ENGR222 Assignment 2

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1. The following questions are concerned with the function

$$f(x,y) = -2x^3 + 3x^2y + 2y^3 - 9y + 5$$

   (a) Determine the first order partial derivative of $f(x,y)$

$$f_x = -6x^2 + 6xy$$
$$f_y = 6y^2 + 3x^2 - 9$$

   (b) Determine the second order partial derivatives of $f(x,y)$

$$f_{xx} = -12x + 6y$$
$$f_{yy} = 12y$$
$$f_{xy} = 6x$$

   (c) Find all of the critical points of $f(x,y)$

By inspection we know $(x = y = -1, 1)$

Let $x = 0$

$$f_x = 0$$
$$f_y = 6y^2 - 9 = 0$$
$$\therefore y = \sqrt{\frac{9}{6}} = \sqrt{\frac{3}{2}}$$

Let $y = 0$

$$f_x = -6x^2 = 0$$
$$f_y = 3x^2 - 9$$

No solution for x when $y = 0$

The critical points are $\rightarrow [(1,1),\ (-1,-1),\ (0, \sqrt{\frac{3}{2}})]$

   (d) Classify the critical point $(0, \sqrt{\frac{3}{2}})$

$$D = f_{xx}(0, \sqrt{\frac{3}{2}}) \times f_{yy}(0, \sqrt{\frac{3}{2}}) - f_{xy}^2(0, \sqrt{\frac{3}{2}})$$

$$f_{xx}(0, \sqrt{\frac{3}{2}}) = 3\sqrt{6}$$
$$f_{yy}(0, \sqrt{\frac{3}{2}}) = 6\sqrt{6}$$
$$f_{xy}(0, \sqrt{\frac{3}{2}}) = 0$$

$$D = 3\sqrt{6} \times 6\sqrt{6} - 0^2 = 108$$

Since $D > 0$ and $f_{xx} > 0$ we know this critical point is a local minimum

2. Quick questions

   (a) Determine the directional derivative of $f(x,y,z) = e^x \cdot \cos(y) \cdot (1-z)^2$ in direction $\bar{u} = (0.36, 0.48, 0.8)$ from the origin:

$$D_u f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)\bar{u}_1 + f_y(x_0,y_0,z_0)\bar{u}_2 + f_z(x_0,y_0,z_0)\bar{u}_3$$

$$f_x = e^x cos(y)(1-z)^2$$
$$f_y = -e^x sin(y)(1-z)^2$$
$$f_z = e^x cos(y)(2z-2)$$

$$f_x(0,0,0) = 1 \cdot 1 \cdot 1 = 1$$
$$f_y(0,0,0) = -1 \cdot 0 \cdot 1 = 0$$
$$f_z(0,0,0) = 1 \cdot 1 \cdot -2 = -2$$

$$D_u f(0,0,0) = 0.36 - 1.6 = -1.24$$

   (b) Determine the local linear approximation of $f(x,y,z) = (1+x)(1-y^2)(1-z)^2$ at the point $(1,2,3)$

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

$$f_x = (1-y^2)(1-z)^2$$
$$f_y = (1+x)(-2y)(1-z)^2$$
$$f_z = 2(1+x)(1-y^2)(z-1)$$

$$f(1,2,3) = (1+1)(1-2^2)(1-3)^2 = -24$$
$$f_x(1,2,3) = (1-2^2)(1-3)^2 = -12$$
$$f_y(1,2,3) = (1+1)(-2(2))(1-3)^2 = -32$$
$$f_z(1,2,3) = 2(1+1)(1-2^2)(3-1) = -24$$

$$L(1,2,3) = -24 - 12(x-1) - 32(y-2) - 24(z-3)$$
$$L(1,2,3) = 124 - 12x - 32y - 24z$$

   (c) Determine the 2nd degree Taylor polynomial of $f(x,y) = e^{-x^2}e^{-y^2}$ as the point (1,1)

