ENGR222 Assignment 1

Niels Clayton : 300437590

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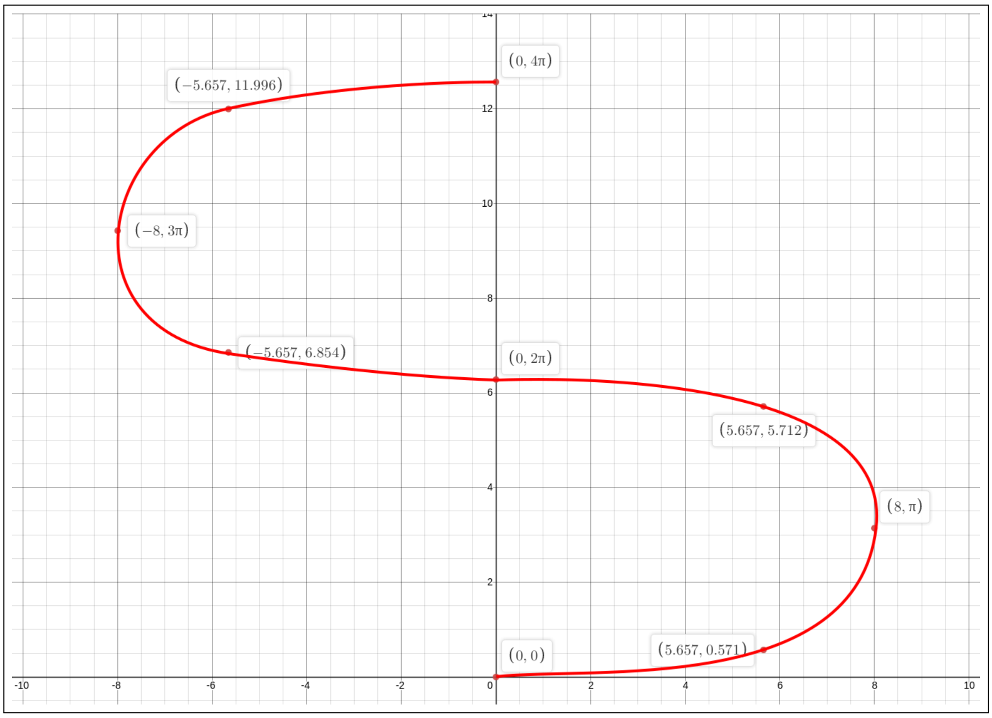
1. Consider the parametric equation:

$$(x, y) = (8sin(t), 2t - sin(2t))$$

over the interval $0 \leq t \leq 2\pi$

(a) Determine the location at $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$ and use this to draw a rough sketch of the curve.



(b) Find the unit tangent vector to the curve when $t = \frac{\pi}{6}$

$$\big(f'(t), g'(t)\big) = (8cos(t), 2 - 2cos(2t))$$
$$t = \frac{\pi}{6} : \big(f'(t), g'(t)\big) = (6.928203, 1)$$

Calculate the unit tangent vector:

$$\frac{(f'(t), g'(t))}{||(f'(t), g'(t))||} = \frac{(6.928203, 1)}{\sqrt{6.928203^2 + 1}} = \left(\frac{6.928203}{7}, \frac{1}{7}\right)$$

(c) Determine an equation describing the tangent line at $t = \frac{\pi}{6}$

$$\begin{aligned} &= (f(t), g(t)) + t\frac{(f'(t), g'(t))}{||(f'(t), g'(t))||} \\ &= (4, 0.181) + t \cdot \left(\frac{6.928203}{7}, \frac{1}{7}\right) \\ &= \left(\frac{6.928203 \cdot t}{7} + 4, \frac{t}{7} + 0.181\right) \end{aligned}$$

(d) Determine an equation describing the normal line at $t = \frac{\pi}{6}$

$$\begin{aligned}
&= (f(t), g(t)) + t\frac{(-g'(t), f'(t))}{||(f'(t), g'(t))||} \\
&= (4, 0.181) + t \cdot \left(\frac{-1}{7}, \frac{6.928203}{7}\right) \\
&= \left(4 - \frac{t}{7}, 0.181 + \frac{6.928203 \cdot t}{7}\right)
\end{aligned}$$

