ENGR222 Assignment 2

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1. The following questions are concerned with the function

$$f(x,y) = -2x^3 + 3x^2y + 2y^3 - 9y + 5$$

(a) Determine the first order partial derivative of $f(x,y)$

$$\begin{aligned} f_x &= -6x^2 + 6xy \\ f_y &= 6y^2 + 3x^2 - 9 \end{aligned}$$

(b) Determine the second order partial derivatives of $f(x,y)$

$$\begin{aligned} f_{xx} &= -12x + 6y \\ f_{yy} &= 12y \\ f_{xy} &= 6x \end{aligned}$$

(c) Find all of the critical points of $f(x,y)$

By inspection we know $(x = y = -1, 1)$

Let $x = 0$

$$\begin{aligned} f_x &= 0 \\ f_y &= 6y^2 - 9 = 0 \\ \therefore y &= \sqrt{\frac{9}{6}} = \sqrt{\frac{3}{2}} \end{aligned}$$

Let $y = 0$

$$\begin{aligned} f_x &= -6x^2 = 0 \\ f_y &= 3x^2 - 9 \end{aligned}$$

No solution for x when $y = 0$

The critical points are $\rightarrow [(1,1),\ (-1,-1),\ (0, \sqrt{\tfrac{3}{2}})]$

(d) Classify the critical point $(0, \sqrt{\tfrac{3}{2}})$

$$D = f_{xx}(0, \sqrt{\frac{3}{2}}) \times f_{yy}(0, \sqrt{\frac{3}{2}}) - f_{xy}^2(0, \sqrt{\frac{3}{2}})$$

$$\begin{aligned} f_{xx}(0, \sqrt{\frac{3}{2}}) &= 3\sqrt{6} \\ f_{yy}(0, \sqrt{\frac{3}{2}}) &= 6\sqrt{6} \\ f_{xy}(0, \sqrt{\frac{3}{2}}) &= 0 \end{aligned}$$

$$D = 3\sqrt{6} \times 6\sqrt{6} - 0^2 = 108$$

Since $D > 0$ and $f_{xx} > 0$ we know this critical point is a local minimum

2. Quick questions

(a) Determine the directional derivative of $f(x,y,z) = e^x \cdot \cos(y) \cdot (1-z)^2$ in direction $\bar{u} = (0.36, 0.48, 0.8)$ from the origin:

$$D_u f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)\bar{u}_1 + f_y(x_0,y_0,z_0)\bar{u}_2 + f_z(x_0,y_0,z_0)\bar{u}_3$$

$$\begin{aligned} f_x &= e^x cos(y)(1-z)^2 \\ f_y &= -e^x sin(y)(1-z)^2 \\ f_z &= e^x cos(y)(2z-2) \end{aligned}$$

$$\begin{aligned} f_x(0,0,0) &= 1 \cdot 1 \cdot 1 = 1 \\ f_y(0,0,0) &= -1 \cdot 0 \cdot 1 = 0 \\ f_z(0,0,0) &= 1 \cdot 1 \cdot -2 = -2 \end{aligned}$$

$$D_u f(0,0,0) = 0.36 - 1.6 = -1.24$$

(b) Determine the local linear approximation of $f(x,y,z) = (1+x)(1-y^2)(1-z)^2$ at the point $(1,2,3)$

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

$$\begin{aligned} f_x &= (1-y^2)(1-z)^2 \\ f_y &= (1+x)(-2y)(1-z)^2 \\ f_z &= 2(1+x)(1-y^2)(z-1) \end{aligned}$$

$$\begin{aligned} f(1,2,3) &= (1+1)(1-2^2)(1-3)^2 = -24 \\ f_x(1,2,3) &= (1-2^2)(1-3)^2 = -12 \\ f_y(1,2,3) &= (1+1)(-2(2))(1-3)^2 = -32 \\ f_z(1,2,3) &= 2(1+1)(1-2^2)(3-1) = -24 \end{aligned}$$

$$\begin{aligned} L(1,2,3) &= -24 - 12(x-1) - 32(y-2) - 24(z-3) \\ L(1,2,3) &= 124 - 12x - 32y - 24z \end{aligned}$$

