Wave-Based Modeling for Jointed Structures

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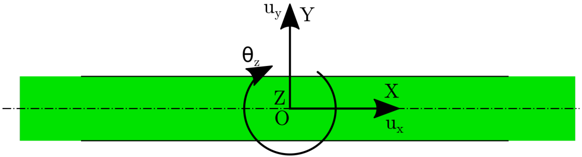
Outline

Section 1

Governing Equations and System of Interest

Dispersion of the Timoshenko Beam I



- The coordinate systems that will be used here are given in the figure above. The beam is taken to be in the XY plane and the degrees of freedom for each point on the beam are taken to be the displacements u_x, u_y, and rotation θ_z.
- The equations of motion used for the Timoshenko beam are:

$$\begin{bmatrix} \rho A & 0 & 0 \\ 0 & \rho A & 0 \\ 0 & 0 & \rho I_z \end{bmatrix} \frac{\partial^2}{\partial t^2} \begin{Bmatrix} u_x \\ u_y \\ \theta_z \end{Bmatrix} - \begin{bmatrix} EA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & EI_z \end{bmatrix} \frac{\partial^2}{\partial x^2} \begin{Bmatrix} u_x \\ u_y \\ \theta_z \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & GA \\ 0 & -GA & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{Bmatrix} u_x \\ u_y \\ \theta_z \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & GA \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \\ \theta_z \end{Bmatrix} = \begin{Bmatrix} f_x \\ f_y \\ m_z \end{Bmatrix}.$$

Dispersion of the Timoshenko Beam II

- Dispersion relationships will be discussed using the wave representation $\bar{u} = \bar{U} e^{i(kx - \omega t)}$.
- The dispersion relationship for the axial vibration (longitudinal waves) is

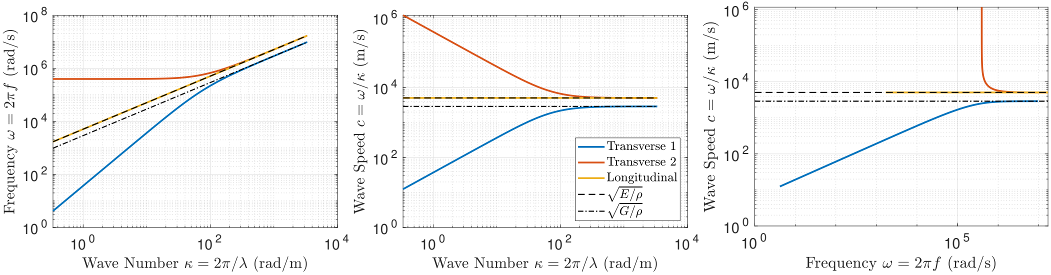
$$\omega = \sqrt{\frac{E}{\rho}} k$$

- For bending vibration (transverse waves), the dispersion relationship is

$$(\rho^2 A I_z)\omega^4 - \left[\rho G A^2 + k^2 \rho A I_z (G + E)\right] \omega^2 + (EGAI_z) k^4 = 0$$

Dispersion of the Timoshenko Beam III

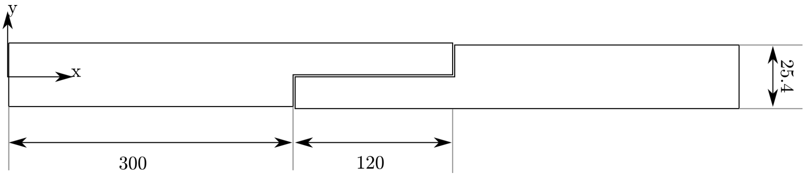
- The dispersion relationship is presented below graphically for a case corresponding to $E = 200GPa$, $\nu = 0.3$, $\rho = 7800kgm^{-3}$ and a square cross-section of $25.4mm$.



- Asymptotes corresponding to the case $\omega \to \infty$ are depicted using dashed lines in the figure

Brake-Reuß Beam Benchmark I

- The following is a schematic representation of the Brake-Reuß Beam (BRB) benchmark that the techniques will be applied to first. All dimensions are in mm and the in-plane thickness is also 25.4 mm.

