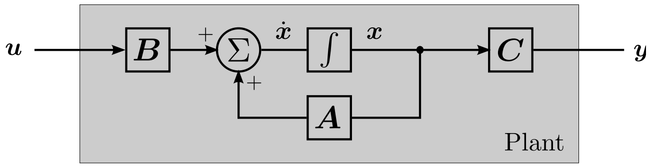
1 Open Loop System

This system is a simple state space model for a plant. It assumes that there is no feedforward from the input $\boldsymbol{u}$ to the output $\boldsymbol{y}$. That is, it assumes $\boldsymbol{D} = 0$.



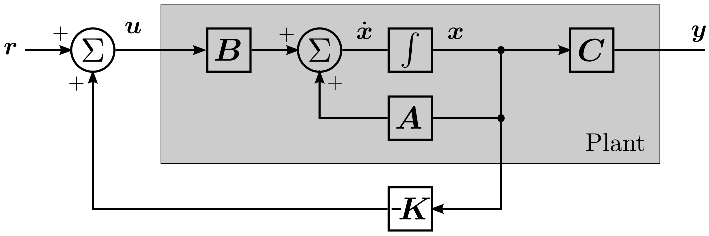
$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} \end{cases}$$

Matlab:

```
ss(A,B,C,0)
```

2 Closed Loop System

This system includes feedback from the state vector $\boldsymbol{x}$ to the input.



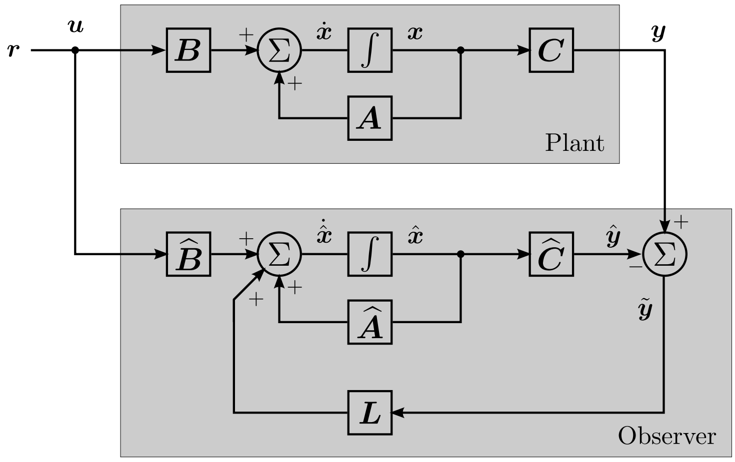
$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{r} - \boldsymbol{B}\boldsymbol{K}\boldsymbol{x} \\ \quad = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\,\boldsymbol{x} + \boldsymbol{B}\boldsymbol{r} \\ \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} \end{cases}$$

Matlab:

```
ss(A-B*K,B,C,0)
```

3 Open Loop System with unused Observer

This system is useful when you want to assess the observer's behaviour in forming its estimate of the system's state vector. In this system the observer is fully functional, but the plant is open loop. The presence of the observer consequently does not affect the response of either $\boldsymbol{x}$ or $\boldsymbol{y}$.



$$
\begin{cases}
\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{r} \\
\dot{\hat{\boldsymbol{x}}} = \widehat{\boldsymbol{A}}\hat{\boldsymbol{x}} + \boldsymbol{L}(\boldsymbol{y} - \hat{\boldsymbol{y}}) + \widehat{\boldsymbol{B}}\boldsymbol{r} \\
\quad = \widehat{\boldsymbol{A}}\hat{\boldsymbol{x}} + \boldsymbol{L}\boldsymbol{C}\boldsymbol{x} - \boldsymbol{L}\widehat{\boldsymbol{C}}\hat{\boldsymbol{x}} + \widehat{\boldsymbol{B}}\boldsymbol{r} \\
\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}
\end{cases}
$$

