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theory Problem-1
  imports Complex-Main
begin

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0.1 Problem 1

Solve the equation in the integers:

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theorem problem1:
  fixes x y :: int
  assumes x ≠ 0 and y ≠ 0
  shows 1 / x2 + 1 / (x*y) + 1 / y2 = 1
    ⟷ x = 1 ∧ y = -1 ∨ x = -1 ∧ y = 1
    (is ?eqn ⟷ ?sols)
proof
  — Unfortunately, removing the conversions between int and real takes a few lines
  let ?x = real-of-int x and ?y = real-of-int y
  assume ?eqn
  then have 1 / ?x2 + 1 / (?x*?y) + 1 / ?y2 = 1 by auto
  hence ?x2*?y2 / ?x2 + ?x2*?y2 / (?x*?y) + ?x2*?y2 / ?y2 = ?x2*?y2
    by algebra
  hence ?x2 + ?x*?y + ?y2 = ?x2 * ?y2 using ⟨x ≠ 0⟩ ⟨y ≠ 0⟩
    by (simp add: power2-eq-square)
  hence inteq: x2 + x*y + y2 = x2 * y2
    using of-int-eq-iff by fastforce

  define g where g = gcd x y
  then have g ≠ 0 and g > 0 using ⟨x ≠ 0⟩ ⟨y ≠ 0⟩ by auto
  define x' y' where x' = x div g and y' = y div g
  then have x' * g = x and y' * g = y using g-def by auto
  from inteq and this have g2 * (x'2 + x' * y' + y'2) = x'2 * y'2 * g4
    by algebra
  hence reduced: x'2 + x' * y' + y'2 = x'2 * y'2 * g2 using ⟨g ≠ 0⟩ by algebra

  hence x' dvd y'2 and y' dvd x'2
    by algebra+
  moreover have coprime x' (y'2) coprime (x'2) y'
    unfolding x'-def y'-def g-def
    using assms div-gcd-coprime by auto
  ultimately have is-unit x' is-unit y'
    unfolding coprime-def by auto
  hence abs1: |x'| = 1 ∧ |y'| = 1 using assms by auto
  then consider (same-sign) x' = y' | (diff-sign) x' = -y' by fastforce
  thus ?sols
proof cases
  case same-sign
  then have x' * y' = 1
    using abs1 and zmultip-eq-1-iff by fastforce
  hence g2 = 3
    using abs1 same-sign and reduced by algebra
  hence 12 < g2 and g2 < 22 by auto
  hence 1 < g and g < 2
    using ⟨g > 0⟩ and power2-less-imp-less by fastforce+
  hence False by auto
  thus ?sols by auto
next
  case diff-sign
  then have x' * y' = -1
    using abs1
    by (smt mult-cancel-left2 mult-cancel-right2)
  hence g2 = 1
    using abs1 diff-sign and reduced by algebra
  hence g = 1 using ⟨g > 0⟩

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      by (smt power2-eq-1-iff)
    hence  $x = x'$  and  $y = y'$ 
      unfolding  $x'$ -def and  $y'$ -def by auto
      thus ?sols using abs1 and diff-sign by auto
    qed
  next
    assume ?sols
    then show ?eqn by auto
  qed
end

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