the symmetric difference of event A and B, it contains the outcomes in either A or B, but not in both A and B.
Example: A={the selected card is King}
B={the selected card is heart}
, then $A \Delta B$ ={the selected card is either King or heart, except King of heart}.

Venn diagram $A \Delta B = (A \cup B) \setminus AB.$



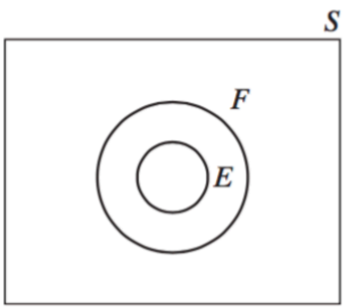
Remember that S is the sample space

2.3 Operations on Events

Sometimes, $E \subseteq F$ $x \in E$

Definitions (Inclusion of events). *For any two events E and F, if all of the outcomes in E are also in F, then we say that E is contained in F and write $E \subset F$, or $F \supset E$. Thus, if $E \subset F$, the occurrence of E necessarily implies the occurrence of F.*

If event E occurs, then event F also occurs, e.g., E={card is king of heart}, F={card is heart}



Remark 2.2.7. If $E \subset F$ and $F \subset E$, we have $E = F$.

Remember that S is the sample space

2.3 Operations on Events

$\emptyset$

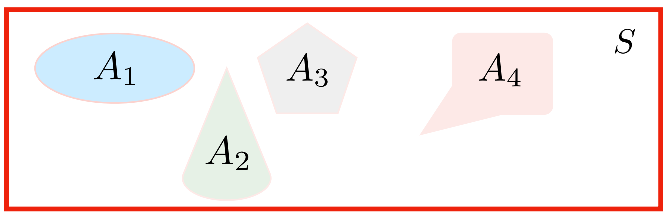
Additional Concepts

Disjoint: if two events share no common outcomes, that is their intersection is an empty set $\emptyset$, they are called disjoint.

Example: A={the selected card is King}, B={the selected card is Queen}. Then, $A \cap B = \emptyset$

Mutually exclusive: if events $A_1, A_2, \cdots, A_k$ are pairwise disjoint, then they are called mutually exclusive.

$A_i \cap A_j = \phi$ for $i \neq j$, $1 \leq i, j \leq k$

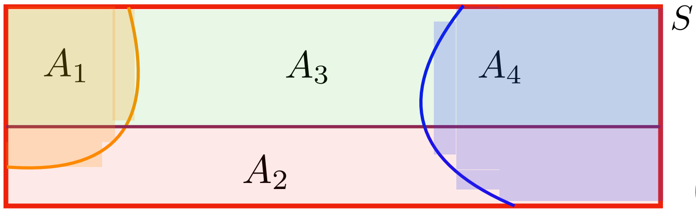


Can you represent them by Venn diagram?

2.3 Operations on Events

Additional Concepts

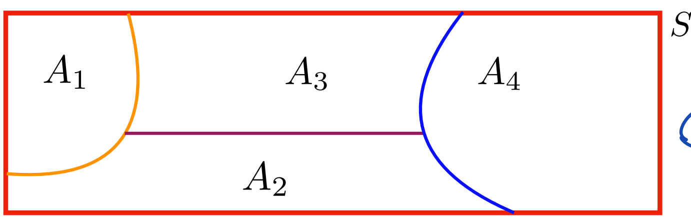
Exhaustive: if the union of some events $A_1, A_2, \cdots, A_k$ is the sample space, then they are called exhaustive.



$A_1 \cup \cdots \cup A_k = \bigcup_{i=1}^{k} A_i = S$

(they can overlap).

Partition: if the sample of events $A_1, A_2, \cdots, A_k$ are both exclusive and exhaustive, we call them a partition.



$A_1, \cdots, A_k$ is a partition:

① $A_i \cap A_j = \phi$ for $i \neq j$, $1 \leq i, j \leq k$

② $\bigcup_{i=1}^{k} A_i = S$.

