ENGR122 Assignment 7 Solutions

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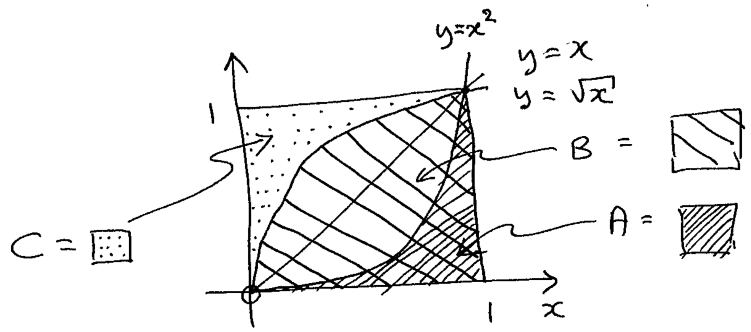
1. Show using geometry that if A is the area between $y = x^2$ and the x-interval [0,1], and if B is the area between $y = \sqrt{x}$ and the same x-interval, then A+B = 1. *Hint: since the two functions are inverses of each other in the first quadrant, if you reflect one about the line $y = x$, you get the other function.*

   Solution:



   Since reflecting $y = x^2$ about the line $y = x$ gives $y = \sqrt{x}$, the shape of region A in the figure is a mirror image of the region C, and they must have the same area by symmetry, so that A=C (using A and C to also mean the area of these regions without ambiguity). Hence the areas A+B = C+B, and since C and B combined give a square with side of length one, C+B=1 in area, hence A+B=1.

2. On Mars, gravity causes objects to accelerate downwards at 3.9m/s^2. Robin drops a laptop from a bridge on Mars. It hits the ground after 20s.

   (a) How far did it fall?

   Solution: $a(t) = 3.9\text{m/s}^2$. Therefore

$$v(t) = \int 3.9\, dt = 3.9t + c,$$

   however $c = 0$ since $v(0) = 0$, so $v(t) = 3.9t$. Next, observe that

$$d(t) = \int v(t)\, dt = \int 3.9t\, dt = \frac{3.9}{2}t^2 + c,$$

   where $c = 0$ since $d(0) = 0$ (that is, we conveniently define the height from which the laptop is dropped to be height zero). Finally, plugging in $t = 20$ seconds we obtain $d(20) = \frac{3.9}{2}400 = 780$ meters.

   (b) How fast was the laptop going when it hit the ground?

   Solution: $v(t) = 3.9t$, hence $v(20) = 3.9(20) = 78$ meters per second.

(c) If Robin threw the laptop downwards at 12m/s (from the same height as in part a), how long would it take to to hit the ground? What would its speed be when it hit?

Solution: Here, we have that

$$v(t) = 3.9t + c$$

and $v(0) = 12$, so $c = 12$ and $v(t) = 3.9t + 12$. Therefore

$$d(t) = \int 3.9t + 12 \; dt = \frac{3.9}{2}t^2 + 12t + c_2$$

where $c_2 = 0$ as before. The ground is 780 meters down by part (a). We can compute how long by solving

$$d(t) = 780 \qquad \text{for } t$$

That is,

$$\frac{3.9}{2}t^2 + 12t - 780 = 0$$

Recalling the that the solutions to $a^2 + bx + c = 0$ are of the form $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we obtain that $t = 17.16$ or $t = -23.31$. We reject the negative answer for physical reasons.

When the laptop hits the ground, its speed is

$$v(17.16) = 3.9(17.16) + 12 = 78.9 \text{ meters per second.}$$

3. Suppose you are on Jupiter, and that you want to drop a laptop off a building so that it hits the "ground" at Usain Bolt's top speed. Gravitational acceleration on Jupiter is 23.6m/s^2. Usain Bolt's top running speed is 12m/s.

   (a) How long will it take for a falling laptop to reach Usain Bolt's speed?

