Signalbehandling for computer-ingeniører

COMTEK-5, E20

&

Signalbehandling

EIT-5, E20

14. DFT Analysis and The Short Time Fourier Transform (STFT)

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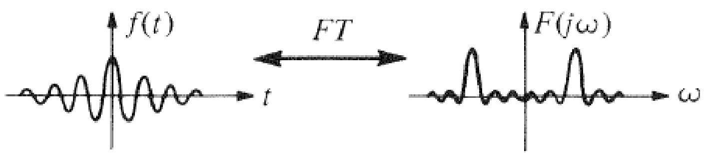
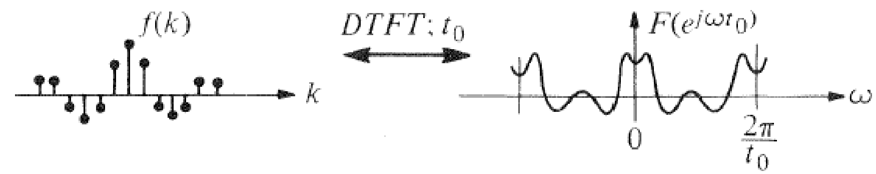
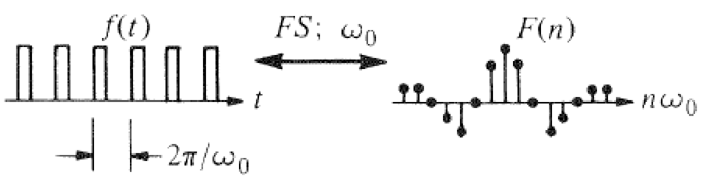
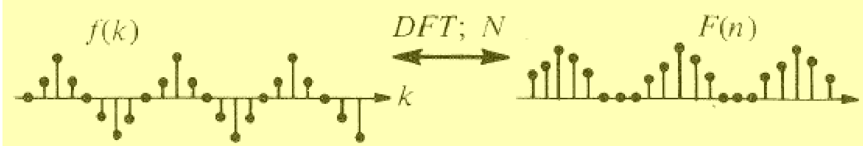
The outline of today's lecture

- Some considerations on practical calculation of the DFT
  - Doing DFT on an infinite length sequence
  - Spectral resolution
  - The impact of the window function

- The Short Time Fourier Transformation
  - Time varying signals
  - Simultaneous time-and-frequency analysis
  - Heisenberg

Fourier Transform – a classification

	Continuous in time	Discrete in time – Periodic in frequency
Continuous in frequency	 $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega$ $F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$ **Fourier transform**	 $f(k) = \frac{t_0}{2\pi}\int_{-\pi/t_0}^{\pi/t_0} F(e^{j\omega t_0})\, e^{jk\omega t_0}\, d\omega$ $F(e^{j\omega t_0}) = \sum_{k=-\infty}^{\infty} f(k)\, e^{-jk\omega t_0}$ **Discrete-time Fourier transform**
Discrete in frequency – Periodic in time	 $f(t) = \sum_{n=-\infty}^{\infty} F(n)\, e^{jn\omega_0 t}$ $F(n) = \frac{\omega_0}{2\pi}\int_{-\pi/\omega_0}^{\pi/\omega_0} f(t)\, e^{-jn\omega_0 t}\, dt$ **Fourier series**	 $f(k) = \frac{1}{N}\sum_{n=0}^{N-1} F(n)\, (e^{j2\pi/N})^{kn}$ $F(n) = \sum_{k=0}^{N-1} f(k)\, (e^{j2\pi/N})^{-kn}$ **Discrete Fourier transform**

Relation between the DFS and DTFT

Assume the infinite periodic sequence $\tilde{x}[n]$.

We have seen that the Discrete Fourier Series coefficients which corresponds to $\tilde{x}[n]$can be found by sampling one period of the DTFT. That is;

$$\tilde{X}[k] = X(e^{j\omega})|_{\omega=(2\pi/N)k} = X(e^{j(2\pi/N)k})$$

which corresponds to sample $X(z)$ in *N* equally spaced angles on the unit circle.

…and similarly for the Discrete Fourier Transform (DFT)

Relation between DFS and DFT

Basically, what we do is that we sample the DTFT at equividistant frequency values over one period. Since we have just argued that the Fourier Series coefficients $\tilde{X}[k]$ can be represented by the DTFT, it follows that

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \leqslant k \leqslant N-1 \\ 0, & \text{otherwise} \end{cases}$$

…and therefore we can write the Discrete Fourier Transform as;

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad \text{and}$$

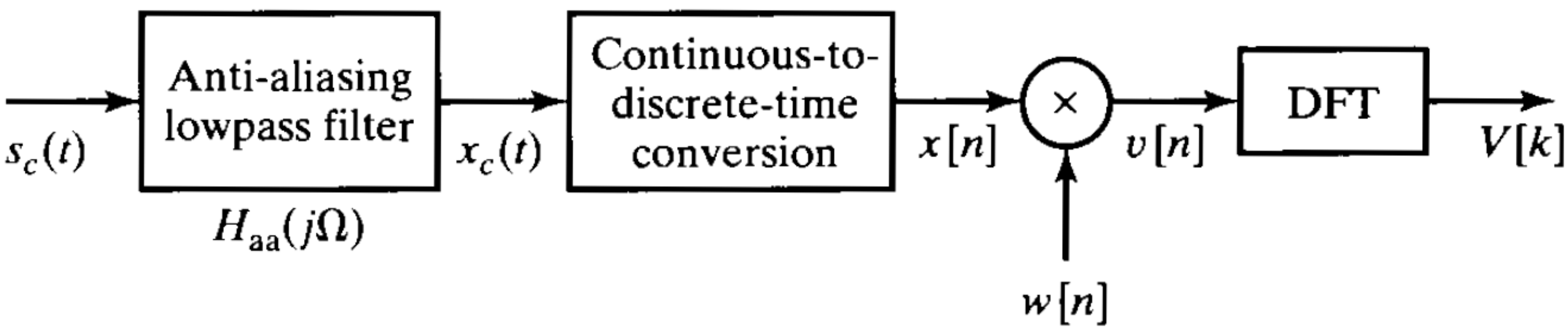
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$$

Here it is important though to notice that both $x[n]$ and $X[k]$ equals zero outside the interval $0..N-1$.

Fourier Analysis using the DFT

The challenge with the DFT is that it requires as input a finite lenght sequence which contradicts the assumption that the signal being an infinite and sequence, normally an observable signal provided directly from an ADC.

In many cases this is accomplished by partitioning the signal using a finite duration ”window”, i.e., a sequence which is identically zero outside the interval *0..N-1*.



Processing steps in the discrete-time Fourier analysis of a continuous-time signal.

Now, let’s discuss what happens throughout this signal chain...

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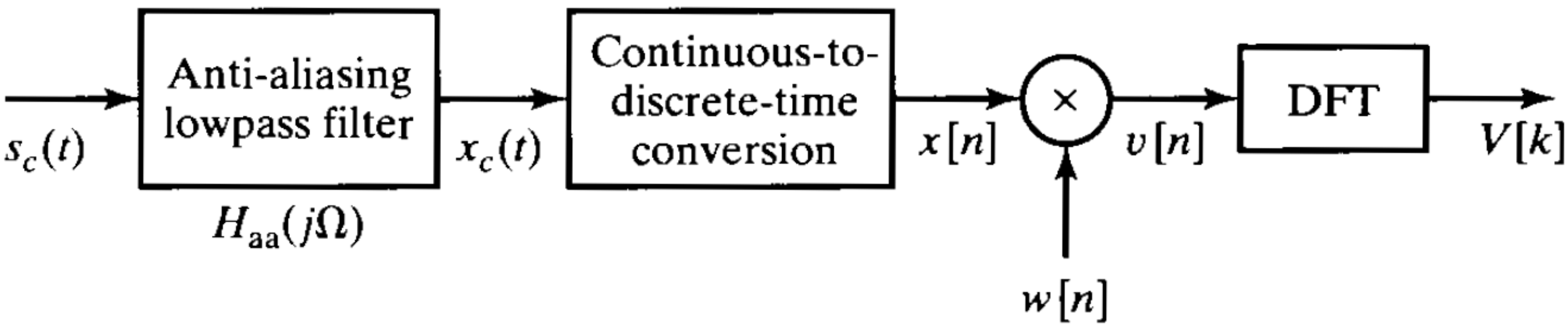
Due to sampling of $x_c(t)$, the Fourier Transform of the infinite sequence $x[n]$ is periodic in frequency



$$X(e^{j\omega}) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} + j\frac{2\pi r}{T}\right)$$

We now partition $x[n]$ into finite duration sequences using a window – but what is a window actually, and what does it looks like in the frequency domain..??

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Typical window functions

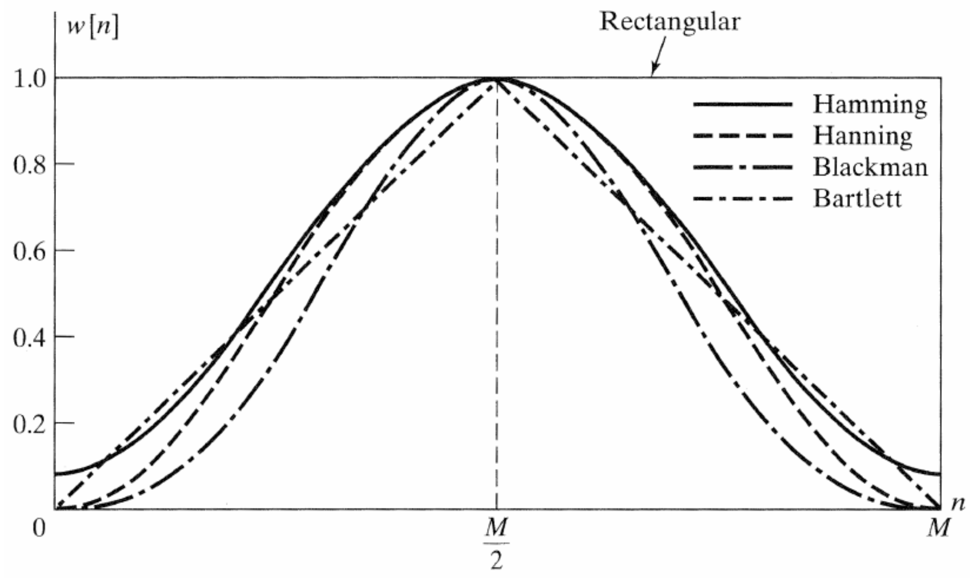
Normally, we would think of the Rectangular window as being the most ”obvious” function to truncate a sequence.
The problem is however, that at the edges of the window, we have discontinuities which may impact negatively the overall performance of the DFT analysis.

Hanning ($\alpha = 0.5$)
& Hamming ($\alpha = 0.46$):

$$w[n] = \begin{cases} \alpha - \alpha\cos(2\pi n/M), & 0 \le n \le M \\ 0, \text{ otherwise} \end{cases}$$

Blackman:

$$w[n] = \begin{cases} 0.42 - 0.5\cos(2\pi n/M) + 0.08\cos(4\pi n/M), & 0 \le n \le M \\ 0, \text{ otherwise} \end{cases}$$



What happens when we multiplying with the window function..??

From the Fourier Transform theorem pair no. 7 on p. 60 in O&S 3rd ed.;

$$7.\ x[n]y[n] \longleftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$$

So, one might think that limiting $x[n]$ to a finite duration sequence is ”just a matter” of preparing the sequence for being suitable as input to the DFT, but the fact is that multiplication (significantly) impact the spectral analysis we are conducting.

