COMP 2011: Data Structures

Lecture 10. Priority Queues and Heapsort

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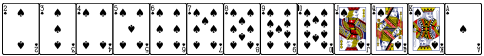
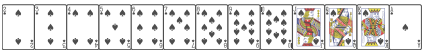
November, 2021

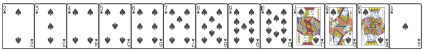


Review of Lecture 9

- Self-balancing binary search trees: AVL trees and red-black trees.
- Binary heaps.
- Operations `insert` and `removeMax` (shape, then bubble up/down).
- Again, all the modifications are restricted to a path
  though you need to check the other child of every node on the path for deletion.
- Also `peek` and `changeKey`.



Priority Queues



Priority queues (🐢)

- A priority queue is an ADT that has `insert` and `removeMax`/`removeMin`.
- Sometimes it supports operations `getMax`/`getMin` and `size` as well.
- Warning: A priority queue is not a queue![1]
- Can be implemented using (un)ordered arrays, (un)ordered linked lists.

- Recall the exercise on Queen Elizabeth Hospital.

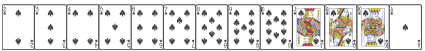
An abstract data type (ADT) defines
- a state of an object and
- operations that act on the object, possibly changing the state.

[1] Indeed, both stacks and queues can be viewed as priority queues.

Using a priority queue to sort

- Insert the elements, one by one, into a priority queue.
- Remove the elements, one by one, from the priority queue.

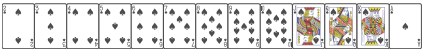
It takes O(_____) time?



Which of the following can be used to implement a priority queue.

- [T] an unsorted array
- [ ] a sorted array
- [ ] a maximum heap
- [ ] a minimum heap
- [ ] a binary search tree
- [ ] a self-balancing binary search tree

If yes, what's the running time for `insert` and `removeMax`?



Implementation	`insert`	`removeMax`	Algorithm	Best	Worst
Unordered Array	$O(1)$	$O(n)$	selection	$O(n^2)$	$O(n^2)$
Ordered Array	$O(n)$	$O(1)$	insertion	$O(n)$	$O(n^2)$
Heap	$O(\log n)$	$O(\log n)$	Heapsort[2]	$O(n \log n)$	$O(n \log n)$
Leonardo Heap	$O(\log n)$	$O(\log n)$	Smoothsort	$O(n \log n)$	$O(n \log n)$
Self-balancing binary search tree	$O(\log n)$	$O(\log n)$		$O(n \log n)$	$O(n \log n)$

[2] Not really

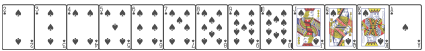
Priority queues and heaps

- In practice, a priority queue is almost always implemented using a heap
- but they're conceptually different!

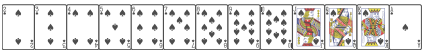
Check java.util.PriorityQueue and its source codes.

Priority queues $\neq$ heaps

Freestyle swimming (🌍) vs. Front crawl (🌍). (Tokyo 2020)

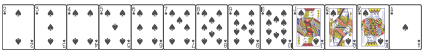


In-place Heapsort



Sorting with a heap

```
1  void heapSort(int[] keys, T[] data) {
2      Heap<T> heap = new Heap<T>(keys.length);
3      for (int i = 0; i < keys.length; i++)
4          heap.insert(keys[i], data[i]);
5      for (int i = keys.length - 1; i >= 0; i--)
6          data[i] = heap.removeMax();
7  }
```



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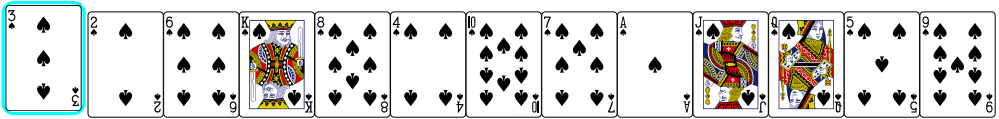

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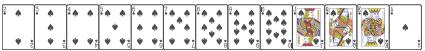
We add the cards into the heap one by one, which initially has only the first card.



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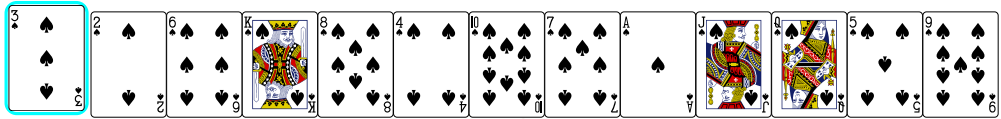

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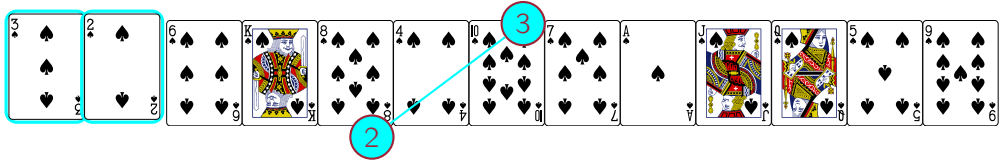

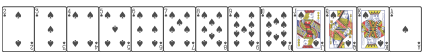

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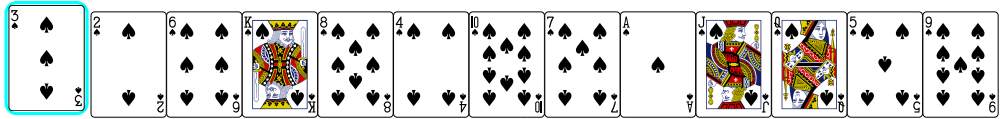

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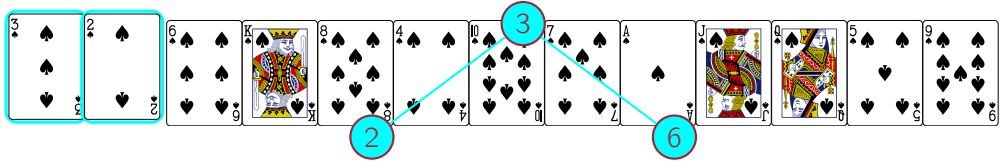
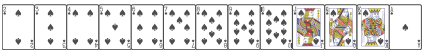
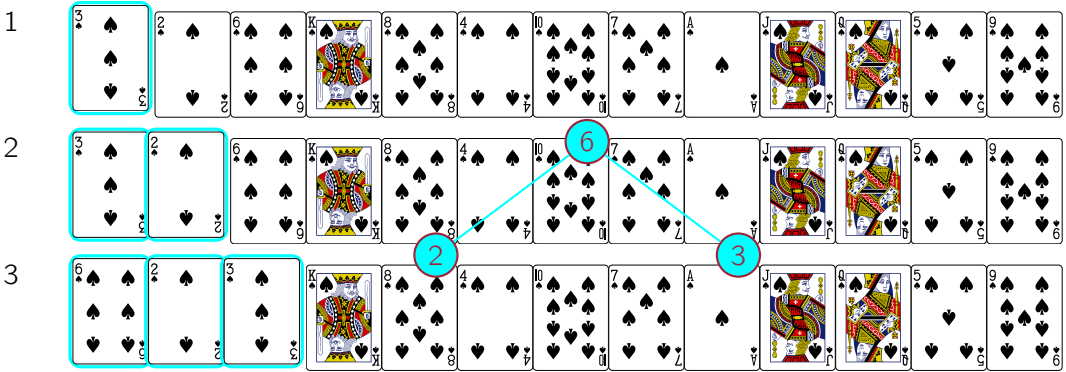




The correct position of the next card (6) is?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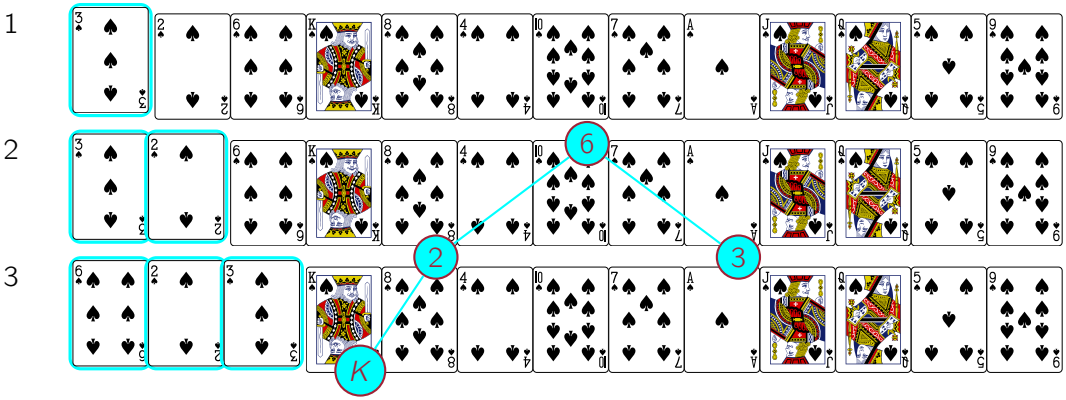
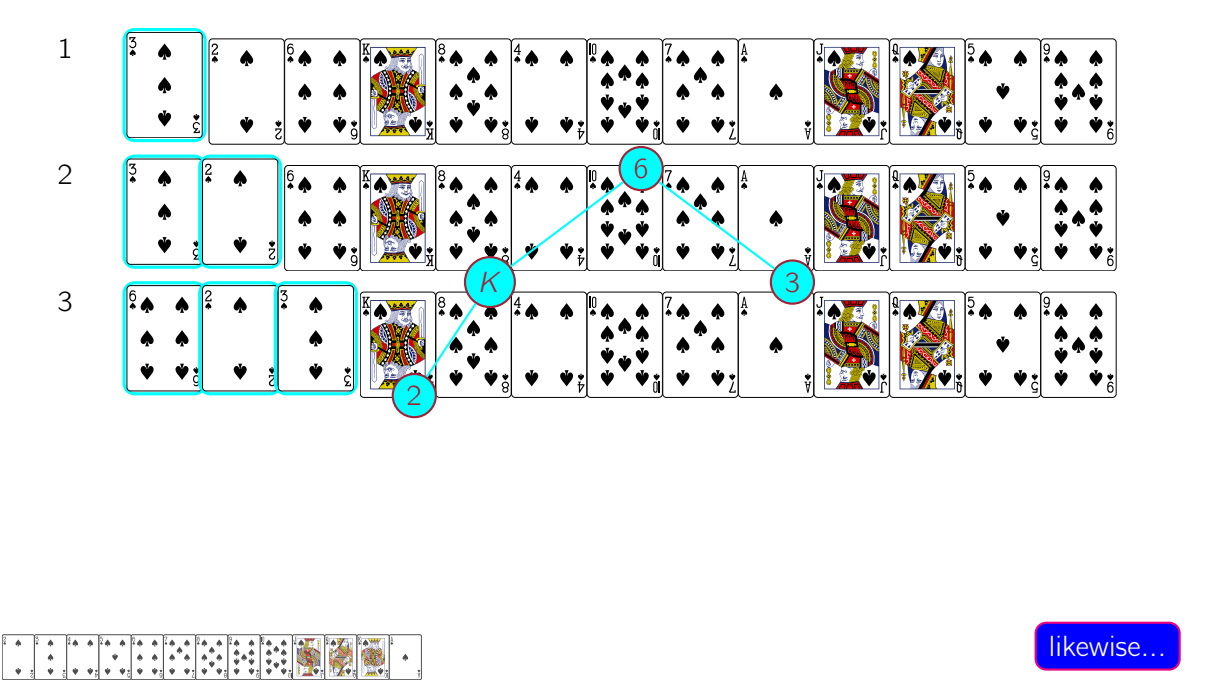


