

OM 1969 — Stage 1

Jakub Kądziołka

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1 Warmup problems (Series I)

Long ago, the Polish Math Olympiad published, apart from 12 problems to be solved and mailed over 3 months, a set of 12 warmup problems, which were similar in spirit, but easier.

```
theory WarmupI
imports
  Complex-Main
  Future-Library.Future-Library
  HOL-Library.Sum-of-Squares
  HOL-Library.Quadratic-Discriminant
  HOL-Number-Theory.Cong
  HOL-Analysis.Analysis
begin
```

1.1 Warmup 1

Solve the equation $3^x = 4y + 5$ in the integers.

We begin with the following lemma:

lemma *even-power-3*: $[3^k = 1::\text{int}] \pmod{4} \longleftrightarrow \text{even } k$

proof —

```

have [ $3^k = (-1::int)^k \pmod{4}$ ] (mod 4)
  by (intro cong-pow) (auto simp: cong-def)
thus ?thesis
  by (auto simp: cong-def minus-one-power-iff)
qed

```

Here is an alternative proof — hopefully it will be instructive in doing calculations mod n .

```

lemma [ $3^k = 1::int \pmod{4} \iff \text{even } k$ ]
proof (cases even k)
  case True
  then obtain l where  $2 * l = k$  by auto
  then have [ $3^k = (3^2)^l \pmod{4}$ ] (is cong - ... -)
    by (auto simp add: power-mult)
  also have [ $\dots = (1::int)^l \pmod{4}$ ] (is cong - ... -)
    by (intro cong-pow) (simp add: cong-def)
  also have [ $\dots = 1 \pmod{4}$ ] by auto
  finally have [ $3^k = 1::int \pmod{4}$ ].
  thus ?thesis using ⟨even k⟩ by blast
next
  case False
  then obtain l where  $2 * l + 1 = k$ 
    using oddE by blast
  then have [ $3^k = 3^{(2 * l + 1)} \pmod{4}$ ] (is cong - ... -) by auto
  also have [ $\dots = (3^2)^l * 3 \pmod{4}$ ] (is cong - ... -)
    by (metis power-mult power-add power-one-right cong-def)
  also have [ $\dots = (1::int)^l * 3 \pmod{4}$ ] (is cong - ... -)
    by (intro cong-mult cong-pow) (auto simp add: cong-def)
  also have [ $\dots = 3 \pmod{4}$ ] by auto
  finally have [ $3^k \neq 1::int \pmod{4}$ ] by (auto simp add: cong-def)
  then show ?thesis using ⟨odd k⟩ by blast
qed

```

This allows us to prove the theorem, provided we assume x is a natural number.

```

theorem warmup1-natx:
  fixes x :: nat and y :: int
  shows  $3^x = 4 * y + 5 \iff \text{even } x \wedge y = (3^x - 5) \text{ div } 4$ 
proof -
  have  $\text{even } x \wedge y = (3^x - 5) \text{ div } 4$  if  $3^x = 4 * y + 5$ 
  proof -
    from that have [ $3^x = 4 * y + 5 \pmod{4}$ ] by auto
    also have [ $4 * y + 5 = 5 \pmod{4}$ ]
      by (metis cong-mult-self-left cong-add-rcancel-0)
    also have [ $5 = 1::int \pmod{4}$ ] by (auto simp add: cong-def)
    finally have [ $(3::int)^x = 1 \pmod{4}$ ].
    hence even x using even-power-3 by auto
    thus ?thesis using that by auto
  qed
qed

```

moreover have $3^x = 4 * y + 5$ **if** *even* $x \wedge y = (3^x - 5) \text{ div } 4$
proof –
from that have *even* x **and** *y-form*: $y = (3^x - 5) \text{ div } 4$ **by** *auto*
then have $[3^x = 1::\text{int}] \text{ (mod } 4)$ **using** *even-power-3* **by** *blast*
then have $((3::\text{int})^x - 5) \text{ mod } 4 = 0$
by (*simp add: cong-def mod-diff-cong*)
thus *?thesis* **using** *y-form* **by** *auto*
qed
ultimately show *?thesis* **by** *blast*
qed

