ENGR222 Assignment 2

Niels Clayton : 300437590

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1. Multiple Integrals

   (a) Evaluate the integral of $f(x,y,z) = xyz$ over the region:

$$G = \{(x,y,z) : xy \le z \le 1,\ 0 \le x \le y,\ 0 \le y \le 1\}$$

$$\begin{aligned}
\iiint_G f(x,y,z)dV &= \int_0^1 \int_0^y \int_{xy}^1 xyz\ dz\, dx\, dy \\
&= \int_0^1 \int_0^y xy\left|\frac{z^2}{2}\right|_{z=xy}^{z=1} dx\, dy \\
&= \int_0^1 \int_0^y xy\left(\frac{1}{2} - \frac{x^2y^2}{2}\right) dx\, dy \\
&= \int_0^1 \int_0^y \frac{1}{2}(xy - x^3y^3)\ dx\, dy \\
&= \int_0^1 \frac{1}{2}\left|\frac{x^2y}{2} - \frac{x^4y^3}{4}\right|_0^y dy \\
&= \frac{1}{8}\left|\frac{y^4}{2} - \frac{y^8}{8}\right|_{y=0}^{y=1} \\
&= \frac{1}{8}\left(\frac{1}{2} - \frac{1}{8}\right) = \frac{3}{64}
\end{aligned}$$

   (b) Using spherical coordinates, determine the integral of $f(x,y,z) = x$ over the region G described by the inequalities $x, y, z \ge 0$ and $x^2 + y^2 + z^2 \le 1$

   In spherical ordinates we have:
   $f(r,\theta,\phi) = r\cos(\theta)\sin(\phi)$ for $G = \{(r,\theta,\phi) : 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{2}\}$

$$\begin{aligned}
\iiint_G f(r,\theta,\phi)dV &= \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} r\cos(\theta)\sin(\phi)\ d\phi\, d\theta\, dr \\
&= \int_0^1 r\ dr \int_0^{\frac{\pi}{2}} \cos(\theta)\ d\theta \int_0^{\frac{\pi}{2}} \sin(\phi)\ d\phi \\
&= \frac{r^2}{2}\Big|_{r=0}^{r=1} \times \sin(\theta)\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} \times -\cos\phi\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} \\
&= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}
\end{aligned}$$

   (c) Calculate the integral of $f(x,y) = y^{-2}e^{-x}$ over the region

$$R = \{(x,y) : x \in [0,\infty], y \in [2,\infty]\}$$

$$\begin{aligned}
\iint_R f(x,y)\ dA &= \int_2^\infty y^{-2}\ dy \int_0^\infty e^{-x}\ dx \\
&= -\frac{1}{y}\Big|_2^\infty \times -e^{-x}\Big|_0^\infty \\
&= \left(0 + \frac{1}{2}\right) \times (0+1) \\
&= \frac{1}{2}
\end{aligned}$$

   (d) Determine the centroid of the two dimensional object described in polar coordinates by

$$R = \{(r,\theta) : 0 \le r \le \theta, \theta \in [0, 2\pi]\}$$

$$\begin{aligned}
\bar{x} &= \frac{1}{\text{area of R}} \iint_R r^2\cos(\theta)\ dr\ d\theta \\
\bar{y} &= \frac{1}{\text{area of R}} \iint_R r^2\sin(\theta)\ dr\ d\theta
\end{aligned}$$

$$\begin{aligned}
\text{area of R} &= \int_{\theta_0}^{\theta_1} \frac{1}{2}r^2\ d\theta \\
&= \int_0^{2\pi} \frac{1}{2}\theta^2\ d\theta \\
&= \frac{1}{6}\theta^3\Big|_0^{2\pi} \\
&= \frac{1}{6}8\pi^3 = \frac{4\pi^3}{3}
\end{aligned}$$

