theory *Problem-2*
  imports
    HOL−Analysis.Analysis
begin

0.1 Problem 2

Prove that a sequence is bounded, converges, and find the limit.

```
context
  fixes a :: real
  assumes a-bounds: 0 < a a < 1
begin
fun c :: nat ⇒ real where
c 0 = a / 2 |
c (Suc n) = (a + (c n)^2) / 2

abbreviation x1 ≡ 1 − sqrt (1 − a)
abbreviation x2 ≡ 1 + sqrt (1 − a)

lemma c-pos: 0 < c n
  using a-bounds
  by (induction n, auto, smt zero-less-power)

lemma c-bounded: c n < x1
proof (induction n)
  case 0
  have (1 − a/2)^2 = 1 − a + (a/2)^2
    by (simp add: power2-diff)
  hence 1 − a < (1 − a/2)^2 using a-bounds by auto
  hence sqrt (1 − a) < 1 − a/2
    using a-bounds and real-less-lsqrt by auto
  thus ?case by auto
next
  case (Suc n)
  then have (c n)^2 < (1 − sqrt (1−a))^2 using c-pos
    by (smt power-less-imp-less-base real-sqrt-abs)
  also have ... = 2 − 2 ∗ sqrt (1−a) − a
    using a-bounds by (simp add: power2-diff)
  finally have (a + (c n)^2)/2 < 1 − sqrt (1−a) by auto
  then show ?case by auto
qed

lemma c-incseq: incseq c
proof (rule incseq-SucI)
  fix n
  from c-bounded have c n < x1 by auto
  have c n < x1 c n < x2
    using c-bounded
    by (smt a-bounds real-sqrt-lt-0-iff)+
  moreover have (c n)^2 − 2∗c n + a = (c n − x1)∗(c n − x2)
    using a-bounds
    by (auto simp add: algebra-simps power2-eq-square)
  ultimately have (c n)^2 − 2∗c n + a > 0
    by (smt nonzero-mult-div-cancel-right zero-le-divide-iff)
  thus c n ≤ c (Suc n) by auto
qed

theorem problem2: c ⟶ x1
proof −
  obtain L where c ⟶ L
    using c-incseq c-bounded incseq-convergent
    by (metis less-imp-le)
```

then have $(\lambda n.\ c\ (Suc\ n)) \longrightarrow L$
  using *LIMSEQ-Suc* **by** *blast*
hence $(\lambda n.\ (a + (c\ n)^2)\ /\ 2 * 2) \longrightarrow L{*}2$
  using *tendsto-mult-right* **by** *fastforce*
hence $(\lambda n.\ a + (c\ n)^2) \longrightarrow L{*}2$ **by** (*simp del*: *distrib-right-numeral*)
hence $(\lambda n.\ a + (c\ n)^2 - a) \longrightarrow L{*}2 - a$
  using *tendsto-diff LIMSEQ-const-iff* **by** *blast*
hence $(\lambda n.\ (c\ n)^2) \longrightarrow L{*}2 - a$
  by *auto*
moreover from ‹$c \longrightarrow L$›
have $(\lambda n.\ (c\ n)^2) \longrightarrow L^2$
  unfolding *power2-eq-square*
  using *tendsto-mult* **by** *blast*
ultimately have $L{*}2 - a = L^2$
  by (*rule LIMSEQ-unique*)
hence $L^2 - 2{*}L + a = 0$ **by** *auto*
moreover have $L^2 - 2{*}L + a = (L - x1){*}(L - x2)$
  using *a-bounds*
  by (*auto simp add*: *algebra-simps power2-eq-square*)
ultimately have $L = x1 \vee L = x2$
  by *auto*
moreover from *c-bounded* **and** ‹$c \longrightarrow L$› **have** $L \le x1$
  by (*meson LIMSEQ-le-const2 le-less-linear less-imp-triv*)
moreover from *a-bounds* **have** $x1 < x2$ **by** *auto*
ultimately have $L = x1$ **by** *auto*
thus *?thesis* **using** ‹$c \longrightarrow L$› **by** *auto*
qed

end

end