IMAC 2024, Orlando, FL, USA

#16556 The Impact of Non-Unique Residual Tractions on the Nonlinear Dynamics of Jointed Structures: Probabilistic Perspectives

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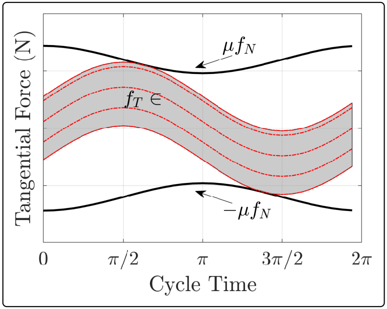
Outline


μf_N
$f_T \in$
$-\mu f_N$
Tangential Force (N)
0
$\pi/2$
π
$3\pi/2$
2π
Cycle Time

1. Introduction

1.1. Informal Introduction to Non-Unique Frictional Tractions

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Elastic Dry Friction/Jenkins Element

$$\mathrm{d}f = \begin{cases} k_t \mathrm{d}u, & |f + \int k_t \mathrm{d}u| < \mu f_n \text{ (stick)} \\ 0 & |f + \int k_t \mathrm{d}u| \geq \mu f_n \text{ (slip)} \end{cases}$$

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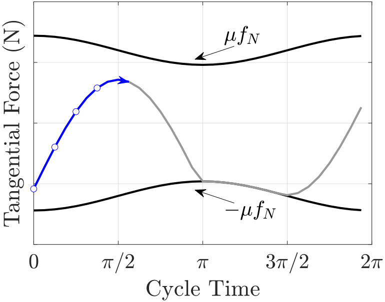

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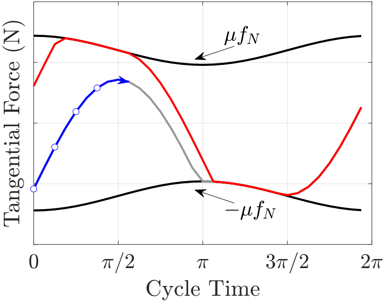

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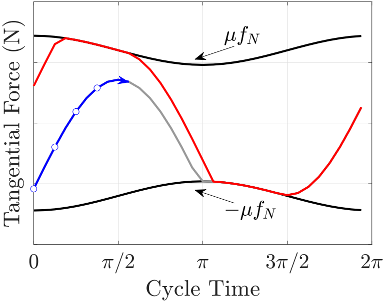

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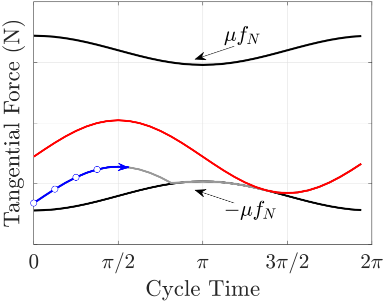

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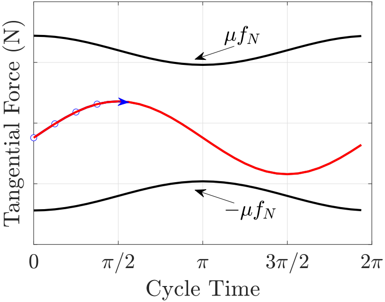

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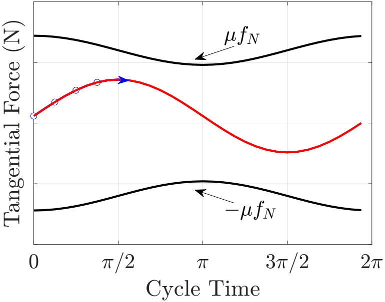

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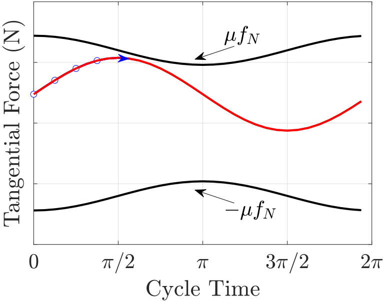

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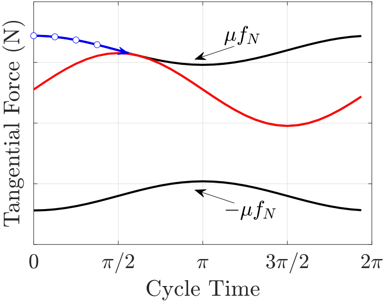

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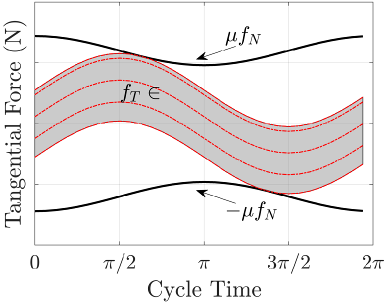

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Does this actually make any sense?

- There exist mathematical frameworks to handle set-valued nonlinearities.
- Is it merely **just a quirk of modeling** or a physical phenomenon that has real consequences?

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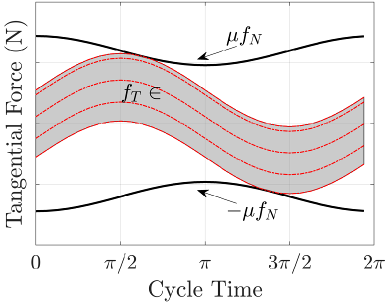

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- This was formalized as **non-uniqueness in residual tractions** [Ferhatoglu and Zucca, *JSV*,. 2021]

Residual traction is...

...static, self-equilibrating dry-friction forces in contacting interfaces.

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- Physical relevance is imminent in structures with **tangent-normal coupling** across contact interfaces

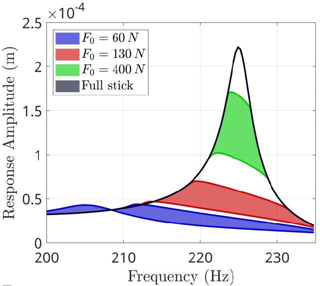
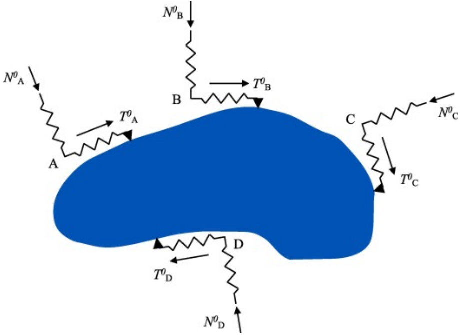


Figure: from [Ferhatoglu and Zucca, *JSV*,. 2021]



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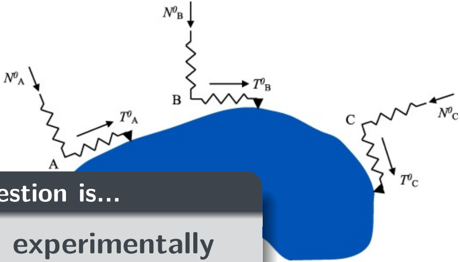
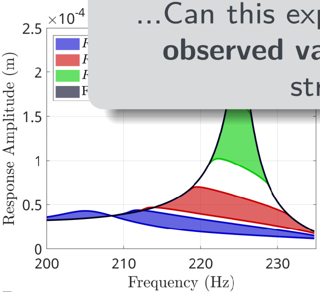


The real question is…

…Can this explain **experimentally observed variability** in jointed structures?



