

ENGR222 Assignment 1

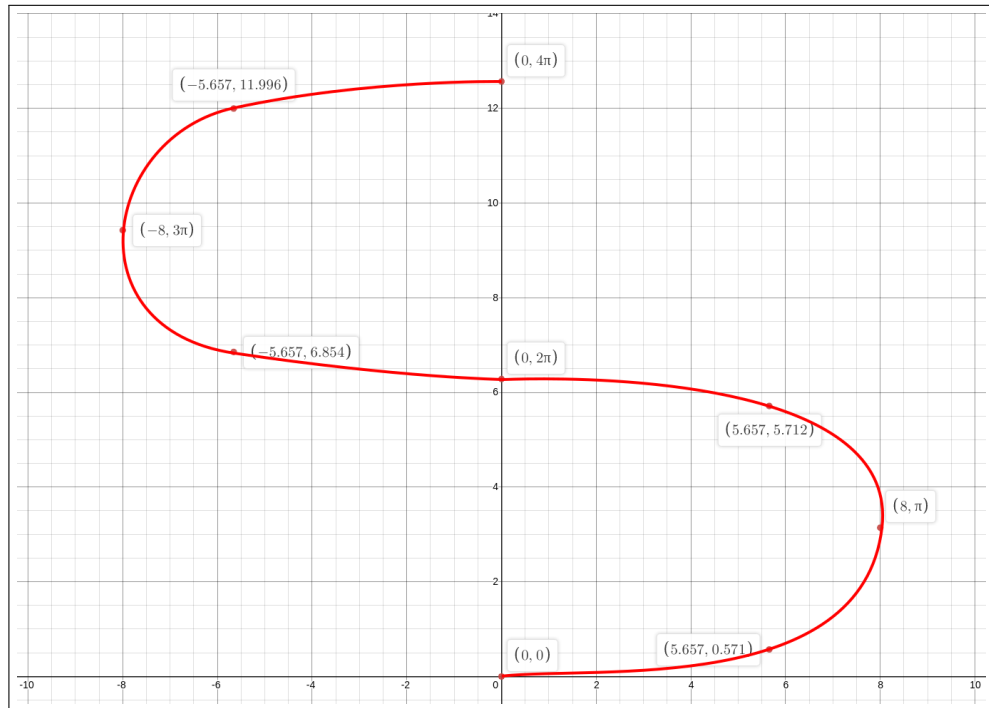
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1. Consider the parametric equation:

$$(x, y) = (8\sin(t), 2t - \sin(2t))$$

over the interval $0 \leq t \leq 2\pi$

- (a) Determine the location at $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$ and use this to draw a rough sketch of the curve.



- (b) Find the unit tangent vector to the curve when $t = \frac{\pi}{6}$

$$(f'(t), g'(t)) = (8\cos(t), 2 - 2\cos(2t))$$

$$t = \frac{\pi}{6} : (f'(t), g'(t)) = (6.928203, 1)$$

Calculate the unit tangent vector:

$$\frac{(f'(t), g'(t))}{\|(f'(t), g'(t))\|} = \frac{(6.928203, 1)}{\sqrt{6.928203^2 + 1}} = \left(\frac{6.928203}{7}, \frac{1}{7} \right)$$

- (c) Determine an equation describing the tangent line at $t = \frac{\pi}{6}$

$$= (f(t), g(t)) + t \frac{(f'(t), g'(t))}{\|(f'(t), g'(t))\|}$$

$$= (4, 0.181) + t \cdot \left(\frac{6.928203}{7}, \frac{1}{7} \right)$$

$$= \left(\frac{6.928203 \cdot t}{7} + 4, \frac{t}{7} + 0.181 \right)$$

(d) Determine an equation describing the normal line at $t = \frac{\pi}{6}$

$$\begin{aligned}
 &= (f(t), g(t)) + t \frac{(-g'(t), f'(t))}{\| (f'(t), g'(t)) \|} \\
 &= (4, 0.181) + t \cdot \left(\frac{-1}{7}, \frac{6.928203}{7} \right) \\
 &= \left(4 - \frac{t}{7}, 0.181 + \frac{6.928203 \cdot t}{7} \right)
 \end{aligned}$$

