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theory Problem-2
  imports
    HOL-Analysis.Analysis
begin

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0.1 Problem 2

Prove that a sequence is bounded, converges, and find the limit.

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context
  fixes a :: real
  assumes a-bounds: 0 < a a < 1
begin
  fun c :: nat => real where
    c 0 = a / 2 |
    c (Suc n) = (a + (c n)2) / 2

  abbreviation x1 ≡ 1 - sqrt (1 - a)
  abbreviation x2 ≡ 1 + sqrt (1 - a)

  lemma c-pos: 0 < c n
  using a-bounds
  by (induction n, auto, smt zero-less-power)

  lemma c-bounded: c n < x1
  proof (induction n)
    case 0
    have (1 - a/2)2 = 1 - a + (a/2)2
    by (simp add: power2-diff)
    hence 1 - a < (1 - a/2)2 using a-bounds by auto
    hence sqrt (1 - a) < 1 - a/2
    using a-bounds and real-less-lsqr by auto
    thus ?case by auto
  next
    case (Suc n)
    then have (c n)2 < (1 - sqrt (1-a))2 using c-pos
    by (smt power-less-imp-less-base real-sqrt-abs)
    also have ... = 2 - 2 * sqrt (1-a) - a
    using a-bounds by (simp add: power2-diff)
    finally have (a + (c n)2)/2 < 1 - sqrt (1-a) by auto
    then show ?case by auto
  qed

  lemma c-incseq: incseq c
  proof (rule incseq-SucI)
    fix n
    from c-bounded have c n < x1 by auto
    have c n < x1 c n < x2
    using c-bounded
    by (smt a-bounds real-sqrt-lt-0-iff)+
    moreover have (c n)2 - 2*c n + a = (c n - x1)*(c n - x2)
    using a-bounds
    by (auto simp add: algebra-simps power2-eq-square)
    ultimately have (c n)2 - 2*c n + a > 0
    by (smt nonzero-mult-div-cancel-right zero-le-divide-iff)
    thus c n ≤ c (Suc n) by auto
  qed

  theorem problem2: c ⟶ x1
  proof -
    obtain L where c ⟶ L
    using c-incseq c-bounded incseq-convergent
    by (metis less-imp-le)

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then have $(\lambda n. c (Suc\ n)) \longrightarrow L$
 using *LIMSEQ-Suc* by *blast*
 hence $(\lambda n. (a + (c\ n)^2) / 2 * 2) \longrightarrow L*2$
 using *tendsto-mult-right* by *fastforce*
 hence $(\lambda n. a + (c\ n)^2) \longrightarrow L*2$ by (*simp del: distrib-right-numeral*)
 hence $(\lambda n. a + (c\ n)^2 - a) \longrightarrow L*2 - a$
 using *tendsto-diff LIMSEQ-const-iff* by *blast*
 hence $(\lambda n. (c\ n)^2) \longrightarrow L*2 - a$
 by *auto*
 moreover from $\langle c \longrightarrow L \rangle$
 have $(\lambda n. (c\ n)^2) \longrightarrow L^2$
 unfolding *power2-eq-square*
 using *tendsto-mult* by *blast*
 ultimately have $L*2 - a = L^2$
 by (*rule LIMSEQ-unique*)
 hence $L^2 - 2*L + a = 0$ by *auto*
 moreover have $L^2 - 2*L + a = (L - x1)*(L - x2)$
 using *a-bounds*
 by (*auto simp add: algebra-simps power2-eq-square*)
 ultimately have $L = x1 \vee L = x2$
 by *auto*
 moreover from *c-bounded* and $\langle c \longrightarrow L \rangle$ have $L \leq x1$
 by (*meson LIMSEQ-le-const2 le-less-linear less-imp-triv*)
 moreover from *a-bounds* have $x1 < x2$ by *auto*
 ultimately have $L = x1$ by *auto*
 thus ?thesis using $\langle c \longrightarrow L \rangle$ by *auto*
 qed

 end

 end