TE WHARE WĀNANGA O TE ŪPOKO O TE IKA A MĀUI
VICTORIA
UNIVERSITY OF WELLINGTON

ENGR 122 Laboratory Instructions 2018

Lab 4: Differential Equations and Integrating Factors

4.1 Aim

In this lab you will examine applications of ordinary differential equations (ODEs) in electrical engineering.

4.2 Ordinary Differential Equations

Differential equations largely describe how the world works. Newton's second law for example is a differential equation (DE). DEs are used throughout electrical engineering to describe a variety of physical phenomena. For example, we can calculate the behaviour of electromagnetic waves (including cell phone signals, radio waves, visible light and x-rays) using Maxwell's equations, a collection of four related differential equations. We can calculate the propagation of acoustic waves using another differential equation, and we can control the motion of complicated electromechanical systems such as the Hubble space telescope by understanding the differential equations that govern their motion.

4.2.1 DEs

Do you remember how differential equations work? If the analysis that follows is unfamiliar, follow along and complete steps in the algebra yourself. The answer is a function (not a variable) that makes the DE work. We find that function in a three step process called GPA:
G: We "guess" at the general form of the solution
P: We plug it into the DE
A: We adjust the parameters to make it work.

For example, the DE for a mass on a spring is

$$ma = F$$

$$m\frac{d^2x}{dt^2} + kx = 0$$

Where $x = x(t)$ is a function of time that describes the motion of the mass.

G: We expect oscillations so we guess

$$x(t) = A\sin(\omega t + \varphi).$$

This is an oscillation with amplitude A, angular frequency ω and phase Φ.

P: We plug that into the DE and find

$$-\omega^2 A\sin(\omega t + \phi) + A\frac{k}{m}\sin(\omega t + \phi) = 0$$

A: This works if we adjust ω:

$$\omega = \sqrt{\frac{k}{m}}$$

The amplitude and phase are determined by the initial conditions, e.g., $x(0) = 0$ m and $v(0) = 1$ m/s.

4.2.2. DEs for Circuits

Kirchhoff's laws applied to a circuit with inductors and/or capacitors result in DEs. For example, RLC circuits which exhibit resonance are used in FM radios to tune to a particular station or in the equalisers of audio amplifiers. On a larger scale, the power transmission network and the homes and businesses that it connects can be modeled as circuits similar to the RLC resonant circuit and its cousins, and can be analysed using similar differential equations. In this lab we will model the power distribution network as a number of simple, resistor-inductor-capacitor (RLC) circuits, that we can analyse using the ordinary differential equation skills and techniques covered in class. The models we will be slightly familiar to you from Lab 2: we have already studied the frequency and phase responses of such circuits to some extent.

http://www.teara.govt.nz/en/photograph/32675/haywards-substation

Figure 4.1: The Haywards substation and power lines supplying the greater Wellington region

4.3 Differential Equations for the Power Distribution Network

As a crude approximation we can model the Haywards substation and the power lines that carry power from it to the University as a voltage supply, V, connected to a large inductor, L. This is a very simple model: the resistance, inductance, and capacitance in the real power grid are distributed, and there is a connection to ground. We will improve our model later, but even this simple model will give useful results. The substation and power lines are shown in Fig. 4.1, with our equivalent model shown in Fig. 4.2.

The voltage across the power lines (the inductor) is proportional to the rate of current flow through them, di/dt:

$$V = L\frac{di}{dt} \tag{4.1}$$

That is, the behaviour of the circuit is governed by the ordinary differential equation shown in Eq. (4.1).

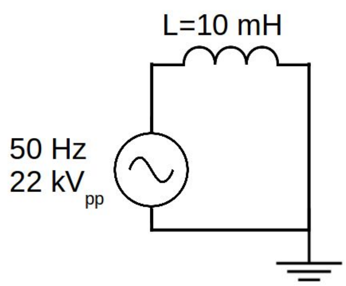

Figure 4.2: Circuit diagram for an inductor connected to a voltage source.

4.3.1 Exact Differential Equations

Some DEs can be solved simply by integrating both sides of the equation. Such DEs are called "exact differential equations". Equation 4.1 with

$$V = V_0 \cos(\omega t)$$

is exact. Let's solve it by integrating.

- First make di/dt the subject of the equation:

$$\frac{di}{dt} = \frac{V_0 \cos(\omega t)}{L}$$

CORE 1 (10 marks)

- Now integrate both sides and find the current.

<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>

$$V_0 \cos(\omega t) - L \frac{di}{dt} = 0$$

$$\frac{di}{dt} = \frac{V_0}{L} \cos(\omega t)$$

$$i = \frac{V_0}{L\omega} \sin(\omega t)$$

<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>

For our circuit, $L = 10$ mH and ω = 2 × π × 50 Hz.

In order to know the current flowing in the circuit at a specific instant of time we need to know the initial conditions of the circuit. In other words, we need to define the value of the constant, *c*, introduced during the previous integration.

