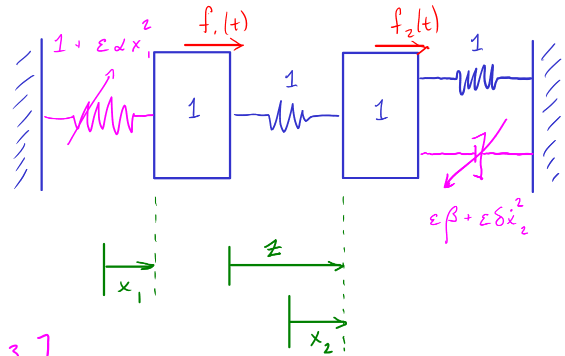
# MULTIPLE SCALE ANALYSIS FOR NONLINEAR NORMAL MODES




$$\ddot{x}_1 + 2x_1 - x_2 + \varepsilon\alpha x_1^3 = f_1(t)$$

$$\ddot{x}_2 + \varepsilon\left(\beta\dot{x}_2 + \delta\dot{x}_2^3\right) - x_1 + 2x_2 = f_2(t)$$

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + \varepsilon\begin{bmatrix}0 & 0\\ 0 & \beta\end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} + \begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \varepsilon\begin{bmatrix}\alpha x_1^3\\ \delta\dot{x}_2^3\end{bmatrix} = \underline{F}(t)$$

$$\underline{\underline{M}} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad \underline{\underline{C}} = \varepsilon\beta\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},\quad \underline{\underline{K}} = \begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix},\quad \underline{N}(\underline{x},\dot{\underline{x}}) = \begin{bmatrix}\alpha x_1^3\\ \delta\dot{x}_2^3\end{bmatrix}$$

FOR THE UNDAMPED LINEAR SYSTEM

$$\underline{\phi}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} \qquad \underline{\phi}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix} \qquad \underline{\phi}_i^T \underline{\underline{M}}\,\underline{\phi}_i = 1$$

$$\lambda_1 = 1 \qquad \lambda_2 = 3 \qquad \underline{\phi}_i^T \underline{\underline{K}}\,\underline{\phi}_i = \lambda_i$$

$$\underline{x}(t) = \sum_{i=1}^{2}\underline{\phi}_i\, q_i(t) \qquad x_1(t) = \frac{q_1(t)+q_2(t)}{\sqrt{2}}, \quad x_2(t) = \frac{q_1(t)-q_2(t)}{\sqrt{2}}$$

$$\underline{\phi}_i^T\left\{\underline{\underline{M}}\,\ddot{\underline{x}} + \underline{\underline{C}}\,\dot{\underline{x}} + \underline{\underline{K}}\,\underline{x} + \underline{N} = \underline{F}(t)\right\}$$

$$\underline{\phi}_i^T\underline{N} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & \pm 1\end{bmatrix}\begin{bmatrix}\alpha/2\sqrt{2}\,(q_1+q_2)^3\\ \delta/2\sqrt{2}\,(\dot{q}_1-\dot{q}_2)^3\end{bmatrix} = \frac{1}{4}\left\{\alpha(q_1+q_2)^3 \pm \delta(\dot{q}_1-\dot{q}_2)^3\right\}$$

$$\ddot{q}_1 + \frac{\varepsilon\beta}{2}(\dot{q}_1-\dot{q}_2) + q_1 + \frac{\varepsilon}{4}\left(\delta(\dot{q}_1-\dot{q}_2)^3 + \alpha(q_1+q_2)^3\right) = \frac{f_1(t)+f_2(t)}{\sqrt{2}} \equiv g_1(t)$$

$$\ddot{q}_2 + \frac{\varepsilon\beta}{2}(-\dot{q}_1+\dot{q}_2) + 3q_1 + \frac{\varepsilon}{4}\left(-\delta(\dot{q}_1-\dot{q}_2)^3 + \alpha(q_1+q_2)^3\right) = \frac{f_1(t)-f_2(t)}{\sqrt{2}} \equiv g_2(t)$$

# METHOD OF MULTIPLE SCALES

$$t \to (\tau, \eta) \qquad \eta = [\eta_i]$$

$$\frac{d}{dt} = \frac{\partial}{\partial\tau}\cdot\frac{\partial\tau}{\partial t} + \frac{\partial}{\partial\eta_1}\frac{\partial\eta_1}{\partial t} + \frac{\partial}{\partial\eta_2}\frac{\partial\eta_2}{\partial t} + \cdots \qquad \frac{\partial\tau}{\partial t} = 1, \quad \frac{\partial\eta_i}{\partial t} = \varepsilon^i$$

$$= \frac{\partial}{\partial\tau} + \varepsilon\frac{\partial}{\partial\eta_1} + \varepsilon^2\frac{\partial}{\partial\eta_2} + \cdots \qquad \frac{d^2}{dt^2} = \frac{\partial^2}{\partial\tau^2} + 2\varepsilon\frac{\partial^2}{\partial\tau\partial\eta_1} + \varepsilon^2\frac{\partial^2}{\partial\eta_1^2} + 2\varepsilon^2\frac{\partial^2}{\partial\tau\partial\eta_2} + \cdots$$

$$q_1(t) = q_{10}(t) + \varepsilon q_{11}(t) + \varepsilon^2 q_{12}(t) + \cdots$$

$$q_2(t) = q_{20}(t) + \varepsilon q_{21}(t) + \varepsilon^2 q_{22}(t) + \cdots$$

$$\left[\frac{\partial^2}{\partial\tau^2} + 2\varepsilon\frac{\partial^2}{\partial\tau\partial\eta_1} + \cdots\right]\left(q_{10} + \varepsilon q_{11} + \cdots\right) + \frac{\varepsilon\beta}{2}\left[\frac{\partial}{\partial\tau} + \cdots\right]\left(q_{10} - q_{20} + \cdots\right) + \left(q_{10} + \varepsilon q_{11} + \cdots\right)$$

