theory *Warmup-Problem-B*
imports
Complex-Main
HOL−Library.Sum-of-Squares
HOL−Analysis.Analysis
begin

0.1 Warmup problem B

Prove that, for all real a and b we have

$$(a+b)^4 \leq 8(a^4+b^4).$$

This problem is simple enough for Isabelle to solve it automatically — with the Sum of Squares decision procedure.

```
theorem
  (a+b)^4 ≤ 8*(a^4 + b^4) for a b :: real
  by sos
```

Of course, we would rather elaborate. We will make use of the inequality known as *sum-squares-bound*:

$(2::'a) * x * y \leq x^2 + y^2$

```
theorem
  (a+b)^4 ≤ 8*(a^4 + b^4) for a b :: real
proof -
  have lemineq: 2*x^3*y ≤ x^4 + x^2*y^2 for x y :: real
    using sum-squares-bound [of x y]
      and mult-left-mono [where c=x^2]
    by (force simp add: numeral-eq-Suc algebra-simps)

  have (a+b)^4 = a^4 + 4*a^3*b + 6*a^2*b^2 + 4*a*b^3 + b^4 by algebra
  also have ... ≤ a^4 + 2*(a^4 + a^2*b^2) + 6*a^2*b^2 + 2*(b^4 + a^2*b^2) + b^4
    using lemineq [of a b]
      and lemineq [of b a]
    by (simp add: algebra-simps)
  also have ... = 3*a^4 + 3*b^4 + 10*a^2*b^2 by (simp add: algebra-simps)
  also have ... ≤ 8*(a^4 + b^4)
    using sum-squares-bound [of a^2 b^2]
    by simp
  finally show ?thesis.
qed
```

Another interesting proof is by Jensen's inequality. In Isabelle, it's known as the *convex-on* lemma:

```
convex S ⟹
convex-on S f =
(∀ k u x.
   (∀ i∈{1..k}. 0 ≤ u i ∧ x i ∈ S) ∧ sum u {1..k} = 1 ⟶
   f (∑ i = 1..k. u i *R x i) ≤ (∑ i = 1..k. u i * f (x i)))
```

Note that the sequences u and x are modeled as functions *nat* ⇒ *real*, thus instead of u_i we have $u\ i$.

Make sure not to confuse the *convex-on* lemma with the *convex-on* predicate, which is defined by *convex-on-def*:

```
convex-on S f =
(∀ x∈S. ∀ y∈S. ∀ u≥0. ∀ v≥0. u + v = 1 ⟶
                        f (u *R x + v *R y) ≤ u * f x + v * f y)
```

The bulk of the work, of course, is in showing that our function, $x \mapsto x^4$, is convex.

```
theorem warmup2:
  (a+b)^4 ≤ 8*(a^4 + b^4) for a b :: real
proof -
  let ?f = λx. x^4
  have convex-on UNIV ?f
  proof (rule f''-ge0-imp-convex)
    show convex UNIV by auto
    let ?f' = λx. 4*x^3
    show (?f has-real-derivative ?f' x) (at x) for x :: real
      using DERIV-pow [where n=4] by fastforce
    let ?f'' = λx. 12*x^2
    show (?f' has-real-derivative ?f'' x) (at x) for x :: real
      using DERIV-pow [where n=3]
        and DERIV-cmult [where c=4]
      by fastforce
    show 0 ≤ ?f'' x for x :: real
      by auto
  qed
  hence (a/2 + b/2)^4 ≤ a^4/2 + b^4/2 (is ?lhs ≤ ?rhs)
    using convex-onD [where t=1/2] by fastforce
  also have ?lhs = ((a + b)/2)^4 by algebra
  also have ... = (a+b)^4/16 using power-divide [of a+b 2, where n=4] by fastforce
  finally show ?thesis by auto
qed

end
```