CORE 2 (10 marks)

- Find *c* assuming $i(t{=}0) = 0$ kA

- Find $i(t = 10.001\ \text{s})$ and the maximum (peak) current in the circuit.

<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>

$i = \frac{V_0}{L\omega}\sin(\omega t) + C$ where $i(t=0) = 0$

$0 = \frac{V_0}{L\omega}\sin(0)$ from this we know $+C = 0$

$\cancel{i(10.001) = \frac{22}{1000\pi}\sin(100\pi \times 10.001) = 0.00216\ \text{kA}}$

~~The~~ maximum

$= \frac{22000}{0.01 \times 100\pi}\left(\sin(100\pi \times 10.001)\right) = 2164$ amps

The max is 7000 amps

<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>

4.4 Adding a Load to the Network

The University connects to the power distribution network via its own step-down transformer, shown in Fig. 4.3. We can model the University's total power consumption by adding a resistive load, R, to our model. We connect R in series with the transmission lines, L, and Haywards substation, V, as shown in Fig. 4.4. This circuit is similar to one you studied in Lab 2.

Figure 4.3: The step-down transformer connecting VUW to the power distribution network.

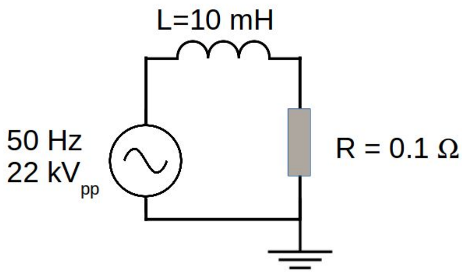

Figure 4.4: Circuit diagram for an inductor and resistor connected in series to a voltage source, $V=V_0\cos(\omega t)$.

4.4.1 Exercise: Integrating Factors

Some differential equations that cannot be integrated immediately can be made integrable using an **integrating factor**. This is essentially a recipe for solving DEs with the form

$$\frac{df}{dt} + P(t) f(t) = Q(t)$$

where $P(t)$ and $Q(t)$ are functions of time but not functions of current. This is the template for DEs we can solve using our integrating factor (defined below).

CORE 3 (10 marks)
Derive a differential equation for the circuit in Fig. 4.4. Hint: Kirchhoff. Check your answer with a tutor before continuing.
<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>

$$V = L\frac{di}{dt} + Ri \qquad \therefore\ V_0 \cos(\omega t) - L\frac{di}{dt} - Ri = 0$$

<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>

CORE 4 (10 marks)
Rearrange the above equation to match the template form and note the expressions you get for P and Q.
<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>
Expression for P(t) is R/L

$$\underline{\frac{di}{dt} + \frac{R}{L} i = \frac{V_0}{L}\cos(\omega t)}$$

<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>

CORE 5 (10 marks)
Try to integrate this equation and explain why this does not work.
<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>
We don't know what f(t) is so we can't integrate, there is both the original function and its derivative within the equation so we are unable to isolate f(t) to find it's value.

<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>

The Integrating Factor

The integrating factor that makes your DE integrable is

$$\mu(t)=\exp(\int P(t)\,dt)$$

COMPLETION 1 (10 marks)

Find the integrating factor.

<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>

$$\mu(t) = e^{\int P(t)\,dt} = e^{\int \frac{R}{L}\,dt} = e^{\frac{Rt}{L}}$$

<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>

COMPLETION 2 (10 marks)

Show that the following holds by plugging in the definition of the integrating factor, not your specific integrating factor.

$$[\mu(t)\frac{di}{dt}+P(t)\mu(t)i(t)]=\frac{d}{dt}[\mu(t)i(t)]$$

<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><

$$e^{\int P(t)\,dt}\frac{di}{dt} + P(t)\,e^{\int P(t)\,dt}\,i(t) = \frac{d}{dt}\left[e^{\int P(t)\,dt}\,i(t)\right] \quad \text{(chain rule)}$$

$$e^{\int P(t)\,dt}\cdot\frac{di}{dt} + P(t)\,e^{\int P(t)\,dt}\cdot i(t) = e^{\int P(t)\,dt}\cdot\frac{di}{dt} + P(t)\,e^{\int P(t)\,dt}\cdot i(t)$$

after chain rule both side are the same.

<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>

COMPLETION 3 (20 marks)

Multiply both sides of your current equation above by your $\mu(t)$ and integrate both sides with respect to t to find a solution for $i(t)$.

Hint: This integration is very easy for the left-hand side of your equation, thanks to the simplification above, but relatively hard for the right-hand side of the equation: essentially, we are trying to integrate $\cos(\omega t)$ multiplied by $\exp(\alpha t)$. To find this integral we need to use integration by parts twice. If we do this correctly, we will end up with the same integral on the left- and right-hand sides of our equation, with sundry multiplicative constants and additive functions thrown in. From this point, we need to rearrange our equation to have all of the integrals on the left-hand side, before we can shift a multiplicative constant or two about to find an expression for the integral we are trying to solve. Also be aware this is the hardest part of the lab - the rest is easier!

