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theory Problem-1
  imports Complex-Main
begin

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Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$xf(x) + f(-x) = 1.$$

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theorem
  fixes f :: real  $\Rightarrow$  real
  shows ( $\forall x. x * f x + f (-x) = 1$ )
     $\longleftrightarrow$  ( $\forall x. f x = (1 + x) / (x^2 + 1)$ )
    (is ( $\forall x. ?eq x$ )  $\longleftrightarrow$  ( $\forall x. ?def x$ ))
proof
  assume  $\forall x. ?eq x$ 
  then have  $?eq x$  for x..
  hence f-negx:  $f (-x) = 1 - x*f x$  for x by smt
  have f-x:  $f x = 1 + x*f (-x)$  for x
    using f-negx[where x=-x] by simp
  have  $f x = 1 + x - x*x*f x$  for x
    using f-x[of x]
    by (simp add: f-negx) algebra
  hence  $f x + x^2 * f x = 1 + x$  for x
    unfolding power2-eq-square by smt
  hence  $(x^2 + 1)*f x = 1 + x$  for x
    by (simp add: Rings.ring-distrib(2) add.commute)
  moreover have  $x^2 + 1 \neq 0$  for x :: real
    unfolding power2-eq-square by (smt zero-le-square)
  ultimately have  $f x = (1 + x) / (x^2 + 1)$  for x
    apply (intro eq-divide-imp)
    by (auto simp add: ac-simps)
  thus  $\forall x. ?def x$ ..
next
  assume  $\forall x. ?def x$ 
  then have [simp]:  $?def x$  for x..
  have [simp]:  $x*x + 1 \neq 0$  for x :: real
    by (smt zero-le-square)
  show  $\forall x. ?eq x$ 
    apply (auto simp add: power2-eq-square
      simp flip: add-divide-distrib)
    by algebra
qed

end

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