ENGR122 Assignment 9 Solutions

Marking Schedule, 25 marks max total

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1. Use partial fractions to find

   (a) $\int \frac{6t+3}{2t^2-5t+2} dt$ **[2]**

   Solution:

$$\int \frac{6t+3}{2t^2-5t+2}\, dt = \int \frac{5}{t-2} - \frac{4}{2t-1}\, dt$$
$$= 5\ln|t-2| - 2\ln|2t-1| + c$$

   (b) $\int_1^2 \frac{3-3x}{2x^2+6x} dx$ **[2]**

   Solution:

$$\begin{aligned}
\frac{3}{2}\int_1^2 \frac{1-x}{x^2+3x}\, dx &= \frac{3}{2}\int_1^2 \left(\frac{1}{3x} - \frac{4}{3}\left(\frac{1}{x+3}\right)\right)\, dx \\
&= \int_1^2 \frac{1}{2x} dx - \int_1^2 \frac{2}{x+3}\, dx \\
&= \left[\frac{1}{2}\ln|x| - 2\ln|x+3|\right]_1^2 \\
&= \left[\ln\left(\frac{x^{\frac{1}{2}}}{(x+3)^2}\right)\right]_1^2 \\
&= \ln\left(\frac{16\sqrt{2}}{25}\right)
\end{aligned}$$

2. Consider the DE

$$\frac{dy}{dx} = 3y(11-y) \quad \text{subject to } y(0) = 5.$$

   (a) Solve the equation. **[2]**

   Solution: Separate the variables to obtain

$$\int \frac{dy}{3y(11-y)} = \int dx = x + c$$

   Solve

$$\frac{A}{3y} + \frac{B}{11-y} = \frac{1}{3y(11-y)}$$

   to obtain

$$11A = 1$$
$$33B = 1$$

so that $A = \frac{1}{11}$ and $B = \frac{1}{33}$. Integrating we obtain

$$\frac{1}{33}\int\left(\frac{1}{y} + \frac{1}{11-y}\right)\, dy = \frac{1}{33}\ln\left|\frac{y}{11-y}\right| = x + c$$

With some re-arranging, we obtain

$$y(1 + Ce^{33x}) = 11Ce^{33x}$$

where $C = \pm e^c$. Finally,

$$y(x) = \frac{11}{1 + \frac{1}{C}e^{-33x}}$$

(b) As $x \to -\infty$, what happens to $y(x)$? [1]

Solution: As $x \to -\infty$, $y(x) \to 0$

(c) As $x \to \infty$, what happens to $y(x)$? [1]

Solution: As $x \to \infty$, $y(x) \to 11$

3. Solve the following separable – or, nearly separable – ODEs

(a)

$$\frac{dx}{dt} = \frac{\cos 3t}{x^2} \quad \text{where } x(0) = 1$$

[2]

Solution:

$$\begin{aligned}
\int x^2\, dx &= \int \cos 3t\, dt \\
\frac{x^3}{3} &= \frac{\sin 3t}{3} + c \\
x^3 &= \sin 3t + c \\
x &= \sqrt[3]{\sin 3t + c}
\end{aligned}$$

Plug in the initial condition $x(0) = 1$ to obtain

$$\sqrt[3]{\sin 0 + c} = \sqrt[3]{c} = 1$$

so $c = 1$ and

$$x(t) = (\sin 3t + 1)^{\frac{1}{3}}$$

(b) *[integrate by parts along the way]* **[2]**

$$\frac{dy}{dx} = \frac{x \ln x}{e^y}$$

Solution:

$$\begin{aligned}
\int e^{-y}\, dy &= \int x \ln x \, dx \\
-e^{-y} &= \frac{1}{2}x^2 \ln x - \frac{x^2}{4} + c \\
e^{-y} &= -\frac{x^2}{2} \ln x + \frac{x^2}{4} + c \\
-y &= \ln\left(\frac{x^2}{4} - \frac{x^2}{2} \ln x + c\right) \\
y &= -\ln\left(\frac{x^2}{4} - \frac{x^2}{2} \ln x + c\right)
\end{aligned}$$

(c)

$$\frac{dy}{dt} = e^{y+t}$$

[2]

Solution:

$$\begin{aligned}
\int \frac{dy}{e^y} &= \int e^t \, dt \\
-e^{-y} &= e^t + c \\
y &= -\ln\left(c - e^t\right)
\end{aligned}$$

(d)

$$\frac{dy}{dx} = \frac{y}{x} + \frac{y^4}{x^4}$$

[3]