2.3 Operations on Events

Fundamental Laws

Let A, B, C be any three events. Let $A_1, A_2, \cdots, A_k$ be events

Commutative law

$$1). \quad A \cap B = B \cap A, \quad A \cup B = B \cup A$$

Associative law

$$2). \quad A \cap (B \cap C) = (A \cap B) \cap C, \quad A \cup (B \cup C) = (A \cup B) \cup C$$

Distributive law

$$3). \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

DeMorgan's law

$$4). \quad \left(\cup_{i=1}^{k} A_i \right)^{c} = \cap_{i=1}^{k} A_i^{c}, \quad \left(\cap_{i=1}^{k} A_i \right)^{c} = \cup_{i=1}^{k} A_i^{c}$$

The complement of unions = intersection of the complements
The complement of intersections = the union of complements

2.3 Operations on Events

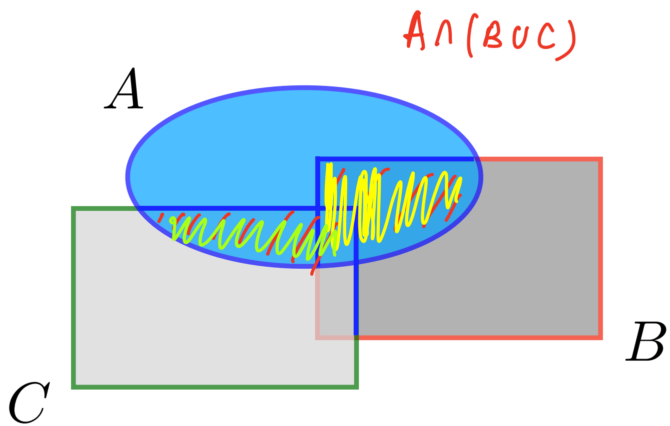
$a \times (b+c) = a\times b + a \times c$

Fundamental Laws

Distributive law

3). $A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Illustration by Venn diagram



2.3 Operations on Events

$(A_1 \cup A_2)^c = A_1^c \cap A_2^c$

Fundamental Laws

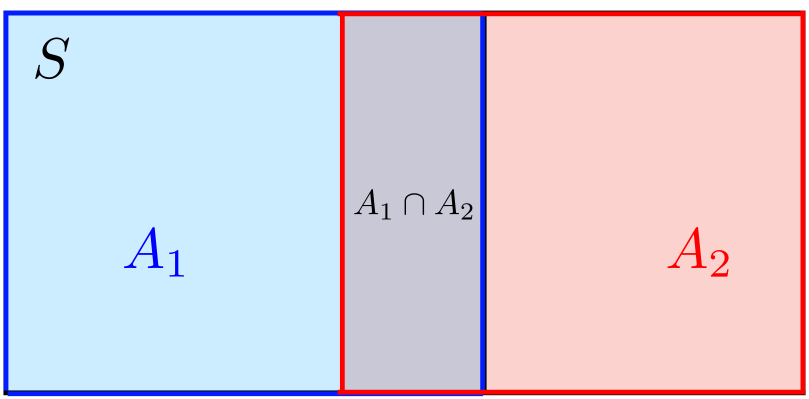
$(A_1 \cap A_2)^c = A_1^c \cup A_2^c$

DeMorgan's law

4). $\left(\cup_{i=1}^k A_i\right)^c = \cap_{i=1}^k A_i^c, \quad \left(\cap_{i=1}^k A_i\right)^c = \cup_{i=1}^k A_i^c$

The complement of unions = intersection of the complements
The complement of intersections = the union of complements