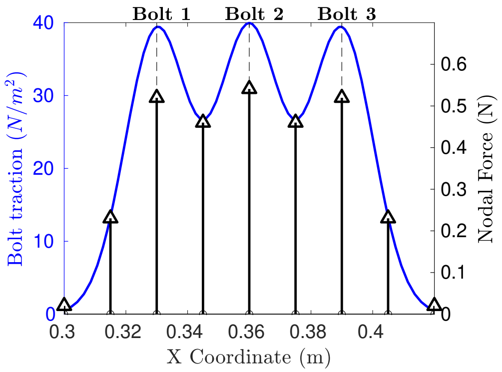


- The interface region is discretized using 4, 8, 16, and 32 elements to study the influence of mesh density.

Brake-Reuß Beam Benchmark II

- The bolt prestress is modeled by simulating a normal pressure distribution described by a sum of three Gaussian distributions with means at equidistant locations and standard deviations equal to 10.265 mm (fixed through washer dimensions and 33 degree pressure cone).
- The following shows the "bolt-traction" distribution and the static prestress solution of the system.

Brake-Reuß Beam Benchmark III



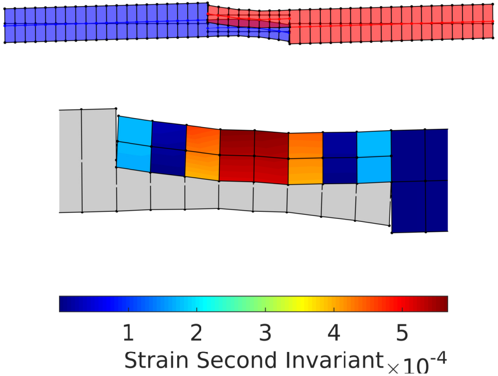


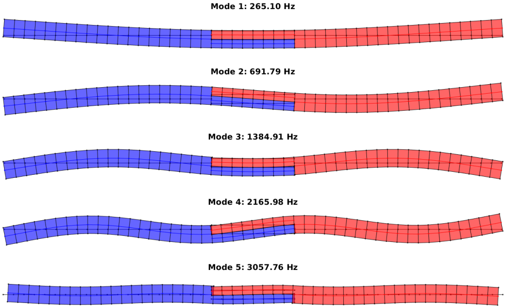
Figure: Bolt traction distribution and Static Prestress solution (displacements magnified 500x). The second strain invariant is $(\sigma_1^2 + \sigma_2^2)/2$.

Brake-Reuß Beam Benchmark IV

- The first five modal frequencies and mode shapes of this structure are given here

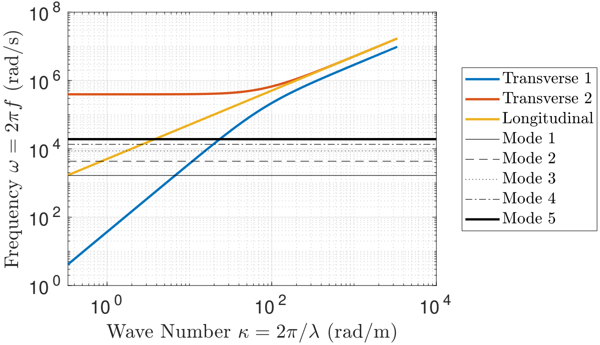
Sno.	Natural Frequency (Hz)
1	265.10
2	691.79
3	1384.91
4	2165.98
5	3057.76

Table: First Five Natural Frequencies



Brake-Reuß Beam Benchmark V

- Below the natural frequencies are plotted against the Dispersion plot:



- Observation: The second transverse wave does not seem to be engaged in the first five modes that might be interesting later on.

Wave Propagation Calculations I

Longitudinal Waves

Equations of Motion

$$\begin{bmatrix}\rho A & 0 & 0\\ 0 & \rho A & 0\\ 0 & 0 & \rho I_z\end{bmatrix}\frac{\partial^2}{\partial t^2}\begin{Bmatrix}u_x\\ u_y\\ \theta_z\end{Bmatrix}-\begin{bmatrix}EA & 0 & 0\\ 0 & GA & 0\\ 0 & 0 & EI_z\end{bmatrix}\frac{\partial^2}{\partial x^2}\begin{Bmatrix}u_x\\ u_y\\ \theta_z\end{Bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & GA\\ 0 & -GA & 0\end{bmatrix}\frac{\partial}{\partial x}\begin{Bmatrix}u_x\\ u_y\\ \theta_z\end{Bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & GA\end{bmatrix}\begin{Bmatrix}u_x\\ u_y\\ \theta_z\end{Bmatrix}=\begin{Bmatrix}f_x\\ f_y\\ m_z\end{Bmatrix}.$$

