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theory Problem-2
imports
  HOL-Analysis.Analysis
begin

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0.1 Problem 2

Prove that a sequence is bounded, converges, and find the limit.

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context
  fixes  $a :: \text{real}$ 
  assumes  $a\text{-bounds}$ :  $0 < a \wedge a < 1$ 
begin
fun  $c :: \text{nat} \Rightarrow \text{real}$  where
 $c\ 0 = a / 2$  |
 $c\ (\text{Suc } n) = (a + (c\ n)^2) / 2$ 

abbreviation  $x1 \equiv 1 - \text{sqrt } (1 - a)$ 
abbreviation  $x2 \equiv 1 + \text{sqrt } (1 - a)$ 

lemma  $c\text{-pos}$ :  $0 < c\ n$ 
  using  $a\text{-bounds}$ 
  by ( $\text{induction } n, \text{auto}, \text{smt zero-less-power}$ )

lemma  $c\text{-bounded}$ :  $c\ n < x1$ 
proof ( $\text{induction } n$ )
  case 0
  have  $(1 - a/2)^2 = 1 - a + (a/2)^2$ 
    by ( $\text{simp add: power2-diff}$ )
  hence  $1 - a < (1 - a/2)^2$  using  $a\text{-bounds}$  by  $\text{auto}$ 
  hence  $\text{sqrt } (1 - a) < 1 - a/2$ 
    using  $a\text{-bounds}$  and  $\text{real-less-lsqrt}$  by  $\text{auto}$ 
  thus ?case by  $\text{auto}$ 
next
  case ( $\text{Suc } n$ )
  then have  $(c\ n)^2 < (1 - \text{sqrt } (1-a))^2$  using  $c\text{-pos}$ 
    by ( $\text{smt power-less-imp-less-base real-sqrt-abs}$ )
  also have  $\dots = 2 - 2 * \text{sqrt } (1-a) - a$ 
    using  $a\text{-bounds}$  by ( $\text{simp add: power2-diff}$ )
  finally have  $(a + (c\ n)^2)/2 < 1 - \text{sqrt } (1-a)$  by  $\text{auto}$ 
  then show ?case by  $\text{auto}$ 
qed

lemma  $c\text{-incseq}$ :  $\text{incseq } c$ 
proof ( $\text{rule incseq-SucI}$ )
  fix  $n$ 
  from  $c\text{-bounded}$  have  $c\ n < x1$  by  $\text{auto}$ 
  have  $c\ n < x1 \wedge c\ n < x2$ 
    using  $c\text{-bounded}$ 
    by ( $\text{smt a-bounds real-sqrt-lt-0-iff}$ ) +
  moreover have  $(c\ n)^2 - 2*c\ n + a = (c\ n - x1)*(c\ n - x2)$ 
    using  $a\text{-bounds}$ 
    by ( $\text{auto simp add: algebra-simps power2-eq-square}$ )
  ultimately have  $(c\ n)^2 - 2*c\ n + a > 0$ 
    by ( $\text{smt nonzero-mult-div-cancel-right zero-le-divide-iff}$ )
  thus  $c\ n \leq c\ (\text{Suc } n)$  by  $\text{auto}$ 
qed

theorem  $\text{problem2}$ :  $c \longrightarrow x1$ 
proof -
  obtain  $L$  where  $c \longrightarrow L$ 
    using  $c\text{-incseq } c\text{-bounded incseq-convergent}$ 
    by ( $\text{metis less-imp-le}$ )

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then have $(\lambda n. c (Suc\ n)) \longrightarrow L$
 using *LIMSEQ-Suc* by *blast*
 hence $(\lambda n. (a + (c\ n)^2) / 2 * 2) \longrightarrow L*2$
 using *tendsto-mult-right* by *fastforce*
 hence $(\lambda n. a + (c\ n)^2) \longrightarrow L*2$ by *(simp del: distrib-right-numeral)*
 hence $(\lambda n. a + (c\ n)^2 - a) \longrightarrow L*2 - a$
 using *tendsto-diff LIMSEQ-const-iff* by *blast*
 hence $(\lambda n. (c\ n)^2) \longrightarrow L*2 - a$
 by *auto*
 moreover from $\langle c \longrightarrow L \rangle$
 have $(\lambda n. (c\ n)^2) \longrightarrow L^2$
 unfolding *power2-eq-square*
 using *tendsto-mult* by *blast*
 ultimately have $L*2 - a = L^2$
 by *(rule LIMSEQ-unique)*
 hence $L^2 - 2*L + a = 0$ by *auto*
 moreover have $L^2 - 2*L + a = (L - x1)*(L - x2)$
 using *a-bounds*
 by *(auto simp add: algebra-simps power2-eq-square)*
 ultimately have $L = x1 \vee L = x2$
 by *auto*
 moreover from *c-bounded* and $\langle c \longrightarrow L \rangle$ have $L \leq x1$
 by *(meson LIMSEQ-le-const2 le-less-linear less-imp-triv)*
 moreover from *a-bounds* have $x1 < x2$ by *auto*
 ultimately have $L = x1$ by *auto*
 thus *?thesis* using $\langle c \longrightarrow L \rangle$ by *auto*
 qed

 end

 end