Illustration by Venn diagram



2.3 Operations on Events

useful

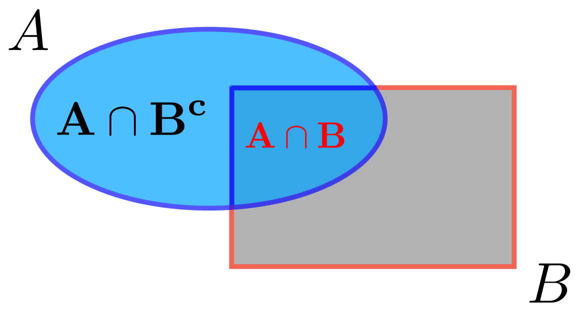
Lemma For any events A, B, we have

$$A = (A \cap B) \cup (A \cap B^c)$$

$A = AB \cup AB^c$

and $AB \cap AB^c = \phi$.

Illustration by Venn diagram



2.3 Operations on Events

Lemma For any events $A, B,$ we have

$$A = (A \cap B) \cup (A \cap B^c)$$

Proof

By distributive law

$$\begin{aligned}(A \cap B) \cup (A \cap B^c) &= A \cap (B \cup B^c) \\ &= A \cap S \\ &= A\end{aligned}$$

2.4 Axioms of Probability

What is probability ?

Intuitively, a **probability** is a function that assigns numbers to events,

Example the probability of "sum of 2 dices is 7" is 1/6.

This number can characterize how likely this event occurs.

Appetizer

If I randomly roll this dice,
how likely I will get 5?
and how to describe this likelihood?

An intuitive idea is to roll this dice for 1 million times and count the number of times when 5 is observed.

Then, the relative frequency can be regarded as a measure of how likely 5 is observed.

2.4 Axioms of Probability

Definitions (primitive definition via limiting frequency). *Suppose that an experiment, with sample space S, is repeatedly performed. For each event $E \subset S$, we define $n(E)$ to be the number of times in the first n repetitions of the experiment that the event E occurs. Then $P(E)$, the probability of the event E, is defined by*

$$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}.$$

That is, $P(E)$ is defined as the (limiting) proportion of time that E occurs.

2.4 Axioms of Probability

The issues of primitive definitions

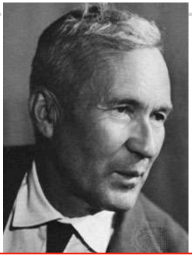
However, the above definition has some serious drawback:

1. How do we know if the limit of $\frac{n(E)}{n}$ exists or not for a sequence of repetitions of the experiment?

2. Even if the limits exist for all sequences, how do we know that the limits are the same?

For a mathematical probability model, we can certainly make an axiom to assume that $\frac{n(E)}{n}$ will converge to the same constant value. However, this statement seems too complicated to be an axiom.

Modern probability theory is built on Kolmogorov Axiom



Andrey Nikolaevich Kolmogorov was a Soviet mathematician who made significant contributions to the mathematics of probability theory, topology, intuitionistic logic, turbulence, classical mechanics, algorithmic information theory and computational complexity.

2.4 Axioms of Probability

Kolmogorov Axioms Andrey Kolmogorov in 1933

Probability, denoted by P, is a function on the collection of events satisfying

(i) For any event A,

$$0 \leq P(A) \leq 1.$$

(ii) Let S be the sample space, then

$S = \cup \{\text{outcome}\}.$

$$P(S) = 1.$$

(iii) For any sequence of mutually exclusive events $A_1, A_2, \ldots$ (that is, $A_i \cap A_j$ $\emptyset$ when $i \neq j$),

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

$$\text{event } A \xrightarrow{P(\cdot)} [0, 1]$$

2.4 Axioms of Probability

Remarks

We call $P(A)$ the **probability** of the event A.

Axiom 1 states that the probability that the outcome is in E is some number between 0 and 1. Axiom 2 states that, with probability 1, the outcome will be a point in the sample space S. Axiom 3 states that for any sequence of mutually exclusive events the probability of at least one of these events occurring is just the sum their respective probabilities.