To consider negative values of x , we'll need to venture into the reals:

lemma *powr-int-pos*:
fixes $x \ y :: \text{int}$
assumes $*$: $3^{\text{powr } x} = y$
shows $x \geq 0$
proof (*rule ccontr*)
assume *neg-x*: $\neg x \geq 0$
then have *y-inv*: $y = \text{inverse } ((3::\text{nat})^{\text{nat } (-x)})$ (*is* $y = \text{inverse } (?n::\text{nat})$)
using *powr-real-of-int* **and** $*$ **by** *auto*
hence *real ?n * of-int* $y = 1$ **by** *auto*
hence $?n * y = 1$ **using** *of-int-eq-iff* **by** *fastforce*
hence $?n = 1$
by (*metis nat-1-eq-mult-iff nat-int nat-numeral-as-int numeral-One of-nat-mult zmult-eq-1-iff*)
hence $\text{nat } (-x) = 0$ **by** *auto*
thus *False* **using** *neg-x* **by** *auto*
qed

corollary *warmup1*:
fixes $x \ y :: \text{int}$
shows $3^{\text{powr } x} = 4 * y + 5 \iff x \geq 0 \wedge \text{even } x \wedge y = (3^{(\text{nat } x)} - 5) \text{ div } 4$
proof
assume *assm*: $3^{\text{powr } x} = 4 * y + 5$
then have $x \geq 0$ **using** *powr-int-pos* **by** *fastforce*
hence $3^{\text{powr } (\text{nat } x)} = 4 * y + 5$ **using** *assm* **by** *simp*
hence $(3::\text{real})^{(\text{nat } x)} = 4 * y + 5$ **using** *powr-realpow* **by** *auto*
hence *with-nat*: $3^{(\text{nat } x)} = 4 * y + 5$ **using** *of-int-eq-iff* **by** *fastforce*
hence *even* $(\text{nat } x) \wedge y = (3^{(\text{nat } x)} - 5) \text{ div } 4$ **using** *warmup1-natx* **by** *auto*
thus $x \geq 0 \wedge \text{even } x \wedge y = (3^{(\text{nat } x)} - 5) \text{ div } 4$ **using** $\langle x \geq 0 \rangle$ **and** *even-nat-iff*
by *auto*
next
assume *assm*: $x \geq 0 \wedge \text{even } x \wedge y = (3^{(\text{nat } x)} - 5) \text{ div } 4$
then have $3^{(\text{nat } x)} = 4 * y + 5$ **using** *warmup1-natx* **and** *even-nat-iff* **by** *blast*
thus $3^{\text{powr } x} = 4 * y + 5$ **using** *assm powr-real-of-int* **by** *fastforce*
qed

1.2 Warmup 2

Prove that, for all real a and b we have

$$(a + b)^4 \leq 8(a^4 + b^4).$$

This problem is simple enough for Isabelle to solve it automatically — with the Sum of Squares decision procedure.

theorem

$(a+b)^4 \leq 8*(a^4 + b^4)$ **for** $a\ b :: real$
by *sos*

Of course, we would rather elaborate. We will make use of the inequality known as *sum-squares-bound*:

$$(2::'a) * x * y \leq x^2 + y^2$$

theorem

$(a+b)^4 \leq 8*(a^4 + b^4)$ **for** $a\ b :: real$

proof –

have *lemineq*: $2*x^3*y \leq x^4 + x^2*y^2$ **for** $x\ y :: real$

using *sum-squares-bound* [of $x\ y$]

and *mult-left-mono* [where $c=x^2$]

by (*force simp add: numeral-eq-Suc algebra-simps*)