- Longitudinal waves are decoupled from transverse waves and are therefore considered separately.
- Choosing the symbol a_ℓ^+ and a_ℓ^- to denote complex amplitudes of forward and backward propagating waves, the longitudinal-only solution to the equations is:

$$u_x(x,t) = a_\ell^+ e^{-i(k_\ell x-\omega t)} + a_\ell^- e^{i(k_\ell x+\omega t)}.$$

Longitudinal Waves

- The dispersion relationship for k_ℓ and ω is,

$$\omega^2 = \underbrace{\frac{E}{\rho}}_{C_\ell^2} k_\ell^2.$$

- The wave-speed $C_\ell = \sqrt{\frac{E}{\rho}}$ is a constant for this case.
- Considering two points x_A and x_B on the beam with wave amplitudes $a_\ell^\pm$ and $b_\ell^\pm$ respectively, the following propagation relationship may be written for a fully linear beam:

$$b_\ell^+ = e^{-ik_\ell(x_A - x_B)} a_\ell^+ \qquad b_\ell^- = e^{ik_\ell(x_A - x_B)} a_\ell^-$$

Wave Propagation Calculations I

Transverse Waves

Equations of Motion

$$\begin{bmatrix} \rho A & 0 & 0 \\ 0 & \rho A & 0 \\ 0 & 0 & \rho I_z \end{bmatrix} \frac{\partial^2}{\partial t^2} \begin{Bmatrix} u_x \\ u_y \\ \theta_z \end{Bmatrix} - \begin{bmatrix} EA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & EI_z \end{bmatrix} \frac{\partial^2}{\partial x^2} \begin{Bmatrix} u_x \\ u_y \\ \theta_z \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & GA \\ 0 & -GA & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{Bmatrix} u_x \\ u_y \\ \theta_z \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & GA \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \\ \theta_z \end{Bmatrix} = \begin{Bmatrix} f_x \\ f_y \\ m_z \end{Bmatrix}.$$

- Transverse waves are studied independent of longitudinal waves since the equations are naturally decoupled (for the linear case)
- From theory (see Mei and Mace, 2005) and the wave speed plots before, it is known that two waves exist for transverse vibrations of Timoshenko beams
- The wave numbers of these are denoted as k_1 and $-ik_2$ respectively.
- The second wave is not harmonic unless the frequency is above a critical cut-off.

Transverse Waves

- The cut-off is found by evaluating the solution of the dispersion relationship in the limit $k \to 0$ as follows:

$$\omega^2 \left[(\rho^2 A I_z)\omega^2 - (\rho G A^2) \right] = 0$$

$$\implies \omega = \underbrace{0, 0}_{\text{first kind}} \quad ; \quad \underbrace{\sqrt{\frac{GA}{\rho I_z}}, -\sqrt{\frac{GA}{\rho I_z}}}_{\text{second kind}}$$

$$\text{denote } \omega_c = \sqrt{\frac{GA}{\rho I_z}}.$$

- For frequencies $\omega < \omega_c$, the waves of the second kind occur as pairs of growing and decaying components. The current study will be restricted to this region.

Wave Propagation Calculations III

Transverse Waves

- In these cases, the wave number will be complex and therefore $-ik_2$ is used to denote the wave number of convenience.
- With forward and backward components for each kinds of waves $a_1^{\pm}$, $a_2^{\pm}$, the transverse displacement is represented as,

$$u_y(x,t) = \underbrace{a_1^+ e^{-i(k_1x-\omega t)} + a_1^- e^{i(k_1x+\omega t)}}_{\text{first kind waves}} + \underbrace{a_2^+ e^{-(k_2x-i\omega t)} + a_2^- e^{(k_2x+i\omega t)}}_{\text{second kind waves}}$$

- Similarly, with amplitudes $\overline{a_1^{\pm}}$, $\overline{a_2^{\pm}}$, the rotations are written as,

$$\theta_z(x,t) = \underbrace{\overline{a_1^+} e^{-i(k_1x-\omega t)} + \overline{a_1^-} e^{i(k_1x+\omega t)}}_{\text{first kind waves}} + \underbrace{\overline{a_2^+} e^{-(k_2x-i\omega t)} + \overline{a_2^-} e^{(k_2x+i\omega t)}}_{\text{second kind waves}}$$

