theory *Problem-1*
imports *Complex-Main*
begin

0.1 Problem 1

Solve the equation in the integers:

theorem *problem1*:
fixes $x\ y :: int$
assumes $x \neq 0$ **and** $y \neq 0$
shows $1\ /\ x^2 + 1\ /\ (x{*}y) + 1\ /\ y^2 = 1$
$\longleftrightarrow x = 1 \wedge y = -1 \vee x = -1 \wedge y = 1$
(**is** $?eqn \longleftrightarrow ?sols$)
proof
— Unfortunately, removing the conversions between int and real takes a few lines
let $?x = \textit{real-of-int}\ x$ **and** $?y = \textit{real-of-int}\ y$
assume $?eqn$
then have $1\ /\ ?x^2 + 1\ /\ (?x{*}?y) + 1\ /\ ?y^2 = 1$ **by** *auto*
hence $?x^2{*}?y^2\ /\ ?x^2 + ?x^2{*}?y^2\ /\ (?x{*}?y) + ?x^2{*}?y^2\ /\ ?y^2 = ?x^2{*}?y^2$
by *algebra*
hence $?x^2 + ?x{*}?y + ?y^2 = ?x^2 * ?y^2$ **using** $\langle x \neq 0\rangle$ $\langle y \neq 0\rangle$
by (*simp add*: *power2-eq-square*)
hence *inteq*: $x^2 + x{*}y + y^2 = x^2 * y^2$
using *of-int-eq-iff* **by** *fastforce*

define g **where** $g = gcd\ x\ y$
then have $g \neq 0$ **and** $g > 0$ **using** $\langle x \neq 0\rangle$ $\langle y \neq 0\rangle$ **by** *auto*
define $x'\ y'$ **where** $x' = x\ div\ g$ **and** $y' = y\ div\ g$
then have $x' * g = x$ **and** $y' * g = y$ **using** *g-def* **by** *auto*
from *inteq* **and** *this* **have** $g^2 * (x'^2 + x' * y' + y'^2) = x'^2 * y'^2 * g\char`^4$
by *algebra*
hence *reduced*: $x'^2 + x' * y' + y'^2 = x'^2 * y'^2 * g^2$ **using** $\langle g \neq 0\rangle$ **by** *algebra*

hence $x'\ dvd\ y'^2$ **and** $y'\ dvd\ x'^2$
by *algebra+*
moreover have *coprime* x' (y'^2) *coprime* (x'^2) y'
unfolding *x′-def y′-def g-def*
using *assms div-gcd-coprime* **by** *auto*
ultimately have *is-unit* x' *is-unit* y'
unfolding *coprime-def* **by** *auto*
hence *abs1*: $|x'| = 1 \wedge |y'| = 1$ **using** *assms* **by** *auto*
then consider (*same-sign*) $x' = y'$ | (*diff-sign*) $x' = -y'$ **by** *fastforce*
thus $?sols$
proof *cases*
case *same-sign*
then have $x' * y' = 1$
using *abs1* **and** *zmult-eq-1-iff* **by** *fastforce*
hence $g^2 = 3$
using *abs1 same-sign* **and** *reduced* **by** *algebra*
hence $1^2 < g^2$ **and** $g^2 < 2^2$ **by** *auto*
hence $1 < g$ **and** $g < 2$
using $\langle g > 0\rangle$ **and** *power2-less-imp-less* **by** *fastforce+*
hence *False* **by** *auto*
thus $?sols$ **by** *auto*
next
case *diff-sign*
then have $x' * y' = -1$
using *abs1*
by (*smt mult-cancel-left2 mult-cancel-right2*)
hence $g^2 = 1$
using *abs1 diff-sign* **and** *reduced* **by** *algebra*
hence $g = 1$ **using** $\langle g > 0\rangle$

      by (*smt power2-eq-1-iff*)
    hence $x = x'$ **and** $y = y'$
      unfolding *x′-def* **and** *y′-def* **by** *auto*
    thus *?sols* **using** *abs1* **and** *diff-sign* **by** *auto*
  qed
next
  assume *?sols*
  then show *?eqn* **by** *auto*
qed

end