have $(a+b)^4 = a^4 + 4*a^3*b + 6*a^2*b^2 + 4*a*b^3 + b^4$ **by** *algebra*

also have $\dots \leq a^4 + 2*(a^4 + a^2*b^2) + 6*a^2*b^2 + 2*(b^4 + a^2*b^2)$
 $+ b^4$

using *lemineq* [of $a\ b$]

and *lemineq* [of $b\ a$]

by (*simp add: algebra-simps*)

also have $\dots = 3*a^4 + 3*b^4 + 10*a^2*b^2$ **by** (*simp add: algebra-simps*)

also have $\dots \leq 8*(a^4 + b^4)$

using *sum-squares-bound* [of $a^2\ b^2$]

by *simp*

finally show *?thesis*.

qed

Another interesting proof is by Jensen's inequality. In Isabelle, it's known as the *convex-on* lemma:

convex $S \implies$

convex-on $S\ f =$

$(\forall k\ u\ x.$

$(\forall i \in \{1..k\}. 0 \leq u\ i \wedge x\ i \in S) \wedge \text{sum } u\ \{1..k\} = 1 \longrightarrow$

$f\ (\sum i = 1..k. u\ i * x\ i) \leq (\sum i = 1..k. u\ i * f\ (x\ i)))$

Note that the sequences u and x are modeled as functions $nat \Rightarrow real$, thus instead of u_i we have $u\ i$.

Make sure not to confuse the *convex-on* lemma with the *convex-on* predicate, which is defined by *convex-on-def*:

convex-on s f =
 $(\forall x \in s. \forall y \in s. \forall u \geq 0. \forall v \geq 0. u + v = 1 \longrightarrow$
 $f (u *_R x + v *_R y) \leq u * f x + v * f y)$

The bulk of the work, of course, is in showing that our function, $x \mapsto x^4$, is convex.

theorem *warmup2*:

$(a+b)^4 \leq 8*(a^4 + b^4)$ **for** $a\ b :: \text{real}$

proof –

let $?f = \lambda x. x^4$

have *convex-on UNIV* $?f$

proof (*rule f''-ge0-imp-convex*)

show *convex UNIV* **by** *auto*

let $?f' = \lambda x. 4*x^3$

show ($?f$ *has-real-derivative* $?f' x$) (*at* x) **for** $x :: \text{real}$

using *DERIV-pow* [**where** $n=4$] **by** *fastforce*

let $?f'' = \lambda x. 12*x^2$

show ($?f'$ *has-real-derivative* $?f'' x$) (*at* x) **for** $x :: \text{real}$

using *DERIV-pow* [**where** $n=3$]

and *DERIV-cmult* [**where** $c=4$]

by *fastforce*

show $0 \leq ?f'' x$ **for** $x :: \text{real}$

by *auto*

qed

hence $(a/2 + b/2)^4 \leq a^4/2 + b^4/2$ (**is** $?lhs \leq ?rhs$)

using *convex-onD* [**where** $t=1/2$] **by** *fastforce*

also have $?lhs = ((a + b)/2)^4$ **by** *algebra*

also have $\dots = (a+b)^4/16$ **using** *power-divide* [*of* $a+b$ 2, **where** $n=4$] **by** *fastforce*

finally show $?thesis$ **by** *auto*

qed

1.3 Warmup 3

This one is a straight-forward equation:

theorem *warmup3*:

$|x-1|*|x+2|*|x-3|*|x+4| = |x+1|*|x-2|*|x+3|*|x-4|$

$\longleftrightarrow x \in \{0, \text{sqrt } 7, -\text{sqrt } 7,$

$\text{sqrt } ((13 + \text{sqrt } 73) / 2),$

$-\text{sqrt } ((13 + \text{sqrt } 73) / 2),$

$\text{sqrt } ((13 - \text{sqrt } 73) / 2),$

$-\text{sqrt } ((13 - \text{sqrt } 73) / 2)\}$

(**is** $?eqn \longleftrightarrow ?sols$)

proof –

have $?eqn \longleftrightarrow |(x-1)*(x+2)*(x-3)*(x+4)| = |(x+1)*(x-2)*(x+3)*(x-4)|$