Remarks

Axiom 3 is sometimes written as

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} P(A_i), \quad \text{for all } N = 1, 2, \cdots,$$

See a proof on Slide 26

2.4 Axioms of Probability

Example

We roll a fair six-faced dice randomly and we consider the number of the random outcome.
Let A={1,3} and B={2,4} be two events.
*By **Kolmogorov's axiom of probabilities**, what are P(A) and P(B) ?*

Sample space $S = \{1, 2, 3, 4, 5, 6\}$.

Since the die is fair, $P(\{1\}) = P(2) = P(3) = \cdots = P(6)$

$$1 = P(S) = P(1) + P(2) + \cdots + P(6)$$

$$\Rightarrow P(1) = P(2) = \cdots = P(6) = \frac{1}{6}.$$

$$P(A) = P(1) + P(3) = \frac{2}{6} = \frac{1}{3}$$

$$P(B) = \frac{1}{3}.$$

2.5 Properties of Probability

Theorem

$$P(\emptyset) = 0.$$

Proof

Take $A_1 = S$, $A_2 = A_3 = \cdots = \phi$.

Then $A_1, A_2, \cdots$ are mutually exclusive.

By Axiom 3,

$$P(S) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \underbrace{P(A_1)}_{P(S)} + \sum_{i=2}^{\infty} P(\phi)$$

Thus $P(\phi) = 0$.

2.5 Properties of Probability

Theorem

For any finite sequence of mutually exclusive events $A_1, A_2, \ldots, A_n$,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i).$$

Proof

Let $A_{n+1} = A_{n+2} = \cdots = \phi$,

Thus $A_1, A_2, \cdots$ is mutually exclusive,

and $P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i\right)$

$= \sum_{i=1}^{\infty} P(A_i)$

$= \sum_{i=1}^{n} P(A_i)$ since $P(A_i) = 0$ for $i \geq n+1$.

2.5 Properties of Probability

Theorem (The complement rule).

Let A be an event, then

$$P(A^c) = 1 - P(A).$$

Proof

Since $S = A \cup A^c$, A, A^c are mutually exclusive,

$$1 = P(S) = P(A) + P(A^c)$$

$$\Rightarrow \quad P(A^c) = 1 - P(A).$$

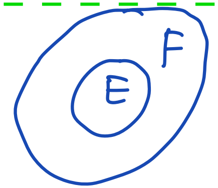
2.5 Properties of Probability

Theorem (Monotonicity)

If $E \subset F$, *then*

$$P(E) \leq P(F).$$

Proof



$F = FE \cup FE^c = E \cup FE^c$.

$FE = E$ since $E \subset F$.

E, FE^c are mutually exclusive.

Hence, $P(F) = P(E) + P(FE^c)$

Since $P(FE^c) \geq 0$,

$P(F) \geq P(E)$.

2.5 Properties of Probability

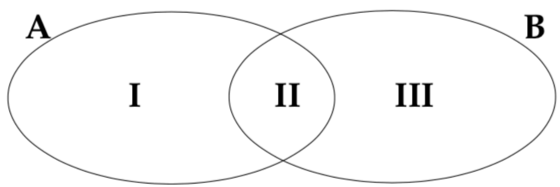
Theorem (Sum rule).

Let A and B be any two events, then

$$P(A \cup B) = P(A) + P(B) - P(AB).$$

Proof

Illustration by Venn Diagram



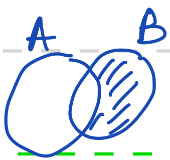
$$\begin{aligned} P(A) &= P(I) + P(II) & P(AB) &= P(II) \\ P(B) &= P(II) + P(III) & P(A \cup B) &= P(I) + P(II) + P(III). \end{aligned}$$

2.5 Properties of Probability

Theorem → Later, we will generalize it to "inclusion-exclusion formula".

Let A and B be any two events, then

$$P(A \cup B) = P(A) + P(B) - P(AB).$$



Proof

$A \cup B = A \cup (B \setminus A) = A \cup BA^c$, A and BA^c mutually exclusive.