Solution: Let $z = \frac{y}{x}$. It follows that the RHS is $z + z^4$ and the LHS is

$$\frac{dy}{dx} = \frac{d}{dx}(xz) = z + x\frac{dz}{dx}$$

Thus,

$$\begin{aligned}
z + x\frac{dz}{dx} &= z + z^4 \\
\int z^{-4}\, dz &= \int \frac{1}{x}\, dx \\
-\frac{z^{-3}}{3} &= \ln|x| + c \\
z^3 &= \frac{-1}{3\ln|x| + c} \\
y &= x\left(\frac{-1}{3\ln|x| + c}\right)^{\frac{1}{3}}
\end{aligned}$$

4. Solve the following first order linear ODEs using integrating factors

(a)

$$\frac{dy}{dx} + \frac{y}{x} = 1$$

[2]

Solution: Observe that $P(x) = \frac{1}{x}$ and $Q(x) = 1$. Thus

$$\mu(x) = e^{\int P(x)\, dx} = e^{\int \frac{1}{x}\, dx} = e^{\ln x} = x$$

Even though it looks like this requires $x > 0$, since there is a term $\ln x$, we will see the only restriction on our solution is that $x \neq 0$. Next, after multiplying through the DE by $\mu = x$ and integrating the resulting exact DE, we obtain

$$\mu(x)y = \int \mu(x)Q(x)\, dx = \int x\, dx = \frac{x^2}{2} + c$$

Finally,

$$y = \frac{\frac{x^2}{2} + c}{\mu(x)} = \frac{\frac{x^2}{2} + c}{x} = \frac{x}{2} + \frac{c}{x}$$

(b) *[parts along the way]*

$$x^2\frac{dy}{dx} + xy - x^2e^x = 0$$

[2]

Solution: Rewrite the equation as

$$\frac{dy}{dx} + \frac{y}{x} = e^x$$

so $P(x) = \frac{1}{x}$ and $Q(x) = e^x$. Then

$$\mu(x) = e^{\int P(x)\,dx} = e^{\ln x} = x$$

as in the previous part (a). Next, after multiplying through the DE by $\mu = x$ and integrating the resulting exact DE, we obtain

$$\mu(x)y = \int \mu(x)Q(x)\,dx = \int xe^x\,dx = xe^x - e^x + c$$

Finally,

$$y = e^x - \frac{e^x}{x} + \frac{c}{x}$$

(c) *[use parts twice]*

$$\frac{dx}{dt} + \frac{2x}{t} = \sin t$$

[2]

Solution: $P(t) = \frac{2}{t}$ and $Q(t) = \sin t$.

$$\mu(t) = e^{\int \frac{2}{t}\,dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

Then after multiplying through the DE by μ and integrating the resulting exact DE, we obtain

$$\begin{aligned}
\mu(t)x &= \int t^2 \sin t\,dt \\
&= -t^2\cos t + 2\int t\cos t\,dt \\
&= -t^2\cos t + 2\left(t\sin t - \int \sin t\,dt\right) \\
&= -t^2\cos t + 2t\sin t + 2\cos t + c
\end{aligned}$$

Finally,

$$x = -\cos t + \frac{2\sin t}{t} + \frac{2\cos t}{t^2} + \frac{c}{t^2}$$

where $t > 0$.

(d) *[parts]*

$$\frac{dx}{dt} + 6t^2x = t^2 + 2t^5$$

[2]

Solution: $P(t) = 6t^2$ and $Q(t) = t^2 + 2t^5$. Thus

$$\mu(t) = e^{\int 6t^2\,dt} = e^{2t^3}$$

Moreover after multiplying through the DE by μ and integrating the resulting exact DE, we obtain

$$\mu(t)x = \int \left(t^2 e^{2t^3} + 2t^5 e^{2t^3}\right)\, dt$$

Let $u = 2t^3$ so $\frac{du}{dt} = 6t^2$. We then obtain

$$\begin{aligned}
\frac{1}{6}\int \left(e^u + ue^u\right)\, du &= \frac{1}{6}\int \left(e^u + ue^u\right)\, du \\
&= \frac{1}{6}e^u + \frac{1}{6}(ue^u - e^u) + c \\
&= \frac{1}{6}ue^u + c \\
&= \frac{1}{6}2t^3 e^{2t^3} + c
\end{aligned}$$

Finally

$$x(t) = \frac{\frac{1}{3}t^3 e^{2t^3} + c}{e^{2t^3}} = \frac{1}{3}t^3 + ce^{-2t^3}$$