$$\omega_{10}^2 + \varepsilon\sigma_1 + \cdots$$

$$+\varepsilon N_{10}\left(q_{10}, q_{20}\right) = \varepsilon g_1(\tau)$$

$$\left[\frac{\partial^2}{\partial\tau^2} + 2\varepsilon\frac{\partial^2}{\partial\tau\partial\eta_1} + \cdots\right]\left(q_{20} + \varepsilon q_{21} + \cdots\right) + \frac{\varepsilon\beta}{2}\left[\frac{\partial}{\partial\tau} + \cdots\right]\left(-q_{10} + q_{20} + \cdots\right) + 3\left(q_{20} + \varepsilon q_{21} + \cdots\right)$$

$$+\varepsilon N_{20}\left(q_{10}, q_{20}\right) = \varepsilon g_2(\tau)$$

WITH

$$N_{10} = \frac{1}{4}\left\{\delta\left(\left[\frac{\partial}{\partial\tau} + \cdots\right]\left(q_{10} - q_{20} + \cdots\right)\right)^3 + \alpha\left(\left(q_{10} + q_{20} + \cdots\right)\right)^3\right\}$$

$$N_{20} = \frac{1}{4}\left\{-\delta\left(\left[\frac{\partial}{\partial\tau} + \cdots\right]\left(q_{10} - q_{20} + \cdots\right)\right)^3 + \alpha\left(\left(q_{10} + q_{20} + \cdots\right)\right)^3\right\}$$

$O(1)$:

$$\frac{\partial^2 q_{10}}{\partial\tau^2} + q_{10} = 0 \qquad q_{10}(\tau, \eta_1) = A_{10}(\eta_1)\, S_{\psi_{10}} \qquad \psi_{10} \equiv \tau + \phi_{10}(\eta_1)$$

$$\frac{\partial^2 q_{20}}{\partial\tau^2} + 3q_{20} = 0 \qquad q_{20}(\tau, \eta_1) = A_{20}(\eta_1)\, S_{\sqrt{3}\psi_{20}} \qquad \psi_{20} \equiv \tau + \phi_{20}(\eta_1)$$

$O(\varepsilon)$:

$$\frac{\partial^2 q_{11}}{\partial\tau^2} + q_{11} = g_1(\tau) - \left\{2\frac{\partial^2 q_{10}}{\partial\tau\partial\eta_1} + \frac{\beta}{2}\left(\frac{\partial q_{10}}{\partial\tau} - \frac{\partial q_{20}}{\partial\tau}\right) + N_{10}\left(q_{10}, q_{20}\right)\right\}$$

$$\frac{\partial^2 q_{21}}{\partial\tau^2} + 3q_{21} = g_2(\tau) - \left\{2\frac{\partial^2 q_{20}}{\partial\tau\partial\eta_1} - \frac{\beta}{2}\left(\frac{\partial q_{10}}{\partial\tau} - \frac{\partial q_{20}}{\partial\tau}\right) + N_{20}\left(q_{10}, q_{20}\right)\right\}$$

$$\frac{\partial^2 q_{11}}{\partial\tau^2} + q_{11} = g_1(\tau) - \left\{2\left(\frac{\partial A_{10}}{\partial\eta_1} C_{\psi_{10}} - A_{10}\frac{\partial\phi_{10}}{\partial\eta_1} S_{\psi_{10}}\right) + \frac{\beta}{2}\left(A_{10} C_{\psi_{10}} - \sqrt{3} A_{20} C_{\sqrt{3}\psi_{20}}\right)\right.$$
$$\left. + N_{10}\left(A_{10} S_{\psi_{10}}, A_{20} S_{\sqrt{3}\psi_{20}}\right)\right.$$

$$\frac{\partial^2 q_{21}}{\partial\tau^2} + 3q_{21} = g_2(\tau) - \left\{2\left(\sqrt{3}\frac{\partial A_{20}}{\partial\eta_1} C_{\sqrt{3}\psi_{20}} - 3A_{20}\frac{\partial\phi_{20}}{\partial\eta_1} S_{\sqrt{3}\psi_{20}}\right) - \frac{\beta}{2}\left(A_{10} C_{\psi_{10}} - \sqrt{3} A_{20} C_{\sqrt{3}\psi_{20}}\right)\right.$$
$$\left. + N_{20}\left(A_{10} S_{\psi_{10}}, A_{20} S_{\sqrt{3}\psi_{20}}\right)\right.$$

SECULAR DYNAMICS ARISE FROM RESONANT EXCITATION, WHICH CAN BE IDENTIFIED WITH A FOURIER EXPANSION

$$\langle N_{10} \rangle_{\psi_{10}} \equiv \left[ \frac{1}{\pi} \int_0^{2\pi} N_{10} \cdot S_\psi \, d\psi \right] S_{\psi_{10}} + \left[ \frac{1}{\pi} \int_0^{2\pi} N_{10} \cdot C_\psi \, d\psi \right] C_{\psi_{10}}$$

