Solutions to Assignment #3

1. (a) $A+D$ not possible since A is 2×2 while D is 1×3

$$C-A=\begin{pmatrix}-7 & 1\\ 0 & 4\end{pmatrix}-\begin{pmatrix}1 & 1\\ 3 & 4\end{pmatrix}=\begin{pmatrix}-8 & 0\\ -3 & 0\end{pmatrix}$$

$D-E$ is not possible

(b) $$AB=\begin{pmatrix}1 & 1\\ 3 & 4\end{pmatrix}\begin{pmatrix}2\\ 1\end{pmatrix}=\begin{pmatrix}3\\ 10\end{pmatrix}$$

BA is not possible

$$CA=\begin{pmatrix}-7 & 1\\ 0 & 4\end{pmatrix}\begin{pmatrix}1 & 1\\ 3 & 4\end{pmatrix}=\begin{pmatrix}-7\cdot 1+1\cdot 3 & -7\cdot 1+1\cdot 4\\ 0\cdot 1+4\cdot 3 & 0\cdot 1+4 4\end{pmatrix}=$$

$$=\begin{pmatrix}-4 & -3\\ 12 & 16\end{pmatrix}$$

$$AC=\begin{pmatrix}1 & 1\\ 3 & 4\end{pmatrix}\begin{pmatrix}-7 & 1\\ 0 & 4\end{pmatrix}=\begin{pmatrix}1(-7)+1\cdot 0 & 1\cdot 1+1\cdot 4\\ 3(-7)+4\cdot 0 & 1\cdot 3+4 4\end{pmatrix}=$$

$$=\begin{pmatrix}-7 & 5\\ -21 & 19\end{pmatrix}$$

DA is not possible

DB is not possible

BD is possible $BD=\begin{pmatrix}2\\ 1\end{pmatrix}\begin{pmatrix}3 & 2 & 1\end{pmatrix}=\begin{pmatrix}6 & 4 & 3\\ 3 & 2 & 1\end{pmatrix}$

$2\times\boxed{1 \quad 1}\times 2$ $\quad 2\times 2$

$EB=\begin{pmatrix}2 & 3 & 4\\ 1 & 2 & -1\end{pmatrix}\begin{pmatrix}2\\ 1\end{pmatrix}$ is not possible

$2\times\boxed{3 \quad\quad 2}\times 1$

Do not match

$BE=\begin{pmatrix}2\\ 1\end{pmatrix}\begin{pmatrix}2 & 3 & 4\\ 1 & 2 & -1\end{pmatrix}$ do not match

$2\times\boxed{1 \quad\quad 2}\times 3$

$$AE=\begin{pmatrix}1&1\\3&4\end{pmatrix}\begin{pmatrix}2&3&4\\1&2&-1\end{pmatrix}=\begin{pmatrix}3&5&3\\10&17&8\end{pmatrix}$$

b) $$7C=7\cdot\begin{pmatrix}-7&1\\0&4\end{pmatrix}=\begin{pmatrix}-49&7\\0&28\end{pmatrix}$$

$$-3D=-3\cdot\begin{pmatrix}3&2&1\end{pmatrix}=\begin{pmatrix}-9&-6&-3\end{pmatrix}$$

$$kE=k\cdot\begin{pmatrix}2&3&4\\1&2&-1\end{pmatrix}=\begin{pmatrix}2k&3k&4k\\1k&2k&-k\end{pmatrix}$$

2. $$A^2=A\cdot A=\begin{pmatrix}4&2\\1&3\end{pmatrix}\begin{pmatrix}4&2\\1&3\end{pmatrix}=\begin{pmatrix}4\cdot4+2\cdot1&4\cdot2+2\cdot3\\1\cdot4+3\cdot1&1\cdot2+3\cdot3\end{pmatrix}=$$

$$=\begin{pmatrix}18&14\\7&11\end{pmatrix}$$

$$A^3=A^2\cdot A=\begin{pmatrix}18&14\\7&11\end{pmatrix}\cdot\begin{pmatrix}4&2\\1&3\end{pmatrix}=\begin{pmatrix}4\cdot18+14\cdot1&18\cdot2+14\cdot3\\7\cdot4+11\cdot1&7\cdot2+11\cdot3\end{pmatrix}$$

$$=\begin{pmatrix}86&78\\39&47\end{pmatrix}$$

3. $$A\cdot B=\begin{pmatrix}1&3&2\\-1&0&4\\5&1&-1\end{pmatrix}\begin{pmatrix}5&2&1\\0&3&4\\1&3&5\end{pmatrix}=$$