(e) Calculate the arc length over the interval $0 \leq t \leq 2\pi$

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{f'\left(t\right)^2 \ + \ g'\left(t\right)^2}\,\mathrm{dt} \\
&= \int_0^{2\pi} \sqrt{\left(8cos\left(t\right)\right)^2 + \left(2 - 2cos\left(2t\right)\right)^2}\,\mathrm{dt} \\
&= \int_0^{2\pi} \sqrt{64\cos\left(t\right)^2 + \left(2 - 2\cos\left(2t\right)\right)^2}\,\mathrm{dt} \\
&= \int_0^{2\pi} \sqrt{\left(32\cos\left(2t\right) + 32\right) + \left(4 + 4\cos\left(2t\right)^2 - 8\cos\left(2t\right)\right)}\,\mathrm{dt} \\
&= \int_0^{2\pi} \sqrt{4\cos\left(2t\right)^2 + 24\cos\left(2t\right) + 36}\,\mathrm{dt} \\
&= \int_0^{2\pi} \sqrt{4\left(\cos\left(2t\right)^2 + 6\cos\left(2t\right) + 9\right)}\,\mathrm{dt} \\
&= \int_0^{2\pi} \sqrt{4\left(\cos\left(2t\right) + 3\right)^2}\,\mathrm{dt} \\
&= \ 2\int_0^{2\pi} \left(\cos\left(2t\right) + 3\right)\mathrm{dt} \\
&= \Big| sin(2t) + 6t \Big|_0^{2\pi} \\
&= (0 + 12\pi) - (0 + 0) \\
&= 12\pi \approx 37.6991
\end{aligned}$$

2. Consider the curve described by the vector valued function

$$\bar{r}(t) = \frac{1}{4}(e^{2t} - 2t)\bar{i} + e^t\bar{j}$$

(a) Find a point on the curve for which $\bar{r}(t) \cdot \bar{j} = 2$

$$\begin{aligned}
\bar{r}(t) \cdot \bar{j} &= e^t = 2 \\
t &= \ln(2) = 0.693147 \\
\bar{r}(\ln(2)) &= (0.635, 2)
\end{aligned}$$

(b) Determine the unit tangent vector to the curve (for arbitrary t)

$$\bar{r}'(t) = \frac{1}{4}(2e^{2t} - 2)\bar{i} + e^t\bar{j}$$
$$\bar{T}(t) = \frac{\bar{r}'(t)}{||\bar{r}'(t)||}$$
$$= \frac{\frac{1}{4}(2e^{2t} - 2)\bar{i} + e^t\bar{j}}{\sqrt{(\frac{1}{4}(2e^{2t} - 2))^2 + e^{2t}}}$$
$$= \frac{(e^{2t} - 1)}{(e^{2t} + 1)}\bar{i} + \frac{2e^t}{(e^{2t} + 1)}\bar{j}$$

(c) Determine the principal unit normal vector to the curve (for arbitrary t)

$$\bar{N}(t) = \frac{\bar{T}'(t)}{||\bar{T}'(t)||}$$

$$\bar{T}'(t) = \frac{2e^{2t}(e^{2t} + 1) - 2e^{2t}(e^{2t} - 1)}{(e^{2t} + 1)^2}\bar{i} + \frac{2e^t(e^{2t} + 1) - 2e^t \cdot 2e^{2t}}{(e^{2t} + 1)^2}\bar{j}$$

$$\bar{T}'(t) = \frac{4e^{2t}}{(e^{2t} + 1)^2}\bar{i} + \frac{2e^t - 2e^{3t}}{(e^{2t} + 1)^2}\bar{j} \equiv \operatorname{sech}^2(t)\bar{i} + (-\operatorname{sech}(t)\tanh(t))\bar{j}$$

$$||\bar{T}(t)|| = \sqrt{\operatorname{sech}^4(t) + (-\operatorname{sech}(t)\tanh(t))^2} = \sqrt{\operatorname{sech}^2(t)}$$

$$\bar{N}(t) = \frac{\operatorname{sech}(t)^2\,\bar{i} + (-\operatorname{sech}(t)\tanh(t))\,\bar{j}}{\sqrt{\operatorname{sech}^2(t)}}$$

(d) Determine the curvature of the curve (for arbitrary t)

$$\kappa(t) = \frac{||\bar{T}'(t)||}{||\bar{r}'(t)||}$$
$$= \frac{2\sqrt{\operatorname{sech}^2(t)}}{(e^{2t} + 1)}$$

