1 Series II

theory *SeriesII*
  imports
    Complex-Main
    HOL−Analysis.Analysis
begin

1.1 Problem 5

Real numbers $M, a_1, a_2, \ldots, a_{10}$ are given. Prove that, if $a_1x_1 + a_2x_2 + \cdots + a_{10}x_{10} \leq M$ for all x_i such that $|x_i| = 1$, then

$$\sqrt{a_1^2 + a_2^2 + \cdots + a_{10}^2} \leq M.$$

lemma *sqr-sum-ineq*:
  list-all $(\lambda x.\ x \geq 0)$ *xs* $\Longrightarrow$ *sum-list* (*map power2 xs*) $\leq$ (*sum-list xs*)^2
  for *xs* :: *real list*
proof (*induction xs*)
  case *Nil*
  then show *?case* **by** *auto*
next
  case (*Cons x xs*)
  note *IH* = ‹*list-all* $(\lambda x.\ x \geq 0)$ *xs* $\Longrightarrow$ *sum-list* (*map power2 xs*) $\leq$ (*sum-list xs*)^2›
  note *nonneg* = ‹*list-all* $(\lambda x.\ x \geq 0)$ (*x # xs*)›
  then have $x \geq 0$ **and** *nonneg'*: *list-all* $(\lambda x.\ x \geq 0)$ *xs* **by** *auto*
  hence *sum-list xs* ≥ 0 **using** *sum-list-nonneg* **unfolding** *list-all-def* **by** *auto*

  have *sum-list* (*map power2* (*x # xs*)) = x^2 + *sum-list* (*map power2 xs*) **by** *auto*
  also have ... $\leq x^2$ + (*sum-list xs*)^2 **using** *IH* **and** *nonneg'* **by** *auto*
  also have ... $\leq x^2$ + 2∗x∗(*sum-list xs*) + (*sum-list xs*)^2
    using ‹$x \geq 0$› **and** ‹*sum-list xs* ≥ 0› **by** *auto*
  also have ... = (x + *sum-list xs*)^2 **by** *algebra*
  also have ... = (*sum-list* (*x # xs*))^2 **by** *auto*
  finally show *sum-list* (*map power2* (*x # xs*)) $\leq$ (*sum-list* (*x # xs*))^2**.**
qed

definition *sgn'* :: *real* ⇒ *real* **where**
sgn' x = (*if* $x \geq 0$ *then 1 else* −*1*)

lemma [*simp*]: *x* ∗ *sgn' x* = |*x*|
  unfolding *sgn'-def* **by** *auto*

lemma [*simp*]: |*sgn' x*| = *1*
  unfolding *sgn'-def* **by** *auto*

theorem *problem5*:
  fixes *M* :: *real* **and** *as* :: *real list*

assumes $*$: $\bigwedge xs.\ \textit{list-all}\ (\lambda x.\ |x| = 1)\ xs \Longrightarrow \textit{sum-list}\ (\textit{map2}\ (*)\ \textit{as}\ xs) \le M$
shows $\textit{sqrt}\ (\textit{sum-list}\ (\textit{map}\ \textit{power2}\ \textit{as})) \le M$
proof $-$
define *xs* **where** $xs = \textit{map}\ \textit{sgn}'\ \textit{as}$
then have $\textit{list-all}\ (\lambda x.\ |x| = 1)\ xs$ **unfolding** *list-all-def* **by** *auto*
with $*$ [*of xs*] **have** *sum-abs-below-M*: $\textit{sum-list}\ (\textit{map}\ \textit{abs}\ \textit{as}) \le M$
unfolding *xs-def* **by** (*auto simp add*: *map2-map-map* [**where** $f{=}id$, *simplified*])
moreover have *sum-abs-nonneg*: $\textit{sum-list}\ (\textit{map}\ \textit{abs}\ \textit{as}) \ge 0$
using *sum-list-abs abs-ge-zero order-trans* **by** *blast*
ultimately have $M \ge 0$ **by** *auto*

have [*simp*]: $\textit{power2} \circ \textit{abs} = (\textit{power2} :: {'a} \Rightarrow ({'a} :: \textit{linordered-idom}))$
by *auto*
have $\textit{list-all}\ (\lambda x.\ x \ge 0)\ (\textit{map}\ \textit{abs}\ \textit{as})$ **unfolding** *list-all-def* **by** *auto*
from *sqr-sum-ineq* [*OF this*]
have $\textit{sum-list}\ (\textit{map}\ \textit{power2}\ \textit{as}) \le (\textit{sum-list}\ (\textit{map}\ \textit{abs}\ \textit{as}))^2$
by *auto*
also have $\ldots \le M^2$ **using** *sum-abs-below-M sum-abs-nonneg* **by** *auto*
finally have $\textit{sum-list}\ (\textit{map}\ \textit{power2}\ \textit{as}) \le M^2$**.**
with $\langle M \ge 0\rangle$ **show** $\textit{sqrt}\ (\textit{sum-list}\ (\textit{map}\ \textit{power2}\ \textit{as})) \le M$
by (*metis abs-of-nonneg real-sqrt-abs real-sqrt-le-iff*)
qed

end