

```

theory Warmup-Problem-B
imports
  Complex-Main
  HOL-Library.Sum-of-Squares
  HOL-Analysis.Analysis
begin

```

0.1 Warmup problem B

Prove that, for all real a and b we have

$$(a + b)^4 \leq 8(a^4 + b^4).$$

This problem is simple enough for Isabelle to solve it automatically — with the Sum of Squares decision procedure.

theorem

```

(a+b)^4 ≤ 8*(a^4 + b^4) for a b :: real
by sos

```

Of course, we would rather elaborate. We will make use of the inequality known as *sum-squares-bound*:

$$(2::'a) * x * y \leq x^2 + y^2$$

theorem

```

(a+b)^4 ≤ 8*(a^4 + b^4) for a b :: real

```

proof —

```

have lemineq: 2*x^3*y ≤ x^4 + x^2*y^2 for x y :: real
using sum-squares-bound [of x y]
and mult-left-mono [where c=x^2]
by (force simp add: numeral-eq-Suc algebra-simps)

```

```

have (a+b)^4 = a^4 + 4*a^3*b + 6*a^2*b^2 + 4*a*b^3 + b^4 by algebra
also have ... ≤ a^4 + 2*(a^4 + a^2*b^2) + 6*a^2*b^2 + 2*(b^4 + a^2*b^2) + b^4
using lemineq [of a b]
and lemineq [of b a]
by (simp add: algebra-simps)
also have ... = 3*a^4 + 3*b^4 + 10*a^2*b^2 by (simp add: algebra-simps)
also have ... ≤ 8*(a^4 + b^4)
using sum-squares-bound [of a^2 b^2]
by simp
finally show ?thesis.

```

qed

Another interesting proof is by Jensen's inequality. In Isabelle, it's known as the *convex-on* lemma:

```

convex S ⇒
convex-on S f =
(∀ k u x.
  (∀ i ∈ {1..k}. 0 ≤ u i ∧ x i ∈ S) ∧ sum u {1..k} = 1 ⟶
  f (∑ i = 1..k. u i *R x i) ≤ (∑ i = 1..k. u i * f (x i)))

```

Note that the sequences u and x are modeled as functions $\text{nat} \Rightarrow \text{real}$, thus instead of u_i we have $u\ i$.

Make sure not to confuse the *convex-on* lemma with the *convex-on* predicate, which is defined by *convex-on-def*:

```

convex-on s f =
(∀ x ∈ s. ∀ y ∈ s. ∀ u ≥ 0. ∀ v ≥ 0. u + v = 1 ⟶
  f (u *R x + v *R y) ≤ u * f x + v * f y)

```

The bulk of the work, of course, is in showing that our function, $x \mapsto x^4$, is convex.

theorem *warmup2*:

$(a+b)^4 \leq 8*(a^4 + b^4)$ **for** $a\ b :: \text{real}$

proof –

let $?f = \lambda x. x^4$

have *convex-on UNIV* $?f$

proof (*rule f''-ge0-imp-convex*)

show *convex UNIV* **by** *auto*

let $?f' = \lambda x. 4*x^3$

show (*?f has-real-derivative ?f' x*) (*at x*) **for** $x :: \text{real}$

using *DERIV-pow* [**where** $n=4$] **by** *fastforce*

let $?f'' = \lambda x. 12*x^2$

show (*?f' has-real-derivative ?f'' x*) (*at x*) **for** $x :: \text{real}$

using *DERIV-pow* [**where** $n=3$]

and *DERIV-cmult* [**where** $c=4$]

by *fastforce*

show $0 \leq ?f''\ x$ **for** $x :: \text{real}$

by *auto*

qed

hence $(a/2 + b/2)^4 \leq a^4/2 + b^4/2$ (**is** $?lhs \leq ?rhs$)

using *convex-onD* [**where** $t=1/2$] **by** *fastforce*

also have $?lhs = ((a + b)/2)^4$ **by** *algebra*

also have $\dots = (a+b)^4/16$ **using** *power-divide* [*of* $a+b\ 2$, **where** $n=4$] **by** *fastforce*

finally show *?thesis* **by** *auto*

qed

end