(e) Determine the arc length of the curve over $0 \leq t \leq 3$

$$
\begin{aligned}
L &= \int_0^3 ||\bar{r}'(t)|| \\
&= 2\int_0^3 (e^{2t} + 1)\,\mathrm{dt} \\
&= \left|\frac{1}{4}(e^{2t} + 2t)\right|_0^3 \\
&= \frac{1}{4}((e^6 + 6) - (1 + 0)) \\
&= 102.107
\end{aligned}
$$

3. Quick questions

(a) Determine the arc length parametrisation of:

$$(f(t), g(t), h(t)) = (3t, t - 2, -5t + 7)$$

with $t = 0$ as the starting/reference point

$$
\begin{aligned}
\left(f'(t), g'(t), h'(t)\right) &= (3, 1, -5) \\
||\left(f'(t), g'(t), h'(t)\right)|| &= \sqrt{3^2 + 1^2 + (-5)^2} = \sqrt{35}
\end{aligned}
$$

$$
\begin{aligned}
s &= \int_0^t ||\bar{r}'(u)||\,\mathrm{du} = \int_0^t \sqrt{35}\,\mathrm{du} \\
&= \left|\sqrt{35}\,\mathrm{u}\right|_0^t = \sqrt{35}t
\end{aligned}
$$

$$
\begin{aligned}
t &= \frac{1}{\sqrt{35}}s \\
(f(s), g(s), h(s)) &= \left(\frac{3}{\sqrt{35}}s, \frac{1}{\sqrt{35}}s - 2, \frac{-5}{\sqrt{35}}s + 7\right)
\end{aligned}
$$

(b) Determine the arc length parametrisation of:

$$\bar{r}(t) = (5\cos(t) + 3)\bar{i} + (-5\sin(t) + 2)\bar{j}$$

using $t = 0$ as the starting/reference point

$$\begin{aligned}
\bar{r}'(t) &= (-5\sin(t))\bar{i} + (-5\cos(t))\bar{j} \\
||\bar{r}'(t)|| &= \sqrt{(-5\sin(t))^2 + (-5\cos(t))^2} \\
&= \sqrt{25(\sin(t)^2 + \cos(t))^2} \\
&= \sqrt{25}\sqrt{1} = 5
\end{aligned}$$

$$\begin{aligned}
s &= \int_0^t ||\bar{r}'(t)||\,\mathrm{du} = \int_0^t 5\,\mathrm{du} \\
&= \left| 5\,\mathrm{u} \right|_0^t = 5t
\end{aligned}$$

$$\begin{aligned}
t &= \frac{s}{5} \\
\bar{r}(s) &= \left(5\cos\left(\frac{s}{5}\right) + 3\right)\bar{i} + \left(-5\sin\left(\frac{s}{5}\right) + 2\right)\bar{j}
\end{aligned}$$

(c) Find the unit tangent vector to:

$$\bar{r}(t) = (\sqrt{2}\cos(t))\bar{i} + (\sin(t))\bar{j} + (\sin(t))\bar{k}$$

at $t = \frac{\pi}{3}$

$$\bar{T}(t) = \frac{\bar{r}'(t)}{||\bar{r}'(t)||}$$

$$\begin{aligned}
\bar{r}'(t) &= (-\sqrt{2}\sin(t))\bar{i} + (\cos(t))\bar{j} + (\cos(t))\bar{k} \\
||\bar{r}'(t)|| &= \sqrt{(-\sqrt{2}\sin(t))^2 + (\cos(t))^2 + (\cos(t))^2} \\
&= \sqrt{2\sin^2(t) + 2\cos^2(t)} \\
&= \sqrt{2}\sqrt{1} = \sqrt{2}
\end{aligned}$$

$$\begin{aligned}
\bar{T}(t) &= \frac{(-\sqrt{2}\sin(t))\bar{i} + (\cos(t))\bar{j} + (\cos(t))\bar{k}}{\sqrt{2}} \\
\bar{T}\left(\frac{\pi}{3}\right) &= \frac{(-\sqrt{2}\sin(\frac{\pi}{3}))\bar{i} + (\cos(\frac{\pi}{3}))\bar{j} + (\cos(\frac{\pi}{3}))\bar{k}}{\sqrt{2}}
\end{aligned}$$

$$\bar{T}\left(\frac{\pi}{3}\right) = \frac{\left(-\frac{\sqrt{6}}{2}\right)\bar{i} + \left(\frac{1}{2}\right)\bar{j} + \left(\frac{1}{2}\right)\bar{k}}{\sqrt{2}}$$