<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>

$$\int \frac{d}{dt}\left[e^{\frac{R}{L}t} \cdot i(t)\right] = \int e^{\frac{R}{L}t} \cdot \frac{V_0}{L}\cos(\omega t)$$

$$e^{\frac{R}{L}t} \cdot i(t) = \frac{V_0}{L}\int e^{\frac{R}{L}t} \cdot \cos(\omega t)\, dt$$

$$\cancel{\int e^{\frac{R}{L}t} \cdot \cos(\omega t) = \frac{R}{L} e^{\frac{R}{L}t} \cdot \cos(\omega t) - \int -\omega\sin(\omega t) \cdot e^{\frac{R}{L}t}\, dt}$$

$$\cancel{-\int \omega\sin(\omega t) \cdot e^{\frac{R}{L}t} = \frac{R}{L} e^{\frac{R}{L}t}\, \omega\cos(\omega t) - \int \omega^2\cos(\omega t) \cdot e^{\frac{R}{L}t}\, dt}$$

$$\int e^{\frac{R}{L}t} \cdot \cos(\omega t) = \frac{L}{R} e^{\frac{R}{L}t} \cdot \cos(\omega t) - \int -\omega\sin(\omega t) \cdot \frac{L}{R} e^{\frac{R}{L}t}$$

$$-\int \omega\sin(\omega t) \cdot \frac{L}{R} e^{\frac{R}{L}t} = \frac{L^2}{R^2} e^{\frac{R}{L}t} \cdot \omega\sin(\omega t) - \int \omega^2\cos(\omega t) \cdot \frac{L^2}{R^2} e^{\frac{R}{L}t}$$

$$\int e^{\frac{R}{L}t} \cdot \cos(\omega t) = \frac{L}{R} e^{\frac{R}{L}t} \cdot \cos(\omega t) + \omega\sin(\omega t) \cdot \frac{L^2}{R^2} e^{\frac{R}{L}t} - \left(\omega^2 \frac{L^2}{R^2}\right)\int e^{\frac{R}{L}t} \cdot \cos(\omega t$$

$$\left(1 + \omega^2 \frac{L^2}{R^2}\right)\int e^{\frac{R}{L}t} \cdot \cos(\omega t) = \frac{L}{R} e^{\frac{R}{L}t} \cdot \cos(\omega t) + \omega\sin(\omega t) \cdot \frac{L^2}{R^2} e^{\frac{R}{L}t}$$

$$\int e^{\frac{R}{L}t} \cdot \cos(\omega t) = \frac{\frac{L}{R} e^{\frac{R}{L}t} \cdot \cos(\omega t) + \omega\sin(\omega t) \cdot \frac{L^2}{R^2} e^{\frac{R}{L}t}}{\left(1 + \omega^2 \frac{L^2}{R^2}\right)}$$

$$i(t) = \frac{\frac{1}{e^{\frac{R}{L}t}} \cdot \frac{V_0}{L}\left[\frac{L}{R} e^{\frac{R}{L}t} \cdot \cos(\omega t) + \omega\sin(\omega t) \cdot \frac{L^2}{R^2} e^{\frac{R}{L}t}\right]}{\left(1 + \omega^2 \frac{L^2}{R^2}\right)} + C$$

<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>

CHALLENGE 1 (5 marks) (Note you can do Challenge 2 without doing Challenge 1)

Assume the same parameters for V and L used above and $R = 0.1\ \Omega$. Again assume $i(t = 0\text{ s}) = 0$. Find $i(t = 10.001\text{ s})$ and the maximum (peak) current in the circuit. How does the peak current differ from the peak current in the circuit without L? Is this what you expected?

<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>

$V_0 = V_P * \frac{1}{\sqrt{2}} = 7778.2V$

L = 0.001

R = 0.1

+c = -0.0111

I = 0.00996

<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>

4.5 Capacitive Compensation

As we discussed in Lab 2, any inductance in our circuit increases the distribution costs for the power generator. Adding compensating capacitances at the load (i.e. households and businesses, like the University) is seldom practical, but including a bank of capacitors at the substation is a feasible and oft-used approach. We can model the addition of a compensation network by adding a capacitor, *C*, to our model circuit, as shown in Fig. 4.5.

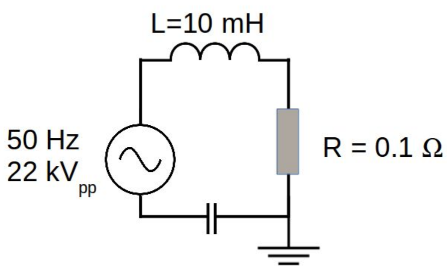

Figure 4.5: A resistor, an inductor, and a capacitor connected in series to a voltage source.

CHALLENGE 2 (5 marks)

Explain how adding a capacitor can reduce the total impedance while keeping the resistance constant. Find the value of the capacitance that would do an optimal job of reducing the impedance of the circuit in Figure 4.5.

<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>

<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>