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theory Problem-1
  imports Main
begin

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Find all functions  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  satisfying

$$f(2a) + 2f(b) = f(f(a + b)).$$

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theorem problem1:
  fixes f :: int ⇒ int
  obtains k where
    (∀ a b. f (2*a) + 2*f b = f (f (a + b))) ⟷
    (∀ x. f x = 2*x + k) ∨ (∀ x. f x = 0)
proof (rule, rule)
  assume ∀ a b. f (2*a) + 2*f b = f (f (a + b))
  then have eq: f (2*a) + 2*f b = f (f (a + b)) for a b by auto
  have f (2*a) + 2*f b = f (2*b) + 2*f a for a b
    using eq[of a b] and eq[of b a]
    by (simp add: add.commute)
  from this[of 0] have [simp]: f (2*a) = 2*f a - f 0 for a by simp
  have eq': 2*f a + 2*f b - f 0 = f (f (a + b)) for a b
    using eq[of a b] by simp
  have 2*f a + f 0 = f (f a) for a
    using eq'[of a 0] by simp
  hence [simp]: f (f a) = 2*f a + f 0 for a..
  from eq' have 2*f a + 2*f b - f 0 = 2*f (a+b) + f 0 for a b by simp
  hence 2*f a + 2*f b - 2*f 0 = 2*f (a + b) for a b by (simp add: ac-simps)
  hence eq'': f a + f b - f 0 = f (a + b) for a b by smt

define m c where
  m = f 1 - f 0 and
  c = f 0
have nat-linear: f (int n) = m*(int n) + c for n :: nat
proof (induction n)
  case 0
  then show ?case unfolding m-def c-def by simp
next
  case (Suc n)
  then show ?case
    unfolding m-def c-def
    by (simp flip: eq''[of 1 int n] add: ac-simps distrib-right)
qed

have f-neg: f (-a) = 2*f 0 - f a for a
  using eq''[of a -a] by simp

have linear: f x = m*x + c for x
proof (cases x ≥ 0)
  case True
  then show ?thesis
    using nat-linear[of nat x] by simp
next
  case False
  then show ?thesis
    using nat-linear[of nat (-x)] f-neg by (simp add: c-def)
qed

hence params: 2*m*(a+b) + 3*c = m*m*(a+b)+m*c+c for a b :: int
  using eq[of a b] by (simp add: algebra-simps)

from params[of 0 0] and params[of 1 0] have 2*m = m*m by algebra
then consider m = 2 | m = 0 by auto
then show (∀ x. f x = 2*x + c) ∨ (∀ x. f x = 0)

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proof cases
  case 1
    then have  $f\ x = 2*x + c$  for  $x$ 
      using linear by simp
    then show ?thesis by simp
  next
    case 2
      with params[of 0 0] have  $c = 0$  by simp
      with linear and  $\langle m = 0 \rangle$  have  $f\ x = 0$  for  $x$  by simp
      then show ?thesis by simp
    qed
  next
    define  $c$  where  $c = f\ 0$ 
    assume  $(\forall x. f\ x = 2*x + c) \vee (\forall x. f\ x = 0)$ 
    then show  $(\forall a\ b. f\ (2*a) + 2*f\ b = f\ (f\ (a + b)))$ 
      by auto
    qed
end

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