ENGR222 Assignment 1

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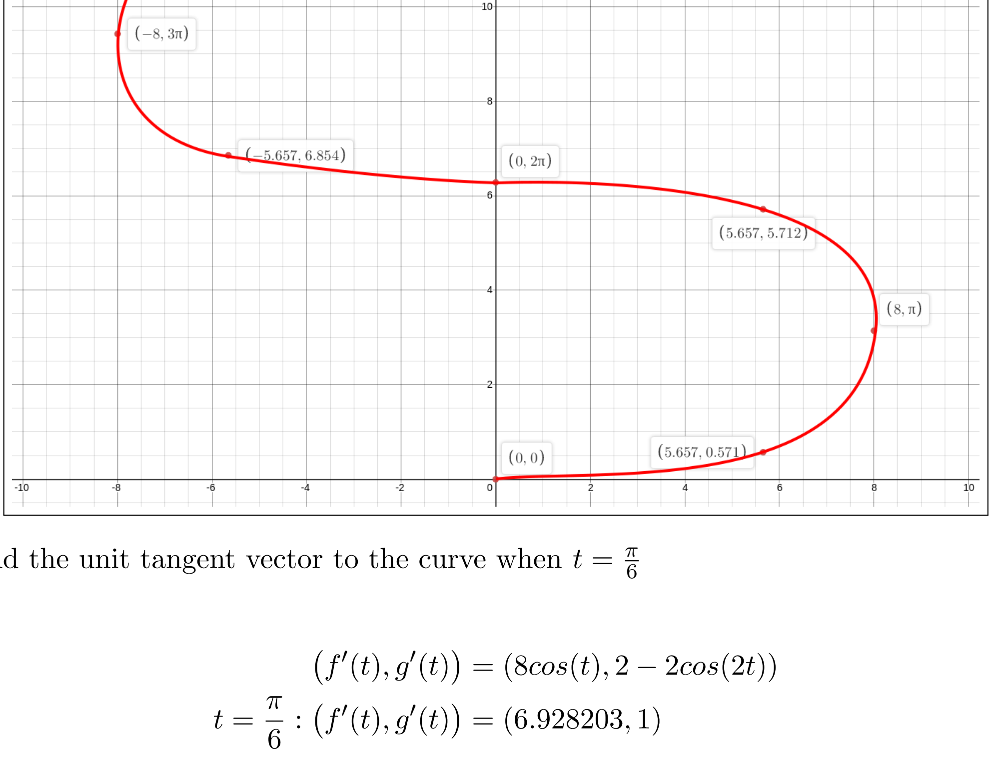
1. Consider the parametric equation:

$$(x, y) = (8sin(t), 2t - sin(2t))$$

over the interval $0 \le t \le 2\pi$

(a) Determine the location at $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$ and use this to draw a rough sketch of the curve.



(b) Find the unit tangent vector to the curve when $t = \frac{\pi}{6}$

$$\big(f'(t), g'(t)\big) = (8cos(t), 2 - 2cos(2t))$$
$$t = \frac{\pi}{6} : \big(f'(t), g'(t)\big) = (6.928203, 1)$$

Calculate the unit tangent vector:

$$\frac{(f'(t), g'(t))}{||(f'(t), g'(t))||} = \frac{(6.928203, 1)}{\sqrt{6.928203^2 + 1}} = \left(\frac{6.928203}{7}, \frac{1}{7}\right)$$

(c) Determine an equation describing the tangent line at $t = \frac{\pi}{6}$

$$\begin{aligned} &= (f(t), g(t)) + t\frac{(f'(t), g'(t))}{||(f'(t), g'(t))||} \\ &= (4, 0.181) + t \cdot \left(\frac{6.928203}{7}, \frac{1}{7}\right) \\ &= \left(\frac{6.928203 \cdot t}{7} + 4, \frac{t}{7} + 0.181\right) \end{aligned}$$

