6
Airplane Response and Closed-Loop Control

6.1 Introduction

In Chapter 4, we studied various coordinate systems used in airplane dynamics and derived equations of motion applicable for small disturbances. We also discussed the methods of evaluating various stability and control derivatives appearing in those equations. Under the assumption of small disturbance, the equations of motion could be grouped into two sets of three equations each: one set for longitudinal motion and another set for lateral-directional motion of the aircraft. This kind of decoupling enables us to study separately the longitudinal and lateral-directional response and closed-loop control. In Chapter 5, we studied the basic principles of linear system theory and design of closed-loop control systems.

In this chapter we will discuss the solution of the small-disturbance equations to determine the airplane response. The airplane response depends on the initial conditions and the input time history. The response to a given set of initial conditions with zero input is called the natural or free response. The free response is indicative of the transient behavior or the dynamic stability of the system. The initial conditions are equivalent to suddenly imposed disturbances. For example, the response with $u(0) = 0$, $\Delta\alpha(0) = 5$ deg, and $q = \Delta\theta = 0$ is equivalent to the response for a suddenly imposed vertical gust that momentarily increases the airplane angle of attack by 5 deg.

The forced response is the solution of equations of motion with zero initial conditions and a given input time history. The forced response is indicative of an airplane's steady-state behavior. The common input test functions used to obtain the forced response are the unit-step and impulse functions. For example, airplane longitudinal response to a unit-step function describes the motion of the airplane to a sudden unit deflection of the elevator. The steady-state solution gives the corresponding steady-state values of forward velocity, angle of attack, attitude, and pitch rate.

Finally, we will discuss the closed-loop control of the airplane to obtain desired level of handling qualities. The closed-loop control systems used for obtaining the specified level of free response are called stability augmentation systems, and those used to establish and hold the desired flight conditions are called autopilots. In this chapter, we will discuss some of the important stability augmentation systems and autopilots.

6.2 Longitudinal Response

In this section we will discuss the longitudinal response of the airplane. We will discuss two types of responses: 1) the free response and 2) the forced response. The free response corresponds to the solution with a given set of initial conditions and zero input and is indicative of the dynamic stability of the system. The forced response corresponds to the solution with zero initial conditions and a given input time history. For free response, we assume that the elevator is held fixed (stick-fixed), and, for forced response, we assume that the elevator is moved in a specified manner to a new position and subsequently held fixed at that position.

The longitudinal equations of motion for elevator control are given by Eqs. (4.417–4.419) from Chapter 4 as,

$$\left(m_1\frac{\mathrm{d}}{\mathrm{d}t} - C_{xu}\right)u - \left(C_{x\dot{\alpha}}c_1\frac{\mathrm{d}}{\mathrm{d}t} + C_{x\alpha}\right)\Delta\alpha - \left(C_{xq}c_1\frac{\mathrm{d}}{\mathrm{d}t} + C_{x\theta}\right)\Delta\theta = C_{x\delta e}\Delta\delta e \tag{6.1}$$

$$-C_{zu}u + \left[\left(m_1\frac{\mathrm{d}}{\mathrm{d}t} - C_{z\dot{\alpha}}c_1\frac{\mathrm{d}}{\mathrm{d}t}\right) - C_{z\alpha}\right]\Delta\alpha - \left(m_1\frac{\mathrm{d}}{\mathrm{d}t} + C_{zq}c_1\frac{\mathrm{d}}{\mathrm{d}t} + C_{z\theta}\right)\Delta\theta$$
$$= C_{z\delta e}\Delta\delta e \tag{6.2}$$

$$-C_{mu}u - \left(C_{m\dot{\alpha}}c_1\frac{\mathrm{d}}{\mathrm{d}t} + C_{m\alpha}\right)\Delta\alpha + \frac{\mathrm{d}}{\mathrm{d}t}\left(I_{y1}\frac{\mathrm{d}}{\mathrm{d}t} - C_{mq}c_1\right)\Delta\theta = C_{m\delta e}\Delta\delta e \tag{6.3}$$

For the study of airplane response, it is convenient to express Eqs. (6.1–6.3) in the state-space form as follows:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{m_1}\big[(C_{xu} + \xi_1 C_{zu})u + (C_{x\alpha} + \xi_1 C_{z\alpha})\Delta\alpha + [C_{xq}c_1 + \xi_1(m_1 + C_{zq}c_1)]\,q$$
$$+ (C_{x\theta} + \xi_1 C_{z\theta})\Delta\theta + \left(C_{x\delta_e} + \xi_1 C_{z\delta_e}\right)\Delta\delta_e\big] \tag{6.4}$$

$$\frac{\mathrm{d}\Delta\alpha}{\mathrm{d}t} = \frac{1}{(m_1 - C_{z\dot{\alpha}}c_1)}[C_{zu}u + C_{z\alpha}\Delta\alpha + (m_1 + C_{zq}c_1)q + C_{z\theta}\Delta\theta + C_{z\delta e}\Delta\delta e] \tag{6.5}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{1}{I_{y1}}\big[(C_{mu} + \xi_2 C_{zu})u + (C_{m\alpha} + \xi_2 C_{z\alpha})\Delta\alpha + [C_{mq}c_1 + \xi_2(m_1 + C_{zq}c_1)]\,q$$
$$+ \xi_2 C_{z\theta}\Delta\theta + \left(C_{m\delta_e} + \xi_2 C_{z\delta_e}\right)\Delta\delta_e\big] \tag{6.6}$$

$$\frac{\mathrm{d}\Delta\theta}{\mathrm{d}t} = q \tag{6.7}$$

where

$$\xi_1 = \frac{C_{x\dot{\alpha}}c_1}{m_1 - C_{z\dot{\alpha}}c_1} \tag{6.8}$$

$$\xi_2 = \frac{C_{m\dot{\alpha}}c_1}{m_1 - C_{z\dot{\alpha}}c_1} \tag{6.9}$$

Let

$$x_1 = u \qquad x_2 = \Delta\alpha \qquad x_3 = q \qquad x_4 = \Delta\theta \tag{6.10}$$

Then, Eqs. (6.4–6.7) can be expressed in the state-space form as

$$\dot{X} = AX + BU \tag{6.11}$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad U = \delta_e \tag{6.12}$$

and

$$a_{11} = \frac{C_{xu} + \xi_1 C_{zu}}{m_1} \qquad a_{12} = \frac{C_{x\alpha} + \xi_1 C_{z\alpha}}{m_1}$$

$$a_{13} = \frac{C_{xq} c_1 + \xi_1 (m_1 + C_{zq} c_1)}{m_1} \qquad a_{14} = \frac{C_{x\theta} + \xi_1 C_{z\theta}}{m_1}$$

$$a_{21} = \frac{C_{zu}}{m_1 - C_{z\dot{\alpha}} c_1} \qquad a_{22} = \frac{C_{z\alpha}}{m_1 - C_{z\dot{\alpha}} c_1}$$

$$a_{23} = \frac{m_1 + C_{zq} c_1}{m_1 - C_{z\dot{\alpha}} c_1} \qquad a_{24} = \frac{C_{z\theta}}{m_1 - c_1 C_{z\dot{\alpha}}}$$

$$a_{31} = \frac{C_{mu} + \xi_2 C_{zu}}{I_{y1}} \qquad a_{32} = \frac{C_{m\alpha} + \xi_2 C_{z\alpha}}{I_{y1}}$$

$$a_{33} = \frac{C_{mq} c_1 + \xi_2 (m_1 + C_{zq} c_1)}{I_{y1}} \qquad a_{34} = \frac{\xi_2 C_{z\theta}}{I_{y1}}$$

$$a_{41} = 0 \qquad a_{42} = 0 \qquad a_{43} = 1 \qquad a_{44} = 0$$

$$b_1 = \frac{C_{x\delta_e} + \xi_1 C_{z\delta_e}}{m_1} \qquad b_2 = \frac{C_{z\delta_e}}{m_1 - c_1 C_{z\dot{\alpha}}}$$

$$b_3 = \frac{C_{m\delta_e} + \xi_2 C_{z\delta_e}}{I_{y1}} \qquad b_4 = 0$$

$$m_1 = \frac{2m}{\rho U_o S} \qquad c_1 = \frac{\bar{c}}{2U_o}$$

$$I_{y1} = \frac{I_y}{\frac{1}{2}\rho U_o^2 S \bar{c}}$$

For free response, $U = 0$ so that

$$\dot{X} = AX \tag{6.13}$$

A solution to Eq. (6.13) can be obtained in the usual way by assuming

$$X = X_o e^{\lambda t} \tag{6.14}$$

so that

$$\dot{X} = X_o \lambda e^{\lambda t} \tag{6.15}$$

and

$$X_o \lambda e^{\lambda t} - A X_o e^{\lambda t} = 0 \tag{6.16}$$

or

$$(\lambda I - A) X_o = 0 \tag{6.17}$$

where I is the identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.18}$$

For nontrivial solutions, the determinant of $(\lambda I - A)$ must be zero

$$|\lambda I - A| = 0 \tag{6.19}$$

An expansion of the determinant in Eq. (6.19) results in a fourth-order algebraic equation of the form

$$A_\delta \lambda^4 + B_\delta \lambda^3 + C_\delta \lambda^2 + D_\delta \lambda + E_\delta = 0 \tag{6.20}$$

where

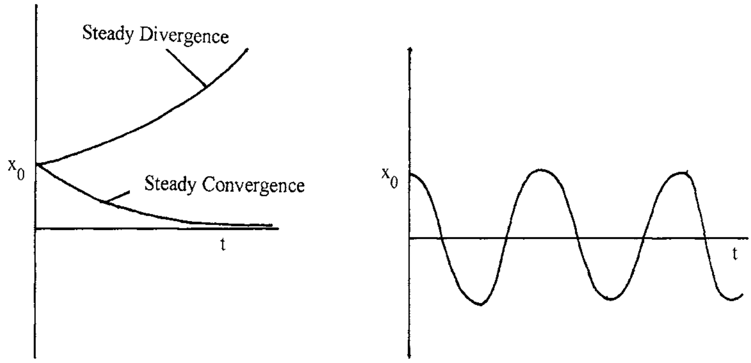
$$A_\delta = m_1 I_{y1} (m_1 - C_{z\dot{\alpha}} c_1) \tag{6.21}$$

$$\begin{aligned} B_\delta = {} & m_1(-I_{y1} C_{z\alpha} - C_{mq} c_1 [m_1 - C_{z\dot{\alpha}} c_1] - C_{m\dot{\alpha}} c_1 [m_1 + C_{zq} c_1]) \\ & - C_{xu} I_{y1} (m_1 - C_{z\dot{\alpha}} c_1) - C_{x\dot{\alpha}} c_1 C_{zu} I_{y1} \end{aligned} \tag{6.22}$$

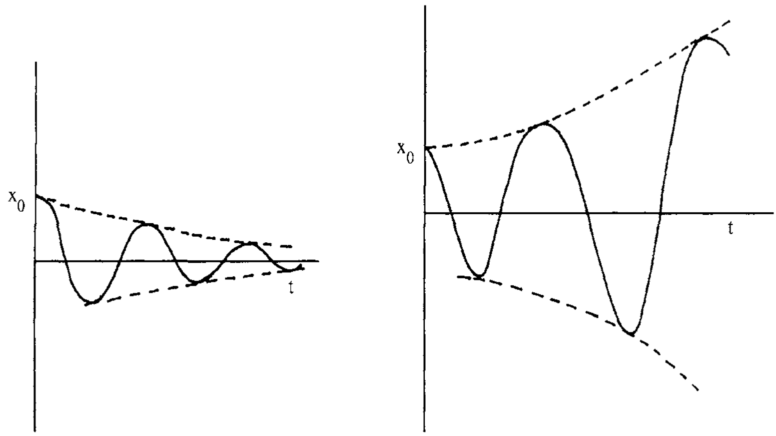
$$\begin{aligned} C_\delta = {} & m_1 \big(C_{z\alpha} C_{mq} c_1 - C_{m\alpha} [m_1 + C_{zq} c_1] - C_{z\theta} C_{m\dot{\alpha}} c_1 \big) \\ & + C_{xu} (I_{y1} C_{z\alpha} + C_{mq} c_1 [m_1 - C_{z\dot{\alpha}} c_1] + C_{m\dot{\alpha}} c_1 [m_1 + C_{zq} c_1]) \\ & - C_{x\alpha} C_{zu} I_{y1} + C_{x\dot{\alpha}} c_1 (C_{zu} C_{mq} c_1 - C_{mu} [m_1 + C_{zq} c_1]) \\ & - C_{xq} c_1 (C_{zu} C_{m\dot{\alpha}} c_1 + C_{mu} [m_1 - C_{z\dot{\alpha}} c_1)]) \end{aligned} \tag{6.23}$$

$$\begin{aligned} D_\delta = {} & -C_{xu} \big(C_{z\alpha} C_{mq} c_1 - C_{m\alpha} [m_1 + C_{zq} c_1] - C_{z\theta} C_{m\dot{\alpha}} c_1 \big) \\ & - m_1 C_{m\alpha} C_{z\theta} + C_{x\alpha} \big(C_{zu} C_{mq} c_1 - C_{mu} [m_1 + C_{zq} c_1] \big) \\ & - C_{x\dot{\alpha}} c_1 C_{mu} C_{z\theta} - C_{xq} c_1 (C_{zu} C_{m\alpha} - C_{z\alpha} C_{mu}) \\ & - C_{x\theta} (C_{zu} C_{m\dot{\alpha}} c_1 + C_{mu} [m_1 - C_{z\dot{\alpha}} c_1]) \end{aligned} \tag{6.24}$$

$$E_\delta = C_{xu} C_{m\alpha} C_{z\theta} - C_{x\alpha} C_{mu} C_{z\theta} - C_{x\theta} (C_{zu} C_{m\alpha} - C_{z\alpha} C_{mu}) \tag{6.25}$$



a) Steady convergence and divergence **b) Undamped (constant amplitude) oscillation**



c) Damped oscillation **d) Diverging oscillation**

Fig. 6.1 Typical dynamic motions.

Equation (6.20) is also called the characteristic equation of the longitudinal motion and gives four values of the root λ. The nature of the free response can be any combination of the typical free responses shown in Fig. 6.1. A steady convergence corresponds to a real negative root, a steady divergence if the real root is positive, an oscillatory motion of constant amplitude if the real part of the complex root is zero, a damped oscillatory motion if the real part of the complex root is negative, and a divergent oscillatory motion if the real part of the complex root is positive. Thus for dynamic stability, the roots of the characteristic equation must be negative if real or must have negative real parts if complex.

For a conventional, statically stable airplane ($C_{m\alpha} < 0$), the longitudinal characteristic equation usually has a pair of complex conjugate roots of the form

$$\lambda_{1,2} = -r_1 \pm js_1 \tag{6.26}$$

$$\lambda_{3,4} = -r_2 \pm js_2 \tag{6.27}$$

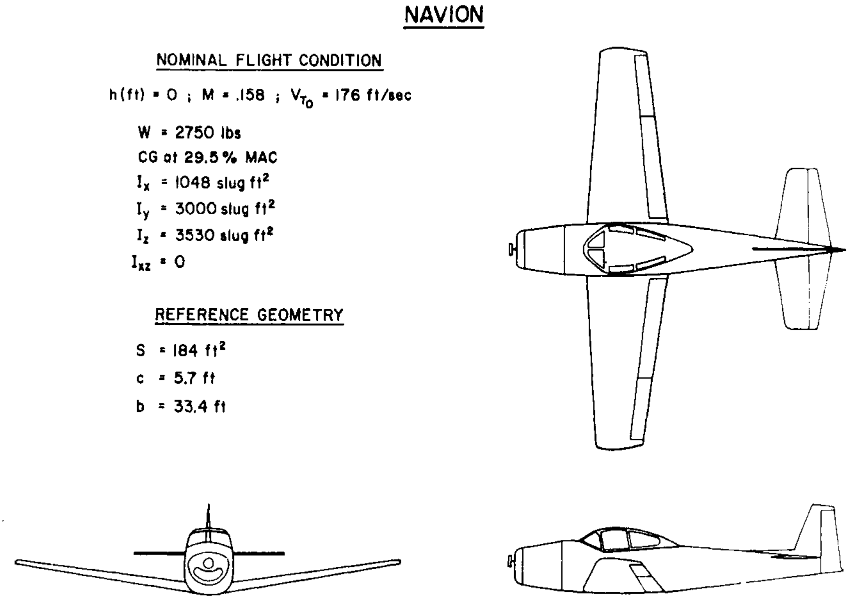


Fig. 6.2 Three-view drawing of the general aviation airplane.[2]

so that

$$\Delta_{\text{long}}(\lambda) = (-r_1 + js_1)(-r_1 - js_1)(-r_2 + js_2)(-r_2 - js_2) \tag{6.28}$$

The roots of the characteristic equation are also called the eigenvalues of the system represented by the matrix A. To determine the eigenvalues λ_i, $i = 1, 4$, we can either solve the fourth-order characteristic equation or obtain them directly using the methods of matrix algebra. However, either of these tasks can be easily performed using commercially available matrix software tools like MATLAB.[1]

To illustrate the nature of free response and the concept of airplane dynamic stability, let us determine the response of a general aviation airplane,[2] which is shown in Fig. 6.2. The mass and aerodynamic properties of this airplane are as follows.[2]

Wing area $S = 16.7225$ m^2, weight $W = 12{,}232.6$ N, wing span $b = 10.1803$ m, wing mean aerodynamic chord $\bar{c} = 1.7374$ m, distance of center of gravity from wing leading edge in terms of mean aerodynamic chord $\bar{x}_{cg} = 0.295$, $I_x = 1420.8973$ kg·m^2, $I_y = 4067.454$ kg·m^2, $I_z = 4786.0375$ kg·m^2, and $I_{xy} = I_{yz} = I_{zx} = 0$. $C_L = 0.41$, $C_D = 0.05$, $C_{L\alpha} = 4.44$, $C_{L\dot{\alpha}} = 0$, $C_{D\dot{\alpha}} = 0$, $C_{LM} = 0$, $C_{L\delta_e} = 0.355$, $C_{m\delta_e} = -0.9230$, $C_{D\alpha} = 0.33$, $C_{DM} = 0$, $C_{D\delta_e} = 0$, $C_{m\alpha} = -0.683$, $C_{m\dot{\alpha}} = -4.36$, $C_{mM} = 0$, $C_{mq} = -9.96$, $M = 0.158$, and $\rho = 1.225$ kg/m^3. All the derivatives are per radian.

We assume that the airplane is in level flight with $\theta_o = 0$ before it encounters any disturbance. This assumption gives us $C_{x\theta} = -C_L$ and $C_{z\theta} = 0$. In Ref. 2, the data on C_{Lq} is not given. For the purpose of this text, we assume $C_{Lq} = 3.8$ per radian.

Recall from Chapter 4 that $C_{xu} = -2C_D - C_{Du}$, $C_{x\alpha} = C_L - C_{D\alpha}$, $C_{x\theta} = -C_L \cos\theta$, $C_{z\theta} = -C_L \sin\theta$, $C_{x\dot{\alpha}} = -C_{D\dot{\alpha}}$, $C_{xq} = -C_{Dq}$, $C_{z\alpha} = -C_{L\alpha} - C_D$, $C_{zu} = -2C_L - C_{Lu}$, $C_{z\theta} = -C_L \sin\theta$, $C_{x\delta_e} = -C_{D\delta_e}$ and $C_{z\delta_e} = -C_{L\delta_e}$.

Substituting the required quantities in Eq. (6.12), we get

$$A = \begin{bmatrix} -0.0453 & 0.0363 & 0 & -0.1859 \\ -0.3717 & -2.0354 & 0.9723 & -0 \\ 0.3398 & -7.0301 & -2.9767 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{6.29}$$

$$B = \begin{bmatrix} 0 \\ -0.1609 \\ -11.8674 \\ 0 \end{bmatrix} \tag{6.30}$$

Note that the units in B matrix for $C_{x\delta_e}$, $C_{z\delta_e}$, and $C_{m\delta_e}$ are per radian. Using MATLAB,[1] we obtain the eigenvalues of the matrix A as

$$\lambda_{1,2} = -2.5118 \pm j2.5706 \tag{6.31}$$

$$\lambda_{3,4} = -0.0169 \pm j0.2174 \tag{6.32}$$

Because both the complex roots have negative real parts, the free response of this airplane is of stable nature. It consists of two decaying oscillatory motions, which are superposed one on another. Thus, the general aviation airplane considered here is dynamically stable.

An alternative way of examining the dynamic stability of the airplane without actually solving for the roots of the characteristic polynomial or the eigenvalues of the matrix A is to make use of the Routh's stability criterion we discussed in Chapter 5. Substituting in Eq. (6.20), the characteristic equation for the general aviation airplane is given by

$$0.3739\,\lambda^4 + 1.9002\,\lambda^3 + 4.9935\,\lambda^2 + 0.1642\,\lambda + 0.2296 = 0 \tag{6.33}$$

The necessary condition for stability is that all the coefficients of the characteristic equation must be positive or have the same sign. This condition is satisfied here. To examine whether the sufficiency condition is satisfied, we form Routh's array as follows:

$$\begin{array}{llll} s^4: & 0.3739 & 4.9935 & 0.2296 \\ s^3: & 1.9002 & 0.1642 & 0 \\ s^2: & 4.9612 & 0.2296 & \\ s^1: & 0.0763 & & \\ s^0: & 0.2296 & & \end{array}$$

We observe that all the elements of the first column of Routh's array are positive or have the same sign; hence the sufficiency condition is also satisfied. Hence, the characteristic polynomial has no positive real root or a complex root with positive real part. Therefore, the given system is stable as we have noted from direct determination of eigenvalues of matrix A.

The standard form of the characteristic equation of a second-order system is

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \tag{6.34}$$

Let

$$\lambda = -r \pm js \tag{6.35}$$

$$= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \tag{6.36}$$

where ζ is the damping ratio and ω_n is the natural frequency of the system. Then,

$$\zeta = \frac{r}{\sqrt{r^2+s^2}} \tag{6.37}$$

$$\omega_n = \sqrt{r^2+s^2} \tag{6.38}$$

The period T and time for the amplitude t_a to become either half or double the initial amplitude are given by

$$T = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{6.39}$$

$$t_a = \frac{0.6931}{|r|} \tag{6.40}$$

Here, t_a is the time for half amplitude if r is positive, and it is the time for the amplitude to double if r is negative.

For the general aviation airplane, we get $\zeta_{1,2} = 0.6997$ and $\omega_{1,2} = 3.6054$, corresponding to $\lambda_{1,2}$ and $\zeta_{3,4} = 0.0281$, and $\omega_{3,4} = 0.2134$, corresponding to $\lambda_{3,4}$. The motion corresponding to $\lambda_{1,2}$ is heavily damped and is of higher frequency or a shorter period. The other motion corresponding to $\lambda_{3,4}$ is lightly damped and is of lower frequency or longer period. These values of $\lambda_{1,2}$ and $\lambda_{3,4}$ are typical of conventional, statically stable airplanes. The high-frequency, heavily damped oscillatory motion is called the short-period mode, and the lightly damped, long-period oscillatory mode is known as phugoid or long-period mode. For the general aviation airplane, we find that the periods of short-period and phugoid modes are 1.7424 and 29.4432 s, respectively. The corresponding values of the time for half amplitude are 0.2735 and 115 s. Because the short-period mode is fast and heavily damped, it is just felt as a bump by the pilot or the passengers. The pilot does not have to take any action to kill this mode. The phugoid mode is very lightly damped and usually persists for a long time. It can be quite annoying if left to die by itself. Because the period is quite long, the pilot can easily operate the longitudinal control (elevator) and kill the phugoid mode.

Physically, the motion of the airplane during the phugoid motion can be depicted as shown in Fig. 6.3. Beginning at the bottom of one cycle, we observe that the pitch angle increases and the airplane gains altitude and loses forward speed. During phugoid motion, the angle of attack remains constant so that a drop in forward speed amounts to a decrease in lift and flattening of the pitch attitude. As a result, at the top of the cycle, the pitch angle goes to zero. Beyond this point, the airplane begins to lose altitude, the pitch angle goes negative, and the air speed increases. At the bottom of the cycle, the pitch attitude is nearly level, the air speed is at its maximum, and the cycle repeats once again.

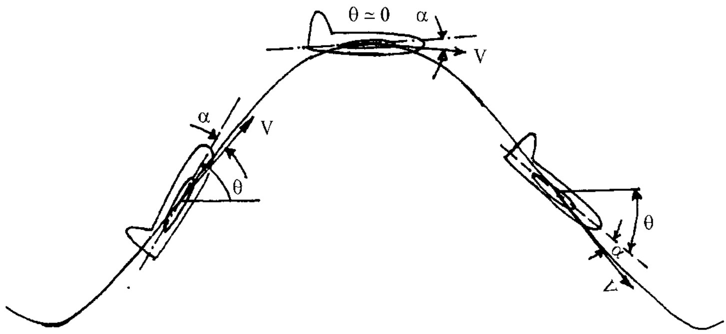


Fig. 6.3 Schematic illustration of the physical motion of the airplane during the phugoid motion.

In view of the fact that the longitudinal response of statically stable airplanes consists of two distinct oscillatory motions, it is customary to introduce the short-period and phugoid approximations as follows.

6.2.1 Short-Period Approximation

Of the two oscillatory modes, the short-period mode is the more heavily damped oscillatory motion with a higher frequency. This oscillatory motion lasts just for a few seconds, usually fewer than 10 s, during which the angle of attack, pitch angle, and pitch rate vary rapidly and the forward speed nearly remains constant. Therefore, we can assume $u = \dot{u} = 0$ during short-period oscillation. With this assumption, we can ignore the force equation in the x direction. Then, the other two equations assume the following form:

$$\left(m_1\frac{\mathrm{d}}{\mathrm{d}t} - C_{z\dot{\alpha}}c_1\frac{\mathrm{d}}{\mathrm{d}t} - C_{z\alpha}\right)\Delta\alpha - \left(m_1\frac{\mathrm{d}}{\mathrm{d}t} + C_{zq}c_1\frac{\mathrm{d}}{\mathrm{d}t} + C_{z\theta}\right)\Delta\theta = C_{z\delta e}\Delta\delta e \tag{6.41}$$

$$-\left(C_{m\dot{\alpha}}c_1\frac{\mathrm{d}}{\mathrm{d}t} + C_{m\alpha}\right)\Delta\alpha + \frac{\mathrm{d}}{\mathrm{d}t}\left(I_{y1}\frac{\mathrm{d}}{\mathrm{d}t} - C_{mq}c_1\right)\Delta\theta = C_{m\delta_e}\Delta\delta_e \tag{6.42}$$

Let

$$x_1 = \Delta\alpha \tag{6.43}$$

$$x_2 = q = \tfrac{d\Delta\theta}{dt} \tag{6.44}$$

$$x_3 = \Delta\theta \tag{6.45}$$

Then rearranging, we have

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}\Delta\delta_e \tag{6.46}$$

Let

$$A_{sp} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{6.47}$$

$$B_{sp} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{6.48}$$

where

$$a_{11} = \frac{C_{z\alpha}}{m_1 - C_{z\dot{\alpha}}c_1} \qquad a_{12} = \frac{m_1 + C_{zq}c_1}{m_1 - C_{z\dot{\alpha}}c_1}$$

$$a_{13} = \frac{C_{z\theta}}{m_1 - C_{z\dot{\alpha}}c_1} \qquad a_{21} = \left(\frac{1}{I_{y1}}\right)\left(C_{m\alpha} + \frac{C_{m\dot{\alpha}}c_1 C_{z\alpha}}{m_1 - C_{z\dot{\alpha}}c_1}\right)$$

$$a_{22} = \left(\frac{1}{I_{y1}}\right)\left[C_{mq}c_1 + \left(\frac{C_{m\dot{\alpha}}c_1}{m_1 - C_{z\dot{\alpha}}c_1}\right)(m_1 + c_1 C_{zq})\right]$$

$$a_{23} = \frac{C_{m\dot{\alpha}}c_1 C_{z\theta}}{I_{y1}(m_1 - C_{z\dot{\alpha}}c_1)} \qquad a_{31} = 0$$

$$a_{32} = 1 \qquad a_{33} = 0$$

$$b_1 = \frac{C_{z\delta_e}}{m_1 - C_{z\dot{\alpha}}c_1} \qquad b_2 = \left(\frac{1}{I_{y1}}\right)\left(C_{m\delta_e} + \frac{C_{m\dot{\alpha}}c_1 C_{z\delta_e}}{m_1 - C_{z\dot{\alpha}}c_1}\right)$$

$$b_3 = 0$$

We have

$$C_{z\theta} = -C_L \sin\theta_o \tag{6.49}$$

For equilibrium level flight, $\theta_o = 0$, so that $C_{z\theta} \simeq 0$, $a_{13} = 0$, and $a_{23} = 0$. With this assumption, Eq. (6.46) reduces to the following form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\Delta\delta_e \tag{6.50}$$

To get an idea of the physical parameters that have a major influence on the short-period mode, we have to introduce some more simplifications. We assume $C_{zq} = C_{z\dot{\alpha}} = 0$ because they are usually small. With these assumptions, it can be shown that the characteristic equation of the short-period mode is given by

$$\lambda^2 + B\lambda + C = 0 \tag{6.51}$$

where

$$B = -\left(\frac{C_{z\alpha}}{m_1} + \frac{c_1}{I_{y1}}(C_{mq} + C_{m\dot{\alpha}})\right) \tag{6.52}$$

$$C = \frac{C_{z\alpha} c_1 C_{mq}}{m_1 I_{y1}} - \frac{C_{m\alpha}}{I_{y1}} \tag{6.53}$$

Comparing this equation with the standard form of a characteristic equation of a second-order system [Eq. (6.34)], we get

$$\omega_n = \sqrt{C} \tag{6.54}$$

$$= \sqrt{\frac{C_{z\alpha} c_1 C_{mq}}{m_1 I_{y1}} - \frac{C_{m\alpha}}{I_{y1}}} \tag{6.55}$$

$$\zeta = \frac{B}{2\omega_n} \tag{6.56}$$

$$= \frac{-\left(\frac{C_{z\alpha}}{m_1} + \frac{c_1}{I_{y1}}(C_{mq} + C_{m\dot{\alpha}})\right)}{2\sqrt{\frac{C_{z\alpha} c_1 C_{mq}}{m_1 I_{y1}} - \frac{C_{m\alpha}}{I_{y1}}}} \tag{6.57}$$

From the above analysis we observe that 1) the frequency of the short-period mode depends directly on the magnitude of the static stability parameter $C_{m\alpha}$ and 2) damping of the short-period mode directly depends on the damping-in-pitch derivative C_{mq} and the acceleration derivative $C_{m\dot{\alpha}}$. We know that the static stability parameter $C_{m\alpha}$ is directly related to the center of gravity position. As the center of gravity moves forward, $C_{m\alpha}$ decreases (becomes more stable), and the frequency of the short-period mode also increases. Conversely, if the center of gravity moves aft, $C_{m\alpha}$ increases (becomes less stable), and the frequency of the short-period mode decreases. At one point, as we see later, when $C_{m\alpha} \geq 0$, the short-period mode breaks up into two exponential modes.

The damping of the short-period mode improves with an increase in the stable values of C_{mq} and $C_{m\dot{\alpha}}$. Recall that the major contribution to C_{mq} and $C_{m\dot{\alpha}}$ comes from the horizontal tail. The higher the tail-volume ratio, the larger will be the horizontal tail contribution to C_{mq} and $C_{m\dot{\alpha}}$ and the higher will be the short-period damping ratio.

6.2.2 Phugoid Approximation

Because the disturbance in angle of attack quickly decays to zero during the short-period oscillation and subsequently remains close to zero, we assume $\Delta\alpha = \Delta\dot{\alpha} = 0$ for the phugoid motion. Furthermore, we assume that the pitching motion is quite slow so that pitch acceleration can be ignored, i.e., $\dot{q} = \ddot{\theta} = 0$. In view of this, the pitching moment Eq. (6.3) can be ignored. Then, Eqs. (6.1) and (6.2) reduce to the following form:

$$m_1 \frac{\mathrm{d}u}{\mathrm{d}t} = C_{xu}u + C_{x\theta}\Delta\theta + C_{xq}c_1\left(\frac{\mathrm{d}\Delta\theta}{\mathrm{d}t}\right) + C_{x\delta_e}\Delta\delta_e \tag{6.58}$$

$$(m_1 + C_{zq}c_1)\frac{\mathrm{d}\Delta\theta}{\mathrm{d}t} = -C_{zu}u - C_{z\theta}\Delta\theta - C_{z\delta_e}\Delta\delta_e \tag{6.59}$$

Rearranging in a form suitable for state-space representation, we obtain

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \left(\frac{C_{xu} - \xi_3 C_{zu}}{m_1}\right)u + \left(\frac{C_{x\theta} - \xi_3 C_{z\theta}}{m_1}\right)\Delta\theta + \left(\frac{C_{x\delta_e} - \xi_3 C_{z\delta_e}}{m_1}\right)\Delta\delta_e \tag{6.60}$$

$$\frac{\mathrm{d}\Delta\theta}{\mathrm{d}t} = \left(\frac{-C_{zu}}{m_1 + C_{zq}c_1}\right)u + \left(\frac{-C_{z\theta}}{m_1 + C_{zq}c_1}\right)\Delta\theta + \left(\frac{-C_{z\delta_e}}{m_1 + C_{zq}c_1}\right)\Delta\delta_e \tag{6.61}$$

where

$$\xi_3 = \frac{C_{xq}c_1}{m_1 + C_{zq}c_1} \tag{6.62}$$

Let

$$x_1 = u \tag{6.63}$$

$$x_2 = \Delta\theta \tag{6.64}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{6.65}$$

$$U = \Delta\delta_e \tag{6.66}$$

so that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} U \tag{6.67}$$

Let

$$A_{ph} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{6.68}$$

$$B_{ph} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{6.69}$$

where

$$a_{11} = \left(\frac{C_{xu} - \xi_3 C_{zu}}{m_1}\right) \qquad a_{12} = \left(\frac{C_{x\theta} - \xi_3 C_{z\theta}}{m_1}\right)$$

$$a_{21} = \left(\frac{-C_{zu}}{m_1 + C_{zq}c_1}\right) \qquad a_{22} = \left(\frac{-C_{z\theta}}{m_1 + C_{zq}c_1}\right)$$

$$b_1 = \left(\frac{C_{x\delta_e} - \xi_3 C_{z\delta_e}}{m_1}\right) \qquad b_2 = \left(\frac{-C_{z\delta_e}}{m_1 + C_{zq}c_1}\right)$$

Equations (6.60) and (6.61) are approximate equations for the phugoid or long-period mode and are applicable for conventional statically stable airplanes. These equations may not be applicable if the airplane is statically unstable or incorporates relaxed static stability concepts wherein the aircraft is rendered statically unstable to improve the performance. For such airplanes, as we will discuss later, the conventional short-period and phugoid modes do not exist.