$$\begin{aligned}
\iint_R r^2\cos(\theta)\ dr\ d\theta &= \int_0^{2\pi}\int_0^\theta r^2\cos(\theta)\ dr\ d\theta \\
&= \frac{1}{3}\int_0^{2\pi} r^3\cos(\theta)\Big|_0^\theta\ d\theta \\
&= \frac{1}{3}\int_0^{2\pi} \theta^3\cos(\theta)\ d\theta \\
&= \frac{1}{3}(12\pi^2) = 4\pi^2
\end{aligned}$$

$$\begin{aligned}
\iint_R r^2\sin(\theta)\ dr\ d\theta &= \int_0^{2\pi}\int_0^\theta r^2\sin(\theta)\ dr\ d\theta \\
&= \frac{1}{3}\int_0^{2\pi} r^3\sin(\theta)\Big|_0^\theta\ d\theta \\
&= \frac{1}{3}\int_0^{2\pi} \theta^3\sin(\theta)\ d\theta \\
&= \frac{1}{3}(12\pi - 8\pi^3)
\end{aligned}$$

$$\begin{aligned}
\bar{x} &= \frac{1}{\frac{4\pi^3}{3}}4\pi^2 = \frac{3}{4\pi^3}4\pi^2 \\
&= \frac{3}{\pi} \approx 0.955
\end{aligned}$$

$$\begin{aligned}
\bar{y} &= \frac{1}{\frac{4\pi^3}{3}}\frac{(12\pi - 8\pi^3)}{3} = \frac{(12\pi - 8\pi^3)}{4\pi^3} \\
&= \frac{3}{\pi^2} - 2 \approx -1.696
\end{aligned}$$

   The centroid can be found at $(x,y) = (\frac{3}{\pi}, \frac{3}{\pi^2} - 2)$

2. Vector Fields

   (a) Calculate the divergence of the vector field $\mathbf{F} = x^2y^3z^4\mathbf{i} - xyz\mathbf{j} + (x+y+z)\mathbf{k}$

$$\begin{aligned}
\text{div }\mathbf{F} &= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \\
&= 2xy^3z^4 - xz + 1
\end{aligned}$$

   (b) Calculate the curl of the vector field $\mathbf{F} = x^2y^3z^4\mathbf{i} - xyz\mathbf{j} + (x+y+z)\mathbf{k}$

$$\text{curl }\mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} - \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}$$

$$= (1 + xy)\,\mathbf{i} - \left(4x^2y^3z^3 - 1\right)\mathbf{j} - \left(yz + 3x^2y^2x^4\right)\mathbf{k}$$

   (c) Determine the gradient field of $\phi(x,y,z) = xz^2 + sin(y)e^x$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

$$= \left(z^2 + sin(y)e^x\right)\mathbf{i} + \left(\cos(y)e^x\right)\mathbf{j} + (2xz)\,\mathbf{k}$$

   (d) Calculate the Laplacian of $\phi(x,y,z) = xz^2 + sin(y)e^x$

$$\nabla\phi^2 = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\begin{aligned}
&= \sin(y)e^x - \sin(y)e^x + 2x \\
&= 2x
\end{aligned}$$

3. Line integrals

   (a) Calculate the value of the line integral $\int_C f\ ds$ where

$$f(x,y,z) = \frac{y}{x}e^z$$

   and C is described by

$$(x,y,z) = (2t,\ t^2,\ ln(t))\ \text{ for }\ t \in [1,4]$$

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\ ||r'(t)||\ dt$$

$$\begin{aligned}
f(2t, t^2, ln(t)) &= \frac{t^2}{2t}e^{ln(t)} \\
&= \frac{t^2}{2}
\end{aligned}$$

$$\begin{aligned}
r'(t) &= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \\
&= 2\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}
\end{aligned}$$

$$\begin{aligned}
||r'(t)|| &= \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2} \\
&= \sqrt{4 + 4t^2 + \frac{1}{t^2}} \\
&= \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}} \\
&= \frac{\sqrt{4t^4 + 4t^2 + 1}}{t} \\
&= \frac{\sqrt{(2t^2+1)^2}}{t} \\
&= \frac{(2t^2+1)}{t}
\end{aligned}$$