(**is** $\longleftrightarrow |?lhs| = |?rhs|$)

```

    by (simp add: abs-mult)
    also have ...  $\longleftrightarrow$  ?lhs - ?rhs = 0  $\vee$  ?lhs + ?rhs = 0 by (auto simp add:
abs-eq-iff)
    also have ...  $\longleftrightarrow$   $x*(x^2 - 7) = 0 \vee x^4 - 13*x^2 + 24 = 0$  by algebra
    also have  $x*(x^2 - 7) = 0 \longleftrightarrow x \in \{0, \text{sqrt } 7, -\text{sqrt } 7\}$  using plus-or-minus-sqrt
by auto
    also have  $x^4 - 13*x^2 + 24 = 0 \longleftrightarrow x^2 \in \{(13 + \text{sqrt } 73) / 2, (13 - \text{sqrt }
73) / 2\}$ 
    using discriminant-nonneg [where  $x=x^2$ , of 1 -13 24]
    by (auto simp add: algebra-simps discrimin-def)
    also have ...  $\longleftrightarrow x \in \{\text{sqrt } ((13 + \text{sqrt } 73) / 2),$ 
 $-\text{sqrt } ((13 + \text{sqrt } 73) / 2),$ 
 $\text{sqrt } ((13 - \text{sqrt } 73) / 2),$ 
 $-\text{sqrt } ((13 - \text{sqrt } 73) / 2)\}$ 

proof -
  have  $0 \leq (13 - \text{sqrt } 73) / 2$  by (auto simp add: real-le-lsqrt)
  hence  $x^2 = (13 - \text{sqrt } 73) / 2$ 
 $\longleftrightarrow x \in \{\text{sqrt } ((13 - \text{sqrt } 73) / 2),$ 
 $-\text{sqrt } ((13 - \text{sqrt } 73) / 2)\}$ 
    using plus-or-minus-sqrt
    by blast
  moreover have  $x^2 = (13 + \text{sqrt } 73) / 2$ 
 $\longleftrightarrow x \in \{\text{sqrt } ((13 + \text{sqrt } 73) / 2),$ 
 $-\text{sqrt } ((13 + \text{sqrt } 73) / 2)\}$ 
    by (smt insert-iff power2-minus power-divide real-sqrt-abs real-sqrt-divide
real-sqrt-pow2 singletonD)
    ultimately show ?thesis by blast
qed
ultimately show ?thesis by blast
qed

```

1.4 Warmup 4

There is a set of n points on a plane with the property that, in each triplet of points, there's a pair with distance at most 1. Prove that the set can be covered with two circles of radius 1.

There's nothing special about the case of points on a plane, the theorem can be proved without additional difficulties for any metric space:

theorem *warmup4-generic:*

fixes $S :: 'a::\text{metric-space set}$

assumes *finite S*

assumes *property:* $\bigwedge T. T \subseteq S \wedge \text{card } T = 3 \implies \exists p \in T. \exists q \in T. p \neq q \wedge \text{dist } p \ q \leq 1$

obtains $O_1 \ O_2$ **where** $S \subseteq \text{cball } O_1 \ 1 \cup \text{cball } O_2 \ 1$