To get an idea of the physical parameters that have major influence on the frequency and damping of the phugoid mode, we need to introduce some additional simplifications. As said before, $C_{z\theta} \simeq 0$. Furthermore, assume that C_{xq} and C_{zq} are small and can be ignored. With these assumptions, it can be shown that the characteristic equation of the phugoid mode is given by

$$\lambda^2 - \frac{C_{xu}}{m_1}\lambda + \frac{C_{x\theta}C_{zu}}{m_1^2} = 0 \tag{6.70}$$

Comparing this equation with that of the standard second-order system [Eq. (6.34)], we obtain

$$\omega_n = \frac{1}{m_1}\sqrt{C_{x\theta}C_{zu}} \tag{6.71}$$

$$\zeta = -\frac{C_{xu}}{2m_1\omega_n} \tag{6.72}$$

We have

$$C_{xu} = -2C_D - C_{Du} \tag{6.73}$$

$$C_{x\theta} = -C_L \cos\theta_o \tag{6.74}$$

$$C_{zu} = -2C_L - C_{Lu} \tag{6.75}$$

For low subsonic speeds, $C_{Du} \simeq C_{Lu} \simeq 0$. For level flight, $\theta_o = 0$ so that

$$C_{xu} = -2C_D \tag{6.76}$$

$$C_{x\theta} = -C_L \tag{6.77}$$

$$C_{zu} = -2C_L \tag{6.78}$$

Furthermore, using $C_L = 2W/\rho U_o^2 S$, we obtain

$$\omega_n = \frac{\sqrt{2}g}{U_o} \tag{6.79}$$

$$T = \frac{\sqrt{2}\pi U_o}{g} \tag{6.80}$$

$$\zeta = \frac{1}{\sqrt{2}}\left(\frac{C_D}{C_L}\right) \tag{6.81}$$

$$= \frac{1}{\sqrt{2}E} \tag{6.82}$$

Thus, we observe that the damping of the phugoid mode is inversely proportional to the aerodynamic efficiency E. Because E varies with angle of attack, the phugoid damping will vary with angle of attack, becoming minimum when the airplane flies at that angle of attack when $E = E_{\max}$ or $C_L = \sqrt{C_{DO}/k}$ [see Eqs. (2.11) and (2.20) of Chapter 2]. We also note that the period of the phugoid motion increases with forward speed.

6.2.3 Accuracy of Short-Period and Phugoid Approximations

It is instructive to assess the accuracy of the short-period and the phugoid approximations. For this purpose, let us consider the general aviation airplane once again as follows.

Substituting the values of mass, inertia, and aerodynamic parameters in Eqs. (6.46) and (6.67), we get

$$A_{sp} = \begin{bmatrix} -2.0354 & 0.9723 \\ -7.0301 & -2.9767 \end{bmatrix} \tag{6.83}$$

$$B_{sp} = \begin{bmatrix} -0.0028 \\ -0.2071 \end{bmatrix} \tag{6.84}$$

$$A_{ph} = \begin{bmatrix} -0.0453 & -0.183 \\ 0.3823 & 0 \end{bmatrix} \tag{6.85}$$

$$B_{ph} = \begin{bmatrix} 0 \\ 0.0029 \end{bmatrix} \tag{6.86}$$

Using MATLAB,[1] we obtain the eigenvalues for the short-period and phugoid approximations as

$$\lambda_{sp} = -2.5060 \pm j2.5717 \tag{6.87}$$

$$\lambda_{ph} = -0.0227 \pm j0.2656 \tag{6.88}$$

Comparing these eigenvalues with those of the complete fourth-order system obtained earlier [Eqs. (6.31) and (6.32)], we observe that the short-period approximation is quite satisfactory for the prediction of free response of the general aviation aircraft, whereas the damping and frequency of phugoid motion are in error. In general, for typical statically stable aircraft, this type of comparison is usually obtained.

The free responses of the general aviation airplane based on complete fourth-order system and based on short-period and phugoid approximations with assumed initial conditions $\Delta\alpha = 5$ deg and $u = 0.1$ are shown in Figs. 6.4 (a) and 6.4 (b). The MATLAB[1] was used for these calculations. We observe that the disturbances in angle of attack and pitch rate q decay quickly and come close to zero within 3–5 s, whereas the disturbances in forward velocity and pitch angle persist for a long time and decay slowly. In other words, disturbances in angle of attack and pitch rate decay rapidly during the short-period mode. During the phugoid mode, which continues after the decay of the short-period mode, the value of angle of attack nearly remains constant, and the pitch rate is close to zero.

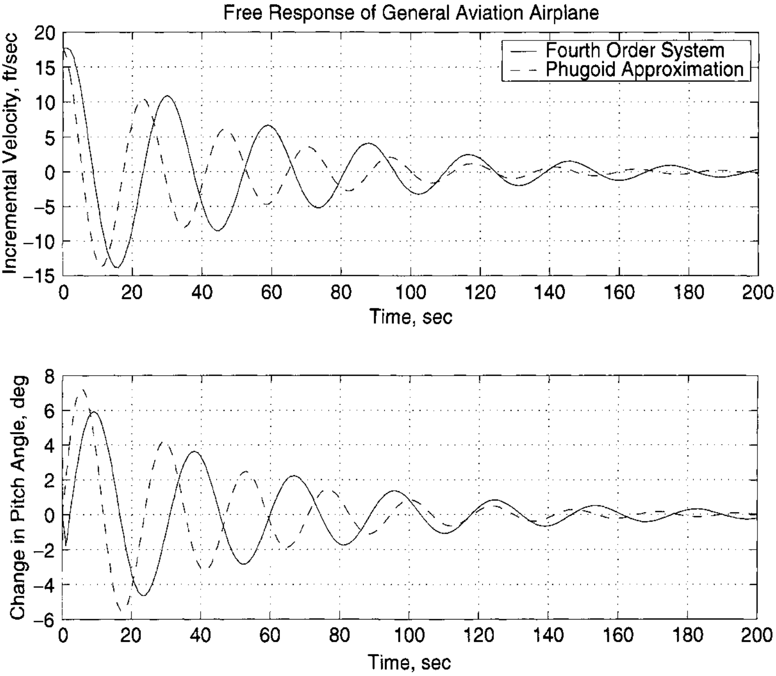


Fig. 6.4a Longitudinal response of general aviation airplane.

Next, let us examine the accuracy of the short-period and the phugoid approximations for forced response. For this purpose, we have computed the unit-step response of the general aviation airplane, and the results are presented in Figs. 6.5–6.7. It is interesting to observe that the step responses differ considerably. The short-period response agrees well with that of the complete fourth-order system initially, but the steady-state values differ. The short-period approximation results in a quick decay of the oscillatory motion, whereas the complete system continues to oscillate for a much longer time. The step response based on phugoid approximation differs considerably from that for the complete system. There is a significant difference in the transient as well as steady-state values. Thus, for the general aviation airplane, the forced response based on short-period approximation is somewhat satisfactory, whereas the phugoid approximation is not adequate at all. Therefore, it is always a good practice to check any control law design based on such approximations by a simulation of the complete system using the same control law. If the two simulations are in fair agreement, then the design is satisfactory. If not, the exercise has to be repeated using the complete system instead of a reduced-order system based on either short-period or phugoid approximations.

6.2.4 *Effect of Static Stability on Longitudinal Response*

When the center of gravity moves aft, the static stability level of the airplane decreases. When it goes aft of the stick-fixed neutral point, the aircraft becomes

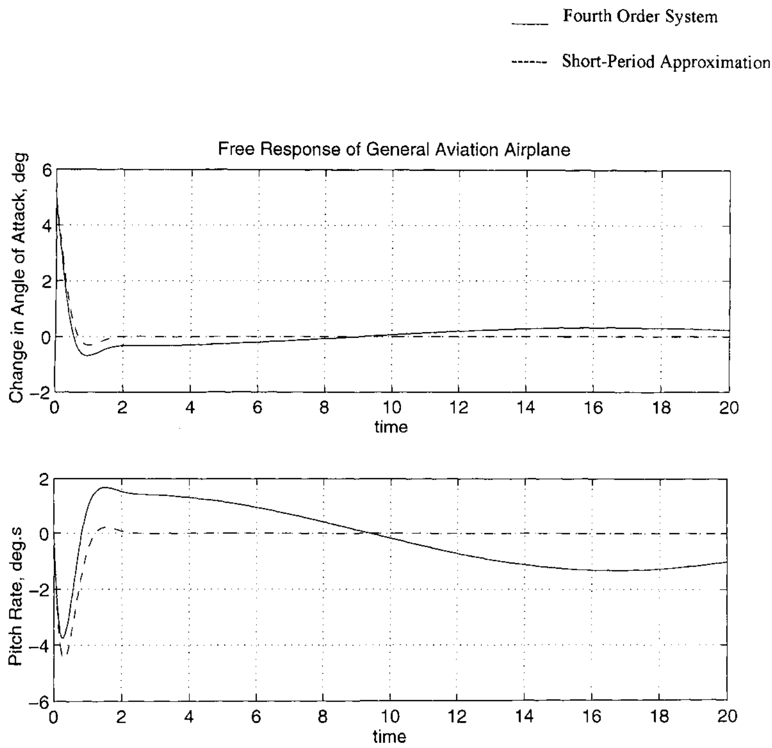


Fig. 6.4b Longitudinal response of general aviation airplane, continued.

statically unstable. When this happens, in the s-plane, the short-period and phugoid roots move towards the real axis as shown in Fig. 6.8. Here, AA and BB denote the short-period and phugoid roots for a stable location of the center of gravity. When the center of gravity moves forward, the roots move onto the real axis, and the conventional short-period and phugoid approximations break down. The right moving branch of the short-period mode meets the left moving branch of the phugoid mode and a new oscillatory mode emerges. This mode has the short-period like damping and phugoid like frequency. This mode is sometimes known as the third oscillatory mode. At the same time, the right moving branch of the phugoid mode crosses the imaginary axis and moves into the right half of the s-plane. This indicates that the airplane will exhibit an exponential instability in pitch.

The condition when the real root crosses the imaginary axis and moves into the right half of the s-plane can be determined as follows.

According to Routh's criterion, the necessary condition for stability is that all coefficients of the characteristic polynomial must be positive or have the same sign. When any one of the coefficients becomes negative while the rest are positive, then the necessary condition is violated and the system becomes unstable. By examining the expression of all the coefficients of the characteristic polynomial, we observe that the coefficient that is most likely to change sign due to a change in static

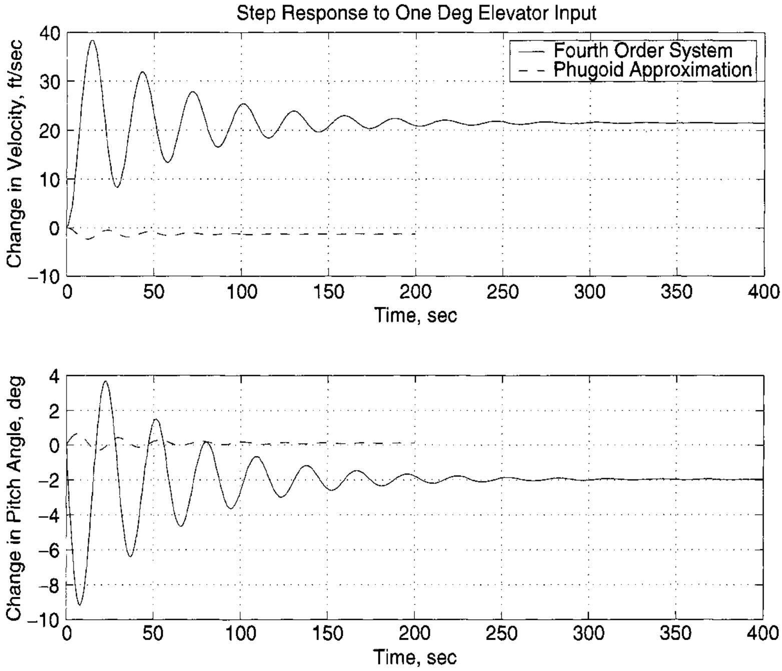


Fig. 6.5 Unit-step response of general aviation airplane: forward velocity and pitch angle.

stability level is the coefficient E_δ as given by Eq. (6.25). Therefore, the condition for the root-locus to cross the imaginary axis is

$$\begin{aligned} E_\delta &= C_{z\theta}(C_{xu}C_{m\alpha} - C_{x\alpha}C_{mu}) - C_{x\theta}(C_{zu}C_{m\alpha} - C_{z\alpha}C_{mu}) \\ &= 0 \end{aligned} \tag{6.89}$$

or

$$C_{m\alpha} = \frac{C_{mu}(C_{x\alpha}C_{z\theta} - C_{x\theta}C_{z\alpha})}{C_{xu}C_{z\theta} - C_{x\theta}C_{zu}} \tag{6.90}$$

We can express the derivative C_{mu} as

$$C_{mu} = \frac{\partial C_m}{\partial\left(\frac{\Delta V}{V}\right)} \tag{6.91}$$

$$= M\frac{\partial C_m}{\partial M} \tag{6.92}$$

$$= M\alpha\frac{\partial C_{m\alpha}}{\partial M} \tag{6.93}$$

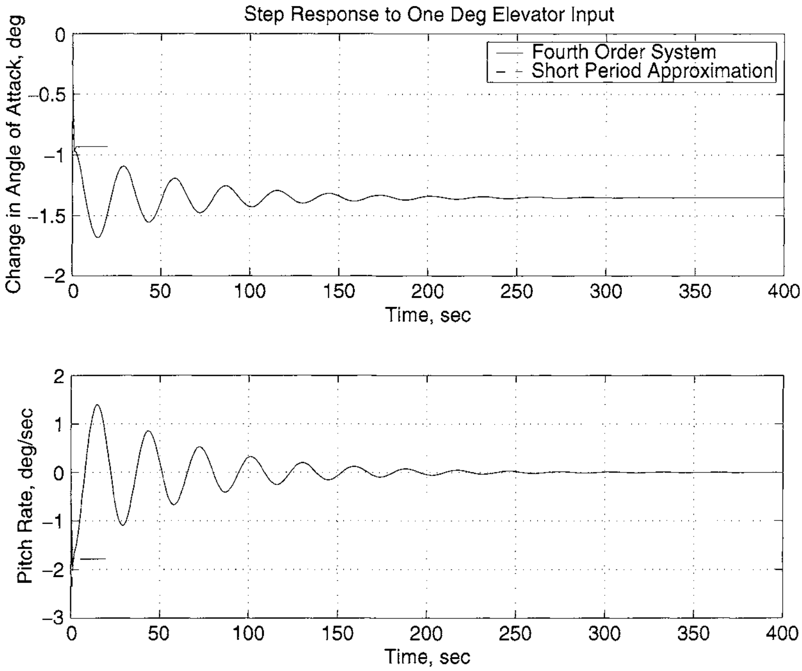


Fig. 6.6 Unit-step response of general aviation airplane: angle of attack and pitch rate.

Furthermore, we have

$$C_{x\alpha} = C_L - C_{D\alpha} \tag{6.94}$$

$$C_{x\theta} = -C_L \cos\theta_o \tag{6.95}$$

$$C_{z\theta} = -C_L \sin\theta_o \tag{6.96}$$

$$C_{xu} = -2C_D - C_{Du}, \quad C_{zu} = -2C_L - C_{Lu} \tag{6.97}$$

$$C_{z\alpha} = -C_{L\alpha} - C_D \tag{6.98}$$

However, if we assume $\theta_o = 0$ (equilibrium level flight) so that $C_{z\theta} = 0$, then we have

$$C_{m\alpha} = \frac{C_{z\alpha} C_{mu}}{C_{zu}} \tag{6.99}$$

Let us ignore the compressibility effects, i.e., we assume that the flight Mach number is below 0.5, so that $C_{mu} = 0$. Then, for such cases, the above condition reduces to

$$C_{m\alpha} = 0 \tag{6.100}$$

From Chapter 3, Eq. (3.85), we have

$$C_{m\alpha} = (\bar{x}_{cg} - N_o) C_{L\alpha} \tag{6.101}$$

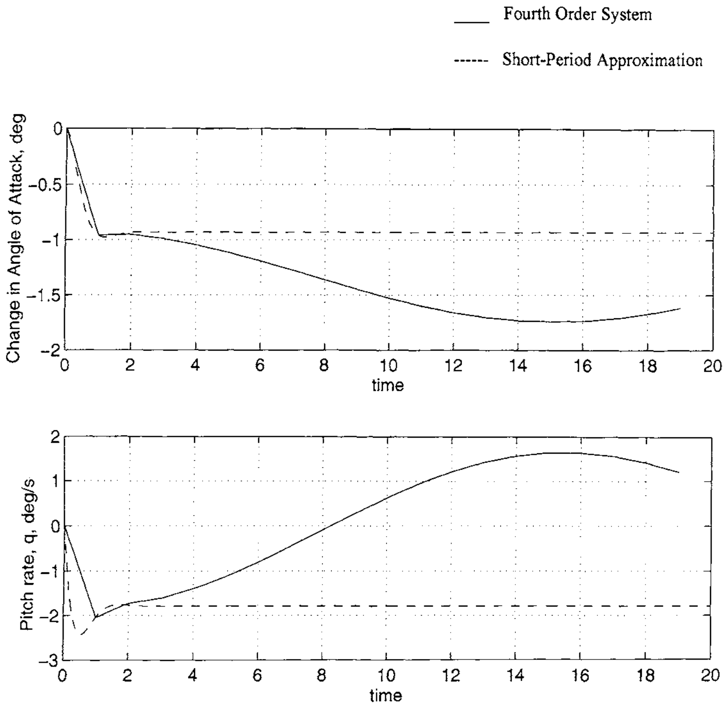


Fig. 6.7 Unit-step response of general aviation airplane: angle of attack and pitch rate for first 20 s.

so that

$$\bar{x}_{cg} = N_o \tag{6.102}$$

Thus, when the center of gravity coincides with the stick-fixed neutral point, the root-locus crosses the imaginary axis. For further aft locations of the center of gravity, the root-locus moves into the right half of the s-plane, and the aircraft becomes dynamically unstable as well. To have an idea of the nature of free response of a statically unstable airplane, let us consider the general aviation airplane once again with the assumption that the center of gravity is moved aft to make it unstable in pitch. We know that the center of gravity has a direct effect on the damping terms C_{mq} and $C_{m\dot{\alpha}}$ through horizontal tail terms. To account for these effects, let $C_{m\alpha} = 0.07$/rad and both $C_{m\dot{\alpha}}$ and C_{mq} be reduced by 50%.

The eigenvalues of this statically unstable version of the general aviation airplane were calculated as

$$\lambda_1 = -3.1303 \tag{6.103}$$

$$\lambda_{2,3} = -0.2965 \pm j0.2062 \tag{6.104}$$

$$\lambda_4 = 0.1542 \tag{6.105}$$

Notice that the pair of complex roots $\lambda_{2,3}$ has a short-period-like damping and phugoidlike frequency. This is the so-called third oscillatory mode.

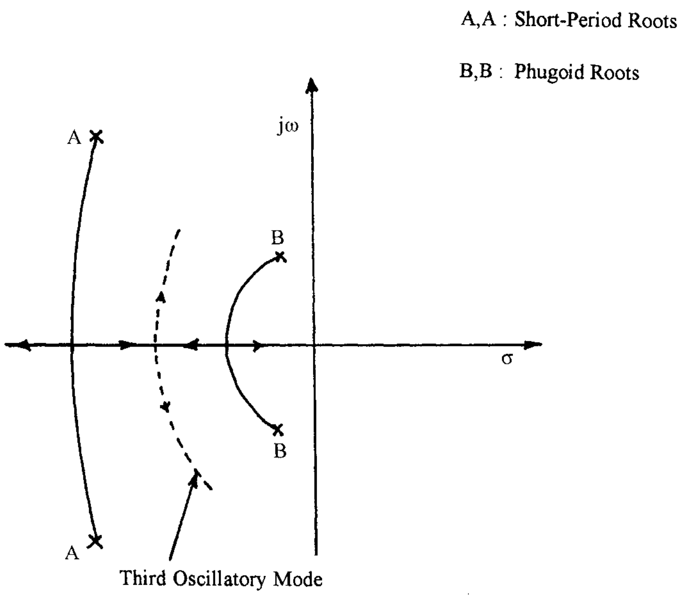


Fig. 6.8 Effect of center of gravity location on longitudinal modes.

The free response to an initial disturbance in angle of attack of 5 deg is shown in Fig. 6.9. We note that the disturbance in angle of attack decays initially because of the damping effect provided by the negative real root and to some extent by the negative real part of the complex root. However, the angle of attack starts building up subsequently due to the real positive root λ_4. Along with this, pitch rate also builds up as shown. For such airplanes, it is desirable to have an automatic stability augmentation system that can take care of this problem so that the airplane has the conventional short-period and phugoid modes in the closed-loop sense. We will discuss the design of such a pitch augmentation system later in this chapter.

6.2.5 Longitudinal Transfer Functions

The transfer function gives a relation between the input and the output of a control system in the Laplace domain. It forms the basis of analysis and design using frequency-domain methods. Here, we will derive longitudinal transfer functions for the elevator input for the complete fourth-order system as well as those based on short-period and phugoid approximations.

Transfer functions of a complete system. Assuming that an elevator is the only longitudinal control and taking the Laplace transform of Eqs. (6.1–6.3), we obtain

$$(m_1 s - C_{xu})\bar{u}(s) - (C_{x\alpha} + C_{x\dot{\alpha}}c_1 s)\Delta\bar{\alpha}(s) - (C_{xq}c_1 s + C_{x\theta})\Delta\bar{\theta}(s) = C_{x\delta_e}\Delta\bar{\delta}_e(s) \tag{6.106}$$

$$-C_{zu}\bar{u}(s) + (m_1 s - C_{z\dot{\alpha}}c_1 s - C_{z\alpha})\Delta\bar{\alpha}(s) - (m_1 s + C_{zq}c_1 s + C_{z\theta})\Delta\bar{\theta}(s) = C_{z\delta e}\Delta\bar{\delta}_e(s) \tag{6.107}$$

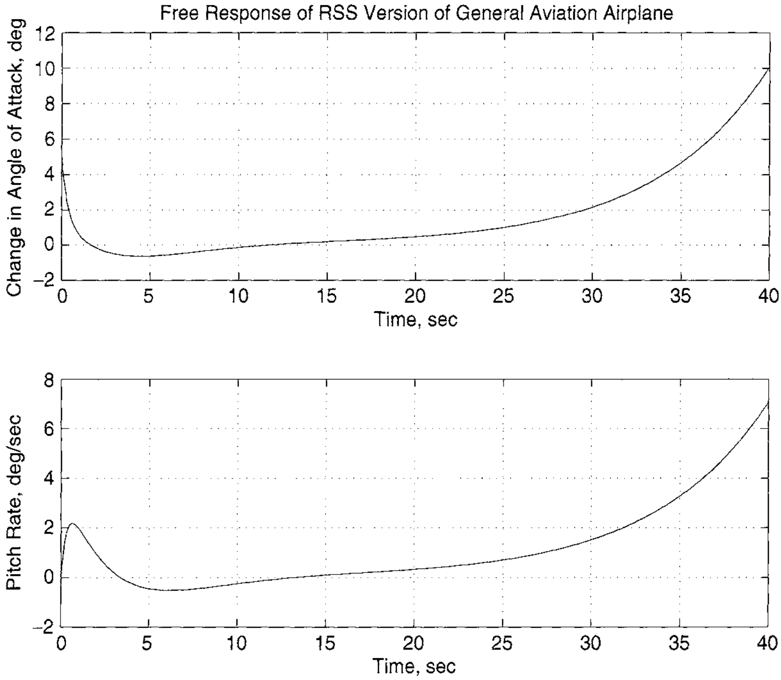


Fig. 6.9 Free (longitudinal) response of statically unstable version of the general aviation airplane.

$$-C_{mu}\bar{u}(s) - (C_{m\dot{\alpha}}c_1 s + C_{m\alpha})\Delta\bar{\alpha}(s) + s(I_{y1}s - C_{mq}c_1)\Delta\bar{\theta}(s) = C_{m\delta e}\Delta\bar{\delta}_e(s) \tag{6.108}$$

Here, s is the Laplace variable, and a bar over the symbol denotes its Laplace transform. Dividing throughout by $\Delta\bar{\delta}_e(s)$ and using Cramer's rule, we obtain

$$\frac{\bar{u}(s)}{\Delta\bar{\delta}_e} = \frac{\begin{vmatrix} C_{x\delta_e} & -(C_{x\alpha} + C_{x\dot{\alpha}}c_1 s) & -(C_{xq}c_1 s + C_{x\theta}) \\ C_{z\delta_e} & (m_1 s - C_{z\dot{\alpha}}c_1 s - C_{z\alpha}) & -(m_1 s + C_{zq}c_1 s + C_{z\theta}) \\ C_{m\delta_e} & -(C_{m\dot{\alpha}}c_1 s + C_{m\alpha}) & s(I_{y1}s - C_{mq}c_1) \end{vmatrix}}{\begin{vmatrix} (m_1 s - C_{xu}) & -(C_{x\alpha} + C_{x\dot{\alpha}}c_1 s) & -(C_{xq}c_1 s + C_{x\theta}) \\ -C_{zu} & (m_1 s - C_{z\dot{\alpha}}c_1 s - C_{z\alpha}) & -(m_1 s + C_{zq}c_1 s + C_{z\theta}) \\ -C_{mu} & -(C_{m\dot{\alpha}}c_1 s + C_{m\alpha}) & s(I_{y1}s - C_{mq}c_1) \end{vmatrix}} \tag{6.109}$$

Let $N_{\bar{u}}$ denote the determinant in the numerator and $\Delta_{\text{long}}(s)$ denote the determinant in the denominator of Eq. (6.109). Then, expanding the determinant in the numerator, we obtain

$$N_{\bar{u}} = A_{\bar{u}}s^3 + B_{\bar{u}}s^2 + C_{\bar{u}}s + D_{\bar{u}} \tag{6.110}$$

where

$$A_{\bar{u}} = I_{y1}\big(C_{x\delta_e}m_1 - C_{x\delta_e}C_{z\dot{\alpha}}c_1 + C_{x\dot{\alpha}}c_1C_{z\delta_e}\big)$$

$$\begin{aligned}B_{\bar{u}} = {} & -C_{x\delta_e}\big(C_{z\alpha}I_{y1} + m_1C_{mq}c_1 - C_{z\dot{\alpha}}C_{mq}c_1^2 + m_1C_{m\dot{\alpha}}c_1 + C_{zq}C_{m\dot{\alpha}}c_1^2\big)\\ & - C_{z\delta e}C_{x\dot{\alpha}}c_1^2C_{mq} + C_{z\delta_e}I_{y1}C_{x\alpha} + C_{x\dot{\alpha}}c_1C_{m\delta_e}(m_1 + C_{zq}c_1)\\ & + C_{xq}c_1^2C_{z\delta_e}C_{m\dot{\alpha}} + C_{xq}c_1C_{m\delta_e}(m_1 - C_{z\dot{\alpha}}c_1)\end{aligned}$$

$$\begin{aligned}C_{\bar{u}} = {} & C_{x\delta_e}(C_{z\alpha}C_{mq}c_1 - m_1C_{m\alpha} - C_{zq}c_1C_{m\alpha} - C_{z\theta}C_{m\dot{\alpha}}c_1)\\ & - C_{mq}C_{z\delta_e}C_{x\alpha}c_1 + C_{x\alpha}C_{m\delta_e}(m_1 + C_{zq}c_1)\\ & + C_{x\dot{\alpha}}c_1C_{z\theta}C_{m\delta_e} + C_{xq}c_1C_{z\delta_e}C_{m\alpha}\\ & - C_{xq}c_1C_{m\delta_e}C_{z\alpha} + C_{x\theta}C_{z\delta_e}C_{m\dot{\alpha}}c_1\\ & + C_{x\theta}C_{m\delta_e}(m_1 - C_{z\dot{\alpha}}c_1)\end{aligned}$$

$$D_{\bar{u}} = C_{z\theta}(C_{x\alpha}C_{m\delta_e} - C_{m\alpha}C_{x\delta_e}) + C_{x\theta}(C_{z\delta_e}C_{m\alpha} - C_{m\delta_e}C_{z\alpha})$$

Furthermore, expanding the determinant in the denominator, we obtain the longitudinal characteristic polynomial

$$\Delta_{\text{long}}(s) = A_\delta s^4 + B_\delta s^3 + C_\delta s^2 + D_\delta s + E_\delta \tag{6.111}$$

where the coefficients A_δ, B_δ, C_δ, D_δ, and E_δ are identical to those given in Eqs. (6.21–6.25) and are reproduced here in the following:

$$A_\delta = m_1I_{y1}(m_1 - C_{z\dot{\alpha}}c_1) \tag{6.112}$$

$$\begin{aligned}B_\delta = {} & m_1(-I_{y1}C_{z\alpha} - C_{mq}c_1[m_1 - C_{z\dot{\alpha}}c_1] - C_{m\dot{\alpha}}c_1[m_1 + C_{zq}c_1])\\ & - C_{xu}I_{y1}(m_1 - C_{z\dot{\alpha}}c_1) - C_{x\dot{\alpha}}c_1C_{zu}I_{y1}\end{aligned} \tag{6.113}$$

$$\begin{aligned}C_\delta = {} & m_1\big(C_{z\alpha}C_{mq}c_1 - C_{m\alpha}[m_1 + C_{zq}c_1] - C_{z\theta}C_{m\dot{\alpha}}c_1\big)\\ & + C_{xu}(I_{y1}C_{z\alpha} + C_{mq}c_1[m_1 - C_{z\dot{\alpha}}c_1] + C_{m\dot{\alpha}}c_1[m_1 + C_{zq}c_1])\\ & - C_{x\alpha}C_{zu}I_{y1} + C_{x\dot{\alpha}}c_1(C_{zu}C_{mq}c_1 - C_{mu}[m_1 + C_{zq}c_1])\\ & - C_{xq}c_1(C_{zu}C_{m\dot{\alpha}}c_1 + C_{mu}[m_1 - C_{z\dot{\alpha}}c_1)])\end{aligned} \tag{6.114}$$

$$\begin{aligned}D_\delta = {} & -C_{xu}\big(C_{z\alpha}C_{mq}c_1 - C_{m\alpha}[m_1 + C_{zq}c_1] - C_{z\theta}C_{m\dot{\alpha}}c_1\big)\\ & - m_1C_{m\alpha}C_{z\theta} + C_{x\alpha}\big(C_{zu}C_{mq}c_1 - C_{mu}[m_1 + C_{zq}c_1]\big)\\ & - C_{x\dot{\alpha}}c_1C_{mu}C_{z\theta} - C_{xq}c_1(C_{zu}C_{m\alpha} - C_{z\alpha}C_{mu})\\ & - C_{x\theta}(C_{zu}C_{m\dot{\alpha}}c_1 + C_{mu}[m_1 - C_{z\dot{\alpha}}c_1])\end{aligned} \tag{6.115}$$

$$E_\delta = C_{xu}C_{m\alpha}C_{z\theta} - C_{x\alpha}C_{mu}C_{z\theta} - C_{x\theta}(C_{zu}C_{m\alpha} - C_{z\alpha}C_{mu}) \tag{6.116}$$

The transfer function for angle of attack is given by

$$\frac{\Delta\bar{\alpha}(s)}{\Delta\bar{\delta}_e} = \frac{\begin{vmatrix} m_1 s - C_{xu} & C_{x\delta_e} & -(C_{xq}c_1 s + C_{x\theta}) \\ -C_{zu} & C_{z\delta_e} & -(m_1 s + C_{zq}c_1 s + C_{z\theta}) \\ -C_{mu} & C_{m\delta_e} & s(I_{y1}s - C_{mq}c_1) \end{vmatrix}}{\Delta_{\text{long}}(s)} \tag{6.117}$$