SECULAR

$$\psi_{10} \equiv \tau + \phi_{10} \; ; \quad \psi_{20} \equiv \tau + \phi_{20} = \psi_{10} + (\phi_{20} - \phi_{10})$$

$$\equiv \left[ \frac{1}{\pi} \int_0^{2\pi} N_{10}\left( A_{10} S_\psi \, , A_{20} S_{\sqrt{3}(\psi + (\phi_{20} - \phi_{10}))} \right) \cdot S_\psi \, d\psi \right] S_{\psi_{10}}$$

$$+ \left[ \frac{1}{\pi} \int_0^{2\pi} N_{10}\left( A_{10} S_\psi , A_{20} S_{\sqrt{3}(\psi + (\phi_{20} - \phi_{10}))} \right) \cdot C_\psi \, d\psi \right] C_{\psi_{10}}$$

$$\equiv \hat{N}_{10}^{s} \, S_{\psi_{10}} + \hat{N}_{10}^{c} \, C_{\psi_{10}}$$

$$\langle N_{20} \rangle_{\sqrt{3}\psi_{20}} \equiv \left[ \frac{\sqrt{3}}{\pi} \int_0^{2\pi/\sqrt{3}} N_{20} \cdot S_{\sqrt{3}\psi} \, d\tau \right] S_{\sqrt{3}\psi_{20}} + \left[ \frac{\sqrt{3}}{\pi} \int_0^{2\pi} N_{20} \cdot C_\psi \, d\psi \right] C_{\sqrt{3}\psi_{20}}$$

$$\psi_{10} \equiv \tau + \phi_{10} = \psi_{20} + (\phi_{10} - \phi_{20}) \; ; \quad \psi_{20} \equiv \tau + \phi_{20}$$

$$\equiv \left[ \frac{\sqrt{3}}{\pi} \int_0^{2\pi/\sqrt{3}} N_{20}\left( A_{10} S_{\psi + (\phi_{10} - \phi_{20})} , A_{20} S_{\sqrt{3}\psi} \right) \cdot S_{\sqrt{3}\psi} \, d\psi \right] S_{\sqrt{3}\psi_{20}}$$

$$+ \left[ \frac{\sqrt{3}}{\pi} \int_0^{2\pi/\sqrt{3}} N_{20}\left( A_{10} S_{\psi + (\phi_{10} - \phi_{20})} , A_{20} S_{\sqrt{3}\psi} \right) \cdot C_{\sqrt{3}\psi} \, d\psi \right] C_{\sqrt{3}\psi_{20}}$$

$$\equiv \hat{N}_{20}^{s} \, S_{\sqrt{3}\psi_{20}} + \hat{N}_{20}^{c} \, C_{\sqrt{3}\psi_{20}}$$

$$\hat{N}_{i0}^{s} \equiv \frac{\omega_i}{\pi} \int_0^{2\pi/\omega_i} N_{i0}\left( \underline{A}_0 \, S_{\underline{\omega}(\psi + (\underline{\phi} - \phi_{i0}))} \right) S_{\omega_i \psi} \, d\psi$$

$$\hat{N}_{i0}^{c} \equiv \frac{\omega_i}{\pi} \int_0^{2\pi/\omega_i} N_{i0}\left( \underline{A}_0 \, S_{(\underline{\omega}(\psi + \underline{\phi} - \phi_{i0}))} \right) C_{\omega_i \psi} \, d\psi$$

SECULAR COEFFICIENTS

DEFINE NON-RESONANT TERMS AS $[\,\cdot\,]_{\omega_i \psi_{i0}}$, SO THAT

$$N_{i0} \equiv \left[ N_{i0} \right]_{\omega_i \psi} + \langle N_{i0} \rangle_{\omega_i \psi}$$

$$\left[ N_{10} \right]_{\psi_{10}} = N_{10}\left( A_{10} S_{\psi_{10}} , A_{20} S_{\sqrt{3}\psi_{20}} \right) - \left( \hat{N}_{10}^{s} S_{\psi_{10}} + \hat{N}_{10}^{c} C_{\psi_{10}} \right)$$

$$\left[ N_{20} \right]_{\sqrt{3}\psi_{20}} = N_{20}\left( A_{10} S_{\psi_{10}} , A_{20} S_{\sqrt{3}\psi_{20}} \right) - \left( \hat{N}_{20}^{s} S_{\sqrt{3}\psi_{20}} + \hat{N}_{20}^{c} C_{\sqrt{3}\psi_{20}} \right)$$

## FORCING

FOURIER COMPONENT AT FREQUENCY $\omega$

$$\sim \left[ \frac{1}{\pi} \int_0^{2\pi} g\left(\frac{\sigma}{\omega}\right) S_\sigma \, d\sigma \right] S_{\omega\tau} + \left[ \frac{1}{\pi} \int_0^{2\pi} g\left(\frac{\sigma}{\omega}\right) C_\sigma \, d\sigma \right] C_{\omega\tau} = G_{\omega,s} \cdot S_{\omega\tau} + G_{\omega,c} \cdot C_{\omega\tau}$$

$$= G_{\omega,s}\left( S_{\omega(\tau+\phi)} C_{\omega\phi} - C_{\omega(\tau+\phi)} S_{\omega\phi} \right) + G_{\omega,c}\left( C_{\omega(\tau+\phi)} C_{\omega\phi} + S_{\omega(\tau+\phi)} S_{\omega\phi} \right)$$

$$= \left( -G_{\omega,s} S_{\omega\phi} + G_{\omega,c} C_{\omega\phi} \right) C_{\omega\psi} + \left( G_{\omega,s} C_{\omega\phi} + G_{\omega,c} S_{\omega\phi} \right) S_{\omega\psi}$$