$$\begin{aligned}
\int_C f(x,y,z)ds &= \int_1^4 \frac{t^2}{2}\frac{(2t^2+1)}{t}\ dt \\
&= \int_1^4 \frac{t(2t^2+1)}{2}\ dt \\
&= \int_1^4 \frac{(2t^3+t)}{2}\ dt \\
&= \frac{t^4}{4} + \frac{t^2}{4}\Big|_1^4 \\
&= \frac{4^4}{4} + \frac{4^2}{4} - \frac{1^4}{4} - \frac{1^2}{4} \\
&= \frac{135}{2} = 67.5
\end{aligned}$$

   (b) Calculate the value of the line integral $\int_C \mathbf{F}\cdot\ d\mathbf{r}$ where

$$\mathbf{F}(x,y,z) = x\mathbf{i} - e^z\mathbf{j} + y\mathbf{k}$$

   and C is described by

$$(x,y,z) = (2t,\ t^2,\ ln(t))\ \text{ for }\ t \in [1,4]$$

$$\int_C \mathbf{F}\cdot\ d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\ dt$$

$$\begin{aligned}
\mathbf{F}(\mathbf{r}(t)) &= 2t\mathbf{i} - e^{ln(t)}\mathbf{j} + t^2\mathbf{k} \\
&= 2t\mathbf{i} - t\mathbf{j} + t^2\mathbf{k}
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}'(t) &= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \\
&= 2\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}
\end{aligned}$$

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = 5t - 2t^2$$

$$\begin{aligned}
\int_C \mathbf{F}\cdot\ d\mathbf{r} &= \int_1^4 5t - 2t^2\ dt \\
&= \frac{5}{2}t^2 - \frac{2}{3}t^3\Big|_1^4 \\
&= \left(\frac{5}{2}\times 16\right) - \left(\frac{2}{3}\times 64\right) - \left(\frac{5}{2}\right) + \left(\frac{2}{3}\right) \\
&= -4.5
\end{aligned}$$

   (c) Calculate the value of the line integral $\int_C \mathbf{F}\cdot\ d\mathbf{r}$ where F is the gradient field of

$$\phi = \cos(x\sin(ye^z))$$

   and C is described by the vector-valued function

$$\mathbf{r}(t) = \left(\pi\cos(\frac{\pi t}{2})\right)\mathbf{i} + \left(\frac{\pi}{2} + \sin(8\pi t)\right)\mathbf{j} + \left(t - t^2\right)\mathbf{k} \quad \text{for} \quad t \in [0,1]$$

$$\int_C \mathbf{F}(x,y,z)\cdot\ d\mathbf{r} = \phi(x_1,y_1,z_1) - \phi(x_0,y_0,z_0)$$

$$\begin{aligned}
\mathbf{r}(0) &= (\pi\cos(0))\,\mathbf{i} + \left(\frac{\pi}{2} + \sin(0)\right)\mathbf{j} + (0)\,\mathbf{k} \\
&= \pi\mathbf{i} + \frac{\pi}{2}\mathbf{j} + 0\mathbf{k} \\
&\therefore (x_0,y_0,z_0) = (\pi, \frac{\pi}{2}, 0)
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}(1) &= \left(\pi\cos(\frac{\pi}{2})\right)\mathbf{i} + \left(\frac{\pi}{2} + \sin(8\pi)\right)\mathbf{j} + \left(1 - 1^2\right)\mathbf{k} \\
&= 0\mathbf{i} + \frac{\pi}{2}\mathbf{j} + 0\mathbf{k} \\
&\therefore (x_1,y_1,z_1) = (0, \frac{\pi}{2}, 0)
\end{aligned}$$

$$\begin{aligned}
\int_C \mathbf{F}(x,y,z)\cdot\ d\mathbf{r} &= \phi(0,\frac{\pi}{2},0) - \phi(\pi,\frac{\pi}{2},0) \\
&= \cos(0\times\sin(\frac{\pi}{2}e^0)) - \cos(\pi\times\sin(\frac{\pi}{2}e^0)) \\
&= 2
\end{aligned}$$