(e) Calculate the arc length over the interval $0 \leq t \leq 2\pi$

$$\begin{aligned}
 L &= \int_0^{2\pi} \sqrt{f'(t)^2 + g'(t)^2} dt \\
 &= \int_0^{2\pi} \sqrt{(8\cos(t))^2 + (2 - 2\cos(2t))^2} dt \\
 &= \int_0^{2\pi} \sqrt{64\cos(t)^2 + (2 - 2\cos(2t))^2} dt \\
 &= \int_0^{2\pi} \sqrt{(32\cos(2t) + 32) + (4 + 4\cos(2t)^2 - 8\cos(2t))} dt \\
 &= \int_0^{2\pi} \sqrt{4\cos(2t)^2 + 24\cos(2t) + 36} dt \\
 &= \int_0^{2\pi} \sqrt{4(\cos(2t)^2 + 6\cos(2t) + 9)} dt \\
 &= \int_0^{2\pi} \sqrt{4(\cos(2t) + 3)^2} dt \\
 &= 2 \int_0^{2\pi} (\cos(2t) + 3) dt \\
 &= \left| \sin(2t) + 6t \right|_0^{2\pi} \\
 &= (0 + 12\pi) - (0 + 0) \\
 &= 12\pi \approx 37.6991
 \end{aligned}$$

2. Consider the curve described by the vector valued function

$$\vec{r}(t) = \frac{1}{4}(e^{2t} - 2t)\vec{i} + e^t\vec{j}$$

(a) Find a point on the curve for which $\vec{r}(t) \cdot \vec{j} = 2$

$$\begin{aligned}
 \vec{r}(t) \cdot \vec{j} &= e^t = 2 \\
 t &= \ln(2) = 0.693147 \\
 \vec{r}(\ln(2)) &= (0.635, 2)
 \end{aligned}$$

(b) Determine the unit tangent vector to the curve (for arbitrary t)

$$\begin{aligned}\bar{r}'(t) &= \frac{1}{4}(2e^{2t} - 2)\bar{i} + e^t\bar{j} \\ \bar{T}(t) &= \frac{\bar{r}'(t)}{\|\bar{r}'(t)\|} \\ &= \frac{\frac{1}{4}(2e^{2t} - 2)\bar{i} + e^t\bar{j}}{\sqrt{(\frac{1}{4}(2e^{2t} - 2))^2 + e^{2t}}} \\ &= \frac{(e^{2t} - 1)}{(e^{2t} + 1)}\bar{i} + \frac{2e^t}{(e^{2t} + 1)}\bar{j}\end{aligned}$$

(c) Determine the principal unit normal vector to the curve (for arbitrary t)

$$\begin{aligned}\bar{N}(t) &= \frac{\bar{T}'(t)}{\|\bar{T}'(t)\|} \\ \bar{T}'(t) &= \frac{2e^{2t}(e^{2t} + 1) - 2e^{2t}(e^{2t} - 1)}{(e^{2t} + 1)^2}\bar{i} + \frac{2e^t(e^{2t} + 1) - 2e^t \cdot 2e^{2t}}{(e^{2t} + 1)^2}\bar{j} \\ \bar{T}'(t) &= \frac{4e^{2t}}{(e^{2t} + 1)^2}\bar{i} + \frac{2e^t - 2e^{3t}}{(e^{2t} + 1)^2}\bar{j} \equiv \text{sech}^2(t)\bar{i} + (-\text{sech}(t)\tanh(t))\bar{j} \\ \|\bar{T}'(t)\| &= \sqrt{\text{sech}^4(t) + (-\text{sech}(t)\tanh(t))^2} = \sqrt{\text{sech}^2(t)} \\ \bar{N}(t) &= \frac{\text{sech}(t)^2\bar{i} + (-\text{sech}(t)\tanh(t))\bar{j}}{\sqrt{\text{sech}^2(t)}}\end{aligned}$$

(d) Determine the curvature of the curve (for arbitrary t)

$$\begin{aligned}\kappa(t) &= \frac{\|\bar{T}'(t)\|}{\|\bar{r}'(t)\|} \\ &= \frac{2\sqrt{\text{sech}^2(t)}}{(e^{2t} + 1)}\end{aligned}$$