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$
$$p_2(x,y) = L(x,y) + \frac{1}{2}\left[f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2\right]$$

$$f_x = (-2x)e^{-x^2}e^{-y^2}$$
$$f_y = (-2y)e^{-x^2}e^{-y^2}$$

$$f_{xx} = (4x^2-2)e^{-x^2-y^2}$$
$$f_{yy} = (4y^2-2)e^{-x^2-y^2}$$
$$f_{xy} = (4xy)e^{-x^2}e^{-y^2}$$

$$L(1,1) = e^{-2} - 2e^{-2}(x-1) - 2e^{-2}(y-1) = e^{-2}(5-2x-2y)$$
$$p_2(1,1) = e^{-2}(5-2x-2y) + \frac{1}{2}\left[2e^{-2}(x-2)^2 + 2e^{-2}(y-2)^2 + 8e^{-2}(x-1)(y-1)\right]$$
$$= e^{-2}(x^2+y^2-8x-8y+4xy+11)$$

   (d) Determine the gradient of $f(x,y) = x^3 + y^3 - 4x - 2y$ along the curve $(x(t), y(t)) = (t^3 - 2t, t^2)$ when $t = 1$

$$\bigtriangledown f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$

$$f_x = 3x^2 - 4$$
$$f_y = 3y^2 - 2$$
$$\therefore \bigtriangledown f(x,y) = (3x^2-4)\mathbf{i} + (3y^2-2)\mathbf{j}$$

$$(x(1), y(1)) = (-1,1)$$
$$\bigtriangledown f(-1,1) = f_x(-1,1)\mathbf{i} + f_y(-1,1)\mathbf{j}$$
$$= -1\mathbf{i} + 1\mathbf{j}$$

   (e) Determine the tangent plane to the surface $z = x^2+xy-y^4$ at the point $(x,y) = (2,1)$

$$F(x,y,z) = z - x^2 - xy + y^4 = 0$$

Find z at the point $(x,y) = (2,1)$

$$z = 2^2 + 2 - 1^4 = 5$$

$$\bigtriangledown F(x,y,z) = (-2x-y)\mathbf{i} + (4y^3-x)\mathbf{j} + \mathbf{k}$$
$$\bigtriangledown F(2,1,5) = -5\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

$$\text{TangentPlane} = -5(x-2) + 2(y-1) + (z-5)$$
$$z = 5x - 2y - 3$$

3. Double Integrals

   (a) Determine the integral of $f(x,y) = e^{-x}\cos(y)$ over the rectangular region $R = (x,y) : x \in [0,2], y \in [\frac{-\pi}{2}, \frac{\pi}{2}]$'

$$\int_{-\pi/2}^{\pi/2}\int_0^2 e^{-x}\cos(y)\,dx\,dy$$

$$= \int_{-\pi/2}^{\pi/2}\cos(y)\int_0^2 e^{-x}\,dx\,dy$$
$$= \int_{-\pi/2}^{\pi/2}\cos(y)\Big| - e^{-x}\Big|_{x=0}^{x=2} dy$$
$$= \int_{-\pi/2}^{\pi/2}\cos(y)(-e^{-2} - -e^0)\,dy$$

$$= \int_{-\pi/2}^{\pi/2}\cos(y)(-e^{-2}+1)\,dy$$
$$= (-e^{-2}+1)\Big|\sin(y)\Big|_{y=-\pi/2}^{y=\pi/2}$$
$$= (-e^{-2}+1)(\sin(\pi/2) - \sin(-\pi/2))$$

$$= -2e^{-2} + 2$$

   (b) Determine the integral of $f(x,y) = \sin(x+y)$ over the triangular region for which $x \geq 0,\ y \geq 0$ and $x + y \leq \pi$

$$\int_0^\pi\int_0^{\pi-x}\sin(x+y)\,dy\,dx$$

$$= \int_0^\pi \Big| - \cos(x+y)\Big|_0^{\pi-x} dx$$
$$= \int_0^\pi - \cos(\pi) + cos(x)\,dx$$
$$= \int_0^\pi 1 + \cos(x)\,dx$$

$$= \Big|x + \sin(x)\Big|_0^\pi$$
$$= (\pi + \sin(\pi)) - (0 + \sin(0))$$
$$= \pi$$

