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theory Warmup-Problem-A
imports
  Complex-Main
  HOL-Number-Theory.Cong
begin

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## 0.1 Warmup problem A

Solve the equation  $3^x = 4y + 5$  in the integers.

We begin with the following lemma:

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lemma even-power-3:  $[3^k = 1::int] \pmod 4 \longleftrightarrow \text{even } k$ 
proof -
  have  $[3^k = (-1::int)^k] \pmod 4$ 
    by (intro cong-pow) (auto simp: cong-def)
  thus ?thesis
    by (auto simp: cong-def minus-one-power-iff)
qed

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Here is an alternative proof — hopefully it will be instructive in doing calculations mod  $n$ .

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lemma  $[3^k = 1::int] \pmod 4 \longleftrightarrow \text{even } k$ 
proof (cases even k)
  case True
    then obtain l where  $2 * l = k$  by auto
    then have  $[3^k = (3^2)^l] \pmod 4$  (is cong - ... -)
      by (auto simp add: power-mult)
    also have  $[... = (1::int)^l] \pmod 4$  (is cong - ... -)
      by (intro cong-pow) (simp add: cong-def)
    also have  $[... = 1] \pmod 4$  by auto
    finally have  $[3^k = 1::int] \pmod 4$ .
    thus ?thesis using (even k) by blast
  next
    case False
    then obtain l where  $2 * l + 1 = k$ 
      using oddE by blast
    then have  $[3^k = 3^{(2 * l + 1)}] \pmod 4$  (is cong - ... -) by auto
    also have  $[... = (3^2)^l * 3] \pmod 4$  (is cong - ... -)
      by (metis power-mult power-add power-one-right cong-def)
    also have  $[... = (1::int)^l * 3] \pmod 4$  (is cong - ... -)
      by (intro cong-mult cong-pow) (auto simp add: cong-def)
    also have  $[... = 3] \pmod 4$  by auto
    finally have  $[3^k \neq 1::int] \pmod 4$  by (auto simp add: cong-def)
    then show ?thesis using (odd k) by blast
qed

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This allows us to prove the theorem, provided we assume  $x$  is a natural number.

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theorem warmupA-natx:
  fixes x :: nat and y :: int
  shows  $3^x = 4 * y + 5 \longleftrightarrow \text{even } x \wedge y = (3^x - 5) \text{ div } 4$ 
proof -
  have  $\text{even } x \wedge y = (3^x - 5) \text{ div } 4$  if  $3^x = 4 * y + 5$ 
  proof -
    from that have  $[3^x = 4 * y + 5] \pmod 4$  by auto
    also have  $[4 * y + 5 = 5] \pmod 4$ 
      by (metis cong-mult-self-left cong-add-rcancel-0)
    also have  $[5 = 1::int] \pmod 4$  by (auto simp add: cong-def)
    finally have  $[(3::int)^x = 1] \pmod 4$ .
    hence even x using even-power-3 by auto
    thus ?thesis using that by auto
  qed
  moreover have  $3^x = 4 * y + 5$  if  $\text{even } x \wedge y = (3^x - 5) \text{ div } 4$ 

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proof –
  from that have even x and y-form:  $y = (3^x - 5) \text{ div } 4$  by auto
  then have  $[3^x = 1::\text{int}] \text{ (mod } 4)$  using even-power-3 by blast
  then have  $((3::\text{int})^x - 5) \text{ mod } 4 = 0$ 
    by (simp add: cong-def mod-diff-cong)
  thus ?thesis using y-form by auto
qed
ultimately show ?thesis by blast
qed

To consider negative values of  $x$ , we'll need to venture into the reals:

lemma powr-int-pos:
  fixes  $x\ y :: \text{int}$ 
  assumes  $3^{\text{powr } x} = y$ 
  shows  $x \geq 0$ 
proof (rule ccontr)
  assume neg-x:  $\neg x \geq 0$ 
  then have y-inv:  $y = \text{inverse } ((3::\text{nat})^{\text{nat } (-x)})$  (is  $y = \text{inverse } (?n::\text{nat})$ )
    using powr-real-of-int and  $*$  by auto
  hence  $\text{real } ?n * \text{of-int } y = 1$  by auto
  hence  $?n * y = 1$  using of-int-eq-iff by fastforce
  hence  $?n = 1$ 
    by (metis nat-1-eq-mult-iff nat-int nat-numeral-as-int numeral-One of-nat-mult zmult-eq-1-iff)

  hence  $\text{nat } (-x) = 0$  by auto
  thus False using neg-x by auto
qed

corollary warmupA:
  fixes  $x\ y :: \text{int}$ 
  shows  $3^{\text{powr } x} = 4*y + 5 \iff x \geq 0 \wedge \text{even } x \wedge y = (3^{(\text{nat } x)} - 5) \text{ div } 4$ 
proof
  assume assm:  $3^{\text{powr } x} = 4*y + 5$ 
  then have  $x \geq 0$  using powr-int-pos by fastforce
  hence  $3^{\text{powr } (\text{nat } x)} = 4*y + 5$  using assm by simp
  hence  $(3::\text{real})^{(\text{nat } x)} = 4*y + 5$  using powr-realpow by auto
  hence with-nat:  $3^{(\text{nat } x)} = 4*y + 5$  using of-int-eq-iff by fastforce
  hence  $\text{even } (\text{nat } x) \wedge y = (3^{(\text{nat } x)} - 5) \text{ div } 4$  using warmupA-natx by auto
  thus  $x \geq 0 \wedge \text{even } x \wedge y = (3^{(\text{nat } x)} - 5) \text{ div } 4$  using  $\langle x \geq 0 \rangle$  and even-nat-iff by auto
next
  assume assm:  $x \geq 0 \wedge \text{even } x \wedge y = (3^{(\text{nat } x)} - 5) \text{ div } 4$ 
  then have  $3^{(\text{nat } x)} = 4*y + 5$  using warmupA-natx and even-nat-iff by blast
  thus  $3^{\text{powr } x} = 4*y + 5$  using assm powr-real-of-int by fastforce
qed

end

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