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theory Problem-2
  imports
    HOL−Number-Theory.Number-Theory
    Common.NT-Facts
begin
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Positive integers p, a and b satisfy the equation $p^2 + a^2 = b^2$. Prove that if p is a prime greater than 3, then a is a multiple of 12 and $2(p + a + 1)$ is a perfect square.

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theorem problem2:
  fixes p a b :: int
  assumes p² + a² = b²
    and p: prime p   p > 3
    and pos: a > 0   b > 0
  shows 12 dvd a
    and ∃k. k² = 2*(p + a + 1)
proof −
  from ‹p² + a² = b²› have *: p * p = (b + a) * (b − a)
    by (simp add: power2-eq-square flip: square-diff-square-factored)
  hence b + a dvd p * p
    by auto
  have b + a ∈ {1, p, p*p}
  proof −
    from ‹b + a dvd p*p› obtain x y where
      b + a = x * y and x dvd p and y dvd p
      using dvd-productE by blast
    with ‹prime p› have |x| = 1 ∨ |x| = p and |y| = 1 ∨ |y| = p
      by (auto simp add: prime-int-iff)
    with pos and ‹b + a = x * y› show b + a ∈ {1, p, p*p}
      by (cases x ≥ 0; cases y ≥ 0; auto; smt zero-less-mult-iff)
  qed
  moreover have b + a ≠ 1 using ‹a > 0› ‹b > 0› by auto
  moreover have b + a ≠ p
  proof
    assume b + a = p
    with * and pos have b − a = p
      by auto
    from ‹b − a = p› and ‹b + a = p› have a = 0 by auto
    thus False using pos by auto
  qed
  ultimately have 1: b + a = p * p by auto
  with * pos have 2: b − a = 1 by auto

  from 1 and 2 have **: 2 * a = p * p − 1 by auto
  moreover have [p * p = 1] (mod 24) using p by (intro pp-mod24)
    — This property is proved in Common.NT-Facts

  ultimately have 24 dvd 2*a
    unfolding cong-def using mod-eq-dvd-iff by fastforce
  thus 12 dvd a
    by auto

  from ** have 2*(p + a + 1) = (p + 1)²
    by (auto simp add: ac-simps power2-sum) (simp add: power2-eq-square)
  thus ∃k. k² = 2*(p + a + 1)
    by auto
qed

end
```