IN THE ABSENCE OF RESONANT FORCING

$$\frac{\partial^2 q_{11}}{\partial \tau^2} + q_{11} = g_1(\tau) - \Bigg\{ 2\left( \frac{\partial A_{10}}{\partial \eta_1} C_{\psi_{10}} - A_{10} \frac{\partial \phi_{10}}{\partial \eta_1} S_{\psi_{10}} \right) + \frac{\beta}{2}\left( A_{10} C_{\psi_{10}} - \sqrt{3} A_{20} C_{\sqrt{3}\psi_{20}} \right)$$

$$+ \hat{N}_{10}^{s} S_{\psi_{10}} + \hat{N}_{10}^{c} C_{\psi_{10}} + \left[ N_{10}\left( A_{10} S_{\psi_{10}}, A_{20} S_{\sqrt{3}\psi_{20}} \right) \right]_{\psi_{10}} \Bigg\}$$

$$\frac{\partial^2 q_{21}}{\partial \tau^2} + 3 q_{21} = g_2(\tau) - \Bigg\{ 2\left( \sqrt{3} \frac{\partial A_{20}}{\partial \eta_1} C_{\sqrt{3}\psi_{20}} - 3 A_{20} \frac{\partial \phi_{20}}{\partial \eta_1} S_{\sqrt{3}\psi_{20}} \right) - \frac{\beta}{2}\left( A_{10} C_{\psi_{10}} - \sqrt{3} A_{20} C_{\sqrt{3}\psi_{20}} \right)$$

$$+ \hat{N}_{20}^{s} S_{\sqrt{3}\psi_{20}} + \hat{N}_{20}^{c} C_{\sqrt{3}\psi_{20}} + \left[ N_{20}\left( A_{10} S_{\psi_{10}}, A_{20} S_{\sqrt{3}\psi_{20}} \right) \right]_{\sqrt{3}\psi_{20}} \Bigg\}$$

THE GENERAL SLOW FLOW EQUATIONS ARE

$$2 \frac{\partial A_{10}}{\partial \eta} + \frac{\beta}{2} A_{10} + \hat{N}_{10}^{c} = 0 \qquad\qquad 2\sqrt{3} \frac{\partial A_{20}}{\partial \eta} + \frac{\sqrt{3}\beta}{2} A_{10} + \hat{N}_{20}^{c} = 0$$

$$-2 A_{10} \frac{\partial \phi_{10}}{\partial \eta} + \hat{N}_{10}^{s} = 0 \qquad\qquad -6 A_{20} \frac{\partial \phi_{20}}{\partial \eta} + \hat{N}_{20}^{s} = 0$$

SO THAT

$$\frac{\partial^2 q_{11}}{\partial \tau^2} + q_{11} = g_1(\tau) - \left\{ -\frac{\sqrt{3}\beta}{2} A_{20} C_{\sqrt{3}\psi_{20}} + \left[ N_{10}\left( A_{10} S_{\psi_{10}}, A_{20} S_{\sqrt{3}\psi_{20}} \right) \right]_{\psi_{10}} \right\}$$

$$\frac{\partial^2 q_{21}}{\partial \tau^2} + 3 q_{21} = g_2(\tau) - \left\{ -\frac{\beta}{2} A_{10} C_{\psi_{10}} + \left[ N_{20}\left( A_{10} S_{\psi_{10}}, A_{20} S_{\sqrt{3}\psi_{20}} \right) \right]_{\sqrt{3}\psi_{20}} \right\}$$

+ RETURNING TO THE EXPANSION

$$q_i(t) = A_{i0}(\eta) S_{\omega_i \psi_i(\tau,\eta)} + \varepsilon q_{i1}(\tau, \eta) \qquad\qquad \tau = t, \quad \eta = \varepsilon t$$

FOR THIS SYSTEM

$$N_{10}(q_{10},q_{20}) = \frac{\delta}{4}\left(\frac{\partial q_{10}}{\partial \tau} - \frac{\partial q_{20}}{\partial \tau}\right)^3 + \frac{\alpha}{4}\left(q_{10}+q_{20}\right)^3$$

$$= \frac{\delta}{4}\left(A_{10}C_{\psi_{10}} - \sqrt{3}A_{20}C_{\sqrt{3}\psi_{20}}\right)^3 + \frac{\alpha}{4}\left(A_{10}S_{\psi_{10}} + A_{20}S_{\sqrt{3}\psi_{20}}\right)^3$$

$$N_{20}(q_{10},q_{20}) = -\frac{\delta}{4}\left(\frac{\partial q_{10}}{\partial \tau} - \frac{\partial q_{20}}{\partial \tau}\right)^3 + \frac{\alpha}{4}\left(q_{10}+q_{20}\right)^3$$

$$= -\frac{\delta}{4}\left(A_{10}C_{\psi_{10}} - \sqrt{3}A_{20}C_{\sqrt{3}\psi_{20}}\right)^3 + \frac{\alpha}{4}\left(A_{10}S_{\psi_{10}} + A_{20}S_{\sqrt{3}\psi_{20}}\right)^3$$

$$\left(A_{10}C_{\psi_{10}} - \sqrt{3}A_{20}C_{\sqrt{3}\psi_{20}}\right)^3 = A_{10}^3 C_{\psi_{10}}^3 - 3\sqrt{3}A_{10}^2A_{20}C_{\psi_{10}}^2 C_{\sqrt{3}\psi_{20}} + 9A_{10}A_{20}^2 C_{\psi_{10}}C_{\sqrt{3}\psi_{20}}^2 - 9\sqrt{3}A_{20}^3 C_{\sqrt{3}\psi_{20}}^3$$

$$\left(A_{10}S_{\psi_{10}} + A_{20}S_{\sqrt{3}\psi_{20}}\right)^3 = A_{10}^3 S_{\psi_{10}}^3 + 3A_{10}^2A_{20}S_{\psi_{10}}^2 S_{\sqrt{3}\psi_{20}} + 3A_{10}A_{20}^2 S_{\psi_{10}}S_{\sqrt{3}\psi_{20}}^2 + A_{20}^3 S_{\sqrt{3}\psi_{20}}^3$$

