Solutions of Assignment 2

1. (a) $z = 3 - i$

$|z| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$

$\theta = \tan^{-1}\left(\frac{-1}{3}\right) \cong -18.43^\circ \cong -0.3218 \text{ rad}$

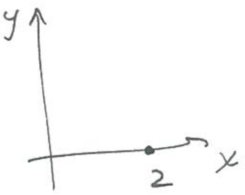
So $z = \sqrt{10}\left(\cos(-18.43^\circ) + i\sin(-18.43^\circ)\right)$

(b) $z = 2$

$\theta = 0$

$|z| = 2$

So $z = 2(\cos 0 + i \sin 0)$



(c) $z = -i$

$\theta = -\frac{\pi}{2}$

$|z| = 1$ So $z = 1\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

(d) $z = -5 + 12i$

$|z| = \sqrt{(-5)^2 + 12^2} = \sqrt{169} = 13$

$\theta = \tan^{-1}\left(\frac{12}{-5}\right) + \pi \cong 1.96$

So $z = 13(\cos(1.96) + i\sin(1.96))$

2. (a) $z_1 = -\sqrt{3} + i$

$|z_1| = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

$\theta_1 = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) + \pi \cong 2.6164$

(b) $z_2 = 4+4i$

$|z_2| = \sqrt{4^2+4^2} = \sqrt{32}$

$\theta_2 = \tan^{-1}\left(\frac{4}{4}\right) = \frac{\pi}{4} \simeq 0.7854$

(c) $z_3 = z_1 z_2 =$

$= |z_1||z_2| \cdot \left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$

So $|z_3| = 2\sqrt{32}$

$\theta_3 \cong 3.4018$

(d) $z_4 = z_1/z_2 = \frac{|z_1|}{|z_2|}\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right) \Rightarrow$

So

$|z_4| = \frac{2}{\sqrt{32}}$ and $\theta_4 = 1.831.$

3. (a) $z = 3 \cdot e^{i\pi/4}$ $\quad |z| = 3 \quad \theta = \frac{\pi}{4}$

(b) $z = 2e^{-i\pi/6}$ $\quad |z| = 2 \quad \theta = -\pi/6$

4. (a) $z = 5e^{i\pi/3} = 5 \cdot (\cos \pi/3 + i \sin \pi/3) =$

$= 5 \cdot \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{5}{2} + i\frac{5\sqrt{3}}{2}$

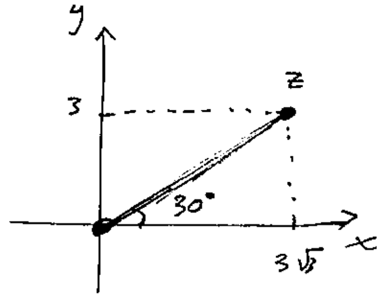
So $Re(z) = \frac{5}{2}$ and $Im(z) = \frac{5\sqrt{3}}{2}$

(b) $z = 11e^{i\pi} = 11 \cdot (\cos(\pi) + i\sin(\pi)) =$

$= 11(-1+0i) = -11$

5. $z = 6(\cos 30^\circ + i \sin 30^\circ) = 6 \cdot e^{i\pi/6}$

$= 6\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = 3\sqrt{3} + i3$



6. (a) $z = 7+5i$

$|z| = \sqrt{7^2+5^2} = \sqrt{74}$

$\theta = \tan^{-1}\left(\frac{5}{7}\right) \cong 0.62 \simeq 35.54°$

$z = \sqrt{74}\, e^{i\,0.62}$

(b) $z = \frac{1}{2} - \frac{1}{3}i$

$|z| = \sqrt{\left(\frac{1}{2}\right)^2+\left(-\frac{1}{3}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{9}} = \sqrt{\frac{9+4}{4\cdot 9}} = \frac{1}{6}\sqrt{13}$

$\theta = \tan^{-1}\left(\frac{-\frac{1}{3}}{\frac{1}{2}}\right) = \tan^{-1}\left(-\frac{2}{3}\right) \simeq -0.588$

So $z = \frac{1}{6}\sqrt{13}\cdot e^{-i\,0.588}$

7. $$\frac{\cos 8\theta + i\sin 8\theta}{\cos 2\theta - i\sin 2\theta} = \frac{(\cos\theta + i\sin\theta)^8}{\cos(-2\theta) + i\sin(-2\theta)} =$$

$$= \frac{(\cos\theta + i\sin\theta)^8}{(\cos(-2\theta) + i\sin(-2\theta))} = \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta + i\sin\theta)^{-2}} =$$

$$= (\cos\theta + i\sin\theta)^{10} = \cos(10\theta) + i\sin(10\theta).$$

(8) (a) $z^3 = -1$

$-1 = -1 + 0i = 1(\cos(\pi + 2\pi n) + i\sin(\pi + 2\pi n))$

If $z = r(\cos\theta + i\sin\theta)$ then $z^3 = -1$ implies

$r^3 = 1 \Rightarrow r = 1$

and ~~$3\theta = 2\pi n$~~ $3\theta = \pi + 2\pi n$, $n = 0, 1, 2$

So $\theta = \frac{\pi}{3}, \frac{\pi}{3} + \frac{2\pi}{3}, \frac{\pi}{3} + \frac{4\pi}{3}$