Consequently, in the frequency domain we are convolving the FT of the input sequence $x[n]$ with the FT of the window function – and to be precisely, we conduct a periodic convolution.

For this reason, it is important to study the FT of the various window functions…

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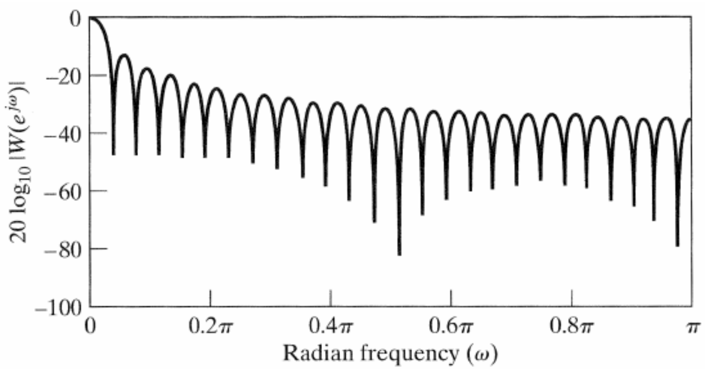
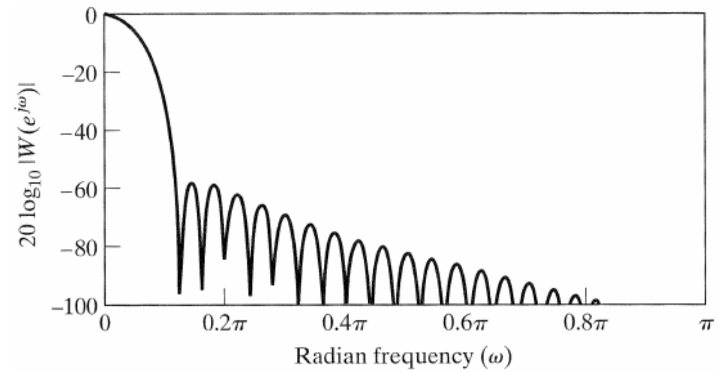
Amplitude response of the window functions

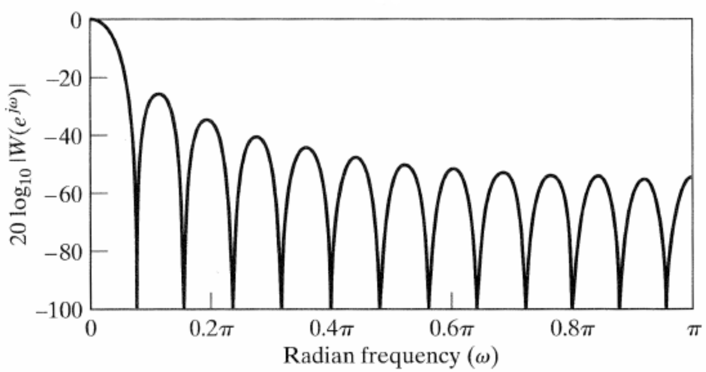
Rectangular



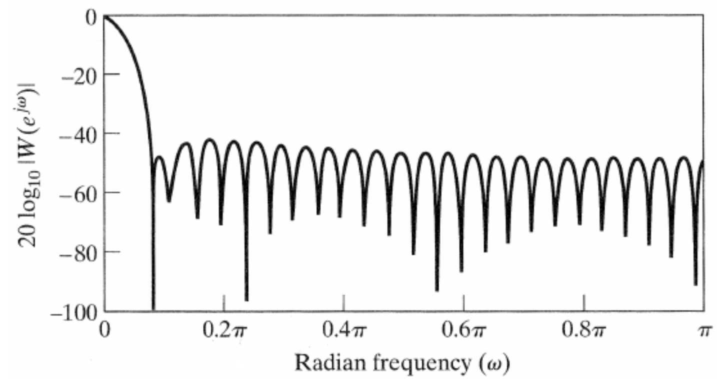
Blackman



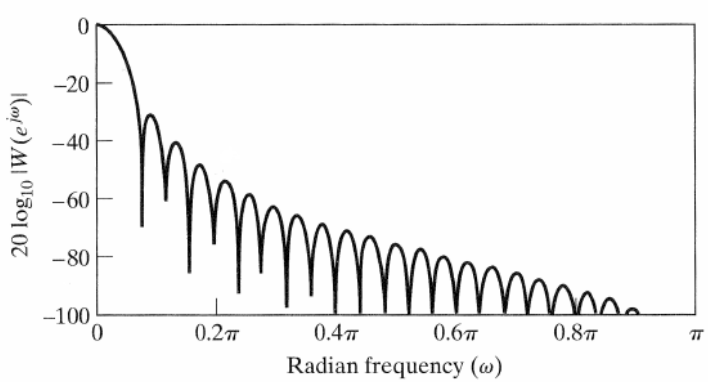
Bartlett



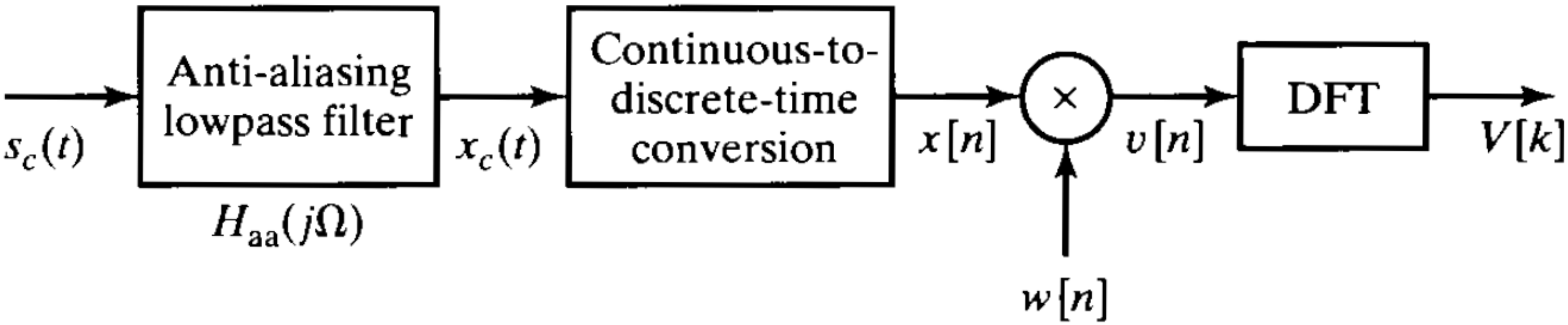
Hamming



Hanning



Type of window	Peak side lobe	Width of main lobe
Rectangular	-13 dB	$4\pi/(M+1)$
Bartlett	-25 dB	$8\pi/M$
Hanning	-31 dB	$8\pi/M$
Blackman	-57 dB	$12\pi/M$
Hamming	-41 dB	$8\pi/M$



So, according to the overall signal chain, the input of the DFT block, $v[n]$, has a spectral representation which is the periodic convolution between $X(e^{j\omega})$ and $W(e^{j\omega})$, i.e.,

$$V(e^{j\omega}) = X(e^{j\omega}) * W(e^{j\omega})$$

Now, since from a functional interpretation, the DFT basically samples the spectrum on its input and presents it as a ”frequency discrete” representation on its output, we can state that

$$V[k] = \mathrm{V}(\mathrm{e}^{j\omega})\big|_{\omega=2\pi k/N}$$

Since $V[k]$ represents one period, $[-\pi;\pi]$ or $[0;2\pi]$, where 2π is the sample frequency, we can easily calculate ”the spectral accuracy”…

Frequency resolution

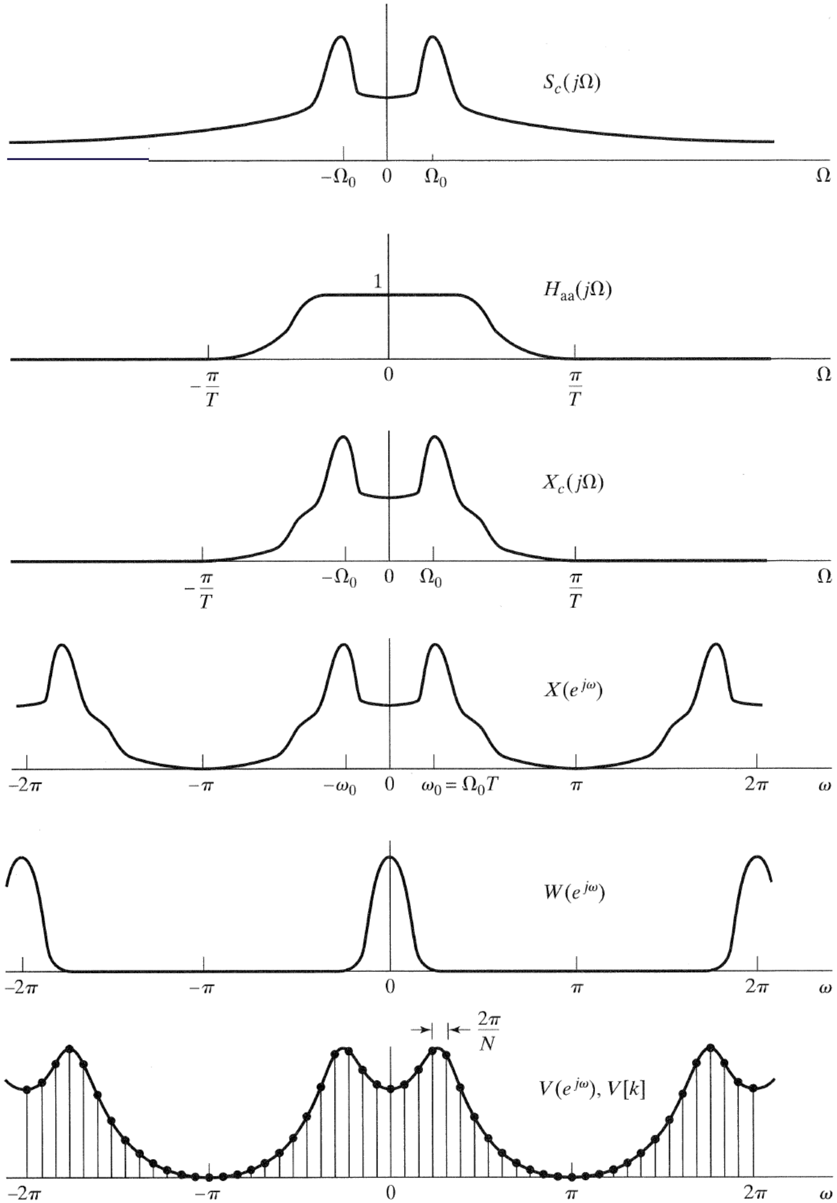
$$V[k] = \mathrm{V}(\mathrm{e}^{j\omega})\big|_{\omega=2\pi k/N}$$

$$\omega = 2\pi k/N = \Omega T$$

$$\Omega_k = \frac{2\pi k}{NT}$$

One important lesson to learn here is that frequency resolution is proportional with the order of the DFT, and inverse proportional with the sample frequency.

The overall picture



The spectrum of the ”wide band” input signal

The amplitude response of the anti-aliasing filter

The time continuous input signal to the S/H circuit.