$$C_a^2 C_b = \frac{C_b}{2} + \frac{C_{(2a-b)}}{4} + \frac{C_{(2a+b)}}{4} \qquad S_a^2 S_b = \frac{S_b}{2} + \frac{S_{(2a-b)}}{4} - \frac{S_{(2a+b)}}{4}$$

$$\left(A_{10}C_{\psi_{10}} - \sqrt{3}A_{20}C_{\sqrt{3}\psi_{20}}\right)^3 = \frac{A_{10}^3}{4}\left(3C_{\psi_{10}} + C_{3\psi_{10}}\right) - \frac{3\sqrt{3}A_{10}^2A_{20}}{4}\left(2C_{\sqrt{3}\psi_{10}} + C_{(2\psi_{10}-\sqrt{3}\psi_{20})} + C_{(2\psi_{10}+\sqrt{3}\psi_{20})}\right)$$

$$+ \frac{9A_{10}A_{20}^2}{4}\left(2C_{\psi_{10}} + C_{(2\sqrt{3}\psi_{20}-\psi_{10})} + C_{(2\sqrt{3}\psi_{20}+\psi_{10})}\right) - \frac{9\sqrt{3}A_{20}^3}{4}\left(3C_{\sqrt{3}\psi_{20}} + C_{3\sqrt{3}\psi_{20}}\right)$$

$$\left(A_{10}S_{\psi_{10}} + A_{20}S_{\sqrt{3}\psi_{20}}\right)^3 = \frac{A_{10}^3}{4}\left(3S_{\psi_{10}} - S_{3\psi_{10}}\right) + \frac{3A_{10}^2A_{20}}{4}\left(2S_{\sqrt{3}\psi_{10}} + S_{(2\psi_{10}-\sqrt{3}\psi_{20})} - S_{(2\psi_{10}+\sqrt{3}\psi_{20})}\right)$$

$$+ \frac{3A_{10}A_{20}^2}{4}\left(2S_{\psi_{10}} + S_{(2\sqrt{3}\psi_{20}-\psi_{10})} - S_{(2\sqrt{3}\psi_{20}+\psi_{10})}\right) + \frac{A_{20}^3}{4}\left(3S_{\sqrt{3}\psi_{20}} - S_{3\sqrt{3}\psi_{20}}\right)$$

SECULAR TERMS $(\psi \equiv \tau + \phi)$

$$\langle N_{10}\rangle_{\psi} = \left[\frac{\delta}{4}\left(\frac{3}{4}A_{10}^3 + \frac{9}{2}A_{10}A_{20}^2\right)\right]C_{\psi_{10}} + \left[-A_{10}\frac{\partial \phi_{10}}{\partial \eta_1} + \frac{\alpha}{4}\left(\frac{3}{4}A_{10}^3 + \frac{3}{2}A_{10}A_{20}^2\right)\right]S_{\psi_{10}}$$

$$\langle N_{20}\rangle_{\sqrt{3}\psi} = \left[\frac{\delta}{4}\left(\frac{3\sqrt{3}}{2}A_{10}^2A_{20} + \frac{27\sqrt{3}}{4}A_{20}^3\right)\right]C_{\sqrt{3}\psi_{20}} + \left[-6A_{20}\frac{\partial \phi_{20}}{\partial \eta_1} + \frac{\alpha}{4}\left(\frac{3}{2}A_{10}^2A_{20} + \frac{3}{4}A_{20}^3\right)\right]S_{\sqrt{3}\psi_{20}}$$

## SLOW FLOW EQUATIONS

$$\frac{\partial A_{10}}{\partial \eta_1} + \left[\frac{\beta}{4} + \frac{3\delta}{32}\left(A_{10}^2 + 6A_{20}^2\right)\right] A_{10} = 0$$

$$\frac{\partial \phi_{10}}{\partial \eta_1} = \frac{3\alpha}{32}\left(A_{10}^2 + 2A_{20}^2\right)$$

$$\frac{\partial A_{20}}{\partial \eta_1} + \left[\frac{\beta}{4} + \frac{3\delta}{32}\left(2A_{10}^2 + 9A_{20}^2\right)\right] A_{20} = 0$$

$$\frac{\partial \phi_{20}}{\partial \eta_1} = \frac{\alpha}{32}\left(2A_{10}^2 + A_{20}^2\right)$$

NOTE THAT THE COUPLING IS PARAMETRIC → THROUGH EFFECTIVE DAMPING RATIO & FREQUENCY

## SINGLE NONLINEAR NORMAL MODE

SYSTEM RESPONDS IN MODE $k$ IF $A_{k0} \neq 0$, $A_{i0}\big|_{i\neq k} = 0$

CONSIDER MODE 1 $\left(A_{20} \equiv 0\right)$

$$\frac{\partial A_{10}}{\partial \eta_1} + \left[\frac{\beta}{4} + \frac{3\delta}{32} A_{10}^2\right] A_{10} = 0$$

AMPLITUDE DYNAMICS

$$\frac{\partial \phi_{10}}{\partial \eta_1} = \frac{3\alpha}{32} A_{10}^2$$

PERIOD-AMPLITUDE DEPENDENCE

& RETURNING TO THE $\mathcal{O}(\epsilon)$ CORRECTIONS

$$\frac{\partial^2 q_{11}}{\partial \tau^2} + q_{11} = -\left\{\frac{\delta}{16} A_{10}^3 C_{3\psi_{10}} - \frac{\alpha}{16} A_{10}^3 S_{3\psi_{10}}\right\}$$

$$\frac{\partial^2 q_{21}}{\partial \tau^2} + 3q_{21} = -\left\{-\frac{\beta}{2} A_{10} C_{\psi_{10}} - \frac{\delta}{16} A_{10}^3\left(3C_{\psi_{10}} + C_{3\psi_{10}}\right) + \frac{\alpha}{16} A_{10}^3\left(3S_{\psi_{10}} - S_{3\psi_{10}}\right)\right\}$$