   Solution: Gravitational acceleration is $a(t) = 23.6$ m/s^2. It follows that $v(t) = 23.6t + c_1$ where $c_1 = 0$ since we know that $v(0) = 0$. Thus, $v(t) = 23.6t$.

   We need to solve $v(t) = 23.6t = 12$, so $t = 0.51$ seconds.

   (b) How high should the building be?

   Solution: We need the object to fall for 0.51 seconds. The distance travelled in 0.51 seconds is

   $$d(t) = \int v(t) \; dt = \int 23.6t \; dt = 11.8t^2 + c_1$$

   Since $d(0) = 0$, it follows that $c_1 = 0$, and therefore $d(t) = 11.8t^2$. Finally,

   $$d(0.51) = 3.1 \text{ meters.}$$

4. The velocity of an object is described by

(a) $v(t) = e^{-2t}$

Solution:

- $d(t) = \int v(t)\,dt = \int e^{-2t}\,dt = -\frac{1}{2}e^{-2t} + c$
- $\int_1^2 v(t)\,dt = d(2) - d(1) = -\frac{1}{2}e^{-4} + \frac{1}{2}e^{-2} = 0.0585$ meters.
- $\int_2^3 v(t)\,dt = d(3) - d(2) = -\frac{1}{2}e^{-6} + \frac{1}{2}e^{-4} = 0.008$ meters.

(b) $v(t) = \frac{t}{2} - 3t^2$

Solution:

- $\int v(t)\,dt = \frac{t^2}{4} - t^3 + c$
- $\int_1^2 v(t)\,dt = \left[\frac{2^2}{4} - 2^3\right] - \left[\frac{1}{4} - 1\right] = -6\frac{1}{4}$ meters
- $\int_2^3 v(t)\,dt = \left[\frac{3^2}{4} - 3^3\right] - \left[\frac{2^2}{4} - 2^3\right] = -17\frac{1}{4}$ meters

(c) $v(t) = 2t - e^t$

Solution:

- $\int v(t)\,dt = t^2 - e^t + c$
- $\int_1^2 v(t)\,dt = [2^2 - e^2] - [1^2 - e] = -1.67$ meters
- $\int_2^3 v(t)\,dt = [3^2 - e^3] - [2^2 - e^2] = -7.7$ meters

5. A car rolls along a street. The **vertical** velocity of a point on the wheel is given by

$$v(t) = \pi \sin\left(\frac{\pi t}{4}\right)\, m/s.$$

(a) Compute the *indefinite integral*

$$\int \pi \sin\left(\frac{\pi t}{4}\right)\, dt.$$

Solution:

$$h(t) = \int \pi \sin\left(\frac{\pi t}{4}\right)\, dt = -4\cos\left(\frac{\pi t}{4}\right) + c$$

(b) What is the value of the constant of integration, c, if the object's height at time 0 is $0m$?

Solution: $h(0) = 0 = -4\cos(0) + c = -4 + c$ (since $\cos(0) = 1$). Therefore, $c = 4$.

(c) What is the object's height after $2s$?

Solution: $h(2) = -4\cos\left(\frac{\pi}{2}\right) + 4 = 4$ meters

(d) And after $4s$?

Solution: $h(4) = -4\cos(\pi) + 4 = -4(-1) + 4 = 8$ meters

(e) What is the wheel's diameter?

Solution: The height varies between 0 and 8 meters (are you sure those are the max and min ?!?). Therefore, the diameter is 8 meters.

6. Compute the following indefinite integrals:

(a)

$$\int \Big[\sin(3x-1) - x\Big]\, dx$$

Solution:

$$\int \sin(3x-1) - x \; dx = -\frac{\cos(3x-1)}{3} - \frac{x^2}{2} + c$$

(b)

$$\int \cos^2(4x) \cdot dx$$

Solution:

$$\int \cos^2(4x)\; dx = \frac{1}{2}\int 1 + \cos(8x)\; dx = \frac{1}{2}x + \frac{\sin(8x)}{16} + c$$