proof

let $?pairs = S \times S$

let $?dist = \lambda(a, b). \text{dist } a \ b$

define *widest-pair* **where** $\text{widest-pair} = \text{arg-max-on } ?dist \ ?pairs$

```

let ?O1 = (fst widest-pair)
let ?O2 = (snd widest-pair)
show  $S \subseteq \text{cball } ?O_1 \ 1 \cup \text{cball } ?O_2 \ 1$ 
proof
  fix x
  assume  $x \in S$ 

  from ⟨finite S⟩ and ⟨ $x \in S$ ⟩
  have finite ?pairs and ?pairs ≠ {} by auto
  hence OinS: widest-pair ∈ ?pairs
    unfolding widest-pair-def by (simp add: arg-max-if-finite)

  have  $\forall (P,Q) \in ?pairs. \text{dist } ?O_1 \ ?O_2 \geq \text{dist } P \ Q$ 
    unfolding widest-pair-def
    using ⟨finite ?pairs⟩ and ⟨?pairs ≠ {}⟩
    by (metis (mono-tags, lifting) arg-max-greatest prod.case-eq-if)
  hence greatest:  $\text{dist } P \ Q \leq \text{dist } ?O_1 \ ?O_2$  if  $P \in S$  and  $Q \in S$  for  $P \ Q$ 
    using that by blast

  let ?T = {?O1, ?O2, x}
  have TinS: ?T ⊆ S using OinS and ⟨ $x \in S$ ⟩ by auto

  {
    presume ?O1 ≠ ?O2 and  $x \notin \{?O_1, ?O_2\}$ 
    then have card ?T = 3 by auto
  }
  then consider
    (primary) card ?T = 3 |
    (limit)  $x \in \{?O_1, ?O_2\}$  |
    (degenerate) ?O1 = ?O2 by blast
  thus  $x \in \text{cball } ?O_1 \ 1 \cup \text{cball } ?O_2 \ 1$ 
  proof cases
    case primary
    obtain p and q where  $p \neq q$  and  $\text{dist } p \ q \leq 1$  and  $p \in ?T$  and  $q \in ?T$ 
      using property [of ?T] and ⟨card ?T = 3⟩ TinS
      by auto
    then have
       $\text{dist } ?O_1 \ ?O_2 \leq 1 \vee \text{dist } ?O_1 \ x \leq 1 \vee \text{dist } ?O_2 \ x \leq 1$ 
      by (metis dist-commute insertE singletonD)
    thus  $x \in \text{cball } ?O_1 \ 1 \cup \text{cball } ?O_2 \ 1$ 
      using greatest and TinS
      by fastforce
    next
    case limit
    then have  $\text{dist } x \ ?O_1 = 0 \vee \text{dist } x \ ?O_2 = 0$  by auto
    thus ?thesis by auto
    next
    case degenerate
    with greatest and TinS have  $\text{dist } ?O_1 \ x = 0$  by auto
  
```

```

    thus ?thesis by auto
  qed
qed
qed

```

Let's make sure that the particular case of points on a plane also works out:

```

corollary warmup4:
  fixes S :: (real ^ 2) set
  assumes finite S
  assumes property:  $\bigwedge T. T \subseteq S \wedge \text{card } T = 3 \implies \exists p \in T. \exists q \in T. p \neq q \wedge \text{dist } p \ q \leq 1$ 
  obtains O1 O2 where  $S \subseteq \text{cball } O_1 \ 1 \cup \text{cball } O_2 \ 1$ 
  using warmup4-generic and assms by auto

end

```

2 Series I

```

theory SeriesI
imports
  Complex-Main
  HOL-Analysis.Analysis
begin

```

2.1 Problem 1

Solve the equation in the integers:

```

theorem problem1:
  fixes x y :: int
  assumes x ≠ 0 and y ≠ 0
  shows  $1 / x^2 + 1 / (x*y) + 1 / y^2 = 1$ 
     $\longleftrightarrow x = 1 \wedge y = -1 \vee x = -1 \wedge y = 1$ 
    (is ?eqn  $\longleftrightarrow$  ?sols)
proof
  — Unfortunately, removing the conversions between int and real takes a few lines
  let ?x = real-of-int x and ?y = real-of-int y
  assume ?eqn
  then have  $1 / ?x^2 + 1 / (?x*?y) + 1 / ?y^2 = 1$  by auto
  hence  $?x^2*?y^2 / ?x^2 + ?x^2*?y^2 / (?x*?y) + ?x^2*?y^2 / ?y^2 = ?x^2*?y^2$ 
    by algebra
  hence  $?x^2 + ?x*?y + ?y^2 = ?x^2 * ?y^2$  using ⟨x ≠ 0⟩ ⟨y ≠ 0⟩
    by (simp add: power2-eq-square)
  hence inteq:  $x^2 + x*y + y^2 = x^2 * y^2$ 
    using of-int-eq-iff by fastforce

  define g where  $g = \text{gcd } x \ y$ 
  then have  $g \neq 0$  and  $g > 0$  using ⟨x ≠ 0⟩ ⟨y ≠ 0⟩ by auto
  define x' y' where  $x' = x \text{ div } g$  and  $y' = y \text{ div } g$ 