(d) An alternative formula for the binormal vector is:

$$\bar{B}(t) = \frac{\bar{r}'(t) \times \bar{r}''(t)}{||\bar{r}'(t) \times \bar{r}''(t)||}$$

Use this to find the binormal vector to:

$$\bar{r}(t) = t\bar{i} - t^3\bar{j} + t^2\bar{k}$$

At the point $t = 1$

$$\begin{aligned}
\bar{r}'(t) &= \bar{i} - 3t^2\bar{j} + 2t\bar{k} \\
\bar{r}''(t) &= 0\bar{i} - 6t\bar{j} + 2\bar{k}
\end{aligned}$$

$$\begin{aligned}
\bar{r}'(t) \times \bar{r}''(t) &= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -3t^2 & 2t \\ 0 & -6t & 2 \end{vmatrix} \\
&= \begin{vmatrix} -3t^2 & 2t \\ -6t & 2 \end{vmatrix} \bar{i} - \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix} \bar{j} + \begin{vmatrix} 1 & -3t^2 \\ 0 & -6t \end{vmatrix} \bar{k} \\
&= 6t^2\bar{i} - 2\bar{j} - 6t\bar{k}
\end{aligned}$$

$$||\bar{r}'(t) \times \bar{r}''(t)|| = \sqrt{36t^4 + 4 + 36t^2}$$

$$\bar{B}(t) = \frac{6t^2\bar{i} - 2\bar{j} - 6t\bar{k}}{\sqrt{36t^4 + 4 + 36t^2}}$$

$$\bar{B}(1) = \frac{6\bar{i} - 2\bar{j} - 6\bar{k}}{\sqrt{76}}$$

(e) Find the minimum and maximum curvature for the curve described by:

$$\bar{r}(t) = (3\sin(t) + 2)\bar{i} + (2\cos(t) + 1)\bar{j}$$
$$\kappa(t) = \frac{||\bar{r}'(t) \times \bar{r}''(t)||}{||\bar{r}'(t)||^3}$$

$$\bar{r}'(t) = 3\cos(t)\bar{i} - 2\sin(t)\bar{j}$$
$$\bar{r}''(t) = -3\sin(t)\bar{i} - 2\cos(t)\bar{j}$$

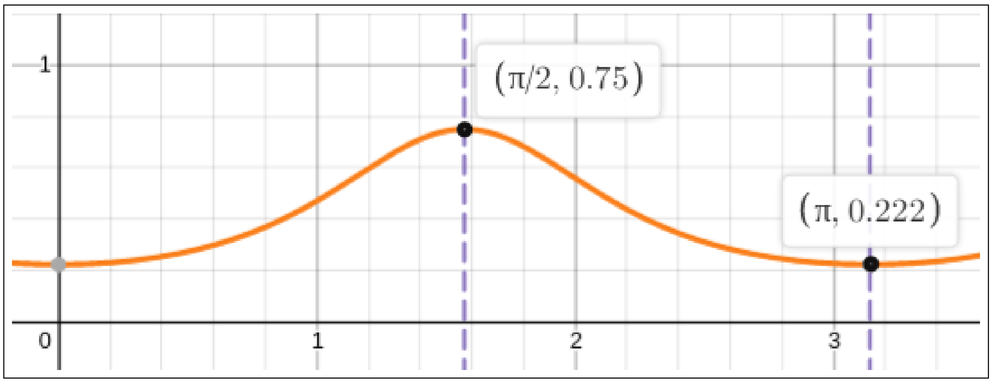
$$\begin{aligned}\bar{r}'(t) \times \bar{r}''(t) &= \begin{vmatrix} \bar{i} & \bar{j} \\ 3\cos(t) & -2\sin(t) \\ -3\sin(t) & -2\cos(t) \end{vmatrix} \bar{k} \\ &= \left(-6\cos^2(t) - 6\sin^2(t)\right)\bar{k} = -6\bar{k}\end{aligned}$$

$$||\bar{r}'(t) \times \bar{r}''(t)|| = \sqrt{-6^2} = 6$$

$$\begin{aligned}||\bar{r}'(t)|| &= \sqrt{(3\cos(t))^2 + (-2\sin(t))^2} \\ &= \sqrt{9\cos^2(t) + 4\sin^2(t)} \\ &= \sqrt{5\cos^2(t) + 4}\end{aligned}$$

$$\kappa(t) = \frac{6}{\left(\sqrt{5\cos^2(t) + 4}\right)^3}$$

Since the curvature κ is periodic on π due to the $\cos^2(t)$, we know that the minimum and maximum will fall on $\frac{\pi}{2}$ and π. In the following plot we can see that $t = \frac{\pi}{2}$ is the maximum, and $t = \pi$ is the minimum.



