ENGR222 Assignment 2

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1. Multiple Integrals

   (a) Evaluate the integral of $f(x,y,z) = xyz$ over the region:

   $$G = \{(x,y,z) : xy \le z \le 1,\ 0 \le x \le y,\ 0 \le y \le 1\}$$

   $$\begin{aligned}
   \iiint_G f(x,y,z)dV &= \int_0^1 \int_0^y \int_{xy}^1 xyz\ dz\, dx\, dy \\
   &= \int_0^1 \int_0^y xy \Big| \frac{z^2}{2} \Big|_{z=xy}^{z=1}\ dx\, dy \\
   &= \int_0^1 \int_0^y xy \left(\frac{1}{2} - \frac{x^2y^2}{2}\right)\ dx\, dy \\
   &= \int_0^1 \int_0^y \frac{1}{2}(xy - x^3y^3)\ dx\, dy \\
   &= \int_0^1 \frac{1}{2} \Big| \frac{x^2y}{2} - \frac{x^4y^3}{4} \Big|_0^y\ dy \\
   &= \frac{1}{8} \Big| \frac{y^4}{2} - \frac{y^8}{8} \Big|_{y=0}^{y=1} \\
   &= \frac{1}{8}\left(\frac{1}{2} - \frac{1}{8}\right) = \frac{3}{64}
   \end{aligned}$$

   (b) Using spherical coordinates, determine the integral of $f(x,y,z) = x$ over the region G described by the inequalities $x, y, z \ge 0$ and $x^2 + y^2 + z^2 \le 1$

   In spherical ordinates we have:
   $f(r,\theta,\phi) = r\cos(\theta)\sin(\phi)$ for $G = \{(r,\theta,\phi) : 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{2}\}$

   $$\begin{aligned}
   \iiint_G f(r,\theta,\phi)dV &= \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} r\cos(\theta)\sin(\phi)\ d\phi\, d\theta\, dr \\
   &= \int_0^1 r\ dr \int_0^{\frac{\pi}{2}} \cos(\theta)\ d\theta \int_0^{\frac{\pi}{2}} \sin(\phi)\ d\phi \\
   &= \frac{r^2}{2}\Big|_{r=0}^{r=1} \times \sin(\theta)\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} \times -\cos\phi\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} \\
   &= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}
   \end{aligned}$$

   (c) Calculate the integral of $f(x,y) = y^{-2}e^{-x}$ over the region

   $$R = \{(x,y) : x \in [0,\infty], y \in [2,\infty]\}$$

   $$\begin{aligned}
   \iint_R f(x,y)\ dA &= \int_2^\infty y^{-2}\ dy \int_0^\infty e^{-x}\ dx \\
   &= -\frac{1}{y}\Big|_2^\infty \times -e^{-x}\Big|_0^\infty \\
   &= \left(0 + \frac{1}{2}\right) \times (0+1) \\
   &= \frac{1}{2}
   \end{aligned}$$

   (d) Determine the centroid of the two dimensional object described in polar coordinates by

   $$R = \{(r,\theta) : 0 \le r \le \theta, \theta \in [0, 2\pi]\}$$

2. Vector Fields

   (a) Calculate the divergence of the vector field $\mathbf{F} = x^2y^3z^4\mathbf{i} - xyz\mathbf{j} + (x+y+z)\mathbf{k}$

   $$\begin{aligned}
   \text{div } \mathbf{F} &= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \\
   &= 2xy^3z^4 - yz = 1
   \end{aligned}$$

   (b) Calculate the curl of the vector field $\mathbf{F} = x^2y^3z^4\mathbf{i} - xyz\mathbf{j} + (x+y+z)\mathbf{k}$

   $$\text{curl } \mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}$$

   $$= (1+xy)\,\mathbf{i} + \left(4x^2y^3z^3 - 1\right)\mathbf{j} - \left(yz + 3x^2y^2x^4\right)\mathbf{k}$$

