Solutions to the problems #5

1. (a) $y'(x) = \left(\sin^2(5+x)\right)' = 2\sin(5+x)\cdot\left(\sin(5+x)\right)' =$

$= 2\sin(5+x)\cdot\cos(5+x)\cdot(5+x)' =$

$= 2\sin(5+x)\cdot\cos(5+x)$

(b) $y'(x) = \left(e^{2\sin x}\right)' = (2\sin x)'\cdot e^{2\sin x} =$

$= 2\cos x\cdot e^{2\sin x}$

(c) $y'(x) = \left((4x+7)^5\right)' = 5\cdot(4x+7)^4\cdot(4x+7)' =$

$= 5\cdot(4x+7)^4\cdot 4 = 20(4x+7)^4$

(d) $y'(x) = (e^{-x}\cdot\cos 5x)' = (e^{-x})'\cdot\cos 5x + e^{-x}\cdot(\cos 5x)' =$

$= -e^{-x}\cdot\cos 5x + e^{-x}\cdot\left(5(-\sin 5x)\right) =$

$= -e^{-x}\cos 5x - 5e^{-x}\sin 5x$

$= -e^{-x}(\cos 5x + 5\sin 5x)$

(e) $y'(x) = (\ln\cos 4x)' = \dfrac{(\cos 4x)'}{\cos 4x} = \dfrac{-4\sin 4x}{\cos 4x} = -4\tan 4x$

(f) $y'(x) = \left(\dfrac{1}{x^2+1}\right)' = -\dfrac{(x^2+1)'}{(x^2+1)^2} = -\dfrac{2x}{(x^2+1)^2}$

(g) $y'(x) = \left(\dfrac{x^3\sin 2x}{\cos x}\right)' = \dfrac{(x^3\sin 2x)'\cos x - (x^3\sin 2x)(\cos x)'}{\cos^2 x} =$

$= \dfrac{(x^3)'\sin 2x\cos x + x^3(\sin 2x)'\cos x + x^3\sin 2x\cdot\sin x}{\cos^2 x}$

$= \dfrac{3x^2\sin 2x\cos x + 2x^3\cos 2x\cos x + x^3\sin 2x\sin x}{\cos^2 x}$

(h) $y'(x) = (x^3 e^{-x} \tan x)' =$

$= (x^3)' \cdot e^{-x} \tan x + x^3 (e^{-x})' \tan x + x^3 e^{-x} (\tan x)' =$

$= 3x^2 \cdot e^{-x} \tan x - x^3 e^{-x} \tan x + x^3 e^{-x} \sec^2 x =$

$= x^2 e^{-x} (3\tan x - x \tan x + x \sec^2 x)$

(i) $y'(x) = \left(\dfrac{x e^{5x}}{\sin x}\right)' = \dfrac{(x e^{5x})' \sin x - x e^{5x} (\sin x)'}{\sin^2 x} =$

$= \dfrac{x' \cdot e^{5x} \sin x + x (e^{5x})' \cdot \sin x - x e^{5x} \cos x}{\sin^2 x} =$

$= \dfrac{e^{5x} \sin x + 5x e^{5x} \sin x - x e^{5x} \cos x}{\sin^2 x} =$

$= \dfrac{e^{5x} (\sin x + 5x \sin x - x \cos x)}{\sin^2 x}$

2. $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt}$

$\dfrac{dy}{dt} = \left(\dfrac{2-t}{1-t}\right)' = \dfrac{(2-t)'(1-t) - (2-t)(1-t)'}{(1-t)^2} =$

$= \dfrac{-(1-t) + (2-t)}{(1-t)^2} = \dfrac{-1+t+2-t}{(1-t)^2} = \dfrac{1}{(1-t)^2}$

$$\frac{dx}{dt} = \left(\frac{5+3t}{1-t}\right)' = \frac{(5+3t)'(1-t)-(5+3t)(1-t)'}{(1-t)^2} =$$

$$= \frac{3(1-t)+(5+3t)}{(1-t)^2} = \frac{3-3t+5+3t}{(1-t)^2} = \frac{8}{(1-t)^2}$$