   (d) Confirm that the vector field is conservative

$$\mathbf{F}(x,y) = \left(-2xe^{-x^2}\sin(y)\right)\mathbf{i} + \left(1 + e^{-x^2}\cos(y)\right)\mathbf{j}$$

   Then determine the potential function of $\mathbf{F}$

$$\begin{aligned}
f_y &= -2xe^{-x^2}\cos(y) \\
g_x &= -2xe^{-x^2}\cos(y)
\end{aligned}$$

   Since $\frac{df}{dy} = \frac{dg}{dx}$ the vector field is conservative

   Since the vector field is conservative, there exists a function where

$$\frac{\partial\phi}{\partial x} = -2xe^{-x^2}\sin(y) \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = 1 + e^{-x^2}\cos(y)$$

$$\begin{aligned}
\phi &= \int f(x,y)\ dx \\
&= \int -2xe^{-x^2}\sin(y)\ dx \\
&= e^{-x^2}\sin(y) + k(y)
\end{aligned}$$

   To find $k(y)$ we differentiate ϕ with respect to y and compare it to what we know $\frac{\partial\phi}{\partial y}$ must be.

$$\frac{\partial\phi}{\partial y} = 1 + e^{-x^2}\cos(y)$$

$$\begin{aligned}
\phi\frac{d}{dy} &= e^{-x^2}\sin(y) + k(y)\ \frac{d}{dy} \\
&= e^{-x^2}\cos(y) + k'(y)
\end{aligned}$$

$$\therefore k'(y) = 1$$

$$k(y) = \int k'(y)\ dy = \int 1\ dy = y$$

$$\phi = e^{-x^2}\sin(y) + y + K$$

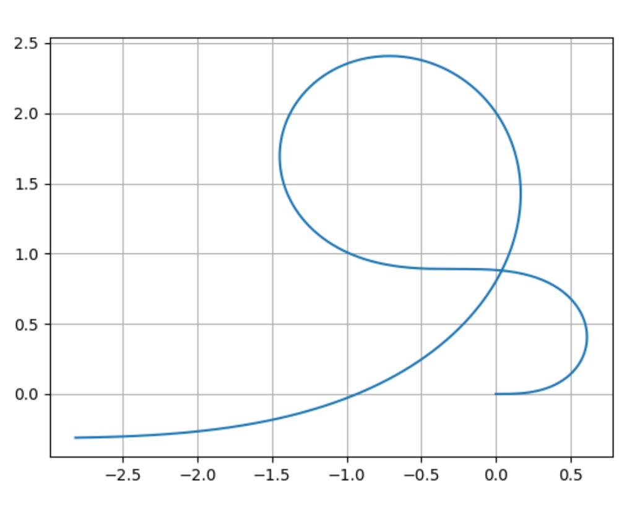
   For any closed curve C the circular integral is defined as

$$\oint \mathbf{F}\ d\mathbf{r} = \iint_R (g_x - f_y)\ dA$$

   Since $\mathbf{F}(x,y)$ is conservative, $g_x = f_y$, meaning that the circular integral is 0