likewise...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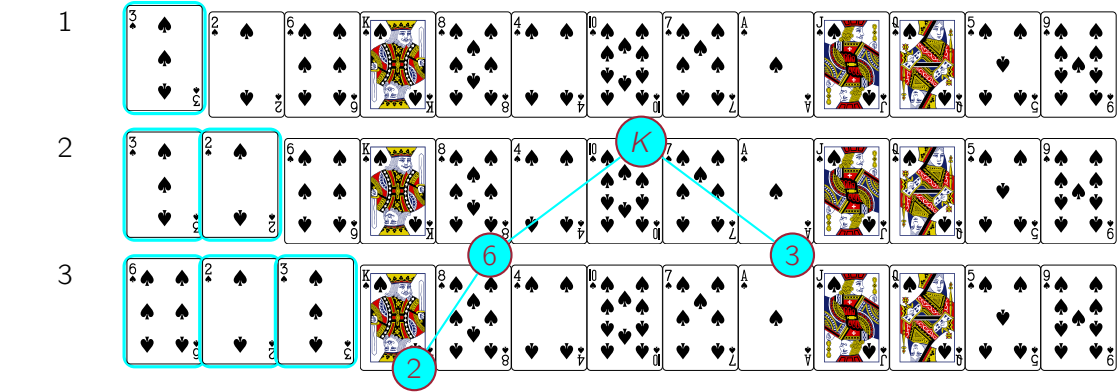
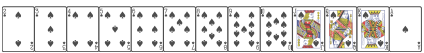


This time we don't swap, so j moves but i stays.



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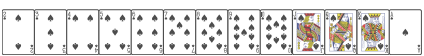

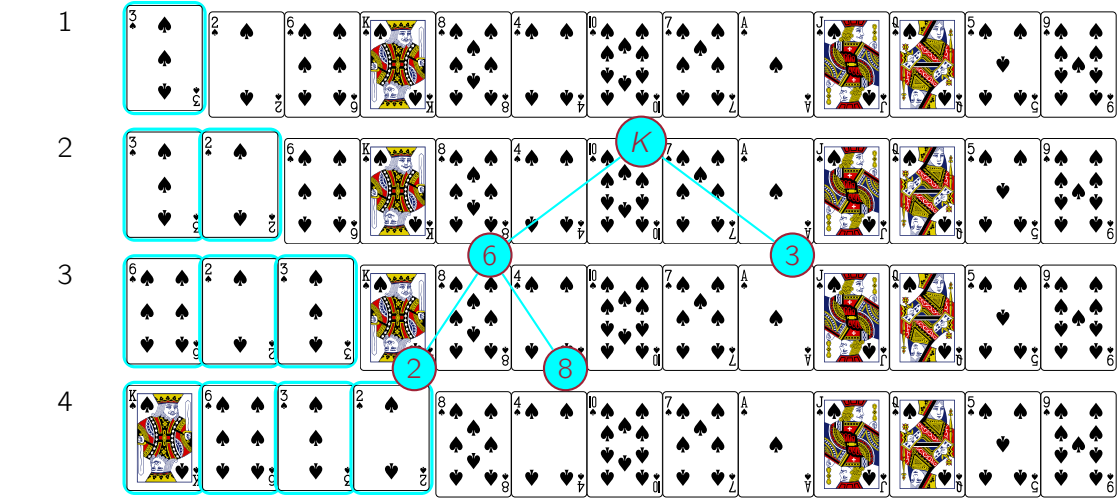
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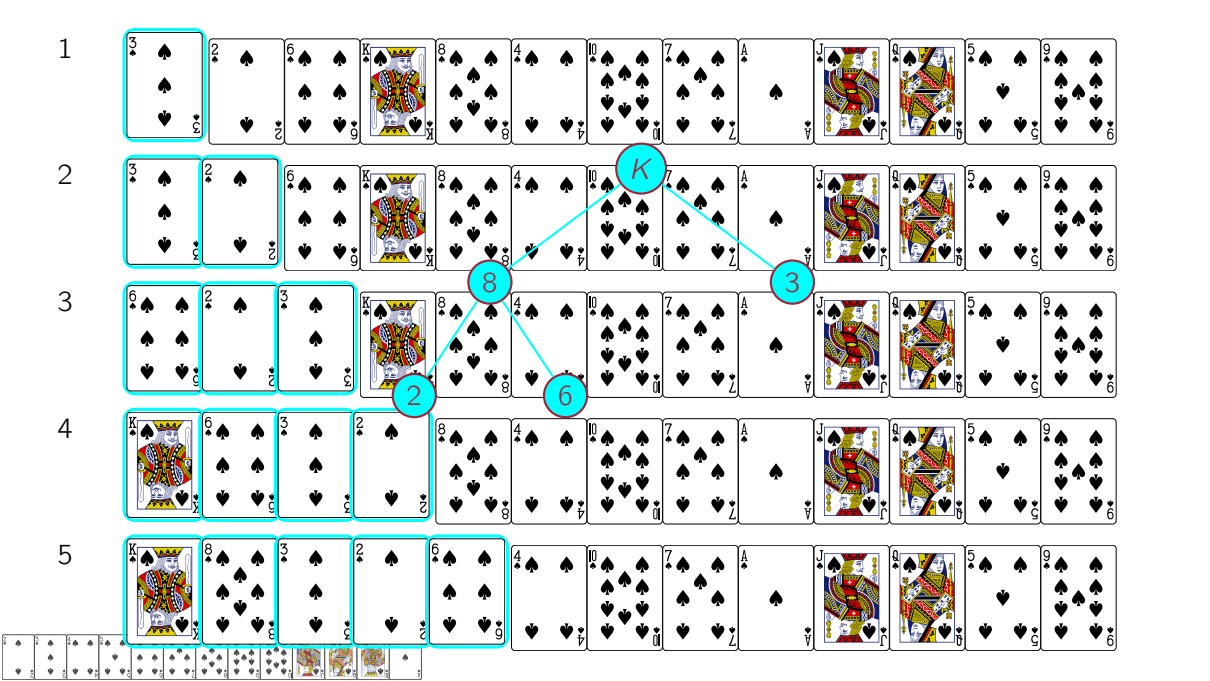
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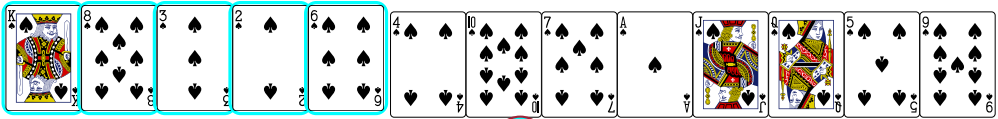

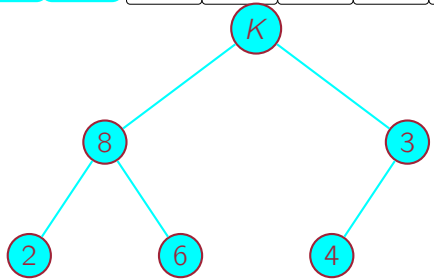

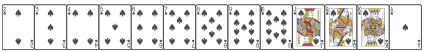

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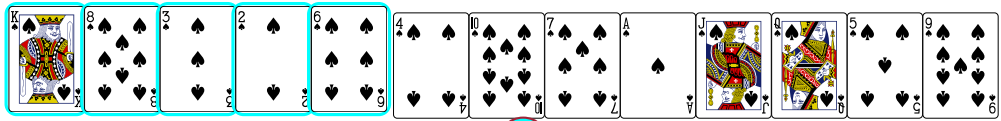

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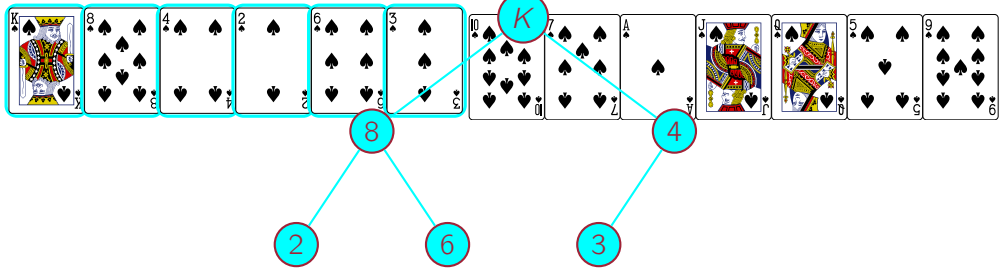

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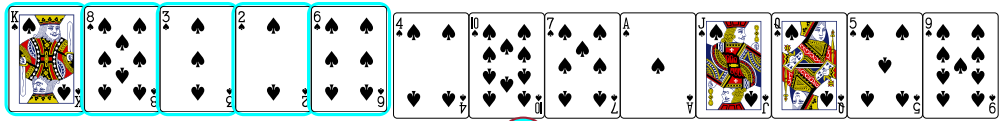