(c) Determine the 2nd degree Taylor polynomial of $f(x,y) = e^{-x^2}e^{-y^2}$ as the point (1,1)

$$\begin{aligned} L(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ p_2(x,y) &= L(x,y) + \frac{1}{2}\left[f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2\right] \end{aligned}$$

$$\begin{aligned} f_x &= (-2x)e^{-x^2}e^{-y^2} \\ f_y &= (-2y)e^{-x^2}e^{-y^2} \end{aligned}$$

$$\begin{aligned} f_{xx} &= (4x^2-2)e^{-x^2-y^2} \\ f_{yy} &= (4y^2-2)e^{-x^2-y^2} \\ f_{xy} &= (4xy)e^{-x^2}e^{-y^2} \end{aligned}$$

$$\begin{aligned} L(1,1) &= e^{-2} - 2e^{-2}(x-1) - 2e^{-2}(y-1) = e^{-2}(5-2x-2y) \\ p_2(1,1) &= e^{-2}(5-2x-2y) + \frac{1}{2}\left[2e^{-2}(x-2)^2 + 2e^{-2}(y-2)^2 + 8e^{-2}(x-1)(y-1)\right] \\ &= e^{-2}(x^2+y^2-8x-8y+4xy+11) \end{aligned}$$

(d) Determine the gradient of $f(x,y) = x^3 + y^3 - 4x - 2y$ along the curve $(x(t), y(t)) = (t^3 - 2t, t^2)$ when $t = 1$

$$\triangledown f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$

$$\begin{aligned} f_x &= 3x^2 - 4 \\ f_y &= 3y^2 - 2 \\ \therefore \triangledown f(x,y) &= (3x^2-4)\mathbf{i} + (3y^2-2)\mathbf{j} \end{aligned}$$

$$\begin{aligned} (x(1), y(1)) &= (-1, 1) \\ \triangledown f(-1,1) &= f_x(-1,1)\mathbf{i} + f_y(-1,1)\mathbf{j} \\ &= -1\mathbf{i} + 1\mathbf{j} \end{aligned}$$

(e) Determine the tangent plane to the surface $z = x^2 + xy - y^4$ at the point $(x,y) = (2,1)$

$$F(x,y,z) = z - x^2 - xy + y^4 = 0$$

Find z at the point $(x,y) = (2,1)$

$$z = 2^2 + 2 - 1^4 = 5$$

$$\begin{aligned} \triangledown F(x,y,z) &= (-2x-y)\mathbf{i} + (4y^3-x)\mathbf{j} + \mathbf{k} \\ \triangledown F(2,1,5) &= -5\mathbf{i} + 2\mathbf{j} + \mathbf{k} \end{aligned}$$

$$\begin{aligned} \text{TangentPlane} &= -5(x-2) + 2(y-1) + (z-5) \\ z &= 5x - 2y - 3 \end{aligned}$$

3. Double Integrals

(a) Determine the integral of $f(x,y) = e^{-x}\cos(y)$ over the rectangular region $R = (x,y) : x \in [0,2], y \in [\frac{-\pi}{2}, \frac{\pi}{2}]$'

$$\int_{-\pi/2}^{\pi/2}\int_0^2 e^{-x}\cos(y)\,dx\,dy$$

$$\begin{aligned} &= \int_{-\pi/2}^{\pi/2}\cos(y)\int_0^2 e^{-x}\,dx\,dy \\ &= \int_{-\pi/2}^{\pi/2}\cos(y)\Big| -e^{-x}\Big|_{x=0}^{x=2} dy \\ &= \int_{-\pi/2}^{\pi/2}\cos(y)(-e^{-2} - -e^0)\,dy \end{aligned}$$

$$\begin{aligned} &= \int_{-\pi/2}^{\pi/2}\cos(y)(-e^{-2}+1)\,dy \\ &= (-e^{-2}+1)\Big|\sin(y)\Big|_{y=-\pi/2}^{y=\pi/2} \\ &= (-e^{-2}+1)(\sin(\pi/2) - \sin(-\pi/2)) \end{aligned}$$

$$= -2e^{-2} + 2$$

(b) Determine the integral of $f(x,y) = \sin(x+y)$ over the triangular region for which $x \ge 0,\ y \ge 0$ and $x + y \le \pi$

$$\int_0^\pi\int_0^{\pi-x}\sin(x+y)\,dy\,dx$$

$$\begin{aligned} &= \int_0^\pi\Big| -\cos(x+y)\Big|_0^{\pi-x} dx \\ &= \int_0^\pi -\cos(\pi) + cos(x)\,dx \\ &= \int_0^\pi 1 + \cos(x)\,dx \end{aligned}$$

$$\begin{aligned} &= \Big|x + \sin(x)\Big|_0^\pi \\ &= (\pi + \sin(\pi)) - (0 + \sin(0)) \\ &= \pi \end{aligned}$$