   (c) Determine the gradient field of $\phi(x,y,z) = xz^2 + sin(y)e^x$

   $$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

   $$= \left(z^2 + sin(y)e^x\right)\mathbf{i} + \left(\cos(y)e^x\right)\mathbf{j} + (2xz)\,\mathbf{k}$$

   (d) Calculate the Laplacian of $\phi(x,y,z) = xz^2 + sin(y)e^x$

   $$\nabla\phi^2 = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

   $$\begin{aligned}
   &= \sin(y)e^x - \sin(y)e^x + 2x \\
   &= 2x
   \end{aligned}$$

3. Line integrals

   (a) Calculate the value of the line integral $\int_C f\ ds$ where

   $$f(x,y,z) = \frac{y}{x}e^z$$

   and C is described by

   $$(x,y,z) = (2t,\ t^2,\ ln(t)) \text{ for } t \in [1,4]$$

   $$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\ ||r'(t)||\ dt$$

   $$\begin{aligned}
   f(2t, t^2, ln(t)) &= \frac{t^2}{2t}e^{ln(t)} \\
   &= \frac{t^2}{2}
   \end{aligned}$$

   $$\begin{aligned}
   r'(t) &= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \\
   &= 2\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}
   \end{aligned}$$

   $$\begin{aligned}
   ||r'(t)|| &= \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2} \\
   &= \sqrt{4 + 4t^2 + \frac{1}{t^2}} \\
   &= \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}} \\
   &= \frac{\sqrt{4t^4 + 4t^2 + 1}}{t} \\
   &= \frac{\sqrt{(2t^2+1)^2}}{t} \\
   &= \frac{(2t^2+1)}{t}
   \end{aligned}$$

   $$\begin{aligned}
   \int_C f(x,y,z)ds &= \int_1^4 \frac{t^2}{2}\frac{(2t^2+1)}{t}\ dt \\
   &= \int_1^4 \frac{t(2t^2+1)}{2}\ dt \\
   &= \int_1^4 \frac{(2t^3+t)}{2}\ dt
   \end{aligned}$$

   $$\begin{aligned}
   &= \frac{t^4}{4} + \frac{t^2}{4}\Big|_1^4 \\
   &= \frac{4^4}{4} + \frac{4^2}{4} - \frac{1^4}{4} - \frac{1^2}{4} \\
   &= \frac{135}{2} = 67.5
   \end{aligned}$$

   (b) Calculate the value of the line integral $\int_C \mathbf{F} \cdot\ d\mathbf{r}$ where

   $$\mathbf{F}(x,y,z) = x\mathbf{i} - e^z\mathbf{j} + y\mathbf{k}$$

   and C is described by

   $$(x,y,z) = (2t,\ t^2,\ ln(t)) \text{ for } t \in [1,4]$$

   $$\int_C \mathbf{F} \cdot\ d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt$$

   $$\begin{aligned}
   \mathbf{F}(\mathbf{r}(t)) &= 2t\mathbf{i} - e^{ln(t)}\mathbf{j} + t^2\mathbf{k} \\
   &= 2t\mathbf{i} - t\mathbf{j} + t^2\mathbf{k}
   \end{aligned}$$

   $$\begin{aligned}
   \mathbf{r}'(t) &= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \\
   &= 2\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}
   \end{aligned}$$

   $$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 5t - 2t^2$$

   $$\begin{aligned}
   \int_C \mathbf{F} \cdot\ d\mathbf{r} &= \int_1^4 5t - 2t^2\ dt \\
   &= \frac{5}{2}t^2 - \frac{2}{3}t^3\ \Big|_1^4 \\
   &= \left(\frac{5}{2} \times 16\right) - \left(\frac{2}{3} \times 64\right) - \left(\frac{5}{2}\right) + \left(\frac{2}{3}\right) \\
   &= -4.5
   \end{aligned}$$