The time discrete input signal

The amplitude response of the window function

The DTFT of the signal and the sampled version which is the DFT

The effect of windowing – an example

In this example we consider a signal which is a sum of two sinusoids;

$$s_c(t) = A_0 \cos(\Omega_0 t + \theta_0) + A_1 \cos(\Omega_1 t + \theta_1)$$

This signal is now sampled in an ideal manner, i.e., no aliasing and no quantization effects;

$$x[n] = A_0 \cos(\omega_0 n + \theta_0) + A_1 \cos(\omega_1 n + \theta_1)$$

$$\text{where } \omega_0 = \Omega_0 T \text{ og } \omega_1 = \Omega_1 T$$

The sequence is now multiplied with a window function $w[n]$, $n=0..N-1$

$$v[n] = A_0 w[n] \cos(\omega_0 n + \theta_0) + A_1 w[n] \cos(\omega_1 n + \theta_1)$$

Next we apply 1) the cosine addition formula;

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

and 2) Euler's formulas for cosine and sine;

$$\cos(x) = (e^{jx} + e^{-jx})/2$$

$$\sin(x) = (e^{jx} - e^{-jx})/2j$$

The effect of windowing – an example

$$v[n]=A_0 w[n]\cos(\omega_0 n+\theta_0)+A_1 w[n]\cos(\omega_1 n+\theta_1)$$

$$v[n]=\frac{A_0}{2}w[n]\mathrm{e}^{j\theta_0}\mathrm{e}^{j\omega_0 n}+\frac{A_0}{2}w[n]\mathrm{e}^{-j\theta_0}\mathrm{e}^{-j\omega_0 n}$$
$$+\frac{A_1}{2}w[n]\mathrm{e}^{j\theta_1}\mathrm{e}^{j\omega_1 n}+\frac{A_1}{2}w[n]\mathrm{e}^{-j\theta_1}\mathrm{e}^{-j\omega_1 n}$$

Now, let use the Fourier Transform theorem pair no. 3, O&S 3rd ed., p. 60;

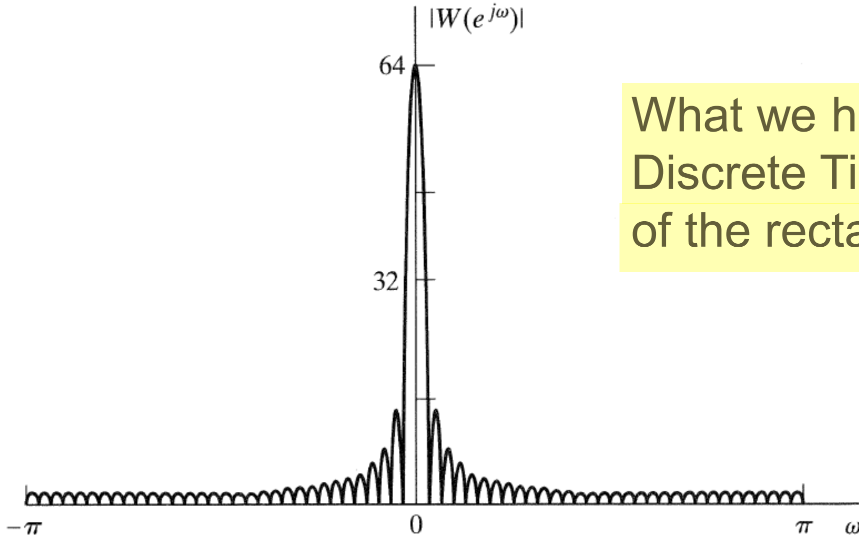
$$3.\quad e^{j\omega_0 n}x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$$

$$\mathrm{V}(\mathrm{e}^{j\omega})=\frac{A_0}{2}\mathrm{e}^{j\theta_0}\mathrm{W}(\mathrm{e}^{j(\omega-\omega_0)})+\frac{A_0}{2}\mathrm{e}^{-j\theta_0}\mathrm{W}(\mathrm{e}^{j(\omega+\omega_0)})$$
$$+\frac{A_1}{2}\mathrm{e}^{j\theta_1}\mathrm{W}(\mathrm{e}^{j(\omega-\omega_1)})+\frac{A_1}{2}\mathrm{e}^{-j\theta_1}\mathrm{W}(\mathrm{e}^{j(\omega+\omega_1)})$$

4 terms…!!
…and each term is the FT of the window

Rectangular window with length 64. Signal amplitudes of $A_0 = 1$ og $A_1 = 0{,}75$

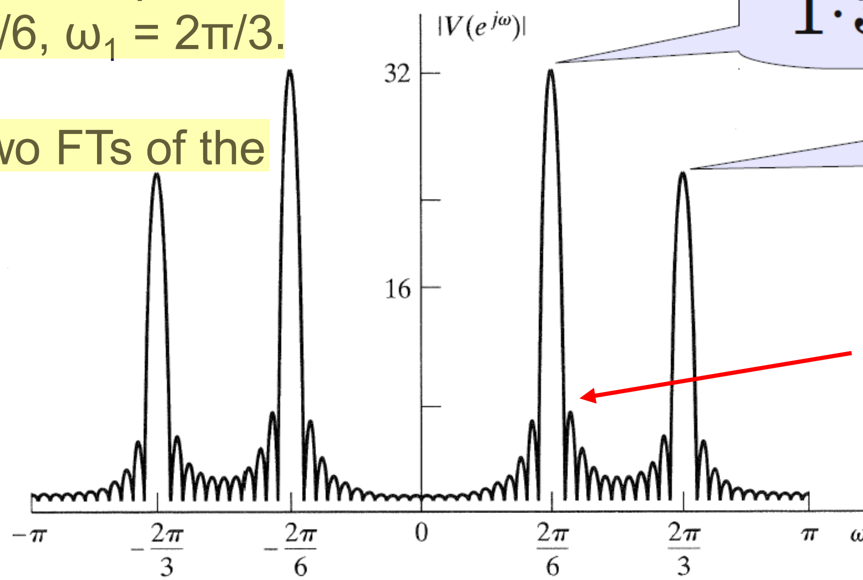
$\omega_0 = 2\pi/6$
$\omega_1 = 2\pi/3$



What we have shown here is the Discrete Time Fourier Transform of the rectangular window

Guess you would have expected two pins at $\omega_0 = 2\pi/6$, $\omega_1 = 2\pi/3$.

But in fact we get two FTs of the window function…

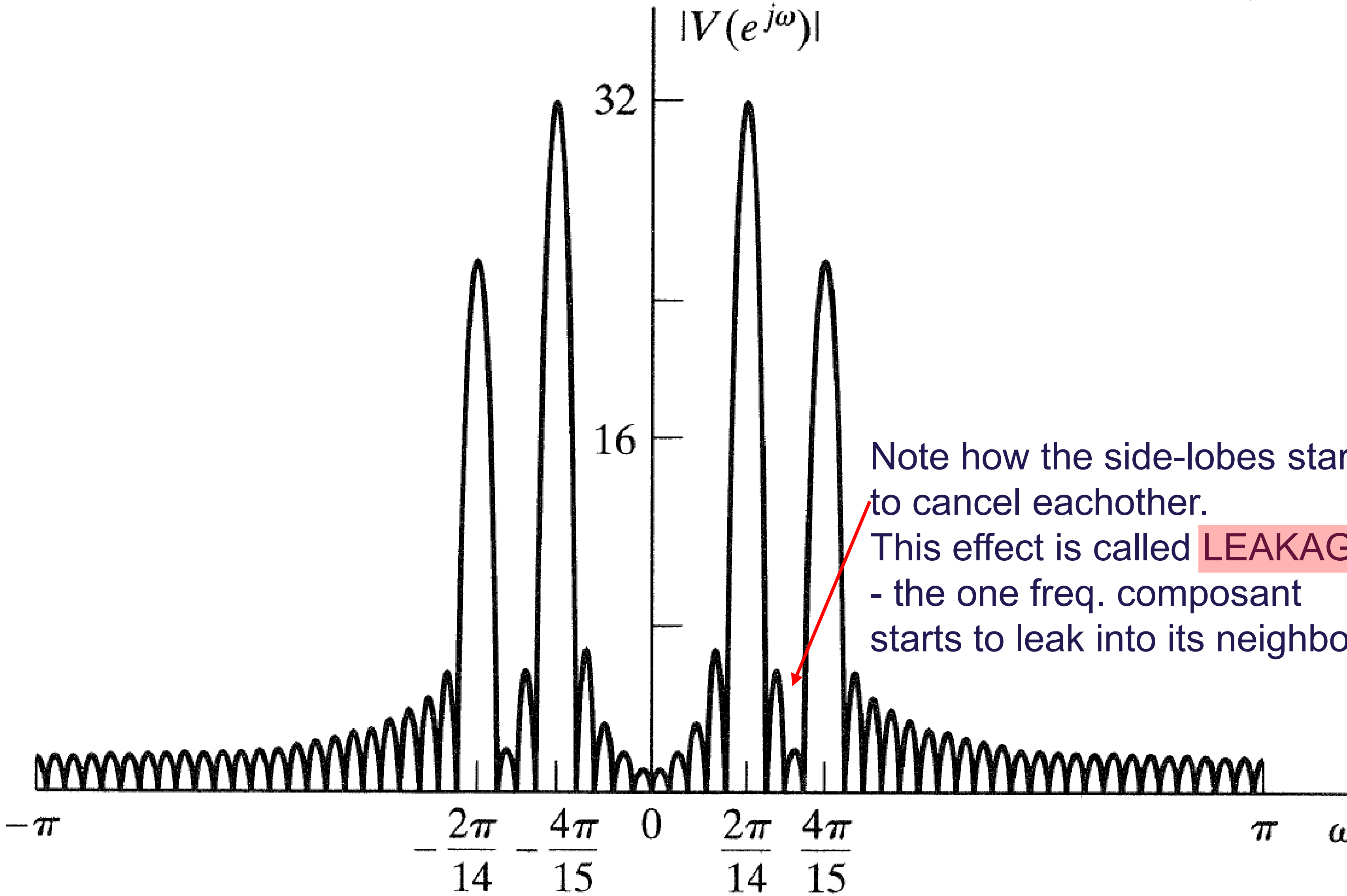


$1 \cdot 32$ (one half)