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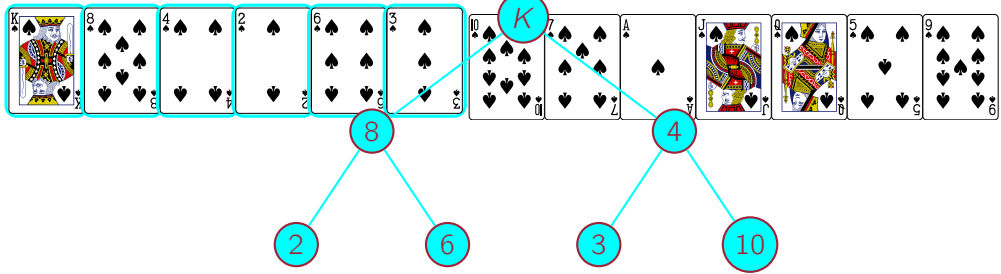

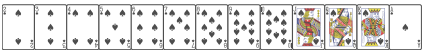

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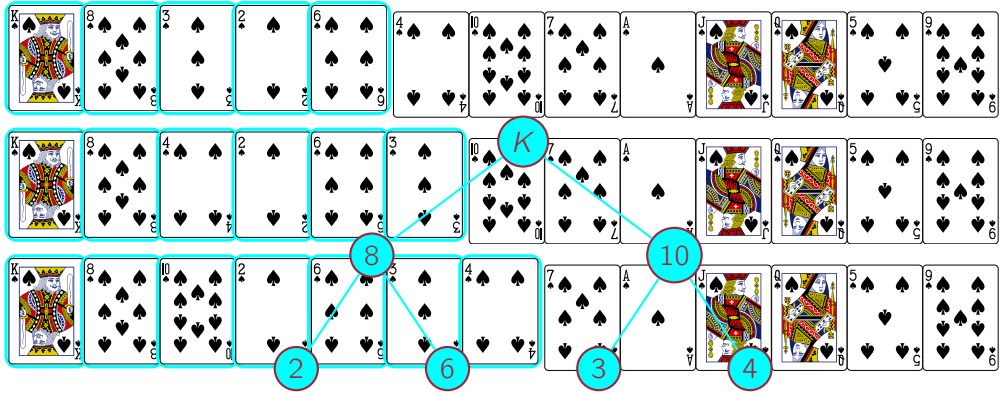

