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theory Problem-3
  imports HOL-Analysis.Analysis
begin

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0.1 Problem 3

Let's assume that a positive integer n has no divisor d that satisfies $\sqrt{n} \leq d \leq \sqrt[3]{n^2}$. Prove that n has a prime divisor $p > \sqrt[3]{n^2}$.

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theorem problem3:
  fixes n :: nat
  assumes [iff]: n ≠ 0
  assumes divrange:  $\bigwedge d :: \text{nat}. \text{sqrt } n \leq d \implies d \leq n \text{ powr } (2/3) \implies \neg d \text{ dvd } n$ 
  obtains p where prime p and p > n powr (2/3)
proof -
  have forbidden-range:  $\neg d \text{ dvd } n$  if  $n \text{ powr } (1/3) \leq d$  and  $d \leq n \text{ powr } (2/3)$  for  $d :: \text{nat}$ 
  proof
    assume d dvd n
    from that consider
      (low)  $n \text{ powr } (1/3) \leq d \leq \text{sqrt } n$  |
      (high)  $\text{sqrt } n \leq d \leq n \text{ powr } (2/3)$ 
    by fastforce
    then show False
  proof cases
    case low
      from ⟨d dvd n⟩ have mirror-divisor:  $(n \text{ div } d) \text{ dvd } n$  by auto

      have  $n/d \leq n / n \text{ powr } (1/3)$ 
        using low by (simp add: frac-le)
      also have ... =  $n \text{ powr } 1 / n \text{ powr } (1/3)$  by auto
      also have ... =  $n \text{ powr } (2/3)$  by (simp del: powr-one flip: powr-diff)
      finally have  $n/d \leq n \text{ powr } (2/3)$ .
      moreover from ⟨d dvd n⟩ have  $n/d = n \text{ div } d$  by auto
      ultimately have upper-bound:  $n \text{ div } d \leq n \text{ powr } (2/3)$  by auto

      from ⟨d dvd n⟩ have d ≠ 0
        by (meson ⟨n ≠ 0⟩ dvd-0-left)
      hence  $n/d \geq n / \text{sqrt } n$ 
        using low by (simp add: frac-le)
      also have  $n / \text{sqrt } n = \text{sqrt } n$ 
        using real-div-sqrt ⟨n ≠ 0⟩ by auto
      finally have  $n/d \geq \text{sqrt } n$ .
      hence lower-bound:  $n \text{ div } d \geq \text{sqrt } n$  using ⟨ $n/d = n \text{ div } d$ ⟩ by auto

      show False using divrange [of n div d] mirror-divisor
        and lower-bound upper-bound by auto
    case high
      then show False using divrange ⟨d dvd n⟩ by auto
  qed
qed

have n > 1
proof -
  {
    assume n = 1
    with divrange [of 1] have  $\neg 1 \text{ dvd } 1$  by auto
    moreover have 1 dvd (1::nat) by auto
    ultimately have False by contradiction
  }
  thus n > 1 using ⟨n ≠ 0⟩
    by fastforce
qed

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let ?smallldivs = {d. d dvd n ∧ d < n powr (1/3)}
have finite ?smallldivs using finite-divisors-nat by fastforce
moreover have ?smallldivs ≠ {} proof -
  have 1 ∈ ?smallldivs using ⟨n > 1⟩ by auto
  thus ?thesis by auto
qed

moreover define a where a = Max ?smallldivs
ultimately have a ∈ ?smallldivs using Max-in by auto
hence a < n powr (1/3) and a dvd n by auto
hence a ≠ 0 using ⟨n ≠ 0⟩ by algebra
have ∧d. d dvd n ⇒ d > a ⇒ d ≥ n powr (1/3)
  using Max-ge ⟨finite ?smallldivs⟩ ⟨?smallldivs ≠ {}⟩ a-def
  by (metis (no-types, lifting) mem-Collect-eq not-le)
hence div-above-a: ∧d. d dvd n ⇒ d > a ⇒ d > n powr (2/3)
  using forbidden-range
  by force

note ⟨a < n powr (1/3)⟩
also have n powr (1/3) < n powr 1 using ⟨n > 1⟩ by (intro powr-less-mono) auto
finally have a < n by auto
hence n div a > 1
  using ⟨a dvd n⟩ by fastforce
then obtain p where prime p and p dvd (n div a)
  by (metis less-irrefl prime-factor-nat)
hence p*a dvd n using ⟨a dvd n⟩ and ⟨n div a > 1⟩
  by (metis div-by-0 dvd-div-iff-mult gr-implies-not-zero)
with div-above-a [of p*a] have p*a > n powr (2/3)
  using ⟨prime p⟩ and prime-nat-iff by fastforce
moreover have a * n powr (1/3) < n powr (1/3) * n powr (1/3)
  using ⟨a < n powr (1/3)⟩ by auto
moreover have ... = n powr (2/3) by (simp flip: powr-add)
ultimately have p*a > a*n powr (1/3) by simp
hence p > n powr (1/3) using ⟨a ≠ 0⟩ by simp
hence p > n powr (2/3) using forbidden-range [of p] and ⟨p * a dvd n⟩ by force
moreover note ⟨prime p⟩
ultimately show ?thesis using that [of p] by auto
qed
end

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