Expanding the determinant in the numerator, we obtain

$$N_{\bar{\alpha}} = A_{\bar{\alpha}}s^3 + B_{\bar{\alpha}}s^2 + C_{\bar{\alpha}}s + D_{\bar{\alpha}} \tag{6.118}$$

where

$$A_{\bar{\alpha}} = m_1 I_{y1} C_{z\delta_e} \tag{6.119}$$

$$\begin{aligned} B_{\bar{\alpha}} = {} & m_1\big(-C_{z\delta_e}C_{mq}c_1 + C_{m\delta_e}m_1 + C_{m\delta_e}C_{zq}c_1\big) \\ & - C_{xu}C_{z\delta_e}I_{y1} + C_{x\delta_e}C_{zu}I_{y1} \end{aligned} \tag{6.120}$$

$$\begin{aligned} C_{\bar{\alpha}} = {} & C_{xu}\big(C_{z\delta_e}C_{mq}c_1 - C_{m\delta_e}m_1 - C_{m\delta_e}C_{zq}c_1\big) + m_1 C_{m\delta_e}C_{z\theta} \\ & - C_{x\delta_e}C_{zu}C_{mq}c_1 + m_1 C_{x\delta_e}C_{mu} + C_{x\delta_e}C_{mu}C_{zq}c_1 \\ & + C_{xq}c_1 C_{zu}C_{m\delta_e} - C_{xq}c_1 C_{mu}C_{z\delta_e} \end{aligned} \tag{6.121}$$

$$D_{\bar{\alpha}} = C_{x\theta}\big(C_{zu}C_{m\delta_e} - C_{mu}C_{z\delta_e}\big) + C_{z\theta}(C_{x\delta_e}C_{mu} - C_{m\delta_e}C_{xu}) \tag{6.122}$$

The transfer function for pitch angle is given by

$$\frac{\Delta\bar{\theta}(s)}{\Delta_{\bar{\delta}_e}(s)} = \frac{\begin{vmatrix} m_1 s - C_{xu} & -(C_{x\alpha} + C_{x\dot{\alpha}}c_1 s) & C_{x\delta_e} \\ -C_{zu} & (m_1 s - C_{z\dot{\alpha}}c_1 s - C_{z\alpha}) & C_{z\delta_e} \\ -C_{mu} & -(C_{m\dot{\alpha}}c_1 s + C_{m\alpha}) & C_{m\delta_e} \end{vmatrix}}{\Delta_{\text{long}}(s)} \tag{6.123}$$

Expanding the determinant in the numerator, we obtain

$$N_{\theta} = A_{\bar{\theta}}s^2 + B_{\bar{\theta}}s + C_{\bar{\theta}} \tag{6.124}$$

where

$$A_{\bar{\theta}} = m_1\big(m_1 C_{m\delta_e} - C_{z\dot{\alpha}}c_1 C_{m\delta_e} + C_{z\delta_e}C_{m\dot{\alpha}}c_1\big) \tag{6.125}$$

$$\begin{aligned} B_{\bar{\theta}} = {} & m_1\big(C_{z\delta_e}C_{m\alpha} - C_{z\alpha}C_{m\delta_e}\big) - C_{xu}\big(m_1 C_{m\delta_e} - C_{z\dot{\alpha}}c_1 C_{m\delta_e} + C_{z\delta_e}C_{m\dot{\alpha}}c_1\big) \\ & + c_1 C_{x\dot{\alpha}}\big(-C_{zu}C_{m\delta_e} + C_{mu}C_{z\delta_e}\big) + C_{x\delta_e}C_{zu}C_{m\dot{\alpha}}c_1 \\ & + C_{x\delta_e}C_{mu}m_1 - C_{x\delta_e}C_{mu}C_{z\dot{\alpha}}c_1 \end{aligned} \tag{6.126}$$

$$\begin{aligned} C_{\bar{\theta}} = {} & -C_{xu}\big(C_{z\delta_e}C_{m\alpha} - C_{z\alpha}C_{m\delta_e}\big) + C_{x\delta_e}(C_{m\alpha}C_{zu} - C_{mu}C_{z\alpha}) \\ & + C_{x\alpha}\big(-C_{zu}C_{m\delta_e} + C_{mu}C_{z\delta_e}\big) \end{aligned} \tag{6.127}$$

We have $q = \dot{\theta} = \Delta\dot{\theta}$ or $\bar{q}(s) = s\bar{\theta}(s)$ so that

$$\frac{\bar{q}(s)}{\Delta\bar{\delta}_e(s)} = \frac{s\Delta\bar{\theta}(s)}{\Delta\bar{\delta}_e(s)} \tag{6.128}$$

Transfer function for short-period approximation. Taking the Laplace transform of Eqs. (6.41) and (6.42), we have

$$(m_1 s - C_{z\dot{\alpha}}c_1 s - C_{z\alpha})\Delta\bar{\alpha}(s) - [s(m_1 + C_{zq}c_1) + C_{z\theta}]\Delta\bar{\theta}(s) = C_{z\delta_e}\Delta\bar{\delta}_e(s) \tag{6.129}$$

$$-(C_{m\alpha} + C_{m\dot{\alpha}}c_1 s)\Delta\bar{\alpha}(s) + (I_{y1}s^2 - C_{mq}c_1 s)\Delta\bar{\theta}(s) = C_{m\delta_e}\Delta\bar{\delta}_e(s) \tag{6.130}$$

Dividing throughout by $\Delta\bar{\delta}_e(s)$ and using Cramer's rule, we obtain the transfer function for angle of attack.

$$\frac{\Delta\bar{\alpha}(s)}{\Delta\bar{\delta}_e} = \frac{\begin{vmatrix} C_{z\delta_e} & -[s(m_1 + C_{zq}c_1) + C_{z\theta}] \\ C_{m\delta_e} & s(I_{y1}s - C_{mq}c_1) \end{vmatrix}}{\begin{vmatrix} s(m_1 - C_{z\dot{\alpha}}) - C_{z\alpha} & -[s(m_1 + C_{zq}c_1) + C_{z\theta}] \\ -(C_{m\dot{\alpha}}c_1 s + C_{m\alpha}) & s(I_{y1}s - C_{mq}c_1) \end{vmatrix}} \tag{6.131}$$

Let $N_{\bar{\alpha},\mathrm{spo}}$ and Δ_{spo} denote the determinants in the numerator and the denominator of Eq. (6.131). Expanding these determinants, we obtain

$$N_{\bar{\alpha},\mathrm{spo}} = A_{\bar{\alpha},\mathrm{spo}}s^2 + B_{\bar{\alpha},\mathrm{spo}}s + C_{\bar{\alpha},\mathrm{spo}} \tag{6.132}$$

where

$$A_{\alpha,\mathrm{spo}} = C_{z\delta_e}I_{y1} \tag{6.133}$$

$$B_{\alpha,\mathrm{spo}} = C_{m\delta_e}(m_1 + C_{zq}c_1) - C_{z\delta_e}C_{mq}c_1 \tag{6.134}$$

$$C_{\alpha,\mathrm{spo}} = C_{m\delta_e}C_{z\theta} \tag{6.135}$$

and

$$\Delta_{\mathrm{spo}} = \begin{vmatrix} s(m_1 - C_{z\dot{\alpha}}) - C_{z\alpha} & -[s(m_1 + C_{zq}c_1) + C_{z\theta}] \\ -(C_{m\dot{\alpha}}c_1 s + C_{m\alpha}) & s(I_{y1}s - C_{mq}c_1) \end{vmatrix} \tag{6.136}$$

$$= a_1 s^3 + a_2 s^2 + a_3 s + a_4 \tag{6.137}$$

where

$$a_1 = (m_1 - C_{z\dot{\alpha}}c_1)I_{y1} \tag{6.138}$$

$$a_2 = -[C_{z\alpha}I_{y1} + C_{mq}c_1(m_1 - C_{z\dot{\alpha}}c_1) + (m_1 + C_{zq}c_1)C_{m\dot{\alpha}}c_1] \tag{6.139}$$

$$a_3 = C_{z\alpha}C_{mq}c_1 - [(m_1 + C_{zq}c_1)C_{m\alpha} + C_{z\theta}C_{m\dot{\alpha}}c_1] \tag{6.140}$$

$$a_4 = -C_{z\theta}C_{m\alpha} \tag{6.141}$$

The transfer function for pitch angle is given by

$$\frac{\Delta\bar{\theta}(s)}{\Delta\bar{\delta}_e} = \frac{\begin{vmatrix} s(m_1 - C_{z\dot{\alpha}}c_1) - C_{z\alpha} & C_{z\delta_e} \\ -(C_{m\alpha} + C_{m\dot{\alpha}}c_1 s) & C_{m\delta_e} \end{vmatrix}}{\Delta_{\text{spo}}(s)} \tag{6.142}$$

Expanding the numerator in Eq. (6.142), we get

$$N_{\bar{\theta},\text{spo}} = A_{\bar{\theta},\text{spo}}s + B_{\bar{\theta},\text{spo}} \tag{6.143}$$

where

$$A_{\bar{\theta},\text{spo}} = C_{m\delta_e}(m_1 - C_{z\dot{\alpha}}c_1) + C_{m\dot{\alpha}}c_1 C_{z\delta_e} \tag{6.144}$$

$$B_{\bar{\theta},\text{spo}} = C_{m\alpha}C_{z\delta_e} - C_{z\alpha}C_{m\delta_e} \tag{6.145}$$

We will be making use of these transfer functions in the design of flight control systems to be discussed later in this chapter.

Transfer function for a phugoid approximation. Taking the Laplace transform of Eqs. (6.58) and (6.59) and rearranging, we get

$$(m_1 s - C_{xu})\bar{u}(s) - (C_{x\theta} + C_{xq}c_1 s)\Delta\bar{\theta}(s) = C_{x\delta_e}\Delta\bar{\delta}_e(s) \tag{6.146}$$

$$-C_{zu}\bar{u}(s) - [C_{z\theta} + s(m_1 + C_{zq}c_1)]\Delta\bar{\theta}(s) = C_{z\delta_e}\Delta\bar{\delta}_e(s) \tag{6.147}$$

Dividing throughout by $\Delta\bar{\delta}_e(s)$ and using Cramer's rule, the transfer function for forward velocity can be obtained as

$$\frac{\bar{u}(s)}{\Delta\bar{\delta}_e(s)} = \frac{\begin{vmatrix} C_{x\delta_e} & -(C_{x\theta} + C_{xq}c_1 s) \\ C_{z\delta_e} & -[C_{z\theta} + s(m_1 + C_{zq}c_1)] \end{vmatrix}}{\begin{vmatrix} m_1 s - C_{xu} & -(C_{x\theta} + C_{xq}c_1 s) \\ -C_{zu} & -[C_{z\theta} + s(m_1 + C_{zq}c_1)] \end{vmatrix}} \tag{6.148}$$

Let $N_{\bar{u},\text{lpo}}$ and Δ_{lpo} denote the determinants in the numerator and denominator of Eq. (6.148). Expanding these determinants, we get

$$N_{\bar{u},\text{lpo}} = A_{\bar{u},\text{lpo}}s + B_{\bar{u},\text{lpo}} \tag{6.149}$$

$$A_{\bar{u},\text{lpo}} = -\left[C_{x\delta_e}(m_1 + C_{zq}c_1) - C_{z\delta_e}C_{xq}c_1\right] \tag{6.150}$$

$$B_{\bar{u},\text{lpo}} = C_{z\delta_e}C_{x\theta} - C_{x\delta_e}C_{z\theta} \tag{6.151}$$

and

$$\Delta_{\text{lpo}} = A_{\delta,\text{lpo}}s^2 + B_{\delta,\text{lpo}}s + C_{\delta,\text{lpo}} \tag{6.152}$$

$$A_{\delta,\text{lpo}} = -m_1(m_1 + C_{zq}c_1) \tag{6.153}$$

$$B_{\delta,\text{lpo}} = -m_1 C_{z\theta} + C_{xu}(m_1 + C_{zq}c_1) - C_{zu}C_{xq}c_1 \tag{6.154}$$

$$C_{\delta,\text{lpo}} = C_{xu}C_{z\theta} - C_{zu}C_{x\theta} \tag{6.155}$$

The transfer function for pitch angle is given by

$$\frac{\bar{\theta}(s)}{\Delta\bar{\delta}_e(s)} = \frac{\begin{vmatrix} m_1 s - C_{xu} & C_{x\delta_e} \\ -C_{zu} & C_{z\delta_e} \end{vmatrix}}{\Delta_{\text{spo}}} \tag{6.156}$$

Expanding the determinant in the numerator of Eq. (6.156), we obtain

$$N_{\bar{\theta},\text{lpo}} = A_{\theta,\text{lpo}} s + B_{\theta,\text{lpo}} \tag{6.157}$$

$$A_{\bar{\theta},\text{lpo}} = m_1 C_{z\delta_e} \tag{6.158}$$

$$B_{\bar{\theta},\text{lpo}} = C_{x\delta_e} C_{zu} - C_{xu} C_{z\delta_e} \tag{6.159}$$

For the general aviation airplane, substitution of various parameters in Eqs. (6.109), (6.123), and (6.117) gives the following longitudinal transfer functions for the complete (fourth-order) system:

$$\frac{\bar{u}(s)}{\Delta\bar{\delta}_e} = \frac{-0.0022\,s^2 + 0.6617\,s + 1.5997}{0.3739\,s^4 + 1.8907\,s^3 + 4.9103\,s^2 + 0.2526\,s + 0.2296} \tag{6.160}$$

$$\frac{\Delta\bar{\theta}(s)}{\Delta\bar{\delta}_e} = \frac{-4.4367\,s^2 - 8.8084\,s - 0.4507}{0.3739\,s^4 + 1.8907\,s^3 + 4.9103\,s^2 + 0.2526\,s + 0.2296} \tag{6.161}$$

$$\frac{\Delta\bar{\alpha}(s)}{\Delta\bar{\delta}_e} = \frac{-0.0602\,s^3 - 4.4954\,s^2 - 0.2037\,s - 0.3103}{0.3739\,s^4 + 1.8907\,s^3 + 4.9103\,s^2 + 0.2526\,s + 0.2296} \tag{6.162}$$

Similarly, the transfer functions of the general aviation airplane based on short-period approximation are given by

$$\frac{\Delta\bar{\alpha}(s)}{\Delta\bar{\delta}_e} = \frac{-0.0273\,s^2 - 2.0366\,s}{0.1695\,s^3 + 0.8494\,s^2 + 2.1851\,s} \tag{6.163}$$

$$\frac{\Delta\bar{\theta}(s)}{\Delta\bar{\delta}_e} = \frac{-2.0112\,s - 3.9018}{0.1695\,s^3 + 0.8494\,s^2 + 2.1851\,s} \tag{6.164}$$

$$\frac{\bar{q}(s)}{\Delta\bar{\delta}_e} = \frac{-2.0112\,s^2 - 3.9018\,s}{0.1695\,s^3 + 0.8494\,s^2 + 2.1851\,s} \tag{6.165}$$

The transfer functions of the general aviation airplane based on phugoid approximation [substitute in Eqs. (6.148) and (6.156)] are given by

$$\frac{\bar{u}(s)}{\Delta\bar{\delta}_e} = \frac{0.1455}{-4.7314\,s^2 - 0.2145\,s - 0.3362} \tag{6.166}$$

$$\frac{\Delta\bar{\theta}(s)}{\Delta\bar{\delta}_e} = \frac{-0.7831\,s - 0.0355}{-4.7314\,s^2 - 0.2145\,s - 0.3362} \tag{6.167}$$

We will be making use of these transfer functions for the design of stability augmentation systems and autopilots later in this chapter.

6.2.6 Longitudinal Frequency Response

The frequency response of the airplane is of interest because the pilot input to a control surface can be considered as a signal with a frequency content. Generally, the bandwidth of a human pilot is about 4 rad/s (Ref. 3). Therefore, it can be expected that a human pilot can control flight path variables that respond to input frequencies within this range. The autopilots have a higher bandwidth. A typical value of autopilot bandwidth is about 20–25 rad/s (Ref. 3).

To illustrate the nature of longitudinal frequency response of conventional, statically stable airplanes, we have presented the longitudinal frequency response of the general aviation airplane based on complete fourth-order transfer function in Figs. 6.10–6.12. Here, G_u, G_α, and G_θ are the longitudinal transfer functions of the general aviation airplane given in Eqs. (6.160–6.162). From the magnitude plot of Fig. 6.10, we observe that $|G_u|$ attains a peak value around the phugoid frequency ω_{ph} and then drops off rapidly. The magnitude crossover frequency (the frequency where the magnitude is 0 db) for $|G_u|$ is about 0.5 rad/s. This implies that elevator input with frequencies beyond 0.5 rad/s have little or no effect on the forward speed. For example at $\omega = 3.6$ rad/s, which corresponds to the short-period frequency ω_{spo}, $20 \log_{10} |G_u| = -40$ db or $|G_u| = 0.01$, i.e., just 1% change in the forward speed for a unit deflection of the elevator with a frequency of 3.6 rad/s.

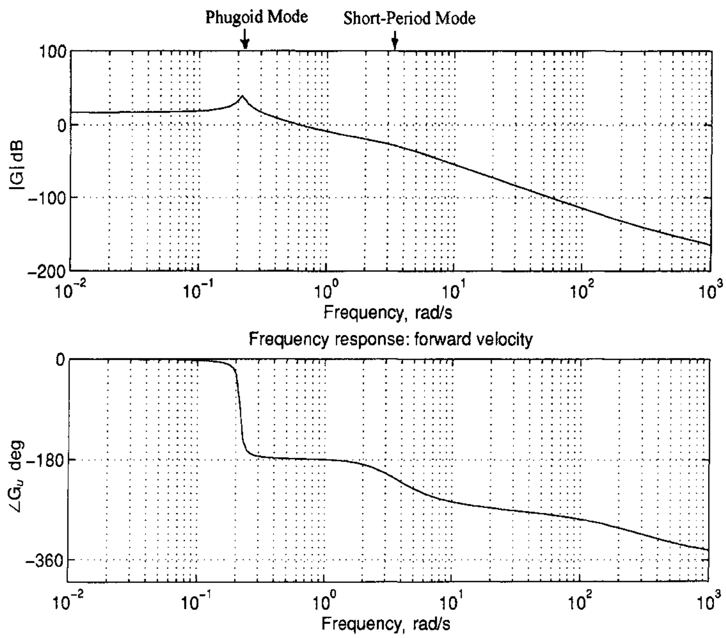


Fig. 6.10 Longitudinal frequency response of the general aviation airplane: forward velocity.

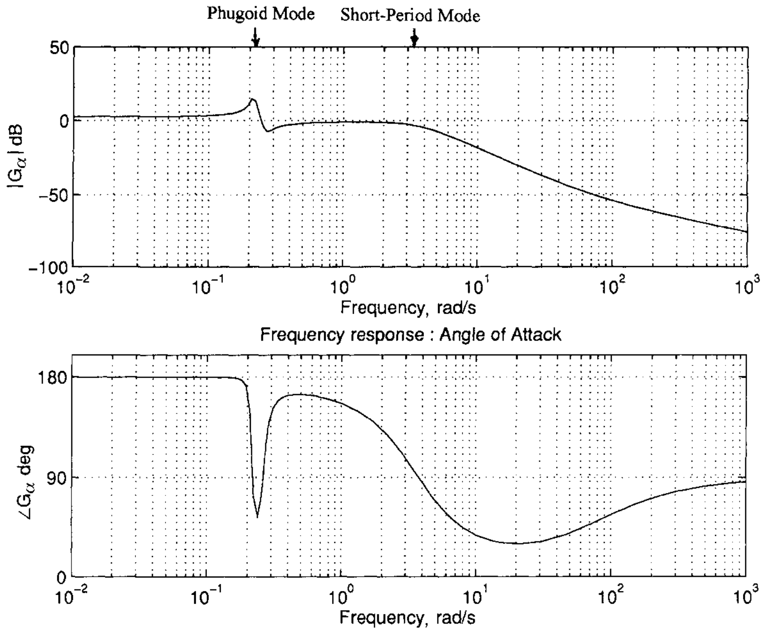


Fig. 6.11 Longitudinal frequency response of the general aviation airplane: angle of attack.

Thus, we understand that forward speed is influenced only by slow elevator inputs and rapid elevator movements have little or no effect on the forward speed. In other words, elevator input is not the best way to quickly change the forward speed.

The phase angle $\angle G_u$ is approximately zero at lower values and then suddenly drops off to -180 around $\omega = 0.21$ rad/s, which corresponds to the phugoid mode. This type of behavior of $\angle G_u$ is characteristic of lightly damped systems. As said above, this implies that only a slow movement of the elevator results in a change of forward speed without any phase lag. For $\omega > \omega_{\text{spo}}$, the phase lag increases, which is characteristic of heavily damped systems.

The frequency response of G_α displays the influence of frequency both at short-period and phugoid frequencies. The $|G_\alpha|$ remains nearly constant at low values of frequency but displays somewhat complex behavior around ω_{ph}. Such a behavior is due to a pole-zero cancellation (close proximity of pole and zero) in the transfer function G_α. The frequency response of G_θ also displays a similar and complex behavior around the phugoid frequency that is characteristic of lightly damped systems. Furthermore, around the short-period frequency, the phase angle of G_θ is close to 90 deg, which is a characteristic feature of second-order systems. We may recall that, for a second-order system, when $\omega = \omega_n$, the phase angle is 90 deg for all values of ζ. The magnitude crossover frequency for G_α and G_θ is about 2.0 rad/s. Therefore, a human pilot with a bandwidth of 4 rad/s can have a good control over the angle of attack and pitch angle.

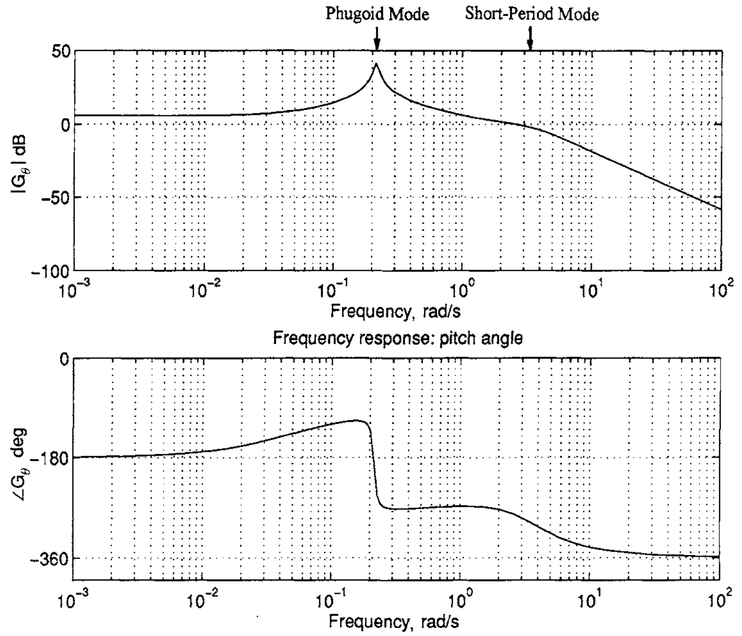


Fig. 6.12 Longitudinal frequency response of the general aviation airplane: pitch angle.

6.3 Lateral-Directional Response

Equations (4.459–4.461) from Chapter 4 which are based on the assumption of small disturbances in lateral-directional degrees of freedom can be expressed as

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = \left(\frac{1}{m_1 - b_1 C_{y\dot{\beta}}}\right)\left[C_{y\beta}\Delta\beta + C_{y\phi}\Delta\phi + b_1 C_{yp} p - (m_1 - b_1 C_{yr})r + C_{y\delta_a}\Delta\delta_a + C_{y\delta_r}\Delta\delta_r\right] \tag{6.168}$$

$$\dot{p} = \frac{1}{I_{x1}}\left(C_{l\beta}\Delta\beta + C_{l\dot{\beta}} b_1 \Delta\dot{\beta} + b_1 C_{lp} p + C_{lr} r b_1 + I_{xz1}\dot{r} + C_{l\delta_a}\Delta\delta_a + C_{l\delta_r}\Delta\delta_r\right) \tag{6.169}$$

$$\dot{r} = \frac{1}{I_{z1}}\left(C_{n\beta}\Delta\beta + C_{n\dot{\beta}} b_1 \Delta\dot{\beta} + b_1 C_{np} p + C_{nr} r b_1 + I_{xz1}\dot{p} + C_{n\delta_a}\Delta\delta_a + C_{n\delta_r}\Delta\delta_r\right) \tag{6.170}$$

where $b_1 = \frac{b}{2U_o}$ and

$$I_{x1} = \frac{I_x}{\frac{1}{2}\rho U_o^2 Sb} \tag{6.171}$$

$$I_{z1} = \frac{I_z}{\frac{1}{2}\rho U_o^2 Sb} \tag{6.172}$$

$$I_{xz1} = \frac{I_{xz}}{\frac{1}{2}\rho U_o^2 Sb} \tag{6.173}$$

Note that some of the other variables appearing in Eqs. (6.168–6.170) are defined in Eqs. (4.462–4.467) of Chapter 4. However, Eqs. (6.169) and (6.170) are not in the standard state-space form because $\dot{\beta}$, $\dot{p}$, and $\dot{r}$ terms appear on their right-hand sides. To get around this problem, we proceed as follows. From Eqs. (4.396) and (4.398), we have

$$\Delta C_l = \dot{p} I_{x1} - I_{xz1}\dot{r} \tag{6.174}$$

$$\Delta C_n = \dot{r} I_{z1} - I_{xz1}\dot{p} \tag{6.175}$$

Multiply Eq. (6.174) by I_{z1} and Eq. (6.175) by I_{xz1}, add the two resulting equations, and simplify to obtain

$$\dot{p} = I'_{z1}\Delta C_l + I'_{xz1}\Delta C_n \tag{6.176}$$

$$\dot{r} = I'_{xz1}\Delta C_l + I'_{x1}\Delta C_n \tag{6.177}$$

where

$$I'_{x1} = \frac{I_{x1}}{I_{x1}I_{z1} - I_{xz1}^2} \tag{6.178}$$

$$I'_{z1} = \frac{I_{z1}}{I_{x1}I_{z1} - I_{xz1}^2} \tag{6.179}$$

$$I'_{xz1} = \frac{I_{xz1}}{I_{x1}I_{z1} - I_{xz1}^2} \tag{6.180}$$

Then,

$$\begin{aligned}\dot{p} = {} & \left(C_{l\beta}I'_{z1} + C_{n\beta}I'_{xz1}\right)\Delta\beta + \left(C_{l\dot{\beta}}b_1 I'_{z1} + C_{n\dot{\beta}}b_1 I'_{xz1}\right)\Delta\dot{\beta} \\ & + \left(C_{lp}b_1 I'_{z1} + C_{np}b_1 I'_{xz1}\right)p + \left(C_{lr}b_1 I'_{z1} + C_{nr}b_1 I'_{xz1}\right)r \\ & + \left(C_{l\delta_a}I'_{z1} + C_{n\delta_a}I'_{xz1}\right)\Delta\delta_a + \left(C_{l\delta_r}I'_{z1} + C_{n\delta_r}I'_{xz1}\right)\Delta\delta_r\end{aligned} \tag{6.181}$$

$$\begin{aligned}\dot{r} = {} & \left(C_{n\beta}I'_{x1} + C_{l\beta}I'_{xz1}\right)\Delta\beta + \left(C_{n\dot{\beta}}b_1 I'_{x1} + C_{l\dot{\beta}}b_1 I'_{xz1}\right)\Delta\dot{\beta} \\ & + \left(C_{np}b_1 I'_{x1} + C_{lp}b_1 I'_{xz1}\right)p + \left(C_{nr}b_1 I'_{x1} + C_{lr}b_1 I'_{xz1}\right)r \\ & + \left(C_{n\delta_a}I'_{x1} + C_{l\delta_a}I'_{xz1}\right)\Delta\delta_a + \left(C_{n\delta_r}I'_{x1} + C_{l\delta_r}I'_{xz1}\right)\Delta\delta_r\end{aligned} \tag{6.182}$$

Now we have to substitute for $\dot{\beta}$ from Eq. (6.168) in Eqs. (6.181) and (6.182) and do some simplification. With this, Eqs. (6.168), (6.181), and (6.182) can be expressed in the standard state-space form as follows:

Let

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \Delta\beta \\ \Delta\phi \\ p \\ \Delta\psi \\ r \end{bmatrix} \tag{6.183}$$

$$U = \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix} \tag{6.184}$$

so that

$$\dot{X} = AX + BU \tag{6.185}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} \tag{6.186}$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} \tag{6.187}$$

and

$$a_{11} = \frac{C_{y\beta}}{m_1 - b_1 C_{y\dot{\beta}}} \qquad a_{12} = \frac{C_{y\phi}}{m_1 - b_1 C_{y\dot{\beta}}}$$

$$a_{13} = \frac{C_{yp} b_1}{m_1 - b_1 C_{y\dot{\beta}}} \qquad a_{14} = 0 \qquad a_{15} = -\left(\frac{m_1 - b_1 C_{yr}}{m_1 - b_1 C_{y\dot{\beta}}}\right)$$

$$a_{21} = 0 \qquad a_{22} = 0$$

$$a_{23} = 1 \qquad a_{24} = 0 \qquad a_{25} = 0$$

$$a_{31} = C_{l\beta} I'_{z1} + C_{n\beta} I'_{xz1} + \xi_1 b_1 a_{11} \qquad a_{32} = \xi_1 b_1 a_{12}$$

$$a_{33} = C_{lp} b_1 I'_{z1} + C_{np} I'_{xz1} b_1 + \xi_1 b_1 a_{13} \qquad a_{34} = 0$$

$$a_{35} = C_{lr} b_1 I'_{z1} + C_{nr} I'_{xz1} b_1 + \xi_1 b_1 a_{15}$$

$$a_{41} = a_{42} = a_{43} = a_{44} = 0 \qquad a_{45} = 1$$

$$a_{51} = I'_{x1} C_{n\beta} + I'_{xz1} C_{l\beta} + b_1 \xi_2 a_{11}$$

$$a_{52} = \xi_2 b_1 a_2 \qquad a_{53} = b_1 \left(C_{np} I'_{x1} + C_{lp} I'_{xz1} + \xi_2 a_{13}\right)$$

$$a_{54} = 0 \qquad a_{55} = b_1\left(I'_{x1}C_{nr} + I'_{xz1}C_{lr} + \xi_2 a_{15}\right)$$

$$b_{11} = \frac{C_{y\delta_a}}{(m_1 - b_1 C_{y\dot{\beta}})} \qquad b_{12} = \frac{C_{y\delta_r}}{(m_1 - b_1 C_{y\dot{\beta}})}$$

$$b_{21} = 0 \qquad b_{22} = 0$$

$$b_{31} = C_{l\delta_a} I'_{z1} + C_{n\delta_a} I'_{xz1} + \xi_1 b_1 b_{11} \qquad b_{32} = C_{l\delta_r} I'_{z1} + C_{n\delta_r} I'_{xz1} + \xi_1 b_1 b_{12}$$

$$b_{41} = 0 \qquad b_{42} = 0$$

$$b_{51} = C_{n\delta_a} I'_{x1} + C_{l\delta_a} I'_{xz1} + \xi_2 b_1 b_{11} \qquad b_{52} = C_{n\delta_r} I'_{x1} + C_{l\delta_r} I'_{xz1} + \xi_2 b_1 b_{12}$$

where

$$\xi_1 = I'_{z1} C_{l\dot{\beta}} + I'_{xz1} C_{n\dot{\beta}}$$

$$\xi_2 = I'_{x1} C_{n\dot{\beta}} + I'_{xz1} C_{l\dot{\beta}}$$

For free response, $U = 0$ so that the above equation reduces to

$$\dot{X} = AX \tag{6.188}$$

As said before, for nontrivial solution, the determinant of $(\lambda I - A)$ must be zero

$$|\lambda I - A| = 0 \tag{6.189}$$

An expansion of the determinant results in a fifth-order algebraic equation of the form

$$A_{\delta,\mathrm{lat}}\lambda^5 + B_{\delta,\mathrm{lat}}\lambda^4 + C_{\delta,\mathrm{lat}}\lambda^3 + D_{\delta,\mathrm{lat}}\lambda^2 + E_{\delta,\mathrm{lat}}\lambda = 0 \tag{6.190}$$

where

$$A_{\delta_{\mathrm{lat}}} = (m_1 - b_1 C_{y\dot{\beta}})\left(I_{x1} I_{z1} - I_{xz1}^2\right) \tag{6.191}$$

$$\begin{aligned} B_{\delta_{\mathrm{lat}}} &= (-C_{y\beta})\left(I_{x1} I_{z1} - I_{xz1}^2\right) - (m_1 - b_1 C_{y\dot{\beta}})(I_{x1} C_{nr} b_1 + I_{z1} C_{lp} b_1 \\ &\quad + I_{xz1} C_{lr} b_1 + I_{xz1} C_{np} b_1) - b_1 C_{yp}(C_{l\dot{\beta}} b_1 I_{z1} + C_{n\dot{\beta}} b_1 I_{xz1}) \\ &\quad + (m_1 - b_1 C_{yr})(C_{l\dot{\beta}} b_1 I_{xz1} + I_{x1} C_{n\dot{\beta}} b_1) \end{aligned} \tag{6.192}$$

$$\begin{aligned} C_{\delta_{\mathrm{lat}}} &= b_1^2(m_1 - b_1 C_{y\dot{\beta}})(C_{lp} C_{nr} - C_{lr} C_{np}) + b_1 C_{y\beta}(I_{x1} C_{nr} + I_{z1} C_{lp} \\ &\quad + I_{xz1} C_{lr} + I_{xz1} C_{np}) - C_{y\phi} b_1 (C_{l\dot{\beta}} I_{z1} + C_{n\dot{\beta}} I_{xz1}) + b_1 C_{yp} \\ &\quad \times \left(C_{l\dot{\beta}} C_{nr} b_1^2 - C_{l\beta} I_{z1} - I_{xz1} C_{n\beta} - C_{lr} C_{n\dot{\beta}} b_1^2\right) + (m_1 - b_1 C_{yr}) \\ &\quad \times \left(C_{l\beta} I_{xz1} + C_{l\dot{\beta}} C_{np} b_1^2 - C_{lp} C_{n\dot{\beta}} b_1^2 + I_{x1} C_{n\beta}\right) \end{aligned} \tag{6.193}$$

$$\begin{aligned} D_{\delta_{\mathrm{lat}}} &= -C_{y\beta} b_1^2 (C_{lp} C_{nr} - C_{lr} C_{np}) + C_{y\phi}\left(C_{l\dot{\beta}} C_{nr} b_1^2 - C_{l\beta} I_{z1} \right. \\ &\quad \left. - I_{xz1} C_{n\beta} - C_{lr} C_{n\dot{\beta}} b_1^2\right) + b_1^2 C_{yp}(C_{l\beta} C_{nr} - C_{lr} C_{n\beta}) \\ &\quad + b_1 (m_1 - b_1 C_{yr})(C_{l\beta} C_{np} - C_{lp} C_{n\beta}) \end{aligned} \tag{6.194}$$

$$E_{\delta_{\mathrm{lat}}} = C_{y\phi} b_1 (C_{l\beta} C_{nr} - C_{n\beta} C_{lr}) \tag{6.195}$$

Equation (6.190) is also called the lateral-directional characteristic equation and gives five values of the root λ. One of them is the zero root corresponding to neutral stability to a disturbance only in angle of yaw. For conventional airplanes, the other roots are two real roots and a pair of complex conjugate roots. One real root is large and negative and corresponds to the heavily damped roll subsidence mode. The other real root is small and may be positive or negative, and the response associated with this root is called the spiral mode. If this real root is positive, then the spiral mode is slowly divergent. The motion associated with the pair of complex roots is called the Dutch-roll oscillation. The damping ratio and frequency of the Dutch roll depend on the type of airplane.