So $$\frac{dy}{dx} = \frac{\frac{1}{(1-t)^2}}{\frac{8}{(1-t)^2}} = \frac{(1-t)^2}{8(1-t)^2} = \frac{1}{8}$$

3. (a) $y(x) = x^2 - x + 6$

$y'(x) = 2x - 1$

$y'(x) = 0 \Longrightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

x		$\frac{1}{2}$	
$y'(x)$	$-$	0	$+$
$y(x)$	$\searrow$		$\nearrow$
		min	

$y(\frac{1}{2})$ is a minimum

(b) $y'(x) = (x-1)' = 1 > 0 \Rightarrow y(x) \nearrow$

$y'(x) = 0$ has no solution

So ther is no turning point.

$y(x)$ is increasing

(c) $y(x) = x^3 - 12x$

$y'(x) = (x^3 - 12x)' = 3x^2 - 12 = 3(x^2 - 4) =$

$= 3(x-2)(x+2)$

$y'(x) = 0 \Rightarrow \begin{cases} x = 2 \\ x = -2 \end{cases}$

x		-2		2	
$y(x)$	$+$	0	$-$	0	$+$
$y'(x)$	$\nearrow$		$\searrow$		$\nearrow$
		max		min	

$y(-2)$ is a maximum

$y(2)$ is a minimum

4. (a) $y(t) = 3t^2 + 6t - 1$

$y'(t) = 6t + 6 = 6(t+1)$

$y''(t) = 6 > 0 \Rightarrow$ no point of inflexion

$y'(t) = 0 \Rightarrow t = -1$

t		-1	
$y'(t)$	$-$	0	$+$
$y''(t)$	$+$		$+$
$y(t)$	$\searrow$	min	$\nearrow$

$y(-1)$ is a minimum

(b) $y(t) = 4 - t - t^2$

$y'(t) = -1 - 2t = -(1+2t)$

$y''(t) = -2 < 0 \Rightarrow$ no inflexion point

$y'(t) = 0 \Rightarrow t = -\frac{1}{2}$

t		$-\frac{1}{2}$	
$y'(t)$	$+$	0	$-$
$y''(t)$	$-$		$-$
$y(t)$	$\nearrow$	max	$\searrow$

$y(-\frac{1}{2})$ is a maximum

(c) $y(x) = x^5 - \frac{5x^3}{3}$

$$y'(x) = \left(x^5 - \frac{5x^3}{3}\right)' = 5x^4 - \frac{15x^2}{3} = 5x^4 - 5x^2$$

$$= 5x^2(x^2-1) = 5x^2(x-1)(x+1)$$

$$y''(x) = 20x^3 - 10x = 10x(2x^2-1)$$

$$y'(x) = 0 \Rightarrow \begin{cases} x=0 \\ x=1 \\ x=-1 \end{cases} \quad \text{turning points}$$

$$y''(x) = 0 \Rightarrow 10x(2x^2-1) = 0 \Rightarrow$$

$$\rightarrow \begin{cases} x=0 \\ x=\frac{1}{\sqrt{2}} \\ x=-\frac{1}{\sqrt{2}} \end{cases} \quad \text{inflexion points}$$

x		-1		$-\frac{1}{\sqrt{2}}$		0		$\frac{1}{\sqrt{2}}$		1	
$y'(x)$	$+$	0	$-$		$-$	0	$-$		$-$	0	$+$
$y''(x)$	$-$		$-$	0	$-$	0	$+$	0	$+$		$+$
$y(x)$	↗	min	↘		↘		↘		↘	max	↗