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1.2. Motivation of Current Study

- Experimental studies[a] have also been trying to capture the variability
- It is becoming clear that the non-uniqueness of residual tractions is **indeed a physical phenomenon** that models have to capture

[a]Ferhatoglu, Botto, and Zucca 2022; Ferhatoglu, Gastaldi, et al. 2022; Gastaldi et al. 2020.

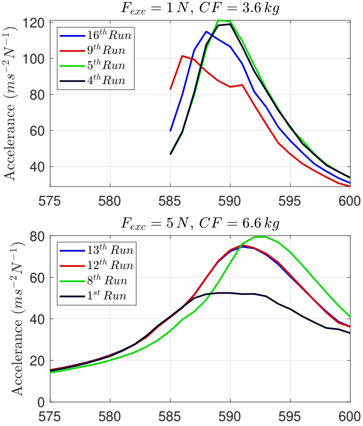

Figure: Data from [Ferhatoglu, Botto, and Zucca, *Journal of Engineering for Gas Turbines and Power*, 2022]

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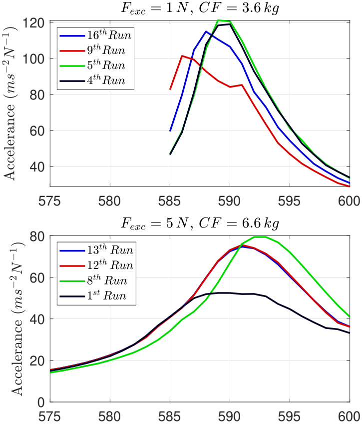
- We here take an **uncertainty propagation approach** in treating the influence of non-uniqueness.
- If contact parameters (stiffness, CoF) are randomly distributed, **what is the relative significance of non-unique tractions on the overall dynamics?**



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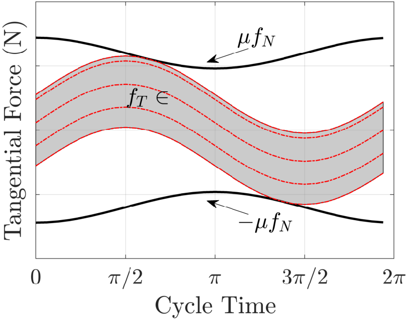


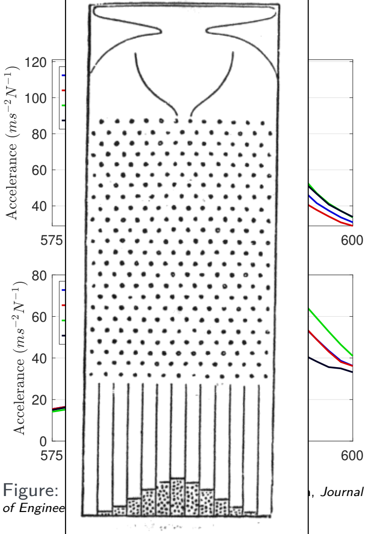


Figure: ... , *Journal of Enginee...*

The Galton Board

Source: Wikipedia

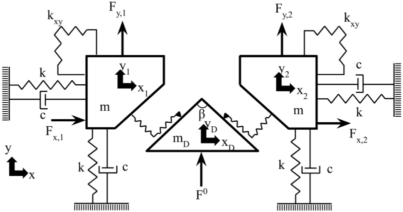
← Galton (1894)

2. Numerical Modeling

2.1. An Exemplary Benchmark System

- A 6DOF lumped-mass oscillator from [Ferhatoglu and Zucca, *JSV*,. 2021] is chosen

- Parameters chosen to induce asymmetry
- Mode shape chosen to **avoid separation and modal interactions**



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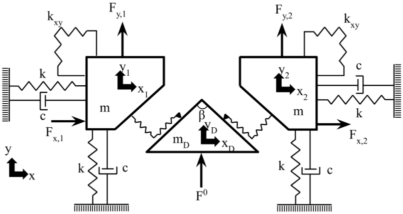

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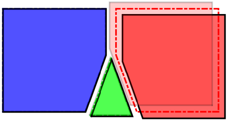

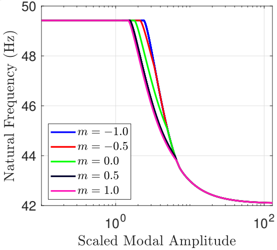

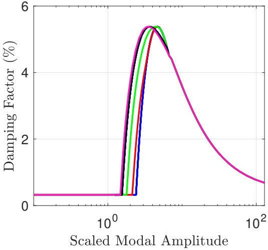

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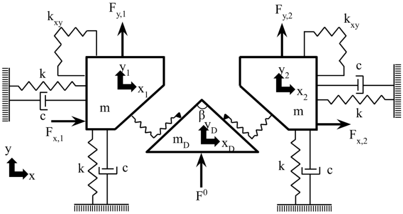

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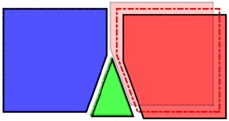

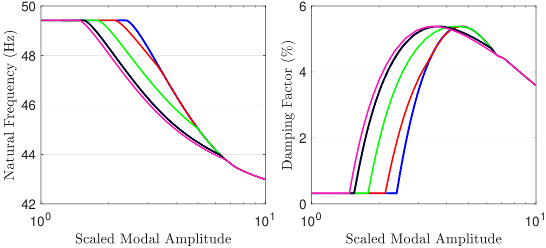

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2.2. Stochastic Modeling Details

- The model is parameterized by 4 parameters:
  Normal Stiffness k_n
  Tangential Stiffness k_t
  Coefficient of Friction μ
  Friction multiplier m