To illustrate the nature of lateral-directional response, let us consider the general aviation airplane once again. In addition to the data given earlier, we have the following data for the lateral-directional aerodynamic coefficients of this airplane:[2]

$C_{y\beta} = -0.564$, $C_{y\delta_a} = 0$, $C_{y\delta_r} = 0.157$, $C_{l\beta} = -0.074$, $C_{lp} = -0.410$, $C_{lr} = 0.107$, $C_{l\delta_a} = 0.1342$, $C_{l\delta_r} = 0.0118$, $C_{n\beta} = 0.0701$, $C_{np} = -0.0575$, $C_{nr} = -0.125$, $C_{n\delta_a} = -0.00346$, $C_{n\delta_r} = -0.0717$, $C_{y\dot{\beta}} = 0$, $C_{l\dot{\beta}} = 0$, $C_{n\dot{\beta}} = 0$, $C_{yp} = 0$, and $C_{yr} = 0$. Note that all the derivatives are per radian. Furthermore, as before, we assume $M = 0.158$ and $\rho = 1.225$ kg/m^3.

Substituting these values and other parameters for the general aviation airplane in Eqs. (6.186) and (6.187) and assuming that the airplane is in equilibrium level flight ($\theta_o = 0$) before encountering a lateral-directional disturbance, we obtain

$$A = \begin{bmatrix} -0.2557 & 0.1820 & 0 & 0 & -1.0000 \\ 0 & 0 & 1.000 & 0 & 0 \\ -16.1572 & 0 & -8.4481 & 0 & 2.2048 \\ 0 & 0 & 0 & 0 & 1.0000 \\ 4.5440 & 0 & -0.3517 & 0 & -0.7647 \end{bmatrix} \tag{6.196}$$

$$B = \begin{bmatrix} 0 & 0.0712 \\ 0 & 0 \\ 29.3013 & 2.5764 \\ 0 & 0 \\ -0.2243 & -4.6477 \end{bmatrix} \tag{6.197}$$

Using MATLAB,[1] we obtain the eigenvalues of the above matrix A as

$$\lambda_1 = 0 \quad \text{(Neutral Stability in Yaw)} \tag{6.198}$$

$$\lambda_2 = -8.4804 \quad \text{(Roll Subsidence)} \tag{6.199}$$

$$\lambda_3 = -0.0087 \quad \text{(Spiral Mode)} \tag{6.200}$$

$$\lambda_{4,5} = -0.4897 \pm j2.3468 \quad \text{(Dutch Roll)} \tag{6.201}$$

The damping ratio ζ, natural frequency ω_n, period T, and time for half amplitude t_a for the Dutch-roll oscillation are obtained using Eqs. (6.35–6.40) as follows:

$$\zeta\omega_n = 0.4897 \tag{6.202}$$

$$\omega_n\sqrt{1-\zeta^2} = 2.3468 \tag{6.203}$$

$$T = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{6.204}$$

$$t_a = \frac{0.69}{\zeta\omega_n} \tag{6.205}$$

so that $\zeta = 0.2043$, $\omega_n = 2.3973$ rad/s, $T = 2.6773$ s, and $t_a = 1.4090$ s.

The free response of the general aviation airplane to various assumed initial conditions is shown in Figs. 6.13–6.16. From Fig. 6.13, we observe that the response to an initial disturbance in sideslip of 5 deg exhibits oscillatory motion due to the Dutch-roll roots. The disturbance in sideslip decays to zero within two or three oscillations lasting for about 5 to 6 s. The disturbance in sideslip induces rolling and yawing motions, which also decay along with the sideslip. However, the angle of yaw does not go to zero. Instead, it assumes a nonzero steady-state value. This is due to the zero root, which makes the motion involving yaw angle neutrally stable.

Figure 6.14 shows the free response to an initial disturbance of 5 deg in bank angle. From a time-history plot for large values of time (not shown here), it is

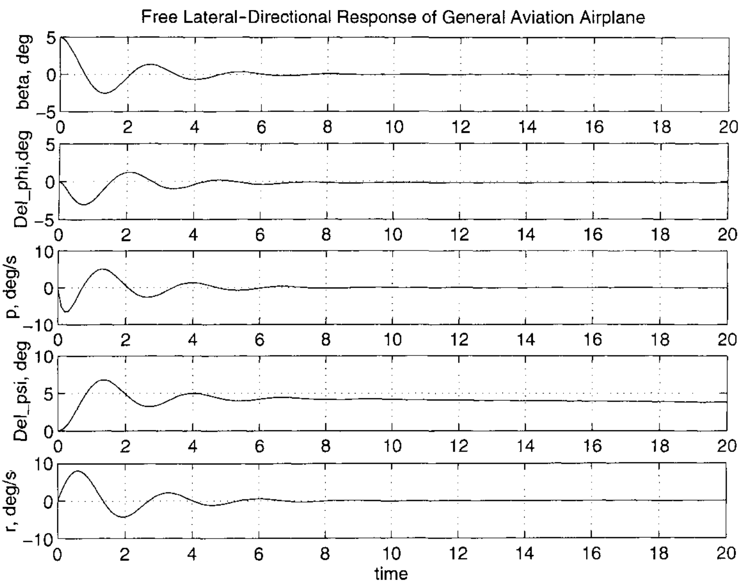


Fig. 6.13 Free (longitudinal) response of the general aviation airplane to a disturbance in sideslip.

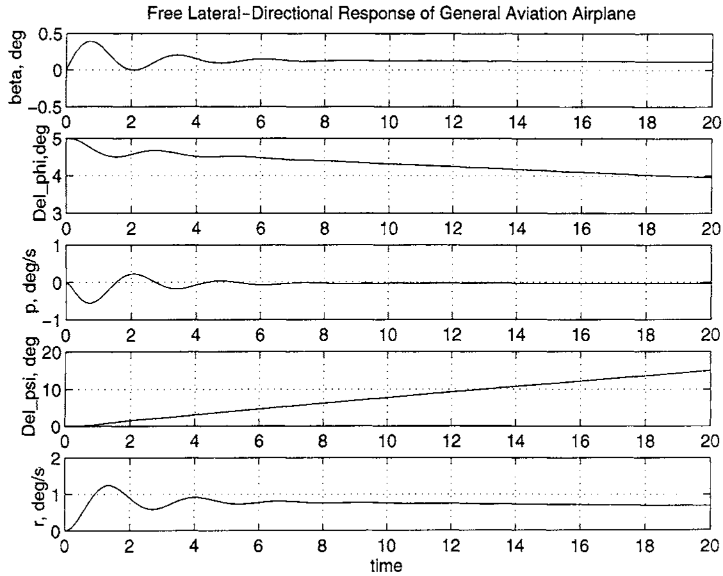


Fig. 6.14 Free (longitudinal) response of the general aviation airplane to a disturbance in bank angle.

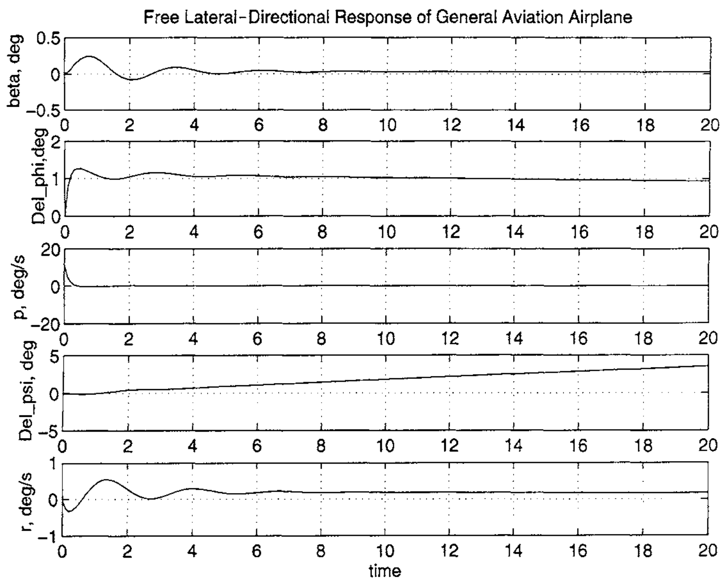


Fig. 6.15 Free (longitudinal) response of the general aviation airplane to a disturbance in roll rate.

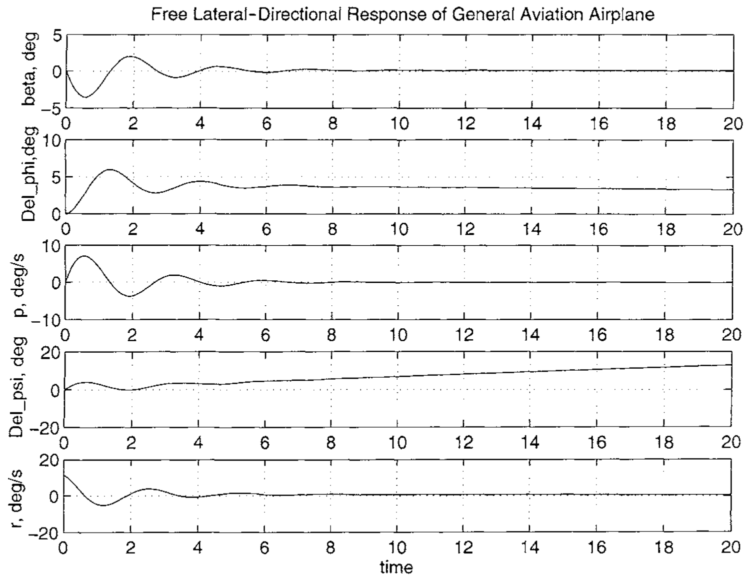


Fig. 6.16 Free (longitudinal) response of the general aviation airplane to a disturbance in yaw rate.

observed that the disturbance in bank angle took as much as 500 s to decay to zero, whereas the roll rate vanishes rapidly. The response following the heavily damped roll subsidence mode is the Dutch roll during which the sideslip, bank angle, and yaw rate display oscillatory behavior. Once the Dutch roll decays, the subsequent motion is a slow convergence and is called spiral mode. During the spiral mode, the bank angle and yaw rates decay to zero slowly. Once again, it is observed that the angle of yaw assumes a nonzero steady-state value.

The free response to an initial disturbance in roll rate of 0.2 rad/s (Fig. 6.15) is similar to the above case of disturbance in roll angle. The disturbance in roll rate quickly decays to zero, and all other variables, except angle of yaw, gradually approach zero.

The simulation of free response to an initial disturbance in angle of yaw of 5 deg (not shown here) shows that a disturbance involving only angle of yaw does not induce any sideslip, banking or roll, or the yaw rate. The disturbance in angle of yaw remains constant and does not decay at all. This is due to the fact that no aerodynamic force or moment depends on the angle of yaw. Mathematically, the associated root is zero, and the airplane is neutrally stable with respect to a disturbance in the angle of yaw.

The free response to a disturbance in yaw rate of 0.2 rad/s is shown in Fig. 6.16. It is observed that this response is very similar to other cases discussed above.

The free response to any arbitrary combination of intial disturbances can be constructed from these basic responses using the principle of superposition because the given system is a linear system.

The above free responses of the general aviation airplane are typical of a majority of the airplanes. In view of this, it is usual to introduce the following lateral-directional approximations.

6.3.1 Lateral-Directional Approximations

Roll-subsidence approximation. The motion immediately following a lateral-directional disturbance is the heavily damped roll subsidence mode during which the airplane motion is predominantly rolling about the x-body axis, and, during this process, other variables vary very slowly so that we can assume $\Delta\beta = \Delta\dot{\beta} = \Delta\psi = r = \dot{r} = 0$. With this assumption, the side force and yawing moment equations can be ignored. Furthermore, the rolling moment equation [Eq. (6.169)] assumes the following form:

$$I_{x1}\dot{p} - C_{lp}b_1 p = C_{l\delta_a}\Delta\delta_a + C_{l\delta_r}\Delta\delta_r \tag{6.206}$$

For free response, $\Delta\delta_a = \Delta\delta_r = 0$ so that

$$I_{x1}\dot{p} - C_{lp}b_1 p = 0 \tag{6.207}$$

or

$$\tau\dot{p} + p = 0 \tag{6.208}$$

where

$$\tau = -\frac{I_{x1}}{C_{lp}b_1} \tag{6.209}$$

The parameter τ is the time constant for the rolling motion of the airplane. Assuming $p = p_o e^{\lambda_r t}$ and substituting in Eq. (6.208), we get

$$\lambda_r = -\frac{1}{\tau} \tag{6.210}$$

$$= \frac{C_{lp}b_1}{I_{x1}} \tag{6.211}$$

Usually, for angles of attack below stall $C_{lp} < 0$. Therefore, λ_r is real. Usually, λ_r has a large negative value. As a result, the roll subsidence mode is well damped and is hardly felt by the pilot or passengers.

Dutch-roll approximation. The oscillatory motion following the heavily damped roll subsidence mode is the lateral-directional oscillatory motion known as the Dutch roll. The aircraft trajectory in a Dutch-roll mode is schematically shown in Fig. 6.17. It involves mainly sideslip and yawing motions. However, it is possible that there is some rolling motion due to aerodynamic roll–yaw coupling as we have observed earlier for the general aviation airplane. If the rolling motion is small and can be neglected ($\Delta\phi = p = 0$), we can ignore the rolling moment equation. Then, Eqs. (6.168) and (6.170) reduce to the following form:

$$(m_1 - b_1 C_{y\dot{\beta}})\frac{\mathrm{d}\beta}{\mathrm{d}t} = C_{y\beta}\Delta\beta - (m_1 - b_1 C_{yr})r + C_{y\delta_a}\Delta\delta_a + C_{y\delta_r}\Delta\delta_r \tag{6.212}$$

$$I_{z1}\dot{r} = C_{n\beta}\Delta\beta + C_{n\dot{\beta}}b_1\Delta\dot{\beta} + C_{nr}b_1 r + C_{n\delta_a}\Delta\delta_a + C_{n\delta_r}\Delta\delta_r \tag{6.213}$$

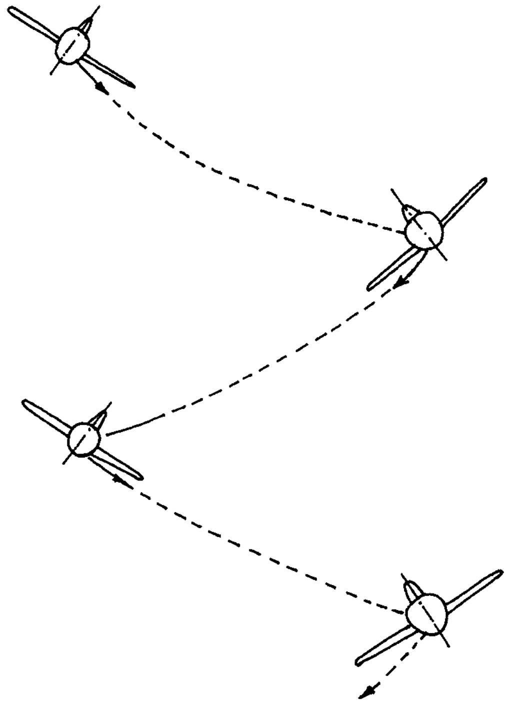


Fig. 6.17 Physical motion of the airplane during a Dutch roll.

Substituting for $\Delta\dot{\beta}$ from Eq. (6.212) in Eq. (6.213) and simplifying, we get these two equations in state-space form

$$\begin{bmatrix} \Delta\dot{\beta} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \Delta\beta \\ r \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix} \tag{6.214}$$

where

$$a_{11} = \frac{C_{y\beta}}{m_1 - C_{y\dot{\beta}}b_1} \tag{6.215}$$

$$a_{12} = -\frac{m_1 - b_1 C_{yr}}{m_1 - C_{y\dot{\beta}}b_1} \tag{6.216}$$

$$a_{21} = \frac{1}{I_{z1}}\left(C_{n\beta} + \frac{C_{n\dot{\beta}}b_1 C_{y\beta}}{m_1 - C_{y\dot{\beta}}b_1}\right) \tag{6.217}$$

$$a_{22} = \frac{1}{I_{z1}}\left(\frac{-C_{n\dot{\beta}}b_1(m_1 - b_1 C_{yr})}{m_1 - C_{y\dot{\beta}}b_1} + b_1 C_{nr}\right) \tag{6.218}$$

and

$$b_{11} = \frac{C_{y\delta_a}}{m_1 - C_{y\dot{\beta}} b_1}$$

$$b_{12} = \frac{C_{y\delta_r}}{m_1 - C_{y\dot{\beta}} b_1}$$

$$b_{21} = C_{n\delta_a} + \frac{C_{n\dot{\beta}} b_1 C_{y\delta_a}}{m_1 - C_{y\dot{\beta}} b_1}$$

$$b_{22} = C_{n\delta_r} + \frac{C_{n\dot{\beta}} b_1 C_{y\delta_r}}{m_1 - C_{y\dot{\beta}} b_1}$$

For free response, $\Delta\delta_a = \Delta\delta_r = 0$.

To have an understanding of the physical parameters that have a major effect on the damping ratio and frequency of the Dutch-roll oscillation, let us introduce some additional simplifications. Usually $C_{y\dot{\beta}}$ and $C_{n\dot{\beta}}$ are small so that we can assume $C_{y\dot{\beta}} = C_{n\dot{\beta}} = 0$. With this, it can be shown that the characteristic equation corresponding to the system given by Eq. (6.214) is given by

$$\lambda^2 + B\lambda + C = 0 \tag{6.219}$$

where

$$B = -\left(\frac{C_{y\beta}}{m_1} + \frac{b_1 C_{nr}}{I_{z1}}\right) \tag{6.220}$$

$$C = \left(\frac{1}{m_1 I_{z1}}\right)[C_{y\beta} C_{nr} b_1 + C_{n\beta}(m_1 - b_1 C_{yr})] \tag{6.221}$$

Comparing Eq. (6.219) with the standard second-order Eq. (6.34), we obtain

$$\omega_n = \sqrt{C} = \sqrt{\left(\frac{1}{m_1 I_{z1}}\right)[C_{y\beta} C_{nr} b_1 + C_{n\beta}(m_1 - b_1 C_{yr})]} \tag{6.222}$$

$$\zeta = \frac{B}{2\omega_n} = -\left(\frac{1}{2\omega_n}\right)\left(\frac{C_{y\beta}}{m_1} + \frac{b_1 C_{nr}}{I_{z1}}\right) \tag{6.223}$$

Usually, $C_{y\beta} < 0$ (side force due to sideslip), $C_{n\beta} > 0$ (static directional stability parameter), $C_{yr} > 0$ (side force due to yaw rate), and $C_{nr} < 0$ (damping in yaw). Generally, C_{yr} is small so that $m_1 > b_1 C_{yr}$. The major contribution to the frequency term comes from $C_{n\beta}$, and that for damping ratio comes from C_{nr}.

Spiral mode. For the general aviation airplane, we have observed that, during the slow convergence corresponding to the small negative real root, the sideslip varies very slowly so that $\Delta\dot{\beta} \simeq 0$. Hence, the side force equation can be ignored. Furthermore, the roll rate is practically zero during this slow spiral motion so that the net rolling moment must be zero. In addition, we ignore the contribution due to the product of inertia term I_{xz1}. With these simplifying assumptions, the rolling moment and yawing moment Eqs. (6.169) and (6.170) reduce to

$$C_{l\beta}\Delta\beta + C_{lr}b_1 r + C_{l\delta a}\Delta\delta_a + C_{l\delta r}\Delta\delta_r = 0 \tag{6.224}$$

$$I_{z1}\dot{r} = C_{n\beta}\Delta\beta + C_{nr}rb_1 + C_{n\delta_a}\Delta\delta_a + C_{n\delta_r}\Delta\delta_r \tag{6.225}$$

For free response, $\Delta\delta_a = \Delta\delta_r = 0$.

Rearranging Eqs. (6.224) and (6.225), we get

$$\dot{r} = \frac{(-C_{n\beta}C_{lr} + C_{nr}C_{l\beta})\,b_1 r}{I_{z1}C_{l\beta}} \tag{6.226}$$

We can obtain an analytical solution to this equation. Let $r = r_o e^{\lambda_{sp}t}$ so that $\dot{r} = r_o\lambda_{sp}e^{\lambda_{sp}t}$. Then,

$$r_o\lambda_{sp}e^{\lambda_{sp}t} = \frac{(-C_{n\beta}C_{lr} + C_{nr}C_{l\beta})}{I_{z1}C_{l\beta}}b_1 r_o e^{\lambda_{sp}t} \tag{6.227}$$

or

$$\lambda_{sp} = \frac{b_1(C_{l\beta}C_{nr} - C_{lr}C_{n\beta})}{I_{z1}C_{l\beta}} \tag{6.228}$$

Usually, $C_{l\beta} < 0$ (stable dihedral effect) and $C_{nr} < 0$ (positive damping in yaw) so that the term $C_{l\beta}C_{nr} > 0$. Also, $C_{n\beta} > 0$ (directionally stable) and $C_{lr} > 0$ (positive roll due to positive yaw) so that $C_{lr}C_{n\beta} > 0$. Therefore, when $|C_{l\beta}C_{nr}| > |C_{lr}C_{n\beta}|$, $\lambda_{sp} < 0$, and the spiral mode will be stable. Therefore, for spiral stability, we must have $C_{l\beta}C_{nr} > C_{lr}C_{n\beta}$. In other words, the airplane should have a relatively higher level of lateral stability or dihedral effect compared to the static directional stability. If not, it is possible that the spiral mode becomes unstable. However, it should be remembered that this inference is based on spiral approximation and not on the complete fifth-order lateral-directional system.

The physical motion of an aircraft experiencing a spiral divergence can be approximately described as follows.

On encountering a gust that raises one wing relative to the other, the aircraft banks, the nose drops ($L\cos\phi < W$), and the aircraft develops sideslip in the direction of the lower wing due to $L\sin\phi$ component. Let us assume that the left wing is raised, the right wing is dropped, and the aircraft sideslips to the right. Because the aircraft has a low level of lateral stability or only a small stable value of $C_{l\beta}$, it cannot come to the wing level condition immediately. Instead, because of the strong directional stability ($C_{n\beta} > 0$), it yaws to the right to orient the nose in the direction of the relative wind. In this process it develops a positive yaw rate and, because of this, it further rolls to the right due to C_{lr} because C_{lr} is usually positive. This rolling motion increases the sideslip further. Because the lateral stability $C_{l\beta}$ is insufficient, the condition worsens, with the bank angle continuously increasing and the nose dropping further. The net result is that the aircraft is losing altitude, gaining air speed, and banking more and more to the right with an ever-increasing turn rate. The aircraft is essentially in a tightening spiral motion. What distinguishes the spiral motion from a spin is that the angle of attack is below the stall and control surfaces are still effective.

To avoid the spiral divergence, we have to improve spiral stability of the aircraft while keeping a check on the level of directional stability. This can be done by increasing the dihedral effect (for example increase wing dihedral) and simultaneously keep the static directional stability level to a minimum.

6.3.2 Accuracy of Lateral-Directional Approximations

Generally, the accuracy of these approximations depends on type of the airplane. It may so happen that for one aircraft these approximations may be satisfactory but for the other they can be considerably in error.

To have an idea of the accuracy of lateral-directional approximations, let us refer to Table 6.1, which gives various values of the lateral-directional roots based on the complete fifth-order system, roll subsidence, Dutch roll, and spiral mode approximations for the general aviation airplane.

We observe that the roll subsidence and Dutch-roll approximations are satisfactory but the spiral approximation is in poor agreement with the complete fifth-order lateral-directional system.

The computed free responses based on the complete fifth-order system and lateral-directional approximations for the general aviation airplane are shown in Figs. 6.18–6.20. It may be observed that the motion corresponding to spiral root differs considerably.

The unit-step responses to the rudder input based on complete fifth-order system and lateral-directional approximations are shown in Figs. 6.21–6.25. It is observed that the transient behaviors are in fair agreement but the steady-state values differ considerably. The reason for large differences in steady-state values is that these lateral-directional approximations are primarily aimed at predicting the free response or the dynamic stability of the lateral-directional motion of the airplane. Therefore, they do a reasonably good job of predicting the poles but not the zeros of the lateral-directional transfer function. We know that the steady-state values also depend on the zeros (see Chapter 5). Because the zeros of the approximate transfer functions are not the same as the zeros of the complete fifth-order transfer function, the steady-state values differ significantly.

6.3.3 Lateral-Directional Transfer Functions

Equations (6.168–6.170) for lateral-directional motion based on the assumption of small disturbance can be expressed as

$$\left(m_1\frac{\mathrm{d}}{\mathrm{d}t} - b_1 C_{y\dot{\beta}}\frac{\mathrm{d}}{\mathrm{d}t} - C_{y\beta}\right)\Delta\beta - \left(b_1 C_{yp}\frac{\mathrm{d}}{\mathrm{d}t} + C_{y\phi}\right)\Delta\phi$$
$$+ (m_1 - b_1 C_{yr})\frac{\mathrm{d}\Delta\psi}{\mathrm{d}t} = C_{y\delta_a}\Delta\delta_a + C_{y\delta_r}\Delta\delta_r \tag{6.229}$$

Table 6.1 Lateral-directional roots of the general aviation airplane

Mode	Approximation	Complete system	Comment
Roll subsidence, λ_r	−8.4481	−8.4804	Excellent
Dutch roll, λ_{dr}	$-0.5102 \pm j2.1164$	-0.4897 ± 2.3468	Good
Spiral mode	−0.1446	−0.0087	Poor

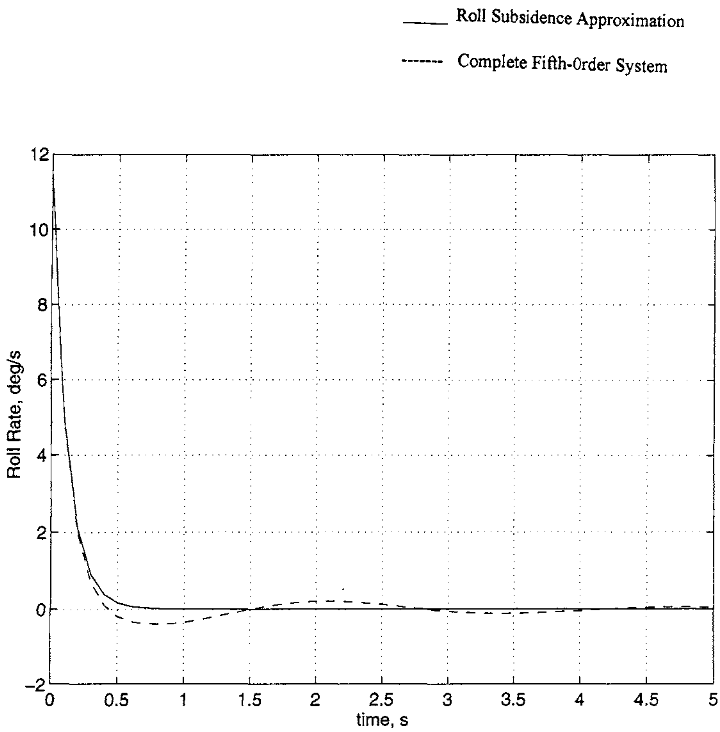


Fig. 6.18 Free response (lateral-directional) of the general aviation airplane.

$$
\left(-C_{l\beta} - b_1 C_{l\dot{\beta}} \frac{\mathrm{d}}{\mathrm{d}t}\right) \Delta\beta + \left(-C_{lp} b_1 \frac{\mathrm{d}}{\mathrm{d}t} + I_{x1} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\right) \Delta\phi - \left(C_{lr} b_1 \frac{\mathrm{d}}{\mathrm{d}t} + I_{xz1} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\right) \Delta\psi = C_{l\delta_a} \Delta\delta_a + C_{l\delta_r} \Delta\delta_r \tag{6.230}
$$

$$
\left(-C_{n\beta} - b_1 C_{n\dot{\beta}} \frac{\mathrm{d}}{\mathrm{d}t}\right) \Delta\beta + \left(-C_{np} b_1 \frac{\mathrm{d}}{\mathrm{d}t} - I_{xz1} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\right) \Delta\phi + \left(-C_{nr} b_1 \frac{\mathrm{d}}{\mathrm{d}t} + I_{z1} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\right) \Delta\psi = C_{n\delta_a} \Delta\delta_a + C_{n\delta_r} \Delta\delta_r \tag{6.231}
$$

In the following, we will derive the transfer function for aileron deflection assuming that the rudder is held fixed, i.e., $\Delta\delta_r = 0$. Similarly, to derive transfer functions for rudder deflection, assume $\delta_a = 0$. Using these transfer functions, the airplane response to a given aileron input can be obtained. To obtain the airplane response to rudder inputs, we need the transfer functions for rudder deflection, which can be obtained by replacing $\Delta\delta_a$ by $\Delta\delta_r$, $C_{y\delta_a}$ by $C_{y\delta_r}$, $C_{l\delta_a}$ by $C_{l\delta_r}$, and $C_{n\delta_a}$ by $C_{n\delta_r}$.

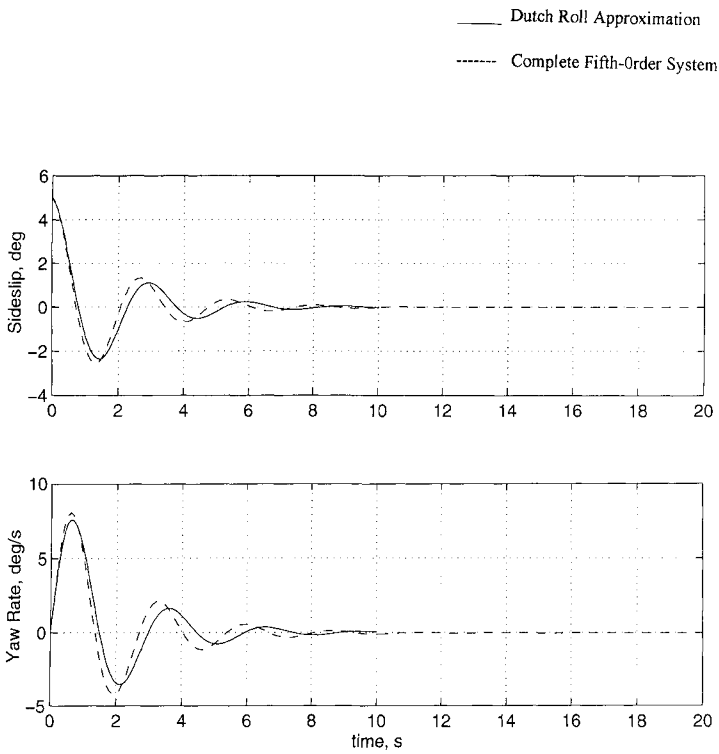


Fig. 6.19 Free response (lateral-directional) of the general aviation airplane.

