```
theory Warmup-Problem-D
  imports
    Complex-Main
    Common.Future-Library
    HOL−Analysis.Analysis
begin
```

0.1 Warmup problem D

There is a set of n points on a plane with the property that, in each triplet of points, there's a pair with distance at most 1. Prove that the set can be covered with two circles of radius 1.

There's nothing special about the case of points on a plane, the theorem can be proved without additional difficulties for any metric space:

```
theorem warmup4-generic:
  fixes S :: 'a::metric-space set
  assumes finite S
  assumes property: ⋀T. T ⊆ S ∧ card T = 3 ⟹ ∃p∈T. ∃q∈T. p ≠ q ∧ dist
p q ≤ 1
  obtains O₁ O₂ where S ⊆ cball O₁ 1 ∪ cball O₂ 1
proof
  let ?pairs = S × S
  let ?dist = λ(a, b). dist a b
  define widest-pair where widest-pair = arg-max-on ?dist ?pairs
  let ?O₁ = (fst widest-pair)
  let ?O₂ = (snd widest-pair)
  show S ⊆ cball ?O₁ 1 ∪ cball ?O₂ 1
  proof
    fix x
    assume x ∈ S

    from ‹finite S› and ‹x ∈ S›
    have finite ?pairs and ?pairs ≠ {} by auto
    hence OinS: widest-pair ∈ ?pairs
      unfolding widest-pair-def by (simp add: arg-max-if-finite)

    have ∀(P,Q)∈?pairs. dist ?O₁ ?O₂ ≥ dist P Q
      unfolding widest-pair-def
      using ‹finite ?pairs› and ‹?pairs ≠ {}›
      by (metis (mono-tags, lifting) arg-max-greatest prod.case-eq-if)
    hence greatest: dist P Q ≤ dist ?O₁ ?O₂ if P ∈ S and Q ∈ S for P Q
      using that by blast

    let ?T = {?O₁, ?O₂, x}
    have TinS: ?T ⊆ S using OinS and ‹x ∈ S› by auto

    have card ?T = 3 if ?O₁ ≠ ?O₂ and x ∉ {?O₁, ?O₂} using that by auto
    then consider
```

  (*primary*) *card ?T* = *3* |
  (*limit*) $x \in \{?O_1, ?O_2\}$ |
  (*degenerate*) $?O_1 = ?O_2$ **by** *blast*
thus $x \in$ *cball* $?O_1$ *1* $\cup$ *cball* $?O_2$ *1*
proof *cases*
  case *primary*
  obtain *p* **and** *q* **where** $p \neq q$ **and** *dist p q* $\leq$ *1* **and** $p \in ?T$ **and** $q \in ?T$
    using *property* [*of ?T*] **and** ‹*card ?T = 3*› *TinS*
    by *auto*
  then have
    dist $?O_1$ $?O_2$ $\leq$ *1* $\vee$ *dist* $?O_1$ *x* $\leq$ *1* $\vee$ *dist* $?O_2$ *x* $\leq$ *1*
    by (*metis dist-commute insertE singletonD*)
  thus $x \in$ *cball* $?O_1$ *1* $\cup$ *cball* $?O_2$ *1*
    using *greatest* **and** *TinS*
    by *fastforce*
next
  case *limit*
  then have *dist x* $?O_1$ = *0* $\vee$ *dist x* $?O_2$ = *0* **by** *auto*
  thus *?thesis* **by** *auto*
next
  case *degenerate*
  with *greatest* **and** *TinS* **have** *dist* $?O_1$ *x* = *0* **by** *auto*
  thus *?thesis* **by** *auto*
qed
qed
qed

Let's make sure that the particular case of points on a plane also works out:

corollary *warmup4*:
fixes *S* :: (*real* ^ *2*) *set*
assumes *finite S*
assumes *property*: $\bigwedge T.\ T \subseteq S \wedge$ *card T* = *3* $\Longrightarrow \exists p \in T.\ \exists q \in T.\ p \neq q \wedge$ *dist*
p q $\leq$ *1*
obtains O_1 O_2 **where** $S \subseteq$ *cball* O_1 *1* $\cup$ *cball* O_2 *1*
using *warmup4-generic* **and** *assms* **by** *auto*

end