Solutions to the Assignment #4

1. $\begin{vmatrix} 1 & 3-\lambda & 4 \\ 4-\lambda & 2 & -1 \\ 1 & \lambda-6 & 2 \end{vmatrix} = 1\cdot\begin{vmatrix} 2 & -1 \\ \lambda-6 & 2 \end{vmatrix} - (3-\lambda)\begin{vmatrix} 4-\lambda & -1 \\ 1 & 2 \end{vmatrix} + 4\begin{vmatrix} 4-\lambda & 2 \\ 1 & \lambda-6 \end{vmatrix} =$

$= (4+\lambda-6) - (3-\lambda)(8-2\lambda+1) + 4((4-\lambda)(\lambda-6)-2) =$

$= -2+\lambda - (3-\lambda)(9-2\lambda) + 4(-\lambda^2+10\lambda-26) =$

$= -2+\lambda - 2\lambda^2 + 15\lambda - 27 - 4\lambda^2 + 40\lambda - 104 =$

$= -\underset{a}{6}\lambda^2 + \underset{b}{56}\lambda - \underset{c}{133} = 0$

$D = b^2 - 4ac = 56^2 - 4\cdot(-6)\cdot(-133) =$
$= 3136 - 3192 = -56 < 0$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-56 \pm i\sqrt{56}}{12} \approx 4.667 \pm i\,0.62361.$$

2. We first form the matrix

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

and we apply linear transformation to the rows:

$R_1 \to R_1$
$R_2 \to R_2$
$R_3 - 2R_1 \to R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{array}\right)$$

$R_1 - 2R_2 \to R_1$
$R_2 \to R_2$
$R_3 + R_2 \to R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -3 & -2 & 1 & 1 \end{array}\right)$$

$R_1 \to R_1$
$R_2 \to R_2$
$-\frac{1}{3}R_3 \to R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right)$$

$R_1 + R_3 \to R_1$
$R_2 - 2R_3 \to R_2$
$R_3 \to R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1+\frac{2}{3} & -2-\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{4}{3} & 1+\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right)$$

So $A^{-1} = \frac{1}{3}\begin{pmatrix} 5 & -7 & -1 \\ -4 & 5 & 2 \\ 2 & -1 & -1 \end{pmatrix}$

3. The augmented matrix is:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 5 \\ 2 & 1 & 1 & 1 & 3 \\ 1 & 2 & 1 & 0 & 4 \\ 0 & 1 & 1 & 2 & 0 \end{array}\right)$$

The steps of the elimination:

$$\begin{array}{l} R_1 \to R_1 \\ R_2 - 2R_1 \to R_2 \\ R_3 - R_1 \to R_3 \\ R_4 \to R_4 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 5 \\ 0 & -3 & -5 & -1 & -7 \\ 0 & 0 & -2 & -1 & -1 \\ 0 & 1 & 1 & 2 & 0 \end{array}\right)$$

$$\begin{array}{l} R_1 \to R_1 \\ R_4 \to R_2 \\ R_3 \to R_3 \\ R_4 \to R_2 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 5 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & -2 & -1 & -1 \\ 0 & -3 & -5 & -1 & -7 \end{array}\right)$$

$$\begin{array}{l} R_1 \to R_1 \\ R_2 \to R_2 \\ R_3 \to R_3 \\ R_4 + 3R_2 \to R_4 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 5 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & -2 & -1 & -1 \\ 0 & 0 & -2 & 5 & -7 \end{array}\right)$$

$R_1 \to R_1$

$R_2 \to R_2$

$R_3 \to R_3$

$R_4 - R_3 \to R_4$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 5 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 6 & -6 \end{array}\right)$$

So the backward substitution steps are:

$$\left.\begin{array}{r} x+2y+3z+t=5 \\ y+z+2t=0 \\ -2z-t=-1 \\ 6t=-6 \end{array}\right\} \quad \begin{array}{l} x=1 \\ y=1 \\ z=1 \\ t=-1 \end{array}$$

4. The eigenvalues are $\lambda = 1, -1, 2$
(see tutorial questions)

For $\lambda = 1$ the eignvectors can be found by solving the System:

$$\begin{pmatrix} 0 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which is

$$\left.\begin{array}{r} y-2z=0 \\ -x+y+z=0 \\ y-2z=0 \end{array}\right\} \quad \begin{array}{l} \text{Let } z=k\in\mathbb{R} \\ y=2k \\ x=3k \end{array}$$

So $e_1 = \begin{pmatrix} 3k \\ 2k \\ k \end{pmatrix} = k\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

Similarly we get:

$$e_2 = k\begin{pmatrix}1\\0\\1\end{pmatrix} \quad \text{and} \quad e_3 = K\begin{pmatrix}1\\3\\1\end{pmatrix}$$

5. $A = \begin{pmatrix}1 & 0 & -1\\ 1 & 0 & 1\\ 2 & 2 & 2\end{pmatrix}$,

$|A^2| = |A|^2$ and $|A+A| = |2A| = 2^3 \cdot |A|$

$$|A| = \begin{vmatrix}1 & 0 & -1\\ 1 & 0 & 1\\ 2 & 2 & 2\end{vmatrix} = -0\begin{vmatrix}1 & 1\\ 2 & 2\end{vmatrix} + 0\begin{vmatrix}1 & -1\\ 2 & 2\end{vmatrix} - 2\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}$$

$$= -2(1+1) = -4$$

So $|A^2| = (-4)^2 = 16$

and $|A+A| = 2^3 \cdot (-4) = -8 \cdot 4 = -32$