The airplane response to combined aileron and rudder deflections can be obtained by a linear addition of the two responses because the system is assumed to be linear.

Taking the Laplace transform of Eqs. (6.229–6.231), we obtain

$$(m_1 s - b_1 C_{y\dot\beta} s - C_{y\beta})\Delta\bar\beta(s) - (b_1 C_{yp} s + C_{y\phi})\Delta\bar\phi(s) + (m_1 - b_1 C_{yr}) s \Delta\bar\psi(s) = C_{y\delta_a}\Delta\bar\delta_a(s) \tag{6.232}$$

$$(-C_{l\beta} - b_1 C_{l\dot\beta} s)\Delta\bar\beta(s) + (-C_{lp} b_1 s + I_{x1} s^2)\Delta\bar\phi(s) - (C_{lr} b_1 s + I_{xz1} s^2)\Delta\bar\psi(s) = C_{l\delta_a}\Delta\bar\delta_a(s) \tag{6.233}$$

$$(-C_{n\beta} - b_1 C_{n\dot\beta} s)\Delta\bar\beta(s) + (-C_{np} b_1 s - I_{xz1} s^2)\Delta\bar\phi(s) + (-C_{nr} b_1 s + I_{z1} s^2)\Delta\bar\psi(s) = C_{n\delta_a}\Delta\bar\delta_a(s) \tag{6.234}$$

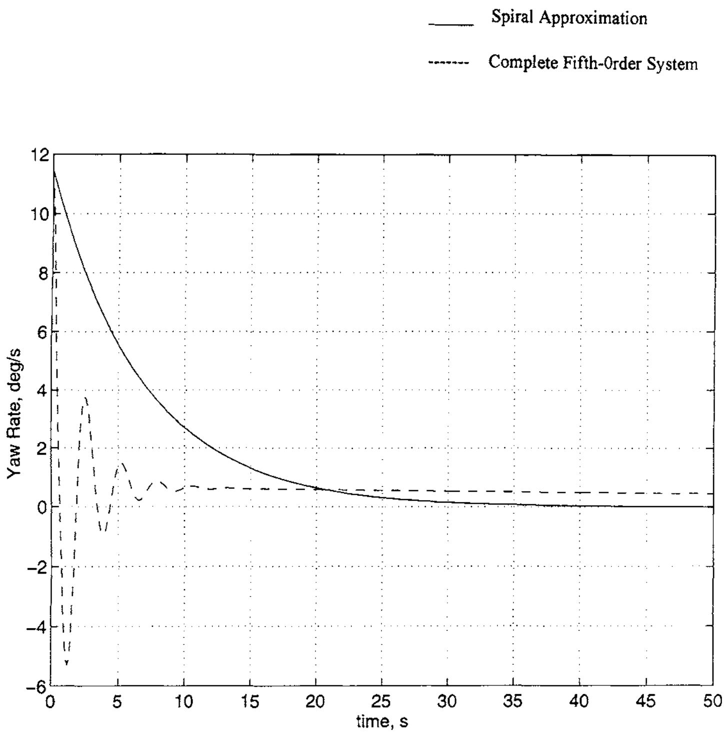


Fig. 6.20 Free response (lateral-directional) of the general aviation airplane.

Dividing throughout by $\Delta\bar{\delta}_a(s)$ and using Cramer's rule, we obtain

$$\frac{\Delta\bar{\beta}(s)}{\Delta\bar{\delta}_a(s)} = \frac{\begin{vmatrix} C_{y\delta_a} & -(b_1C_{yp}s + C_{y\phi}) & s(m_1 - b_1C_{yr}) \\ C_{l\delta_a} & -C_{lp}b_1s + I_{x1}s^2 & -(C_{lr}b_1s + I_{xz1}s^2) \\ C_{n\delta_a} & -C_{np}b_1s - I_{xz1}s^2 & -C_{nr}b_1s + I_{z1}s^2 \end{vmatrix}}{\begin{vmatrix} m_1s - b_1C_{y\dot{\beta}}s - C_{y\beta} & -(b_1C_{yp}s + C_{y\phi}) & s(m_1 - b_1C_{yr}) \\ -C_{l\beta} - b_1C_{l\dot{\beta}}s & -C_{lp}b_1s + I_{x1}s^2 & -(C_{lr}b_1s + I_{xz1}s^2) \\ -C_{n\beta} - b_1C_{n\dot{\beta}}s & -C_{np}b_1s - I_{xz1}s^2 & -C_{nr}b_1s + I_{z1}s^2 \end{vmatrix}} \tag{6.235}$$

Let $N_{\bar{\beta},\delta_a}$ denote the determinant in the numerator and Δ_{lat} denote the determinant in the denominator. Expanding the determinant $N_{\bar{\beta},\delta_a}$, we obtain

$$N_{\bar{\beta},\delta_a} = A_{\beta,\delta_a}s^4 + B_{\beta,\delta_a}s^3 + C_{\beta,\delta_a}s^2 + D_{\beta,\delta_a}s \tag{6.236}$$

where

$$A_{\beta,\delta_a} = C_{y\delta_a}\left(I_{x1}I_{z1} - I_{xz1}^2\right) \tag{6.237}$$

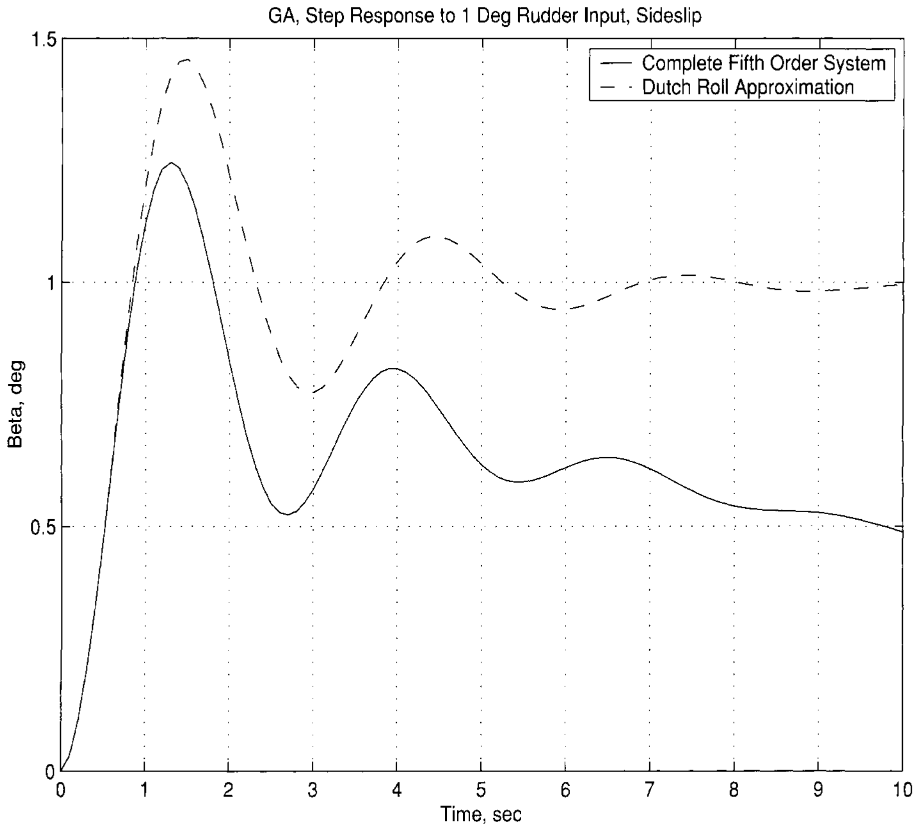


Fig. 6.21 Unit-step response (lateral-directional) of the general aviation airplane: sideslip.

$$B_{\beta,\delta_a} = -b_1 C_{y\delta_a}(I_{x1}C_{nr} + C_{lp}I_{z1} + C_{lr}I_{xz1} + C_{np}I_{xz1})$$
$$+ b_1 C_{yp}\left(I_{z1}C_{l\delta_a} + C_{n\delta_a}I_{xz1}\right) - (m_1 - b_1 C_{yr})\left(C_{l\delta_a}I_{xz1} + C_{n\delta_a}I_{x1}\right) \tag{6.238}$$

$$C_{\beta,\delta_a} = b_1^2 C_{y\delta_a}(C_{lp}C_{nr} - C_{lr}C_{np}) + C_{y\phi}(C_{l\delta_a}I_{z1} + C_{n\delta_a}I_{xz1})$$
$$- b_1^2 C_{yp}\left(C_{l\delta_a}C_{nr} - C_{n\delta_a}C_{lr}\right) + b_1(m_1 - b_1 C_{yr})\left(C_{n\delta_a}C_{lp} - C_{l\delta_a}C_{np}\right) \tag{6.239}$$

$$D_{\beta,\delta_a} = -b_1 C_{y\phi}\left(C_{l\delta_a}C_{nr} - C_{n\delta_a}C_{lr}\right) \tag{6.240}$$

Expanding the determinant Δ_{lat}, we get a fifth-order polynomial in s, which is of the form

$$\Delta_{\text{lat}} = A_{\delta_{\text{lat}}}s^5 + B_{\delta_{\text{lat}}}s^4 + C_{\delta_{\text{lat}}}s^3 + D_{\delta_{\text{lat}}}s^2 + E_{\delta_{\text{lat}}}s \tag{6.241}$$

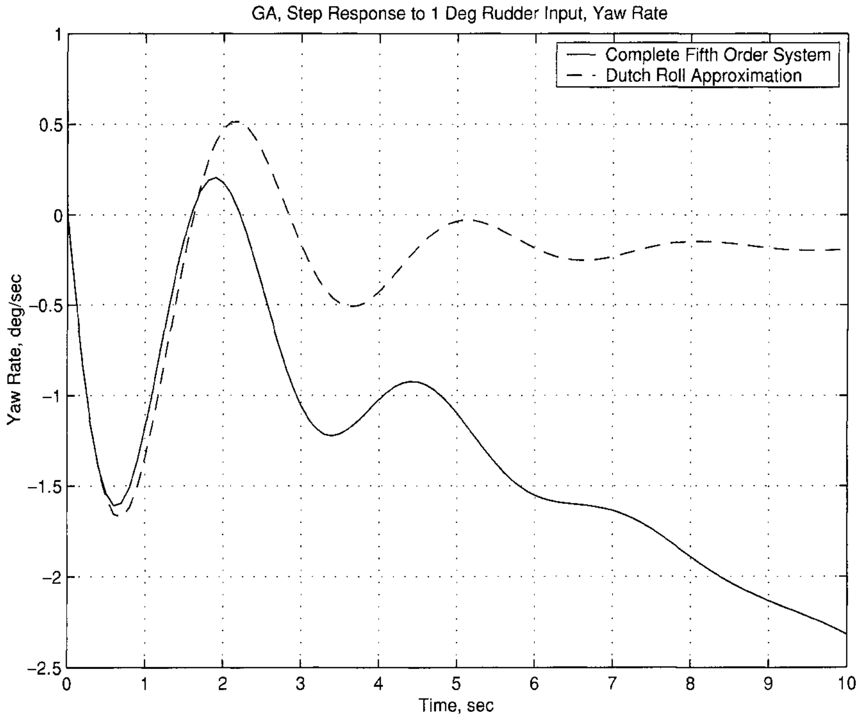


Fig. 6.22 Unit-step response (lateral-directional) of the general aviation airplane: yaw rate.

where

$$A_{\delta_{\text{lat}}} = (m_1 - b_1 C_{y\dot{\beta}})\left(I_{x1}I_{z1} - I_{xz1}^2\right) \tag{6.242}$$

$$\begin{aligned} B_{\delta_{\text{lat}}} &= (-C_{y\beta})\left(I_{x1}I_{z1} - I_{xz1}^2\right) - (m_1 - b_1 C_{y\dot{\beta}})(I_{x1}C_{nr}b_1 + I_{z1}C_{lp}b_1 \\ &\quad + I_{xz1}C_{lr}b_1 + I_{xz1}C_{np}b_1) - b_1 C_{yp}(C_{l\dot{\beta}}b_1 I_{z1} + C_{n\dot{\beta}}b_1 I_{xz1}) \\ &\quad + (m_1 - b_1 C_{yr})(C_{l\dot{\beta}}b_1 I_{xz1} + I_{x1}C_{n\dot{\beta}}b_1) \end{aligned} \tag{6.243}$$

$$\begin{aligned} C_{\delta_{\text{lat}}} &= b_1^2(m_1 - b_1 C_{y\dot{\beta}})(C_{lp}C_{nr} - C_{lr}C_{np}) + b_1 C_{y\beta}(I_{x1}C_{nr} + I_{z1}C_{lp} \\ &\quad + I_{xz1}C_{lr} + I_{xz1}C_{np}) - C_{y\phi}b_1(C_{l\dot{\beta}}I_{z1} + C_{n\dot{\beta}}I_{xz1}) \\ &\quad + b_1 C_{yp}\left(C_{l\dot{\beta}}C_{nr}b_1^2 - C_{l\beta}I_{z1} - I_{xz1}C_{n\beta} - C_{lr}C_{n\dot{\beta}}b_1^2\right) \\ &\quad + (m_1 - b_1 C_{yr})\left(C_{l\beta}I_{xz1} + C_{l\dot{\beta}}C_{np}b_1^2 - C_{lp}C_{n\dot{\beta}}b_1^2 + I_{x1}C_{n\beta}\right) \end{aligned} \tag{6.244}$$

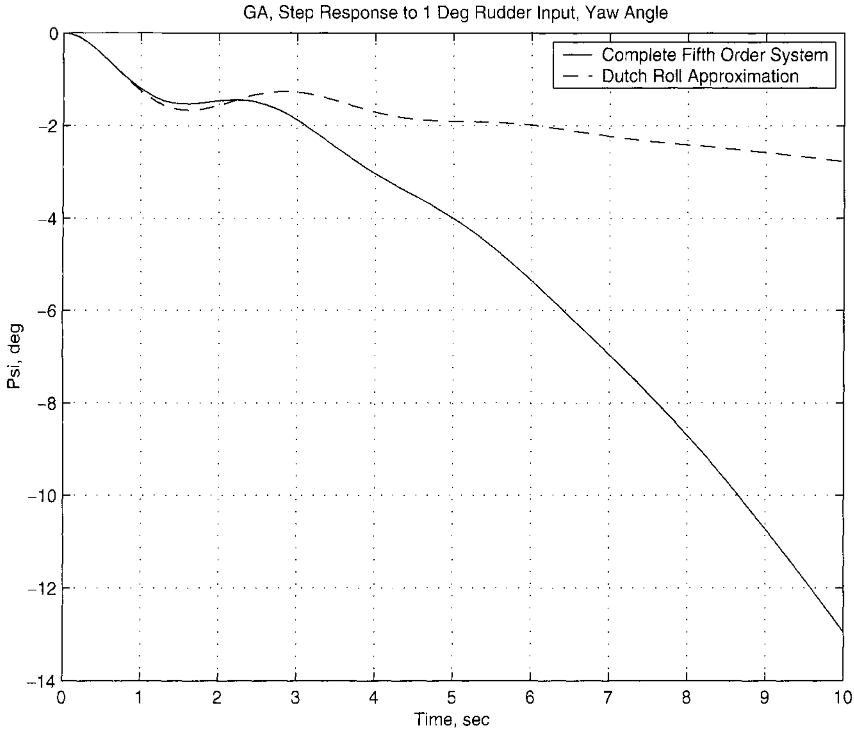


Fig. 6.23 Unit-step response (lateral-directional) of the general aviation airplane: angle of yaw.

$$D_{\delta_{\text{lat}}} = -C_{y\beta}b_1^2(C_{lp}C_{nr} - C_{lr}C_{np}) + C_{y\phi}\big(C_{l\dot\beta}C_{nr}b_1^2 - C_{l\beta}I_{z1} - I_{xz1}C_{n\beta} - C_{lr}C_{n\dot\beta}b_1^2\big) + b_1^2C_{yp}(C_{l\beta}C_{nr} - C_{lr}C_{n\beta}) + b_1(m_1 - b_1C_{yr}) \times (C_{l\beta}C_{np} - C_{lp}C_{n\beta}) \tag{6.245}$$

$$E_{\delta_{\text{lat}}} = C_{y\phi}b_1(C_{l\beta}C_{nr} - C_{n\beta}C_{lr}) \tag{6.246}$$

The transfer function for angle of yaw is given by

$$\frac{\Delta\bar\psi(s)}{\Delta\bar\delta_a(s)} = \frac{\begin{vmatrix} m_1s - b_1C_{y\dot\beta}s - C_{y\beta} & -(b_1C_{yp}s + C_{y\phi}) & C_{y\delta_a} \\ -\big(C_{l\beta} + b_1C_{l\dot\beta}s\big) & -C_{lp}b_1s + I_{x1}s^2 & C_{l\delta_a} \\ -\big(C_{n\beta} + b_1C_{n\dot\beta}s\big) & -C_{np}b_1s - I_{xz1}s^2 & C_{n\delta_a} \end{vmatrix}}{\Delta_{\text{lat}}} \tag{6.247}$$

Let $N_{\bar\psi,\delta_a}$ denote the determinant in the numerator. Expanding, we get

$$N_{\bar\psi,\delta_a} = A_{\bar\psi}s^3 + B_{\bar\psi}s^2 + C_{\bar\psi}s + D_{\bar\psi} \tag{6.248}$$

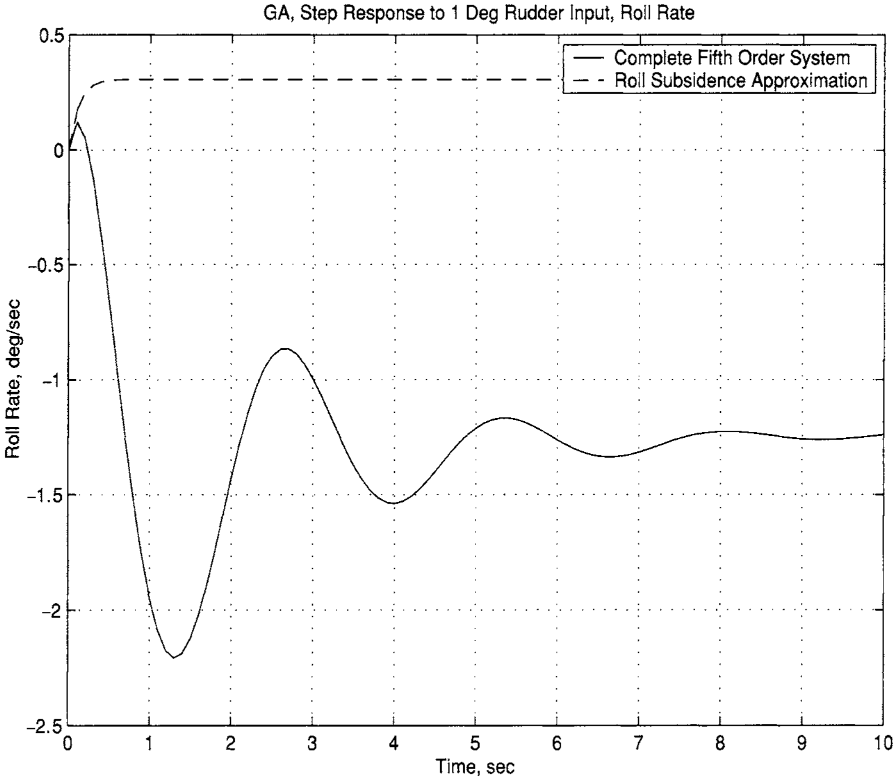


Fig. 6.24 Unit-step response (lateral-directional) of the general aviation airplane: roll rate.

where

$$A_{\bar{\psi}} = (m_1 - b_1 C_{y\dot{\beta}})\big(I_{x1} C_{n\delta_a} + I_{xz1} C_{l\delta_a}\big) + C_{y\delta_a}(I_{xz1} C_{l\dot{\beta}} b_1 + I_{x1} C_{n\dot{\beta}} b_1) \tag{6.249}$$

$$\begin{aligned} B_{\bar{\psi}} = {} & -C_{y\beta}\big(I_{xz1} C_{n\delta_a} + I_{xz1} C_{l\delta_a}\big) + (m_1 - b_1 C_{y\dot{\beta}})\big(C_{l\delta_a} C_{np} b_1 - C_{n\delta_a} C_{lp} b_1\big) \\ & + C_{yp} b_1\big(- C_{n\delta_a} C_{l\dot{\beta}} b_1 + C_{l\delta_a} C_{n\dot{\beta}} b_1\big) + C_{y\delta_a}(I_{xz1} C_{l\beta} + C_{l\dot{\beta}} C_{np} b_1^2 \\ & - C_{n\dot{\beta}} C_{lp} b_1^2 + I_{x1} C_{n\beta}) \end{aligned} \tag{6.250}$$

$$\begin{aligned} C_{\bar{\psi}} = {} & -C_{y\beta} b_1 \left(C_{l\delta_a} C_{np} - C_{n\delta_a} C_{lp}\right) + b_1 C_{yp} \left(C_{n\beta} C_{l\delta_a} - C_{l\beta} C_{n\delta_a}\right) \\ & + C_{y\phi}\big(-C_{n\delta_a} C_{l\dot{\beta}} b_1 + C_{l\delta_a} C_{n\dot{\beta}} b_1\big) + b_1 C_{y\delta_a}(C_{l\beta} C_{np} - C_{lp} C_{n\beta}) \end{aligned} \tag{6.251}$$

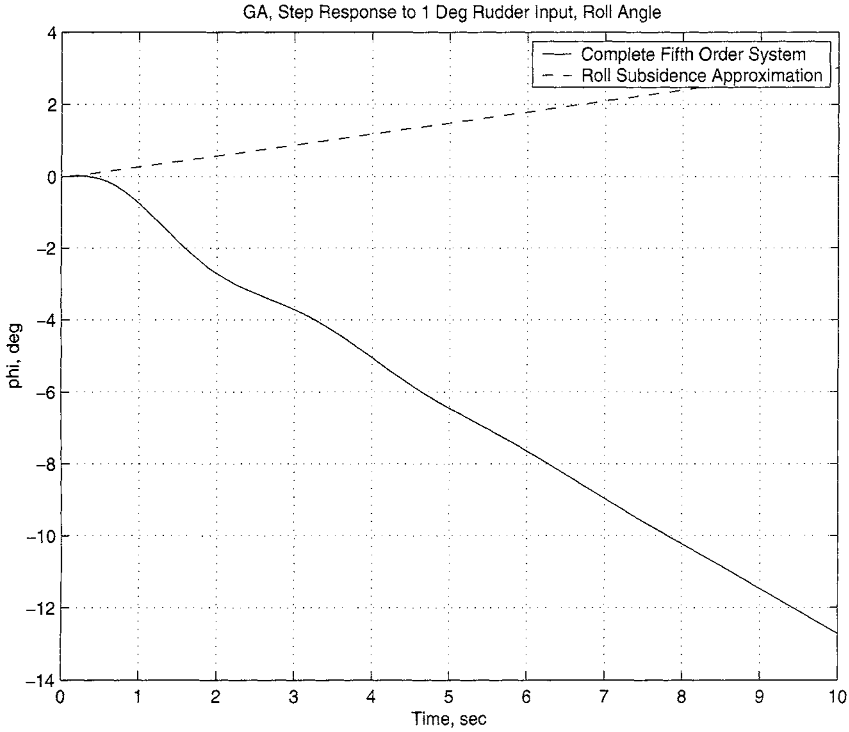


Fig. 6.25 Unit-step response (lateral-directional) of the general aviation airplane: roll angle.

$$D_{\bar{\psi}} = C_{y\phi}\left(C_{n\beta}C_{l\delta_a} - C_{l\beta}C_{n\delta_a}\right) \tag{6.252}$$

The transfer function for the bank angle is given by

$$\frac{\Delta\bar{\phi}(s)}{\Delta\bar{\delta}_a(s)} = \frac{\begin{vmatrix} (m_1 s - b_1 C_{y\dot{\beta}}s - C_{y\beta}) & C_{y\delta_a} & m_1 s - b_1 C_{yr}s \\ -(C_{l\beta} + b_1 C_{l\dot{\beta}}s) & C_{l\delta_a} & -\left(C_{lr}b_1 s + I_{xz1}s^2\right) \\ -(C_{n\beta} + b_1 C_{n\dot{\beta}}s) & C_{n\delta_a} & -\left(C_{nr}b_1 s - I_{z1}s^2\right) \end{vmatrix}}{\Delta_{\text{lat}}} \tag{6.253}$$

Let $N_{\bar{\phi},\delta_a}$ denote the determinant in the numerator. Expanding, we get

$$N_{\bar{\phi},\delta_a} = A_{\bar{\phi}}s^3 + B_{\bar{\phi}}s^2 + C_{\bar{\phi}}s \tag{6.254}$$

where

$$A_{\bar{\phi}} = (m_1 - b_1 C_{y\dot{\beta}})\left(I_{xz1}C_{n\delta_a} + I_{z1}C_{l\delta_a}\right) + C_{y\delta_a}(I_{xz1}C_{n\dot{\beta}}b_1 + I_{z1}C_{l\dot{\beta}}b_1) \tag{6.255}$$

$$B_{\bar{\phi}} = -C_{y\beta}\left(I_{xz1}C_{n\delta_a} + I_{z1}C_{l\delta_a}\right) + (m_1 - b_1 C_{y\dot{\beta}})\left(C_{n\delta_a}C_{lr} - C_{l\delta_a}C_{nr}\right)b_1$$
$$- C_{y\delta_a}\left(C_{l\dot{\beta}}C_{nr}b_1^2 - C_{l\beta}I_{z1} - C_{n\beta}I_{xz1} - C_{n\dot{\beta}}C_{lr}b_1^2\right)$$
$$+ (m_1 - b_1 C_{yr})\left(C_{l\delta_a}C_{n\dot{\beta}}b_1 - C_{n\delta_a}C_{l\dot{\beta}}b_1\right) \tag{6.256}$$

$$C_{\bar{\phi}} = (m_1 - b_1 C_{yr})\left(C_{l\delta_a}C_{n\beta} - C_{n\delta_a}C_{l\beta}\right) + b_1 C_{y\beta}\left(C_{l\delta_a}C_{nr} - C_{n\delta_a}C_{lr}\right)$$
$$+ b_1 C_{y\delta_a}\left(C_{n\beta}C_{lr} - C_{l\beta}C_{nr}\right) \tag{6.257}$$

Substituting the mass and aerodynamic characteristics of the general aviation airplane, we get the transfer functions of the general aviation airplane as follows:

$$\frac{\Delta\bar{\beta}(s)}{\Delta\bar{\delta}_a(s)} = \frac{3.4958 \times 10^{-5}s^3 + 0.0027s^2 + 6.2143 \times 10^{-4}s}{1.5586 \times 10^{-4}s^5 + 0.0015s^4 + 0.0022s^3 + 0.0076s^2 + 6.6266 \times 10^{-5}s} \tag{6.258}$$

$$\frac{\Delta\bar{\psi}(s)}{\Delta\bar{\delta}_a(s)} = \frac{-3.4958 \times 10^{-5}s^3 - 0.0019s^2 - 4.8622 \times 10^{-4}s + 0.0037}{1.5586 \times 10^{-4}s^5 + 0.0015s^4 + 0.0022s^3 + 0.0076s^2 + 6.6266 \times 10^{-5}s} \tag{6.259}$$

$$\frac{\Delta\bar{\phi}(s)}{\Delta\bar{\delta}_a(s)} = \frac{0.0046s^3 + 0.0046s^2 + 0.0211s}{1.5586 \times 10^{-4}s^5 + 0.0015s^4 + 0.0022s^3 + 0.0076s^2 + 6.6266 \times 10^{-5}s} \tag{6.260}$$

We note that

$$\frac{\Delta\bar{r}(s)}{\Delta\bar{\delta}_a(s)} = \frac{s\Delta\bar{\psi}(s)}{\Delta\bar{\delta}_a(s)} \tag{6.261}$$

$$\frac{\Delta\bar{p}(s)}{\Delta\bar{\delta}_a(s)} = \frac{s\Delta\bar{\phi}(s)}{\Delta\bar{\delta}_a(s)} \tag{6.262}$$

The transfer functions for the general aviation airplane (Fig. 6.2) for the rudder input are as follows:

$$\frac{\Delta\bar{\beta}(s)}{\Delta\bar{\delta}_r(s)} = \frac{1.1093 \times 10^{-5}s^3 + 8.2661 \times 10^{-4}s^2 + 0.0064s - 2.3475 \times 10^{-4}}{1.5586 \times 10^{-4}s^5 + 0.0015s^4 + 0.0022s^3 + 0.0076s^2 + 6.6266 \times 10^{-5}s} \tag{6.263}$$

$$\frac{\Delta\bar{\psi}(s)}{\Delta\bar{\delta}_r(s)} = \frac{-7.2441\times10^{-4}s^3 - 0.0064s^2 - 0.0011s - 0.0018}{1.5586\times10^{-4}s^5 + 0.0015s^4 + 0.0022s^3 + 0.0076s^2 + 6.6266\times10^{-5}s} \tag{6.264}$$

$$\frac{\Delta\bar{\phi}(s)}{\Delta\bar{\delta}_r(s)} = \frac{4.0157\times10^{-4}s^3 - 0.0014s^2 - 0.0102s}{1.5586\times10^{-4}s^5 + 0.0015s^4 + 0.0022s^3 + 0.0076s^2 + 6.6266\times10^{-5}s} \tag{6.265}$$

Approximate lateral-directional transfer functions.

Roll subsidence. Taking the Laplace transform of Eq. (6.206) for aileron input with rudder held fixed ($\Delta\delta_r = 0$) and rearranging the terms, we get

$$\frac{\Delta\bar{\phi}(s)}{\Delta\bar{\delta}_a(s)} = \frac{C_{l\delta_a}}{s(I_{x1}s - C_{lp}b_1)} \tag{6.266}$$

so that

$$\frac{\bar{p}(s)}{\Delta\bar{\delta}_a(s)} = \frac{s\Delta\bar{\phi}(s)}{\Delta\bar{\delta}_a(s)} \tag{6.267}$$

Transfer functions for Dutch-roll approximation. Taking the Laplace transform of Eqs. (6.212) and (6.213) and using Cramer's rule, we obtain the following transfer functions for aileron deflections:

$$\frac{\Delta\bar{\beta}(s)}{\Delta\bar{\delta}_a(s)} = \frac{N_{\bar{\beta}}}{\Delta_{\text{lat},dr}} \tag{6.268}$$

$$\frac{\Delta\bar{\psi}(s)}{\Delta\bar{\delta}_a(s)} = \frac{N_{\bar{\psi}}}{\Delta_{\text{lat},dr}} \tag{6.269}$$

where

$$N_{\bar{\beta}} = I_{z1}C_{y\delta_a}s^2 + (b_1C_{n\delta_a}C_{yr} - C_{y\delta_a}C_{nr}b_1 - m_1C_{n\delta_a})s \tag{6.270}$$

$$\begin{aligned}\Delta_{\text{lat},dr} = {} & I_{z1}(m_1 - b_1C_{y\dot{\beta}})s^3 - [C_{y\beta}I_{z1} + C_{nr}b_1(m_1 - b_1C_{y\dot{\beta}}) \\ & + C_{n\dot{\beta}}b_1(m_1 - b_1C_{yr})]s^2 + [C_{y\beta}C_{nr}b_1 + C_{n\beta}(m_1 - b_1C_{yr})]s\end{aligned} \tag{6.271}$$

$$N_{\bar{\psi}} = \left[C_{n\delta_a}(m_1 - b_1C_{y\dot{\beta}}) + C_{y\delta_a}C_{n\dot{\beta}}b_1\right]s + \left(C_{y\delta_a}C_{n\beta} - C_{n\delta_a}C_{y\beta}\right) \tag{6.272}$$

Spiral approximation. Taking the Laplace transform of Eqs. (6.224) and (6.225) and using Cramer's rule, we obtain the following transfer functions for aileron deflections:

$$\frac{\Delta\bar{\beta}(s)}{\Delta\bar{\delta}_a(s)} = \frac{A_{\bar{\beta}}s^2 + B_{\bar{\beta}}s}{A_\delta s^2 + B_\delta s} \tag{6.273}$$

$$\frac{\Delta\bar{\psi}(s)}{\Delta\bar{\delta}_a(s)} = \frac{A_{\bar{\psi}}}{A_\delta s^2 + B_\delta s} \tag{6.274}$$

where

$$A_{\bar{\beta}} = C_{l\delta_a} I_{z1} \tag{6.275}$$

$$B_{\bar{\beta}} = b_1 \left(C_{n\delta_a} C_{lr} - C_{l\delta_a} C_{nr}\right) \tag{6.276}$$

$$A_\delta = -C_{l\beta} I_{z1} \tag{6.277}$$

$$B_\delta = b_1 (C_{l\beta} C_{nr} - C_{n\beta} C_{lr}) \tag{6.278}$$

$$A_{\bar{\psi}} = C_{n\beta} C_{l\delta_a} - C_{l\beta} C_{n\delta_a} \tag{6.279}$$

Substituting the mass and aerodynamic characteristics of the general aviation airplane, we get the approximate transfer functions of the general aviation airplane for aileron and rudder inputs as follows. Note that we replace $\Delta\delta_a$ by $\Delta\delta_r$, $C_{y\delta_a}$ by $C_{y\delta_r}$ and so on ..