$$=\begin{pmatrix}1\cdot5+3\cdot0+2\cdot1&1\cdot2+3\cdot3+2\cdot3&1\cdot1+3\cdot4+2\cdot5\\(-1)5+0\cdot0+4\cdot1&(-1)\cdot2+0\cdot3+4\cdot3&(-1)\cdot1+0\cdot4+4\cdot5\\5\cdot5+1\cdot1+(-1)\cdot1&2\cdot5+1\cdot3+(-1)\cdot3&5\cdot1+4\cdot1+(-1)\cdot5\end{pmatrix}=$$

$$=\begin{pmatrix}7&17&23\\-1&10&19\\24&10&4\end{pmatrix}$$

Similarly

$$B \cdot A = \begin{pmatrix} 8 & 16 & 17 \\ 17 & 4 & 8 \\ 23 & 8 & 9 \end{pmatrix}$$

Observe that $AB \neq BA$.

4. $A^T = \begin{pmatrix} 2 & 4 & -1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ $\quad B^T = \begin{pmatrix} 1 & 0 & 3 \\ -7 & 2 & 4 \\ 0 & 5 & 5 \end{pmatrix}$

$$AB = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ -1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & -7 & 0 \\ 0 & 2 & 5 \\ 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 11 & 0 & 20 \\ 7 & -20 & 15 \\ 5 & 21 & 25 \end{pmatrix}$$

$$(AB)^T = \begin{pmatrix} 11 & 7 & 5 \\ 0 & -20 & 21 \\ 20 & 15 & 25 \end{pmatrix}$$

$$B^T A^T = \begin{pmatrix} 1 & 0 & 3 \\ -7 & 2 & 4 \\ 0 & 5 & 5 \end{pmatrix} \begin{pmatrix} 2 & 4 & -1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 11 & 7 & 5 \\ 0 & -20 & 21 \\ 20 & 15 & 25 \end{pmatrix}$$

$$(AB)^T = B^T A^T$$

5. a)

$$A=\begin{pmatrix}1&2\\3&4\end{pmatrix} \qquad |A|=1\cdot4-2\cdot3=4-6=-2\neq0$$

$$A^{-1}=\frac{1}{-2}\cdot\begin{pmatrix}4&-2\\-3&1\end{pmatrix}=\begin{pmatrix}-2&1\\\frac{3}{2}&-\frac{1}{2}\end{pmatrix}$$

b) $A+aA^{-1}=bI$

$$\begin{pmatrix}1&2\\3&4\end{pmatrix}+a\begin{pmatrix}-2&1\\\frac{3}{2}&-\frac{1}{2}\end{pmatrix}=b\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\begin{pmatrix}1&2\\3&4\end{pmatrix}+\begin{pmatrix}-2a&a\\\frac{3}{2}a&-\frac{1}{2}a\end{pmatrix}=\begin{pmatrix}b&0\\0&b\end{pmatrix}$$

$$\begin{pmatrix}1-2a&2+a\\3+\frac{3}{2}a&4-\frac{1}{2}a\end{pmatrix}=\begin{pmatrix}b&0\\0&b\end{pmatrix}$$

$$\left.\begin{aligned}1-2a&=b\\2+a&=0\\3+\tfrac{3}{2}a&=0\\4-\tfrac{1}{2}a&=b\end{aligned}\right\}\quad \begin{aligned}a&=-2\\b&=5\end{aligned}$$

6.

We choose to use the second row:

$$|A| = \begin{vmatrix} 3 & 7 & 6 \\ -2 & 1 & 0 \\ 4 & 2 & -5 \end{vmatrix} = -(-2)\cdot\begin{vmatrix} 7 & 6 \\ 2 & -5 \end{vmatrix} + 1\cdot\begin{vmatrix} 3 & 6 \\ 4 & -5 \end{vmatrix} - 0\begin{vmatrix} 3 & 7 \\ 4 & 2 \end{vmatrix}$$

$$= 2\cdot(7(-5)-2\cdot 6)+(3\cdot(-5)-6\cdot 4) =$$

$$= 2\cdot(-35-12)+(-15-24) = -433$$

7. $A = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & 1 \\ 2 & -2 & -1 \end{bmatrix}$ $b = \begin{pmatrix} 0 \\ 10 \\ -1 \end{pmatrix}$

$$|A| = \begin{vmatrix} 2 & -3 & 1 \\ 5 & 4 & 1 \\ 2 & -2 & -1 \end{vmatrix} = -43$$

$$|A_x| = \begin{vmatrix} 0 & -3 & 1 \\ 10 & 4 & 1 \\ -1 & -2 & -1 \end{vmatrix} = -43$$

$$|A_y| = \begin{vmatrix} 2 & 0 & 1 \\ 5 & 10 & 1 \\ 2 & -1 & -1 \end{vmatrix} = -43$$

$$|A_z| = \begin{vmatrix} 2 & -3 & 0 \\ 5 & 4 & 10 \\ 2 & 2 & -1 \end{vmatrix} = -43$$