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Lognormal Distribution

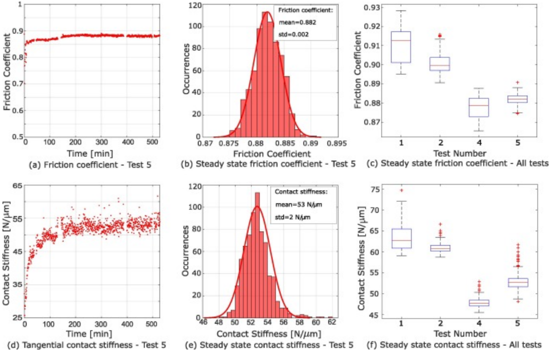
$$K = e^{\mu_\ell + \sigma_\ell Z}$$

$$Z \sim \mathcal{N}(0,1)$$

$$\langle K \rangle = \ln\left(\frac{\mu_\ell^2}{\sqrt{\mu_\ell^2 + \sigma_\ell^2}}\right)$$

$$\langle [K]^2 \rangle = \ln\left(1 + \frac{\sigma_\ell^2}{\mu_\ell^2}\right)$$

Distribution for k_n, k_t inspired from [Yuan et al., *MSSP*,. 2021]



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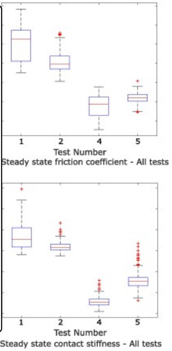

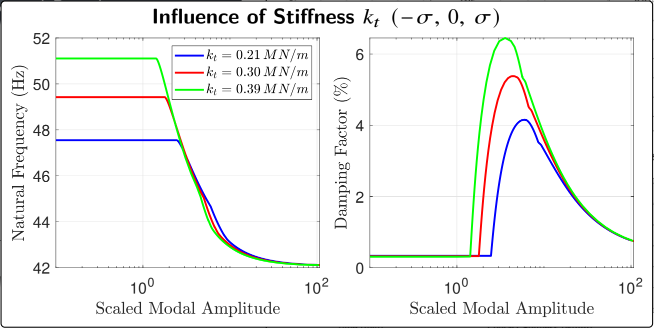

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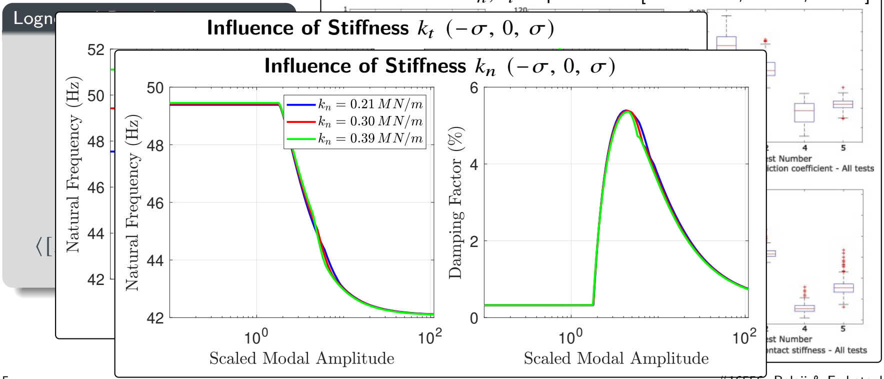

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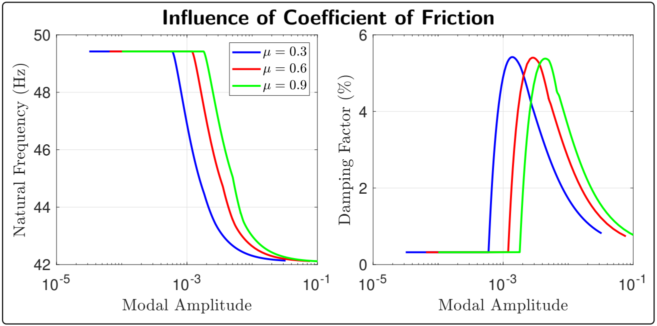

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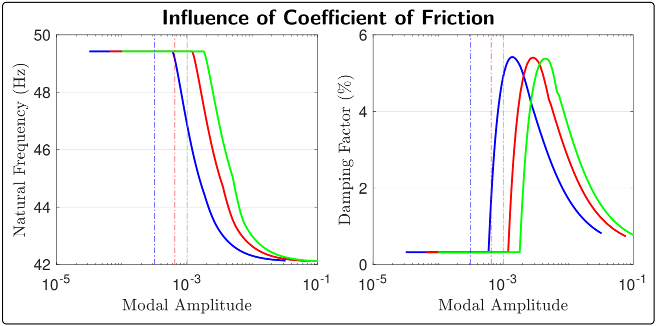

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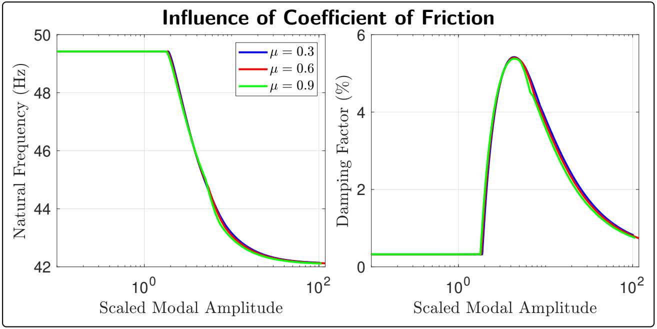

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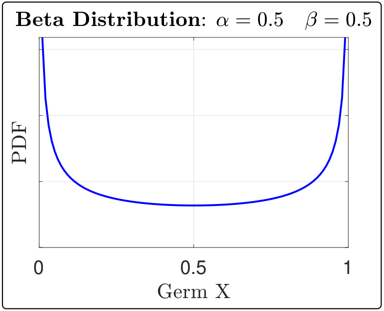

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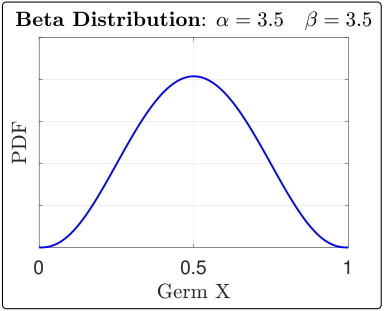

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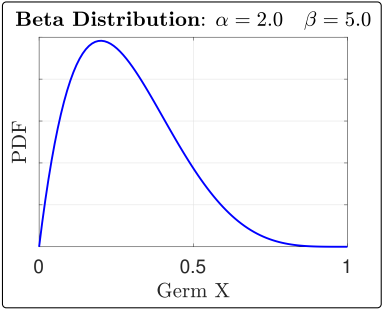

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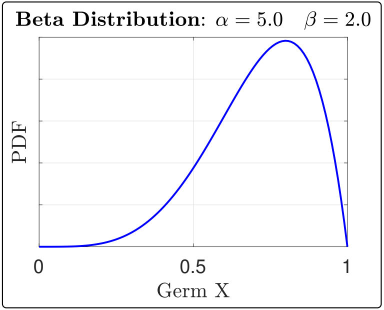