```


then have $x' * g = x$ and $y' * g = y$ using *g-def* by *auto*
 from *inteq* and this have $g^2 * (x'^2 + x' * y' + y'^2) = x'^2 * y'^2 * g^4$
 by *algebra*
 hence reduced: $x'^2 + x' * y' + y'^2 = x'^2 * y'^2 * g^2$ using $\langle g \neq 0 \rangle$ by *algebra*

 hence $x' \text{ dvd } y'^2$ and $y' \text{ dvd } x'^2$
 by *algebra* +
 moreover have *coprime* $x' (y'^2)$ *coprime* $(x'^2) y'$
 unfolding *x'-def* *y'-def* *g-def*
 using *assms div-gcd-coprime* by *auto*
 ultimately have *is-unit* x' *is-unit* y'
 unfolding *coprime-def* by *auto*
 hence *abs1*: $|x'| = 1 \wedge |y'| = 1$ using *assms* by *auto*
 then consider (*same-sign*) $x' = y'$ | (*diff-sign*) $x' = -y'$ by *fastforce*
 thus ?sols
 proof cases
 case *same-sign*
 then have $x' * y' = 1$
 using *abs1* and *zmult-eq-1-iff* by *fastforce*
 hence $g^2 = 3$
 using *abs1 same-sign* and reduced by *algebra*
 hence $1^2 < g^2$ and $g^2 < 2^2$ by *auto*
 hence $1 < g$ and $g < 2$
 using $\langle g > 0 \rangle$ and *power2-less-imp-less* by *fastforce* +
 hence *False* by *auto*
 thus ?sols by *auto*
 next
 case *diff-sign*
 then have $x' * y' = -1$
 using *abs1*
 by (*smt mult-cancel-left2 mult-cancel-right2*)
 hence $g^2 = 1$
 using *abs1 diff-sign* and reduced by *algebra*
 hence $g = 1$ using $\langle g > 0 \rangle$
 by (*smt power2-eq-1-iff*)
 hence $x = x'$ and $y = y'$
 unfolding *x'-def* and *y'-def* by *auto*
 thus ?sols using *abs1* and *diff-sign* by *auto*
 qed
 next
 assume ?sols
 then show ?eqn by *auto*
 qed

2.2 Problem 2

Prove that a sequence is bounded, converges, and find the limit.