(d) Determine an equation describing the normal line at $t = \frac{\pi}{6}$

$$\begin{aligned} &= (f(t), g(t)) + t\frac{(-g'(t), f'(t))}{||(f'(t), g'(t))||} \\ &= (4, 0.181) + t \cdot \left(\frac{-1}{7}, \frac{6.928203}{7}\right) \\ &= \left(4 - \frac{t}{7}, 0.181 + \frac{6.928203 \cdot t}{7}\right) \end{aligned}$$

(e) Calculate the arc length over the interval $0 \le t \le 2\pi$

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{f'(t)^2 \; + \; g'(t)^2}\, \mathrm{dt} \\ &= \int_0^{2\pi} \sqrt{(8cos(t))^2 + (2 - 2cos(2t))^2}\, \mathrm{dt} \\ &= \int_0^{2\pi} \sqrt{64\cos(t)^2 + (2 - 2\cos(2t))^2}\, \mathrm{dt} \\ &= \int_0^{2\pi} \sqrt{(32\cos(2t) + 32) + \left(4 + 4\cos(2t)^2 - 8\cos(2t)\right)}\, \mathrm{dt} \\ &= \int_0^{2\pi} \sqrt{4\cos(2t)^2 + 24\cos(2t) + 36}\, \mathrm{dt} \\ &= \int_0^{2\pi} \sqrt{4\left(\cos(2t)^2 + 6\cos(2t) + 9\right)}\, \mathrm{dt} \\ &= \int_0^{2\pi} \sqrt{4(\cos(2t) + 3)^2}\, \mathrm{dt} \\ &= \; 2\int_0^{2\pi} (\cos(2t) + 3)\, \mathrm{dt} \\ &= \Big|sin(2t) + 6t\Big|_0^{2\pi} \\ &= (0 + 12\pi) - (0 + 0) \\ &= 12\pi \approx 37.6991 \end{aligned}$$

2. Consider the curve described by the vector valued function

$$\bar{r}(t) = \frac{1}{4}(e^{2t} - 2t)\bar{i} + e^t\bar{j}$$

(a) Find a point on the curve for which $\bar{r}(t) \cdot \bar{j} = 2$

$$\begin{aligned} \bar{r}(t) \cdot \bar{j} &= e^t = 2 \\ t &= \ln(2) = 0.693147 \\ \bar{r}(\ln(2)) &= (0.635, 2) \end{aligned}$$

(b) Determine the unit tangent vector to the curve (for arbitrary t)

$$\bar{r}'(t) = \frac{1}{4}(2e^{2t} - 2)\bar{i} + e^t\bar{j}$$
$$\bar{T}(t) = \frac{\bar{r}'(t)}{||\bar{r}'(t)||}$$
$$= \frac{\frac{1}{4}(2e^{2t} - 2)\bar{i} + e^t\bar{j}}{\sqrt{(\frac{1}{4}(2e^{2t} - 2))^2 + e^{2t}}}$$
$$= \frac{(e^{2t} - 1)}{(e^{2t} + 1)}\bar{i} + \frac{2e^t}{(e^{2t} + 1)}\bar{j}$$

(c) Determine the principal unit normal vector to the curve (for arbitrary t)

$$\bar{N}(t) = \frac{\bar{T}'(t)}{||\bar{T}'(t)||}$$

$$\bar{T}'(t) = \frac{2e^{2t}(e^{2t} + 1) - 2e^{2t}(e^{2t} - 1)}{(e^{2t} + 1)^2}\bar{i} + \frac{2e^t(e^{2t} + 1) - 2e^t \cdot 2e^{2t}}{(e^{2t} + 1)^2}\bar{j}$$

$$\bar{T}'(t) = \frac{4e^{2t}}{(e^{2t} + 1)^2}\bar{i} + \frac{2e^t - 2e^{3t}}{(e^{2t} + 1)^2}\bar{j} \equiv \operatorname{sech}^2(t)\bar{i} + (-\operatorname{sech}(t)\tanh(t))\bar{j}$$

$$||\bar{T}(t)|| = \sqrt{\operatorname{sech}^4(t) + (-\operatorname{sech}(t)\tanh(t))^2} = \sqrt{\operatorname{sech}^2(t)}$$

$$\bar{N}(t) = \frac{\operatorname{sech}(t)^2\,\bar{i} + (-\operatorname{sech}(t)\tanh(t))\,\bar{j}}{\sqrt{\operatorname{sech}^2(t)}}$$