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- We approximate this functional relationship using a polynomial expansion,

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2.2. Stochastic Modeling with Polynomial Chaos Expansion

- We introduce **germs** $\{X_j\}_{j=1,\ldots,4}$ such that

$$k_n = e^{\mu_\ell + \sigma_\ell x_1}, \quad k_t = e^{\mu_\ell + \sigma_\ell x_2}, \quad \mu = \mu_{\max} \frac{1 + x_3}{2}, \quad m = 2x_4 - 1$$

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Polynomial Families from the Askey Scheme

$$\mathcal{N}(0,1) \to \underbrace{\text{Hermite;}}_{\mathbb{R}\to\mathbb{R}} \quad \mathcal{U}(-1,1) \to \underbrace{\text{Legendre;}}_{(-1,1)\to\mathbb{R}} \quad \mathrm{Beta}(\alpha,\beta) \to \underbrace{\text{Jacobi}}_{(0,1)\to\mathbb{R}}$$

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- **The statistics of** $\hat{Y}$ can hereby be written as,

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2.2. Sensitivity Analysis based on Variance Decomposition

- Since the variance of the outcomes are written as the sum $\langle [\hat{Y}]^2 \rangle = \sum_{\bar{i} \in \mathcal{I} \setminus \{0,0,0,0\}} \eta_{\bar{i}}^2$, it can be **decomposed**.
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- **First order Sensitivity** of the m^{th} parameter is hereby written as

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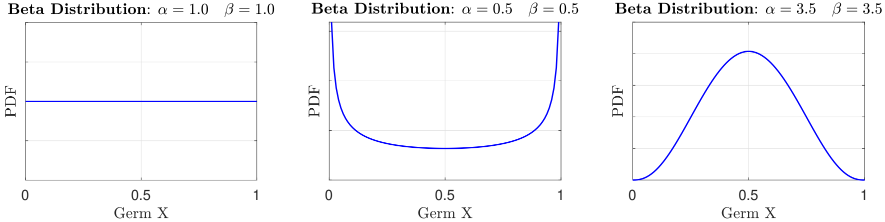
Also referred to as **Sobol' indices**, $S_1^{(m)}, S_2^{(m,n)} \quad \in \quad [0, 1]$.

3. Results

Overview

- Multiple cases were considered for the distribution of the m parameter while the distributions of the others were kept fixed
- 3 cases presented here:



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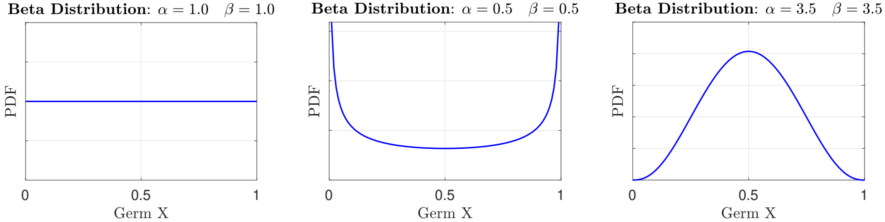

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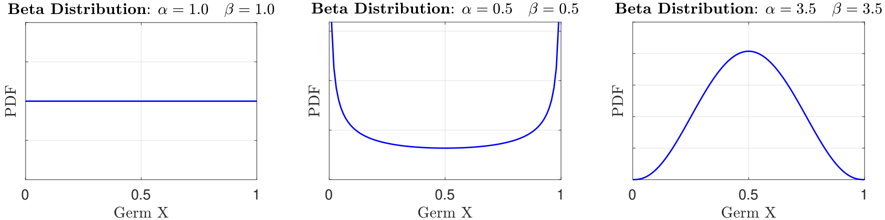

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- **5 quadrature points in each dimension** (totaling 625 points in 4D) were used to fit a **PCE truncated to order 4** for all simulations

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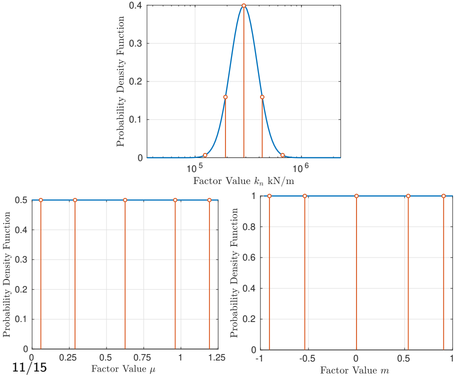
- When $\alpha = \beta = 1$, the distribution of the m parameter is uniform in $[-1, 1]$.



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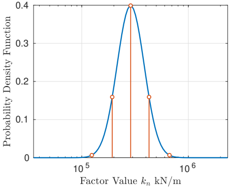

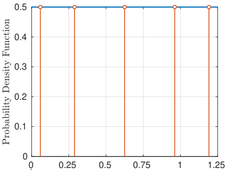

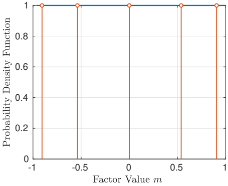

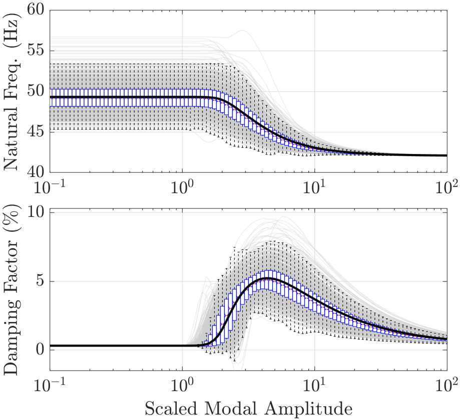

Grey: Realizations, Boxplot: 25-75th percentiles

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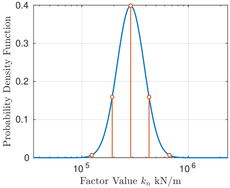

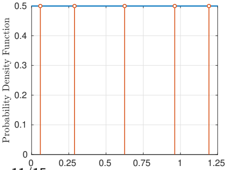

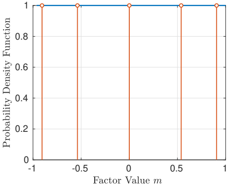

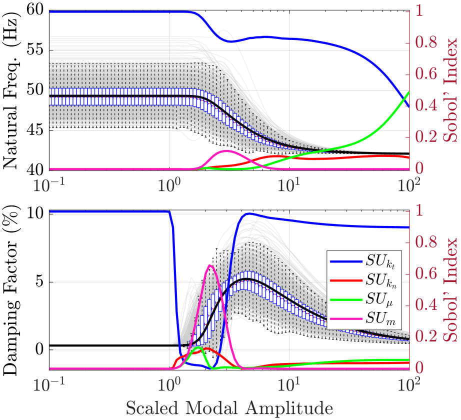

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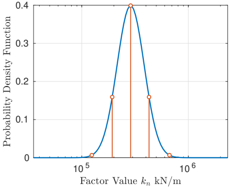

