theory *Warmup-Problem-B*
  imports
    Complex-Main
    HOL−Library.Sum-of-Squares
    HOL−Analysis.Analysis
begin

0.1 Warmup problem B

Prove that, for all real a and b we have

$$(a + b)^4 \leq 8(a^4 + b^4).$$

This problem is simple enough for Isabelle to solve it automatically — with the Sum of Squares decision procedure.

theorem
  $(a+b)\char`^4 \leq 8*(a\char`^4 + b\char`^4)$ **for** $a\ b :: real$
  by *sos*

Of course, we would rather elaborate. We will make use of the inequality known as *sum-squares-bound*:

$(2::'a) * x * y \leq x^2 + y^2$

theorem
  $(a+b)\char`^4 \leq 8*(a\char`^4 + b\char`^4)$ **for** $a\ b :: real$
proof −
  have *lemineq*: $2*x\char`^3*y \leq x\char`^4 + x\char`^2*y\char`^2$ **for** $x\ y :: real$
    using *sum-squares-bound* [*of x y*]
      and *mult-left-mono* [**where** $c=x\char`^2$]
    by (*force simp add*: *numeral-eq-Suc algebra-simps*)

  have $(a+b)\char`^4 = a\char`^4 + 4*a\char`^3*b + 6*a\char`^2*b\char`^2 + 4*a*b\char`^3 + b\char`^4$ **by** *algebra*
  also have $... \leq a\char`^4 + 2*(a\char`^4 + a\char`^2*b\char`^2) + 6*a\char`^2*b\char`^2 + 2*(b\char`^4 + a\char`^2*b\char`^2) + b\char`^4$
    using *lemineq* [*of a b*]
      and *lemineq* [*of b a*]
    by (*simp add*: *algebra-simps*)
  also have $... = 3*a\char`^4 + 3*b\char`^4 + 10*a\char`^2*b\char`^2$ **by** (*simp add*: *algebra-simps*)
  also have $... \leq 8*(a\char`^4 + b\char`^4)$
    using *sum-squares-bound* [*of a\char`^2 b\char`^2*]
    by *simp*
  finally show *?thesis*.
qed

Another interesting proof is by Jensen's inequality. In Isabelle, it's known as the *convex-on* lemma:

convex S $\implies$
convex-on S f =
($\forall k\ u\ x$.
    ($\forall i \in \{1..k\}$. $0 \leq u\ i \wedge x\ i \in S$) $\wedge$ *sum u* $\{1..k\} = 1 \longrightarrow$
    $f\ (\sum i = 1..k.\ u\ i *_R x\ i) \leq (\sum i = 1..k.\ u\ i * f\ (x\ i)))$

Note that the sequences u and x are modeled as functions *nat* $\Rightarrow$ *real*, thus instead of u_i we have $u\ i$.

Make sure not to confuse the *convex-on* lemma with the *convex-on* predicate, which is defined by *convex-on-def*:

convex-on S f =
($\forall x \in S$. $\forall y \in S$. $\forall u \geq 0$. $\forall v \geq 0$. $u + v = 1 \longrightarrow$
                    $f\ (u *_R x + v *_R y) \leq u * f\ x + v * f\ y)$

The bulk of the work, of course, is in showing that our function, $x \mapsto x^4$, is convex.

theorem *warmup2*:
  $(a+b)\hat{\ }4 \leq 8*(a\hat{\ }4 + b\hat{\ }4)$ **for** *a b* :: *real*
proof $-$
  let *?f* = $\lambda x.\ x\hat{\ }4$
  have *convex-on UNIV ?f*
  proof (*rule f''-ge0-imp-convex*)
    show *convex UNIV* **by** *auto*
    let *?f'* = $\lambda x.\ 4*x\hat{\ }3$
    show (*?f has-real-derivative ?f' x*) (*at x*) **for** *x* :: *real*
      using *DERIV-pow* [**where** *n=4*] **by** *fastforce*
    let *?f''* = $\lambda x.\ 12*x\hat{\ }2$
    show (*?f' has-real-derivative ?f'' x*) (*at x*) **for** *x* :: *real*
      using *DERIV-pow* [**where** *n=3*]
        and *DERIV-cmult* [**where** *c=4*]
      by *fastforce*
    show $0 \leq$ *?f''* x **for** *x* :: *real*
      by *auto*
  qed
  hence $(a/2 + b/2)\hat{\ }4 \leq a\hat{\ }4/2 + b\hat{\ }4/2$ (**is** *?lhs* $\leq$ *?rhs*)
    using *convex-onD* [**where** *t=1/2*] **by** *fastforce*
  also have *?lhs* = $((a + b)/2)\hat{\ }4$ **by** *algebra*
  also have ... = $(a+b)\hat{\ }4/16$ **using** *power-divide* [*of a+b 2*, **where** *n=4*] **by** *fastforce*
  finally show *?thesis* **by** *auto*
qed

end