Using the augmented state vector $\begin{bmatrix}\boldsymbol{x} & \hat{\boldsymbol{x}}\end{bmatrix}^\mathsf{T}$ we can express this as

$$
\begin{bmatrix}\dot{\boldsymbol{x}} \\ \dot{\hat{\boldsymbol{x}}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{L}\boldsymbol{C} & \widehat{\boldsymbol{A}} - \boldsymbol{L}\widehat{\boldsymbol{C}}\end{bmatrix} \begin{bmatrix}\boldsymbol{x} \\ \hat{\boldsymbol{x}}\end{bmatrix} + \begin{bmatrix}\boldsymbol{B} \\ \widehat{\boldsymbol{B}}\end{bmatrix} \boldsymbol{r}
$$

$$
\boldsymbol{y} = \begin{bmatrix}\boldsymbol{C} & \boldsymbol{0}\end{bmatrix} \begin{bmatrix}\boldsymbol{x} \\ \hat{\boldsymbol{x}}\end{bmatrix}
$$

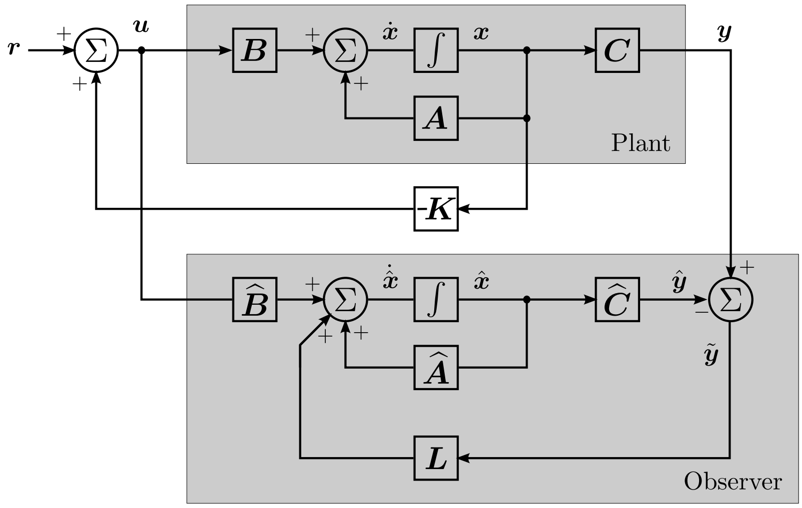
Matlab:

```
ss([A,zeros(size(A));L*C,Ae-L*Ce],[B;Be],[C zeros(size(C))],0)
```

where `Ae`, `Be`, `Ce` and `De` are the estimated model matrices $\widehat{\boldsymbol{A}}$, $\widehat{\boldsymbol{B}}$, $\widehat{\boldsymbol{C}}$ and $\widehat{\boldsymbol{D}}$ respectively.

4 Closed Loop System with unused Observer

This system implements a system, with feedback from the state vector, $\boldsymbol{x}$. An observer is in place, but as the estimate of the state vector is not used to form the feedback signal, its presence has no effect on the system dynamics. You may want to use this system to assess the performance of an observer when the system is running closed loop.



$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{B}\boldsymbol{K}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{r} \\ \dot{\hat{\boldsymbol{x}}} = \widehat{\boldsymbol{A}}\hat{\boldsymbol{x}} + \boldsymbol{L}\boldsymbol{C}\boldsymbol{x} - \boldsymbol{L}\widehat{\boldsymbol{C}}\hat{\boldsymbol{x}} - \widehat{\boldsymbol{B}}\boldsymbol{K}\boldsymbol{x} + \widehat{\boldsymbol{B}}\boldsymbol{r} \\ \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} \end{cases}$$