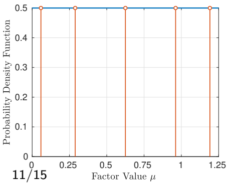

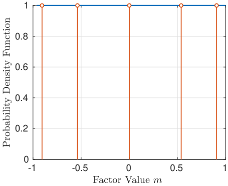

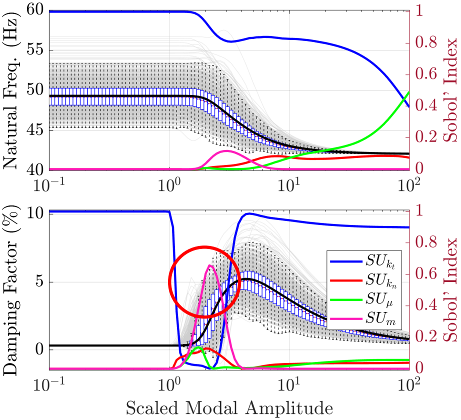

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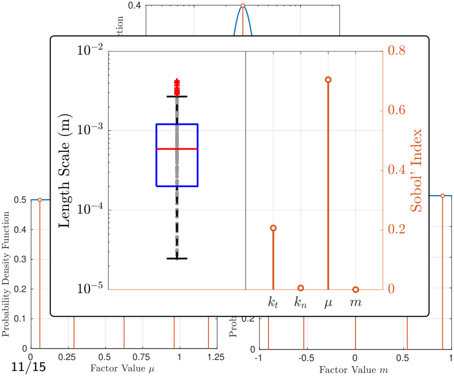
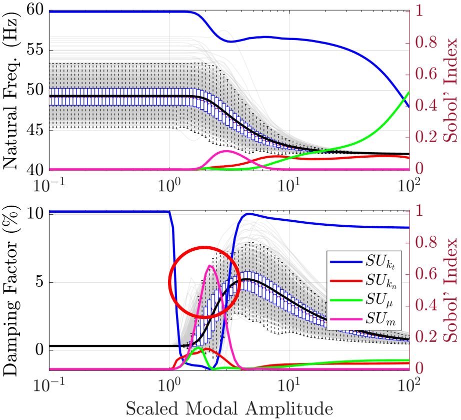
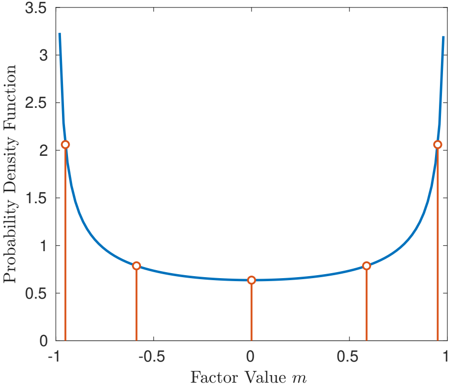




Grey: Realizations, Boxplot: 25-75th percentiles

3. Results

Case 2: $\alpha = 0.5, \beta = 0.5$ and Case 3: $\alpha = 3.5, \beta = 3.5$



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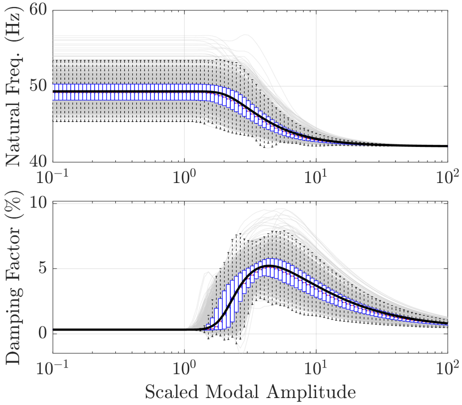

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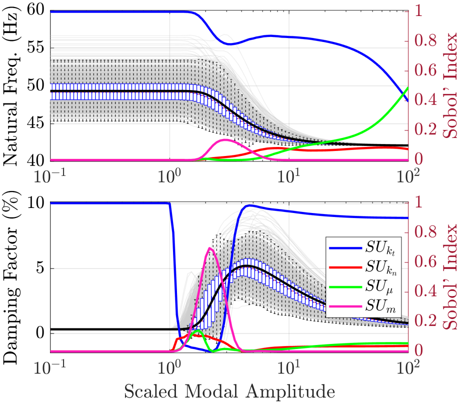

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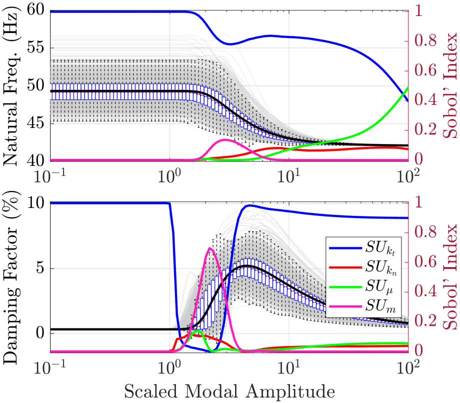

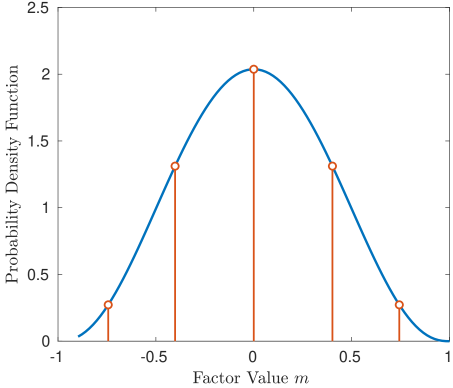

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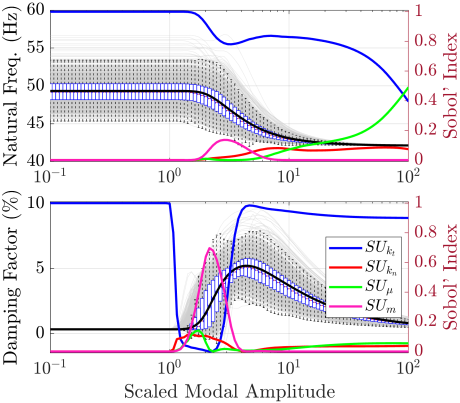

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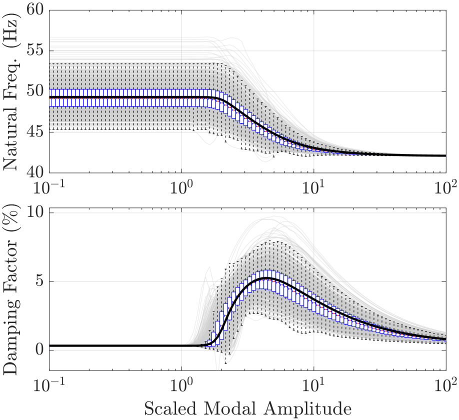

