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theory Warmup-Problem-D
imports
  Complex-Main
  Common.Future-Library
  HOL-Analysis.Analysis
begin

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0.1 Warmup problem D

There is a set of n points on a plane with the property that, in each triplet of points, there's a pair with distance at most 1. Prove that the set can be covered with two circles of radius 1.

There's nothing special about the case of points on a plane, the theorem can be proved without additional difficulties for any metric space:

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theorem warmup4-generic:
  fixes  $S :: 'a::metric-space\ set$ 
  assumes finite S
  assumes property:  $\bigwedge T. T \subseteq S \wedge \text{card } T = 3 \implies \exists p \in T. \exists q \in T. p \neq q \wedge \text{dist } p\ q \leq 1$ 
  obtains  $O_1\ O_2$  where  $S \subseteq \text{cball } O_1\ 1 \cup \text{cball } O_2\ 1$ 
proof
  let  $?pairs = S \times S$ 
  let  $?dist = \lambda(a, b). \text{dist } a\ b$ 
  define widest-pair where widest-pair = arg-max-on ?dist ?pairs
  let  $?O_1 = (\text{fst } \text{widest-pair})$ 
  let  $?O_2 = (\text{snd } \text{widest-pair})$ 
  show  $S \subseteq \text{cball } ?O_1\ 1 \cup \text{cball } ?O_2\ 1$ 
proof
  fix  $x$ 
  assume  $x \in S$ 

  from  $\langle \text{finite } S \rangle$  and  $\langle x \in S \rangle$ 
  have finite ?pairs and  $?pairs \neq \{\}$  by auto
  hence  $O \text{in } S$ :  $\text{widest-pair} \in ?pairs$ 
    unfolding widest-pair-def by  $(\text{simp add: arg-max-if-finite})$ 

  have  $\forall (P, Q) \in ?pairs. \text{dist } ?O_1\ ?O_2 \geq \text{dist } P\ Q$ 
    unfolding widest-pair-def
    using  $\langle \text{finite } ?pairs \rangle$  and  $\langle ?pairs \neq \{\} \rangle$ 
    by  $(\text{metis } (\text{mono-tags, lifting}) \text{ arg-max-greatest prod.case-eq-if})$ 
  hence greatest:  $\text{dist } P\ Q \leq \text{dist } ?O_1\ ?O_2$  if  $P \in S$  and  $Q \in S$  for  $P\ Q$ 
    using that by blast

  let  $?T = \{?O_1, ?O_2, x\}$ 
  have  $T \text{in } S$ :  $?T \subseteq S$  using  $O \text{in } S$  and  $\langle x \in S \rangle$  by auto

  have  $\text{card } ?T = 3$  if  $?O_1 \neq ?O_2$  and  $x \notin \{?O_1, ?O_2\}$  using that by auto
  then consider
     $(\text{primary})\ \text{card } ?T = 3 \mid$ 
     $(\text{limit})\ x \in \{?O_1, ?O_2\} \mid$ 
     $(\text{degenerate})\ ?O_1 = ?O_2$  by blast
  thus  $x \in \text{cball } ?O_1\ 1 \cup \text{cball } ?O_2\ 1$ 
proof cases
  case primary
  obtain  $p$  and  $q$  where  $p \neq q$  and  $\text{dist } p\ q \leq 1$  and  $p \in ?T$  and  $q \in ?T$ 
    using property [of ?T] and  $\langle \text{card } ?T = 3 \rangle\ T \text{in } S$ 
    by auto
  then have
     $\text{dist } ?O_1\ ?O_2 \leq 1 \vee \text{dist } ?O_1\ x \leq 1 \vee \text{dist } ?O_2\ x \leq 1$ 
    by  $(\text{metis } \text{dist-commute insertE singletonD})$ 
  thus  $x \in \text{cball } ?O_1\ 1 \cup \text{cball } ?O_2\ 1$ 
    using greatest and  $T \text{in } S$ 

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      by fastforce
next
  case limit
  then have  $\text{dist } x \text{ ?}O_1 = 0 \vee \text{dist } x \text{ ?}O_2 = 0$  by auto
  thus ?thesis by auto
next
  case degenerate
  with greatest and TinS have  $\text{dist ?}O_1 x = 0$  by auto
  thus ?thesis by auto
qed
qed
qed

```

Let's make sure that the particular case of points on a plane also works out:

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corollary warmup4:
  fixes  $S :: (\text{real} ^ 2)$  set
  assumes finite S
  assumes property:  $\bigwedge T. T \subseteq S \wedge \text{card } T = 3 \implies \exists p \in T. \exists q \in T. p \neq q \wedge \text{dist } p \ q \leq 1$ 
  obtains  $O_1 \ O_2$  where  $S \subseteq \text{cball } O_1 \ 1 \cup \text{cball } O_2 \ 1$ 
  using warmup4-generic and assms by auto
end

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