$$\kappa(\frac{\pi}{2}) = \frac{6}{\left(\sqrt{5\cos^2(\frac{\pi}{2}) + 4}\right)^3} = 0.75$$
$$\kappa(\pi) = \frac{6}{\left(\sqrt{5\cos^2(\pi) + 4}\right)^3} = 0.22\bar{2}$$

4. Suppose a roller coaster follows a path described by:

$$\bar{r}(t) = \frac{1}{5}t(20-t)\bar{i} + \frac{t^2}{50}(20-t)\bar{j} + \frac{t}{50}(10-t)(20-t)\bar{k}$$

(a) Determine the velocity vector of the roller coaster (for arbitrary t)

$$\bar{v}(t) = (4 - \frac{2t}{5})\bar{i} + \frac{t}{50}(40-3t)\bar{j} + \left(\left(\frac{3t^2}{50} - \frac{6t}{5}\right) + 4\right)\bar{k}$$

(b) Determine the speed when $t = 5$

$$\begin{aligned}
\bar{v}(5) &= (4-2)\bar{i} + \frac{5}{50}(40-15)\bar{j} + \left(\left(\frac{75}{50} - \frac{30}{5}\right) + 4\right)\bar{k} \\
&= 2\bar{i} + 2.5\bar{j} - 0.5\bar{k} \\
||\bar{v}(5)|| &= \sqrt{2^2 + 2.5^2 + -0.5^2} = 3.24037
\end{aligned}$$

(c) Determine the acceleration vector of the roller coaster (for arbitrary t)

$$\bar{a}(t) = -\frac{2}{5}\bar{i} + (\frac{4}{5} - \frac{3t}{25})\bar{j} + \frac{3t-30}{25}\bar{k}$$

(d) Determine the curvature of curve followed by the roller coaster when $t = 5$

$$\kappa(t) = \frac{||\bar{r}'(t) \times \bar{r}''(t)||}{||\bar{r}'(t)||^3}$$

$$\begin{aligned}
\bar{v}(5) &= 2\bar{i} + 2.5\bar{j} - 0.5\bar{k} \\
||\bar{v}(5)|| &= \sqrt{10.5} \\
||\bar{v}(5)||^3 &= \sqrt{10.5}^3 = 34.02388
\end{aligned}$$

$$\bar{a}(5) = -0.4\bar{i} + 0.2\bar{j} - 0.6\bar{k}$$

$$\begin{aligned}
\bar{v}(5) \times \bar{a}(5) &= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 2.5 & -0.5 \\ -0.4 & 0.2 & -0.6 \end{vmatrix} \\
&= \begin{vmatrix} 2.5 & -0.5 \\ 0.2 & -0.6 \end{vmatrix}\bar{i} - \begin{vmatrix} 2 & -0.5 \\ -0.4 & -0.6 \end{vmatrix}\bar{j} + \begin{vmatrix} 2 & 2.5 \\ -0.4 & 0.2 \end{vmatrix}\bar{k} \\
&= -1.4\bar{i} - 1.4\bar{k} + 1.4\bar{k}
\end{aligned}$$

$$||\bar{v}(5) \times \bar{a}(5)|| = \sqrt{-1.4^2 - 1.4^2 + 1.4^2} = \sqrt{5.88}$$

$$\kappa(5) = \frac{\sqrt{5.88}}{34.02388} = 0.0712696$$

(e) Determine the normal scalar component of acceleration when $t = 5$
Comparing this with $||\bar{a}(5)||$, what can you say about the tangential scalar component of acceleration when $t = 5$?

$$\bar{a}_N(t) = \frac{||\bar{v}(t) \times \bar{a}(t)||}{||\bar{v}(t)||}$$

$$\bar{a}_N(5) = \frac{\sqrt{5.88}}{\sqrt{10.5}} = \sqrt{0.56}$$

$$||\bar{a}(5)|| = \sqrt{(-0.4)^2 + 0.2^2 + (-0.6)^2} = \sqrt{0.56}$$

It can be seen that the magnitude of the acceleration vector is equal to the normal scalar component of acceleration at $t = 5$.