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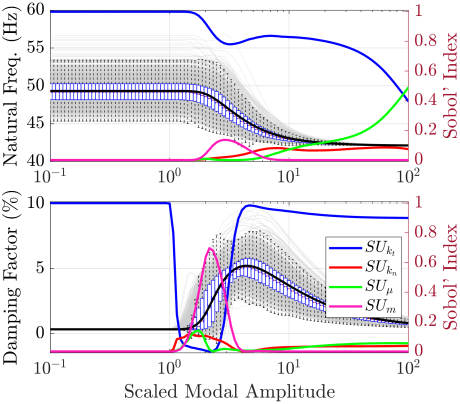

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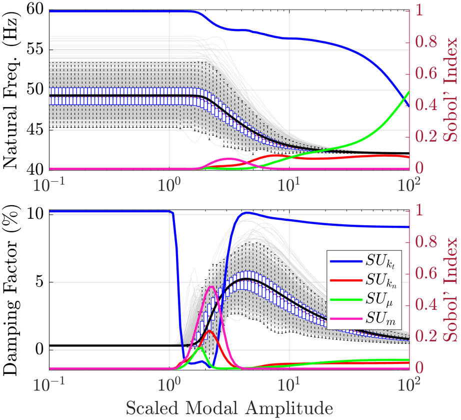

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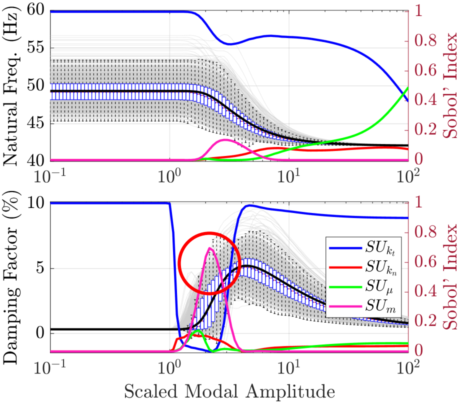

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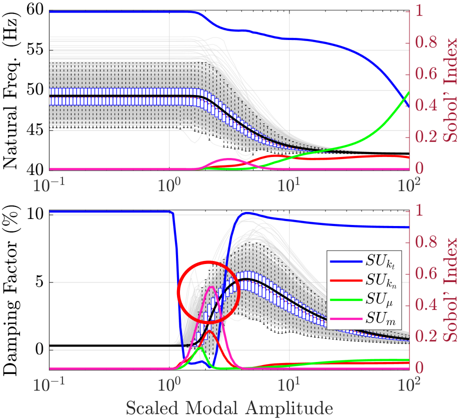

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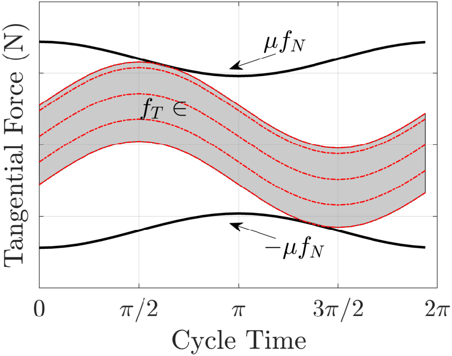
4. Conclusions & Discussions

Conclusions

- Work adds to recent literature on the **non-uniqueness of residual tractions** in frictional contact



4. Conclusions & Discussions

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- Work adds to recent literature on the **non-uniqueness of residual tractions** in frictional contact
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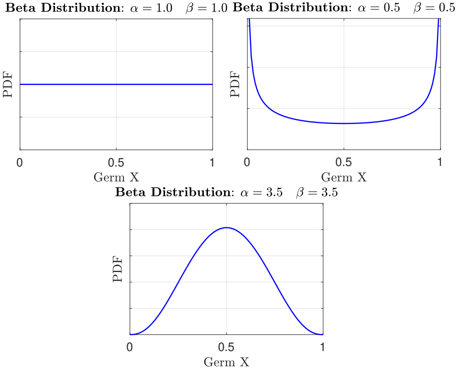
- In the absence of experimental techniques for accurate characterization, different distributions were considered



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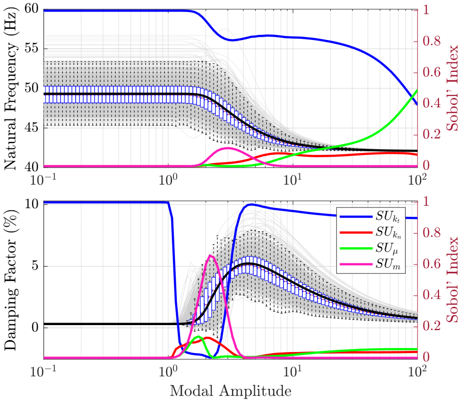
- **Joint-dissipation** observed to have much **more sensitivity than stiffness**



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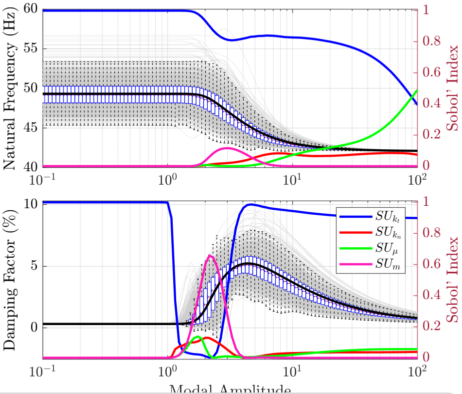


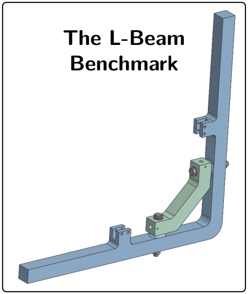
Experimental studies focusing on microslip need to be specially careful of this added source of uncertainty

4. Conclusions & Discussions

Lingering Questions

Can the residual tractions be experimentally measured/controlled?

- In **TRC 2023**, an experimental campaign was undertaken to "induce" residual-traction based uncertainty
- See #16576 (next talk)



4. Conclusions & Discussions

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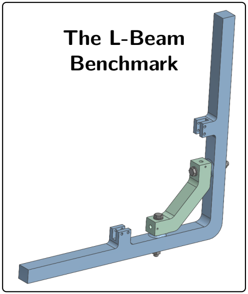
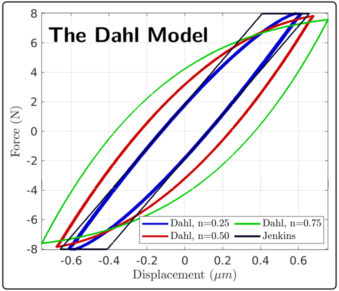
Are micro-slip models "wrong"?