Hence, $P(A \cup B) = P(A) + \underline{P(BA^c)}$.

Since $B = BA \cup BA^c$, BA and BA^c mutually exclusive

$P(B) = P(AB) + \underline{P(BA^c)} \Rightarrow P(BA^c) = P(B) - P(AB)$

Hence $P(A \cup B) = P(A) + P(B) - P(AB)$.

2.5 Properties of Probability

Class Discussion

Now, can you use Math language to answer

"What is probability?"

2.5 Properties of Probability

Class Discussion

Suppose a coin is tossed, and the sample space is

$$S = \{\text{head}, \ \text{tail}\}$$

Are the following functions probabilities ?

Choice A	Choice B	Choice C
$P(\text{head}) = 0.5$	$P(\text{head}) = 0$	$P(\text{head}) = 0.3$
$P(\text{tail}) = 0.6$	$P(\text{tail}) = 1.0$	$P(\text{tail}) = 1.3$

2.5 Properties of Probability

Class Discussion

There is a six-faced die with numbers {1,2,3,4,5,6} on each face. We randomly roll this die. Suppose that the probability of getting a number ≥ 3 is 3/5, and the probability of getting a number ≤ 3 is 1/2.
Then, what is the probability of getting the number 3?

Choose your answer?

Choice A	Choice B	Choice C
$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{10}$

2.5 Properties of Probability

Example

Let A and B be two events such that $P(B) = \dfrac{5}{8}$ and $P(A \cap B) = \dfrac{1}{2}$. Find $P(B \cap A^c) =?$

Solution.

By property 3) of Kolmogorov's Axiom

2.5 Properties of Probability

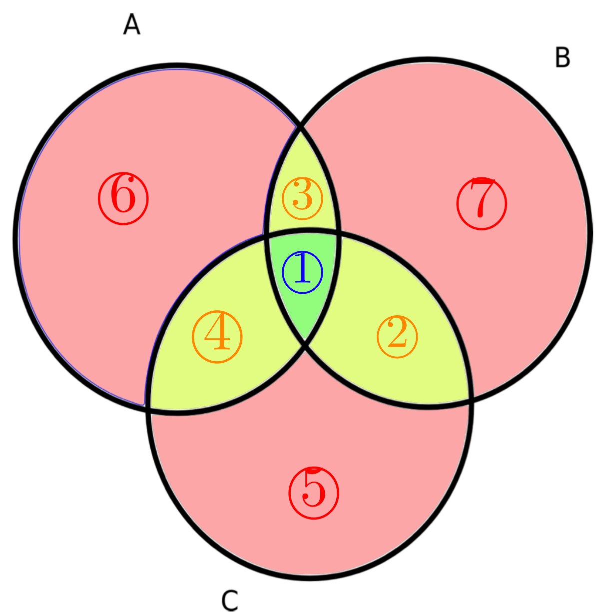
Example J is taking two books along on her holiday vacation. With probability 0.5, she will like the first book; with probability 0.4, she will like the second book; and with probability 0.3, she will like both books. What is the probability that she likes neither book?

Solution.

2.5 Properties of Probability

We already have a formula for $P(A \cup B)$.

Do we have formula for $P(A \cup B \cup C)$?



2.5 Properties of Probability

Theorem (Inclusion-Exclusion Principle). *Let $A_1, A_2, \ldots, A_n$ be any events, then*

$$\begin{aligned} P(A_1 \cup A_2 \cup \cdots \cup A_n) &= \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i_1 < i_2 \le n} P(A_{i_1} A_{i_2}) + \cdots \\ &+ (-1)^{r+1} \sum_{1 \le i_1 < \cdots < i_r \le n} P(A_{i_1} \cdots A_{i_r}) \\ &+ \cdots + (-1)^{n+1} P(A_1 \cdots A_n). \end{aligned}$$