(e) Determine the arc length of the curve over $0 \leq t \leq 3$

$$\begin{aligned} L &= \int_0^3 \|\vec{r}'(t)\| \, dt \\ &= 2 \int_0^3 (e^{2t} + 1) \, dt \\ &= \left| \frac{1}{2} (e^{2t} + 2t) \right|_0^3 \\ &= \frac{1}{2} ((e^6 + 6) - (1 + 0)) \\ &= 102.107 \end{aligned}$$

3. Quick questions

(a) Determine the arc length parametrisation of:

$$(f(t), g(t), h(t)) = (3t, t - 2, -5t + 7)$$

with $t = 0$ as the starting/reference point

$$\begin{aligned} (f'(t), g'(t), h'(t)) &= (3, 1, -5) \\ \|(f'(t), g'(t), h'(t))\| &= \sqrt{3^2 + 1^2 + (-5)^2} = \sqrt{35} \end{aligned}$$

$$\begin{aligned} s &= \int_0^t \|\vec{r}'(u)\| \, du = \int_0^t \sqrt{35} \, du \\ &= \left| \sqrt{35} u \right|_0^t = \sqrt{35} t \end{aligned}$$

$$\begin{aligned} t &= \frac{1}{\sqrt{35}} s \\ (f(s), g(s), h(s)) &= \left(\frac{3}{\sqrt{35}} s, \frac{1}{\sqrt{35}} s - 2, \frac{-5}{\sqrt{35}} s + 7 \right) \end{aligned}$$

(b) Determine the arc length parametrisation of:

$$\vec{r}(t) = (5 \cos(t) + 3)\vec{i} + (-5 \sin(t) + 2)\vec{j}$$

using $t = 0$ as the starting/reference point

$$\begin{aligned}\vec{r}'(t) &= (-5 \sin(t))\vec{i} + (-5 \cos(t))\vec{j} \\ \|\vec{r}'(t)\| &= \sqrt{(-5 \sin(t))^2 + (-5 \cos(t))^2} \\ &= \sqrt{25(\sin(t)^2 + \cos(t)^2)} \\ &= \sqrt{25}\sqrt{1} = 5\end{aligned}$$

$$\begin{aligned}s &= \int_0^t \|\vec{r}'(t)\| \, du = \int_0^t 5 \, du \\ &= \left| 5u \right|_0^t = 5t\end{aligned}$$

$$\begin{aligned}t &= \frac{s}{5} \\ \vec{r}(s) &= \left(5 \cos\left(\frac{s}{5}\right) + 3 \right) \vec{i} + \left(-5 \sin\left(\frac{s}{5}\right) + 2 \right) \vec{j}\end{aligned}$$

(c) Find the unit tangent vector to:

$$\vec{r}(t) = (\sqrt{2} \cos(t))\vec{i} + (\sin(t))\vec{j} + (\sin(t))\vec{k}$$

at $t = \frac{\pi}{3}$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

$$\begin{aligned}\vec{r}'(t) &= (-\sqrt{2} \sin(t))\vec{i} + (\cos(t))\vec{j} + (\cos(t))\vec{k} \\ \|\vec{r}'(t)\| &= \sqrt{(-\sqrt{2} \sin(t))^2 + (\cos(t))^2 + (\cos(t))^2} \\ &= \sqrt{2 \sin^2(t) + 2 \cos^2(t)} \\ &= \sqrt{2}\sqrt{1} = \sqrt{2}\end{aligned}$$

$$\begin{aligned}\vec{T}(t) &= \frac{(-\sqrt{2} \sin(t))\vec{i} + (\cos(t))\vec{j} + (\cos(t))\vec{k}}{\sqrt{2}} \\ \vec{T}\left(\frac{\pi}{3}\right) &= \frac{(-\sqrt{2} \sin(\frac{\pi}{3}))\vec{i} + (\cos(\frac{\pi}{3}))\vec{j} + (\cos(\frac{\pi}{3}))\vec{k}}{\sqrt{2}}\end{aligned}$$

$$\vec{T}\left(\frac{\pi}{3}\right) = \frac{\left(-\frac{\sqrt{6}}{2}\right)\vec{i} + \left(\frac{1}{2}\right)\vec{j} + \left(\frac{1}{2}\right)\vec{k}}{\sqrt{2}}$$