Wave Propagation Calculations IV

Transverse Waves

- Substituting these back into the equations of motions allows one to obtain relationships between $a_{1,2}^{\pm}$ and $\overline{a_{1,2}}^{\pm}$:
  - For $(a_1^+, \overline{a_1^+})$,

$$\begin{bmatrix} (GA)k_1^2 - (\rho A)\omega^2 & -ik_1(GA) \\ ik_1(GA) & (EI_z)k_1^2 - (\rho I_z)\omega^2(GA) \end{bmatrix} \begin{Bmatrix} a_1^+ \\ \overline{a_1^+} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

$$\implies \boxed{\frac{\overline{a_1^+}}{a_1^+} = -ik_1\left(1 - \frac{\omega^2}{C_s^2 k_1^2}\right)} \quad \text{with } C_s^2 = \frac{G}{A}.$$

Transverse Waves

- For $(a_1^-, \overline{a_1^-})$,

$$\begin{bmatrix} (GA)k_1^2 - (\rho A)\omega^2 & ik_1(GA) \\ -ik_1(GA) & (EI_z)k_1^2 - (\rho I_z)\omega^2(GA) \end{bmatrix} \begin{Bmatrix} a_1^- \\ \overline{a_1^-} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

$$\implies \boxed{\frac{\overline{a_1^-}}{a_1^-} = ik_1\left(1 - \frac{\omega^2}{C_s^2 k_1^2}\right)}.$$

- For $(a_2^+, \overline{a_2^+})$,

$$\begin{bmatrix} -(GA)k_2^2 - (\rho A)\omega^2 & -k_2(GA) \\ k_2(GA) & -(EI_z)k_2^2 - (\rho I_z)\omega^2(GA) \end{bmatrix} \begin{Bmatrix} a_2^+ \\ \overline{a_2^+} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

$$\implies \boxed{\frac{\overline{a_2^+}}{a_2^+} = -k_2\left(1 + \frac{\omega^2}{C_s^2 k_2^2}\right)}.$$

Transverse Waves

- For $(a_2^-, \overline{a_2^-})$,

$$\begin{bmatrix} -(GA)k_2^2-(\rho A)\omega^2 & k_2(GA) \\ -k_2(GA) & -(EI_z)k_2^2-(\rho I_z)\omega^2(GA) \end{bmatrix} \begin{Bmatrix} a_2^- \\ \overline{a_2^-} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

$$\implies \boxed{\frac{\overline{a_2^-}}{a_2^-} = k_2\left(1+\frac{\omega^2}{C_s^2 k_2^2}\right)}.$$

- In summary we have,

$$\boxed{\frac{\overline{a_1^+}}{a_1^+} = -iP} \quad \boxed{\frac{\overline{a_1^-}}{a_1^-} = iP} \quad \boxed{\frac{\overline{a_2^+}}{a_2^+} = -N} \quad \boxed{\frac{\overline{a_2^-}}{a_2^-} = N}$$

with, $P = k_1\left(1-\dfrac{\omega^2}{C_s^2 k_1^2}\right) \qquad N = k_2\left(1+\dfrac{\omega^2}{C_s^2 k_2^2}\right).$

Transverse Waves

- Substituting the above, we can express a general wave solution as,

$$\begin{Bmatrix} u_y \\ \theta_z \end{Bmatrix} = \begin{bmatrix} e^{-i(k_1x-\omega t)} & e^{-(k_2x-i\omega t)} \\ -(iP)e^{-i(k_1x-\omega t)} & -(N)e^{-(k_2x-i\omega t)} \end{bmatrix} \begin{Bmatrix} a_1^+ \\ a_2^+ \end{Bmatrix} + \begin{bmatrix} e^{i(k_1x+\omega t)} & e^{(k_2x+i\omega t)} \\ (iP)e^{i(k_1x+\omega t)} & (N)e^{(k_2x+i\omega t)} \end{bmatrix} \begin{Bmatrix} a_1^- \\ a_2^- \end{Bmatrix}$$

