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theory Problem-1
  imports Complex-Main
begin
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Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$xf(x) + f(-x) = 1.$$

```
theorem
  fixes f :: real ⇒ real
  shows (∀x. x * f x + f (−x) = 1)
    ⟷ (∀x. f x = (1 + x) / (x² + 1))
    (is (∀x. ?eq x) ⟷ (∀x. ?def x))
proof
  assume ∀x. ?eq x
  then have ?eq x for x..
  hence f-negx: f (−x) = 1 − x*f x for x by smt
  have f-x: f x = 1 + x*f (−x) for x
    using f-negx[where x=−x] by simp
  have f x = 1 + x − x*x*f x for x
    using f-x[of x]
    by (simp add: f-negx) algebra
  hence f x + x² * f x = 1 + x for x
    unfolding power2-eq-square by smt
  hence (x² + 1)*f x = 1 + x for x
    by (simp add: Rings.ring-distribs(2) add.commute)
  moreover have x² + 1 ≠ 0 for x :: real
    unfolding power2-eq-square by (smt zero-le-square)
  ultimately have f x = (1 + x) / (x² + 1) for x
    apply (intro eq-divide-imp)
    by (auto simp add: ac-simps)
  thus ∀x. ?def x..
next
  assume ∀x. ?def x
  then have [simp]: ?def x for x..
  have [simp]: x*x + 1 ≠ 0 for x :: real
    by (smt zero-le-square)
  show ∀x. ?eq x
    apply (auto simp add: power2-eq-square
        simp flip: add-divide-distrib)
    by algebra
qed

end
```