For the roll subsidence mode,

$$\frac{\Delta\bar{\phi}(s)}{\Delta\bar{\delta}_a(s)} = \frac{0.1342}{s(0.0046\,s + 0.0387)} \tag{6.280}$$

$$\frac{\Delta\bar{\phi}(s)}{\Delta\bar{\delta}_r(s)} = \frac{0.0118}{s(0.0046\,s + 0.0387)} \tag{6.281}$$

For the Dutch-roll approximation,

$$\frac{\Delta\bar{\beta}(s)}{\Delta\bar{\delta}_a(s)} = \frac{0.0076\,s}{0.0340\,s^3 + 0.0347\,s^2 + 0.1613\,s} \tag{6.282}$$

$$\frac{\Delta\bar{\psi}(s)}{\Delta\bar{\delta}_a(s)} = \frac{-0.0076\,s - 0.002}{0.0340\,s^3 + 0.0347\,s^2 + 0.1613\,s} \tag{6.283}$$

$$\frac{\Delta\bar{\beta}(s)}{\Delta\bar{\delta}_r(s)} = \frac{0.0024\,s^2 + 0.16\,s}{0.0340\,s^3 + 0.0347\,s^2 + 0.1613\,s} \tag{6.284}$$

$$\frac{\Delta\bar{\psi}(s)}{\Delta\bar{\delta}_r(s)} = \frac{-0.1582\,s - 0.0294}{0.0340\,s^3 + 0.0347\,s^2 + 0.1613\,s} \tag{6.285}$$

For the spiral approximation,

$$\frac{\Delta\bar{\beta}(s)}{\Delta\bar{\delta}_a(s)} = \frac{0.0021\,s^2 + 0.0016\,s}{0.0011\,s^2 + 1.6508 \times 10^{-4}\,s} \tag{6.286}$$

$$\frac{\Delta\bar{\psi}(s)}{\Delta\bar{\delta}_a(s)} = \frac{0.0092\,s}{0.0011\,s^2 + 1.6508 \times 10^{-4}\,s} \tag{6.287}$$

$$\frac{\Delta\bar{\beta}(s)}{\Delta\bar{\delta}_r(s)} = \frac{1.8204 \times 10^{-4}\,s^2 - 5.4861 \times 10^{-4}\,s}{0.0011\,s^2 + 1.6508 \times 10^{-4}\,s} \tag{6.288}$$

$$\frac{\Delta\bar{\psi}(s)}{\Delta\bar{\delta}_r(s)} = \frac{-0.0045}{0.0011\,s^2 + 1.6508 \times 10^{-4}\,s} \tag{6.289}$$

We note that

$$\bar{p}(s) = s\Delta\bar{\phi}(s) \tag{6.290}$$

$$\bar{r}(s) = s\Delta\bar{\psi}(s) \tag{6.291}$$

Using the above transfer functions of the general aviation airplane and the final value theorem [see Eq. (5.20) of Chapter 5], the steady-state values of the lateral-directional variables $\Delta\beta$, $\Delta\phi$, and p for a given input such as a step input can be obtained. However, this is left as an exercise to the reader.

6.3.4 Lateral-Directional Frequency Response

The lateral-directional frequency response to aileron and rudder inputs is of interest because the pilot input to these two control surfaces can be considered as a signal with a frequency content. The bandwidth of the human pilot is about 4 rad/s (Ref. 3). Therefore, it is reasonable to expect that a human pilot can control any lateral-directional flight path variable using aileron/rudder inputs within this range of frequency. The autopilots have much higher bandwidths, up to 20–25 rad/s (Ref. 3).

As an example of the lateral-directional frequency response, we have presented the frequency responses for sideslip, roll angle, roll rate, yaw angle, and yaw rate to aileron input in Figs. 6.26–6.30 for the general aviation airplane. Here, G_β, G_ψ, G_ϕ, G_p, and G_r are the lateral-directional transfer functions of the general aviation airplane given in Eqs. (6.258–6.262). These frequency-response (Bode magnitude and phase) plots were drawn using MATLAB.[1] Similar frequency responses can be obtained for rudder input using the transfer functions given in Eqs. (6.263–6.265).

For the general aviation airplane, the Dutch-roll frequency is 2.3973 rad/s, and the damping ratio is 0.2043. Thus, the Dutch roll is lightly damped. In view of this, the magnitude responses of all the variables display a peak value around this frequency.

The magnitude crossover frequency for sideslip is around 0.09 rad/s, which means that the aileron inputs of frequencies of 0.09 rad/s and higher will have little or no effect on the sideslip. Only aileron inputs of frequencies much lower than 0.09 rad/s will have some effect on the sideslip. Around the Dutch-roll frequency, the phase angle of $|G_\beta|$ is close to 90 deg, which is a characteristic of lightly damped systems.

The magnitude crossover frequency for roll angle is about 4 rad/s. Hence, the human pilot can have good roll control using the aileron inputs. We observe that the magnitude of the crossover frequency for yaw rate is quite low and is about 0.5 rad/s. Thus, it is not possible to have good yaw control using the aileron inputs.

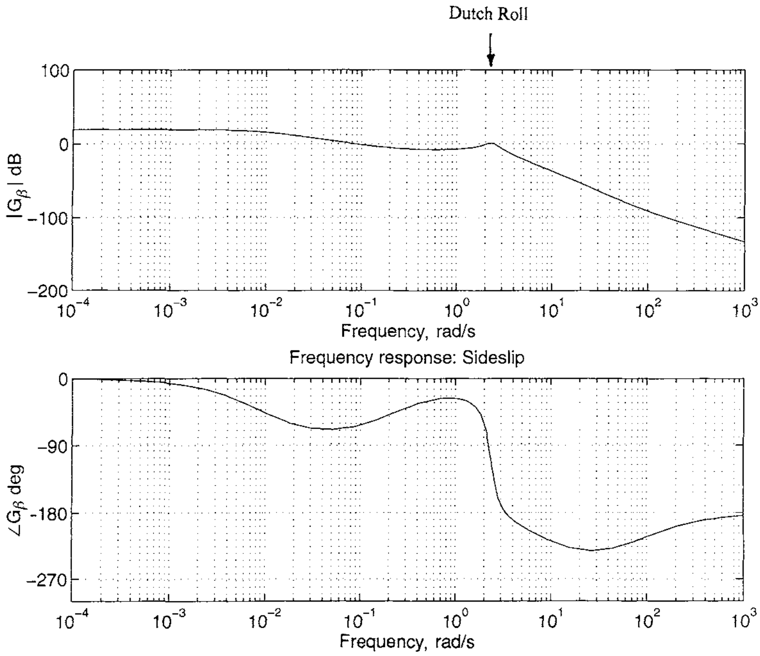


Fig. 6.26 Frequency response of the general aviation airplane: sideslip to aileron input.

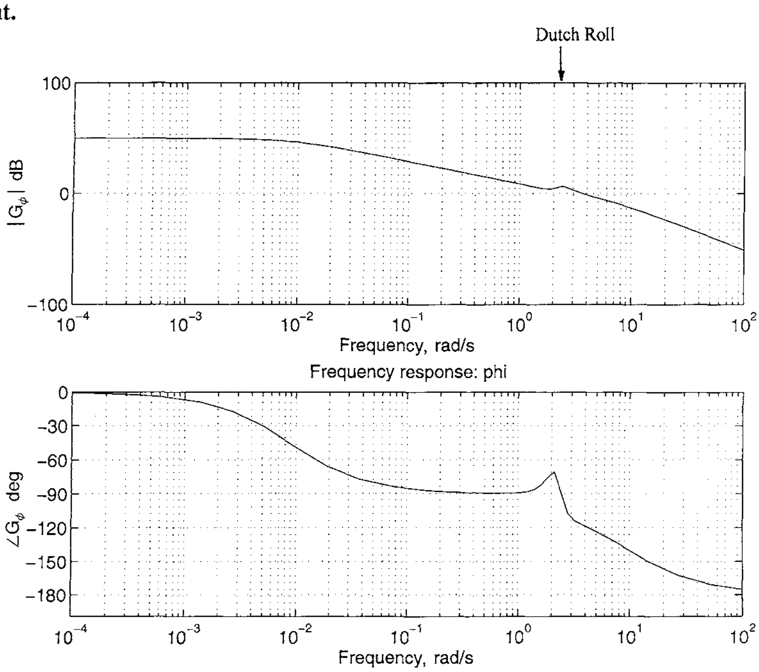


Fig. 6.27 Frequency response of the general aviation airplane: roll angle to aileron input.

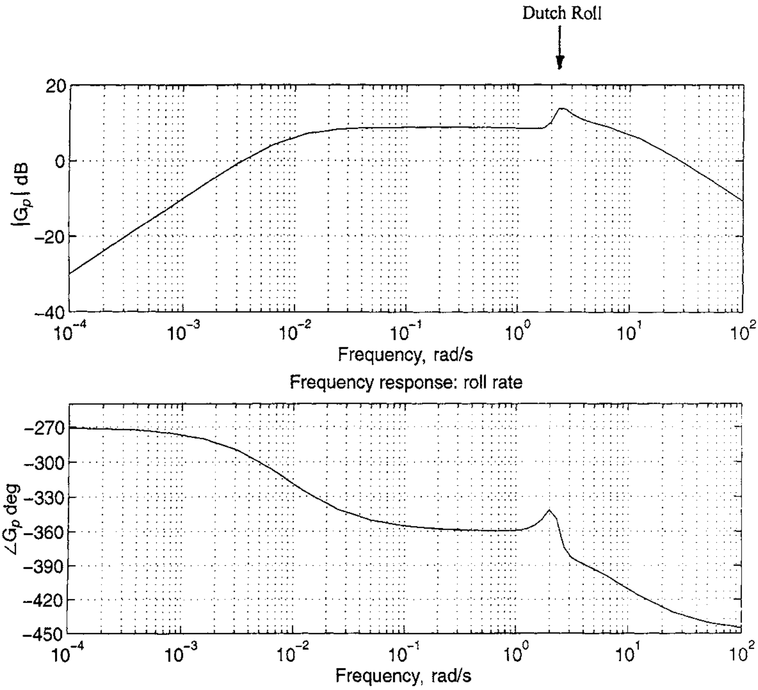


Fig. 6.28 Frequency response of the general aviation airplane: roll rate to aileron input.

6.4 Flying Qualities

The subject of flying qualities attempts to quantify the ease or difficulty with which a pilot can fly an airplane in a given category of flight. Besides the dynamic behavior of the airplane, pilot opinion is also influenced by such factors like stick force gradients, design and layout of the cockpit controls, visibility, weather conditions, physical and emotional condition of the pilot, etc. In view of such widely varying factors that can affect pilot opinion, it is usual to conduct several tests and ask a number of pilots to fly the given airplane for each phase of flight and then take an average of all the test results. The Cooper–Harper scale is a systematic method of quantifying these test results. It assigns three levels to describe the flying qualities of an airplane as follows.[4,5]

Level I. Flying qualities clearly adequate for the mission flight phase.

Level II. Flying qualities adequate to accomplish the mission flight phase when some increase in pilot workload or degradation in mission effectiveness exists or both.

Level III. Flying qualities such that the mission can be controlled safely, but pilot workload is excessive or mission effectiveness is inadequate or both.

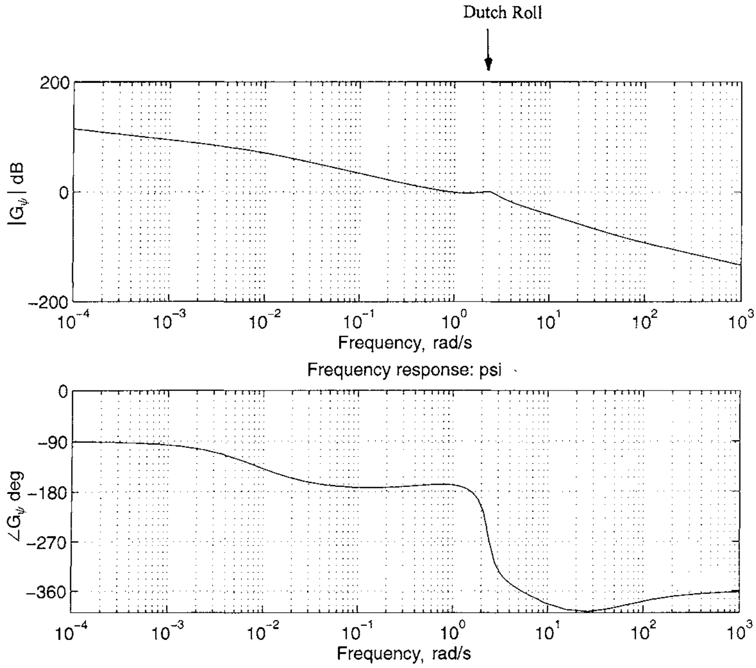


Fig. 6.29 Frequency response of the general aviation airplane: yaw angle to aileron input.

Category A flight phases can be terminated safely, and category B and C flight phases can be completed. The flight phases are divided as follows.

Nonterminal flight phases.

Category A. Nonterminal flight phases that require rapid maneuvering, precision tracking, or precise flight-path control. Included in the category are air-to-air combat, ground attack, weapon delivery/launch, aerial recovery, reconnaissance, in-flight refueling (receiver), terrain following, antisubmarine search, and close-formation flying.

Category B. Nonterminal flight phases that are normally accomplished using gradual maneuvers and without precision tracking, although accurate flight-path control may be required. Included in the category are climb, cruise, loiter, in-flight refueling (tanker), descent, emergency descent, emergency deceleration, and aerial delivery.

Terminal flight phases.

Category C. Terminal flight phases are normally accomplished using gradual maneuvers and usually require accurate flight-path control. Included in this category are takeoff, catapult takeoff, approach, waveoff/go-around, and landing.

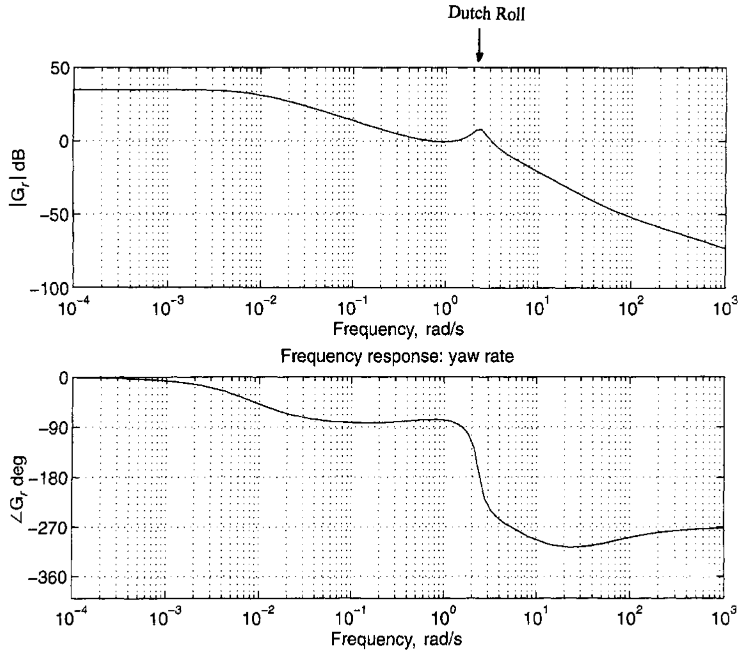


Fig. 6.30 Frequency response of the general aviation airplane: yaw rate to aileron input.

The aircraft are classified as follows.

Class I. Small, light airplanes, such as light utility, primary trainer, and light observation craft.

Class II. Medium-weight, low-to-medium maneuverability airplanes, such as heavy utility/search and rescue, light or medium transport/cargo/tanker, reconnaissance, tactical bomber, heavy attack, and trainer for class II.

Class III. Large, heavy, low-to-medium maneuverability airplanes, such as heavy transport/cargo/tanker, heavy bomber, and trainer for class III.

Class IV. High-maneuverability airplanes, such as fighter/interceptor, attack, tactical reconnaissance, observation, and trainer for class IV.

Extensive research and development activities have been conducted by various government agencies and the aviation industry to relate the Cooper–Harper flying qualities to physical parameters like the frequency and damping ratio of the oscillatory motions of the aircraft. The military specifications MIL-F-8785C[6] (used also for civilian aircraft) are as follows.

Table 6.2 Damping ratios for short-period mode

	Cat A and C $\zeta_{sp,\min}$	Cat A and C $\zeta_{sp,\max}$	Cat B $\zeta_{sp,\min}$	Cat B $\zeta_{sp,\max}$
Level I	0.35	1.30	0.30	2.0
Level II	0.25	2.0	0.20	2.0
Level III	0.15	—	0.15	—

6.4.1 *Longitudinal Flying Qualities*

Phugoid mode.[4,5]

$$\text{Level I} = \zeta > 0.04$$

$$\text{Level II} = \zeta > 0$$

$$\text{Level III} = T_2 > 55\ \text{s}$$

In the level III, the aircraft is assumed to have an unstable (divergent) phugoid mode, and T_2 denotes the time required for the amplitude to double the initial value.

Short-period mode. The damping ratios (ζ) of the short-period mode[4,5] for various levels of flying qualities for category A, B, and C flight phases are given in Table 6.2.

The requirements on the natural frequencies (ω_n) of the short-period mode[4,5] for category A, B, and C flight phases according to MIL-F-8785C[6] should lie within the limits as given in Table 6.3.

In Table 6.3, n is the load factor as given by

$$n = \frac{L}{W} \tag{6.292}$$

$$= \frac{\frac{1}{2}\rho U_o^2 S C_L}{W} \tag{6.293}$$

For linear range of angle of attack,

$$\frac{n}{\alpha} = \left(\frac{1}{2W}\right)\rho U_o^2 S C_{L\alpha} \tag{6.294}$$

Thus, the ratio n/α depends on the flight velocity and flight altitude.

Table 6.3 Limits on $\omega_{n,sp}^2/(n/\alpha)$

	Cat A, min	Cat A, max	Cat B, min	Cat B, max	Cat C, min	Cat C, max
Level I	0.28	3.6	0.085	3.6	0.16	3.6
Level II	0.16	10.0	0.038	10.0	0.096	10.0
Level III	0.16	—	0.038	—	0.096	—

Table 6.4 Time constants for roll subsidence mode

Class	Category	Level I	Level II	Level III
I, IV	A	1.0	1.4	10.0
II, III	A	1.4	3.0	10.0
All	B	1.4	3.0	10.0
I, IV	C	1.0	1.4	10.0
II, III	C	1.4	3.0	10.0

As an example, let us calculate the limiting values of the short-period frequency for the general aviation airplane level I for category B phase of flight at sea level. We have $U_o = Ma = 0.158 * 340.592 = 53.8135$ m/s, $\rho = 1.225$ kg/m^3, $S = 16.7225$ m^2, $C_{L\alpha} = 4.44$ per radian, and $W = 12{,}232.6$ N. Substituting, we get $n/\alpha = 10.6689$. Therefore, for the general aviation airplane, the limits on the short-period frequency for level I category B phase of flight are $0.085 \leq \omega_{n,sp}^2/(n/\alpha) \leq 3.6$ or $0.9523 \leq \omega_n \leq 6.1974$ rad/s.

6.4.2 Lateral-Directional Flying Qualities

Roll mode. The maximum allowable values of the roll-subsidence mode time constant[4] are shown in Table 6.4.

Dutch roll. The Dutch-roll flying qualities[4] are specified in Table 6.5.

The natural frequency (ω_n) and damping ratio (ζ) should be generally better or exceed the minimum values given in Table 6.5 in terms of the minimum time to double the amplitude.

Spiral mode. The spiral mode flying qualities[4] are given in Table 6.6 in terms of minimum time to double the amplitude.

6.5 Closed-Loop Flight Control

In the previous sections, we have discussed longitudinal and lateral-directional free and forced responses of the aircraft. These responses can be classified into two broad categories: 1) those involving mainly rotational degrees of freedom such

Table 6.5 Dutch-roll flying qualities

Level	Category	Class	Min ζ[a]	Min $\zeta\omega_n$,[a] rad/s	Min ω_n, rad/s
I	A	I, IV	0.19	0.35	1.0
I	A	II, III	0.19	0.35	0.4
I	B	All	0.08	0.15	1.0
I	C	I,II-C,[b] IV	0.08	0.15	1.0
I	C	II-L,[b] III	0.08	0.15	0.4
II	All	All	0.02	0.05	0.4
III	All	All	0.02	—	0.4

[a] The governing damping requirement is that yielding the larger value of ζ.
[b] The letters C and L denote the carrier-based and land-based aircraft.

Table 6.6 Time to double amplitude for spiral mode

Class	Category	Level I	Level II	Level III
I, IV	A	12 s	12 s	4 s
I, IV	B and C	20 s	12 s	4 s
II and III	All	20 s	12 s	4 s

as short-period mode, the roll-subsidence mode, and the Dutch-roll oscillation and 2) those involving flight path changes such as phugoid and spiral modes. The responses belonging to first category are of high frequency. Therefore, it is essential that they be adequately damped. If not, it will be difficult for the human pilot to take corrective action especially if the frequencies of the unstable modes are higher than the bandwidth of the human pilot, which is about 4 rad/s. For such cases, it is necessary to provide an automatic control system that ensures that these modes are adequately damped in the closed-loop sense. Such a control system is called a stability augmentation system.

There is another class of flight control systems called autopilots. The autopilots are used for establishing and maintaining desired flight conditions. Once the autopilot is engaged, it continues to hold that flight condition without further intervention from the pilot.

In the following, we will discuss some of the typical stability augmentation systems and autopilots.

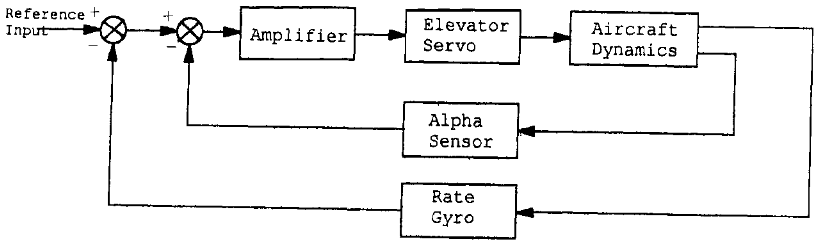
6.5.1 Pitch Stabilization System

The pitch stabilization system is used when the short-period mode is unstable. Typically, modern high-performance aircraft experience these problems because such aircraft are intentionally made statically unstable for better performance. As we have seen earlier for the relaxed static stability version of the general aviation airplane, such aircraft experience exponential instability in pitch. Here, we will discuss the design of a pitch stabilization system to make the aircraft closed-loop stable in pitch.

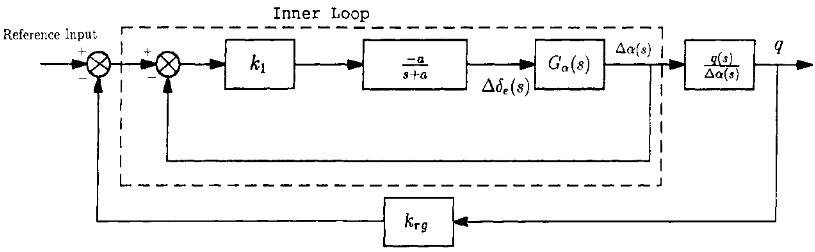
In Fig. 6.31a, the schematic diagram of a typical pitch stabilization system is shown and, in Fig. 6.31b, its block diagram implementation is presented. The elevator servo is approximated as a first-order lag with a break frequency $a = 10$ rad/s. We have an inner loop with angle of attack feedback and an outer loop with pitch rate feedback. The purpose of the inner-loop alpha feedback is to make the aircraft statically stable. The outer-loop pitch rate feedback enables us to select a suitable value of the damping ratio so that the aircraft has the desired level of longitudinal handling qualities in the closed-loop sense.

To illustrate the design procedure, we consider the statically unstable version of the general aviation airplane. Here, we consider the transfer function of the complete (fourth-order) system because the short-period approximation is not valid. For the purpose of this illustration, we assume $C_{m\alpha} = 0.07$/rad and C_{mq} and $C_{m\dot{\alpha}}$ at 50% of the baseline values. All the other stability and control derivatives remain unchanged. With this assumption, we get

$$\frac{\Delta\bar{\alpha}(s)}{\Delta\bar{\delta}_e(s)} = \frac{-(0.0602\,s^3 + 4.4326\,s^2 + 0.2008\,s + 0.3103)}{0.3739\,s^4 + 1.3343\,s^3 + 0.5280\,s^2 + 0.0381\,s - 0.0235} \tag{6.295}$$



a) Schematic diagram



b) Block diagram

Fig. 6.31 Pitch stabilization system.

$$\frac{\Delta\bar{q}(s)}{\Delta\bar{\delta}_e(s)} = \frac{-s(4.4642\, s^2 + 9.3994\, s + 0.4775)}{0.3739\, s^4 + 1.3343\, s^3 + 0.5280\, s^2 + 0.0381\, s - 0.0235} \tag{6.296}$$

In Fig. 6.31b, G_α denotes the transfer function given by Eq. (6.295). Because G_α has a negative sign, we choose a negative sign for the transfer function of the elevator servo so that the loop transfer function is positive, and we can use a negative feedback as usual.

The first step is to design the inner loop. The block diagram of the inner loop is shown in Fig. 6.32a. We draw the root-locus of the inner loop and select a point on the root-locus corresponding to $\zeta = 0.8$ as shown in Fig. 6.33. We get $k_1 = 0.1848$. Next, we proceed to the design of the outer loop. The block diagram of the outer loop is shown in Fig. 6.32b. Here, T_α represents the closed-loop transfer function of the inner loop and is given by

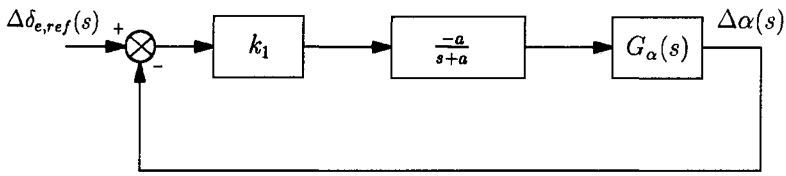
$$T_\alpha = \frac{\frac{-k_1 a G_\alpha}{s+a}}{1 - \frac{k_1 a G_\alpha}{s+a}} \tag{6.297}$$

The transfer function for the pitch rate to angle of attack is obtained as follows:

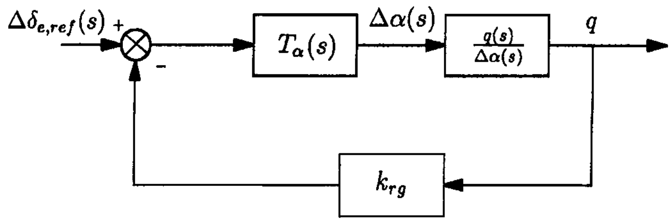
$$\frac{\bar{q}(s)}{\Delta\bar{\alpha}(s)} = \left(\frac{\bar{q}(s)}{\Delta\bar{\delta}_e(s)}\right)\left(\frac{\Delta\bar{\delta}_e(s)}{\Delta\bar{\alpha}(s)}\right) \tag{6.298}$$

Using Eqs. (6.295) and (6.296) and the relation $\bar{q}(s) = s\bar{\theta}(s)$, we get

$$\frac{\bar{q}(s)}{\bar{\alpha}(s)} = \frac{s(4.4642\, s^2 + 9.3994\, s + 0.4775)}{0.0602\, s^3 + 4.4326\, s^2 + 0.2008\, s + 0.3103} \tag{6.299}$$



a) Inner loop



b) Outer loop

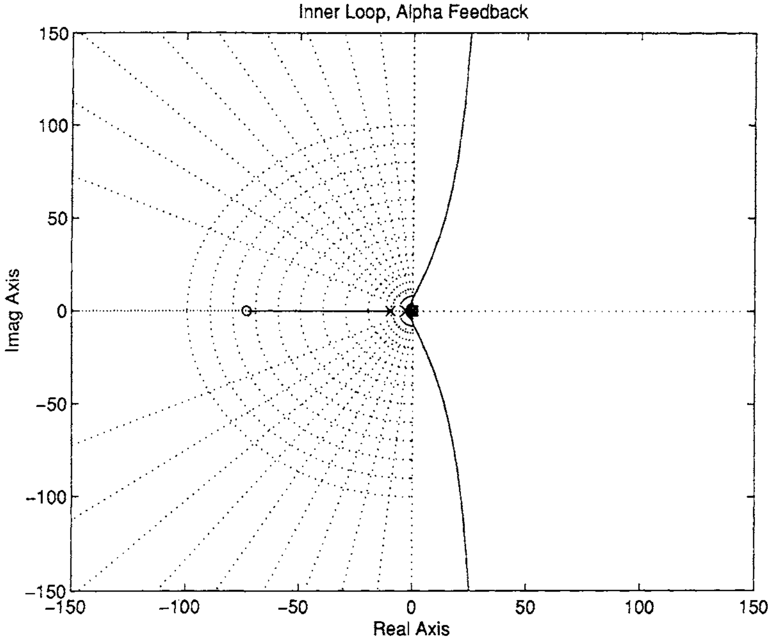
Fig. 6.32 Inner and outer loops of the pitch stabilization system.

The root-locus of the outer loop is shown in Fig. 6.34. We pick a point for $\zeta = 0.9$, which gives $k_{rg} = 0.1194$. We have the closed-loop poles at -9.9710, $-1.7853 \pm j0.8773$ (short-period), and $-0.0197 \pm j0.1502$ (phugoid). For these closed-loop pole locations, we get $\zeta = 0.9$, $\omega_n = 1.9882$ rad/s for the short-period mode and $\zeta = 0.13$, $\omega_n = 0.1515$ rad/s for the phugoid mode. Thus, with alpha and pitch rate feedback, the relaxed static stability version of the general aviation airplane has the conventional short-period and phugoid modes with level I flying qualities.

It may be noted that there is some subjective element involved in picking a point on the root-locus in Matlab using the cursor. In view of this, the values of gain and locations of the closed-loop poles given in this text should be used as guidelines and not as absolute numbers.

6.5.2 Full-State Feedback Design for Longitudinal Stability Augmentation System

To illustrate the design procedure of a full-state feedback stability augmentation system, let us consider the general aviation airplane (see Fig. 6.2) once again. Here, we assume that all four longitudinal states u, $\Delta\alpha$, $\Delta\theta$, and q are accurately measured and are available for feedback. The air speed is usually measured by a pitot-static sensor, the pitch angle and pitch rates are measured by positional and rate gyros, and the angle of attack is measured by a vane-type sensor. Out of these four state variables, angle of attack is the difficult one to measure because the local flow around the airplane is altered due to wing–body upwash in front and downwash behind the wing. The alpha sensor is usually mounted on the wingtips or at the nose of the fuselage. Thus, it is subjected to the upwash field. Hence, it is necessary to correct the measured values of the angle of attack for the induced upwash/downwash effects. For our purpose, we will assume that such corrections are done and that all four states are accurately measured and are available for feedback.