(d) An alternative formula for the binormal vector is:

$$\bar{B}(t) = \frac{\bar{r}'(t) \times \bar{r}''(t)}{\|\bar{r}'(t) \times \bar{r}''(t)\|}$$

Use this to find the binormal vector to:

$$\bar{r}(t) = t\bar{i} - t^3\bar{j} + t^2\bar{k}$$

At the point $t = 1$

$$\bar{r}'(t) = \bar{i} - 3t^2\bar{j} + 2t\bar{k}$$

$$\bar{r}''(t) = 0\bar{i} - 6t\bar{j} + 2\bar{k}$$

$$\begin{aligned}\bar{r}'(t) \times \bar{r}''(t) &= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -3t^2 & 2t \\ 0 & -6t & 2 \end{vmatrix} \\ &= \begin{vmatrix} -3t^2 & 2t \\ -6t & 2 \end{vmatrix} \bar{i} - \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix} \bar{j} + \begin{vmatrix} 1 & -3t^2 \\ 0 & -6t \end{vmatrix} \bar{k} \\ &= 6t^2\bar{i} - 2\bar{j} - 6t\bar{k}\end{aligned}$$

$$\|\bar{r}'(t) \times \bar{r}''(t)\| = \sqrt{36t^4 + 4 + 36t^2}$$

$$\bar{B}(t) = \frac{6t^2\bar{i} - 2\bar{j} - 6t\bar{k}}{\sqrt{36t^4 + 4 + 36t^2}}$$

$$\bar{B}(1) = \frac{6\bar{i} - 2\bar{j} - 6\bar{k}}{\sqrt{76}}$$

(e) Find the minimum and maximum curvature for the curve described by:

$$\bar{r}(t) = (3 \sin(t) + 2)\bar{i} + (2 \cos(t) + 1)\bar{j}$$

$$\kappa(t) = \frac{\|\bar{r}'(t) \times \bar{r}''(t)\|}{\|\bar{r}'(t)\|^3}$$

$$\bar{r}'(t) = 3 \cos(t)\bar{i} - 2 \sin(t)\bar{j}$$

$$\bar{r}''(t) = -3 \sin(t)\bar{i} - 2 \cos(t)\bar{j}$$

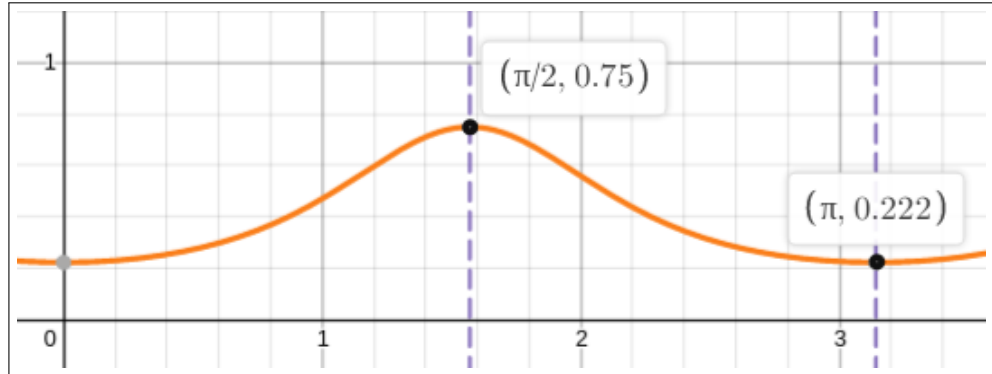
$$\begin{aligned}\bar{r}'(t) \times \bar{r}''(t) &= \begin{vmatrix} \bar{i} & \bar{j} \\ 3 \cos(t) & -2 \sin(t) \\ -3 \sin(t) & -2 \cos(t) \end{vmatrix} \bar{k} \\ &= (-6 \cos^2(t) - 6 \sin^2(t)) \bar{k} = -6\bar{k}\end{aligned}$$