$0.75 \cdot 32$

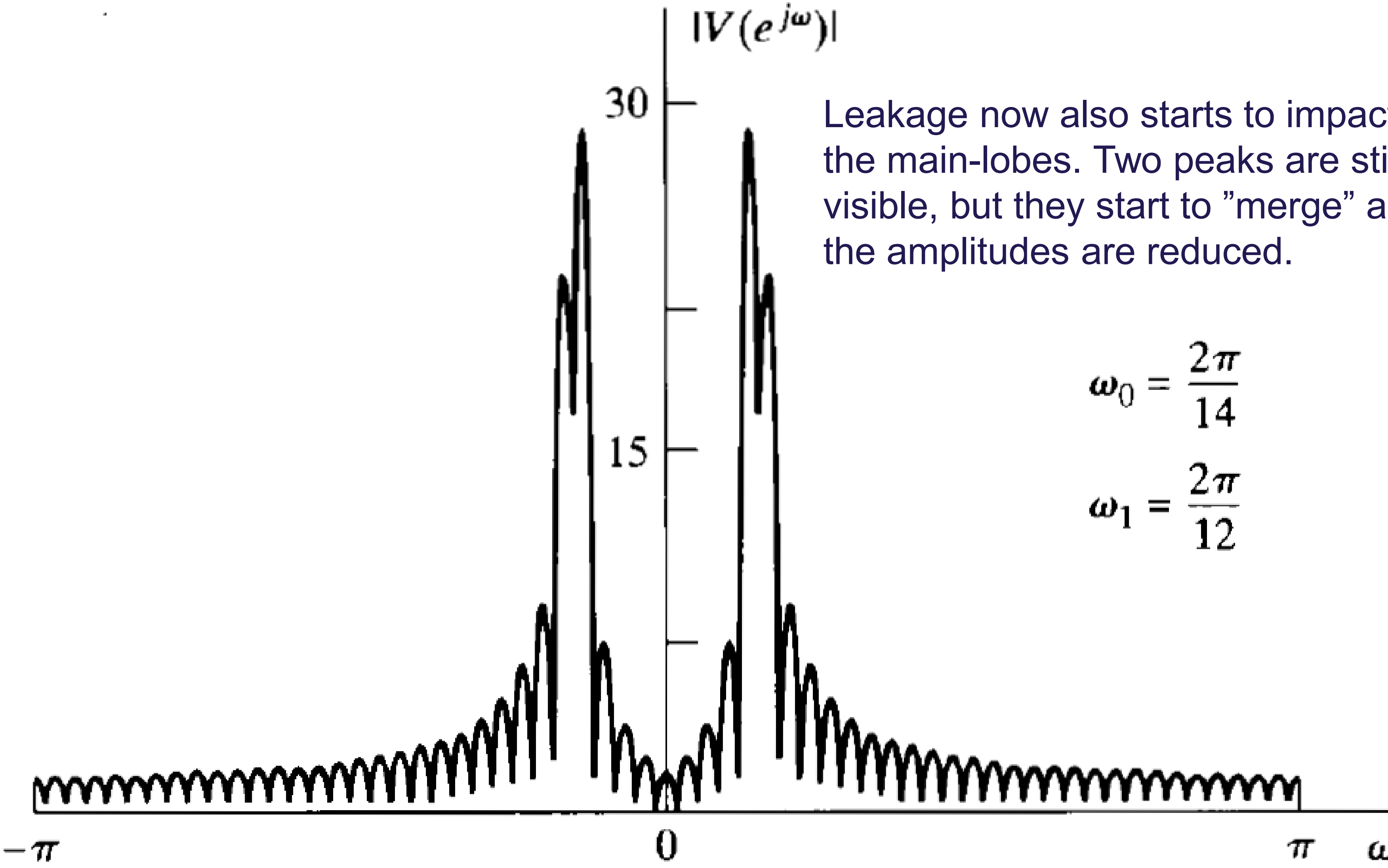
Note that the side lobes are down approx. 13 dB – just as expected…

Now, let's try to move the two cosine signals a bit closer to eachother



Note how the side-lobes start to cancel eachother.
This effect is called LEAKAGE
- the one freq. composant starts to leak into its neighbours

…and even closer



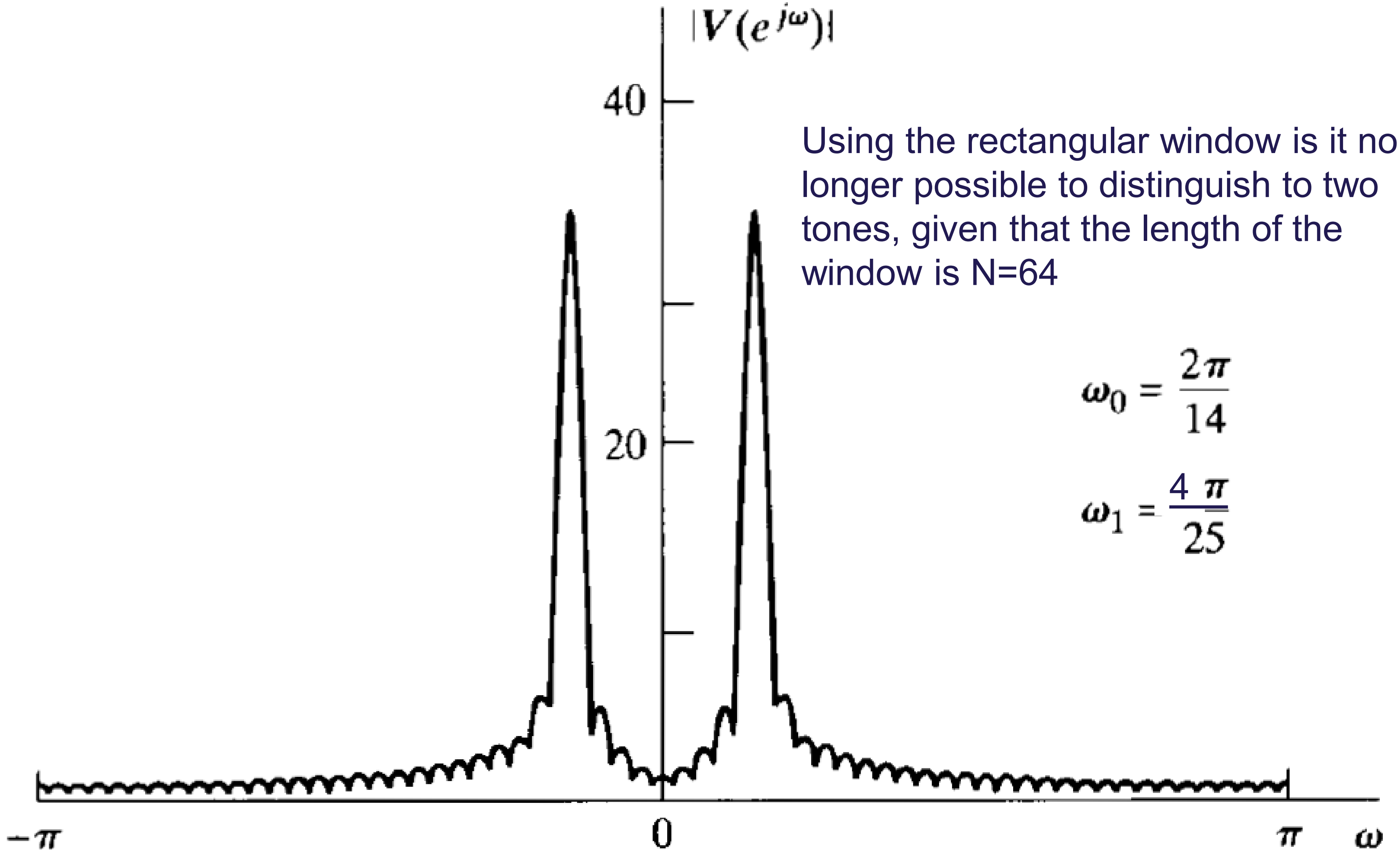
(d)

Leakage now also starts to impact the main-lobes. Two peaks are still visible, but they start to ”merge” and the amplitudes are reduced.

$$\omega_0 = \frac{2\pi}{14}$$

$$\omega_1 = \frac{2\pi}{12}$$

…and finally very close together.



(e)

Using the rectangular window is it no longer possible to distinguish to two tones, given that the length of the window is N=64

$$\omega_0 = \frac{2\pi}{14}$$

$$\omega_1 = \frac{4\pi}{25}$$

Some important considerations

The <u>effective</u> frequency resolution depends on the window's main-lobe width and thus the window length.

The leakage depends on the ratio between main-lobe amplitude and side-lobe amplitudes.

The rectangular window gives the highest possible frequency resolution but also has the largest side-lobes.

The DFT is a "spectral sampling" of the DTFT
What happens if we don't sample in the right frequencies..??

Discrete time frequencies

$$\omega_k = 2\pi k / N , \quad k = 0, 1, \ldots N-1$$

corresponds to the continuous time frequencies:

$$\Omega_k = (2\pi k)/(NT), \quad k = 0, 1, \ldots N/2$$

Example of a sampled (2-tone) signal, truncated using a rectangular window with length 64:

$$v[n] = \begin{Bmatrix} 1\cos\left(\frac{2\pi}{14}n\right) + 0{,}75\cos\left(\frac{4\pi}{15}n\right), \; 0 \le n \le 63 \\ 0, \text{ otherwise} \end{Bmatrix}$$

The 2nd experíment shown on slide no. 17.

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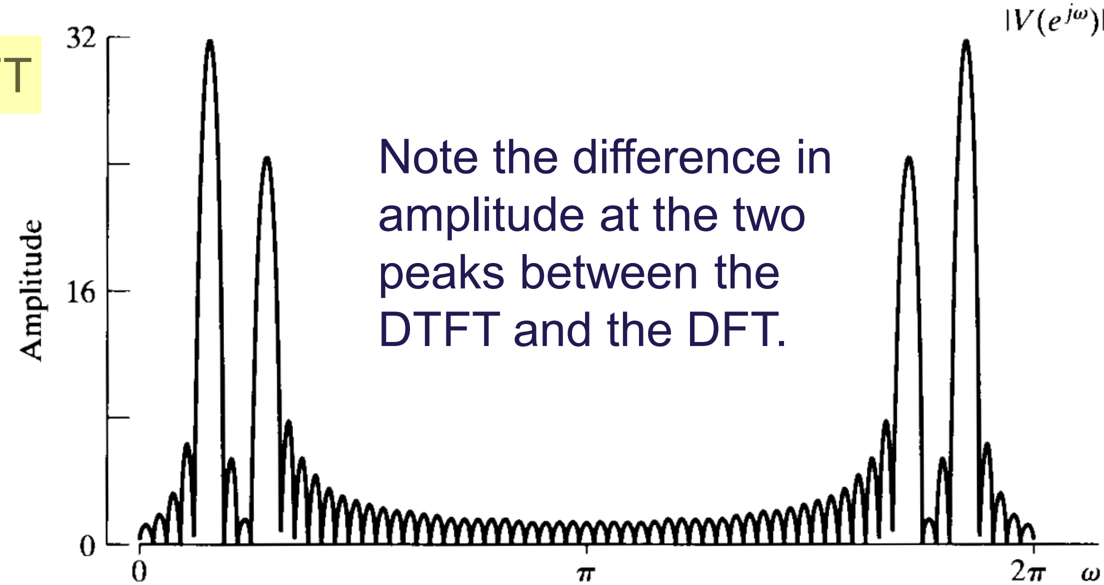
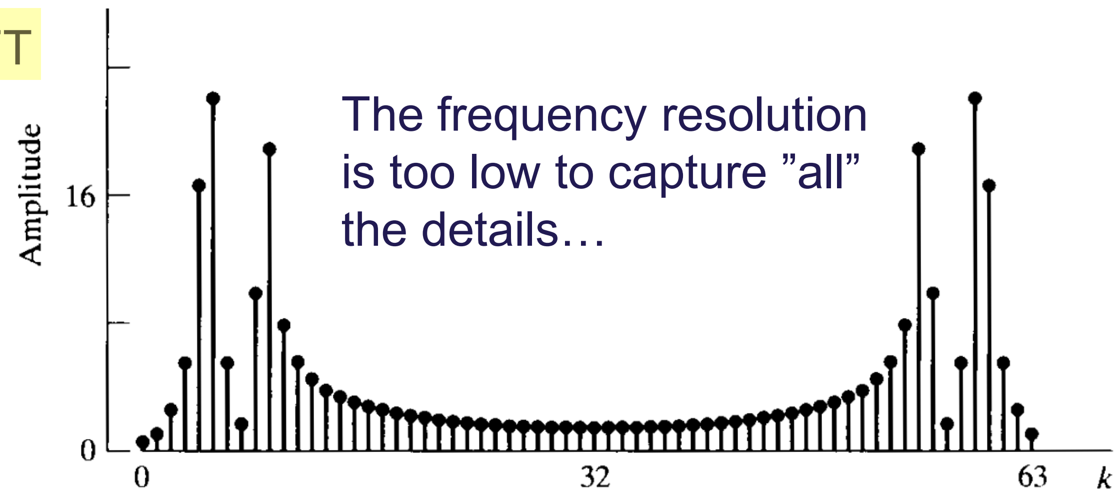
What we would like to have, and what we get…
DTFT vs. DFT if the frequencies don't match