- The elastic-dry friction permits residual traction specification since it is a **macro-slip model**.
- **Micro-slip models**, however, don't support multiple solutions in the fully stuck configuration.
- If **non-unique residual tractions** is a physical phenomenon, every friction model must admit multiple solutions in the fully stuck case.





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— (Nov. 2021b). "On the Non-Uniqueness of Friction Forces and the Systematic Computation of Dynamic Response Boundaries for Turbine Bladed Disks with Contacts". In: *MSSP* 160, p. 107917.

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6. Backup Slides

Summary

- **Cases with Unsymmetric β distributions**
- **Second Order Sobol' Indices**
- **Convergence of PCE**

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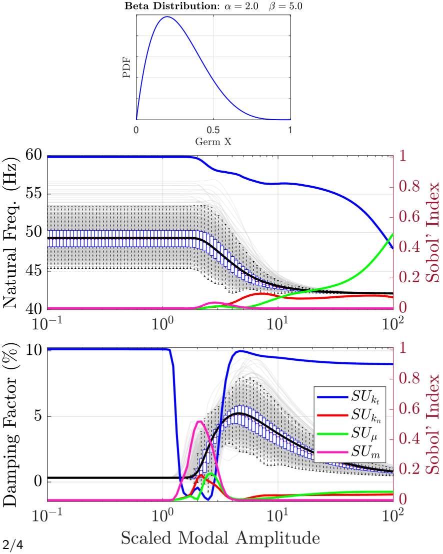
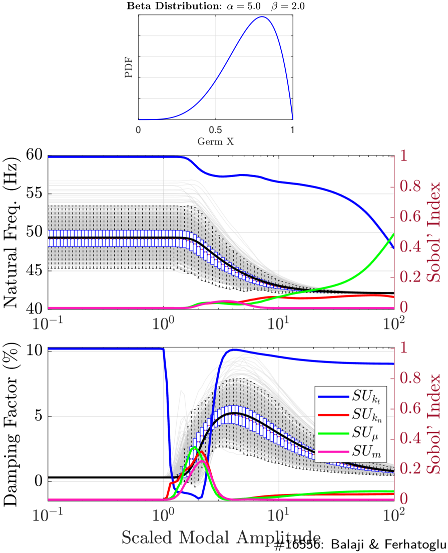
6. Backup Slides

6.1. Cases with Unsymmetric β Distributions





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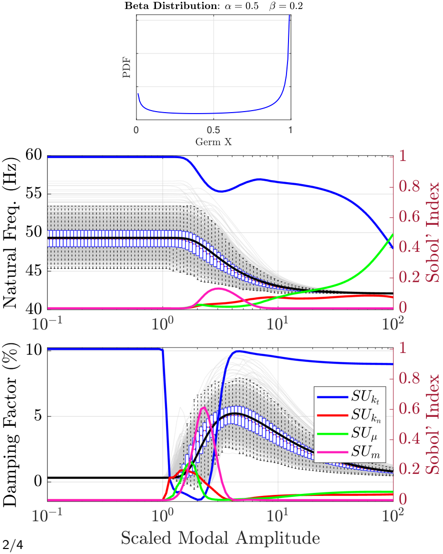
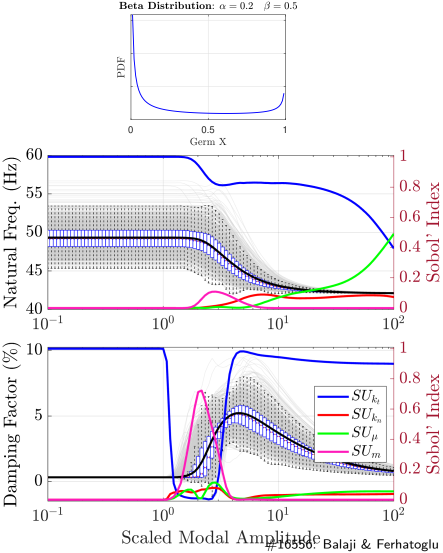
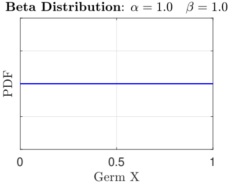


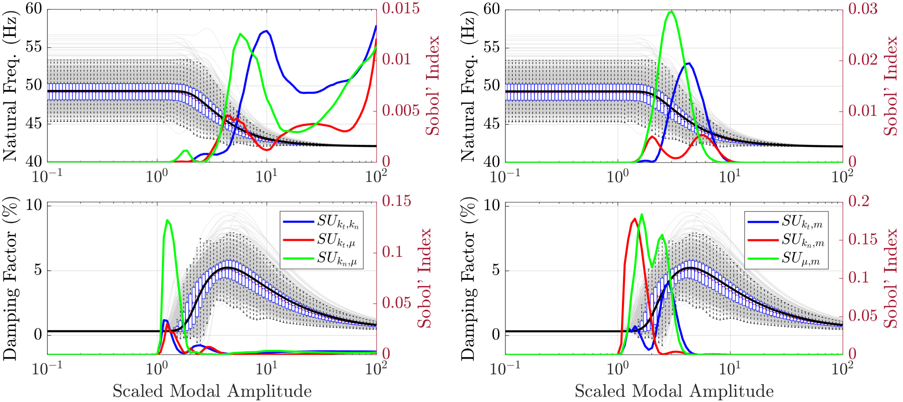


6. Backup Slides

6.2. Second Order Sobol' Indices



Case 1



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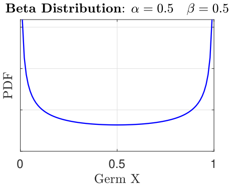
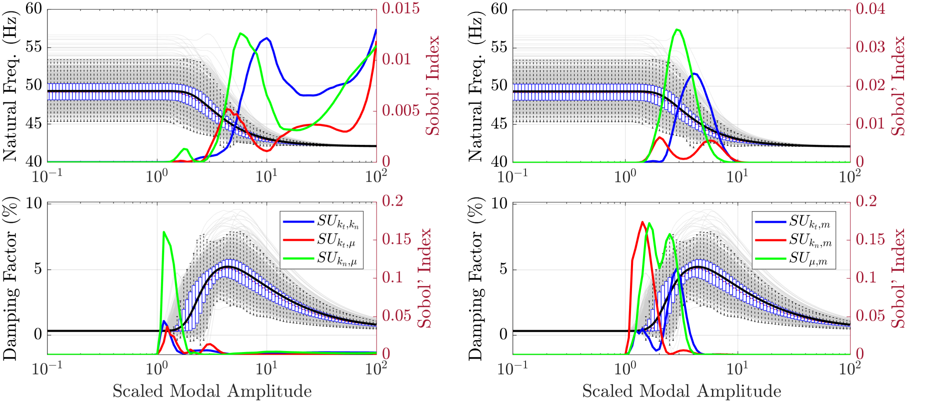


