1.) $\begin{vmatrix} 1 & 3-\lambda & 4 \\ 4-\lambda & 2 & -1 \\ 1 & \lambda-6 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ \lambda-6 & 2 \end{vmatrix} - \left(4-\lambda \begin{vmatrix} 3-\lambda & 4 \\ \lambda-6 & 2 \end{vmatrix}\right) + \begin{vmatrix} 3-\lambda & 4 \\ 2 & -1 \end{vmatrix} = 0$

$(4-(-\lambda+6)) - (4-\lambda(6-2\lambda-4\lambda+24)) - 3 + \lambda - 8 = 0$

$-2 + \lambda - 24 + 6\lambda + 8\lambda - 2\lambda^2 + 16\lambda - 4\lambda^2 - 96 + 24\lambda - 3 + \lambda - 8 = 0$

$-6\lambda^2 + 56\lambda - 133 = 0$

$\lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\qquad \lambda = \dfrac{-56 \pm \sqrt{56^2 - (4 \times {}^-6 \times {}^-133}}{-12} = \dfrac{-14}{-3} \pm \dfrac{\sqrt{14\cancel{4}}}{-6} i$

$\lambda = \dfrac{\cancel{-}14}{3} + \dfrac{\sqrt{14}}{6} i$ or $\dfrac{\cancel{-}14}{3} - \dfrac{\sqrt{14}}{6} i$

2.)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{array}\right] \quad R_3 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -3 & -2 & 1 & 1 \end{array}\right] \xrightarrow{3R_2 + 2R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & -4 & 5 & 2 \\ 0 & 0 & -3 & -2 & 1 & 1 \end{array}\right] \quad R_1 + R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 1 & 1 \\ 0 & 3 & 0 & -4 & 5 & 2 \\ 0 & 0 & -3 & -2 & 1 & 1 \end{array}\right] \xrightarrow{3R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 5 & -7 & -1 \\ 0 & 3 & 0 & -4 & 5 & 2 \\ 0 & 0 & -3 & -2 & 1 & 1 \end{array}\right]$$

Divide each row to get an Identity matrix on the left.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/3 & -7/3 & -1/3 \\ 0 & 1 & 0 & -4/3 & 5/3 & 2/3 \\ 0 & 0 & 1 & 2/3 & -1/3 & -1/3 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 5/3 & -7/3 & -1/3 \\ -4/3 & 5/3 & 2/3 \\ 2/3 & -1/3 & -1/3 \end{bmatrix}$$

3.)

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 5 \\ 2 & 1 & 1 & 1 & 3 \\ 1 & 2 & 1 & 0 & 4 \\ 0 & 1 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 5 \\ 0 & -3 & -5 & -1 & -7 \\ 1 & 2 & 1 & 0 & 4 \\ 0 & 1 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 5 \\ 0 & -3 & -5 & -1 & -7 \\ 0 & 0 & -2 & -1 & -1 \\ 0 & 1 & 1 & 2 & 0 \end{array}\right] \quad 3R_4 + R_2$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 5 \\ 0 & -3 & -5 & -1 & -7 \\ 0 & 0 & -2 & -1 & -1 \\ 0 & 0 & -2 & 5 & -7 \end{array}\right] \xrightarrow{R_4 - R_3} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 5 \\ 0 & -3 & -5 & -1 & -7 \\ 0 & 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 6 & -6 \end{array}\right]$$

$6t = -6$

$t = -1$

$-2z + 6 = -1 \therefore z = 1$

$-3y - 5 + 1 = -7 \therefore y = 1$

$x + 2 + 3 - 1 = 5 \therefore x = 1$

$$\boxed{\begin{aligned} x &= 1 \\ y &= 1 \\ z &= 1 \\ t &= -1 \end{aligned}}$$

4.)

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \qquad \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix}$$

$$\left\| \begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix} \right\| = (1-\lambda)\begin{bmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} + \begin{vmatrix} 1 & -2 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$= (1-\lambda)(\lambda^2 - 3\lambda + 2) + (3-\lambda) = 0$$

$$= \lambda^2 - 3\lambda + 2 - \lambda^3 + 3\lambda^2 - 2\lambda + 3 - \lambda = 0$$

$$= -\lambda^3 + 4\lambda^2 - 6\lambda + 5 = 0$$

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix}$$

$$\left\| \begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix} \right\| = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$= (1-\lambda)(\lambda^2 - \lambda - 3) + 1 - \lambda = 0$$

$$= \lambda^2 - \lambda - 3 - \lambda^3 + \lambda^2 + 3\lambda + 1 - \lambda = 0$$

$$= -\lambda^3 + 2\lambda^2 + \lambda - 2 = 0 \qquad \boxed{\lambda = 1, \text{ or } -1 \text{ or } 2}$$

Eigenvalues

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad \begin{aligned} y - 2z &= 0 \\ -x + y + z &= 0 \\ y - 2z &= 0 \end{aligned} \quad \therefore \quad \begin{aligned} z &= 1 \\ y &= 2 \\ x &= 3 \end{aligned} \qquad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ if } \lambda = 1$$

Eigenvectors

$$\begin{bmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad \begin{aligned} 2x + y - 2z &= 0 \\ -1x + 3y + z &= 0 \\ y &= 0 \end{aligned} \qquad \begin{aligned} z &= 1 \\ y &= 0 \\ x &= 1 \end{aligned} \qquad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ if } \lambda = -1$$

$$\begin{bmatrix} -1 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad \begin{aligned} -x + y - 2z &= 0 \\ -x + z &= 0 \\ y - 3z &= 0 \end{aligned} \qquad \begin{aligned} z &= 1 \\ y &= 3 \\ x &= 1 \end{aligned} \qquad \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \text{ if } \lambda = 2$$

5.)

$$|A^2| = |A|^2$$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

$$|A| = -1 \times \left| \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \right| + \left| \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \right|$$

$$|A| = -2 + -2 = -4$$

$$|A^2| = 16$$

$$|A+A| = \left| \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 2 \end{bmatrix} \right| = \left| \begin{bmatrix} 2 & 0 & -2 \\ 2 & 0 & 2 \\ 4 & 4 & 4 \end{bmatrix} \right|$$

$$|A+A| = -2 \left| \begin{bmatrix} 2 & 0 \\ 4 & 4 \end{bmatrix} \right| + 2 \left| \begin{bmatrix} 0 & 2 \\ 4 & 4 \end{bmatrix} \right| = -32$$

$$|A+A| = -32$$