DTFT

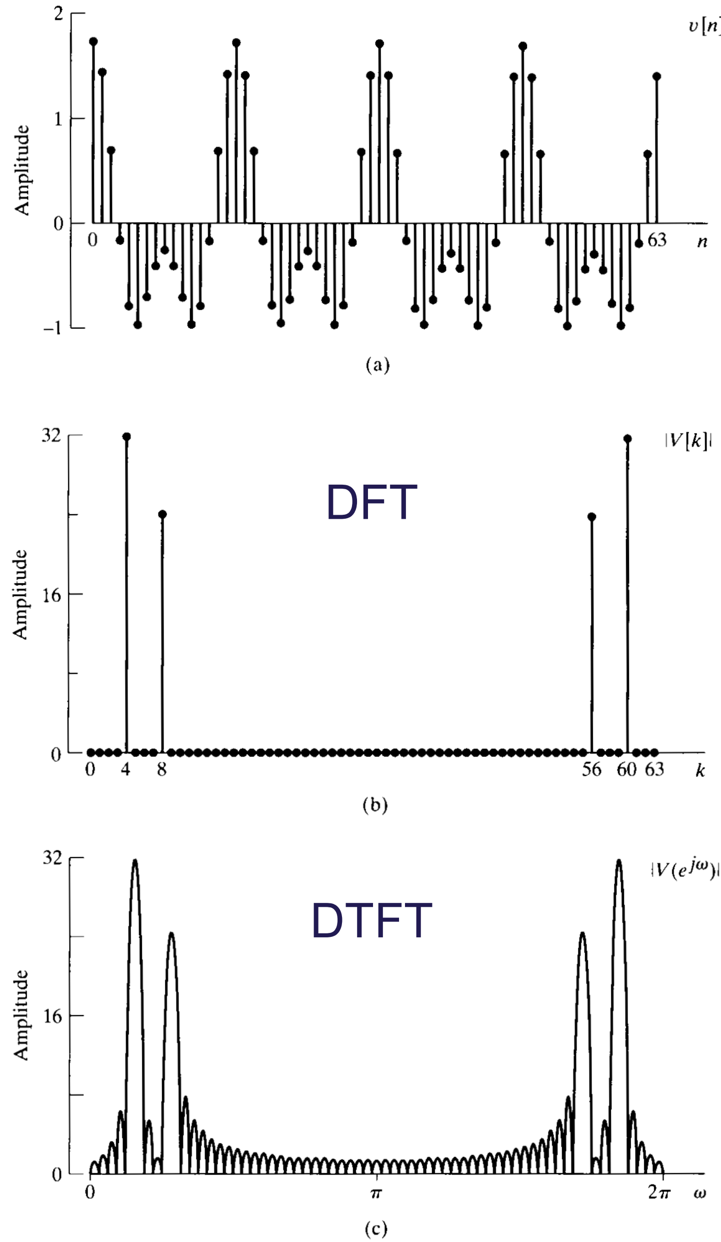


Note the difference in amplitude at the two peaks between the DTFT and the DFT.

DFT



The frequency resolution is too low to capture "all" the details…



Discrete Fourier analysis of the sum of two sinusoids for a case in which the Fourier transform is zero at all DFT frequencies except those corresponding to the frequencies of the two sinusoidal components. (a) Windowed signal. (b) Magnitude of DFT. (c) Magnitude of discrete-time Fourier transform ($|V(e^{j\omega})|$).

Now, we move the frequencies a bit…

$$v[n] = \begin{cases} \cos\left(\frac{2\pi}{16}n\right) + 0.75\cos\left(\frac{2\pi}{8}n\right), & 0 \le n \le 63, \\ 0, & \text{otherwise}, \end{cases}$$

For this particular two-tone sequence and the given window length, N=64, we have a perfect match, and thus we get a correct sampling of the two tones…

In any real-life situation, that is very unlikely…!

Furthermore, the ”perfect sampling” also now hides the impact from the window function.

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It’s now time for a…

…before we discuss the Short-Time Fourier Transform

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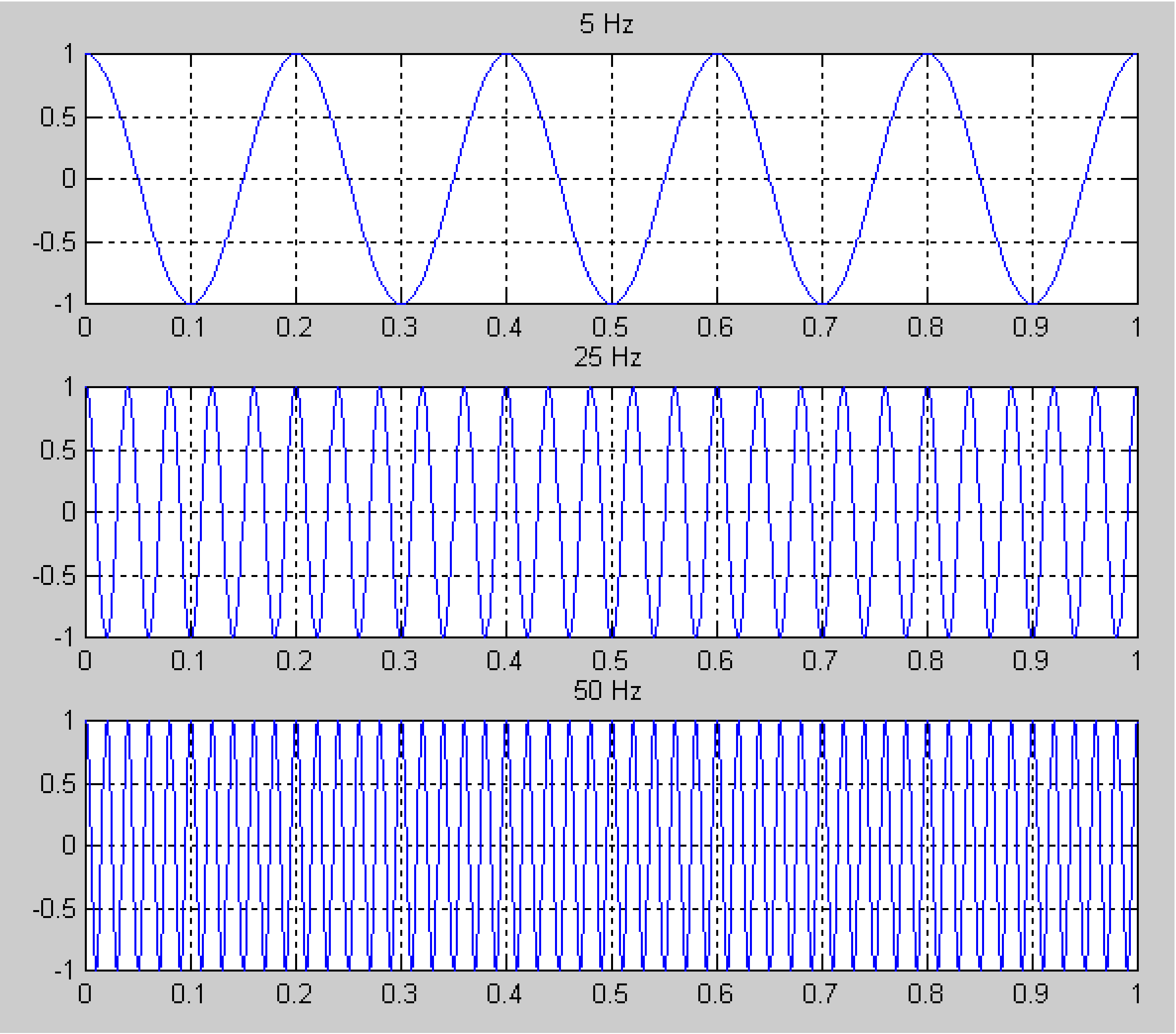
A motivating example – simple sinusoids