4. Lab questions

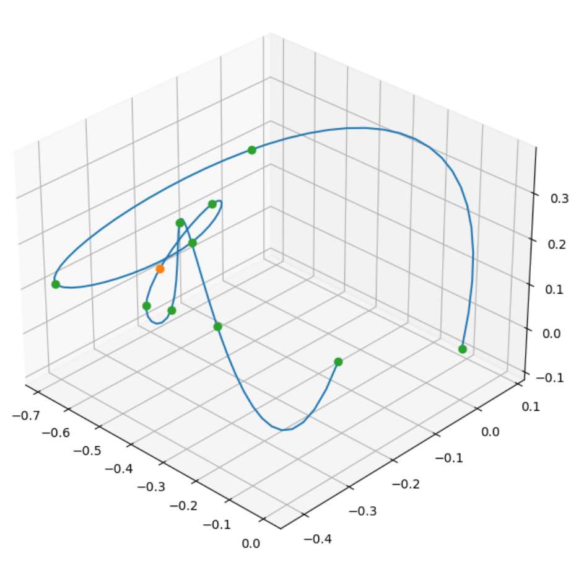
   (a) i. The coordinates at s = 10.0 are:
   $(x,y) = (-2.81794675549219, -0.3112599911165987)$



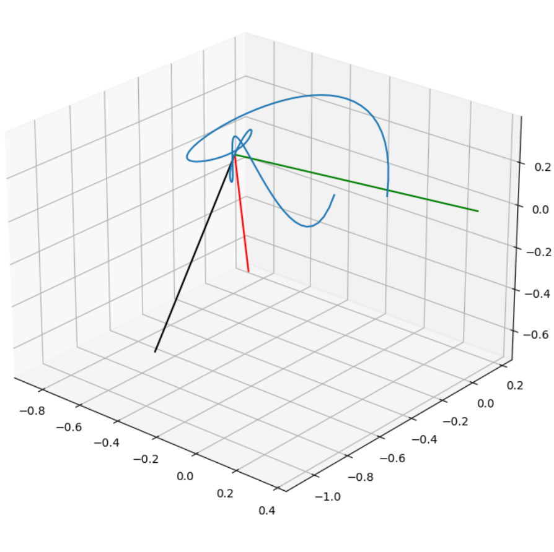
   ii. For $h = 0.01$: $K(s=5) = 1.2001127248505328$
   For $h = 0.005$: $K(s=5) = 1.2001135935941987$
   For $h = 0.001$: $K(s=5) = 1.200113871589581$
   For $h = 0.0005$: $K(s=5) = 1.2001138802762434$

   $K(s=5) \approx 1.20011$

   (b) i. `The coordinates at t = 2 are:`
   $(x,y,z) = (-0.5580567444088435, -0.2720113135900402, 0.11995195426460033)$



   ii. Determine the unit tangent, principal unit normal and binormal vectors at t = 2



   (c) i. The integral evaluates to: 8906.117634354592
   ii. The integral evaluates to: 10.787064853079256

Lab Code

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import cumtrapz
from scipy.interpolate import UnivariateSpline
from scipy.integrate import dblquad
from scipy.integrate import tplquad


def a_i():
    theta = lambda u : np.pi * np.sin( np.log(1 + u**2))

    n = 1001
    s = np.linspace(0, 10, n)
    x = cumtrapz(np.cos(theta(s)), s, initial=0)
    y = cumtrapz(np.sin(theta(s)), s, initial=0)

    print(f"The coordinates at s = {s[-1]} are: $(x,y) = ({x[-1]},{y[-1]})$")

    plt.plot(x,y)
    plt.grid()
    plt.show()


def a_ii():
    theta = lambda u : np.pi * np.sin( np.log(1 + u**2))

    s_0 = 5
    steps = [0.01, 0.005, 0.001, 0.0005]

    for h in steps:
        df = (theta(s_0 + h) - theta(s_0 - h))/(2 * h)

        curvature = np.abs(df)
        print(f"For $h = {h}$ : $K(s=5) = {curvature}$")


def b_i():
    ti = [0.0, 0.6 , 1.1 , 1.5 , 1.8 , 2.1 , 2.3 , 2.5 , 2.8 , 3.2 ]
    xi = [0.0,-0.44,-0.69,-0.63,-0.62,-0.55,-0.63,-0.67,-0.44,-0.10]
    yi = [0.0,-0.15,-0.41,-0.15,-0.11,-0.31,-0.20,-0.15,-0.23,-0.21]
    zi = [0.0, 0.36, 0.11, 0.09, 0.16, 0.06,-0.04, 0.12, 0.01, 0.05]

