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theory Problem-1
  imports Complex-Main
begin

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Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$xf(x) + f(-x) = 1.$$

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theorem
  fixes  $f :: real \Rightarrow real$ 
  shows  $(\forall x. x * f\ x + f\ (-x) = 1)$ 
     $\longleftrightarrow (\forall x. f\ x = (1 + x) / (x^2 + 1))$ 
    (is  $(\forall x. ?eq\ x) \longleftrightarrow (\forall x. ?def\ x)$ )
proof
  assume  $\forall x. ?eq\ x$ 
  then have  $?eq\ x$  for  $x..$ 
  hence  $f\ neg\ x: f\ (-x) = 1 - x * f\ x$  for  $x$  by smt
  have  $f\ x: f\ x = 1 + x * f\ (-x)$  for  $x$ 
    using  $f\ neg\ x$  [where  $x = -x$ ] by simp
  have  $f\ x = 1 + x - x * x * f\ x$  for  $x$ 
    using  $f\ x$  [of  $x$ ]
    by  $(simp\ add: f\ neg\ x)$  algebra
  hence  $f\ x + x^2 * f\ x = 1 + x$  for  $x$ 
    unfolding power2-eq-square by smt
  hence  $(x^2 + 1) * f\ x = 1 + x$  for  $x$ 
    by  $(simp\ add: Rings.\ ring-distrib(2)\ add.\ commute)$ 
  moreover have  $x^2 + 1 \neq 0$  for  $x :: real$ 
    unfolding power2-eq-square by  $(smt\ zero-le-square)$ 
  ultimately have  $f\ x = (1 + x) / (x^2 + 1)$  for  $x$ 
    apply  $(intro\ eq-divide-imp)$ 
    by  $(auto\ simp\ add: ac-simps)$ 
  thus  $\forall x. ?def\ x..$ 
next
  assume  $\forall x. ?def\ x$ 
  then have  $[simp]: ?def\ x$  for  $x..$ 
  have  $[simp]: x * x + 1 \neq 0$  for  $x :: real$ 
    by  $(smt\ zero-le-square)$ 
  show  $\forall x. ?eq\ x$ 
    apply  $(auto\ simp\ add: power2-eq-square$ 
       $simp\ flip: add-divide-distrib)$ 
    by algebra
qed
end

```