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theory Problem-1
imports Complex-Main
begin

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Consider a set S of $n \geq 3$ positive real numbers. Show that at most $n - 2$ distinct integer powers of 3 can be expressed as a sum of three distinct elements of S .

definition *intpow3* **where**

intpow3 $x \longleftrightarrow (\exists k::int. 3 \text{ powr } k = x)$

lemma *card3-distinct[elim]*:

card $\{a, b, c\} = 3 \implies a \neq b \wedge b \neq c \wedge c \neq a$

by (*metis One-nat-def add-Suc-right add-cancel-left-right card.empty card.insert card-2-iff equalityI finite.intros(1) insert-subset nat.simps(3) numeral-eq-Suc one-add-one order-refl pred-numeral-simps(3)*)**+**

definition *threepows* **where**

threepows $S = \{a + b + c \mid a \ b \ c. \{a, b, c\} \subseteq S \wedge \text{card } \{a, b, c\} = 3 \wedge \text{intpow3 } (a + b + c)\}$

lemma *threepows-mono*:

$A \subseteq B \implies \text{threepows } A \subseteq \text{threepows } B$

by (*auto simp: threepows-def*)

lemma *threepows-intpow3*:

$x \in \text{threepows } S \implies \text{intpow3 } x$

by (*auto simp: threepows-def*)

lemma *threepowsI[intro]*:

assumes $\{a, b, c\} \subseteq S$ **and** *card* $\{a, b, c\} = 3$ **and** *intpow3* $(a + b + c)$

shows $a + b + c \in \text{threepows } S$

using *assms* **by** (*auto simp: threepows-def*)

lemma *threepowsE[elim]*:

assumes $x \in \text{threepows } S$

assumes $\bigwedge a \ b \ c. x = a + b + c \implies \{a, b, c\} \subseteq S \implies \text{card } \{a, b, c\} = 3 \implies \text{intpow3 } (a + b + c) \implies P \ x$

shows $P \ x$

using *assms* **by** (*auto simp: threepows-def*)

lemma *threepows-trivial*:

assumes *card* $S < 3$ **and** *finite* S

shows *threepows* $S = \{\}$

proof (*rule ccontr*)

assume *threepows* $S \neq \{\}$

then obtain x **where** $x \in \text{threepows } S$ **by** *auto*

thus *False*

proof (*rule threepowsE*)

fix $a \ b \ c$

assume $\{a, b, c\} \subseteq S$ *card* $\{a, b, c\} = 3$

have *card* $\{a, b, c\} \leq \text{card } S$

using $\langle \text{finite } S \rangle \langle \{a, b, c\} \subseteq S \rangle$ **by** (*intro card-mono*)

thus *False*

using $\langle \text{card } \{a, b, c\} = 3 \rangle \langle \text{card } S < 3 \rangle$ **by** *simp*

qed

qed

lemma *threepows-largest*:

fixes $k::int$

assumes $3 \text{ powr } k \in \text{threepows } S$

obtains a **where** $a \in S$ **and** $a \geq 3 \text{ powr } (k - 1)$

proof –

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assume $\nexists a. a \in S \wedge a \geq 3 \text{ powr } (k - 1)$

then have $*: a \in S \implies a < 3 \text{ powr } (k - 1)$ **for** a **by** *auto*

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have  $x \in \text{threepows } S \implies x < 3 * 3^{\text{powr } (k - 1)}$  for  $x$ 
proof (erule threepowsE)
  fix  $a \ b \ c$ 
  assume  $\{a, b, c\} \subseteq S$  and  $x = a + b + c$ 
  with * have  $a < 3^{\text{powr } (k - 1)}$  and  $b < 3^{\text{powr } (k - 1)}$  and  $c < 3^{\text{powr } (k - 1)}$ 
  by auto
  thus  $x < 3 * 3^{\text{powr } (k - 1)}$ 
  unfolding  $\langle x = a + b + c \rangle$  by simp
qed

with assms have  $3^{\text{powr } k} < 3 * 3^{\text{powr } (k - 1)}$ 
  by simp
hence False by (simp add: powr-mult-base)
}
thus thesis using that by blast
qed

