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theory Problem-789
imports Complex-Main
begin

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Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfying

$$f(f(x) + y) = 4yf(x) + f(x^2 - y).$$

```

theorem
  fixes  $f :: \text{real} \Rightarrow \text{real}$ 
  shows  $(\forall x y. f (f x + y) = 4*y*f x + f (x^2 - y))$ 
     $\longleftrightarrow (\forall x. f x = x^2) \vee (\forall x. f x = 0)$ 
  (is  $(\forall x y. ?eqn x y) \longleftrightarrow -)$ 
proof
  assume  $\forall x y. ?eqn x y$ 
  then have  $eqn: ?eqn x y$  for  $x y$  by auto
  have  $[simp]: f (x^2 + f x) = 4*x^2*f x + f 0$  for  $x$ 
    using  $eqn[where\ y=x^2\ and\ x=x]$ 
    unfolding power2-eq-square by smt
  have  $opts: f x = 0 \vee f x = x^2$  for  $x$ 
    using  $eqn[where\ y=-f x\ and\ x=x, simplified]$ 
    by auto
  {
    fix  $a$ 
    presume  $f a \neq a^2$ 
    then have  $a \neq 0$  and  $[simp]: f a = 0$ 
      using  $opts$  by fastforce+
    fix  $y$ 
    have  $*$ :  $f y = f (a^2 - y)$ 
      using  $eqn[where\ x=a\ and\ y=y]$  by simp
    presume  $f y \neq 0$ 
    hence  $f y = y^2$  using  $opts$  by auto
    moreover from  $*$  and  $\langle f y \neq 0 \rangle$  have  $f (a^2 - y) = (a^2 - y)^2$ 
      using  $opts$  by auto
    ultimately have  $y^2 = (a^2 - y)^2$ 
      using  $*$  by auto
    with  $\langle a \neq 0 \rangle$  have  $a^2 = 2*y$ 
      by  $(simp\ add: power2-eq-square)$  algebra
  }
  hence  $**$ :  $f a \neq a^2 \implies 2*y \neq a^2 \implies f y = 0$  for  $a y$ 
    by fastforce

  {
    fix  $a$  and  $y$ 
    assume  $f a \neq a^2$  and  $2*y = a^2$ 
    moreover obtain  $b$  where  $2*y \neq b^2$  and  $b \neq 0$  and  $b \neq y$ 
      by  $(smt\ four-x-squared\ one-power2)$ 
    ultimately have  $f b \neq b^2$  using  $**$   $[where\ y=b\ and\ a=a]$ 
      by simp
    with  $**$   $[where\ a=b\ and\ y=y]$  and  $\langle 2*y \neq b^2 \rangle$ 
      have  $f y = 0$  by simp
  }
  with  $**$  have  $f a \neq a^2 \implies f y = 0$  for  $a y$ 
    by blast
  thus  $(\forall x. f x = x^2) \vee (\forall x. f x = 0)$ 
    using  $opts$  by blast
next
  assume  $(\forall x. f x = x^2) \vee (\forall x. f x = 0)$ 
  then show  $\forall x y. ?eqn x y$ 
    apply  $(auto\ simp\ add: power2-eq-square)$ 
    apply  $(thin-tac\ \forall x. f x = x * x)$ 
    by algebra
qed

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**end**