$$\boxed{\frac{\partial^2 q}{\partial \tau^2} + \Omega^2 q = F_0 S_{(\omega\psi)}, \quad \psi = \tau + \phi \quad \Rightarrow \quad q(\tau) = \frac{F_0}{\omega^2 - \Omega^2} S_{(\omega\psi)}}$$

$$q_{11}(\tau) = \cancelto{0}{A_{11} S_{\psi_{11}}} - \left\{\frac{\delta A_{10}^3}{16(-9+1)} C_{3\psi_{10}} - \frac{\alpha A_{10}^3}{16(-9+1)} S_{3\psi_{10}}\right\}$$

$$= \frac{\delta}{128} A_{10}^3 C_{3\psi_{10}} - \frac{\alpha}{128} A_{10}^3 S_{3\psi_{10}}$$

$$q_{21}(\tau) = \cancelto{0}{A_{21} S_{\sqrt{3}\psi_{21}}} - \left\{-\frac{\beta A_{10}}{2(-1+3)} C_{\psi_{10}} - \frac{\delta A_{10}^3}{16}\left(\frac{3C_{\psi_{10}}}{(-1+3)} + \frac{C_{3\psi_{10}}}{(-9+3)}\right) + \frac{\alpha A_{10}^3}{16}\left(\frac{3S_{\psi_{10}}}{(-1+3)} - \frac{S_{3\psi_{10}}}{(-9+3)}\right)\right\}$$

$$= \frac{\beta}{4} A_{10} C_{\psi_{10}} + \frac{\delta}{16} A_{10}^3\left(\frac{3}{2} C_{\psi_{10}} - \frac{1}{6} C_{3\psi_{10}}\right) - \frac{\alpha}{16} A_{10}^3\left(\frac{3}{2} S_{\psi_{10}} + \frac{1}{6} S_{3\psi_{10}}\right)$$

$$q_1(t) = q_{10}(t) + \varepsilon q_{11}(t) + \cdots$$

$$q_2(t) = q_{20}(t) + \varepsilon q_{21}(t) + \cdots$$

$$q_1(t) = A_{10} S_{\psi_{10}} + \varepsilon \left\{ -\frac{\alpha}{128} A_{10}^3 S_{3\psi_{10}} + \frac{\delta}{128} A_{10}^3 C_{3\psi_{10}} \right\}$$

$$q_2(t) = \varepsilon \left\{ -\frac{\alpha}{16} A_{10}^3 \left( \frac{3}{2} S_{\psi_{10}} + \frac{1}{6} S_{3\psi_{10}} \right) + \frac{\beta}{4} A_{10} C_{\psi_{10}} + \frac{\delta}{16} A_{10}^3 \left( \frac{3}{2} C_{\psi_{10}} - \frac{1}{6} C_{3\psi_{10}} \right) \right\}$$

THE NONLINEAR NORMAL MODE CAN BE PARAMETERIZED

DEFINE $\quad u_1 \equiv A_{10} S_{\psi_{10}} \; ; \; v_1 \equiv A_{10} C_{\psi_{10}} \qquad \frac{\partial u_1}{\partial \tau} \equiv v_1 \; ; \; \frac{\partial v_1}{\partial \tau} \equiv -u_1 \qquad A_{10}^2 = u_1^2 + v_1^2$

$$S_x^3 = \frac{3}{4} S_x - \frac{1}{4} S_{3x} \qquad C_x^3 = \frac{3}{4} C_x + \frac{1}{4} C_{3x}$$

$$S_{3x} = 3S_x - 4S_x^3 \qquad C_{3x} = -3C_x + 4C_x^3$$

$$q_1 = u_1 + \varepsilon \left\{ -\frac{\alpha}{128} \left( 3A_{10}^2 u_1 - 4u_1^3 \right) + \frac{\delta}{128} \left( -3A_{10}^2 v_1 + 4v_1^3 \right) \right\}$$

$$q_2 = \varepsilon \left\{ -\frac{\alpha}{16} \left( \frac{3}{2} A_{10}^2 u_1 + \frac{1}{6} \left( 3A_{10}^2 u_1 - 4u_1^3 \right) \right) + \frac{\beta}{4} v_1 + \frac{\delta}{16} \left( \frac{3}{2} A_{10}^2 v_1 - \frac{1}{6} \left( -3A_{10}^2 v_1 + 4v_1^3 \right) \right) \right\}$$

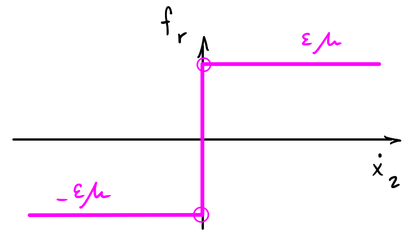
$$= \varepsilon \left\{ -\frac{\alpha}{8} \left( A_{10}^2 u_1 - \frac{1}{3} u_1^3 \right) + \frac{\beta}{4} v_1 + \frac{\delta}{8} \left( A_{10}^2 v_1 - \frac{1}{3} v_1^3 \right) \right\}$$

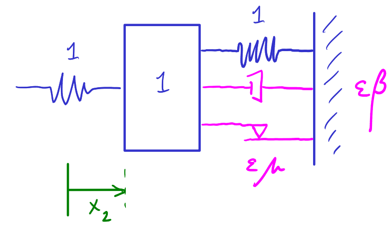
$$\frac{\partial q_1}{\partial \tau} = v_1 + \varepsilon \left\{ -\frac{\alpha}{128} \left( 3A_{10}^2 - 12u_1^2 \right) v_1 - \frac{\delta}{128} \left( -3A_{10}^2 + 12v_1^2 \right) u_1 \right\}$$

$$\frac{\partial q_2}{\partial \tau} = \varepsilon \left\{ -\frac{\alpha}{8} \left( A_{10}^2 - u_1^2 \right) v_1 - \frac{\beta}{4} u_1 - \frac{\delta}{8} \left( A_{10}^2 - v_1^2 \right) u_1 \right\}$$