    f = UnivariateSpline(ti,xi,s=0)
    g = UnivariateSpline(ti,yi,s=0)
    h = UnivariateSpline(ti,zi,s=0)
    t = np.linspace(ti[0],ti[-1],101)

    print(f"The coordinates at t = 2 are: $(x,y,z) = ({f(2)},{g(2)},{h(2)})$")

    fig = plt.figure()
    ax = fig.add_subplot(111, projection='3d')
    ax.plot(f(t),g(t),h(t))
    ax.plot(f(2),g(2),h(2), 'o')
    ax.plot(xi,yi,zi,'o')

    plt.show()

def b_ii():
    ti = [0.0, 0.6 , 1.1 , 1.5 , 1.8 , 2.1 , 2.3 , 2.5 , 2.8 , 3.2 ]
    xi = [0.0,-0.44,-0.69,-0.63,-0.62,-0.55,-0.63,-0.67,-0.44,-0.10]
    yi = [0.0,-0.15,-0.41,-0.15,-0.11,-0.31,-0.20,-0.15,-0.23,-0.21]
    zi = [0.0, 0.36, 0.11, 0.09, 0.16, 0.06,-0.04, 0.12, 0.01, 0.05]

    f = UnivariateSpline(ti,xi,s=0)
    g = UnivariateSpline(ti,yi,s=0)
    h = UnivariateSpline(ti,zi,s=0)
    t = np.linspace(ti[0],ti[-1],101)

    r = lambda t:np.array([f(t),g(t),h(t)]).T
    dfdt = f.derivative()
    dgdt = g.derivative()
    dhdt = h.derivative()
    v = lambda t:np.array([dfdt(t),dgdt(t),dhdt(t)]).T
    d2fdt2 = dfdt.derivative()
    d2gdt2 = dgdt.derivative()
    d2hdt2 = dhdt.derivative()
    a = lambda t:np.array([d2fdt2(t),d2gdt2(t),d2hdt2(t)]).T

    rk = r(2)
    vk = v(2)
    ak = a(2)

    Tk = vk/np.linalg.norm(vk)
    Nk = ak*np.dot(vk,vk)-vk*np.dot(ak,vk)
    Nk /= np.linalg.norm(Nk)
    Bk = np.cross(Tk,Nk)

    fig = plt.figure()
    ax = fig.add_subplot(111, projection='3d')
    ax.plot([rk[0],rk[0]+Tk[0]],[rk[1],rk[1]+Tk[1]],[rk[2],rk[2]+Tk[2]],'k-')
    ax.plot([rk[0],rk[0]+Nk[0]],[rk[1],rk[1]+Nk[1]],[rk[2],rk[2]+Nk[2]],'r-')
    ax.plot([rk[0],rk[0]+Bk[0]],[rk[1],rk[1]+Bk[1]],[rk[2],rk[2]+Bk[2]],'g-')
    ax.plot(f(t),g(t),h(t))
    plt.show()

def c_i():

    f = lambda y, x : np.cos(x) * np.exp(y)
    g1 = lambda x : x**2
    g2 = lambda x : np.sin(x) + 10

    print(f"The integral evaluates to: {dblquad(f,-3, 3, g1, g2)[0]}")


def c_ii():

    f = lambda x,y,z : 4 / (1 + x**2 + y**2 + z**2)
    def F(r,t,p):
        x = r*np.cos(t)*np.sin(p)
        y = r*np.sin(t)*np.sin(p)
        z = r*np.cos(p)
        return f(x,y,z)*r**2*np.sin(p)

    print(f"The integral evaluates to: {tplquad(F,0,np.pi,0,2*np.pi,0,1)[0]}")

c_ii()
```