If n=4

$\sum_{1 \le i_1 < i_2 \le 4} P(A_{i_1} A_{i_2})$ means

$$\begin{aligned} P(A_1A_2) + P(A_1A_3) + P(A_1A_4) + P(A_2A_3) \\ + P(A_2A_4) + P(A_3A_4). \end{aligned}$$

$\sum_{1 \le i_1 < i_2 < i_3 \le 4} P(A_{i_1} A_{i_2} A_{i_3})$ means

$$P(A_1A_2A_3) + P(A_1A_2A_4) + P(A_1A_3A_4) + P(A_2A_3A_4).$$

2.6 Sample Spaces Having Equally Likely Outcomes

For many experiments, it is natural to assume that all outcomes in the sample space (of finite number of elements) are equally likely to occur.

Example

- tossing a fair coin: $P(\{\text{ head }\}) = P(\{\text{ tail }\})$
- tossing a pair of fair dice: $P(\{(1,1)\}) = P(\{(1,2)\}) = \cdots$.
- randomly pick up a card from a standard deck of playing cards

Write $S = \{s_1, s_2, \ldots s_N\}$ where N denotes the number of outcomes of S. (We will use $|S|$ to denote the number of outcomes of S.) Since outcomes are assumed to be equally likely to occur, write $P(\{s_i\}) = c$, for $i = 1, 2, \ldots, N$. As

$$1 = P(S) = P(\cup_{i=1}^{N}\{s_i\}) = \sum_{i=1}^{N} P(\{s_i\}) = Nc,$$

we get $c = 1/N$. In other words, $P(\{s_i\}) = \dfrac{1}{|S|}$.

2.6 Sample Spaces Having Equally Likely Outcomes

Then, it follows from Axiom 3 that for any event $E \subset S$,

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S}.$$

In words, if we assume that all outcomes of an experiment are equally likely to occur, then the probability of any event E equals the proportion of outcomes in the sample space that are contained in E.

2.6 Sample Spaces Having Equally Likely Outcomes

Example

We select a card randomly from a standard deck of 52 playing cards.
Let A={the selected card is diamond},
B={the selected card is Jack}.
Find P(A) and P(B) and P(AUB)?

Solution.

2.6 Sample Spaces Having Equally Likely Outcomes

Example A pair of fair dice is tossed. What is the probability of getting a sum of 7?

Solution.

As the dice are fair, we assume all outcomes are equally likely. So

$$\begin{aligned} A &= \{\text{ sum is } 7\} \\ &= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} \end{aligned}$$

therefore,

$$P(A) = \frac{|A|}{|S|} = \frac{6}{36} = \frac{1}{6}.$$

2.6 Sample Spaces Having Equally Likely Outcomes

Example If 3 balls are "randomly drawn" from an urn containing 6 white and 5 black balls, what is the probability that one of the drawn balls is white and the other two black?

treat all balls are distinct

Solution 1

2.6 Sample Spaces Having Equally Likely Outcomes

Example If 3 balls are "randomly drawn" from an urn containing 6 white and 5 black balls, what is the probability that one of the drawn balls is white and the other two black?

treat all balls are distinct

Solution 2

2.6 Sample Spaces Having Equally Likely Outcomes

Example If 3 balls are "randomly drawn" from an urn containing 6 white and 5 black balls, what is the probability that one of the drawn balls is white and the other two black?

Discussion

In the previous two solutions, we regard all balls as distinct balls.

You may wonder

"Can we treat all white balls as alike, and all black balls as alike?"

Can we?

2.6 Sample Spaces Having Equally Likely Outcomes

Example (Birthday problem II). How large must the group be so that there is a probability of greater than 0.5 that someone will have the same birthday as you do? (Exclude Feb 29 from the calculation)

Solution

2.6 Sample Spaces Having Equally Likely Outcomes

Example *(Birthday problem I)* What is the probability that in a group of n people, at least 2 of them will have the same birthday?

Solution