$y(-1)$ is a minimum

$y(1)$ is a maximum

(d) $y(x) = x^2 \ln x \qquad x > 0$

$y'(x) = (x^2 \ln x)' = 2x \ln x + x = x(2\ln x + 1)$

$y''(x) = (2x \ln x + x)' = (2x \ln x)' + 1$

$= 2x' \ln x + 2x(\ln x)' + 1 =$

$= 2 \ln x + 2x \frac{1}{x} + 1 =$

$= 2 \ln x + 3$

$y'(x) = 0 \Rightarrow$ $x = 0$ rejected since $x > 0$

$2\ln x + 1 = 0 \Rightarrow \ln x = -\frac{1}{2} \Rightarrow \boxed{x = e^{-\frac{1}{2}}}$

turning point $x = e^{-1/2}$

$y''(x) = 0 \Rightarrow 2\ln x + 3 = 0 \Rightarrow \ln x = -\frac{3}{2} \Rightarrow \boxed{x = e^{-\frac{3}{2}}}$

infexion point $x = e^{-3/2}$

x	0		$e^{-3/2}$		$e^{-1/2}$	
$y'(x)$		$-$		$-$	0	$+$
$y''(x)$		$-$	0	$+$		$+$
$y(x)$						

min

$y(e^{-1/2})$ is a mininum.

5. (a) $y(x) = e^x \rightarrow y(0) = 1 \qquad y(1) = e$

$y'(x) = e^x \rightarrow y'(0) = 1 \qquad y'(1) = e$

$$P_1(x) = y(0) + y'(0) \cdot x =$$
$$= 1 + x$$

$$\tilde{P}_1(x) = y(1) + y'(1)(x-1) =$$
$$= e + e(x-1) = e(1 + x - 1) =$$
$$= e \cdot x$$

(b) $y(0.1) = e^{0.1} = 1.105$

$P_1(0.1) = 1. + 0.1 = 1.1$

$\tilde{P}_1(0.1) = e \cdot 0.1 = 0.27$

(c) The approximation $P_1(0.1)$ is better than the $\tilde{P}_1(0.1)$ since the polynomial P_1 is computed around $a=0$ which is close to 0.1.

6. (a) $y(x)=3x^4+1 \rightarrow y(2)=49$

$y'(x)=12x^3 \rightarrow y'(2)=96$

$y''(x)=36x^2 \rightarrow y''(2)=144$

$$P_2(x)=y(2)+y'(2)\cdot(x-2)+\frac{y''(2)}{2!}(x-2)^2=$$

$$=49+96\cdot(x-2)+\frac{144}{2}(x-2)^2=$$

$$=49+96\cdot(x-2)+72(x-2)^2$$

$$=49+96x-192+72x^2-288x+288$$

$$=72x^2-192x+145$$

(b) $y(1.8)=3\cdot(1.8)^4+1=32.4925$

$$P_2(1.8)=72\cdot(1.8)^2-192(1.8)+145=$$

$$=233.28-345.6+145$$

$$=32.68$$

7. $y(x)=\sin(x) \rightarrow y(0)=0$

$y'(x)=\cos(x) \rightarrow y'(0)=1$

$y''(x)=-\sin(x) \rightarrow y''(0)=0$

$y'''(x)=-\cos x \rightarrow y'''(0)=-1$

$y^{(4)}(x)=\sin(x) \rightarrow y^{(4)}(0)=0$

$y^{(5)}(x)=\cos(x) \rightarrow y^{(5)}(0)=1$

$$P_3(x)=y(0)+y'(0)\cdot x+\frac{y''(0)}{2!}x^2+\frac{y'''(0)}{3!}x^3$$

$$= x + \frac{-1}{6}x^3 =$$

$$= x-\frac{1}{6}x^3$$

$$P_4(x)=y(0)+y'(0)\cdot x+\frac{y''(0)}{2!}x^2+\frac{y'''(0)}{3!}x^3+\frac{y^{(4)}(0)}{4!}x^4=$$

$$= x-\frac{1}{6}x^3 = P_3(x)$$

$$P_5(x)=y(0)+y'(0)\cdot x+\frac{y''(0)}{2!}x^2+\frac{y'''(0)}{3!}x^3+\frac{y^{(4)}(0)}{4!}x^4+\frac{y^{(5)}(0)}{5!}x^5$$

$$= x-\frac{1}{6}x^3+\frac{1}{120}x^5$$