- Considering points at x_A, x_B, with wave amplitudes $a_{1,2}^{\pm}$ and $b_{1,2}^{\pm}$ respectively, the propagation of the wave amplitudes may be written as,

$$\boxed{\begin{Bmatrix} b_1^+ \\ b_2^+ \end{Bmatrix} = \begin{bmatrix} e^{-ik_1(x_A-x_B)} & 0 \\ 0 & e^{-k_2(x_A-x_B)} \end{bmatrix} \begin{Bmatrix} a_1^+ \\ a_2^+ \end{Bmatrix}}$$

$$\boxed{\begin{Bmatrix} b_1^- \\ b_2^- \end{Bmatrix} = \begin{bmatrix} e^{ik_1(x_A-x_B)} & 0 \\ 0 & e^{k_2(x_A-x_B)} \end{bmatrix} \begin{Bmatrix} a_1^- \\ a_2^- \end{Bmatrix}}$$

Section 2

Numerical Wave-based Study Methodology

Numerical Wave-based Study Methodology I

Approach 1: Inhomogeneous Response

- Since an analytical solution does not exist for this problem, the study will be conducted using finite element simulations
- 1D linear Timoshenko beam elements are used for this purpose
- Force pulses are generated by windowing sines with a Hanning window. An example is provided in the following slide.
- The width of the pulse will be chosen based on the size of the domain used for simulations.
- Simulations will be carried out for *pulse frequencies* (frequency of the excitation pulse) ranging from 150 Hz up to 350 Hz, so as to cover the first bending mode.

Numerical Wave-based Study Methodology II

Approach 1: Inhomogeneous Response

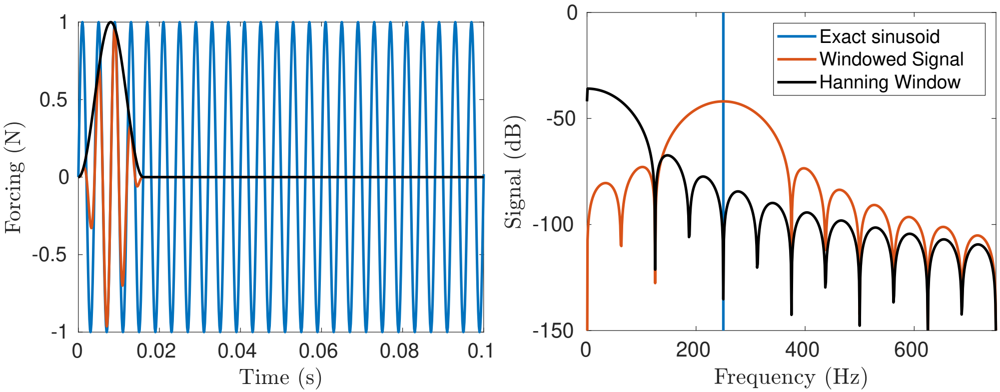


Figure: An exact sinusoid,

Approach 1: Inhomogeneous Response

- Using the dispersion relationships, the wave numbers, wavelengths, phase speeds and group speeds for the (transverse 1, transverse 2, longitudinal) waves are presented in parenthesis (in that order).

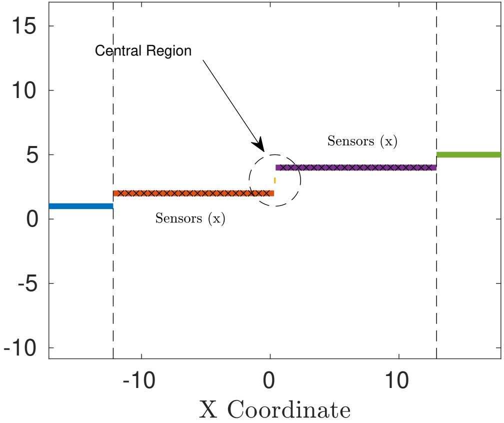
Frequency (Hz)	Wavenumber (rad/m)	Wavelength (m)	Phase Speed (m/s)	Group Speed (m/s)
150.00	(5.04, NaN, 0.19)	(1.25, NaN, 33.76)	(186.79, NaN, 5063.70)	(-1892.33, NaN, 5063.70)
350.00	(7.72, NaN, 0.43)	(0.81, NaN, 14.47)	(284.82, NaN, 5063.70)	(-4444.42, NaN, 5063.70)