theorem problem1:
  fixes  $S :: \text{real set}$ 
  assumes  $\text{card } S \geq 3$ 
  assumes  $\bigwedge x. x \in S \implies x > 0$ 
  shows  $\text{card } (\text{threepows } S) \leq \text{card } S - 2$ 
  using assms
proof (induction  $\text{card } S$  arbitrary:  $S$  rule: less-induct)
  case less
  then have finite  $S$ 
  using card.infinite by fastforce
  show ?case
  proof (cases  $\text{card } (\text{threepows } S) = 0$ )
    case False
    then have finite  $(\text{threepows } S)$  and  $\text{threepows } S \neq \{\}$ 
    by (auto intro: card-ge-0-finite)
    then have  $\text{Max } (\text{threepows } S) \in \text{threepows } S$ 
    by (intro Max-in)
    with threepows-intpow3 intpow3-def obtain  $k::\text{int}$  where  $k: 3^{\text{powr } k} = \text{Max } (\text{threepows } S)$ 
    by blast
    let ?discard =  $\{x \in S. x \geq 3^{\text{powr } (k - 1)}\}$ 
    have  $\text{threepows } S - \{\text{Max } (\text{threepows } S)\} = \text{threepows } (S - ?discard)$ 
    proof -
      have  $\text{Max } (\text{threepows } S) \notin \text{threepows } (S - ?discard)$ 
      proof
        assume  $\text{Max } (\text{threepows } S) \in \text{threepows } (S - ?discard)$ 
        then obtain  $a$  where  $a \in S - ?discard$  and  $a \geq 3^{\text{powr } (k - 1)}$ 
        unfolding  $k[\text{symmetric}]$  by (rule threepows-largest)
        thus False by simp
      qed
    moreover have  $\text{threepows } (S - ?discard) \subseteq \text{threepows } S$ 
    by (intro threepows-mono; auto)
    moreover have  $x = \text{Max } (\text{threepows } S)$ 
    if  $x \in \text{threepows } S$  and not-in-discard:  $x \notin \text{threepows } (S - ?discard)$  for  $x$ 
    using  $\langle x \in \text{threepows } S \rangle$ 
    proof (rule threepowsE)
      fix  $a \ b \ c$ 
      assume  $x = a + b + c$   $\{a, b, c\} \subseteq S$   $\text{card } \{a, b, c\} = 3$   $\text{intpow3 } (a + b + c)$ 
      have  $\{a, b, c\} \cap ?discard \neq \{\}$ 
      proof
        assume  $\{a, b, c\} \cap ?discard = \{\}$ 
        with  $\langle \{a, b, c\} \subseteq S \rangle$  have  $\{a, b, c\} \subseteq S - ?discard$  by simp
        hence  $a + b + c \in \text{threepows } (S - ?discard)$ 
        using  $\langle \text{card } \{a, b, c\} = 3 \rangle$   $\langle \text{intpow3 } (a + b + c) \rangle$  by (intro threepowsI)
        with  $\langle x = a + b + c \rangle$  and not-in-discard show False by simp
      qed
    then have  $a \geq 3^{\text{powr } (k - 1)} \vee b \geq 3^{\text{powr } (k - 1)} \vee c \geq 3^{\text{powr } (k - 1)}$ 

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    by auto
  moreover from less.prem1 and  $\langle \{a, b, c\} \subseteq S \rangle$  have  $a > 0 \wedge b > 0 \wedge c > 0$ 
    by simp
  ultimately have  $a + b + c > 3 \text{ powr } (k - 1)$ 
    by linarith
  have  $a + b + c \geq 3 \text{ powr } k$ 
  proof (intro leI notI)
    assume **:  $a + b + c < 3 \text{ powr } k$ 
    moreover from  $\langle \text{intpow3 } (a + b + c) \rangle$  obtain  $k'::\text{int}$  where  $a + b + c = 3 \text{ powr } k'$ 
      unfolding intpow3-def by metis
    with * and ** have  $3 \text{ powr } (k - 1) < 3 \text{ powr } k'$  and  $3 \text{ powr } k' < 3 \text{ powr } k$ 
      by auto
    hence  $k - 1 < k'$  and  $k' < k$ 
      by auto
    thus False by auto
  qed
  moreover from  $\langle x \in \text{threepows } S \rangle$  have  $x \leq \text{Max } (\text{threepows } S)$ 
    using  $\langle \text{finite } (\text{threepows } S) \rangle$  by (intro Max.coboundedI; auto)
  ultimately show  $x = \text{Max } (\text{threepows } S)$ 
    using  $\langle x = a + b + c \rangle$  and  $k$  by simp
  qed
  ultimately show ?thesis
    by auto
  qed
  hence card-threepows:  $\text{card } (\text{threepows } S) = \text{Suc } (\text{card } (\text{threepows } (S - ?discard)))$ 
    by (metis  $\langle \text{Max } (\text{threepows } S) \in \text{threepows } S \rangle$   $\langle \text{finite } (\text{threepows } S) \rangle$  card-Suc-Diff1)

  have ?discard  $\neq \{\}$ 
  proof -
    from  $k \langle \text{Max } (\text{threepows } S) \in \text{threepows } S \rangle$  threepows-largest
    obtain  $a$  where  $a \in S$  and  $a \geq 3 \text{ powr } (k - 1)$ 
      by metis
    hence  $a \in ?discard$  by simp
    thus ?discard  $\neq \{\}$  by auto
  qed
  hence discard-strict:  $\text{card } (S - ?discard) < \text{card } S$ 
    using  $\langle \text{finite } S \rangle$  by (intro psubset-card-mono; auto)
  show ?thesis
  proof (cases  $\text{card } (S - ?discard) < 3$ )
    case True
    then have threepows  $(S - ?discard) = \{\}$ 
      using  $\langle \text{finite } S \rangle$  by (intro threepows-trivial; auto)
    hence  $\text{card } (\text{threepows } S) = 1$ 
      by (simp add: card-threepows)
    with  $\langle \text{card } S \geq 3 \rangle$  show ?thesis by simp
  next
    case False
    hence  $\text{card } (\text{threepows } (S - ?discard)) \leq \text{card } (S - ?discard) - 2$ 
      using discard-strict  $\langle \text{finite } S \rangle$  by (intro less; auto)
    hence  $\text{card } (\text{threepows } S) \leq \text{Suc } (\text{card } (S - ?discard) - 2)$ 
      unfolding card-threepows by simp
    then show ?thesis
      using discard-strict False by simp
  qed
  qed simp
  qed
end

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