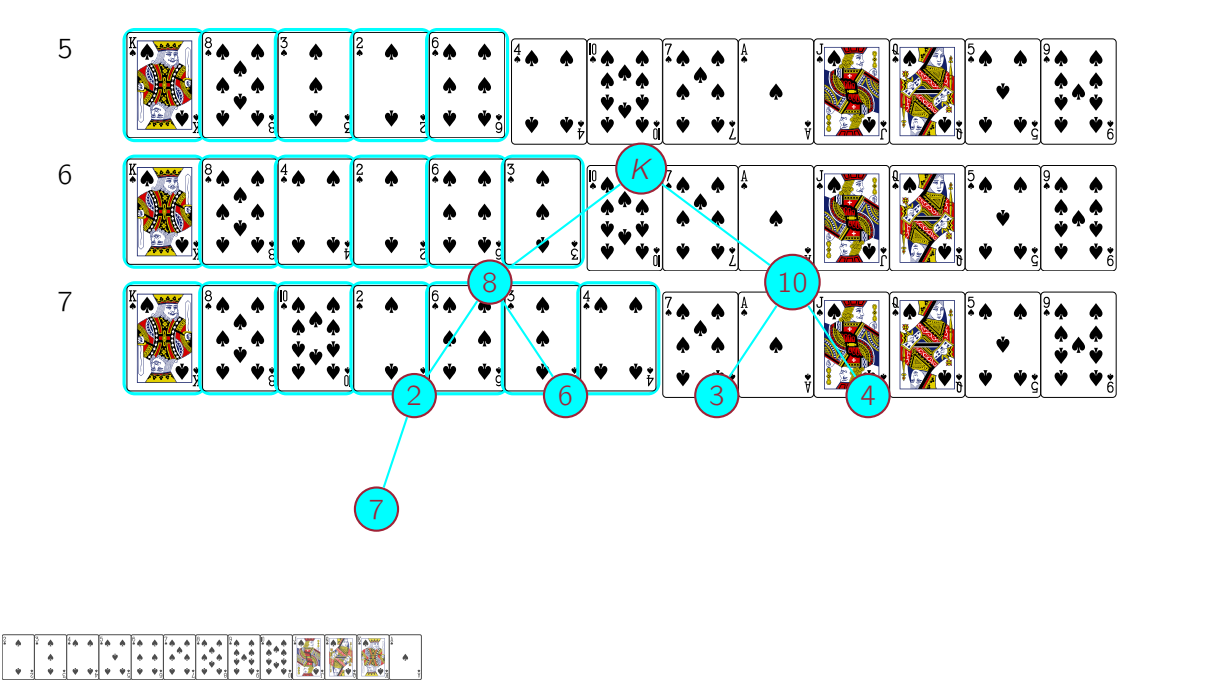
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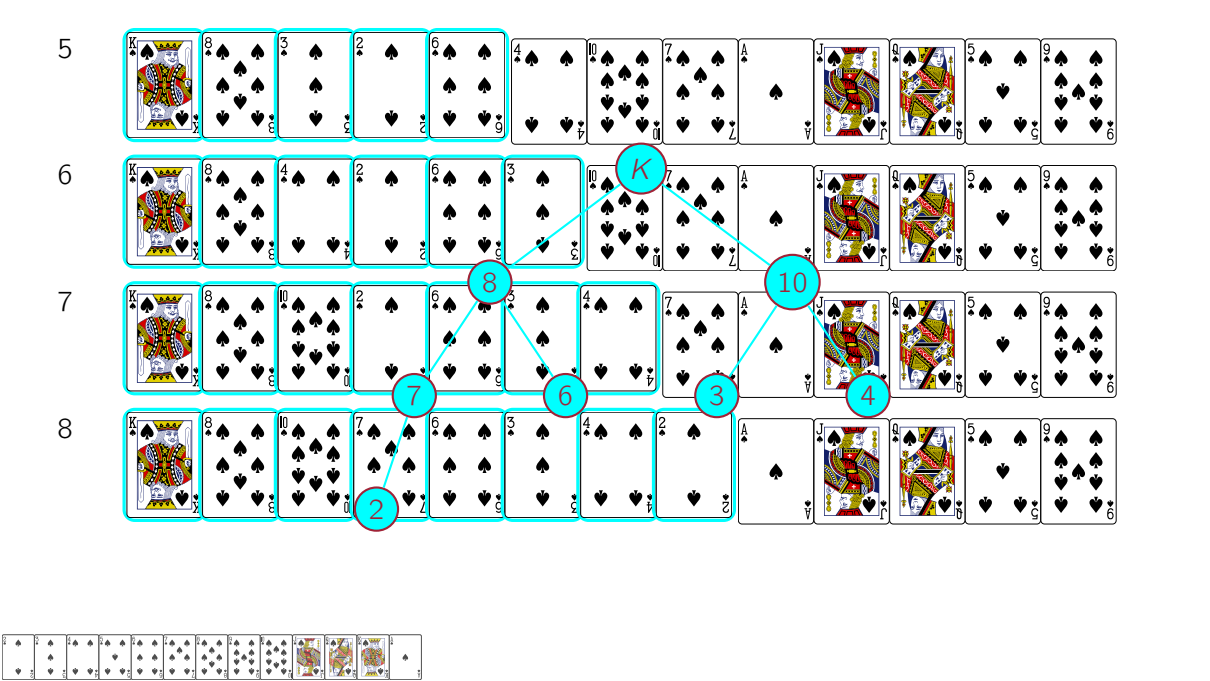
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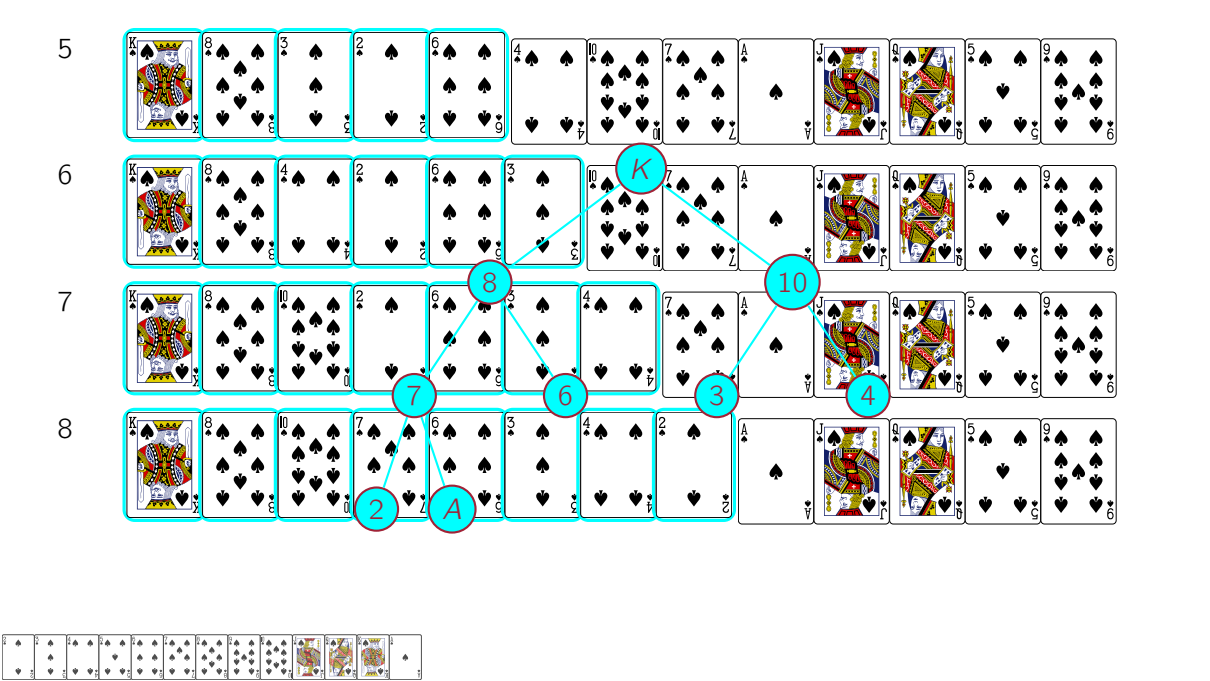
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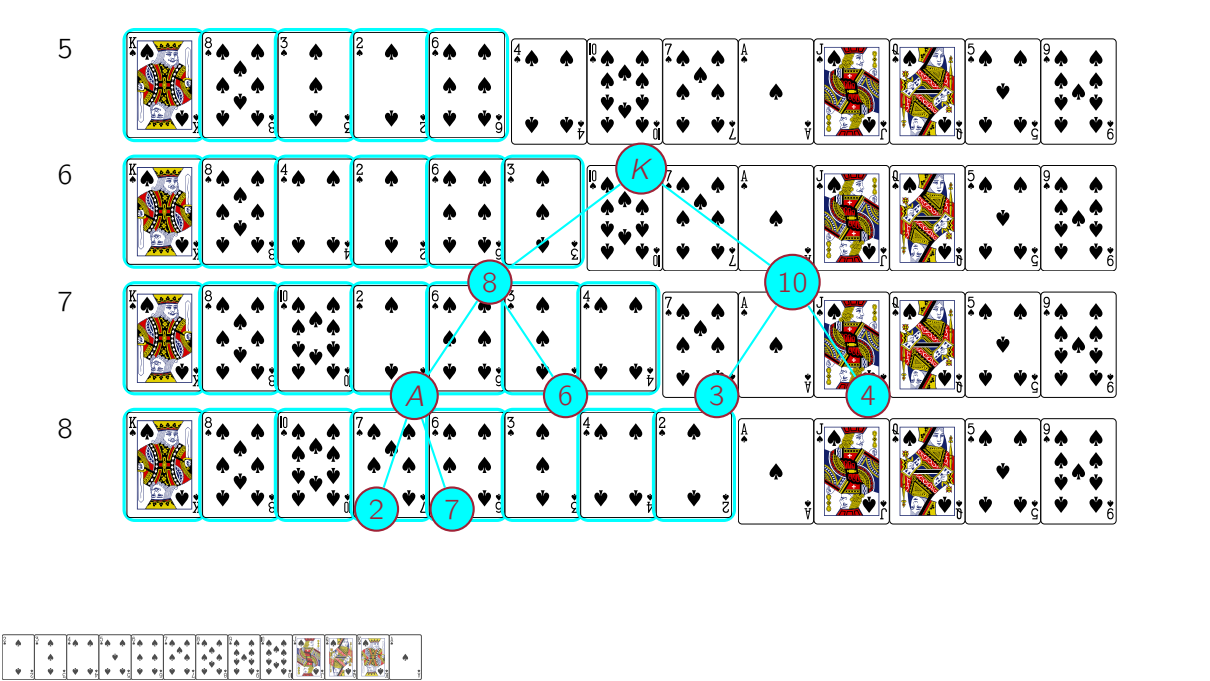
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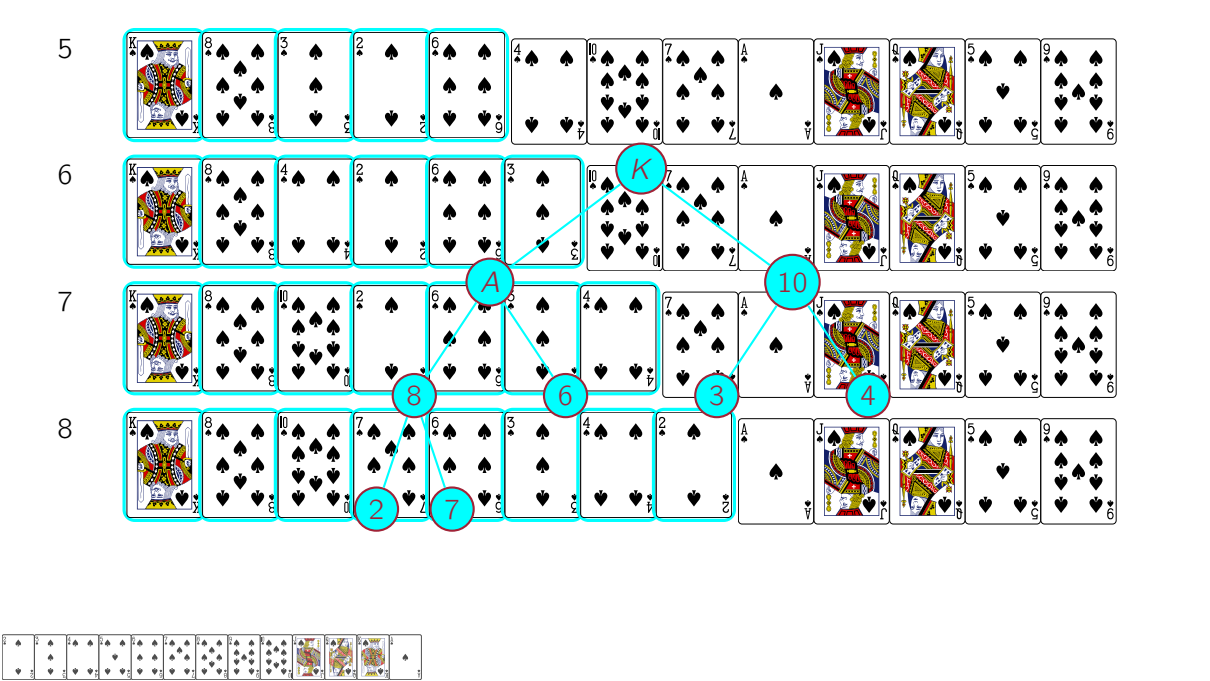
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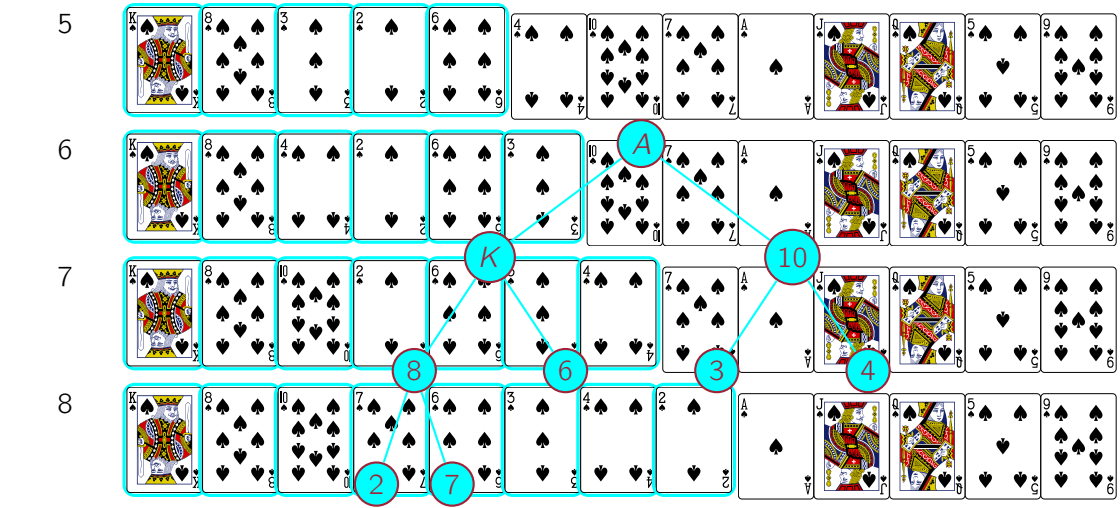
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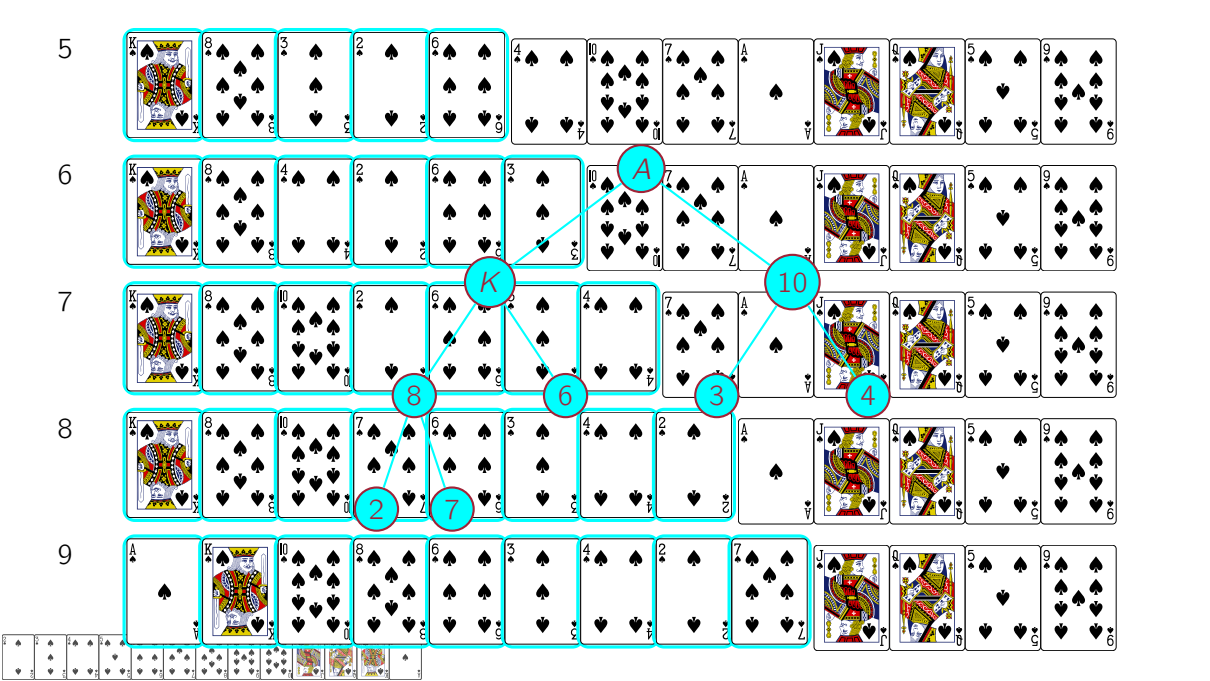
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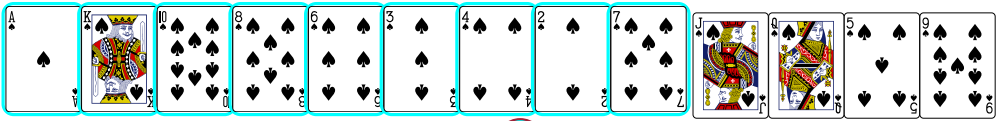

