```
theory Warmup-Problem-D
  imports
    Complex-Main
    Common.Future-Library
    HOL−Analysis.Analysis
begin
```

0.1 Warmup problem D

There is a set of n points on a plane with the property that, in each triplet of points, there's a pair with distance at most 1. Prove that the set can be covered with two circles of radius 1.

There's nothing special about the case of points on a plane, the theorem can be proved without additional difficulties for any metric space:

```
theorem warmup4-generic:
  fixes S :: 'a::metric-space set
  assumes finite S
  assumes property: ⋀T. T ⊆ S ∧ card T = 3 ⟹ ∃p∈T. ∃q∈T. p ≠ q ∧ dist p q ≤ 1
  obtains O₁ O₂ where S ⊆ cball O₁ 1 ∪ cball O₂ 1
proof
  let ?pairs = S × S
  let ?dist = λ(a, b). dist a b
  define widest-pair where widest-pair = arg-max-on ?dist ?pairs
  let ?O₁ = (fst widest-pair)
  let ?O₂ = (snd widest-pair)
  show S ⊆ cball ?O₁ 1 ∪ cball ?O₂ 1
  proof
    fix x
    assume x ∈ S

    from ‹finite S› and ‹x ∈ S›
    have finite ?pairs and ?pairs ≠ {} by auto
    hence OinS: widest-pair ∈ ?pairs
      unfolding widest-pair-def by (simp add: arg-max-if-finite)

    have ∀(P,Q)∈?pairs. dist ?O₁ ?O₂ ≥ dist P Q
      unfolding widest-pair-def
      using ‹finite ?pairs› and ‹?pairs ≠ {}›
      by (metis (mono-tags, lifting) arg-max-greatest prod.case-eq-if)
    hence greatest: dist P Q ≤ dist ?O₁ ?O₂ if P ∈ S and Q ∈ S for P Q
      using that by blast

    let ?T = {?O₁, ?O₂, x}
    have TinS: ?T ⊆ S using OinS and ‹x ∈ S› by auto

    have card ?T = 3 if ?O₁ ≠ ?O₂ and x ∉ {?O₁, ?O₂} using that by auto
    then consider
      (primary) card ?T = 3 |
      (limit) x ∈ {?O₁, ?O₂} |
      (degenerate) ?O₁ = ?O₂ by blast
    thus x ∈ cball ?O₁ 1 ∪ cball ?O₂ 1
    proof cases
      case primary
      obtain p and q where p ≠ q and dist p q ≤ 1 and p ∈ ?T and q ∈ ?T
        using property [of ?T] and ‹card ?T = 3› TinS
        by auto
      then have
        dist ?O₁ ?O₂ ≤ 1 ∨ dist ?O₁ x ≤ 1 ∨ dist ?O₂ x ≤ 1
        by (metis dist-commute insertE singletonD)
      thus x ∈ cball ?O₁ 1 ∪ cball ?O₂ 1
        using greatest and TinS
```

by *fastforce*
next
case *limit*
then have $dist\ x\ ?O_1 = 0 \lor dist\ x\ ?O_2 = 0$ **by** *auto*
thus *?thesis* **by** *auto*
next
case *degenerate*
with *greatest* **and** *TinS* **have** $dist\ ?O_1\ x = 0$ **by** *auto*
thus *?thesis* **by** *auto*
qed
qed
qed

Let's make sure that the particular case of points on a plane also works out:

corollary *warmup4*:
fixes $S :: (real \frown 2)\ set$
assumes *finite S*
assumes *property*: $\bigwedge T.\ T \subseteq S \land card\ T = 3 \Longrightarrow \exists p \in T.\ \exists q \in T.\ p \neq q \land dist\ p\ q \leq 1$
obtains $O_1\ O_2$ **where** $S \subseteq cball\ O_1\ 1 \cup cball\ O_2\ 1$
using *warmup4-generic* **and** *assms* **by** *auto*

end