```
theory Warmup-Problem-B
  imports
    Complex-Main
    HOL−Library.Sum-of-Squares
    HOL−Analysis.Analysis
begin
```

0.1 Warmup problem B

Prove that, for all real a and b we have

$$(a+b)^4 \le 8(a^4 + b^4).$$

This problem is simple enough for Isabelle to solve it automatically — with the Sum of Squares decision procedure.

```
theorem
  (a+b)^4 ≤ 8∗(a^4 + b^4) for a b :: real
  by sos
```

Of course, we would rather elaborate. We will make use of the inequality known as *sum-squares-bound*:

$(2::'a) * x * y \le x^2 + y^2$

```
theorem
  (a+b)^4 ≤ 8∗(a^4 + b^4) for a b :: real
proof −
  have lemineq: 2∗x^3∗y ≤ x^4 + x^2∗y^2 for x y :: real
    using sum-squares-bound [of x y]
      and mult-left-mono [where c=x^2]
    by (force simp add: numeral-eq-Suc algebra-simps)

  have (a+b)^4 = a^4 + 4∗a^3∗b + 6∗a^2∗b^2 + 4∗a∗b^3 + b^4 by algebra
  also have ... ≤ a^4 + 2∗(a^4 + a^2∗b^2) + 6∗a^2∗b^2 + 2∗(b^4 + a^2∗b^2)
+ b^4
    using lemineq [of a b]
      and lemineq [of b a]
    by (simp add: algebra-simps)
  also have ... = 3∗a^4 + 3∗b^4 + 10∗a^2∗b^2 by (simp add: algebra-simps)
  also have ... ≤ 8∗(a^4 + b^4)
    using sum-squares-bound [of a^2 b^2]
    by simp
  finally show ?thesis.
qed
```

Another interesting proof is by Jensen's inequality. In Isabelle, it's known as the *convex-on* lemma:

$convex\ S \Longrightarrow$
$convex\text{-}on\ S\ f =$
$(\forall k\ u\ x.$
$\quad (\forall i{\in}\{1..k\}.\ 0 \le u\ i \wedge x\ i \in S) \wedge sum\ u\ \{1..k\} = 1 \longrightarrow$
$\quad f\ (\sum i = 1..k.\ u\ i *_R x\ i) \le (\sum i = 1..k.\ u\ i * f\ (x\ i)))$

Note that the sequences u and x are modeled as functions $nat \Rightarrow real$, thus instead of u_i we have $u\ i$.

Make sure not to confuse the *convex-on* lemma with the *convex-on* predicate, which is defined by *convex-on-def*:

$convex\text{-}on\ s\ f =$
$(\forall x{\in}s.\ \forall y{\in}s.\ \forall u{\ge}0.\ \forall v{\ge}0.\ u + v = 1 \longrightarrow$
$\qquad f\ (u *_R x + v *_R y) \le u * f\ x + v * f\ y)$

The bulk of the work, of course, is in showing that our function, $x \mapsto x^4$, is convex.

```
theorem warmup2:
  (a+b)^4 ≤ 8*(a^4 + b^4) for a b :: real
proof -
  let ?f = λx. x^4
  have convex-on UNIV ?f
  proof (rule f''-ge0-imp-convex)
    show convex UNIV by auto
    let ?f' = λx. 4*x^3
    show (?f has-real-derivative ?f' x) (at x) for x :: real
      using DERIV-pow [where n=4] by fastforce
    let ?f'' = λx. 12*x^2
    show (?f' has-real-derivative ?f'' x) (at x) for x :: real
      using DERIV-pow [where n=3]
        and DERIV-cmult [where c=4]
      by fastforce
    show 0 ≤ ?f'' x for x :: real
      by auto
  qed
  hence (a/2 + b/2)^4 ≤ a^4/2 + b^4/2 (is ?lhs ≤ ?rhs)
    using convex-onD [where t=1/2] by fastforce
  also have ?lhs = ((a + b)/2)^4 by algebra
   also have ... = (a+b)^4/16 using power-divide [of a+b 2, where n=4] by
fastforce
  finally show ?thesis by auto
qed

end
```