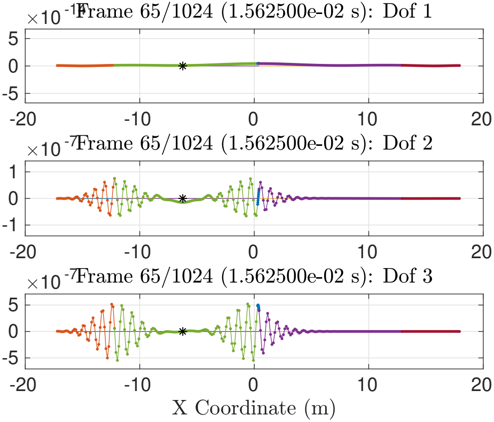
- By conducting time-frequency analyses of the response at specific locations along the beam, it is possible to build transmissibility matrices for pulses.
- A drawback here is that the different wave components in the transverse direction can not be distinguished using just time-frequency analysis.

Numerical Wave-based Study Methodology IV

Approach 1: Inhomogeneous Response

- The following is the setup (left) and a sample response (DOF 1, 2, 3 : u_x, u_y, θ_z).

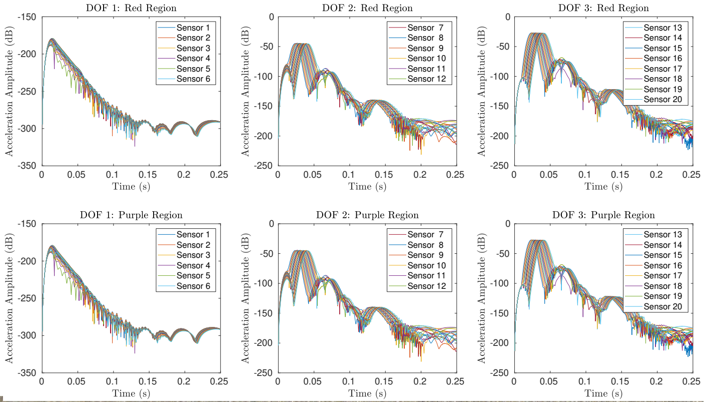




- The forcing location is indicated with $*$ in the plots in the right.

Approach 1: Inhomogeneous Response

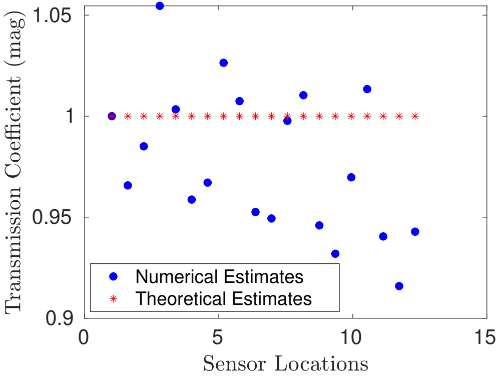
- A time frequency analysis is conducted on 10 “sensor locations” along each beam regions (red and purple in previous slide):



Numerical Wave-based Study Methodology VI

Approach 1: Inhomogeneous Response

- And here are the transmission coefficient estimates



- The theoretical estimates are $e^{-k_1(x_{dest}-x_{source})}$ (which assumes that the response only has the first kind of transverse waves).

Numerical Wave-based Study Methodology VII

Approach 1: Inhomogeneous Response

- However, as can be seen, the numerical estimates are nowhere near exact and I'm not sure **how to decompose a wave solution into the different wave components.**
- Further, at the location of forcing all the "wave components" get added up, thereby making detection even harder.

Approach 2: Homogeneous Wave Response

- A more direct approach would be to solve this as an initial value problem with initial conditions specified as,

$$u_x(x,0) = a_\ell^+ e^{-ik_\ell x} + a_\ell^- e^{ik_\ell x}$$

$$\begin{Bmatrix} u_y \\ \theta_z \end{Bmatrix} = \begin{bmatrix} e^{-i(k_1 x - \omega t)} & e^{-(k_2 x - i\omega t)} \\ -(iP)e^{-i(k_1 x - \omega t)} & -(N)e^{-(k_2 x - i\omega t)} \end{bmatrix} \begin{Bmatrix} a_1^+ \\ a_2^+ \end{Bmatrix} + \begin{bmatrix} e^{i(k_1 x + \omega t)} & e^{(k_2 x + i\omega t)} \\ (iP)e^{i(k_1 x + \omega t)} & (N)e^{(k_2 x + i\omega t)} \end{bmatrix} \begin{Bmatrix} a_1^- \\ a_2^- \end{Bmatrix}$$