CONSIDER A FRICTION DAMPER BETWEEN BLOCK 2 & GROUND

$$\underline{N} = \begin{bmatrix} \alpha x_1^3 \\ f_r[x_2] \end{bmatrix}$$




$$x_1 = \frac{q_1 + q_2}{\sqrt{2}} \qquad x_2 = \frac{q_1 - q_2}{\sqrt{2}}$$

SO THAT

$$\underline{N} = \begin{bmatrix} \frac{\alpha}{2\sqrt{2}}(q_1+q_2)^3 \\ \text{SIGN}(\dot{x}_2)\mu \end{bmatrix} \; ; \qquad \dot{x}_2 = \frac{\dot{q}_1 - \dot{q}_2}{\sqrt{2}}$$

$$\phi_i^T \underline{N} = \frac{1}{\sqrt{2}}\left\{ \frac{\alpha}{2\sqrt{2}}(q_1+q_2)^3 \pm \text{SIGN}(\dot{q}_1 - \dot{q}_2)\mu \right\} \qquad N_{10} \rightarrow + \qquad N_{20} \rightarrow -$$

THE RESONANT COEFFICIENTS REDUCE TO

$$\hat{N}_{10}^S = \frac{3\alpha}{16}\left(A_{10}^2 + 2A_{20}^2\right)A_{10} + \frac{1}{\pi}\int_0^{2\pi} \frac{\mu}{\sqrt{2}}\,\text{SIGN}\left(A_{10}C_\psi - \sqrt{3}A_{20}C_{\sqrt{3}(\psi + (\phi_{20}-\phi_{10}))}\right) S_\psi\, d\psi$$

$$\hat{N}_{10}^C = \frac{1}{\pi}\int_0^{2\pi} \frac{\mu}{\sqrt{2}}\,\text{SIGN}\left(A_{10}C_\psi - \sqrt{3}A_{20}C_{\sqrt{3}(\psi+(\phi_{20}-\phi_{10}))}\right) C_\psi\, d\psi$$

$$\hat{N}_{20}^S = \frac{3\alpha}{16}\left(2A_{10}^2 + A_{20}^2\right)A_{20} + \frac{\sqrt{3}}{\pi}\int_0^{2\pi/\sqrt{3}} -\frac{\mu}{\sqrt{2}}\,\text{SIGN}\left(A_{10}C_{(\psi+(\phi_{10}-\phi_{20}))} - \sqrt{3}A_{20}C_{\sqrt{3}\psi}\right) S_{\sqrt{3}\psi}\, d\psi$$

$$\hat{N}_{20}^C = \frac{\sqrt{3}}{\pi}\int_0^{2\pi/\sqrt{3}} -\frac{\mu}{\sqrt{2}}\,\text{SIGN}\left(A_{10}C_{(\psi+(\phi_{10}-\phi_{20}))} - \sqrt{3}A_{20}C_{\sqrt{3}\psi}\right) C_{\sqrt{3}\psi}\, d\psi$$

CONSIDER A SINGLE MODE RESPONSE $\quad (A_{20} \equiv 0)$

$$\text{SIGN}(A_{10}C_\psi) = \text{SIGN}(C_\psi)$$

$$\hat{N}_{10}^S = \frac{3\alpha}{16}A_{10}^3 \qquad \hat{N}_{20}^S = 0$$

$$N_{20}^C = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \frac{\mu}{\sqrt{2}} C_\psi\, d\psi = \frac{4\mu}{\sqrt{2}\pi} \qquad \hat{N}_{20}^C = 0$$

SO THAT

$$2\frac{\partial A_{10}}{\partial \eta} + \frac{4\mu}{\sqrt{2}\pi} = 0 \qquad -2A_{10}\frac{\partial \phi_{10}}{\partial \eta} + \frac{3\alpha}{16}A_{10}^3 = 0$$

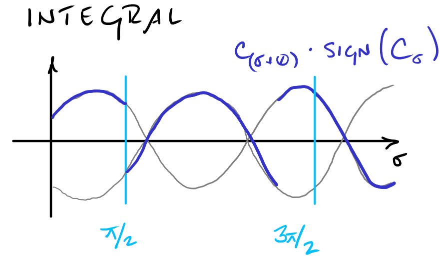
AT $O(\varepsilon)$ THE CORRECTIONS ARE

$$\frac{\partial^2 q_{11}}{\partial \tau^2} + q_{11} = \frac{\alpha}{16} A_{10}^3 S_{3\tau_{10}} + \frac{\mu}{\sqrt{2}}\left[\mathrm{SIGN}\left(A_{10} C_{\tau_{10}}\right) - \frac{4}{\pi} C_{\tau_{10}}\right]$$

$$\frac{\partial^2 q_{21}}{\partial \tau^2} + 3 q_{21} = -\frac{\alpha}{16} A_{10}^3 \left(3 S_{\tau_{10}} - S_{3\tau_{10}}\right) - \frac{\mu}{\sqrt{2}}\left[\mathrm{SIGN}\left(A_{10} C_{\tau_{10}}\right)\right]$$

