theory *Problem-2*
  imports
    HOL−Analysis.Analysis
begin

0.1 Problem 2

Prove that a sequence is bounded, converges, and find the limit.

context
  fixes *a* :: *real*
  assumes *a-bounds*: $0 < a\ a < 1$
begin
fun *c* :: *nat* $\Rightarrow$ *real* **where**
$c\ 0 = a\ /\ 2\ |$
$c\ (Suc\ n) = (a + (c\ n)^2)\ /\ 2$

abbreviation *x1* $\equiv$ $1 - sqrt\ (1 - a)$
abbreviation *x2* $\equiv$ $1 + sqrt\ (1 - a)$

lemma *c-pos*: $0 < c\ n$
  using *a-bounds*
  by (*induction n*, *auto*, *smt zero-less-power*)

lemma *c-bounded*: $c\ n < x1$
proof (*induction n*)
  case *0*
  have $(1 - a/2)^2 = 1 - a + (a/2)^2$
    by (*simp add*: *power2-diff*)
  hence $1 - a < (1 - a/2)^2$ **using** *a-bounds* **by** *auto*
  hence $sqrt\ (1 - a) < 1 - a/2$
    using *a-bounds* **and** *real-less-lsqrt* **by** *auto*
  thus *?case* **by** *auto*
next
  case (*Suc n*)
  then have $(c\ n)^2 < (1 - sqrt\ (1-a))^2$ **using** *c-pos*
    by (*smt power-less-imp-less-base real-sqrt-abs*)
  also have $... = 2 - 2 * sqrt\ (1-a) - a$
    using *a-bounds* **by** (*simp add*: *power2-diff*)
  finally have $(a + (c\ n)^2)/2 < 1 - sqrt\ (1-a)$ **by** *auto*
  then show *?case* **by** *auto*
qed

lemma *c-incseq*: *incseq c*
proof (*rule incseq-SucI*)
  fix *n*
  from *c-bounded* **have** $c\ n < x1$ **by** *auto*
  have $c\ n < x1\ c\ n < x2$
    using *c-bounded*
    by (*smt a-bounds real-sqrt-lt-0-iff*)+
  moreover have $(c\ n)^2 - 2*c\ n + a = (c\ n - x1)*(c\ n - x2)$
    using *a-bounds*
    by (*auto simp add*: *algebra-simps power2-eq-square*)
  ultimately have $(c\ n)^2 - 2*c\ n + a > 0$
    by (*smt nonzero-mult-div-cancel-right zero-le-divide-iff*)
  thus $c\ n \le c\ (Suc\ n)$ **by** *auto*
qed

theorem *problem2*: $c \longrightarrow x1$
proof −
  obtain *L* **where** $c \longrightarrow L$
    using *c-incseq c-bounded incseq-convergent*
    by (*metis less-imp-le*)

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then have $(\lambda n.\ c\ (Suc\ n)) \longrightarrow L$
  using *LIMSEQ-Suc* **by** *blast*
hence $(\lambda n.\ (a + (c\ n)^2)\ /\ 2 * 2) \longrightarrow L*2$
  using *tendsto-mult-right* **by** *fastforce*
hence $(\lambda n.\ a + (c\ n)^2) \longrightarrow L*2$ **by** (*simp del*: *distrib-right-numeral*)
hence $(\lambda n.\ a + (c\ n)^2 - a) \longrightarrow L*2 - a$
  using *tendsto-diff LIMSEQ-const-iff* **by** *blast*
hence $(\lambda n.\ (c\ n)^2) \longrightarrow L*2 - a$
  by *auto*
moreover from $\langle c \longrightarrow L\rangle$
have $(\lambda n.\ (c\ n)^2) \longrightarrow L^2$
  unfolding *power2-eq-square*
  using *tendsto-mult* **by** *blast*
ultimately have $L*2 - a = L^2$
  by (*rule LIMSEQ-unique*)
hence $L^2 - 2*L + a = 0$ **by** *auto*
moreover have $L^2 - 2*L + a = (L - x1)*(L - x2)$
  using *a-bounds*
  by (*auto simp add*: *algebra-simps power2-eq-square*)
ultimately have $L = x1 \lor L = x2$
  by *auto*
moreover from *c-bounded* **and** $\langle c \longrightarrow L\rangle$ **have** $L \leq x1$
  by (*meson LIMSEQ-le-const2 le-less-linear less-imp-triv*)
moreover from *a-bounds* **have** $x1 < x2$ **by** *auto*
ultimately have $L = x1$ **by** *auto*
thus *?thesis* **using** $\langle c \longrightarrow L\rangle$ **by** *auto*
qed

end

end