$$f_1(t) = \cos(2\pi \cdot 5 \cdot t)$$

$$f_2(t) = \cos(2\pi \cdot 25 \cdot t)$$

$$f_3(t) = \cos(2\pi \cdot 50 \cdot t)$$

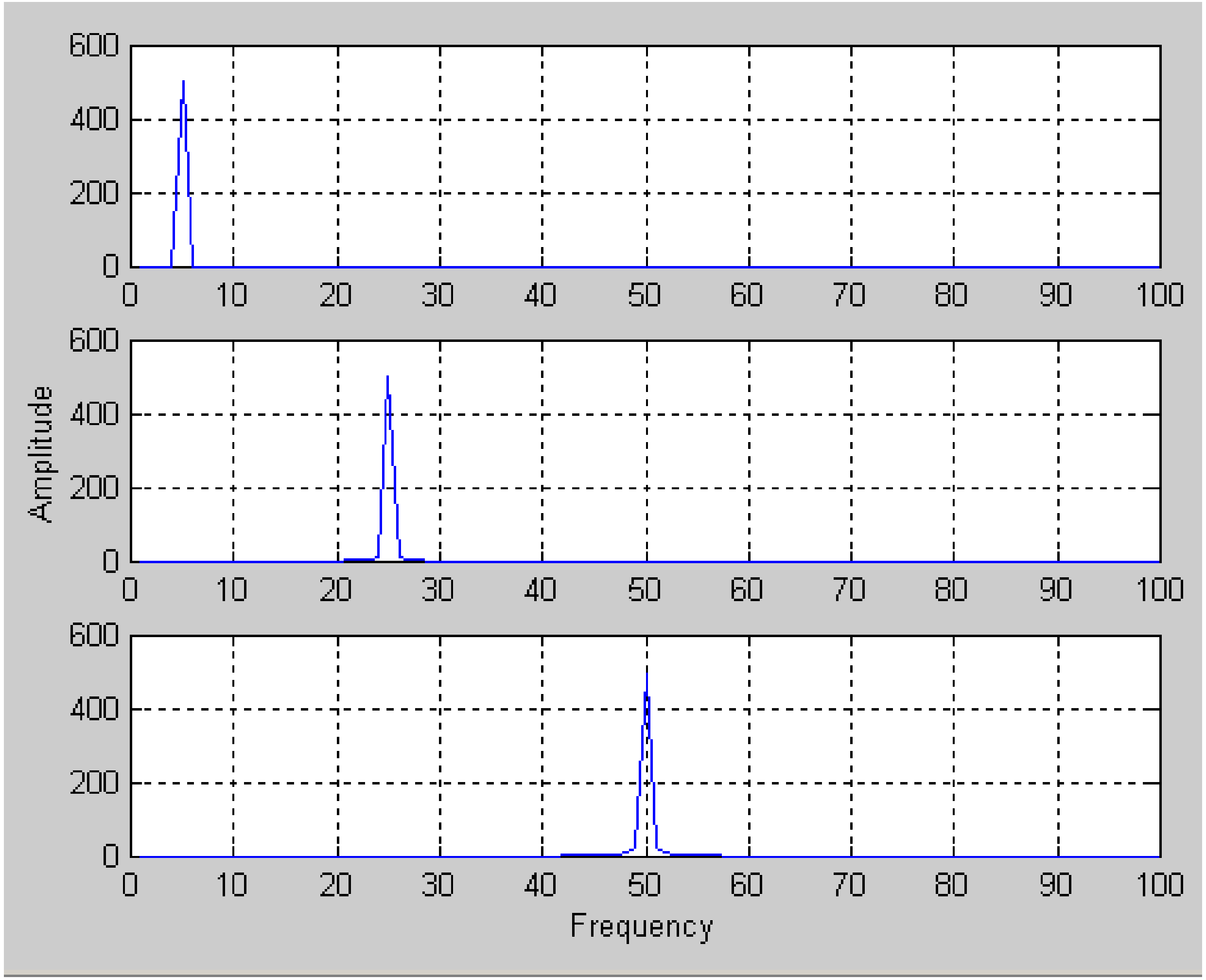


The Amplitude Responses

$F_1(\Omega)$

$F_2(\Omega)$

$F_3(\Omega)$



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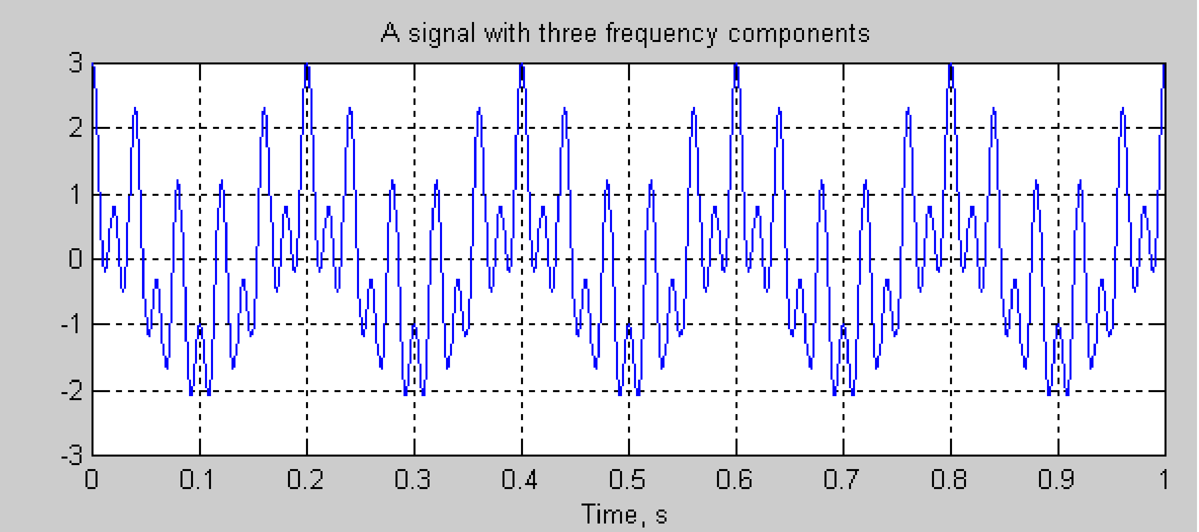
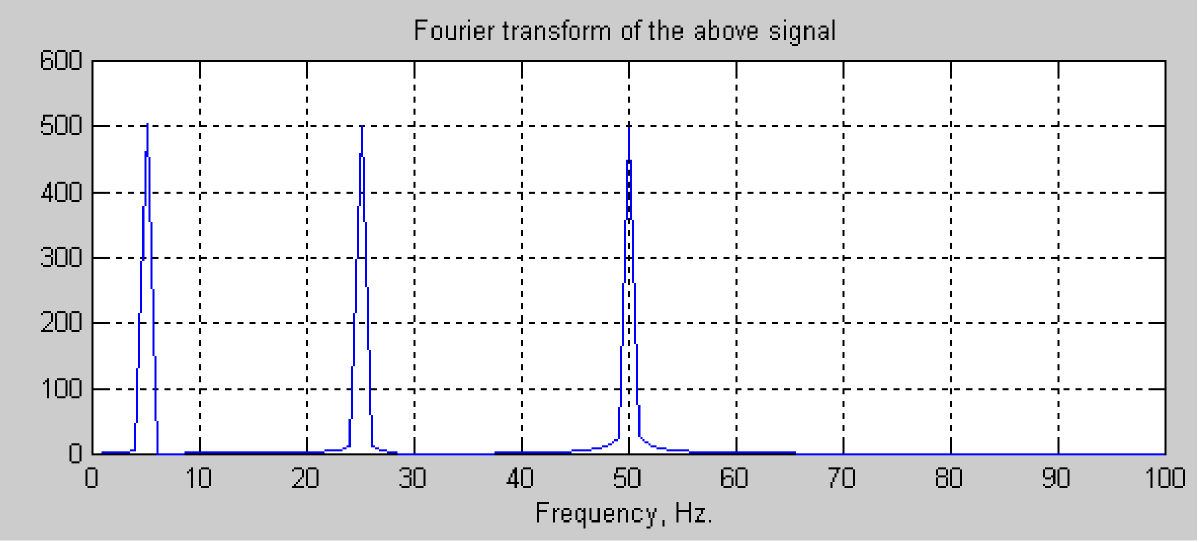
Now, let's add the three sinusoids

$$f_4(t) = \cos(2\pi \cdot 5 \cdot t) + \cos(2\pi \cdot 25 \cdot t) + \cos(2\pi \cdot 50 \cdot t)$$



$F_4(\Omega)$



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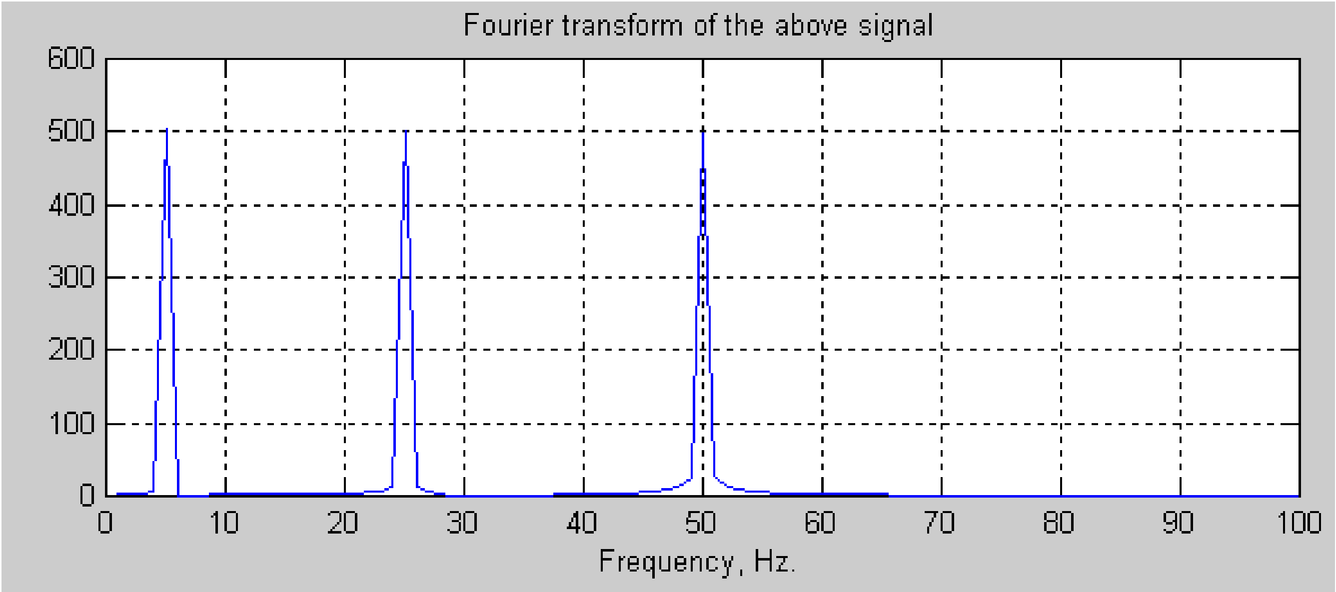
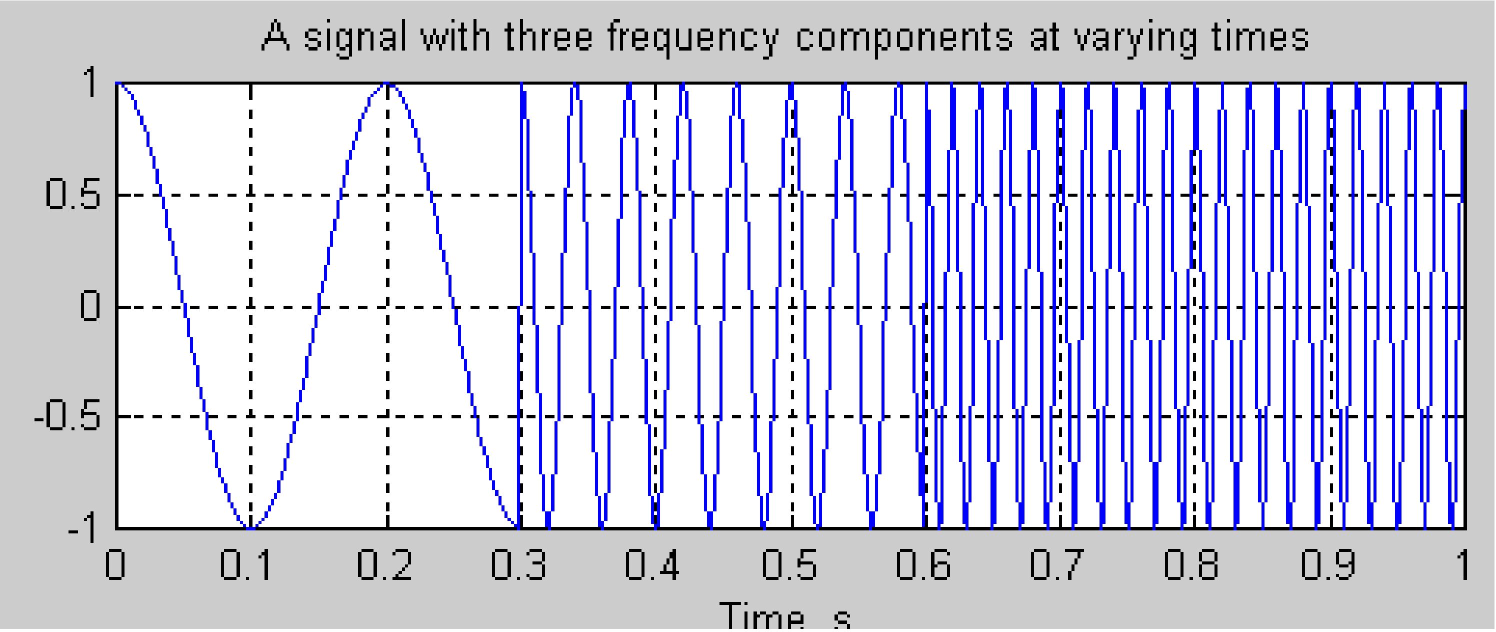
The sum of sinusoids is a Time-Invariant signal

The three frequency components, are present at all times!



