ENGR122 Assignment 10 Solutions

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1. Suppose the effect of friction on a spring is proportional to its velocity.

$$\frac{d^2x}{dt^2} = \overbrace{-2x}^{\text{standard spring}} - \overbrace{2\frac{dx}{dt}}^{\text{friction}}$$

(a) Solve the equation.

Solution: The characteristic equation is

$$m^2 + 2m + 2$$

which has solutions $m = -1 \pm i$. By recipe #3, the solution has the form

$$x(t) = e^{-t}\Big(C_1 \cos(-t) + C_2 \sin(-t)\Big)$$

(b) *Explain how the solution differs from when there is* ***negative*** *friction*: The friction here causes the size of the oscillations to **decay exponentially**, whereas when the friction was negative in the tutorial question, the oscillations grew exponentially in amplitude, as time increased.

2. Consider the linear differential equation

$$\frac{dy}{dx} = \frac{x-y}{x} \qquad \text{with } y(1) = 1.$$

(a) Find the exact solution.

Solution: Rewrite in standard form as

$$\frac{dy}{dx} + \frac{1}{x}y = 1$$

so that $P(x) = \frac{1}{x}$ and $Q(x) = 1$. Then the integrating factor is

$$\mu(x) = e^{\ln x} = x$$

where $x > 0$. Next,

$$\mu(x)y = \int x \, dx = \frac{x^2}{2} + c$$

so that

$$y(x) = \frac{x}{2} + \frac{c}{x}$$

Setting $y(1) = 1$ gives $c = \frac{1}{2}$ and so

$$y(x) = \frac{x}{2} + \frac{1}{2x}$$

(b) Estimate $y(2)$ using Euler's method with $h = 0.5$.

Solution: <u>Euler</u> $(x_0, y_0) = (1, 1)$ with $h = 0.5$

$$
\begin{aligned}
x_1 &= 1.5 \\
y_1 &= y_0 + \frac{1}{2} f(x_0, y_0) = 1 + \frac{1}{2}\frac{1-1}{1} = 1
\end{aligned}
$$

Thus, $(x_1, y_1) = (1.5, 1)$.

$$
\begin{aligned}
x_2 &= 2 \\
y_2 &= y_1 + \frac{1}{2} f(x_1, y_1) = 1 + \frac{1}{2}\frac{1.5-1}{1.5} = 1 + \frac{1}{6}
\end{aligned}
$$

so $(x_2, y_2) = (2, 1\frac{1}{6})$

(c) Estimate $y(2)$ using Euler's **improved** method with $h = 0.5$.

Solution: <u>Improved Euler</u> $(x_0, y_0) = (1, 1)$ with $h = \frac{1}{2}$

$$
\begin{aligned}
x_1 &= 1.5 \\
\hat{y}_1 &= y_0 + \frac{1}{2} f(x_0, y_0) = 1 + \frac{1}{2}\frac{1-1}{1} = 1 \\
y_1 &= y_0 + \frac{h}{2}\Big[f(x_0, y_0) + f(x_1, \hat{y}_1)\Big] = 1 + \frac{1}{4}\left[0 + \frac{1.5-1}{1.5}\right] = 1\frac{1}{12}
\end{aligned}
$$

Next,

$$
\begin{aligned}
x_2 &= 2 \\
\hat{y}_2 &= y_1 + \frac{1}{2} f(x_1, y_1) = 1\frac{1}{12} + \frac{1}{2}\frac{1.5 - 1\frac{1}{12}}{1.5} = 1.222 \\
y_2 &= y_1 + \frac{h}{2}\Big[f(x_1, y_1) + f(x_2, \hat{y}_2)\Big] = 1\frac{1}{12} + \frac{1}{4}\left[\frac{1.5 - 1\frac{1}{12}}{1.5} + \frac{2 - 1.222}{2}\right] = 1.25
\end{aligned}
$$

so $(x_2, y_2) = (2, 1.25)$

(d) Compare the value of $y(2)$ for the three approaches (exact, and the two approximations).

Solution: The exact solution is $(x_2, y_2) = (2, 1.25)$. Thus, improved Euler gives the exact solution whereas Euler is slightly off.

3. Find the first and second partial derivatives (that is $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, and $\frac{\partial^2 f}{\partial x \partial y}$) of

(a) $f(x, y) = \pi x^2 y$.

Solution:

$$
\begin{aligned}
f_x &= 2\pi xy \qquad & f_{xx} &= 2\pi y \qquad & f_{xy} &= 2\pi x \\
f_y &= \pi x^2 \qquad & f_{yy} &= 0
\end{aligned}
$$

(b) $f(x, y) = \cos(x^2 + y^2)$

Solution:

$$\begin{aligned} f_x &= -2x \sin(x^2 + y^2) \\ f_y &= -2y \sin(x^2 + y^2) \\ f_{xx} &= -2\sin(x^2 + y^2) - 4x^2 \cos(x^2 + y^2) \\ f_{yy} &= -2\sin(x^2 + y^2) - 4y^2 \cos(x^2 + y^2) \\ f_{xy} &= -2xy \cos(x^2 + y^2) \end{aligned}$$

(c) $f(x, y) = e^{2x} \cos y$

Solution:

$$\begin{aligned} f_x &= 2e^{2x} \cos y & f_{xx} &= 4e^{2x} \cos y & f_{xy} &= -2e^{2x} \sin y \\ f_y &= -e^{2x} \sin y & f_{yy} &= e^{-2x} \cos y & & \end{aligned}$$

4. You are at position $(1, 2)$. For each function in #3,

(a) What is the steepest direction?

