

```

theory Problem-1
  imports Complex-Main
begin

```

0.1 Problem 1

Solve the equation in the integers:

theorem *problem1*:

```

  fixes  $x\ y :: \text{int}$ 
  assumes  $x \neq 0$  and  $y \neq 0$ 
  shows  $1 / x^2 + 1 / (x*y) + 1 / y^2 = 1$ 
     $\longleftrightarrow x = 1 \wedge y = -1 \vee x = -1 \wedge y = 1$ 
    (is ?eqn  $\longleftrightarrow$  ?sols)

```

proof

— Unfortunately, removing the conversions between int and real takes a few lines

let $?x = \text{real-of-int } x$ and $?y = \text{real-of-int } y$

assume ?eqn

then have $1 / ?x^2 + 1 / (?x * ?y) + 1 / ?y^2 = 1$ by auto

hence $?x^2 * ?y^2 / ?x^2 + ?x^2 * ?y^2 / (?x * ?y) + ?x^2 * ?y^2 / ?y^2 = ?x^2 * ?y^2$

by algebra

hence $?x^2 + ?x * ?y + ?y^2 = ?x^2 * ?y^2$ using $\langle x \neq 0 \rangle \langle y \neq 0 \rangle$

by (simp add: power2-eq-square)

hence inteq: $x^2 + x*y + y^2 = x^2 * y^2$

using of-int-eq-iff by fastforce

define g where $g = \text{gcd } x\ y$

then have $g \neq 0$ and $g > 0$ using $\langle x \neq 0 \rangle \langle y \neq 0 \rangle$ by auto

define $x'\ y'$ where $x' = x \text{ div } g$ and $y' = y \text{ div } g$

then have $x' * g = x$ and $y' * g = y$ using $g\text{-def}$ by auto

from inteq and this have $g^2 * (x'^2 + x' * y' + y'^2) = x'^2 * y'^2 * g^4$

by algebra

hence reduced: $x'^2 + x' * y' + y'^2 = x'^2 * y'^2 * g^2$ using $\langle g \neq 0 \rangle$ by algebra

hence $x' \text{ dvd } y'^2$ and $y' \text{ dvd } x'^2$

by algebra+

moreover have coprime $x' (y'^2)$ coprime $(x'^2) y'$

unfolding $x'\text{-def } y'\text{-def } g\text{-def}$

using *assms div-gcd-coprime* by auto

ultimately have *is-unit* x' *is-unit* y'

unfolding *coprime-def* by auto

hence *abs1*: $|x'| = 1 \wedge |y'| = 1$ using *assms* by auto

then consider (*same-sign*) $x' = y' \mid$ (*diff-sign*) $x' = -y'$ by fastforce

thus ?sols

proof *cases*

case *same-sign*

then have $x' * y' = 1$

using *abs1* and *zmult-eq-1-iff* by fastforce

hence $g^2 = 3$

using *abs1 same-sign* and *reduced* by algebra

hence $1^2 < g^2$ and $g^2 < 2^2$ by auto

hence $1 < g$ and $g < 2$

using $\langle g > 0 \rangle$ and *power2-less-imp-less* by fastforce+

hence *False* by auto

thus ?sols by auto

next

case *diff-sign*

then have $x' * y' = -1$

using *abs1*

by (*smt mult-cancel-left2 mult-cancel-right2*)

hence $g^2 = 1$

using *abs1 diff-sign* and *reduced* by algebra

hence $g = 1$ using $\langle g > 0 \rangle$

```

      by (smt power2-eq-1-iff)
    hence  $x = x'$  and  $y = y'$ 
      unfolding  $x'$ -def and  $y'$ -def by auto
      thus ?sols using abs1 and diff-sign by auto
    qed
  next
    assume ?sols
    then show ?eqn by auto
  qed
end

```