No matter when you perform the Fourier Analysis, you get the same result – the spectra is therefore also time-invariant

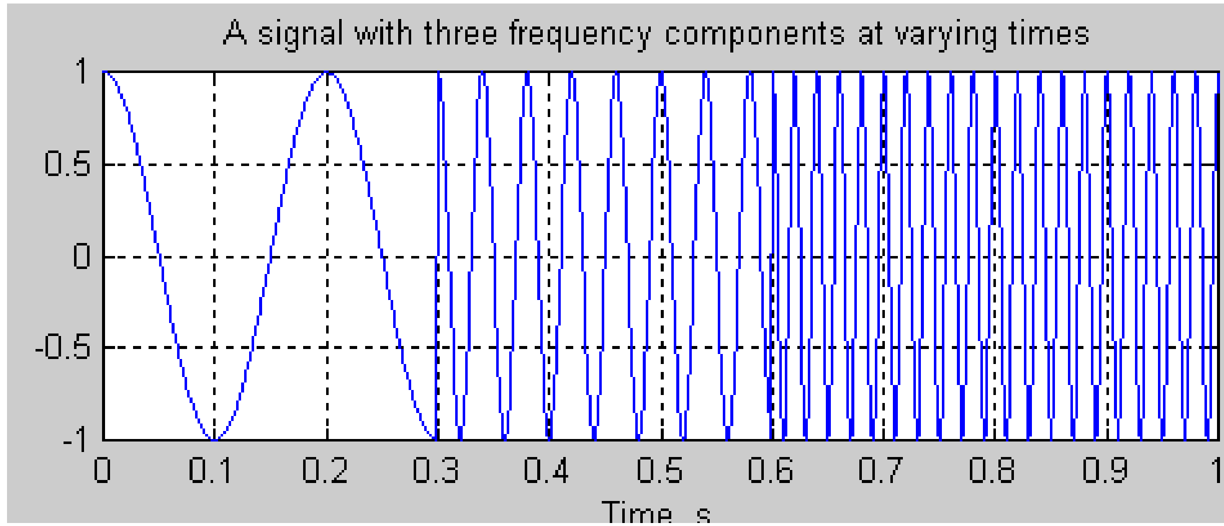
Now, let's append the three sinusoids such that they occur distinctive in time



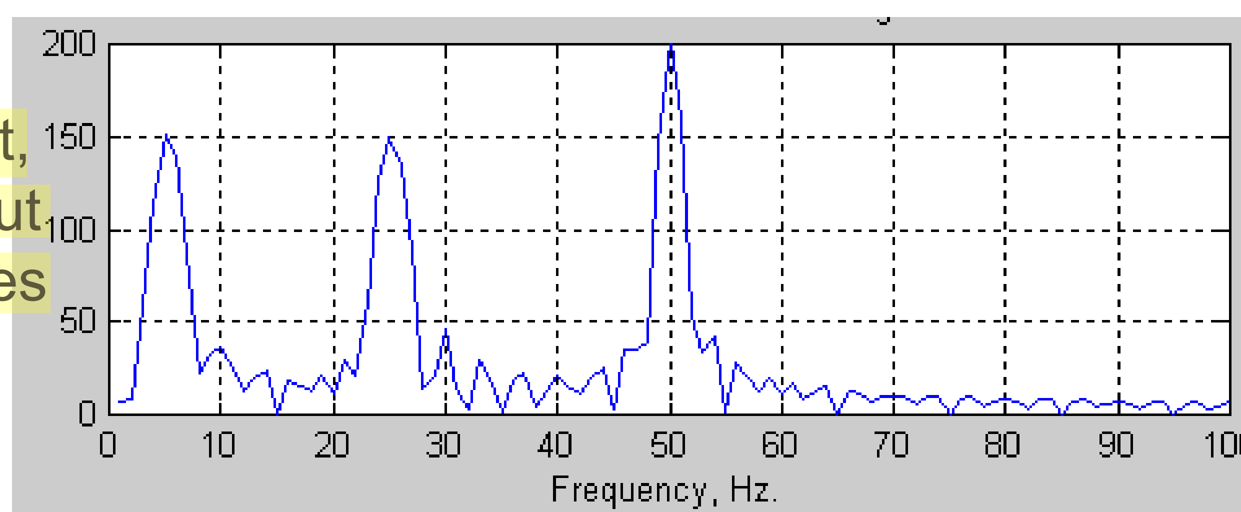
Depending on WHEN you perform the Fourier Analysis, you will see different results…

Spectral analysis of Time-varying signals

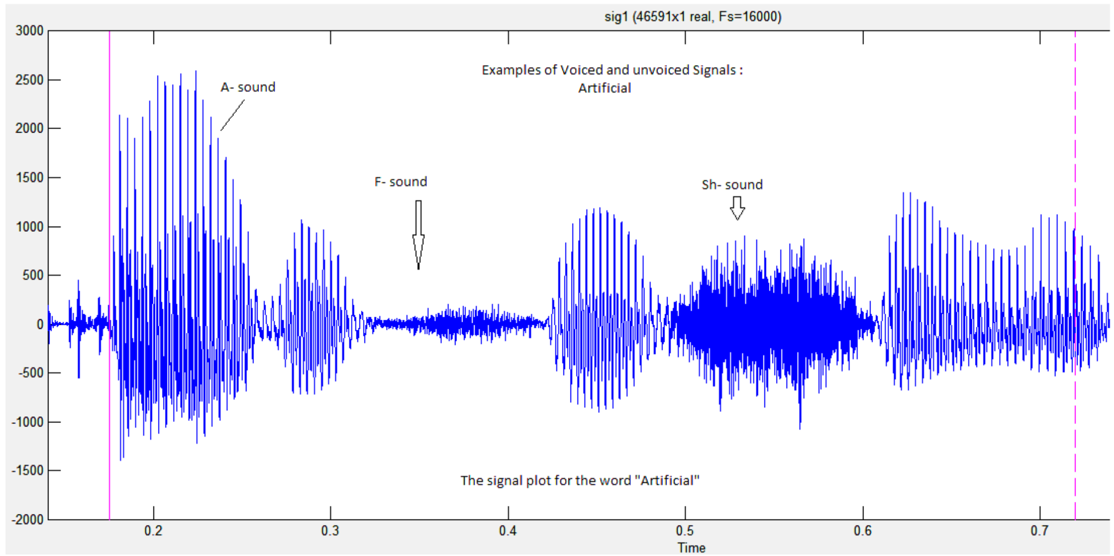
The three frequency components are NOT present at all times!

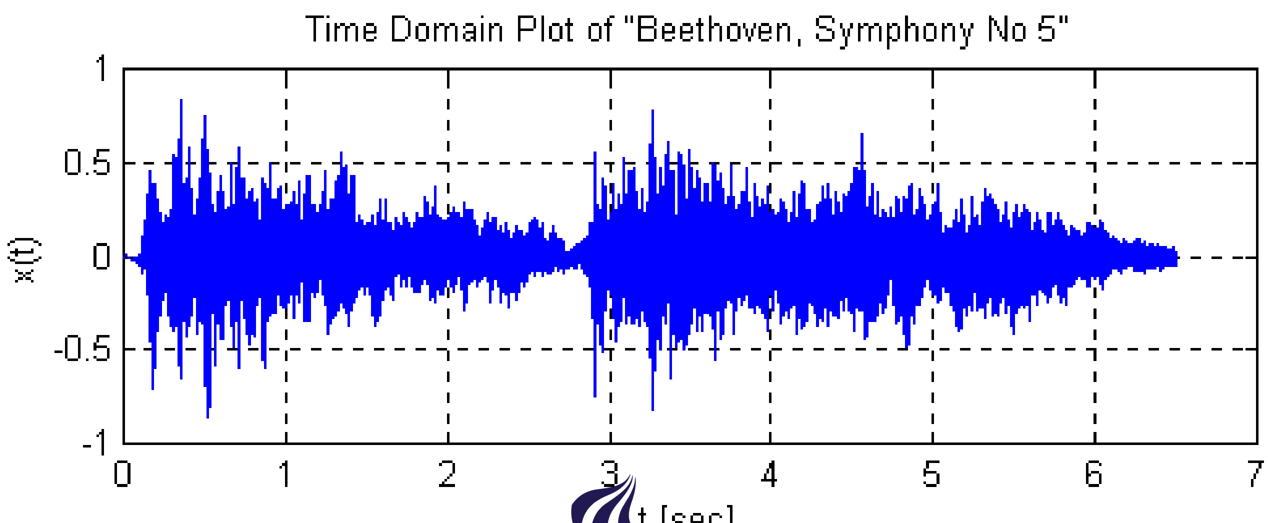


Perfect knowledge of which frequencies exist, but no information about where these frequencies are located in time!



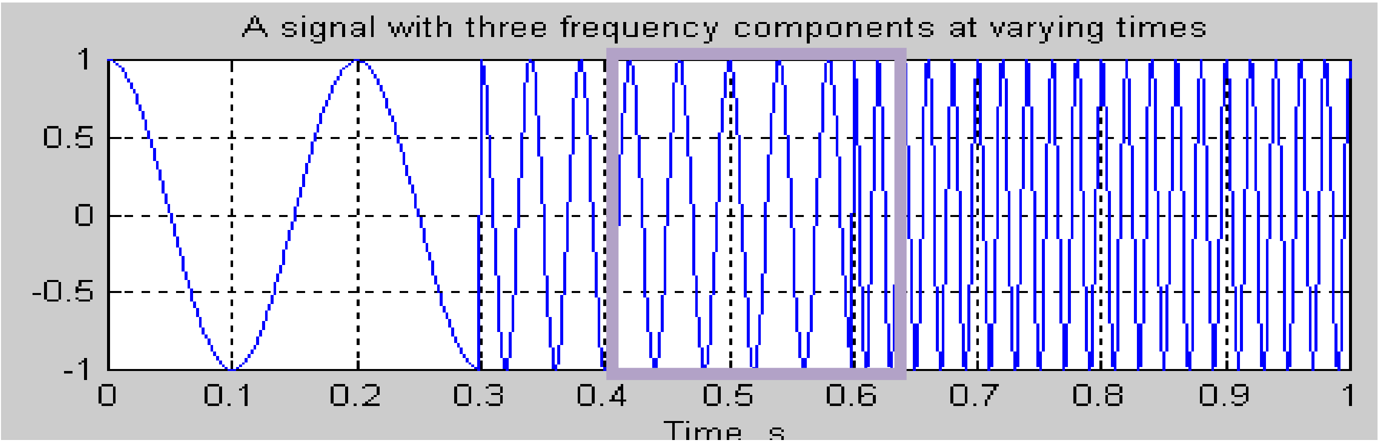
This is the normal situation for real-life signals…





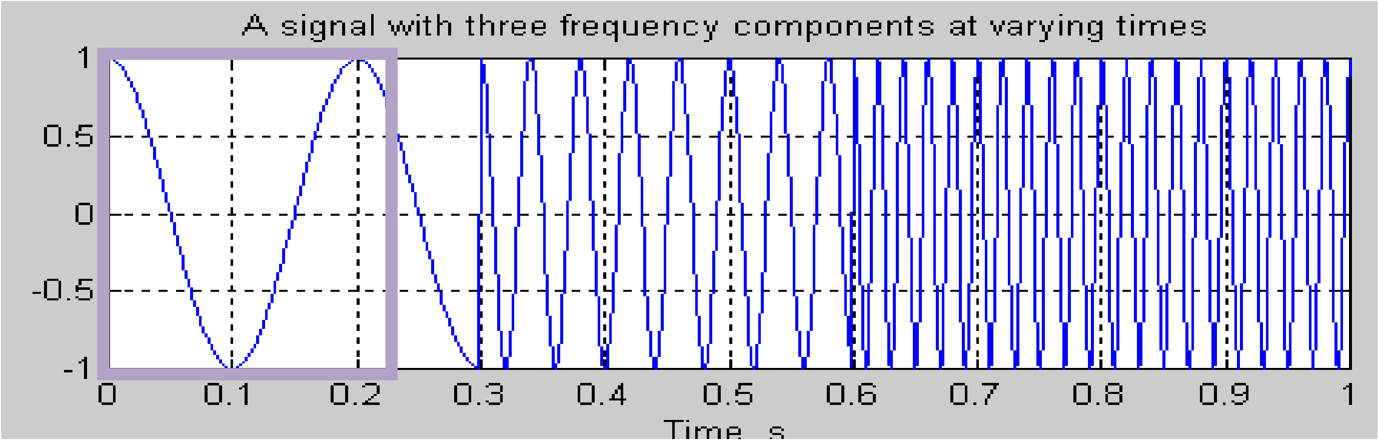
Therefore, we need (yet) another Fourier Transform

- Segment the signal into short time intervals (i.e., short enough for the signal to be considered time-invariant) and take the FT of each segment.
- Each FT provides the spectral information of a separate time-slice of the signal, providing simultaneous time and frequency information.