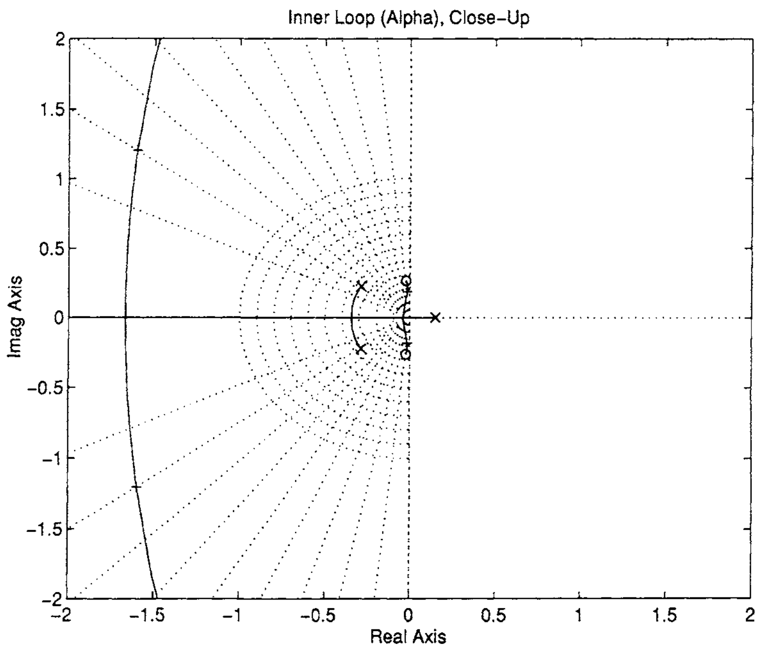
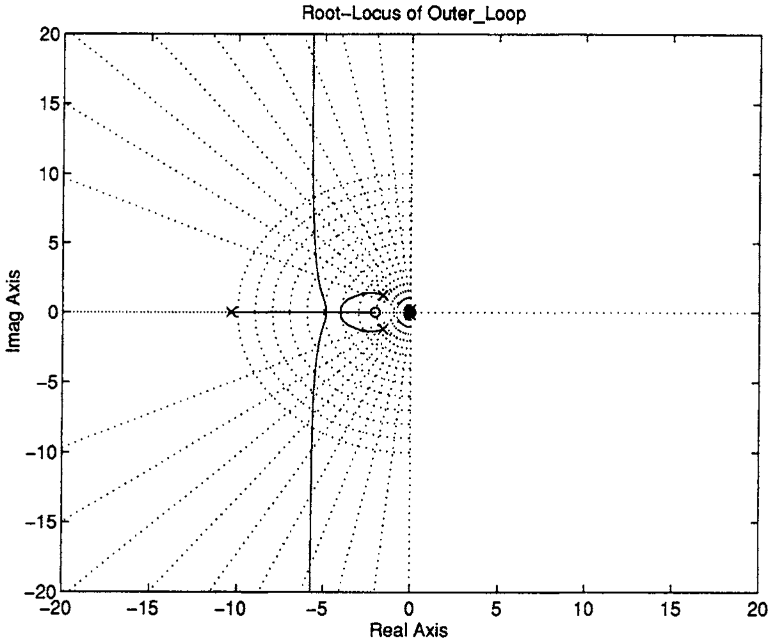
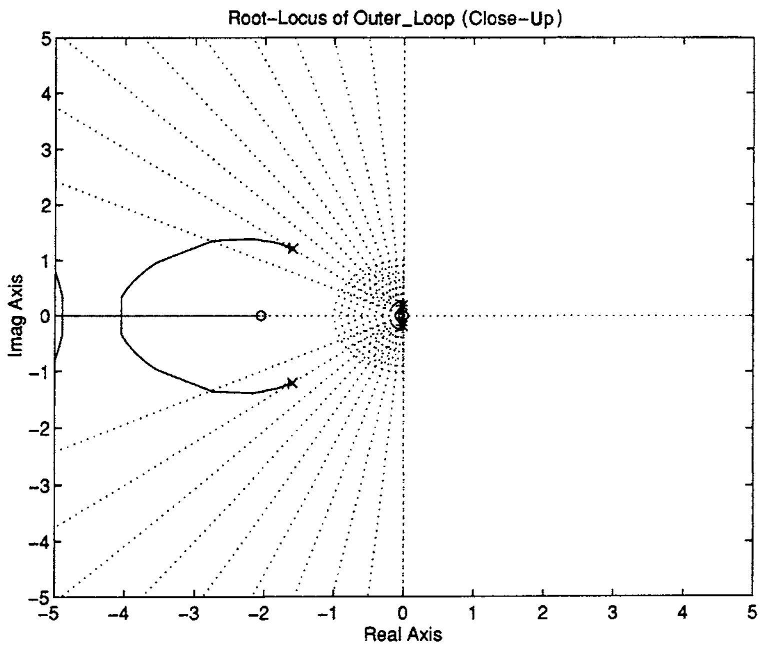


Fig. 6.33 Root-locii of the inner loop of the pitch stabilization system of the general aviation airplane.



a)



b)

Fig. 6.34 Root-locii of the outer loop of the pitch stabilization system of the general aviation airplane.

Let us assume that the desired poles or the roots of the longitudinal characteristic equations that give the specified handling qualities are as follows.

For the short-period mode, let

$$\lambda_{1,2} = -4.8 \pm j2.16 \tag{6.300}$$

which corresponds to a damping ratio of 0.9119 and a natural frequency of 5.2636 rad/s for the short-period mode. As we have seen earlier, these values of damping ratio and natural frequency give us level I short-period flying qualities.

For the phugoid mode, let

$$\lambda_{3,4} = -0.04 \pm j0.1960 \tag{6.301}$$

which corresponds to a damping ratio of 0.15 and a natural frequency of 0.121 rad/s, giving level I phugoid flying qualities.

For the values of the roots given by Eqs. (6.300) and (6.301), the desired characteristic equation is given by

$$s^4 + 9.68\, s^3 + 28.5136\, s^2 + 2.6006\, s + 1.1087 \tag{6.302}$$

The objective is to design a full-state feedback law so that the closed-loop characteristic equation is identical to the desired characteristic equation.

We have the given system in the state-space form

$$\dot{X} = AX + BU \tag{6.303}$$

where

$$X = \begin{bmatrix} u \\ \Delta\alpha \\ q \\ \Delta\theta \end{bmatrix} \tag{6.304}$$

and the matrices A and B given by Eqs. (6.29) and (6.30) are

$$A = \begin{bmatrix} -0.0453 & 0.0363 & 0 & -0.1859 \\ -0.3717 & -2.0354 & 0.9723 & 0 \\ 0.3398 & -7.0301 & -2.9767 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{6.305}$$

$$B = \begin{bmatrix} 0 \\ -0.1609 \\ -11.8674 \\ 0 \end{bmatrix} \tag{6.306}$$

The design procedure for the state feedback was explained in Sec. 5.10.9 of Chapter 5. The first step is to express the given plant in the phase-variable form. Because the given plant is not in the phase-variable form, we have to do a transformation. The advantage is that the elements of the last row of a matrix in phase-variable form are the coefficients of the characteristic equation. Then, we do the full-state feedback design in the transformed space, and then, finally, we do a reverse transformation to obtain the state feedback law in the original state-space.

We express the given plant in the phase-variable form as follows:

$$\dot{Z} = A_p Z + B_p U \tag{6.307}$$

where Z is a new state vector (4×1) and matrices A_p and B_p are in the phase-variable form and are defined as

$$Z = PX \tag{6.308}$$

$$A_p = PAP^{-1} \tag{6.309}$$

$$B_p = PB \tag{6.310}$$

The transformation matrix P is constructed as follows.

1) Obtain the state-controllability matrix Q_c given by

$$Q_c = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} \tag{6.311}$$

2) Form the matrices P_1, P_2, P_3, and P_4 as follows:

$$P_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} Q_c^{-1} \tag{6.312}$$

$$P_2 = P_1 A \tag{6.313}$$

$$P_3 = P_1 A^2 \tag{6.314}$$

$$P_4 = P_1 A^3 \tag{6.315}$$

$$P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \tag{6.316}$$

Then

$$A_p = PAP^{-1} \tag{6.317}$$

$$B_p = PB \tag{6.318}$$

For the general aviation airplane, substitution gives

$$Q_c = \begin{bmatrix} 0 & -0.0058 & 1.7994 & -4.7437 \\ -0.1609 & -11.2112 & 58.2684 & -148.1528 \\ -11.8674 & 36.4568 & -29.7073 & -320.5912 \\ 0 & -11.8674 & 36.4568 & -29.7073 \end{bmatrix} \tag{6.319}$$

$$Q_c^{-1} = \begin{bmatrix} 0.0597 & 0.3006 & -0.0883 & -0.5554 \\ 3.0533 & -0.1042 & 0.0014 & 0.0171 \\ 1.1855 & -0.0490 & 0.0007 & 0.0477 \\ 0.2352 & -0.0184 & 0.0003 & 0.0181 \end{bmatrix} \tag{6.320}$$

$$P = \begin{bmatrix} 0.2352 & -0.0184 & 0.0003 & 0.0181 \\ -0.0037 & 0.0443 & -0.0006 & 0.0437 \\ -0.0165 & -0.0861 & 0.0012 & 0.0007 \\ 0.0332 & 0.1665 & -0.0865 & 0.0031 \end{bmatrix} \tag{6.321}$$

$$A_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.6143 & -0.6754 & -13.1347 & -5.0574 \end{bmatrix} \tag{6.322}$$

$$B_p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{6.323}$$

The input with full-state feedback in the transformed system is

$$U = r(t) - KZ \tag{6.324}$$

where K is given by

$$K = [k_1 \quad k_2 \quad k_3 \quad k_4] \tag{6.325}$$

Then

$$\dot{Z} = (A_p - B_p K)Z + B_p r(t) \tag{6.326}$$

and

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(0.6143 + k_1) & -(0.6754 + k_2) & -(13.1347 + k_3) & -(5.0574 + k_4) \end{bmatrix} \times \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t) \tag{6.327}$$

The characteristic equation of this transformed phase-variable form is

$$s^4 + (5.0574 + k_4)s^3 + (13.1347 + k_3)s^2 + (0.6754 + k_2)s + (0.6143 + k_1) = 0 \tag{6.328}$$

Comparing the coefficient of Eqs. (6.302) and (6.328), we obtain

$$k_1 = 0.4944 \qquad k_2 = 1.9252 \qquad k_3 = 15.3789 \qquad k_4 = 4.6226 \tag{6.329}$$

Then next step is to do a reverse transformation back to the original state-space. Substituting for Z, A_p, and B_p in Eq. (6.307) and rearranging, we obtain

$$\dot{X} = (A - BKP)X + Br(t) \tag{6.330}$$

$$= AX + B\,[r(t) - KPX] \tag{6.331}$$

This gives us the desired full-state feedback control law as

$$U = r(t) - KPX \tag{6.332}$$

To verify the design, we have performed a simulation of the free response $[r(t) = 0]$ using MATLAB[1] for the initial conditions $\Delta\alpha = 5\,\text{deg}$, $u = 0, \theta = q = 0$. The results are shown in Fig. 6.35. We observe that the system with full-state feedback law performs as expected.

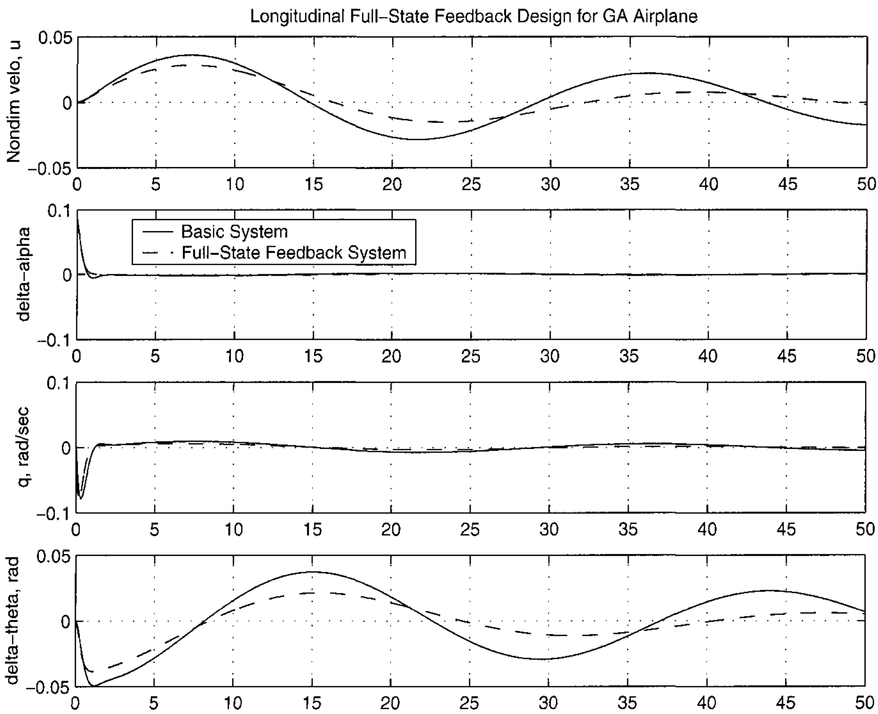


Fig. 6.35 Longitudinal response of the full-state feedback controller for the general aviation airplane.

6.5.3 Yaw Damper

The purpose of the yaw damper is to generate a yawing moment that opposes any yaw rate that builds up from an unstable Dutch-roll mode. However, this type of control augmentation has one drawback. It will oppose any steady-state yaw rate the pilot wants to intentionally generate, for example, during a steady coordinated level turn. In this case the pilot will have to fight the yaw damper system. One way of avoiding this problem is to use a wash-out circuit in the feedback loop. The wash-out circuit will filter out all the steady-state components so that the yaw damper system will not respond to steady-state yaw rates.

A simple yaw damper with yaw rate feedback is shown in Fig. 6.36. To illustrate the design procedure, consider the general aviation airplane and use the Dutch-roll transfer function to rudder input. From Eq. (6.285),

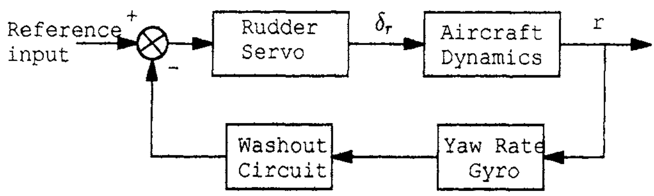
$$\frac{\Delta\bar{\psi}(s)}{\Delta\bar{\delta}_r(s)} = \frac{-(0.1582\, s + 0.0294)}{s(0.0340\, s^2 + 0.0347\, s + 0.1613)} \tag{6.333}$$

so that

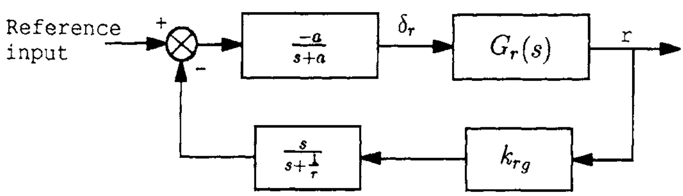
$$\frac{\bar{r}(s)}{\Delta\bar{\delta}_r(s)} = \frac{s\Delta\bar{\psi}(s)}{\Delta\bar{\delta}_r(s)} \tag{6.334}$$

$$= \frac{-(0.1582\, s + 0.0294)}{0.0340\, s^2 + 0.0347\, s + 0.1613} \tag{6.335}$$

We assume that the rudder servo is a first-order system with $a = 10$ so that the time constant $1/a = 0.10$. As said before, we use the negative sign in the numerator of the rudder servo transfer function because we have a negative sign in the numerator of the yaw-rate-to-rudder-input transfer function as given by Eq. (6.335). With this sign adjustment, we can use the negative feedback as shown in Fig. 6.36b.



a) Schematic diagram



b) Block diagram

Fig. 6.36 Yaw damper.

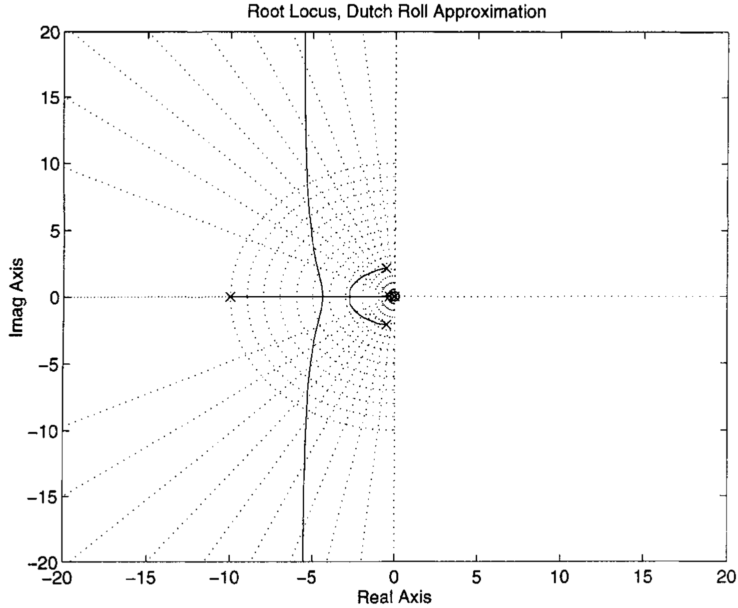


Fig. 6.37 Root-locus of the yaw damper system for the general aviation airplane.

We assume that the transfer function of the wash-out circuit is given by

$$G_c = \frac{s}{s + 0.3333} \tag{6.336}$$

Now we draw the root-locus as shown in Figs. 6.37 and 6.38. The root-locus of Fig. 6.38 is a close-up of the root-locus around the origin. We select a point on root-locus to have $\zeta = 0.8$ for the Dutch-roll mode. We get $k_{rg} = 0.4228$ and the location of the closed-loop poles at -6.9705, $-2.0108 \pm j1.5140$, -0.3619. This gives us a damping ratio of 0.8 and a natural frequency of 2.5170 rad/s, giving us level I Dutch-roll flying qualities for the general aviation airplane, which is a class I airplane.

We can now verify the design by simulating the response to a unit-step function (rudder input) as shown in Fig. 6.39a. We observe that the closed-loop system with yaw damper is performing satisfactory. To check the design further, we perform a simulation for the complete fifth-order system as shown in Fig. 6.39b. We observe that the design is satisfactory. The difference in the steady-state behaviors is due to the approximations involved in deriving Dutch-roll transfer functions as we have discussed earlier.

6.5.4 Full-State Feedback Design for Lateral-Directional Stability Augmentation System

In this section we will design a full-state feedback control law for a lateral-directional stability augmentation system. Such a flight control system needs that

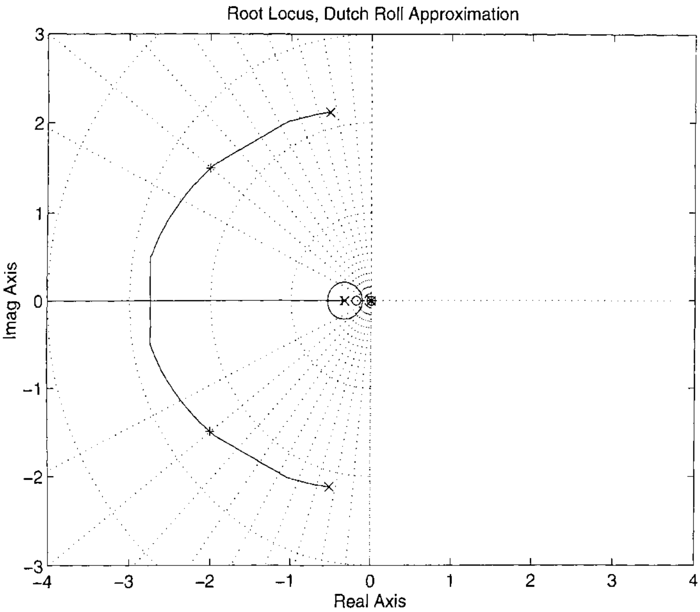


Fig. 6.38 Root-locus of the yaw damper system for the general aviation airplane (close-up around the origin).

all five state variables β, ϕ, p, ψ, and r be accurately measured and be available for feedback. The roll and yaw angles can be accurately measured by positional gyros and the roll and yaw rates by the rate gyros. The sideslip angle is perhaps the most difficult one to measure. A vane-type instrument is usually used to measure the sideslip but, like angle of attack sensor, it is also subject to positional errors due to fuselage sidewash. Another method used to obtain sideslip is the integration of accelerometer outputs. However, this is also subject to measurement noise. In any case, it is essential that the sideslip be accurately estimated and be available for feedback. For our purpose, we will assume that all five state variables are measured accurately and are available for feedback.

Let us consider the general aviation airplane once again with only rudder input and assume that the desired poles of the lateral-directional closed-loop characteristic equations are as follows.

For Dutch-roll mode, let

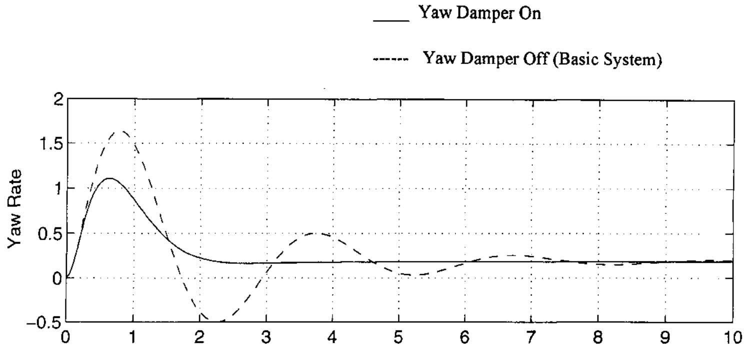
$$\lambda_{dr} = -1.2 \pm j2.75 \tag{6.337}$$

so that we have a damping ratio of 0.4 and a natural frequency ω_n of 3.0 rad/s, corresponding to level I Dutch-roll flying qualities.

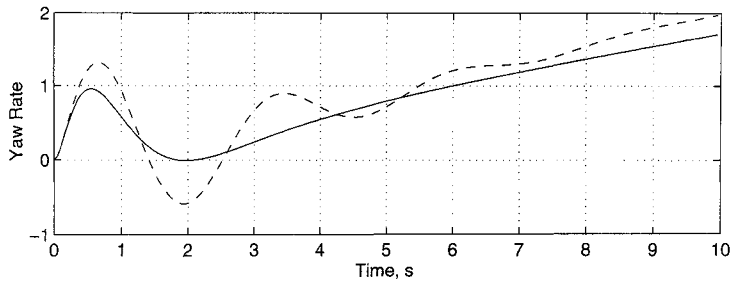
For roll subsidence and spiral mode, let

$$\lambda_r = -8.50 \tag{6.338}$$

$$\lambda_{sp} = -0.0080 \tag{6.339}$$



a) Dutch-roll approximation



b) Complete fifth-order system

Fig. 6.39 Unit-step response of the yaw damper for the general aviation airplane.

The other root is the zero root, $\lambda = 0$.

We have a roll-time constant of 0.1176, which gives level I flying qualities for the roll-subsidence mode. The spiral mode is damped and, therefore, it also has level I flying qualities. Thus, overall, the assumed pole locations give us level I lateral-directional flying qualities.

For these values of five roots, the desired lateral-directional characteristic equation is given by

$$s(s^4 + 10.9080\, s^3 + 29.4897\, s^2 + 76.7565\, s + 0.6122) = 0 \tag{6.340}$$

The objective is to design a full-state feedback law so that the closed-loop lateral-directional characteristic equation of the general aviation airplane is identical to the desired characteristic equation.

From Eqs. (6.196) and (6.197), we have

$$A = \begin{bmatrix} -0.2557 & 0.1820 & 0 & 0 & -1.0000 \\ 0 & 0 & 1.0 & 0 & 0 \\ -16.1572 & 0 & -8.4481 & 0 & 2.2048 \\ 0 & 0 & 0 & 0 & 1.0 \\ 4.5440 & 0 & -0.3517 & 0 & -0.7647 \end{bmatrix} \tag{6.341}$$

$$B = \begin{bmatrix} 0 & 0.0712 \\ 0 & 0 \\ 29.3013 & 2.5764 \\ 0 & 0 \\ -0.2243 & -4.6477 \end{bmatrix} \tag{6.342}$$

Note that for rudder input, $B = B(:, 2)$. Because the given plant is not in phase-variable form, we have to do a transformation and express it in the phase-variable form (see Sec. 5.10.7, Chapter 5). Then, do the full-state feedback design in this new transformed space. Finally, do a reverse transformation to obtain the full-state feedback law in the original state-space.

The phase-variable form is given by

$$\dot{Z} = A_p Z + B_p U \tag{6.343}$$

where Z is the new state vector and matrices A_p and B_p are in the phase-variable form

$$Z = PX \tag{6.344}$$

$$A_p = PAP^{-1} \tag{6.345}$$

$$B_p = PB \tag{6.346}$$

The state vector X was defined in Eq. (6.183). The transformation matrix P is constructed as follows.

1) Obtain the lateral-directional controllability matrix Q_c given by

$$Q_c = [B \quad AB \quad A^2B \quad A^3B \quad A^4B] \tag{6.347}$$

2) Form matrices P_1 to P_5 as follows:

$$P_1 = [0 \quad 0 \quad 0 \quad 0 \quad 1]Q_c^{-1} \tag{6.348}$$

$$P_2 = P_1 A \tag{6.349}$$

$$P_3 = P_1 A^2 \tag{6.350}$$

$$P_4 = P_1 A^3 \tag{6.351}$$

$$P_5 = P_1 A^4 \tag{6.352}$$

$$P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} \tag{6.353}$$

Then the phase-variable forms A_p and B_p are given by

$$A_p = PAP^{-1} \tag{6.354}$$

$$B_p = PB \tag{6.355}$$

For the general aviation airplane, the lateral-directional controllability matrix is given by

$$Q_c = \begin{bmatrix} 0 & 5.0 & -4.0 & -36.0 & 162.0 \\ 0 & 3.0 & -33.0 & 212.0 & -1664.0 \\ 3.0 & -33.0 & 212.0 & -1664.0 & 14,376.0 \\ 0 & -5.0 & 3.0 & 30.0 & -115.0 \\ -5.0 & 3.0 & 30.0 & -115.0 & 511.0 \end{bmatrix} \tag{6.356}$$

so that

$$Q_c^{-1} = \begin{bmatrix} 0.0000 & -0.6953 & -0.0717 & -0.0369 & -0.2549 \\ -3.8467 & 0.3386 & 0.0499 & -4.2354 & -0.0312 \\ -1.1859 & 0.0187 & 0.0063 & -1.2253 & -0.0147 \\ -0.7785 & 0.0926 & 0.0131 & -0.8208 & -0.0046 \\ -0.0815 & 0.0113 & 0.0016 & -0.0867 & -0.0003 \end{bmatrix} \tag{6.357}$$

$$P = \begin{bmatrix} -0.0815 & 0.0113 & 0.0016 & -0.0867 & -0.0003 \\ -0.0070 & -0.0148 & -0.0022 & 0 & -0.0014 \\ 0.0320 & -0.0013 & 0.0046 & 0 & 0.0031 \\ -0.0692 & 0.0058 & -0.0415 & 0 & -0.0241 \\ 0.5791 & -0.0126 & 0.3650 & 0 & -0.0039 \end{bmatrix} \tag{6.358}$$

$$A_p = \begin{bmatrix} 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \\ 0.0000 & -0.4253 & -48.8614 & -14.1354 & -9.4685 \end{bmatrix} \tag{6.359}$$

$$B_p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{6.360}$$

The input with full-state feedback in the transformed system is

$$U = r(t) - KZ \tag{6.361}$$

where K is given by

$$K = [k_1 \quad k_2 \quad k_3 \quad k_4 \quad k_5] \tag{6.362}$$

Then,

$$\dot{Z} = (A_p - B_p K)Z + B_p r(t) \tag{6.363}$$

Substituting, we get

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ k_1 & -(0.4253 + k_2) & -(48.8614 + k_3) & -(14.1354 + k_4) & -(9.4685 + k_5) \end{bmatrix} \times \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t) \tag{6.364}$$

The characteristic equation of this transformed phase-variable form is

$$\begin{aligned} s[s^4 + (9.4685 + k_5)s^3 + (14.1354 + k_4)s^2 + (48.8614 + k_3)s \\ + (0.4253 + k_2) + k_1] = 0 \end{aligned} \tag{6.365}$$

Comparing the coefficient of Eqs. (6.340) and (6.365), we obtain $k_1 = 0$, $k_2 = 0.1869$, $k_3 = 27.8950$, $k_4 = 15.3543$, and $k_5 = 1.4395$.

Then next step is to transform back to the original state-space. Substituting for Z, A_p, and B_p in Eq. (6.363), we obtain

$$\dot{X} = (A - BKP)X + Br(t) \tag{6.366}$$

$$= AX + B[r(t) - KP(X)] \tag{6.367}$$

This gives us the desired state feedback control law as

$$U = r(t) - KPX \tag{6.368}$$

where the feedback gain matrix K is given by Eq. (6.362) with $k_1 \ldots k_5$ obtained as above and the transformation matrix P is given by Eq. (6.358).

To verifiy the design, we obtain the free response $[r(t) = 0]$ using MATLAB[1] for the initial conditions $\Delta\beta = 5$ deg, $\Delta\phi = p = \Delta\psi = r = 0$ as shown in Fig. 6.40. We observe that the system with full-state feedback law performs as expected.

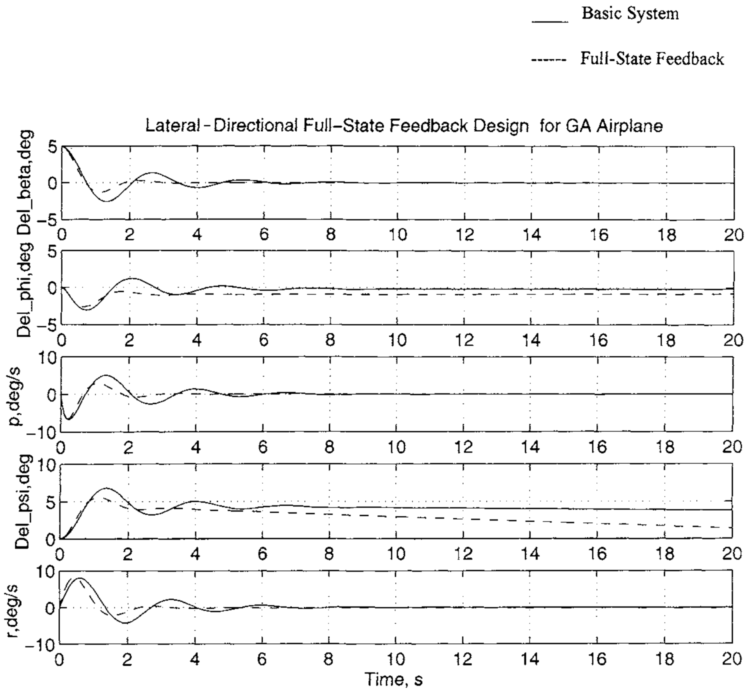


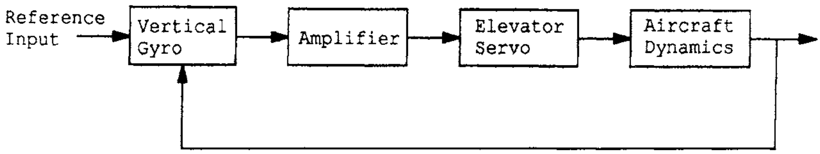
Fig. 6.40 Lateral-directional full-state feedback controller for the general aviation airplane.

6.5.5 Autopilots

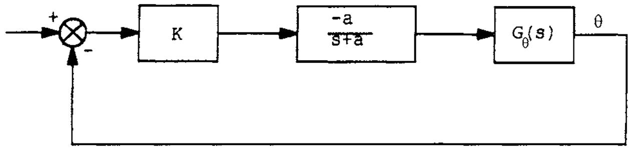
Autopilots come in various types and degrees of sophistication and perform different functions. Autopilots are commonly used for maintaining pitch attitude, heading, altitude, Mach number, prescribed glide slope during approach, and landing.

The principal element of an autopilot is a sensing element called gyro (gyroscope). It may be a positional gyro or a rate gyro. Whenever there is a deviation from the preselected condition, the gyro senses the change and generates electrical signals, which are amplified and are used to drive the servo devices that move the appropriate control surfaces in the desired manner.

Pitch displacement autopilot. The function of the pitch displacement autopilot is to maintain the given pitch attitude of the aircraft. A schematic diagram of a typical pitch displacement autopilot is shown in Fig. 6.41a, and the block diagram implementation is shown in Fig. 6.41b. The vertical gyro is replaced by a summer, and the elevator servo is represented by a first-order lag with a time constant $\tau = 1/a$. The parameter a characterizes the response time of the servo and is usually in the range of 10 to 20. In this design, we will assume $a = 10$ so that $\tau = 0.1$. As said before in Sec. 6.5.1, we choose a negative sign for the elevator servo transfer function.



a) Schematic diagram



b) Block diagram

Fig. 6.41 Pitch displacement autopilot.

Let us consider the general aviation airplane and design a pitch displacement autopilot for it. We will use the short-period approximation because it is found to be adequate for predicting the free and forced longitudinal responses. The short-period transfer function of the general aviation airplane for the pitch angle is given by Eq. (6.164)

$$\frac{\Delta\bar{\theta}(s)}{\Delta\bar{\delta}_e} = \frac{-2.0112\, s - 3.9018}{s(0.1695\, s^2 + 0.8494\, s + 2.1851)} \tag{6.369}$$

The first step is to draw the root-locus of the system with pitch angle feedback. We use MATLAB[1] to draw the root-locus as shown in Fig. 6.42. We observe that the aircraft (open-loop system) has a damping ratio in pitch of about 0.69 and a natural frequency of 2.6 rad/s, which still gives us level I handling qualities. However, as the amplifier gain increases, the closed-loop damping ratio reduces, and the aircraft becomes dynamically unstable. Therefore, we have to select a small value of the gain. We choose $k = 0.3335$, which gives a damping ratio of $\zeta = 0.53$ and still gives us level I flying qualities. With this value of the gain, the closed-loop poles are at -10.4636, $-1.9940 \pm j3.305$, and -0.5597.

To verify the design, we obtain a unit-step response using MATLAB[1] as shown in Fig. 6.43. The steady-state value of the output (pitch angle) is unity indicating that the steady-state error is zero. Thus, the design is found to be satisfactory.

Pitch displacement autopilot with pitch rate feedback. In the above design of pitch displacement autopilot for the general aviation airplane, the open-loop system had adequate damping in pitch; hence, on closing the loop, we had a small but still some margin to select the amplifier gain. However, this may not always be the case. Several aircraft have much lower open-loop damping ratios in pitch so that, on closing the loop, they tend to become unstable even for very small values of the amplifier gain. In such cases, we have to add an inner-loop pitch rate feedback as shown in Fig. 6.44 and select the gain of the rate gyro so that the overall system has adequate damping. To illustrate this design procedure, we will

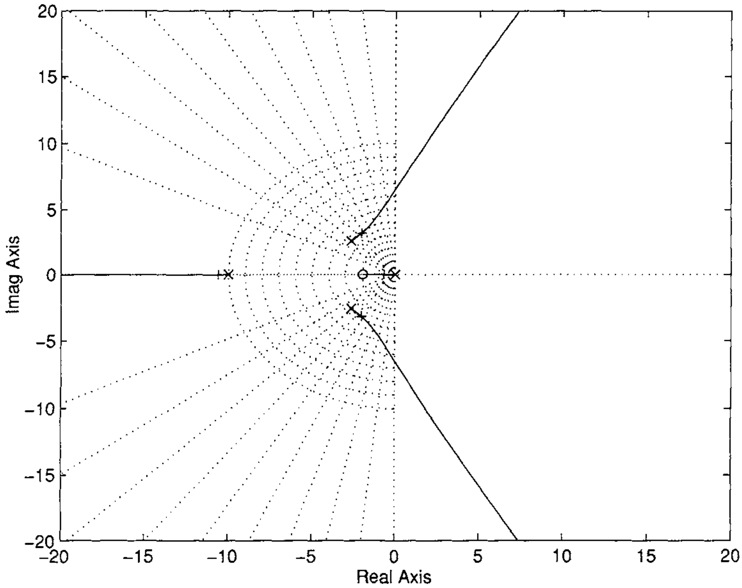


Fig. 6.42 Root-locus of pitch displacement autopilot for the general aviation airplane.