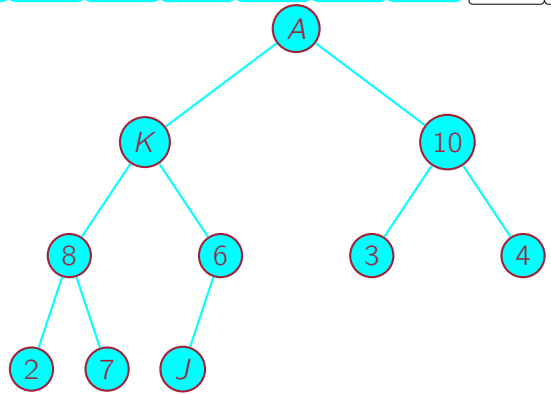

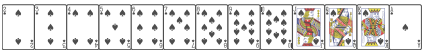

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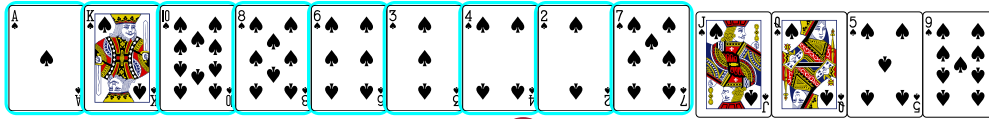

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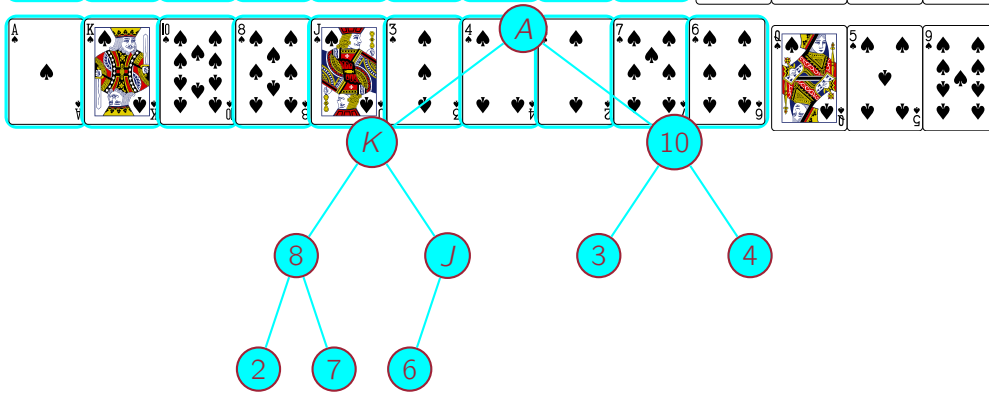

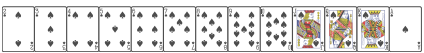

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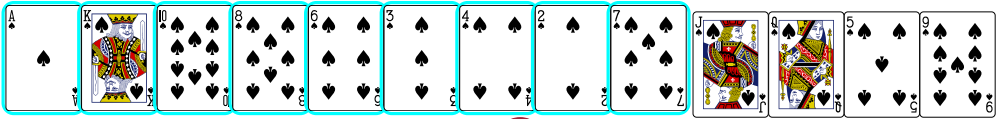

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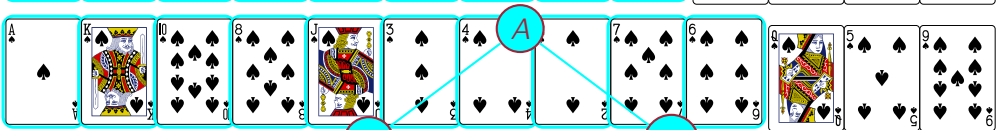