   (c) Determine the area of the region $R = \{(x,y) : e^{y/3} \leq x \leq 10 + \sin(y), y \in [0,5]\}$

$$\int_0^5\int_{e^{y/3}}^{10+\sin(y)} dx\,dy$$

$$\int_0^5 \Big|x\Big|_{e^{y/3}}^{10+\sin(y)} dy$$
$$\int_0^5 10 + \sin(y) - e^{y/3}\,dy$$

$$= \Big|10y - \cos(y) - 3e^{y/3}\Big|_0^5$$
$$= 50 - \cos(5) - 3e^{5/3} + \cos(0) + 3e^0$$
$$= 37.832$$

   (d) Determine the average of $f(x,y) = 3y - 2x$ over the region $R = \{(x,y) : 0 \leq y \leq 4 - x^2, x \in [-2,2]\}$

$$\mu = \frac{1}{|R|}\iint_R f(x,y)dA$$

$$|R| = \int_{-2}^2\int_0^{4-x^2} dy\,dx$$
$$= \int_{-2}^2 |y|_0^{4-x^2} dx$$
$$= \int_{-2}^2 4 - x^2\,dx$$

$$= \Big|4x - \frac{x^3}{3}\Big|_{-2}^2$$
$$= 8 - \frac{2^3}{3} + 8 - \frac{2^3}{3}$$
$$= \frac{32}{3}$$

$$\int_{-2}^2\int_0^{4-x^2} 3y - 2x\,dy\,dx$$
$$= \int_{-2}^2 \Big|\frac{3y^2}{2} - 2xy\Big|_0^{4-x^2} dx$$
$$= \int_{-2}^2 \frac{3(4-x^2)^2}{2} - 2x(4-x^2)\,dx$$
$$= \int_{-2}^2 \frac{3x^4}{2} + 2x^3 - 12x^2 - 8x + 24\,dx$$

$$= \int_{-2}^2 \frac{3x^5}{10} + \frac{x^4}{2} - 4x^3 - 4x^2 + 24x$$
$$= \Big|\frac{3x^5}{10} + \frac{x^4}{2} - 4x^3 - 4x^2 + 24x\Big|_{-2}^2$$
$$= (\frac{3(2)^5}{10} + \frac{(2)^4}{2} - 4(2)^3 - 4(2)^2 + 24(2)) - (\frac{3(-2)^5}{10} + \frac{(-2)^4}{2} - 4(-2)^3 - 4(-2)^2 + 24(-2))$$
$$= \frac{256}{5}$$

$$\mu = \frac{\frac{256}{5}}{\frac{32}{3}}$$
$$= \frac{24}{5} = 4.8$$

   (e) Determine the surface area of the surface described by $z = \sqrt{9-x^2}$ over the region $R = \{(x,y) : 0 \leq y \leq x, x \in [0,3]\}$

$$\text{Surface Area } = \iint_R \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}dA$$

$$\text{Integrand} = \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}$$

$$f_x = -\frac{x}{\sqrt{9-x^2}}$$
$$f_y = 0$$

$$\text{Integrand} = \sqrt{\left(-\frac{x}{\sqrt{9-x^2}}\right)^2 + 1}$$

$$= \sqrt{\frac{x^2}{9-x^2} + 1} = \frac{3}{\sqrt{9-x^2}}$$

$$\text{Surface Area } = \int_0^3\int_0^x \frac{3}{\sqrt{9-x^2}}\,dy\,dx$$
$$= \int_0^3 \Big|\left(\frac{3}{\sqrt{9-x^2}}\right)y\Big|_0^x dx$$
$$= \int_0^3 \left(\frac{3}{\sqrt{9-x^2}}\right)x\,dx$$