context

fixes $a :: \text{real}$

```

    assumes a-bounds:  $0 < a$   $a < 1$ 
  begin
  fun c :: nat  $\Rightarrow$  real where
    c 0 =  $a / 2$  |
    c (Suc n) =  $(a + (c\ n)^2) / 2$ 

  abbreviation x1  $\equiv 1 - \text{sqrt } (1 - a)$ 
  abbreviation x2  $\equiv 1 + \text{sqrt } (1 - a)$ 

  lemma c-pos:  $0 < c\ n$ 
    using a-bounds
    by (induction n, auto, smt zero-less-power)

  lemma c-bounded:  $c\ n < x1$ 
  proof (induction n)
    case 0
      have  $(1 - a/2)^2 = 1 - a + (a/2)^2$ 
        by (simp add: power2-diff)
      hence  $1 - a < (1 - a/2)^2$  using a-bounds by auto
      hence  $\text{sqrt } (1 - a) < 1 - a/2$ 
        using a-bounds and real-less-lsqrt by auto
      thus ?case by auto
    next
      case (Suc n)
      then have  $(c\ n)^2 < (1 - \text{sqrt } (1-a))^2$  using c-pos
        by (smt power-less-imp-less-base real-sqrt-abs)
      also have  $\dots = 2 - 2 * \text{sqrt } (1-a) - a$ 
        using a-bounds by (simp add: power2-diff)
      finally have  $(a + (c\ n)^2)/2 < 1 - \text{sqrt } (1-a)$  by auto
      then show ?case by auto
  qed

  lemma c-incseq: incseq c
  proof (rule incseq-SucI)
    fix n
    from c-bounded have  $c\ n < x1$  by auto
    have  $c\ n < x1$   $c\ n < x2$ 
      using c-bounded
      by (smt a-bounds real-sqrt-lt-0-iff) +
    moreover have  $(c\ n)^2 - 2*c\ n + a = (c\ n - x1)*(c\ n - x2)$ 
      using a-bounds
      by (auto simp add: algebra-simps power2-eq-square)
    ultimately have  $(c\ n)^2 - 2*c\ n + a > 0$ 
      by (smt nonzero-mult-div-cancel-right zero-le-divide-iff)
    thus  $c\ n \leq c\ (Suc\ n)$  by auto
  qed

  theorem problem2:  $c \longrightarrow x1$ 
  proof -

```

```

obtain  $L$  where  $c \longrightarrow L$ 
  using  $c\text{-incseq } c\text{-bounded incseq-convergent}$ 
  by ( $\text{metis less-imp-le}$ )
then have  $(\lambda n. c (Suc\ n)) \longrightarrow L$ 
  using  $LIMSEQ\text{-}Suc$  by  $\text{blast}$ 
hence  $(\lambda n. (a + (c\ n)^2) / 2 * 2) \longrightarrow L * 2$ 
  using  $tendsto\text{-}mult\text{-}right$  by  $\text{fastforce}$ 
hence  $(\lambda n. a + (c\ n)^2) \longrightarrow L * 2$  by ( $\text{simp del: distrib-right-numeral}$ )
hence  $(\lambda n. a + (c\ n)^2 - a) \longrightarrow L * 2 - a$ 
  using  $tendsto\text{-}diff\ LIMSEQ\text{-}const\text{-}iff$  by  $\text{blast}$ 
hence  $(\lambda n. (c\ n)^2) \longrightarrow L * 2 - a$ 
  by  $\text{auto}$ 
moreover from  $\langle c \longrightarrow L \rangle$ 
have  $(\lambda n. (c\ n)^2) \longrightarrow L^2$ 
  unfolding  $\text{power2-eq-square}$ 
  using  $tendsto\text{-}mult$  by  $\text{blast}$ 
ultimately have  $L * 2 - a = L^2$ 
  by ( $\text{rule LIMSEQ-unique}$ )
hence  $L^2 - 2 * L + a = 0$  by  $\text{auto}$ 
moreover have  $L^2 - 2 * L + a = (L - x1) * (L - x2)$ 
  using  $a\text{-bounds}$ 
  by ( $\text{auto simp add: algebra-simps power2-eq-square}$ )
ultimately have  $L = x1 \vee L = x2$ 
  by  $\text{auto}$ 
moreover from  $c\text{-bounded}$  and  $\langle c \longrightarrow L \rangle$  have  $L \leq x1$ 
  by ( $\text{meson LIMSEQ-le-const2 le-less-linear less-imp-triv}$ )
moreover from  $a\text{-bounds}$  have  $x1 < x2$  by  $\text{auto}$ 
ultimately have  $L = x1$  by  $\text{auto}$ 
thus  $?thesis$  using  $\langle c \longrightarrow L \rangle$  by  $\text{auto}$ 
qed

end

end

```