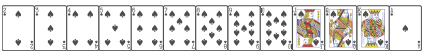

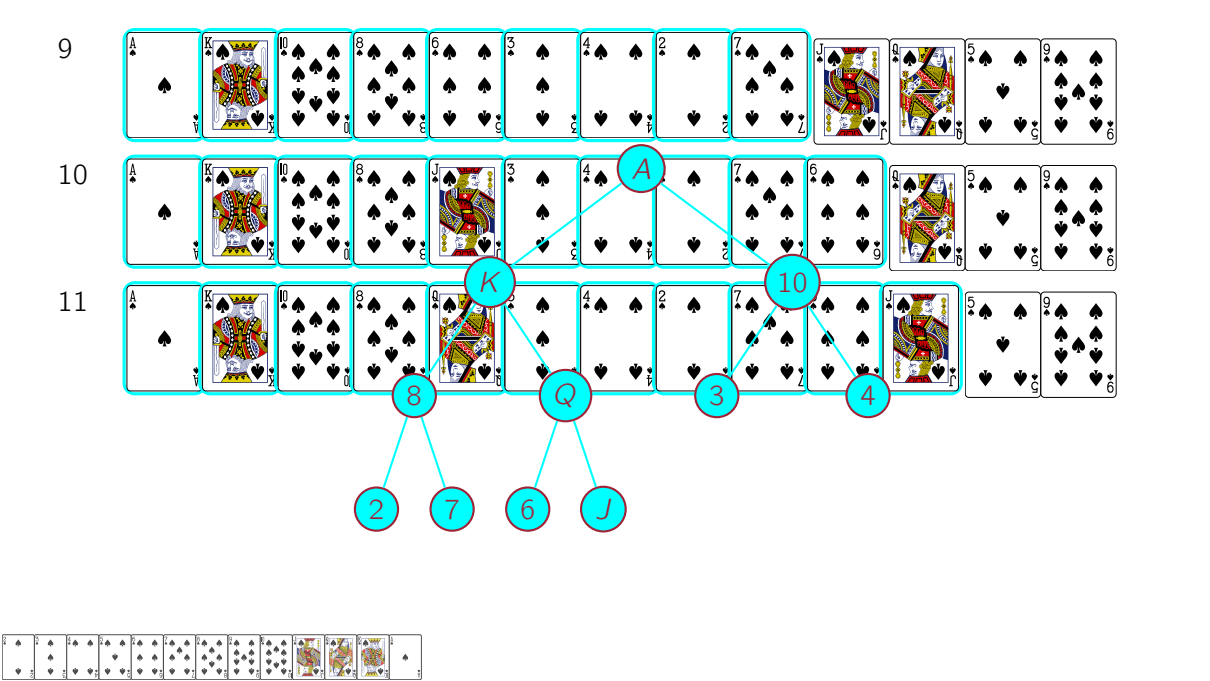
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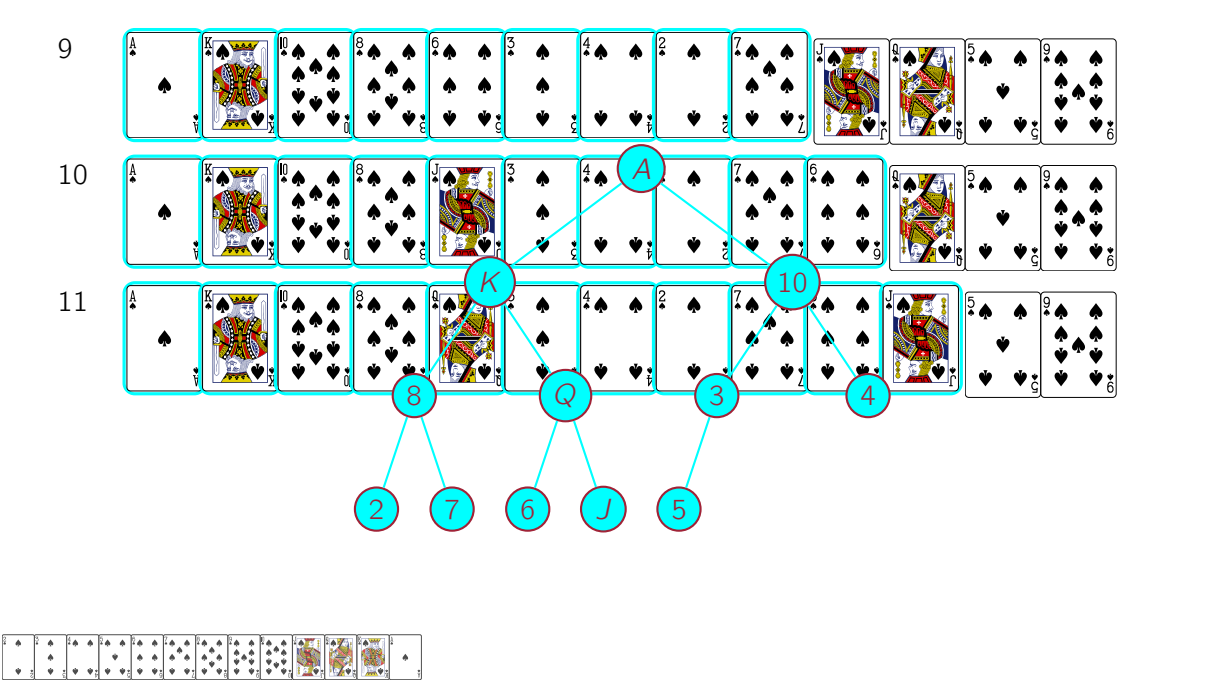
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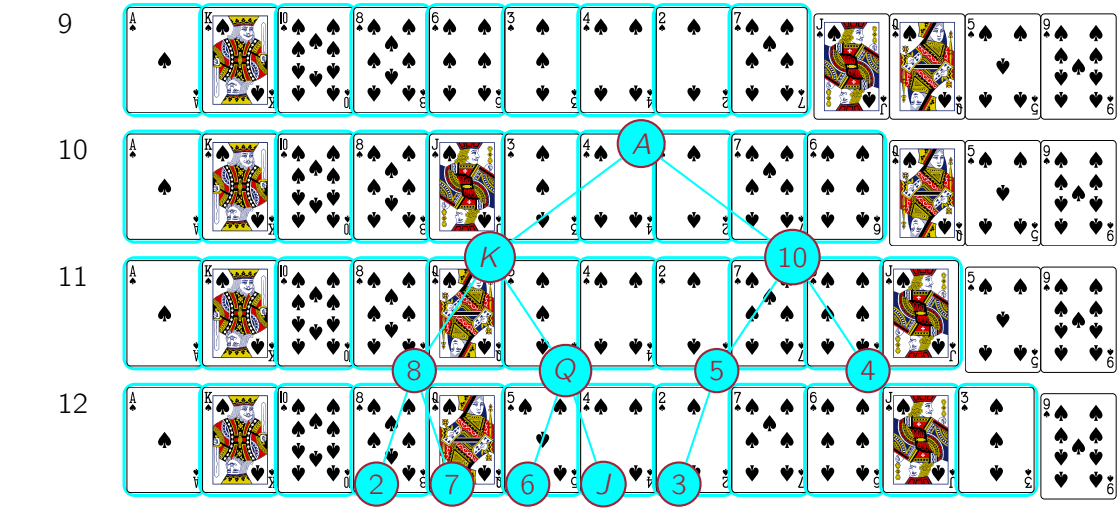
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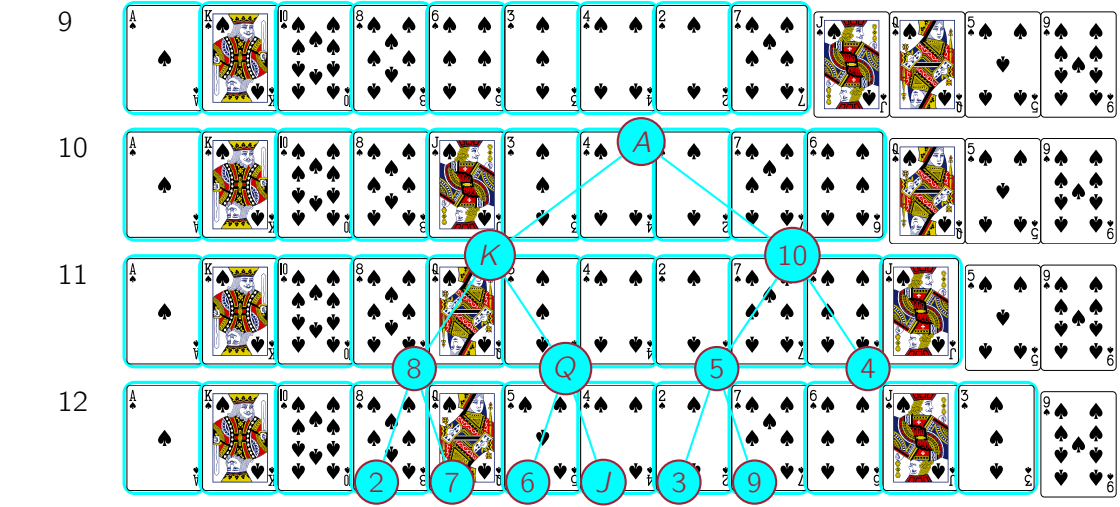
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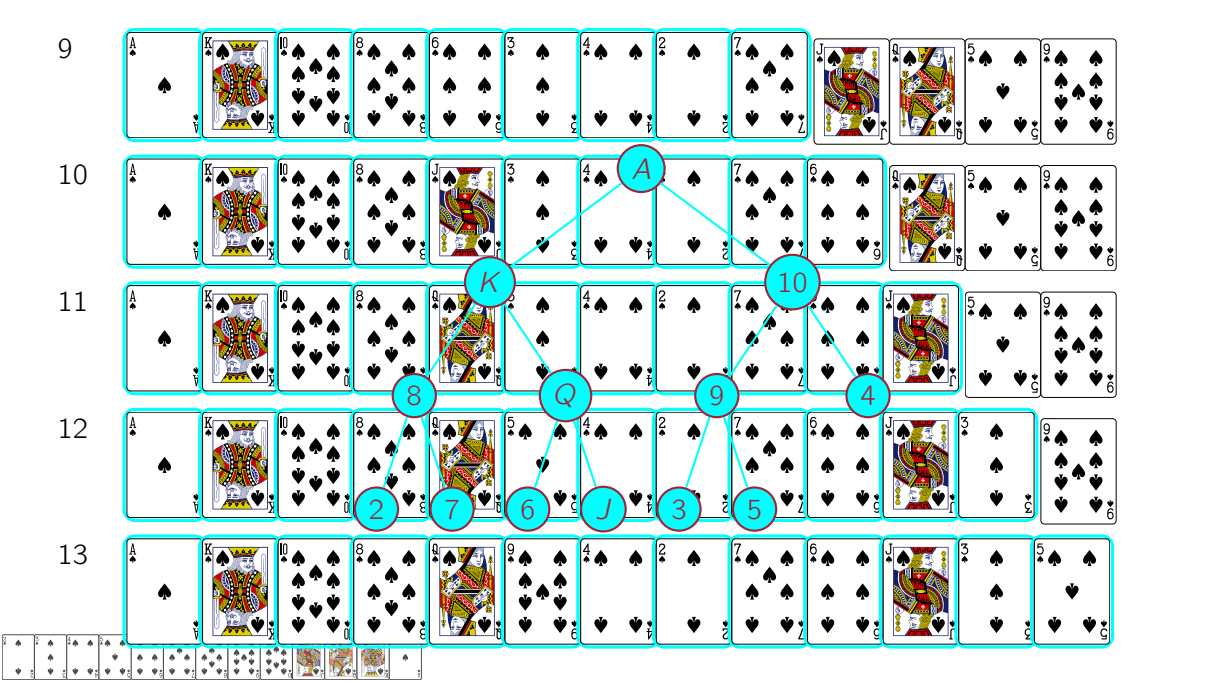
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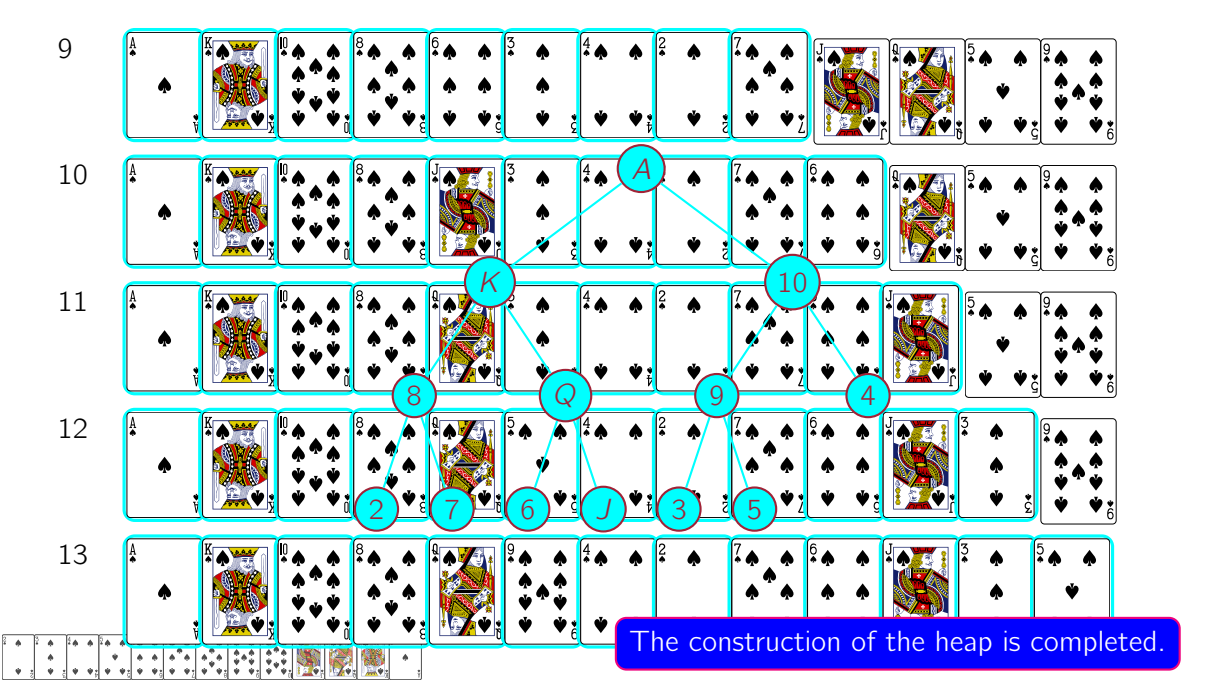
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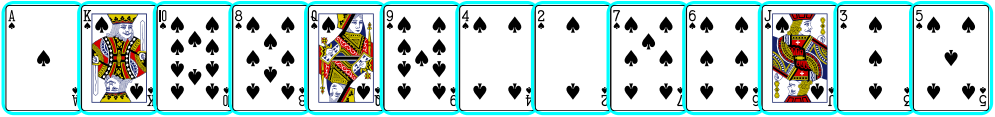
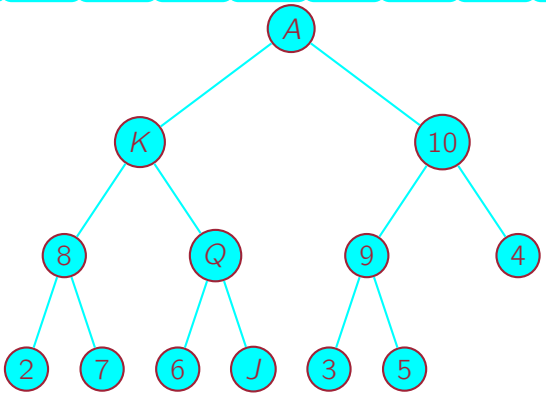
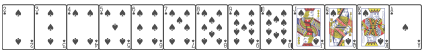
The construction of the heap is completed.







What happens if we delete the root (max)?

