theory *Problem-789*
imports *Complex-Main*
begin

Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x) + y) = 4yf(x) + f(x^2 - y).$$

theorem
fixes f :: *real* $\Rightarrow$ *real*
shows $(\forall x\ y.\ f\ (f\ x + y) = 4*y*f\ x + f\ (x^2 - y))$
$\longleftrightarrow$ $(\forall x.\ f\ x = x^2) \vee (\forall x.\ f\ x = 0)$
(**is** $(\forall x\ y.\ ?eqn\ x\ y) \longleftrightarrow$ -)
proof
assume $\forall x\ y.\ ?eqn\ x\ y$
then have *eqn*: *?eqn x y* **for** *x y* **by** *auto*
have [*simp*]: $f\ (x^2 + f\ x) = 4*x^2*f\ x + f\ 0$ **for** *x*
using *eqn*[**where** $y{=}x^2$ **and** $x{=}x$]
unfolding *power2-eq-square* **by** *smt*
have *opts*: $f\ x = 0 \vee f\ x = x^2$ **for** *x*
using *eqn*[**where** $y{=}-f\ x$ **and** $x{=}x$, *simplified*]
by *auto*
{
fix *a*
presume $f\ a \neq a^2$
then have $a \neq 0$ **and** [*simp*]: $f\ a = 0$
using *opts* **by** *fastforce+*
fix *y*
have $*$: $f\ y = f\ (a^2 - y)$
using *eqn*[**where** $x{=}a$ **and** $y{=}y$] **by** *simp*
presume $f\ y \neq 0$
hence $f\ y = y^2$ **using** *opts* **by** *auto*
moreover from $*$ **and** ‹$f\ y \neq 0$› **have** $f\ (a^2 - y) = (a^2 - y)^2$
using *opts* **by** *auto*
ultimately have $y^2 = (a^2 - y)^2$
using $*$ **by** *auto*
with ‹$a \neq 0$› **have** $a^2 = 2*y$
by (*simp add*: *power2-eq-square*) *algebra*
}
hence $**$: $f\ a \neq a^2 \Longrightarrow 2*y \neq a^2 \Longrightarrow f\ y = 0$ **for** *a y*
by *fastforce*

{
fix *a* **and** *y*
assume $f\ a \neq a^2$ **and** $2*y = a^2$
moreover obtain *b* **where** $2*y \neq b^2$ **and** $b \neq 0$ **and** $b \neq y$
by (*smt four-x-squared one-power2*)
ultimately have $f\ b \neq b^2$ **using** $**$ [**where** $y{=}b$ **and** $a{=}a$]
by *simp*
with $**$ [**where** $a{=}b$ **and** $y{=}y$] **and** ‹$2*y \neq b^2$›
have $f\ y = 0$ **by** *simp*
}
with $**$ **have** $f\ a \neq a^2 \Longrightarrow f\ y = 0$ **for** *a y*
by *blast*
thus $(\forall x.\ f\ x = x^2) \vee (\forall x.\ f\ x = 0)$
using *opts* **by** *blast*
next
assume $(\forall x.\ f\ x = x^2) \vee (\forall x.\ f\ x = 0)$
then show $\forall x\ y.\ ?eqn\ x\ y$
apply (*auto simp add*: *power2-eq-square*)
apply (*thin-tac* $\forall x.\ f\ x = x * x$)
by *algebra*
qed

end