(d) Determine the curvature of the curve (for arbitrary t)

$$\kappa(t) = \frac{||\bar{T}'(t)||}{||\bar{r}'(t)||}$$
$$= \frac{2\sqrt{\operatorname{sech}^2(t)}}{(e^{2t} + 1)}$$

(e) Determine the arc length of the curve over $0 \le t \le 3$

$$\begin{aligned} L &= \int_0^3 ||\bar{r}'(t)|| \\ &= 2\int_0^3 (e^{2t} + 1)\, \mathrm{dt} \\ &= \left|\frac{1}{4}(e^{2t} + 2t)\right|_0^3 \\ &= \frac{1}{4}((e^6 + 6) - (1 + 0)) \\ &= 102.107 \end{aligned}$$

3. Quick questions

(a) Determine the arc length parametrisation of:

$$(f(t), g(t), h(t)) = (3t, t - 2, -5t + 7)$$

with $t = 0$ as the starting/reference point

$$\big(f'(t), g'(t), h'(t)\big) = (3, 1, -5)$$
$$||\big(f'(t), g'(t), h'(t)\big)|| = \sqrt{3^2 + 1^2 + (-5)^2} = \sqrt{35}$$

$$\begin{aligned} s &= \int_0^t ||\bar{r}'(u)||\, \mathrm{du} = \int_0^t \sqrt{35}\, \mathrm{du} \\ &= \Big|\sqrt{35}\, \mathrm{u}\Big|_0^t = \sqrt{35}t \end{aligned}$$

$$t = \frac{1}{\sqrt{35}}s$$
$$(f(s), g(s), h(s)) = \left(\frac{3}{\sqrt{35}}s, \frac{1}{\sqrt{35}}s - 2, \frac{-5}{\sqrt{35}}s + 7\right)$$

(b) Determine the arc length parametrisation of:

$$\bar{r}(t) = (5\cos(t) + 3)\bar{i} + (-5\sin(t) + 2)\bar{j}$$

using $t = 0$ as the starting/reference point

$$\begin{aligned} \bar{r}'(t) &= (-5\sin(t))\bar{i} + (-5\cos(t))\bar{j} \\ ||\bar{r}'(t)|| &= \sqrt{(-5\sin(t))^2 + (-5\cos(t))^2} \\ &= \sqrt{25(\sin(t)^2 + \cos(t))^2} \\ &= \sqrt{25}\sqrt{1} = 5 \end{aligned}$$

$$\begin{aligned} s &= \int_0^t ||\bar{r}'(t)||\, \mathrm{du} = \int_0^t 5\, \mathrm{du} \\ &= \Big|5\, \mathrm{u}\Big|_0^t = 5t \end{aligned}$$

$$t = \frac{s}{5}$$
$$\bar{r}(s) = \left(5\cos\left(\frac{s}{5}\right) + 3\right)\bar{i} + \left(-5\sin\left(\frac{s}{5}\right) + 2\right)\bar{j}$$

(c) Find the unit tangent vector to:

$$\bar{r}(t) = (\sqrt{2}\cos(t))\bar{i} + (\sin(t))\bar{j} + (\sin(t))\bar{k}$$

at $t = \frac{\pi}{3}$

$$\bar{T}(t) = \frac{\bar{r}'(t)}{||\bar{r}'(t)||}$$

$$\begin{aligned} \bar{r}'(t) &= (-\sqrt{2}\sin(t))\bar{i} + (\cos(t))\bar{j} + (\cos(t))\bar{k} \\ ||\bar{r}'(t)|| &= \sqrt{(-\sqrt{2}\sin(t))^2 + (\cos(t))^2 + (\cos(t))^2} \\ &= \sqrt{2\sin^2(t) + 2\cos^2(t)} \\ &= \sqrt{2}\sqrt{1} = \sqrt{2} \end{aligned}$$