Using the augmented state vector $\begin{bmatrix}\boldsymbol{x} & \hat{\boldsymbol{x}}\end{bmatrix}^\mathsf{T}$ we can express this as

$$\begin{bmatrix}\dot{\boldsymbol{x}} \\ \dot{\hat{\boldsymbol{x}}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{L}\boldsymbol{C} - \widehat{\boldsymbol{B}}\boldsymbol{K} & \widehat{\boldsymbol{A}} - \boldsymbol{L}\widehat{\boldsymbol{C}}\end{bmatrix}\begin{bmatrix}\boldsymbol{x} \\ \hat{\boldsymbol{x}}\end{bmatrix} + \begin{bmatrix}\boldsymbol{B} \\ \widehat{\boldsymbol{B}}\end{bmatrix}\boldsymbol{r}$$

$$\boldsymbol{y} = \begin{bmatrix}\boldsymbol{C} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{x} \\ \hat{\boldsymbol{x}}\end{bmatrix}$$

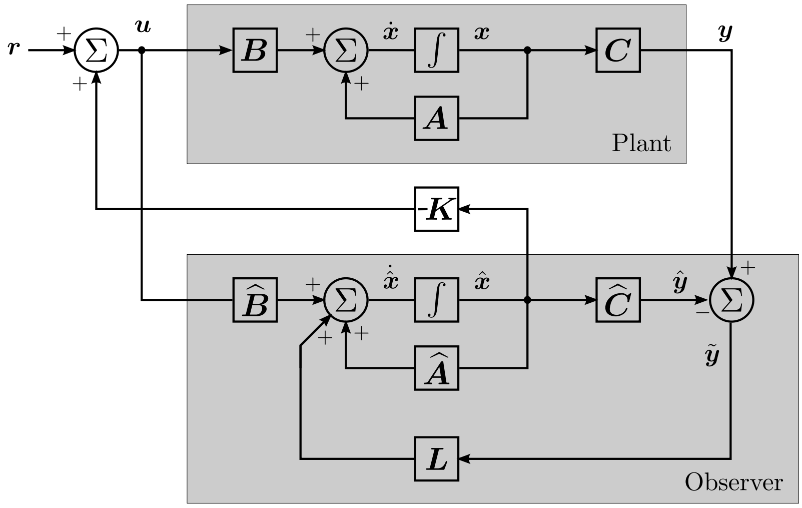
Matlab:

```
ss([A-B*K,zeros(size(A)); L*C-Be*K,Ae-L*Ce],[B;Be],[C zeros(size(C))],0)
```

where `Ae`, `Be`, `Ce` and `De` are the estimated model matrices $\widehat{\boldsymbol{A}}$, $\widehat{\boldsymbol{B}}$, $\widehat{\boldsymbol{C}}$ and $\widehat{\boldsymbol{D}}$ respectively.

5 Closed Loop System using an Observer

This system implements a full controller using the estimate of the state variable ($\hat{\boldsymbol{x}}$) to produce the required feedback signals.



$$
\begin{cases}
\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{B}\boldsymbol{K}\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{r} \\
\dot{\hat{\boldsymbol{x}}} = \widehat{\boldsymbol{A}}\hat{\boldsymbol{x}} + \boldsymbol{L}\boldsymbol{C}\boldsymbol{x} - \boldsymbol{L}\widehat{\boldsymbol{C}}\hat{\boldsymbol{x}} - \widehat{\boldsymbol{B}}\boldsymbol{K}\hat{\boldsymbol{x}} + \widehat{\boldsymbol{B}}\boldsymbol{r} \\
\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}
\end{cases}
$$