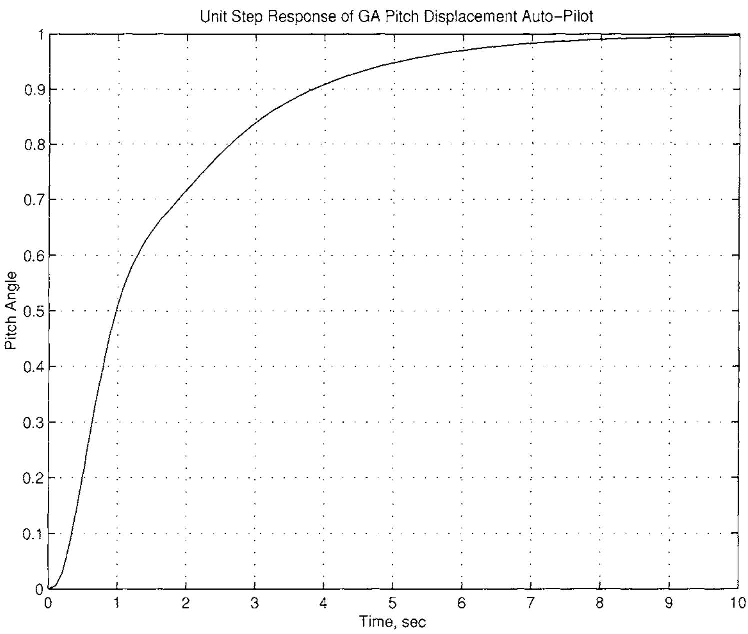


Fig. 6.43 Unit-step response of pitch displacement autopilot of the general aviation airplane.

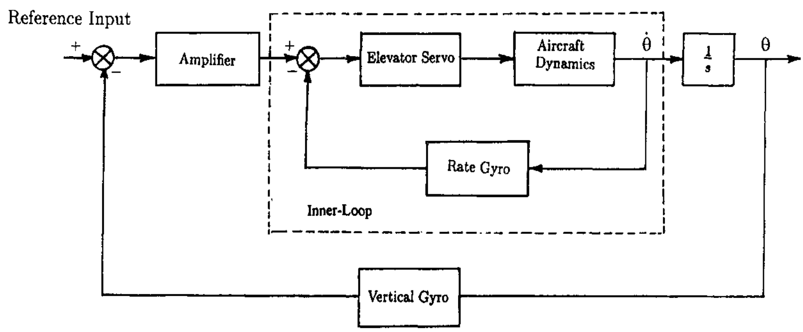


Fig. 6.44 Pitch displacement autopilot with pitch rate feedback.

consider a business jet[4] (see Exercise 6.1) whose short-period transfer function is given by

$$\frac{\Delta\bar{\theta}(s)}{\Delta\bar{\delta}_e(s)} = \frac{d_1 s + d_2}{s\left(a_1 s^2 + a_2 s + a_3\right)} \tag{6.370}$$

$$\frac{\bar{q}(s)}{\Delta\bar{\delta}_e(s)} = \frac{s\Delta\bar{\theta}(s)}{\Delta\bar{\delta}_e(s)} \tag{6.371}$$

$$= \frac{d_1 s + d_2}{a_1 s^2 + a_2 s + a_3} \tag{6.372}$$

where $d_1 = -6.6246$, $d_2 = -3.8069$, $a_1 = 3.1536$, $a_2 = 4.1624$, and $a_3 = 7.5662$.

The first step is to draw the root-locus of the inner loop as shown in Fig. 6.45 and select a suitable value for the rate gyro gain k_{rg}. We select a point on the root-locus for $\zeta = 0.9$ so that the system will have an adequate damping ratio when the outer loop is closed. For $\zeta = 0.9$, we get $k_{rg} = 0.8322$, and the locations of the closed-loop poles for the inner loop are -7.5275, $-1.8959 \pm j0.9623$.

The block diagram of the outer loop is shown in Fig. 6.46. Here, G_{eq} is the equivalent transfer function of the inner loop, which is given by

$$G_{eq} = \frac{-10(d_1 s + d_2)}{a_1 s^3 + (a_2 + 10a_1)s^2 + s(a_3 + 10a_2 - 10k_{rg}d_1) + 10(a_3 - k_{rg}d_2)} \tag{6.373}$$

We draw the outer-loop root-locus using MATLAB[1] as shown in Fig. 6.47. We select a point on the root-locus to have a damping ratio of $\zeta = 0.7$ as shown, which is satisfactory to have level I handling qualities. For this value of $\zeta = 0.70$, we get $k_1 = 0.753$ and closed-loop poles at -7.9213, $-1.5942 \pm j1.7139$, and -0.2095. The pole at -7.9213 is due to the elevator servo and that at -0.2095 is the system pole due to the integrator ($s = 0$) in the forward path. To check the design, we perform a simulation to unit-step (elevator) input as shown in Fig. 6.48. We have zero steady-state error, and the system response is found to be satisfactory.

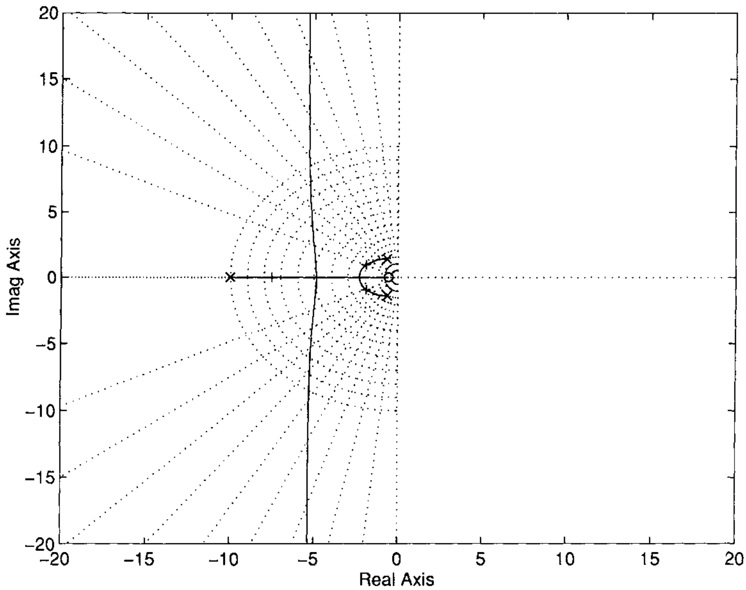


Fig. 6.45 Inner-loop root-locus of the pitch displacement autopilot for the business jet.

Altitude-hold autopilot. Altitude-hold autopilots are generally used by commercial airplanes during the cruise flight when the airplane is in the cruise mode. Usually, with such a system, the airplane is manually operated during the climb and descent portions of the flight. Once the aircraft has reached the desired cruise altitude, the altitude-hold autopilot is engaged. The autopilot will then maintain that altitude, making whatever corrections necessary when updrafts or downdrafts tend to cause the aircraft to gain or lose altitude.

Some autopilots have the feature that enables the pilot to enter a given flight altitude in advance. In this case, the autopilot flies the aircraft, making it climb or descend to attain the desired altitude. On reaching that altitude, the autopilot automatically levels the aircraft and, afterwards, it will maintain that altitude until told to do otherwise.

The schematic diagram of a typical altitude-hold autopilot is shown in Fig. 6.49a. The corresponding block diagram is shown in Fig. 6.49b. We have a lag compensator

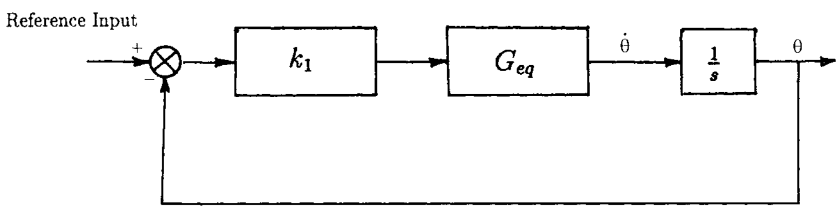


Fig. 6.46 Block diagram of the outer loop of the pitch displacement autopilot.

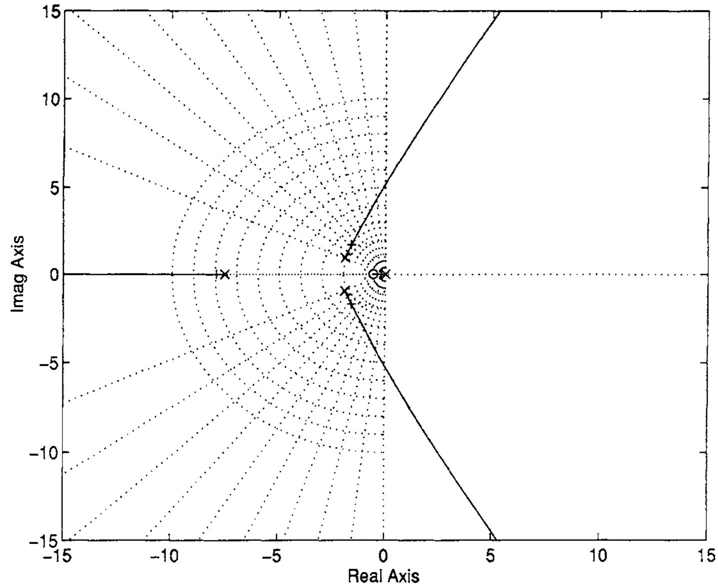


Fig. 6.47 Root-locus of the outer loop of the pitch displacement autopilot for the business jet.

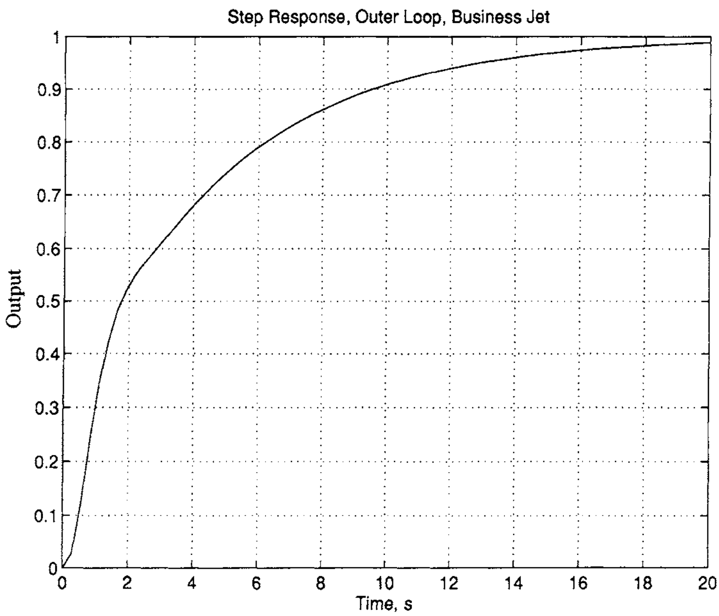
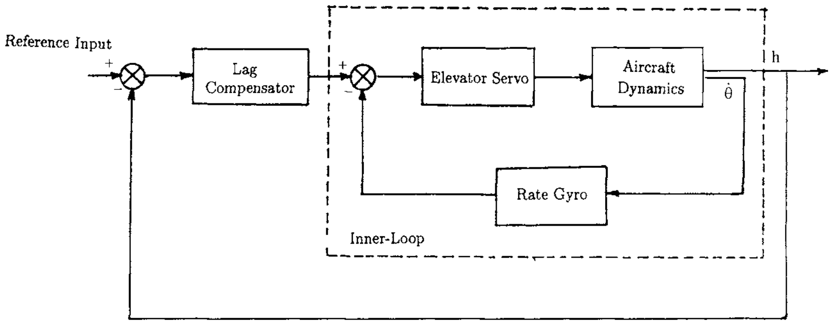
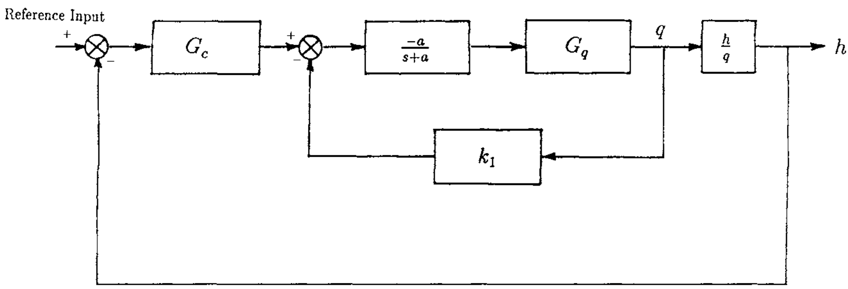


Fig. 6.48 Unit-step response of the pitch displacement autopilot for the business jet.



a) Schematic diagram



b) Block diagram

Fig. 6.49 Altitude-hold autopilot.

in the forward path, which is required to retain stability on closing the outer loop. The transfer function of the lag compensator is assumed as

$$G_c = \frac{k_3(s + 0.8)}{s + 8} \tag{6.374}$$

The transfer function for the altitude-to-elevator input can be obtained as follows:

$$\dot{h} = U_o \sin\gamma \tag{6.375}$$

$$\gamma = \theta - \alpha \tag{6.376}$$

where U_o is the flight velocity and γ is the flight path angle which is assumed to be small. Then,

$$s\bar{h}(s) = U_o\bar{\gamma}(s) \tag{6.377}$$

$$\frac{\bar{h}(s)}{\Delta\bar{\delta}_e(s)} = \frac{U_o\bar{\gamma}(s)}{\Delta\bar{\delta}_e(s)} \tag{6.378}$$

$$= \frac{U_o[\bar{\theta}(s) - \bar{\alpha}(s)]}{s\Delta\bar{\delta}_e(s)} \tag{6.379}$$

To illustrate the design procedure, let us consider the general aviation airplane once again and design an altitude-hold autopilot for it. Using Eqs. (6.163) and (6.164), we get

$$\frac{\bar{h}(s)}{\bar{\Delta}\delta_e(s)} = \frac{U_o\left(-b_1 s^2 + (d_1 - b_2)s + (d_2 - b_3)\right)}{s(a_1 s^3 + a_2 s^2 + a_3 s)} \tag{6.380}$$

where $b_1 = -0.0273$, $b_2 = -2.0366$, $b_3 = 0$, $a_1 = 0.1695$, $a_2 = 0.8494$, $a_3 = 2.1851$, $d_1 = -2.0112$, and $d_2 = -3.9018$.

At first, let us assume that there is no pitch rate feedback. The corresponding root-locus of the altitude-hold autopilot with only altitude feedback is shown in Fig. 6.50. We observe that, as soon as the outer loop is closed, the system becomes unstable for any positive value of the gain k_3. Therefore, we need to add the inner loop with pitch rate feedback as shown in Fig. 6.51. The transfer function G_q is given by

$$G_q = \frac{\bar{q}(s)}{\Delta\bar{\delta}_e(s)} = \frac{d_1 s + d_2}{a_1 s^2 + a_2 s + a_3} \tag{6.381}$$

Now we plot the root-locus of the inner loop as shown in Fig. 6.52 and pick a point on the root-locus corresponding to $\zeta = 0.8$, which gives the gain of rate gyro $k_1 = 0.2296$. This is a small improvement in the damping ratio compared to the basic system (open loop) damping of 0.7 but may be sufficient to make the system stable on closing the outer loop.

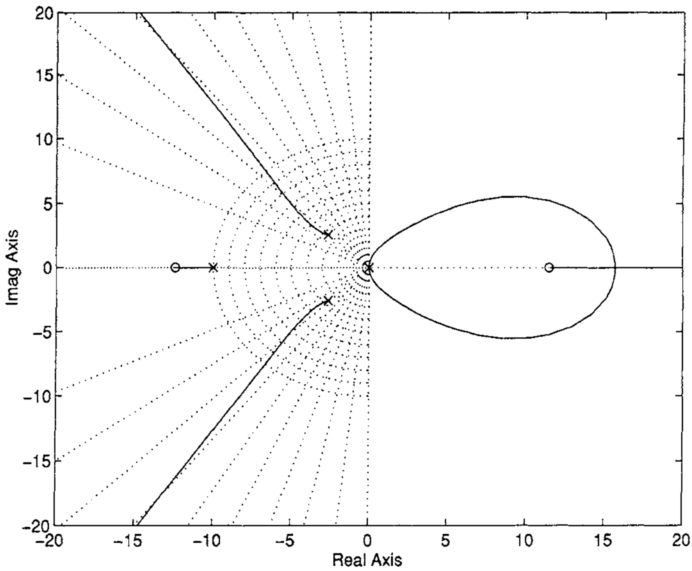


Fig. 6.50 Root-locus of altitude-hold autopilot for the general aviation airplane.

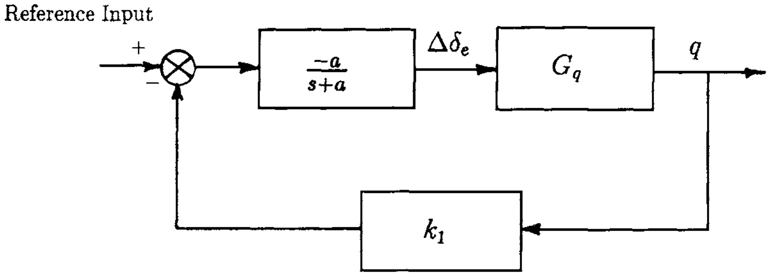


Fig. 6.51 Inner loop of the altitude-hold autopilot.

The next step is to design the outer loop (see Fig. 6.53). The transfer functions G_{eq} and $\bar{h}(s)/\bar{q}(s)$ are given by the following expressions:

$$G_{eq} = \frac{-10(d_1 s + d_2)}{a_1 s^3 + (a_2 + 10a_1)s^2 + (a_3 + 10a_2 - 10k_1 d_1)s + 10\,(a_3 - k_1 d_2)} \tag{6.382}$$

$$\frac{\bar{h}(s)}{\bar{q}(s)} = \left(\frac{U_o}{s^2}\right)\left(\frac{-b_1 s^2 + (d_1 - b_2)s + d_2 - b_3}{d_1 s + d_2}\right) \tag{6.383}$$

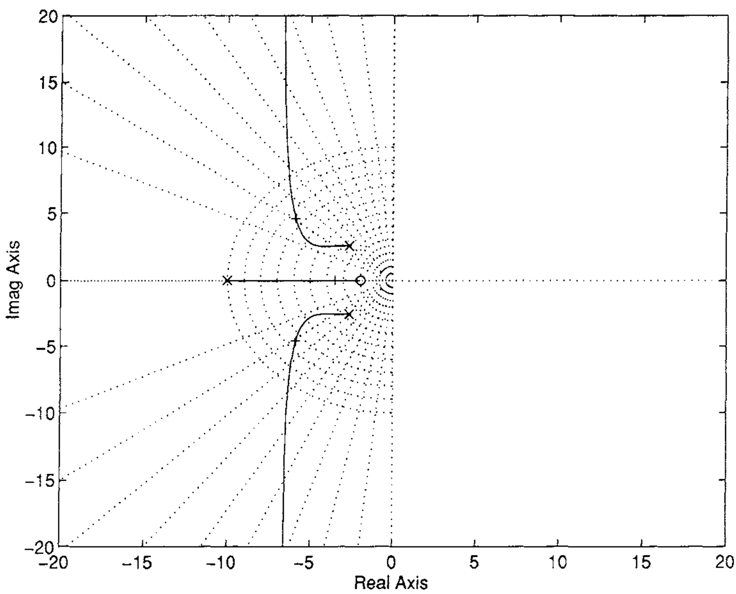


Fig. 6.52 Root-locus of the inner loop of the altitude-hold autopilot for the general aviation airplane.

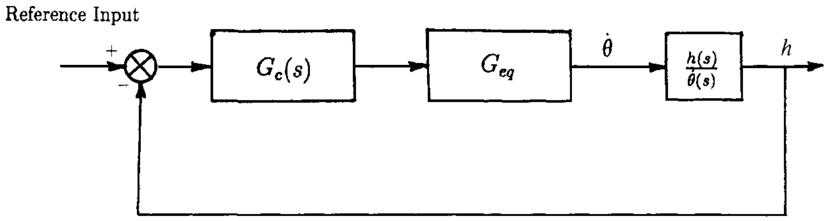


Fig. 6.53 Outer loop of the altitude-hold autopilot.

The lag-compensator pole ($s = -8$) is sufficiently away from the origin so that it has very little effect on the short-period dynamics. The root-locus of the outer loop is shown in Fig. 6.54. We observe that we have to confine ourselves to very low values of the compensator gain k_3 to keep the system stable. A higher value of k_3 will make the system unstable. The effect of the compensator is obvious. It has pulled to the left the branches of the root-locus that originate from the two poles at the origin and migrate to the right half of the plane. This provides a small range of stable values for the complex conjugate poles. With $k_3 = 0.0214$, we get the closed-loop poles at -8.4427, $-6.1427 \pm j4.6397$, $-0.1536 \pm j0.7955$, and -2.2242.

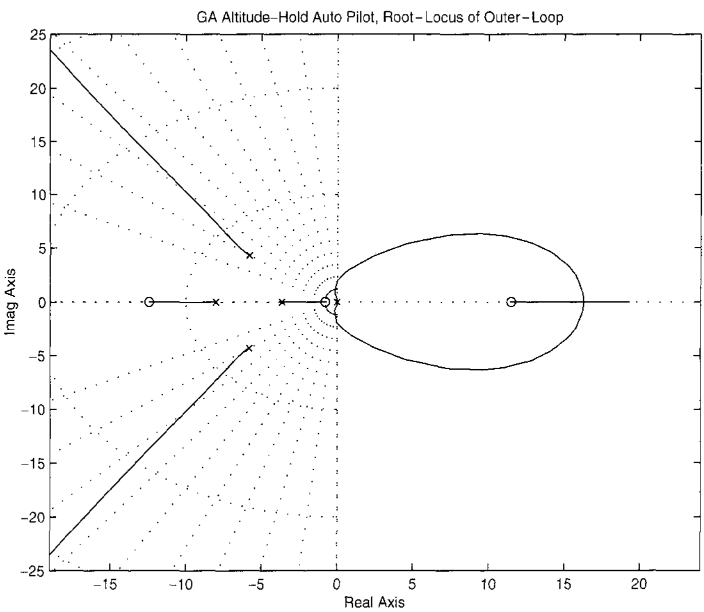


Fig. 6.54 Root-locus of the outer loop of the pitch displacement autopilot for the general aviation airplane.

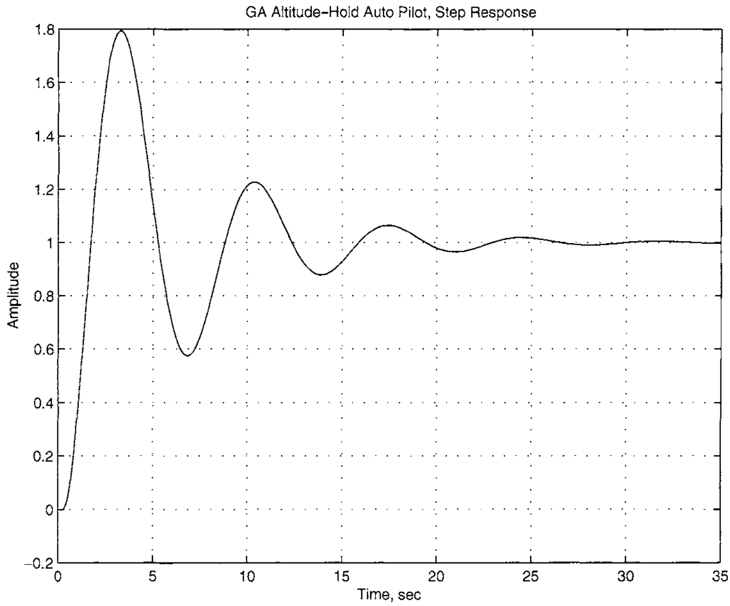


Fig. 6.55 Unit-step response of the pitch displacement autopilot for the general aviation airplane.

To verify the design, we simulate the response of the system to a unit-step elevator input as shown in Fig. 6.55. The large overshoot is due to the low value of the damping ratio. Except for this, the system performance is satisfactory.

6.6 Summary

In this chapter, we have studied the longitudinal and lateral-directional responses of the airplane. The longitudinal response of a statically stable airplane consists of two distinct modes, the short-period mode and the phugoid mode. The lateral-direction motion comprises of roll-subsidence mode, the Dutch-roll oscillation, and the spiral modes. Accordingly, we obtained simplified transfer functions and state-space models to study these modes individually. These simplified transfer functions were found to be of considerable help in designing the flight control systems.

For a statically unstable airplane, the usual short-period and phugoid modes do not exist. Instead, we have two exponential modes and an oscillatory mode. The oscillatory mode is called the third oscillatory mode that has short-period-like damping and phugoidlike frequency. Because of static instability, at least one of the two exponential modes will be unstable.

The handling quality requirements form the basis of the design of stability augmentation systems and autopilots. The longitudinal flying quality requirements were presented in terms of damping ratios and frequencies of the short-period and phugoid modes. Similarly, the lateral-directional flying qualities were presented in terms of damping ratios and frequency for Dutch-roll mode and in terms of time for half or double the amplitude for roll-subsidence and spiral modes.

We have illustrated the basic design procedures for stability augmentation systems and autopilots. It was not possible to cover all types of stability augmentation systems and autopilots that are in use on modern airplanes. The interested reader may refer elsewhere[5,7] for additional information.

References

[1]*Pro-MATLAB for Sun Workstations*, The MathWorks, Natick, MA, Jan. 1990.

[2]Teper, G. L., "Aircraft Stability and Control Data," NASA CR-96008, April 1969.

[3]Roskam, J., *Airplane Flight Dynamics and Automatic Flight Control, Part II*, Roskam Aviation and Engineering Corp., Lawrence, KS, 1979.

[4]Nelson, R. C., *Flight Stability and Automatic Control*, McGraw–Hill, New York, 1989.

[5]Stevens, B. L., and Lewis, F. L., *Aircraft Control and Simulation*, Wiley, New York, 1994.

[6]MIL-F-8785C, "Flying Qualities of Piloted Airplanes," *U.S. Department of Defense Military Specifications*, Nov. 5, 1980.

[7]Blakelock, J. H., *Automatic Control of Aircraft and Missiles*, Wiley, New York, 1965.

Problems

6.1 The mass properties and aerodynamic characteristics of a business jet airplane[4] are as follows.

Mass properties: weight $W = 169{,}921.24$ N, wing area $S = 50.3983$ m^2, wing span $b = 16.383$ m, mean aerodynamic chord $\bar{c} = 3.3315$ m, distance between center of gravity and aerodynamic center in terms of mean aerodynamic chord $\bar{x}_{cg} = 0.25$, $I_x = 161{,}032.43$ kgm^2, $I_y = 184{,}211.19$ kgm^2, $I_z = 330{,}142.72$ kgm^2, $I_{xy} = I_{yz} = 0$, and $I_{xz} = 6861.7038$ kgm^2.

Longitudinal aerodynamic characteristics: $C_L = 0.737$, $C_D = 0.095$, $C_{L\alpha} = 5.0$, $C_{D\alpha} = 0.75$, $C_{m\alpha} = -0.8$, $C_{L\dot{\alpha}} = 0$, $C_{D\dot{\alpha}} = 0$, $C_{m\dot{\alpha}} = -3.0$, $C_{LM} = 0$, $C_{DM} = 0$, $C_{mM} = -0.05$, $C_{L\delta_e} = 0.4$, $C_{D\delta_e} = 0$, $C_{m\delta_e} = -0.81$, $C_{Lq} = 0$, $C_{Dq} = 0$, and $C_{mq} = -8.0$.

Lateral-directional aerodynamic characteristics: $C_{y\beta} = -0.720$, $C_{l\beta} = -0.103$, $C_{n\beta} = 0.137$, $C_{y\delta_a} = 0$, $C_{l\delta_a} = 0.054$, $C_{n\delta_a} = -0.0075$, $C_{y\delta_r} = 0.175$, $C_{l\delta_r} = 0.029$, $C_{n\delta_r} = -0.063$, $C_{yp} = 0$, $C_{lp} = -0.37$, $C_{np} = -0.14$, $C_{yr} = 0$, $C_{lr} = 0.11$, $C_{nr} = -0.16$, $C_{y\dot{\beta}} = 0$, $C_{l\dot{\beta}} = 0$, and $C_{n\dot{\beta}} = 0$.

Note that all derivatives are per radian.

For flight at Mach 0.2, determine the following:

(a) The state-space representations, eigenvalues, free responses, and unit-step responses for the longitudinal and lateral-directional dynamics based on a complete system as well as longitudinal (short-period and phugoid) and lateral-directional (roll-subsidence, Dutch-roll, and spiral) approximations.

(b) Transfer functions, roots of characteristic equations, steady-state values for unit-step inputs for longitudinal and lateral-directional variables based on a complete system as well as longitudinal (short-period and phugoid) and lateral-directional (roll-subsidence, Dutch-roll, and spiral) approximations. Compare these results with those obtained in (a).

(c) The longitudinal (elevator input) and lateral-directional (aileron and rudder inputs) frequency responses based on a complete system.

6.2 The longitudinal characteristic equation of an airplane is given by

$$s^4 + 1.8\,s^3 + 3.5\,s^2 + 0.25\,s + 0.05 = 0$$

Examine the longitudinal stability of the airplane using Routh's criterion.

6.3 The longitudinal transfer functions of an airplane based on the short-period approximations are given by

$$G_\theta(s) = \frac{-(5.25\,s + 3.15)}{s(2.25\,s^2 + 3.15\,s + 6.53)}$$

$$G_\alpha(s) = \frac{-(0.04\,s^2 + 4\,s)}{s(2.25\,s^2 + 3.15\,s + 6.53)}$$

Design (a) a pitch displacement autopilot to give level I handling qualities and (b) an altitude-hold autopilot for $M = 0.2$ at sea level to give acceptable handling qualities.

6.4 A light airplane has the following longitudinal transfer functions:

$$G_\alpha(s) = \frac{-(0.2\,s^3 + 10\,s^2 + s + 1.5)}{(s^4 + 4\,s^3 + 2\,s^2 + 0.15\,s - 0.1)}$$

$$G_\theta(s) = \frac{-(10\,s^2 + 20\,s + 1.5)}{(s^4 + 4\,s^3 + 2\,s^2 + 0.15\,s - 0.1)}$$

Determine the longitudinal eigenvalues. Is the airplane longitudinally stable? If not, design a suitable pitch augmentation system so that the airplane has conventional short-period and phugoid modes in the closed-loop sense.

6.5 The transfer function of an airplane for yaw rate to rudder deflection is given by

$$\frac{\bar{r}(s)}{\Delta\bar{\delta}_r(s)} = \frac{-(4.5\,s + 0.75)}{(s^2 + 1.1\,s + 4.2)}$$

Design a yaw damper to give level I Dutch-roll flying qualities.

6.6 The longitudinal dynamics of an airplane in state-space form is given by

$$\begin{bmatrix} \dot{u} \\ \Delta\dot{\alpha} \\ \dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.01 & 0.05 & 0 & -0.2 \\ -0.50 & -2.5 & 1.0 & -0.05 \\ 0.5 & -5.0 & -3.5 & 0.05 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ \Delta\alpha \\ q \\ \Delta\theta \end{bmatrix} + \begin{bmatrix} 0 \\ -0.5 \\ -7.5 \\ 0 \end{bmatrix} \Delta\delta_e$$

Design a full-state feedback controller to give a natural frequency of 5.0 rad/s and damping ratio of 0.8 for the short-period mode and a natural frequency of 0.15 rad/s and damping ratio of 0.15 for the phugoid mode.

6.7 The lateral-directional dynamics of an airplane in state-space form is given by

$$\begin{bmatrix} \Delta\dot{\beta} \\ \Delta\dot{\phi} \\ \dot{p} \\ \Delta\dot{\psi} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} -0.2 & 0.2 & 0 & 0 & -1.0 \\ 0 & 0 & 1.0 & 0 & 0 \\ -15.0 & 0 & -8.0 & 0 & 2.0 \\ 0 & 0 & 0 & 0 & 1 \\ 3.5 & 0 & -0.5 & 0 & -0.20 \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta\phi \\ p \\ \Delta\psi \\ r \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0.05 \\ 0 & 0 \\ 25.0 & 2.0 \\ 0 & 0 \\ -0.15 & -4.0 \end{bmatrix} \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix}$$

Design a full-state feedback controller to place the closed-loop eigenvalues at $0, -0.6 \pm j1.9079, -0.005$, and -8.0.

This page intentionally left blank