THIS CAN BE SOLVED DIRECTLY WITH A CONVOLUTION INTEGRAL

$$q(\tau) = \int_0^\tau h(\tau-\sigma) f(\sigma)\, d\sigma \qquad h(t) = \frac{S_{\omega t}}{\omega_i}$$




$$\int_0^\tau \frac{S_{\omega(\tau-\sigma)}}{\omega} \cdot \mathrm{SIGN}\left(A C_\sigma\right) d\sigma$$

$$= \frac{S_{\omega\tau}}{\omega} \int_0^\tau C_{\omega\sigma} \cdot \mathrm{SIGN}\left(C_\sigma\right) d\sigma - \frac{C_{\omega\tau}}{\omega} \int_0^\tau S_{\omega\sigma} \cdot \mathrm{SIGN}\left(C_\sigma\right) d\sigma$$

$$\int_0^\tau e^{i\omega\sigma} \cdot \mathrm{SIGN}\left(C_\sigma\right) d\sigma$$

DEFINE $\gamma \equiv \mathrm{FLOOR}\left(\frac{\tau + \pi/2}{\pi}\right)$

$0 \le \tau < \pi/2$, $\gamma = 0$:
$$= \left.\frac{e^{i\omega\sigma}}{i\omega}\right|_0^\tau = \frac{e^{i\omega\tau}}{i\omega} - \frac{1}{i\omega}$$

$\pi/2 \le \tau < 3\pi/2$, $\gamma = 1$:
$$= -\left.\frac{e^{i\omega\sigma}}{i\omega}\right|_{\pi/2}^\tau + \frac{e^{i\omega\pi/2}}{i\omega} - \frac{1}{i\omega} = -\frac{e^{i\omega\tau}}{i\omega} + 2\frac{e^{i\omega\pi/2}}{i\omega} - \frac{1}{i\omega}$$

$3\pi/2 \le \tau < 5\pi/2$, $\gamma = 2$:
$$= \left.\frac{e^{i\omega\sigma}}{i\omega}\right|_{3\pi/2}^\tau + \left(-\frac{e^{i\omega 3\pi/2}}{i\omega} + 2\frac{e^{i\omega\pi/2}}{i\omega} - \frac{1}{i\omega}\right) = \frac{e^{i\omega\tau}}{i\omega} - 2\left(\frac{e^{i\omega 3\pi/2}}{i\omega} - \frac{e^{i\omega\pi/2}}{i\omega}\right) - \frac{1}{i\omega}$$

$$= \frac{1}{i\omega}\left\{(-1)^{\gamma(\tau)} e^{i\omega\tau} - 2\sum_{k=1}^{\gamma(\tau)} (-1)^k e^{i\omega(2j-1)\pi/2}\right\} - \frac{1}{i\omega}$$

$$\int_0^\tau C_{\omega\sigma}\, \mathrm{SIGN}\left(C_\sigma\right) d\sigma = \frac{1}{\omega}\left\{(-1)^{\gamma(\tau)} S_{\omega\tau} - 2\sum_{k=1}^{\gamma(\tau)} (-1)^k S_{\omega(2k-1)\pi/2}\right\}$$

$$\int_0^\tau S_{\omega\sigma}\, \mathrm{SIGN}\left(C_\sigma\right) d\sigma = \frac{1}{\omega}\left\{-(-1)^{\gamma(\tau)} C_{\omega\tau} + 2\sum_{k=1}^{\gamma(\tau)} (-1)^k C_{\omega(2k-1)\pi/2}\right\} + \cancel{\frac{1}{\omega}}$$

HOMOGENEOUS SOLUTION

$$\int_0^\tau \frac{S_{\omega(\tau-\sigma)}}{\omega} \cdot \mathrm{SIGN}\left(A C_\sigma\right) d\sigma = \frac{1}{\omega^2}\left\{(-1)^\gamma S_{\omega\tau}^2 - 2\sum_{k=1}^{\gamma} (-1)^k S_{\omega\tau} S_{\omega(2k-1)\pi/2}\right.$$
$$\left. + (-1)^\gamma C_{\omega\tau}^2 - 2\sum_{k=1}^{\gamma} (-1)^k C_{\omega\tau} C_{\omega(2k-1)\pi/2}\right\}$$

$$= \frac{1}{\omega^2}\left\{(-1)^\gamma - 2\sum_{k=1}^{\gamma} (-1)^k C_{\omega\left(\tau - (2k-1)\pi/2\right)}\right\}$$

NOTE THAT $\mathrm{SIGN}\left(A_{10} C_{\psi_{10}}\right)$ CAN BE WRITTEN IN A FOURIER SERIES AS

$$\mathrm{SIGN}\left(A_{10} C_{\psi_{10}}\right) = \sum_{k=1}^{\infty} a_k C_{k\psi_{10}} \qquad a_k = \begin{cases} \frac{4}{\pi k}(-1)^{(k+1)/2+1} & (k \text{ ODD}) \\ 0 & (k \text{ EVEN}) \end{cases}$$

SO THAT

$$\mathrm{SIGN}\left(A_{10} C_{\psi_{10}}\right) - \frac{4}{\pi} C_{\psi_{10}} = \sum_{j=2}^{\infty} \frac{4}{\pi} (-1)^{(j+1)} C_{(2j-1)\psi_{10}}$$

$$\frac{\partial^2 q_{11}}{\partial \tau^2} + q_{11} = \frac{\alpha}{16} A_{10}^3 S_{3\psi_{10}} + \frac{4\mu}{\sqrt{2}\pi} \sum_{j=2}^{\infty} (-1)^{(j+1)} C_{(2j-1)\psi_{10}}$$

$$\frac{\partial^2 q_{21}}{\partial \tau^2} + 3 q_{21} = -\frac{\alpha}{16} A_{10}^3 \left(3 S_{\psi_{10}} - S_{3\psi_{10}}\right) - \frac{4\mu}{\sqrt{2}\pi} \sum_{j=1}^{\infty} (-1)^{(j+1)} C_{(2j-1)\psi_{10}}$$