- Different choices of $a_{\ell,1,2}^{\pm}$ can be chosen and "spatially windowed" to obtain response
- Analysis of the response is, however, not very clear at the moment. Z

Approach 2: Homogeneous Wave Response

- I have written a code for this but have a few questions:
  1. **How to localize the wave components?** Dr. Leamy had suggested using a Hann window spatially. While this works for the *type 1* wave, it is not very obvious how this will work for the **type 2** wave, which is not harmonic in space.
  2. **How to decompose the transient solution from a finite element model into the different wave components?** I understand how to do this for wave components that are harmonic in space but since the non-harmonic component is also present, I'm not sure if we can do this with just Fourier methods.
  3. **How to model amplitude dependence on the derived model?** When we say amplitude dependence, we need to determine how to choose an amplitude (or set of amplitudes) since we will have the interaction of 3 different kinds of waves for the nonlinear case (1 longitudinal and 2 transversal).

Approach 2: Homogeneous Wave Response

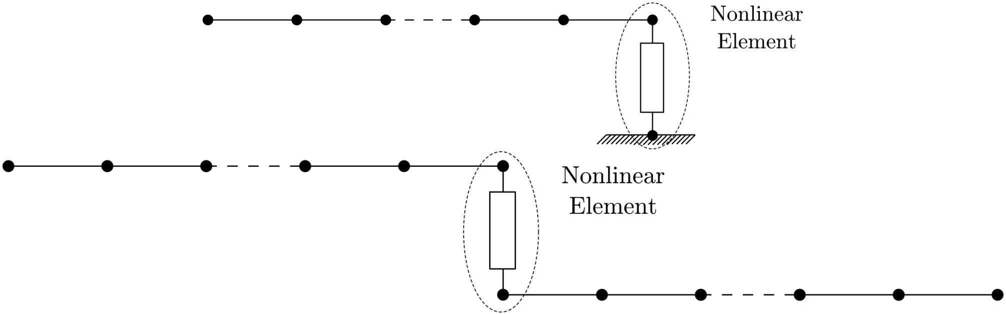
4. I'm currently assuming that we're interested in obtaining a transmission matrix of the form
$$T(a_\ell^+, a_\ell^-, a_1^+, a_1^-, a_2^+, a_2^-) \in \mathbb{C}^{6\times 6},$$
such that
$$\begin{Bmatrix} b_\ell^+ \\ b_\ell^- \\ b_1^+ \\ b_1^- \\ b_2^+ \\ b_2^- \end{Bmatrix} = T(a_\ell^+, a_\ell^-, a_1^+, a_1^-, a_2^+, a_2^-) \begin{Bmatrix} a_\ell^+ \\ a_\ell^- \\ a_1^+ \\ a_1^- \\ a_2^+ \\ a_2^- \end{Bmatrix},$$
where $a_{\ell,1,2}^\pm$ and $b_{\ell,1,2}^\pm$ represent the wave amplitudes "before" and "after" the joint is encountered. **Can this be simplified further?** I know for one that assumptions of symmetry <u>can't be made</u> in the X direction (influence of a_1^+ on b_1^+ will be different from the influence of a_1^- on b_1^- (or other way)) due to the geometry of the BRB joint.

Section 3

Present Directions

Present Directions I

- It may be a good starting step to derive the reflectance and transmittance matrices of a nonlinear element like the following



- Some examples would be

Present Directions II

1. The Bouc-Wen Model:

$$\begin{aligned} F(t) &= ak_i u(t) + (1-a) k_i z(t) \\ \dot{z}(t) &= \dot{u}(t) \left\{ A - \left[\beta \operatorname{sign}(z(t)\dot{u}(t)) + \gamma \right] |z(t)|^n \right\}. \end{aligned}$$

2. The Jenkins model:

$$dF(t) = \begin{cases} k_t du(t) & |F| < F_{slip} \\ F_{slip} sign(\frac{du}{dt}) & slip \end{cases}$$

- Here, u denotes the *relative displacement* experienced in the element and F is the *internal force* developed