$$= \Big| - 3\sqrt{9-x^2}\Big|_0^3$$
$$= -3\sqrt{0} + 3\sqrt{9}$$
$$= 9$$

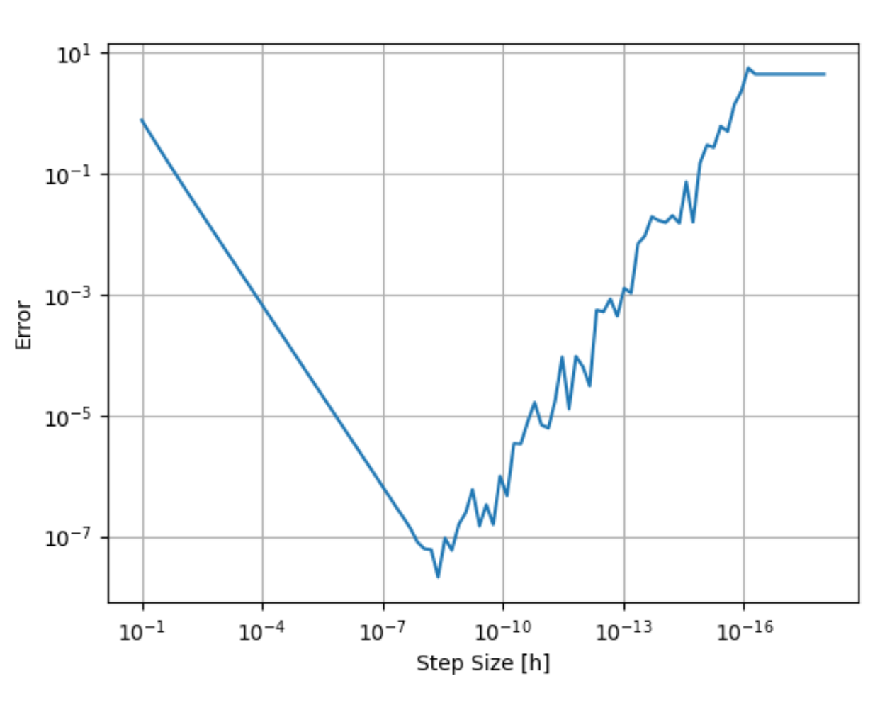
4. Lab Questions

   (a) investigate the numerical approximation of a derivative

      i. Estimate the numerical derivative of $f(x) = e^{\cos(\pi x^2)}$ at the point $x = 1/\sqrt{2}$
Estimate the derivative via:

$$\frac{df}{dx} \approx \frac{f(x+h) - f(x)}{h}$$

for a range of step sizes h plot the error verses the step size:

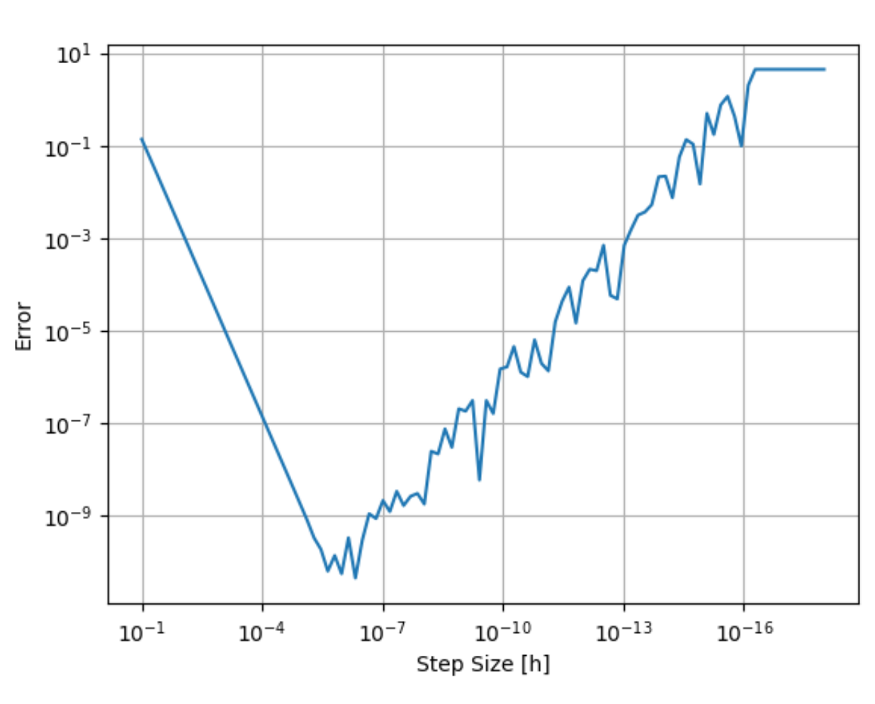


The minimum error of `2.134e-08` was achieved with a step size of `4.132e-09`

      ii. Estimate the numerical derivative of $f(x) = e^{\cos(\pi x^2)}$ at the point $x = 1/\sqrt{2}$
Estimate the derivative via:

$$\frac{df}{dx} \approx \frac{f(x+h) - f(x-h)}{2h}$$

for a range of step sizes h plot the error verses the step size:

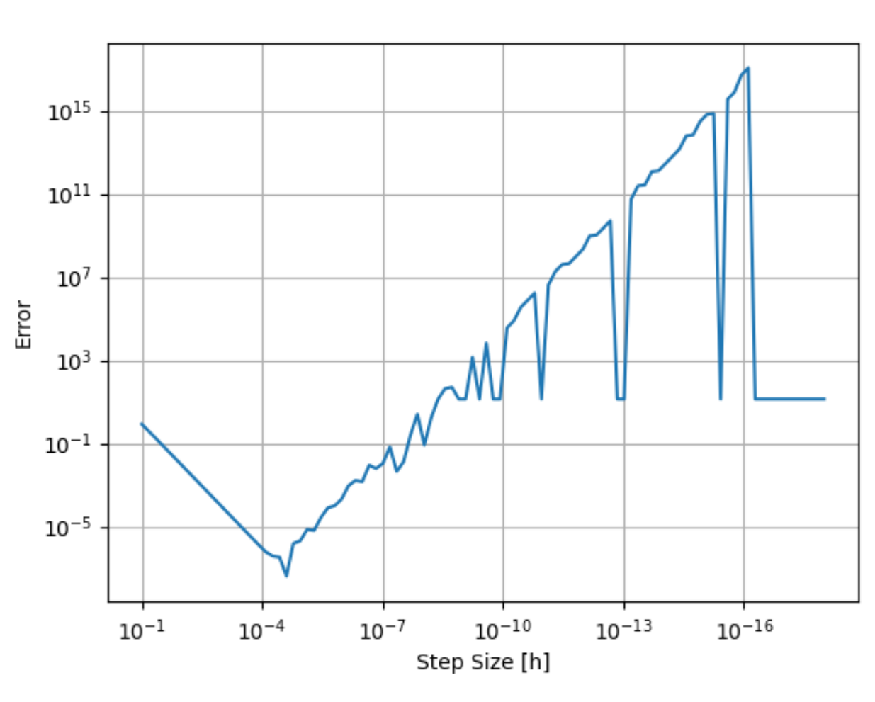


The minimum error of `4.407e-11` was achieved with a step size of `4.75e-07`

      iii. Estimate the numerical second derivative of $f(x) = e^{\cos(\pi x^2)}$ at the point $x = 1/\sqrt{2}$
Estimate the second derivative via:

$$\frac{d^2f}{dx^2} \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

for a range of step sizes h plot the error verses the step size:



The minimum error of `4.08e-08` was achieved with a step size of `2.477e-05`

   (b) investigate the numerical approximation of an integral

      i. Numerically integrate the following integral using the trapezoidal rule, with sub-intervals of $n = 10, 20, 40, 80, 160$

$$\int_0^{10} xe^{-\sqrt{x}}\,dx$$

The following is the numerical integral evaluations:

```
For a subinterval of 10 the integral is 4.608
For a subinterval of 20 the integral is 4.652
For a subinterval of 40 the integral is 4.664
For a subinterval of 80 the integral is 4.668
For a subinterval of 160 the integral is 4.669
```

      ii. Numerically integrate the following integral using the Simpson rule, with sub-intervals of $n = 10, 20, 40, 80, 160$

$$\int_0^{10} xe^{-\sqrt{x}}\,dx$$

The following is the numerical integral evaluations:

```
For a sub-interval of 10 the integral is 4.657102287466295
For a sub-interval of 20 the integral is 4.666621390508981
For a sub-interval of 40 the integral is 4.668462546387442
For a sub-interval of 80 the integral is 4.668804644613161
For a sub-interval of 160 the integral is 4.668866774081337
```

I am comfortable estimating that that 4 digits are correct. This this due to the speed of conversion of the Simpson's rule compared to the trapezoidal rule.

      iii. Use the quad function from scipy.integrate to estimate the value of the integrals:

$$\int_0^{10} xe^{-\sqrt{x}}\,dx\ ,\quad \int_0^{100} xe^{-\sqrt{x}}\,dx\ ,\quad \int_0^{1000} xe^{-\sqrt{x}}\,dx$$

The evaluation of these integrals is as follows:

```
At the upper bound of 10.0 the integral evaluates to
4.668880328350931
At the upper bound of 100.0 the integral evaluates to
11.875967391881685
At the upper bound of 1000.0 the integral evaluates to
11.999999998713989
```

This appears to converge to a value of 12.

