

```

theory Warmup-Problem-B
imports
  Complex-Main
  HOL-Library.Sum-of-Squares
  HOL-Analysis.Analysis
begin

```

## 0.1 Warmup problem B

Prove that, for all real  $a$  and  $b$  we have

$$(a + b)^4 \leq 8(a^4 + b^4).$$

This problem is simple enough for Isabelle to solve it automatically — with the Sum of Squares decision procedure.

```

theorem
   $(a+b)^4 \leq 8*(a^4 + b^4)$  for  $a\ b :: real$ 
by sos

```

Of course, we would rather elaborate. We will make use of the inequality known as *sum-squares-bound*:

$$(2::'a) * x * y \leq x^2 + y^2$$

```

theorem
   $(a+b)^4 \leq 8*(a^4 + b^4)$  for  $a\ b :: real$ 
proof -
  have lemineq:  $2*x^3*y \leq x^4 + x^2*y^2$  for  $x\ y :: real$ 
    using sum-squares-bound [of  $x\ y$ ]
    and mult-left-mono [where  $c=x^2$ ]
    by (force simp add: numeral-eq-Suc algebra-simps)

  have  $(a+b)^4 = a^4 + 4*a^3*b + 6*a^2*b^2 + 4*a*b^3 + b^4$  by algebra
  also have  $\dots \leq a^4 + 2*(a^4 + a^2*b^2) + 6*a^2*b^2 + 2*(b^4 + a^2*b^2) + b^4$ 
    using lemineq [of  $a\ b$ ]
    and lemineq [of  $b\ a$ ]
    by (simp add: algebra-simps)
  also have  $\dots = 3*a^4 + 3*b^4 + 10*a^2*b^2$  by (simp add: algebra-simps)
  also have  $\dots \leq 8*(a^4 + b^4)$ 
    using sum-squares-bound [of  $a^2\ b^2$ ]
    by simp
  finally show ?thesis.
qed

```

Another interesting proof is by Jensen's inequality. In Isabelle, it's known as the *convex-on* lemma:

```

convex S  $\implies$ 
convex-on S f =
 $(\forall k\ u\ x.$ 
   $(\forall i \in \{1..k\}. 0 \leq u\ i \wedge x\ i \in S) \wedge \text{sum } u\ \{1..k\} = 1 \longrightarrow$ 
   $f\ (\sum i = 1..k. u\ i * x\ i) \leq (\sum i = 1..k. u\ i * f\ (x\ i)))$ 

```

Note that the sequences  $u$  and  $x$  are modeled as functions  $nat \Rightarrow real$ , thus instead of  $u_i$  we have  $u\ i$ .

Make sure not to confuse the *convex-on* lemma with the *convex-on* predicate, which is defined by *convex-on-def*:

```

convex-on s f =
 $(\forall x \in s. \forall y \in s. \forall u \geq 0. \forall v \geq 0. u + v = 1 \longrightarrow$ 
   $f\ (u * x + v * y) \leq u * f\ x + v * f\ y)$ 

```

The bulk of the work, of course, is in showing that our function,  $x \mapsto x^4$ , is convex.

**theorem** *warmup2*:

$(a+b)^4 \leq 8*(a^4 + b^4)$  **for**  $a\ b :: \text{real}$

**proof** –

**let**  $?f = \lambda x. x^4$

**have** *convex-on UNIV*  $?f$

**proof** (*rule f''-ge0-imp-convex*)

**show** *convex UNIV* **by** *auto*

**let**  $?f' = \lambda x. 4*x^3$

**show** ( $?f$  *has-real-derivative*  $?f' x$ ) (at  $x$ ) **for**  $x :: \text{real}$

**using** *DERIV-pow* [**where**  $n=4$ ] **by** *fastforce*

**let**  $?f'' = \lambda x. 12*x^2$

**show** ( $?f'$  *has-real-derivative*  $?f'' x$ ) (at  $x$ ) **for**  $x :: \text{real}$

**using** *DERIV-pow* [**where**  $n=3$ ]

**and** *DERIV-cmult* [**where**  $c=4$ ]

**by** *fastforce*

**show**  $0 \leq ?f'' x$  **for**  $x :: \text{real}$

**by** *auto*

**qed**

**hence**  $(a/2 + b/2)^4 \leq a^4/2 + b^4/2$  (**is**  $?lhs \leq ?rhs$ )

**using** *convex-onD* [**where**  $t=1/2$ ] **by** *fastforce*

**also have**  $?lhs = ((a + b)/2)^4$  **by** *algebra*

**also have**  $\dots = (a+b)^4/16$  **using** *power-divide* [*of*  $a+b$  2, **where**  $n=4$ ] **by** *fastforce*

**finally show** *?thesis* **by** *auto*

**qed**

**end**