**theory** *Problem-1*
  **imports** *Complex-Main*
**begin**

Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$xf(x) + f(-x) = 1.$$

**theorem**
  **fixes** $f$ :: *real* $\Rightarrow$ *real*
  **shows** $(\forall x.\ x * f\ x + f\ (-x) = 1)$
      $\longleftrightarrow (\forall x.\ f\ x = (1 + x)\ /\ (x^2 + 1))$
      (**is** $(\forall x.\ \textit{?eq}\ x) \longleftrightarrow (\forall x.\ \textit{?def}\ x)$)
**proof**
  **assume** $\forall x.\ \textit{?eq}\ x$
  **then have** *?eq* $x$ **for** $x$..
  **hence** *f-negx*: $f\ (-x) = 1 - x*f\ x$ **for** $x$ **by** *smt*
  **have** *f-x*: $f\ x = 1 + x*f\ (-x)$ **for** $x$
    **using** *f-negx*[**where** $x{=}{-}x$] **by** *simp*
  **have** $f\ x = 1 + x - x*x*f\ x$ **for** $x$
    **using** *f-x*[*of* $x$]
    **by** (*simp add*: *f-negx*) *algebra*
  **hence** $f\ x + x^2 * f\ x = 1 + x$ **for** $x$
    **unfolding** *power2-eq-square* **by** *smt*
  **hence** $(x^2 + 1)*f\ x = 1 + x$ **for** $x$
    **by** (*simp add*: *Rings.ring-distribs(2)* *add.commute*)
  **moreover have** $x^2 + 1 \neq 0$ **for** $x$ :: *real*
    **unfolding** *power2-eq-square* **by** (*smt zero-le-square*)
  **ultimately have** $f\ x = (1 + x)\ /\ (x^2 + 1)$ **for** $x$
    **apply** (*intro eq-divide-imp*)
    **by** (*auto simp add*: *ac-simps*)
  **thus** $\forall x.\ \textit{?def}\ x$..
**next**
  **assume** $\forall x.\ \textit{?def}\ x$
  **then have** [*simp*]: *?def* $x$ **for** $x$..
  **have** [*simp*]: $x*x + 1 \neq 0$ **for** $x$ :: *real*
    **by** (*smt zero-le-square*)
  **show** $\forall x.\ \textit{?eq}\ x$
    **apply** (*auto simp add*: *power2-eq-square*
        *simp flip*: *add-divide-distrib*)
    **by** *algebra*
**qed**

**end**