Using the augmented state vector $\begin{bmatrix}\boldsymbol{x} & \hat{\boldsymbol{x}}\end{bmatrix}^\mathsf{T}$ we can express this as

$$
\begin{bmatrix}\dot{\boldsymbol{x}} \\ \dot{\hat{\boldsymbol{x}}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & -\boldsymbol{B}\boldsymbol{K} \\ \boldsymbol{L}\boldsymbol{C} & \widehat{\boldsymbol{A}} - \widehat{\boldsymbol{B}}\boldsymbol{K} - \boldsymbol{L}\widehat{\boldsymbol{C}}\end{bmatrix}\begin{bmatrix}\boldsymbol{x} \\ \hat{\boldsymbol{x}}\end{bmatrix} + \begin{bmatrix}\boldsymbol{B} \\ \widehat{\boldsymbol{B}}\end{bmatrix}\boldsymbol{r}
$$

$$
\boldsymbol{y} = \begin{bmatrix}\boldsymbol{C} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{x} \\ \hat{\boldsymbol{x}}\end{bmatrix}
$$

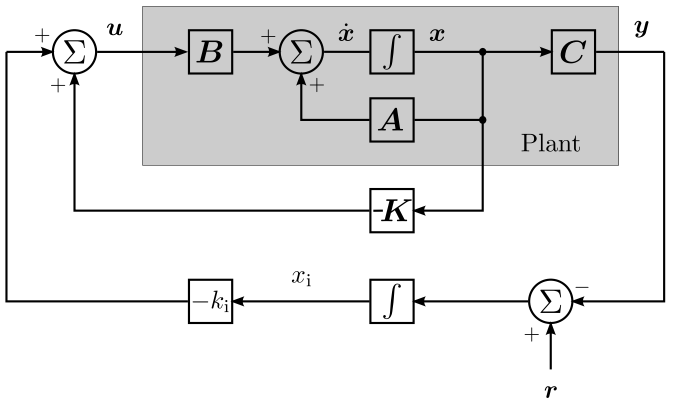
Matlab:

```
ss([A,-B*K;L*C,Ae-Be*K-L*Ce],[B;Be],[C zeros(size(C))],0)
```

where `Ae`, `Be`, `Ce` and `De` are the estimated model matrices $\widehat{\boldsymbol{A}}$, $\widehat{\boldsymbol{B}}$, $\widehat{\boldsymbol{C}}$ and $\widehat{\boldsymbol{D}}$ respectively.

6 Continuous Time Integrator

This system adds a state x_i to the state vector, which is defined to be the integral of the difference between the output and the reference input r. That is, $x_\mathrm{i}(t) = \int_0^t \big(r(\tau) - y(\tau)\big)\,\mathrm{d}\tau$



$$\begin{cases} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{B}\boldsymbol{K}\boldsymbol{x} - \boldsymbol{B}k_\mathrm{i}x_\mathrm{i} \\ \dot{x}_\mathrm{i} &= -\boldsymbol{C}\boldsymbol{x} + r \\ y &= \boldsymbol{C}\boldsymbol{x} \end{cases}$$

Using the augmented state vector $\begin{bmatrix}\boldsymbol{x} & x_\mathrm{i}\end{bmatrix}^\mathsf{T}$, we can express this as

$$\begin{bmatrix}\dot{\boldsymbol{x}} \\ \dot{x}_\mathrm{i}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} & -\boldsymbol{B}k_\mathrm{i} \\ -\boldsymbol{C} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{x} \\ x_\mathrm{i}\end{bmatrix} + \begin{bmatrix}\boldsymbol{0} \\ 1\end{bmatrix} r$$
$$y = \begin{bmatrix}\boldsymbol{C} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{x} \\ x_\mathrm{i}\end{bmatrix}$$

Matlab, assuming `A_i` $\equiv \begin{bmatrix}\boldsymbol{A} & 0 \\ -\boldsymbol{C} & 0\end{bmatrix}$, `B_i` $\equiv \begin{bmatrix}\boldsymbol{B} \\ 0\end{bmatrix}$ and `K_i` $\equiv \begin{bmatrix}\boldsymbol{K} & k_\mathrm{i}\end{bmatrix}$:

```
ss(A_i - B_i * K_i, [zeros(size(B)); 1], [C 0], D);
```