Now evaluate the following integral:

$$\int_0^\infty xe^{-\sqrt{x}}\,dx$$

This evaluates to:

```
At an infinite upper bound the integral evaluates to
12.000000000094914
```

Lab Code

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import quad

# Question a
# Section i
def q_a_i():
    f = lambda x  : np.exp(np.cos(np.pi * x**2))

    x_0 = 1/np.sqrt(2)
    h = 0.1**np.linspace(1, 18, 100)
    df = (f(x_0 + h) - f(x_0))/h
    error = np.abs(df - (-np.sqrt(2) * np.pi))

    print("The step size that provides minimum error is: " + str(h[np.argmin(error)]))
    print("The minimum error is: " + str(error[np.argmin(error)]))

    plt.loglog(h, error)
    plt.gca().invert_xaxis()
    plt.grid()
    plt.xlabel("Step Size [h]")
    plt.ylabel("Error")
    plt.show()

# Section ii
def q_a_ii():
    f = lambda x  : np.exp(np.cos(np.pi * x**2))

    x_0 = 1/np.sqrt(2)
    h = 0.1**np.linspace(1, 18, 100)
    df = (f(x_0 + h) - f(x_0-h))/(2*h)
    error = np.abs(df - (-np.sqrt(2) * np.pi))

    print("The step size that provides minimum error is: " + str(h[np.argmin(error)]))
    print("The minimum error is: " + str(error[np.argmin(error)]))

    plt.loglog(h, error)
    plt.gca().invert_xaxis()
    plt.grid()
    plt.xlabel("Step Size [h]")
    plt.ylabel("Error")
    plt.show()

# Section iii
def q_a_iii():
    f = lambda x  : np.exp(np.cos(np.pi * x**2))

    x_0 = 1/np.sqrt(2)
    h = 0.1**np.linspace(1, 18, 100)
    ddf = ( f(x_0 + h) - 2*f(x_0) + f(x_0-h) )/h**2
    error = np.abs(ddf - (2*np.pi * (np.pi - 1)))

    print("\nThe step size that provides minimum error is: " + str(h[np.argmin(error)]))
    print("The minimum error is: " + str(error[np.argmin(error)]))

    plt.loglog(h, error)
    plt.gca().invert_xaxis()
    plt.grid()
    plt.xlabel("Step Size [h]")
    plt.ylabel("Error")
    plt.show()

# Question B
# Section i

def q_b_i():
    f = lambda x : x * np.exp(-np.sqrt(x))

    a = 0
    b = 10
    steps = np.array([10,20,40,80,160])

    for n in steps:
        x = np.linspace(a, b, n+1)
        h = np.abs(a-b) / n
        y = f(x)
        trap_rule = (y[1:]+y[:-1]).sum()*h/2
        print(f"For a sub-interval of {n} the integral is {trap_rule}")


def q_b_ii():
    f = lambda x : x * np.exp(-np.sqrt(x))

    a = 0
    b = 10
    steps = np.array([10,20,40,80,160])

    for n in steps:
        x = np.linspace(a, b, n+1)
        h = np.abs(a-b) / n
        y = f(x)
        simp_rule = h/3*(y[0]+y[-1])+4*h/3*y[1::2].sum()+2*h/3*y[2:-1:2].sum()
        print(f"For a sub-interval of {n} the integral is {simp_rule}")

def q_b_iii():
    f = lambda x : x * np.exp(-np.sqrt(x))

    a = 0
    upper = np.array([10.0, 100.0, 1000.0])

    for b in upper:
        print(f"At the upper bound of {b} the integral evaluates to {quad(f,a,b)[0]}")

    print(f"At an infinite upper bound the integral evaluates to {quad(f,a,np.inf)[0]}")



q_a_i()
q_a_ii()
q_a_iii()

q_b_i()
q_b_ii()
q_b_iii()
```