$$\|\bar{r}'(t) \times \bar{r}''(t)\| = \sqrt{-6^2} = 6$$

$$\begin{aligned}\|\bar{r}'(t)\| &= \sqrt{(3 \cos(t))^2 + (-2 \sin(t))^2} \\ &= \sqrt{9 \cos^2(t) + 4 \sin^2(t)} \\ &= \sqrt{5 \cos^2(t) + 4}\end{aligned}$$

$$\kappa(t) = \frac{6}{\left(\sqrt{5 \cos^2(t) + 4}\right)^3}$$

Since the curvature κ is periodic on π due to the $\cos^2(t)$, we know that the minimum and maximum will fall on $\frac{\pi}{2}$ and π . In the following plot we can see that $t = \frac{\pi}{2}$ is the maximum, and $t = \pi$ is the minimum.



$$\kappa\left(\frac{\pi}{2}\right) = \frac{6}{\left(\sqrt{5 \cos^2\left(\frac{\pi}{2}\right) + 4}\right)^3} = 0.75$$

$$\kappa(\pi) = \frac{6}{\left(\sqrt{5 \cos^2(\pi) + 4}\right)^3} = 0.22\bar{2}$$

4. Suppose a roller coaster follows a path described by:

$$\bar{r}(t) = \frac{1}{5}t(20-t)\bar{i} + \frac{t^2}{50}(20-t)\bar{j} + \frac{t}{50}(10-t)(20-t)\bar{k}$$

(a) Determine the velocity vector of the roller coaster (for arbitrary t)

$$\bar{r}'(t) = (4 - \frac{2t}{5})\bar{i} + \frac{t}{50}(40 - 3t)\bar{j} + \left(\left(\frac{3t^2}{50} - \frac{6t}{5}\right) + 4\right)\bar{k}$$

(b) Determine the speed when $t = 5$

$$\begin{aligned}\bar{r}'(5) &= (4 - 2)\bar{i} + \frac{5}{50}(40 - 15)\bar{j} + \left(\left(\frac{75}{50} - \frac{30}{5}\right) + 4\right)\bar{k} \\ &= 2\bar{i} + 2.5\bar{j} - 0.5\bar{k} \\ \|\bar{r}'(5)\| &= \sqrt{2^2 + 2.5^2 + (-0.5)^2} = 3.162278\end{aligned}$$

(c) Determine the acceleration vector of the roller coaster (for arbitrary t)

$$\bar{r}''(t) = -\frac{2}{5}\bar{i} + \left(\frac{4}{5} - \frac{3t}{25}\right)\bar{j} + \frac{3t - 30}{25}\bar{k}$$

(d) Determine the curvature of curve followed by the roller coaster when $t = 5$

$$\kappa(t) = \frac{\|\bar{r}'(t) \times \bar{r}''(t)\|}{\|\bar{r}'(t)\|^3}$$

$$\begin{aligned}\bar{r}'(5) &= 2\bar{i} + 2.5\bar{j} - 0.5\bar{k} \\ \bar{r}''(5) &= -0.4\bar{i} + 0.2\bar{j} - 0.6\bar{k} \\ \|\bar{r}'(5)\|^3 &= 31.622787\end{aligned}$$

$$\begin{aligned}\bar{r}'(5) \times \bar{r}''(5) &= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 2.5 & -0.5 \\ -0.4 & 0.2 & -0.6 \end{vmatrix} \\ &= \begin{vmatrix} 2.5 & -0.5 \\ 0.2 & -0.6 \end{vmatrix} \bar{i} - \begin{vmatrix} 2 & -0.5 \\ -0.4 & -0.6 \end{vmatrix} \bar{j} + \begin{vmatrix} 2 & 2.5 \\ -0.4 & 0.2 \end{vmatrix} \bar{k} \\ &= -1.4\bar{i} - 1.4\bar{k} + 1.4\bar{k}\end{aligned}$$

$$\|\bar{r}'(5) \times \bar{r}''(5)\| = \sqrt{-1.4^2 - 1.4^2 + 1.4^2} = \frac{7\sqrt{3}}{5}$$

$$\kappa(5) = \frac{\frac{7\sqrt{3}}{5}}{31.622787} = 0.712696$$