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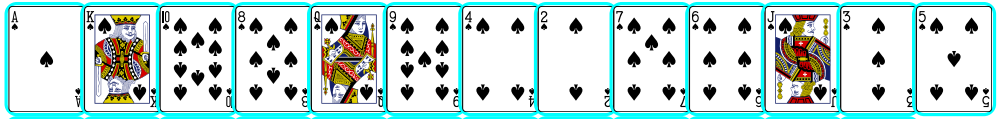

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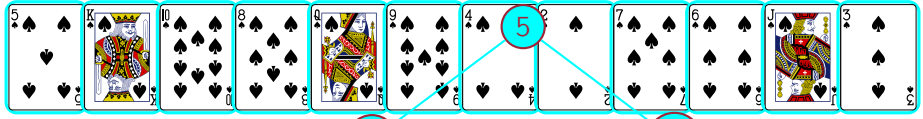
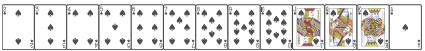
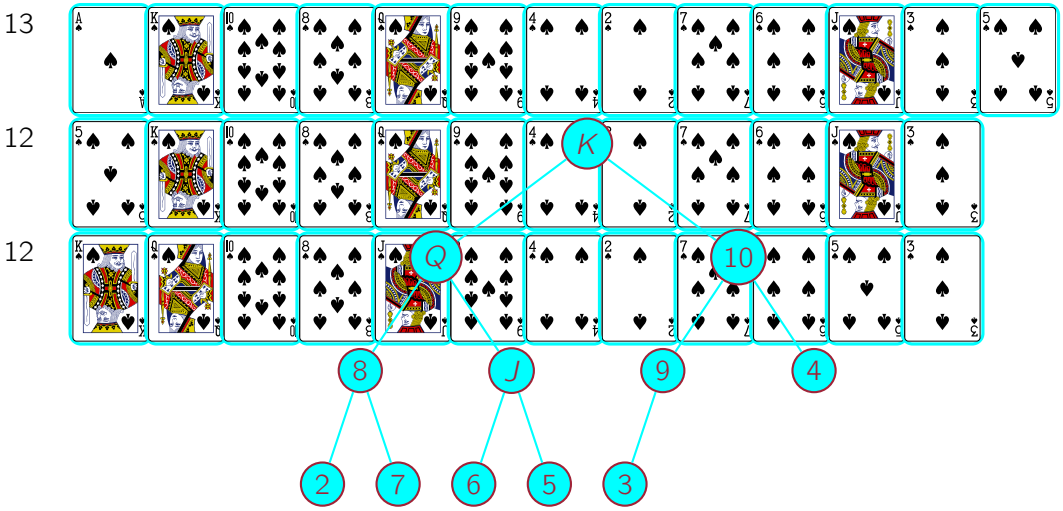
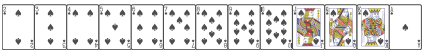


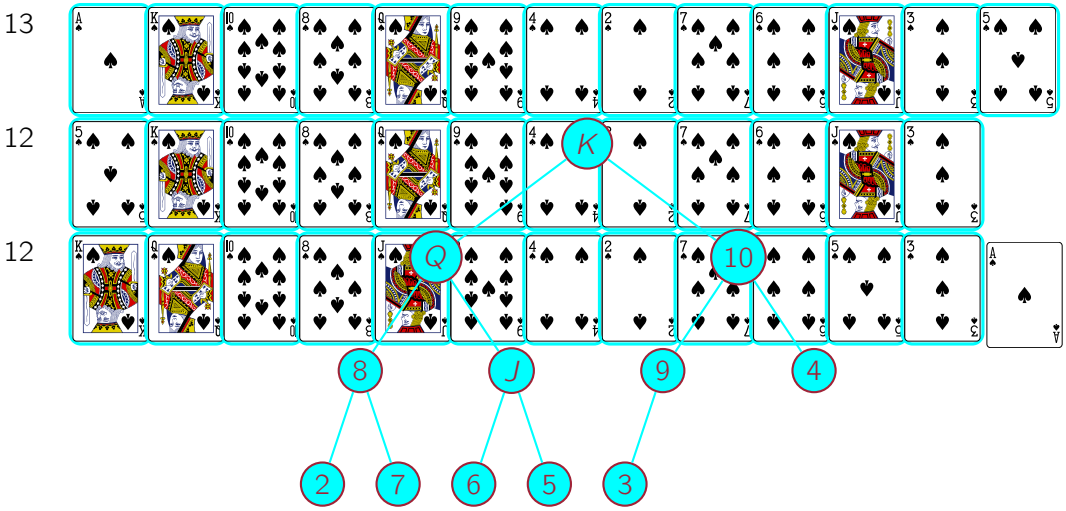


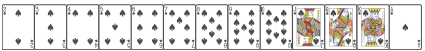
What happens if we delete the root (max)?



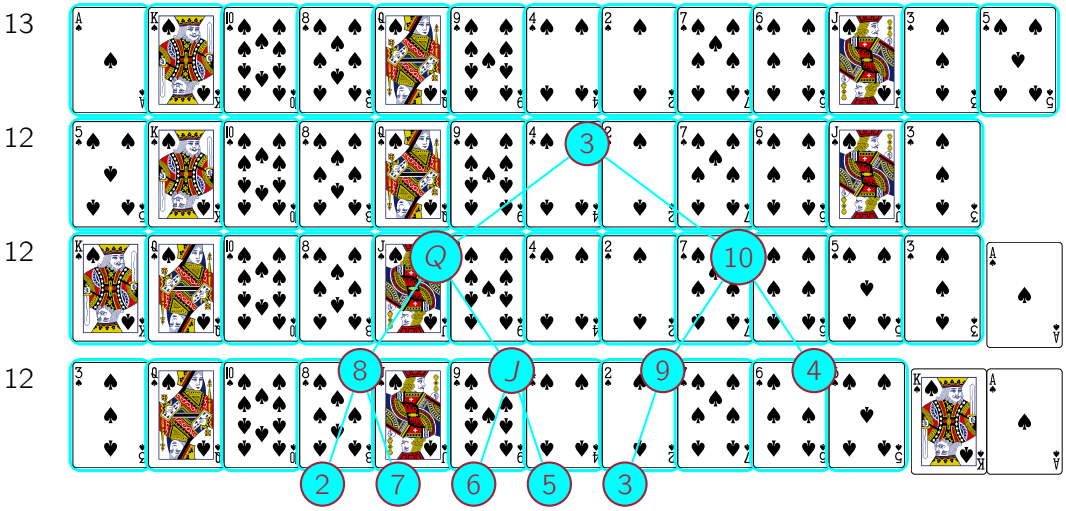


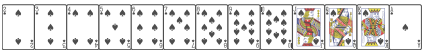
Where do we put A?





The last position is now empty.





In the next turn, we swap 3 and K, so it's in-place.

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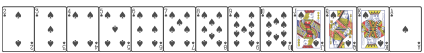

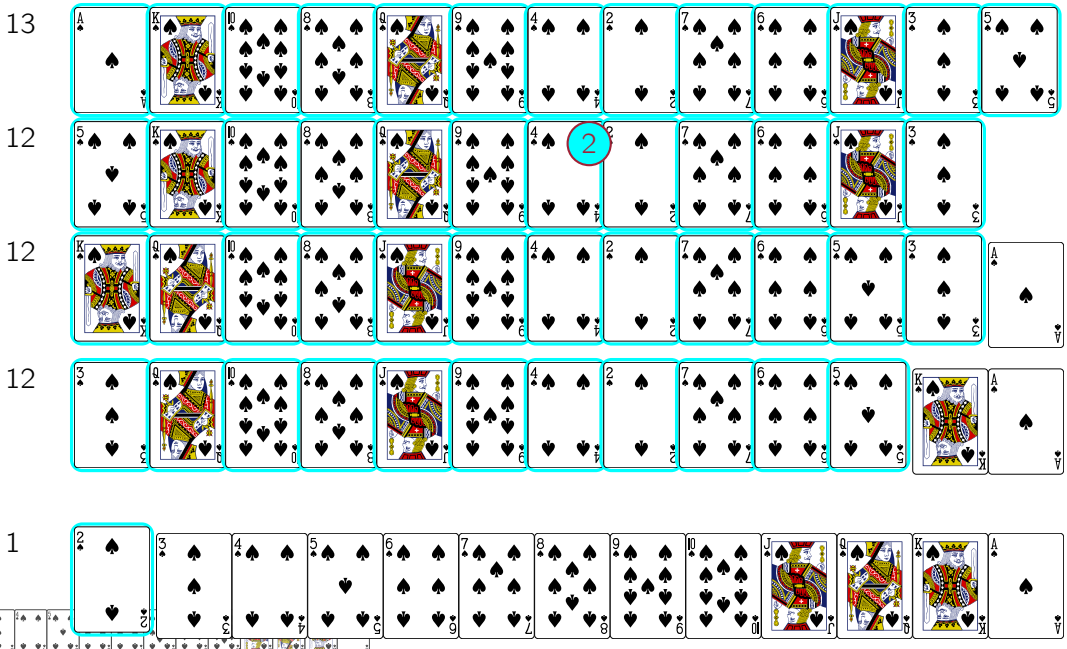
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The heapsort method

```
void void heapsort(int[] a) {
    for (int i = 1; i < a.length; i++)
        up(a, i);
    for (int i = a.length - 1; i > 0; i--) {
        swap(a, 0, i);
        down(a, 0, i);
    }
}
```

```
void up(int[] a, int c) {
    if (c == 0) return;
    int p = (c - 1) / 2;
    if (a[c] <= a[p])
        return;
    swap(a, c, p);
    up(a, p);
}
```

```
void down(int[] a, int p, int s) {
    if (p * 2 + 2 > s) return;
    int l = p * 2 + 1;
    if (l+1<s && a[l]<a[l+1]) l++;
    if (a[p] >= a[l]) return;
    swap(a, p, l);
    down(a, l, s);
}
```

Running time of heapsort

`heapify` (turn something into a heap):

$$\sum_{i=1}^{n-1} \lceil \log(i+1) \rceil < \sum_{i=1}^{n} (\log n + 1) = O(n \log n)$$

`removeMax`:

$$\sum_{i=1}^{n-1} \lceil \log(n+1-i) \rceil < \sum_{i=1}^{n} (\log n + 1) = O(n \log n)$$

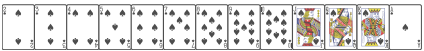
a quick and informal answer

a rigor and detailed analysis

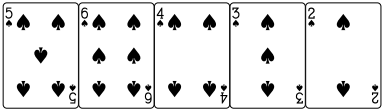
- `heapify` can be done in $O(n)$ time. nontrivial to show
- It's not hard to get an upper bound (O) of the complexity of an algorithm, but it's nontrivial to get a tight one (Θ).
- Analysis of an algorithm is harder than its design.

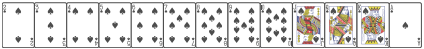
A trivial best-case instance: $[7, 7, 7, \ldots, 7]$.