So $z_1 = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2} i$

$z_2 = \cos(\pi) + i\sin(\pi) = -1$

$z_3 = \cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2} i$

(b) $z^4 = 1 + i$

$$\left.\begin{array}{l} |1+i| = \sqrt{1^2+1^2} = \sqrt{2} \\ \theta = \frac{\pi}{4} \end{array}\right\} \Rightarrow 1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

if $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ then $z^4 = 1+i$ implies:

$r^4 = \sqrt{2} \Rightarrow r = 2^{1/8}$

and $4\theta = \frac{\pi}{4} + 2n\pi$, $n = 0, 1, 2, 3$

So $\theta = \frac{\pi}{16} + \frac{n\pi}{2}$ $n = 0, 1, 2, 3$

(c) Similarly: $r = \sqrt{5}$
$\theta = \frac{\pi}{4} + n\pi/2$

9. We will find the polar form of $z=2+2i$

$$|z| = \sqrt{2^2+2^2} = \sqrt{8}$$

$$\theta = \tan^{-1}\left(\tfrac{2}{2}\right) = \frac{\pi}{4} + 2n\pi \qquad n=0,1,2$$

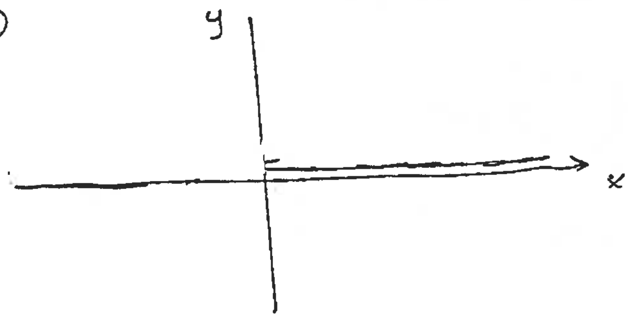
So $z = \sqrt{8}\cdot\left(\cos\left(\frac{\pi}{4}+2n\pi\right) + i\sin\left(\frac{\pi}{4}+2n\pi\right)\right)$

$$\Rightarrow \sqrt[3]{z} = (z)^{1/3} =$$

$$= 8^{1/6}\cdot\left(\cos\left(\frac{\pi}{4}+2n\pi\right) + i\sin\left(\frac{\pi}{4}+2n\pi\right)\right)^{1/3} =$$

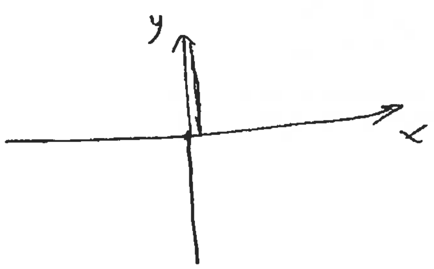
$$= 8^{1/6}\cdot\left(\cos\left(\frac{\pi}{12}+\frac{2\pi n}{3}\right) + i\sin\left(\frac{\pi}{12}+\frac{2\pi n}{3}\right)\right) =$$

10. (a)



$\arg(z)=0 \Rightarrow z\in\mathbb{R}$
$z>0$

(b) $\arg(z) = \frac{\pi}{2} \Rightarrow z \in \mathbb{I}$
$Im(z)>0$



(c)

$$|2z| = |z-1|$$

$$|2(x+iy)| = |x+iy-1|$$

$$|2x+i2y| = |x-1+iy|$$

$$\sqrt{(2x)^2+(2y)^2} = \sqrt{(x-1)^2+y^2}$$

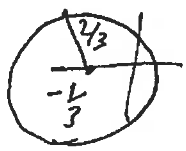
$$(2x)^2+(2y)^2 = (x-1)^2+y^2$$

$$4x^2+4y^2 = x^2-2x+1+y^2$$

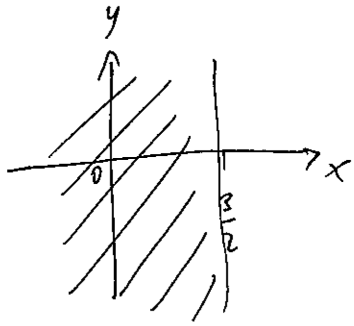
$$3x^2+3y^2+2x=1$$

$$x^2+y^2+2\cdot\frac{1}{3}x = \frac{1}{3}$$

$$\left(x+\frac{1}{3}\right)^2 + y^2 = \frac{9}{3}$$



(d) $|z-1| < |z-2| \Rightarrow x < \frac{3}{2}$



$|z-1| < |z-2|$

$|x-1+iy| < |x-2+iy|$

$\sqrt{(x-1)^2+y^2} < \sqrt{(x-2)^2+y^2}$

$\cancel{x^2}-2x+1+\cancel{y^2} < \cancel{x^2}-4x+4+\cancel{y^2}$

$2x < 3$

$x < \frac{3}{2}$