7 Pole Locations in the z-plane from ζ and ω_{d}

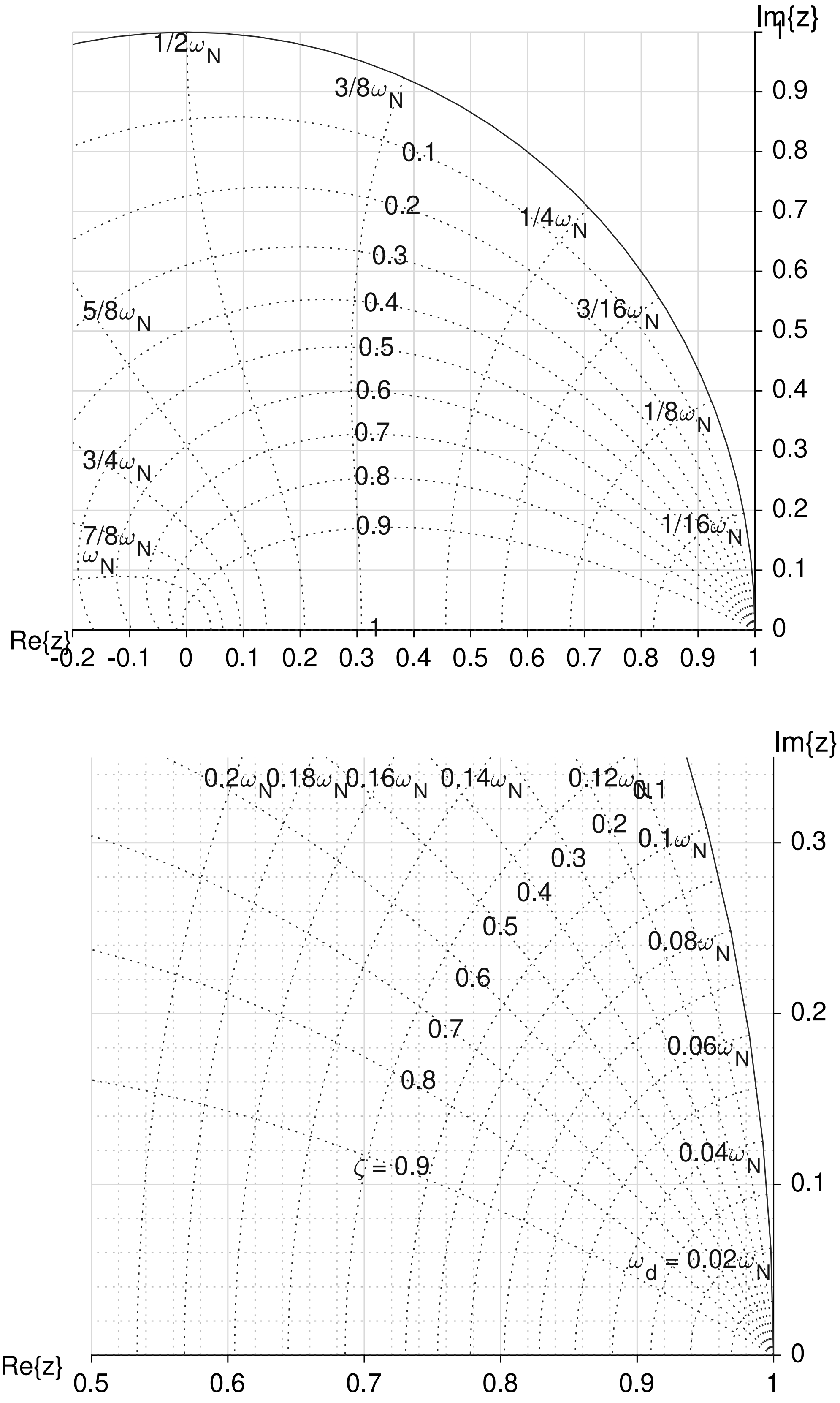


Figure 1: The curved lines show curves having constant damping ratio and damped frequency.