Case 2



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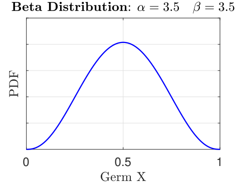

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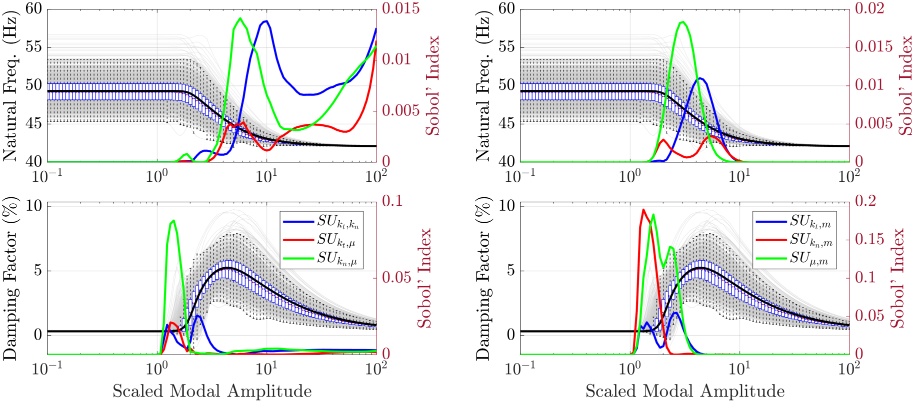

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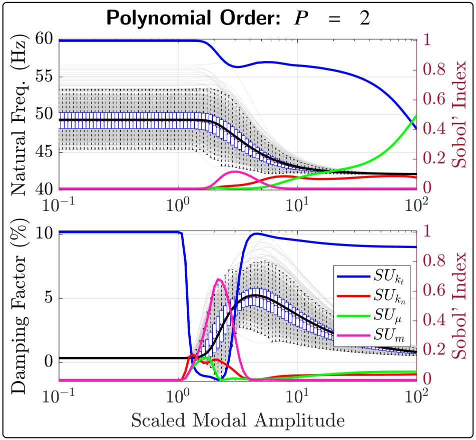
6. Backup Slides

6.3. Convergence of PCE: The "eyeball" norm



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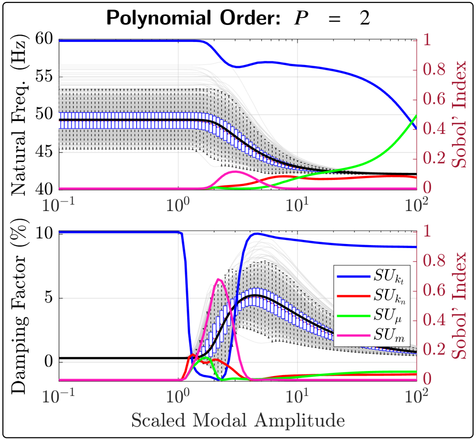

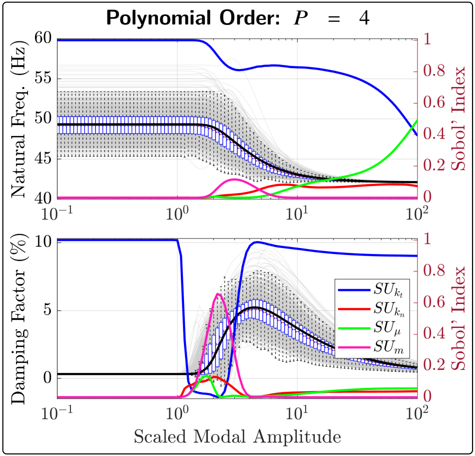

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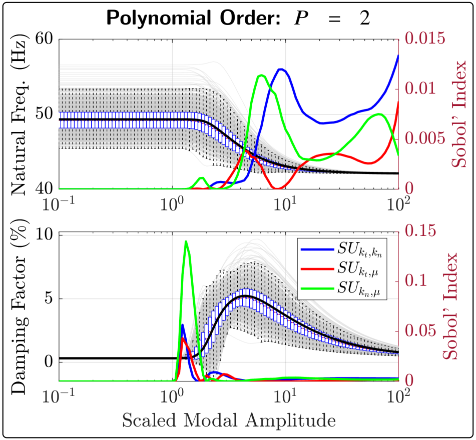

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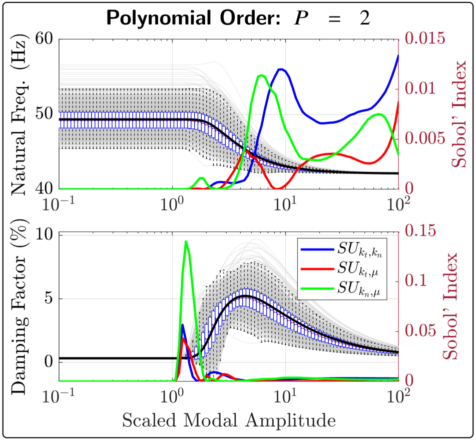

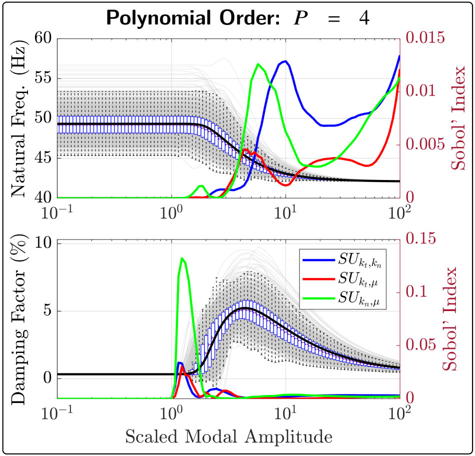

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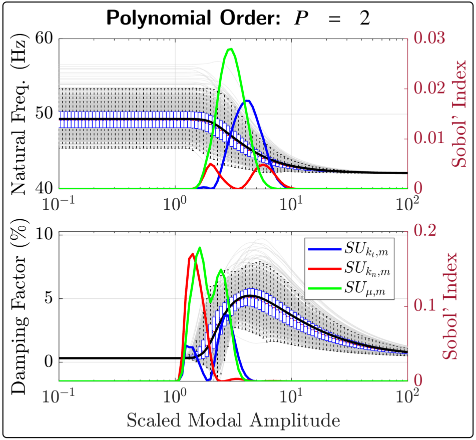

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