How about a sorted array, a reversely sorted array (with distinct keys)?

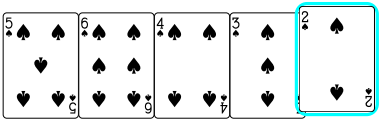


A closer look at selection sort





A closer look at selection sort



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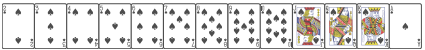
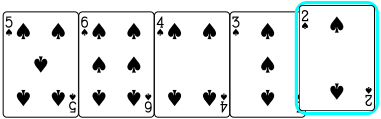
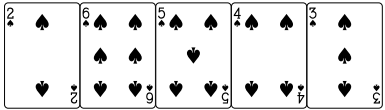
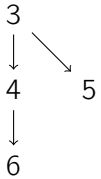
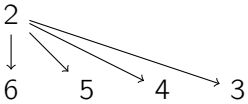


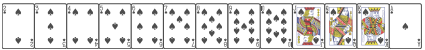
A closer look at selection sort





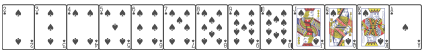






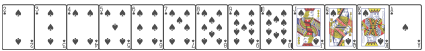
Online resources

- Demo of Sorting Algorithms
- Comparison of Sorting Algorithms
- A slow and interactive version (heapsort others)
- More information about sorting algorithms ()



Stability

- Is heapsort stable?
- If no, can you find an example?

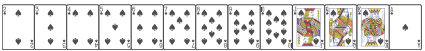


Summary of heapsort

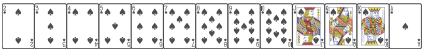
- A heap is usually implemented as an array representing a complete binary tree.
- The root is at index 0 and the last item at index $n - 1$.
- The parent-child relation can be easily decided by the index calculations.

- A heap can be based on a binary tree (not a search tree).
- The last occupied node and the first tree node can be found in $O(\log n)$ time.

- Conceptually, heapsort makes n insertions into a heap, followed by n removals.
- The same array can be used for the initial unordered data, for the heap array, and for the final sorted data.
- Heapsort takes $O(n \log n)$ time and $O(1)$ space (in-place).
- Heapsort can be made stable, then slower. All sorting algorithms can.

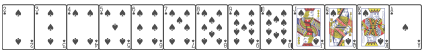


Applications



$[65, 85, 17, 88, 66, 71, 45, 38, 95, 48, 18, 68, 60, 96, 55]$.

The 5th smallest element is ____?

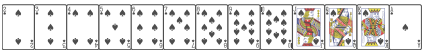


$[65, 85, 17, 88, 66, 71, 45, 38, 95, 48, 18, 68, 60, 96, 55]$.

The 5th smallest element is ____?

Naïve: Sort the array, and return `a[4]`.

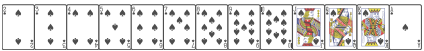
$17, 18, 38, 45, 48, 55, 60, 65, 66, 68, 71, 85, 88, 95, 96$



[65, 85, 17, 88, 66, 71, 45, 38, 95, 48, 18, 68, 60, 96, 55].

The 5th smallest element is ____?

2. Build a min heap on all of them, and do `removeMin` five times.

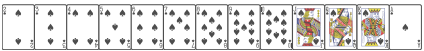


[65, 85, 17, 88, 66, 71, 45, 38, 95, 48, 18, 68, 60, 96, 55].

The 5th smallest element is ____?

3. Keep track of 5 smallest element.

[65, 85, 17, 88, 66]



[65, 85, 17, 88, 66, 71, 45, 38, 95, 48, 18, 68, 60, 96, 55].

The 5th smallest element is ____?

3. Keep track of 5 smallest element.

[65, 85, 17, 88, 66] →[65, 85, 17, 88, 66, 71]

71 replaces 88; there are five elements < 88, so 88 cannot be the answer.

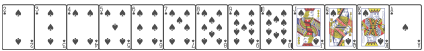
[65, 85, 17, 88, 66, 71, 45, 38, 95, 48, 18, 68, 60, 96, 55].

The 5th smallest element is ____?

3. Keep track of 5 smallest element.

[65, 85, 17, 88, 66] →[65, 85, 17, 88, 66, 71] →[65, 85, 17, 66, 71, 45]

45 replaces 85.

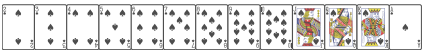


[65, 85, 17, 88, 66, 71, 45, 38, 95, 48, 18, 68, 60, 96, 55].

The 5th smallest element is ____?

3. Keep track of 5 smallest element.

[65, 85, 17, 88, 66] →[65, 85, 17, 88, 66, 71] →[65, 85, 17, 66, 71, 45]
→ ····· →[17, 45, 38, 48, 18, 85]



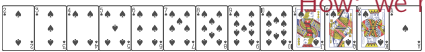
[65, 85, 17, 88, 66, 71, 45, 38, 95, 48, 18, 68, 60, 96, 55].

The 5th smallest element is ____?

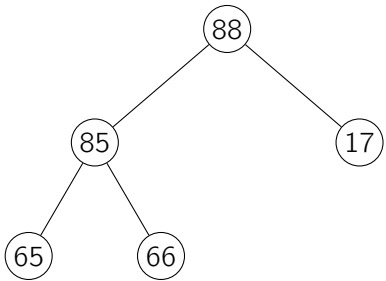
3. Keep track of 5 smallest element.

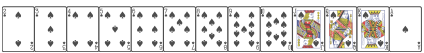
[65, 85, 17, 88, 66] →[65, 85, 17, 88, 66, 71] →[65, 85, 17, 66, 71, 45]
→ ····· →[17, 45, 38, 48, 18, 85]

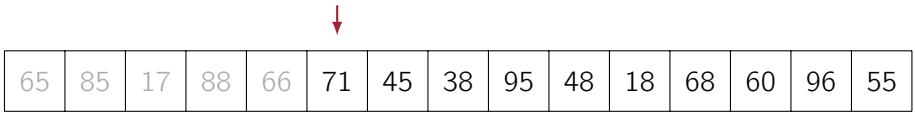
How: we need to `removeMax`, so a maximum heap.



65	85	17	88	66	71	45	38	95	48	18	68	60	96	55



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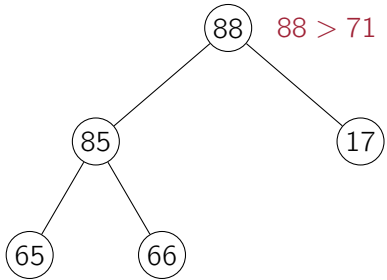
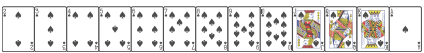
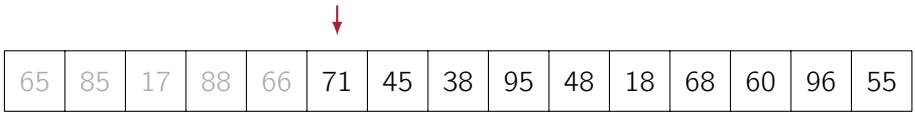

88
88 > 71
85
17
65
66

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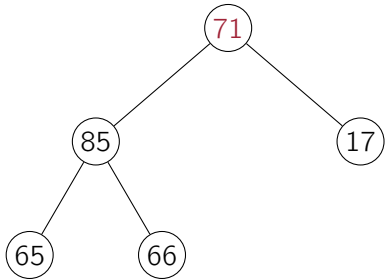
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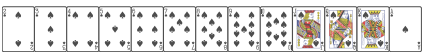

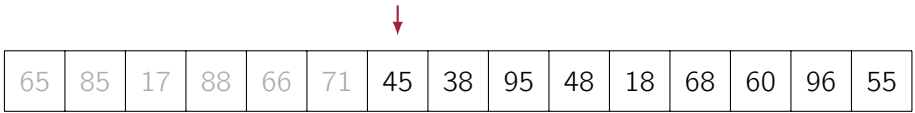
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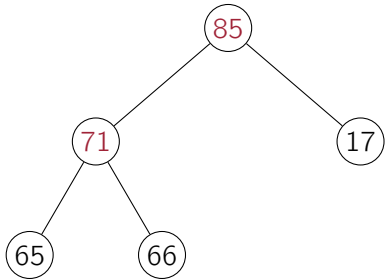
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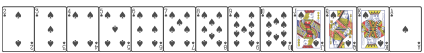
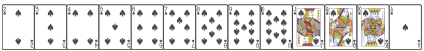


$O(n \log n)$
sort

$O(n + k \log n)$
a min heap (n)

$O(n \log k)$
a max heap (k)



$O(n \log n)$
sort

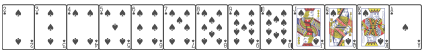
$O(n)$

$O(n + k \log n)$
a min heap (n)

$O(n)$

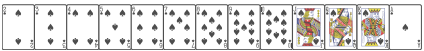
$O(n \log k)$
a max heap (k)

$O(k)$



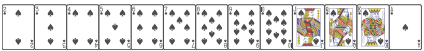
$[10.4m, 35.4b, 66.5k, ..., 120m, 30.5b, ..., 79.8m, 192.7m, 963.5k, ..., 22.1b].$

The wealth gap (HK) between the 1000 richest and the 1000 poorest (🫧)



$[10.4m, 35.4b, 66.5k, \ldots, 120m, 30.5b, \ldots, 79.8m, 192.7m, 963.5k, \ldots\ldots, 22.1b]$.

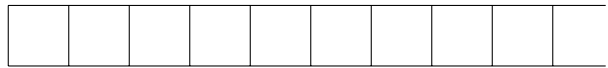
Similar to the previous, we can use a heap to keep track of the 1000 richest.
and another heap to keep track of the 1000 poorest.



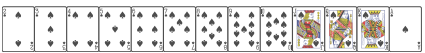
Merging multiple sorted arrays

8	10	14	89
32	38	50	77
16	32	64	128
11	12	13	14

$\Longrightarrow$



First round: 8, 32, 16, 11



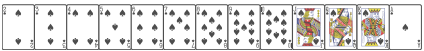
8	10	14	89
32	38	50	77
16	32	64	128
11	12	13	14

$\Longrightarrow$

8									

First round: 8, 32, 16, 11

Next round: 10, 32, 16, 11



Week 12: Hashing