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theory Problem-1
  imports Complex-Main
begin

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## 0.1 Problem 1

Solve the equation in the integers:

**theorem** *problem1*:

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fixes  $x\ y :: \text{int}$ 
assumes  $x \neq 0$  and  $y \neq 0$ 
shows  $1 / x^2 + 1 / (x*y) + 1 / y^2 = 1$ 
   $\longleftrightarrow x = 1 \wedge y = -1 \vee x = -1 \wedge y = 1$ 
  (is ?eqn  $\longleftrightarrow$  ?sols)

```

**proof**

— Unfortunately, removing the conversions between int and real takes a few lines

**let**  $?x = \text{real-of-int } x$  **and**  $?y = \text{real-of-int } y$

**assume** *?eqn*

**then have**  $1 / ?x^2 + 1 / (?x * ?y) + 1 / ?y^2 = 1$  **by** *auto*

**hence**  $?x^2 * ?y^2 / ?x^2 + ?x^2 * ?y^2 / (?x * ?y) + ?x^2 * ?y^2 / ?y^2 = ?x^2 * ?y^2$

**by** *algebra*

**hence**  $?x^2 + ?x * ?y + ?y^2 = ?x^2 * ?y^2$  **using**  $\langle x \neq 0 \rangle \langle y \neq 0 \rangle$

**by** (*simp add: power2-eq-square*)

**hence** *inteq*:  $x^2 + x*y + y^2 = x^2 * y^2$

**using** *of-int-eq-iff* **by** *fastforce*

**define** *g* **where**  $g = \text{gcd } x\ y$

**then have**  $g \neq 0$  **and**  $g > 0$  **using**  $\langle x \neq 0 \rangle \langle y \neq 0 \rangle$  **by** *auto*

**define**  $x' y'$  **where**  $x' = x \text{ div } g$  **and**  $y' = y \text{ div } g$

**then have**  $x' * g = x$  **and**  $y' * g = y$  **using** *g-def* **by** *auto*

**from** *inteq* **and** *this* **have**  $g^2 * (x'^2 + x' * y' + y'^2) = x'^2 * y'^2 * g^4$

**by** *algebra*

**hence** *reduced*:  $x'^2 + x' * y' + y'^2 = x'^2 * y'^2 * g^2$  **using**  $\langle g \neq 0 \rangle$  **by** *algebra*

**hence**  $x' \text{ dvd } y'^2$  **and**  $y' \text{ dvd } x'^2$

**by** *algebra+*

**moreover have** *coprime*  $x' (y'^2)$  *coprime*  $(x'^2) y'$

**unfolding** *x'-def y'-def g-def*

**using** *assms div-gcd-coprime* **by** *auto*

**ultimately have** *is-unit*  $x'$  *is-unit*  $y'$

**unfolding** *coprime-def* **by** *auto*

**hence** *abs1*:  $|x'| = 1 \wedge |y'| = 1$  **using** *assms* **by** *auto*

**then consider** (*same-sign*)  $x' = y'$  | (*diff-sign*)  $x' = -y'$  **by** *fastforce*

**thus** *?sols*

**proof** *cases*

**case** *same-sign*

**then have**  $x' * y' = 1$

**using** *abs1* **and** *zmult-eq-1-iff* **by** *fastforce*

**hence**  $g^2 = 3$

**using** *abs1 same-sign* **and** *reduced* **by** *algebra*

**hence**  $1^2 < g^2$  **and**  $g^2 < 2^2$  **by** *auto*

**hence**  $1 < g$  **and**  $g < 2$

**using**  $\langle g > 0 \rangle$  **and** *power2-less-imp-less* **by** *fastforce+*

**hence** *False* **by** *auto*

**thus** *?sols* **by** *auto*

**next**

**case** *diff-sign*

**then have**  $x' * y' = -1$

**using** *abs1*

**by** (*smt mult-cancel-left2 mult-cancel-right2*)

**hence**  $g^2 = 1$

**using** *abs1 diff-sign* **and** *reduced* **by** *algebra*

**hence**  $g = 1$  **using**  $\langle g > 0 \rangle$

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      by (smt power2-eq-1-iff)
    hence  $x = x'$  and  $y = y'$ 
      unfolding  $x'$ -def and  $y'$ -def by auto
      thus ?sols using abs1 and diff-sign by auto
    qed
  next
    assume ?sols
    then show ?eqn by auto
  qed
end

```