```
theory Problem-789
  imports Complex-Main
begin
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Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x) + y) = 4yf(x) + f(x^2 - y).$$

```
theorem
  fixes f :: real ⇒ real
  shows (∀x y. f (f x + y) = 4*y*f x + f (x² − y))
    ⟷ (∀x. f x = x²) ∨ (∀x. f x = 0)
  (is (∀x y. ?eqn x y) ⟷ -)
proof
  assume ∀x y. ?eqn x y
  then have eqn: ?eqn x y for x y by auto
  have [simp]: f (x² + f x) = 4*x²*f x + f 0 for x
    using eqn[where y=x² and x=x]
    unfolding power2-eq-square by smt
  have opts: f x = 0 ∨ f x = x² for x
    using eqn[where y=−f x and x=x, simplified]
    by auto
  {
    fix a
    presume f a ≠ a²
    then have a ≠ 0 and [simp]: f a = 0
      using opts by fastforce+
    fix y
    have *: f y = f (a² − y)
      using eqn[where x=a and y=y] by simp
    presume f y ≠ 0
    hence f y = y² using opts by auto
    moreover from * and ‹f y ≠ 0› have f (a² − y) = (a² − y)²
      using opts by auto
    ultimately have y² = (a² − y)²
      using * by auto
    with ‹a ≠ 0› have a² = 2*y
      by (simp add: power2-eq-square) algebra
  }
  hence **: f a ≠ a² ⟹ 2*y ≠ a² ⟹ f y = 0 for a y
    by fastforce

  {
    fix a and y
    assume f a ≠ a² and 2*y = a²
    moreover obtain b where 2*y ≠ b² and b ≠ 0 and b ≠ y
      by (smt four-x-squared one-power2)
    ultimately have f b ≠ b² using ** [where y=b and a=a]
      by simp
    with ** [where a=b and y=y] and ‹2*y ≠ b²›
    have f y = 0 by simp
  }
  with ** have f a ≠ a² ⟹ f y = 0 for a y
    by blast
  thus (∀x. f x = x²) ∨ (∀x. f x = 0)
    using opts by blast
next
  assume (∀x. f x = x²) ∨ (∀x. f x = 0)
  then show ∀x y. ?eqn x y
    apply (auto simp add: power2-eq-square)
    apply (thin-tac ∀x. f x = x * x)
    by algebra
qed
```

end