(c) Determine the area of the region $R = \{(x,y) : e^{y/3} \le x \le 10 + \sin(y), y \in [0,5]\}$

$$\int_0^5\int_{e^{y/3}}^{10+\sin(y)} dx\,dy$$

$$\int_0^5\Big|x\Big|_{e^{y/3}}^{10+\sin(y)} dy$$

$$\int_0^5 10 + \sin(y) - e^{y/3}\,dy$$

$$\begin{aligned} &= \Big|10y - \cos(y) - 3e^{y/3}\Big|_0^5 \\ &= 50 - \cos(5) - 3e^{5/3} + \cos(0) + 3e^0 \\ &= 37.832 \end{aligned}$$

(d) Determine the average of $f(x,y) = 3y - 2x$ over the region $R = \{(x,y) : 0 \le y \le 4 - x^2, x \in [-2,2]\}$

$$\mu = \frac{1}{|R|}\iint_R f(x,y)dA$$

$$\begin{aligned} |R| &= \int_{-2}^2\int_0^{4-x^2} dy\,dx \\ &= \int_{-2}^2 \Big|y\Big|_0^{4-x^2} dx \\ &= \int_{-2}^2 4 - x^2\,dx \end{aligned}$$

$$\begin{aligned} &= \Big|4x - \frac{x^3}{3}\Big|_{-2}^2 \\ &= 8 - \frac{2^3}{3} + 8 - \frac{2^3}{3} \\ &= \frac{32}{3} \end{aligned}$$

$$\begin{aligned} &\int_{-2}^2\int_0^{4-x^2} 3y - 2x\,dy\,dx \\ &= \int_{-2}^2\Big|\frac{3y^2}{2} - 2xy\Big|_0^{4-x^2} dx \\ &= \int_{-2}^2 \frac{3(4-x^2)^2}{2} - 2x(4-x^2)\,dx \\ &= \int_{-2}^2 \frac{3x^4}{2} + 2x^3 - 12x^2 - 8x + 24\,dx \end{aligned}$$

$$\begin{aligned} &= \int_{-2}^2 \frac{3x^5}{10} + \frac{x^4}{2} - 4x^3 - 4x^2 + 24x \\ &= \Big|\frac{3x^5}{10} + \frac{x^4}{2} - 4x^3 - 4x^2 + 24x\Big|_{-2}^2 \\ &= (\frac{3(2)^5}{10} + \frac{(2)^4}{2} - 4(2)^3 - 4(2)^2 + 24(2)) - (\frac{3(-2)^5}{10} + \frac{(-2)^4}{2} - 4(-2)^3 - 4(-2)^2 + 24(-2)) \\ &= \frac{256}{5} \end{aligned}$$

$$\begin{aligned} \mu &= \frac{\frac{256}{5}}{\frac{32}{3}} \\ &= \frac{24}{5} = 4.8 \end{aligned}$$

(e) Determine the surface area of the surface described by $z = \sqrt{9 - x^2}$ over the region $R = \{(x,y) : 0 \le y \le x, x \in [0,3]\}$

$$\text{Surface Area } = \iint_R \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}dA$$

$$\text{Integrand} = \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}$$

$$\begin{aligned} f_x &= -\frac{x}{\sqrt{9-x^2}} \\ f_y &= 0 \end{aligned}$$

$$\text{Integrand} = \sqrt{\left(-\frac{x}{\sqrt{9-x^2}}\right)^2 + 1}$$

$$= \sqrt{\frac{x^2}{9-x^2} + 1} = \frac{3}{\sqrt{9-x^2}}$$

$$\begin{aligned} \text{Surface Area } &= \int_0^3\int_0^x \frac{3}{\sqrt{9-x^2}}\,dy\,dx \\ &= \int_0^3\Big|\left(\frac{3}{\sqrt{9-x^2}}\right)y\Big|_0^x dx \\ &= \int_0^3\left(\frac{3}{\sqrt{9-x^2}}\right)x\,dx \end{aligned}$$

$$\begin{aligned} &= \Big| -3\sqrt{9-x^2}\Big|_0^3 \\ &= -3\sqrt{0} + 3\sqrt{9} \\ &= 9 \end{aligned}$$