Solution:

- $$\nabla f = \begin{pmatrix} 2\pi xy \\ \pi x^2 \end{pmatrix} \qquad \nabla f(1, 2) = \begin{pmatrix} 4\pi \\ \pi \end{pmatrix}$$

- $$\nabla f = \begin{pmatrix} -2x \sin(x^2 + y^2) \\ -2y \sin(x^2 + y^2) \end{pmatrix} \qquad \nabla f(1, 2) = \begin{pmatrix} -2\sin(5) \\ -4\sin(5) \end{pmatrix}$$

- $$\nabla f = \begin{pmatrix} 2e^{2x} \cos y \\ -2e^{2x} \sin y \end{pmatrix} \qquad \nabla f(1, 2) = \begin{pmatrix} 2e^2 \cos(2) \\ -e^2 \sin(2) \end{pmatrix}$$

(b) Find the directions $\mathbf{u}$ that cause you to "walk along the side of the mountain without going up or down." Use $\mathbf{u} \cdot \nabla f = 0$ and use the $\nabla f(1, 2)$ values computed above; then normalise to a unit vector:

Solution:

- $\pi x^2 y$

$$\mathbf{u} = \pm \frac{\big(-1, 4\big)}{\sqrt{17}}$$

- $\cos(x^2+y^2)$

$$\mathbf{u} = \pm\frac{\Big(-2,1\Big)}{\sqrt{5}}$$

- $e^{2x}\cos y$

$$\mathbf{u} = \pm\frac{\Big(\sin 2, 2\cos 2\Big)}{\sqrt{\sin^2(2)+4\cos^2(2)}}$$

5. Find the stationary points of the following functions, and figure out whether they are local maxima, minima, or saddle points.

   (a) $f(x,y) = x^2 + x + y^2$

   Solution:

$$\begin{aligned} f_x &= 2x+1 & f_{xx} &= 2 & f_{xy} &= 0 \\ f_y &= 2y & f_{yy} &= -2 \end{aligned}$$

   Stationary point:

$$\begin{aligned} f_x = 0 \iff 2x+1 &= 0 & \implies & & x &= -\frac{1}{2} \\ f_y = 0 \iff 2y &= 0 & \implies & & y &= 0 \end{aligned}$$

   so stationary point is $(-\frac{1}{2},0)$. Check

$$D(-\frac{1}{2},0) = \det\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} = 4 > 0 \qquad \implies \qquad \text{local max/min}$$

   Now, since $f_{xx} > 0$ we have a local min.

   (b) $f(x,y) = y + 3y^2 + xe^x$

   Solution:

$$\begin{aligned} f_x &= e^x + xe^x & f_{xx} &= (x+2)e^x & f_{xy} &= 0 \\ f_y &= 1+6y & f_{yy} &= 6 \end{aligned}$$

   Stationary points:

$$\begin{aligned} f_x = 0 \iff (x+1)e^x &= 0 & \implies & & x &= -1 \\ f_y = 0 \iff 1+6y &= 0 & N \implies & & y &= -\frac{1}{6} \end{aligned}$$

   so stationary point is $(-1,-\frac{1}{6})$. Check:

$$D(-1,-\frac{1}{6}) = \det\begin{pmatrix}e^{-1} & 0\\ 0 & 6\end{pmatrix} = \frac{6}{e} > 0 \qquad \implies \qquad \text{extremum}$$

   Now, since $f_{xx}(-1,-\frac{1}{6}) = \frac{1}{e} > 0$ we have a local min.

6. Compute the first-order Taylor approximation at $(1, 2)$ to each of the functions in #5.

   Solution: Recall the first order Taylor expansion is

$$p_1(x, y) = f(a, b) + (x - a) \cdot f_x(a, b) + (y - b) \cdot f_y(a, b)$$

   - $f(x, y) = x^2 + x + y^2$, with $f_x = 2x + 1$ and $f_y = 2y$. Thus,

$$\text{Taylor at } (1, 2) \text{ is } p_1(x, y) = 6 + (x - 1) \cdot 3 + (y - 2) \cdot 4$$

   - $f(x, y) = y + 3y^2 + xe^x$, with $f_x = e^x + xe^x$ and $f_y = 1 + 6y$. Thus

$$\text{Taylor at } (1, 2) \text{ is } p_1(x, y) = 14 + e + (x - 1) \cdot (2e) + (y - 2) \cdot 13$$

7. Compute the second-order Taylor approximation at $(1, 2)$ to #5b.

   Solution: Recall the second order Taylor expansion is

$$p_2(x, y) = f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b) +$$
$$\frac{1}{2}(x - a)^2 f_{xx}(a, b) + (x - a)(y - b)f_{xy}(a, b) + \frac{1}{2}(y - b)^2 f_{yy}(a, b)$$

   Compute that

$$\begin{aligned} f(1, 2) &= 14 + e \\ f_x(1, 2) &= 2e & f_{xx}(1, 2) &= 3e & f_{xy}(1, 2) &= 0 \\ f_y(1, 2) &= 13 & f_{yy}(1, 2) &= 6 \end{aligned}$$

   Thus,

$$p_2(x, y) = 14 + e + (x - 1)2e + (y - 2)13 + \frac{1}{2}(x - 1)^2 3e + \frac{1}{2}(y - 2)^2 6$$