So

$$x = \frac{|A_x|}{|A|} = 1, \quad y = \frac{|A_y|}{|A|} = 1, \quad z = \frac{|A_z|}{|A|} = 1$$

Solution of problem 8 ii) on page 9

9. $$\left(\begin{array}{ccc|ccc} 4 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ -1 & 1 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{l} 4R_3 + R_1 \to R_3 \\ R_2 \to R_2 \\ R_1 \to R_1 \end{array} \quad \left(\begin{array}{ccc|ccc} 4 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ 0 & 6 & 13 & 1 & 0 & 4 \end{array}\right)$$

$$\begin{array}{l} 3R_1 - 2R_2 \to R_1 \\ R_2 \to R_2 \\ R_3 - 2R_2 \to R_3 \end{array} \quad \left(\begin{array}{ccc|ccc} 12 & 0 & -5 & 3 & -2 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 & -2 & 4 \end{array}\right)$$

$$\begin{array}{l} R_1 + 3R_3 \to R_1 \\ 5R_2 - 4R_3 \to R_2 \\ R_3 \to R_3 \end{array} \quad \left(\begin{array}{ccc|ccc} 12 & 0 & 0 & 4 & -4 & 4 \\ 0 & 15 & 0 & -4 & 13 & -16 \\ 0 & 0 & 5 & 1 & -2 & 4 \end{array}\right)$$

$$\begin{array}{l} \frac{1}{12}R_1 \to R_1 \\ \frac{1}{15}R_2 \to R_2 \\ \frac{1}{5}R_3 \to R_3 \end{array} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{vmatrix} 1/3 & -1/3 & 1/3 \\ -4/15 & 13/15 & -16/15 \\ 1/5 & -2/5 & 4/5 \end{vmatrix}$$

So $A^{-1} = \begin{pmatrix} 1/3 & -1/3 & 1/3 \\ 4/15 & 13/15 & -16/15 \\ 1/5 & -2/5 & 4/5 \end{pmatrix}$

10.

The characteristic equation is:

$$|A-\lambda I| = 0$$

$$|A-\lambda I| = \left| \begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} 5-\lambda & 6 \\ 2 & 1-\lambda \end{vmatrix} =$$

$$= (5-\lambda)(1-\lambda) - 6 \cdot 2 = 5 - 5\cdot\lambda - \lambda + \lambda^2 - 12$$

$$= \lambda^2 - 6\lambda - 7$$

Thus, the characteristic equation is

$$\lambda^2 - 6\lambda - 7 = 0$$

$$\lambda_1 = -1, \quad \lambda_2 = 7$$

The eigenvectors for $\lambda_1 = -1$ satisfy

$$(A-I)x = 0, \quad \begin{pmatrix} 6 & 6 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is:

$6x + 6y = 0 \Rightarrow x = -y$

So if $y = k \in \mathbb{R}$ then $x = -k$

and the eigenvector is $\begin{pmatrix} -k \\ k \end{pmatrix} = k \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Similarely the eigenvectors for $\lambda_2 = 7$ satisfy

$$\begin{pmatrix} -2 & 6 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $-2x + 6y = 0 \Rightarrow x = 3y$

So if $y = k \in \mathbb{R}$ then $x = 3k$ and

the eigenvector is $\begin{pmatrix} 3k \\ k \end{pmatrix} = k \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

8. $\left[\begin{array}{ccc|c} 2 & 1 & -3 & -5 \\ 1 & -1 & 2 & 12 \\ 7 & -2 & 3 & 37 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 12 \\ 2 & 1 & -3 & -5 \\ 7 & -2 & 3 & 37 \end{array}\right]$

$$\xrightarrow[R_3-7R_1\to R_3]{R_2-2R_1\to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 12 \\ 0 & 3 & -7 & -29 \\ 0 & 5 & -11 & -47 \end{array}\right]$$

$$\xrightarrow{R_3-\frac{5}{3}R_2\to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 12 \\ 0 & 3 & -7 & -29 \\ 0 & 0 & 2/3 & 4/3 \end{array}\right]$$

$$\Rightarrow \begin{cases} x - y + 2z = 12 \\ 3y - 7z = -29 \\ \frac{2}{3}z = \frac{4}{3} \end{cases} \Rightarrow \boxed{z=2} \quad \boxed{x=3} \quad \boxed{y=-5}$$