$$\bar{T}(t) = \frac{(-\sqrt{2}\sin(t))\bar{i} + (\cos(t))\bar{j} + (\cos(t))\bar{k}}{\sqrt{2}}$$
$$\bar{T}\left(\frac{\pi}{3}\right) = \frac{(-\sqrt{2}\sin(\frac{\pi}{3}))\bar{i} + (\cos(\frac{\pi}{3}))\bar{j} + (\cos(\frac{\pi}{3}))\bar{k}}{\sqrt{2}}$$

$$\bar{T}\left(\frac{\pi}{3}\right) = \frac{\left(-\frac{\sqrt{6}}{2}\right)\bar{i} + \left(\frac{1}{2}\right)\bar{j} + \left(\frac{1}{2}\right)\bar{k}}{\sqrt{2}}$$

(d) An alternative formula for the binormal vector is:

$$\bar{B}(t) = \frac{\bar{r}'(t) \times \bar{r}''(t)}{||\bar{r}'(t) \times \bar{r}''(t)||}$$

Use this to find the binormal vector to:

$$\bar{r}(t) = t\bar{i} - t^3\bar{j} + t^2\bar{k}$$

At the point $t = 1$

$$\begin{aligned} \bar{r}'(t) &= \bar{i} - 3t^2\bar{j} + 2t\bar{k} \\ \bar{r}''(t) &= 0\bar{i} - 6t\bar{j} + 2\bar{k} \end{aligned}$$

$$\begin{aligned} \bar{r}'(t) \times \bar{r}''(t) &= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -3t^2 & 2t \\ 0 & -6t & 2 \end{vmatrix} \\ &= \begin{vmatrix} -3t^2 & 2t \\ -6t & 2 \end{vmatrix}\bar{i} - \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix}\bar{j} + \begin{vmatrix} 1 & -3t^2 \\ 0 & -6t \end{vmatrix}\bar{k} \\ &= 6t^2\bar{i} - 2\bar{j} - 6t\bar{k} \end{aligned}$$

$$||\bar{r}'(t) \times \bar{r}''(t)|| = \sqrt{36t^4 + 4 + 36t^2}$$

$$\bar{B}(t) = \frac{6t^2\bar{i} - 2\bar{j} - 6t\bar{k}}{\sqrt{36t^4 + 4 + 36t^2}}$$

$$\bar{B}(1) = \frac{6\bar{i} - 2\bar{j} - 6\bar{k}}{\sqrt{76}}$$

(e) Find the minimum and maximum curvature for the curve described by:

$$\bar{r}(t) = (3\sin(t) + 2)\bar{i} + (2\cos(t) + 1)\bar{j}$$
$$\kappa(t) = \frac{||\bar{r}'(t) \times \bar{r}''(t)||}{||\bar{r}'(t)||^3}$$

$$\begin{aligned} \bar{r}'(t) &= 3\cos(t)\bar{i} - 2\sin(t)\bar{j} \\ \bar{r}''(t) &= -3\sin(t)\bar{i} - 2\cos(t)\bar{j} \end{aligned}$$

$$\begin{aligned} \bar{r}'(t) \times \bar{r}''(t) &= \begin{vmatrix} \bar{i} & \bar{j} \\ 3\cos(t) & -2\sin(t) \\ -3\sin(t) & -2\cos(t) \end{vmatrix}\bar{k} \\ &= \left(-6\cos^2(t) - 6\sin^2(t)\right)\bar{k} = -6\bar{k} \end{aligned}$$

$$||\bar{r}'(t) \times \bar{r}''(t)|| = \sqrt{-6^2} = 6$$

$$\begin{aligned} ||\bar{r}'(t)|| &= \sqrt{(3\cos(t))^2 + (-2\sin(t))^2} \\ &= \sqrt{9\cos^2(t) + 4\sin^2(t)} \\ &= \sqrt{5\cos^2(t) + 4} \end{aligned}$$

$$\kappa(t) = \frac{6}{\left(\sqrt{5\cos^2(t) + 4}\right)^3}$$