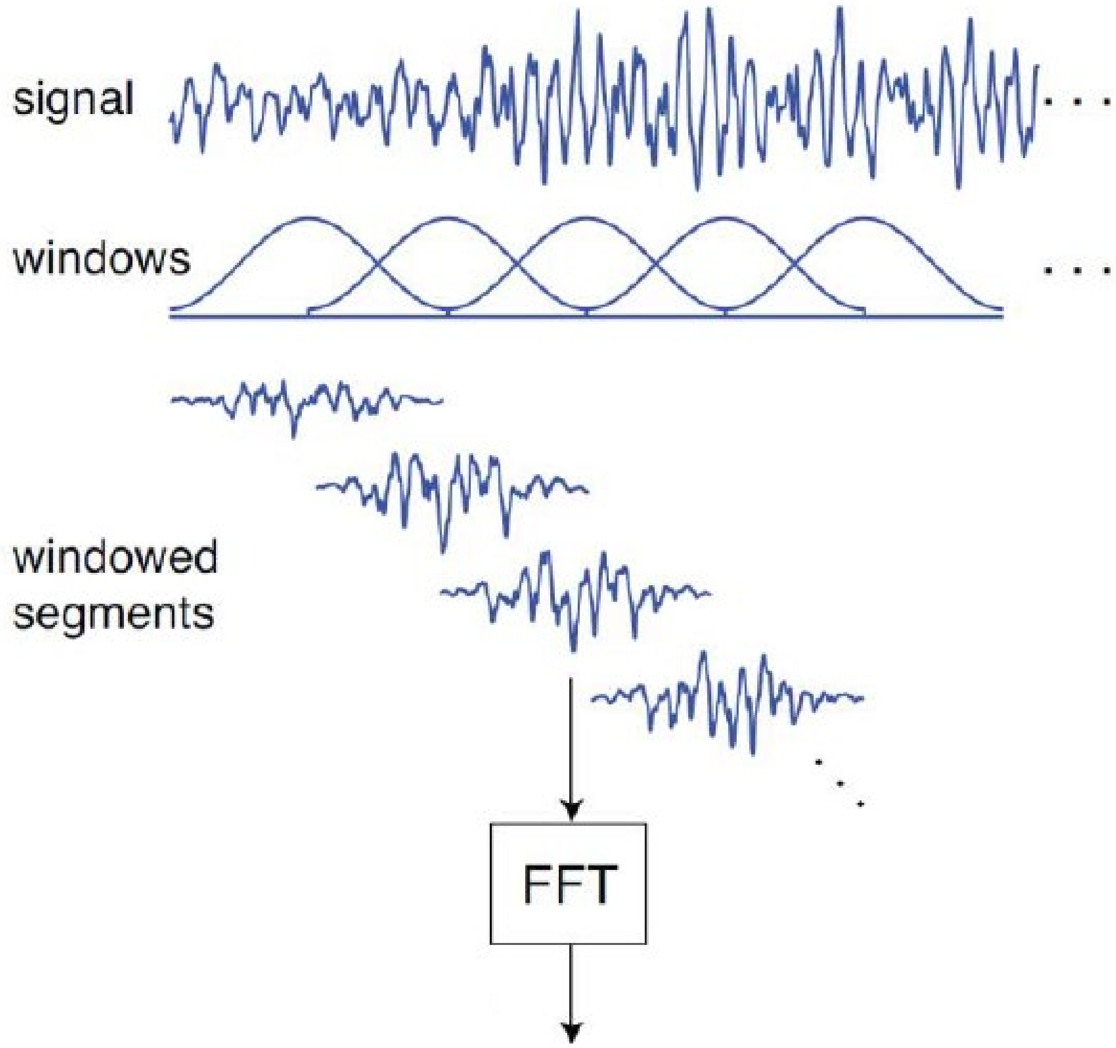


Short-Time Fourier Transform, STFT

(1) Choose a window function of finite length
(2) Place the window on top of the signal at t=0
(3) Truncate the signal using this window
(4) Compute the FT of the truncated signal, save results.
(5) Incrementally slide the window to the right
(6) Go to step 3, until window reaches the end of the signal



The Short-Time Fourier Transform



Definition of STFT

2D function…!!!!

Time parameter

Frequency parameter

Signal to be analyzed

The usual complex exponential function

$$X[n, \lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}$$

STFT of *x*[*n*] computed for each segment starting at time n

Windowing function

Located at *n=0*

$0 \leq \lambda < 2\pi$ or any other interval of length 2π

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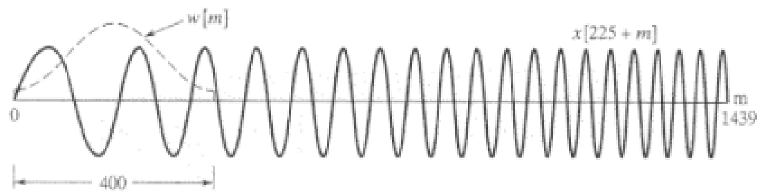
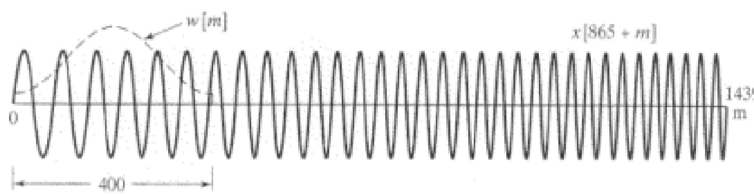
An example - a linear frequency sweep

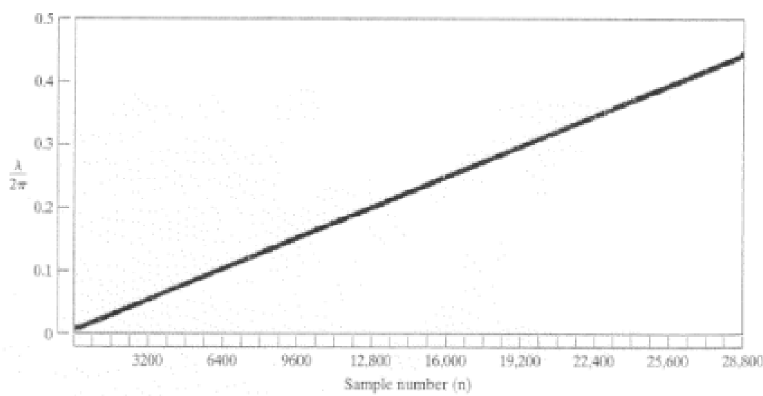
$$x[n]=\cos(\omega_0 n^2),\ \omega_0=2\pi\cdot 7.5\cdot 10^{-6}$$

Time:

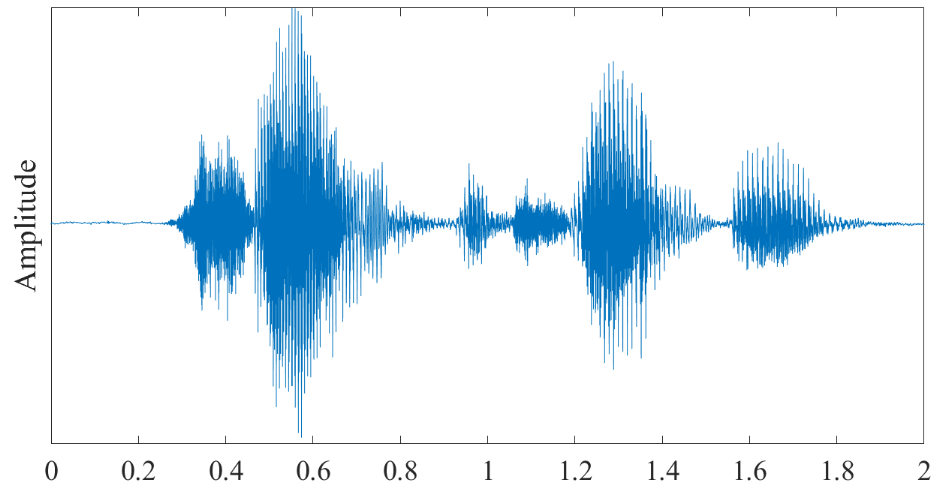


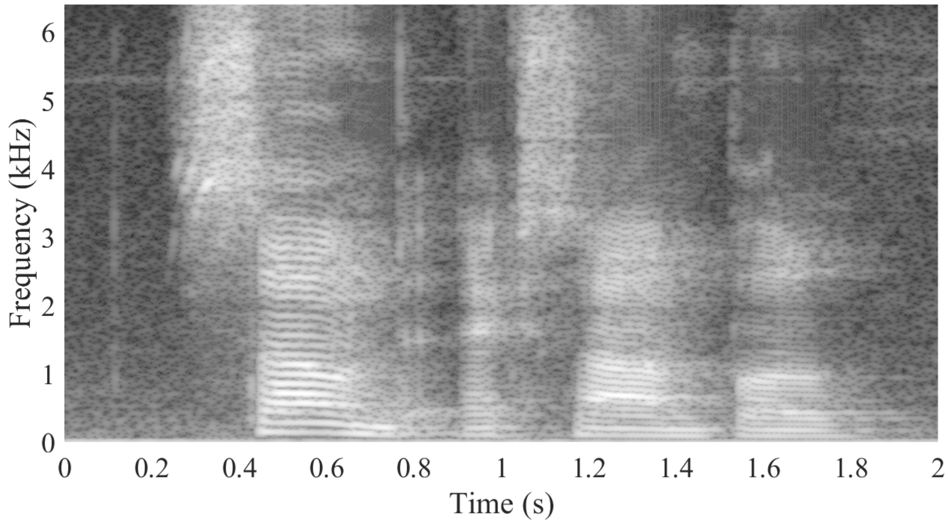


Time/
frequency:



The Spectrogram





Inverse STFT

$$x[n]=\frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$ (the ordinary synthesis formula)

$$x[n+m]w[m]=\frac{1}{2\pi}\int_0^{2\pi} X[n,\lambda)e^{j\lambda m}d\lambda \qquad -\infty < m < \infty$$

i.e. for $m=0$: $$x[n]=\frac{1}{2\pi w[0]}\int_0^{2\pi} X[n,\lambda)d\lambda$$

In conclusion, all samples can be re-constructed as long as they are not multiplied by a zero in the window function.

STFT – an alternative interpretation (not part of the curriculum)

$$X[n,\lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}$$

we make the substitution $m' = n + m$

$$X[n,\lambda) = \sum_{m'=-\infty}^{\infty} x[m']w[-(n-m')]e^{j\lambda(n-m')}.$$

can be interpreted as the convolution

$$X[n,\lambda) = x[n] * h_\lambda[n], \tag{10.23a}$$

where

$$h_\lambda[n] = w[-n]e^{j\lambda n}. \tag{10.23b}$$

we see that the time-dependent Fourier transform as a function of n with λ fixed can be interpreted as the output of a linear time-invariant filter with impulse response $h_\lambda[n]$ or, equivalently, with frequency response

$$H_\lambda(e^{j\omega}) = W(e^{j(\lambda-\omega)}). \tag{10.24}$$

Thus, can be considered as a set of parallel bandpss filters with different center frequencies λ…

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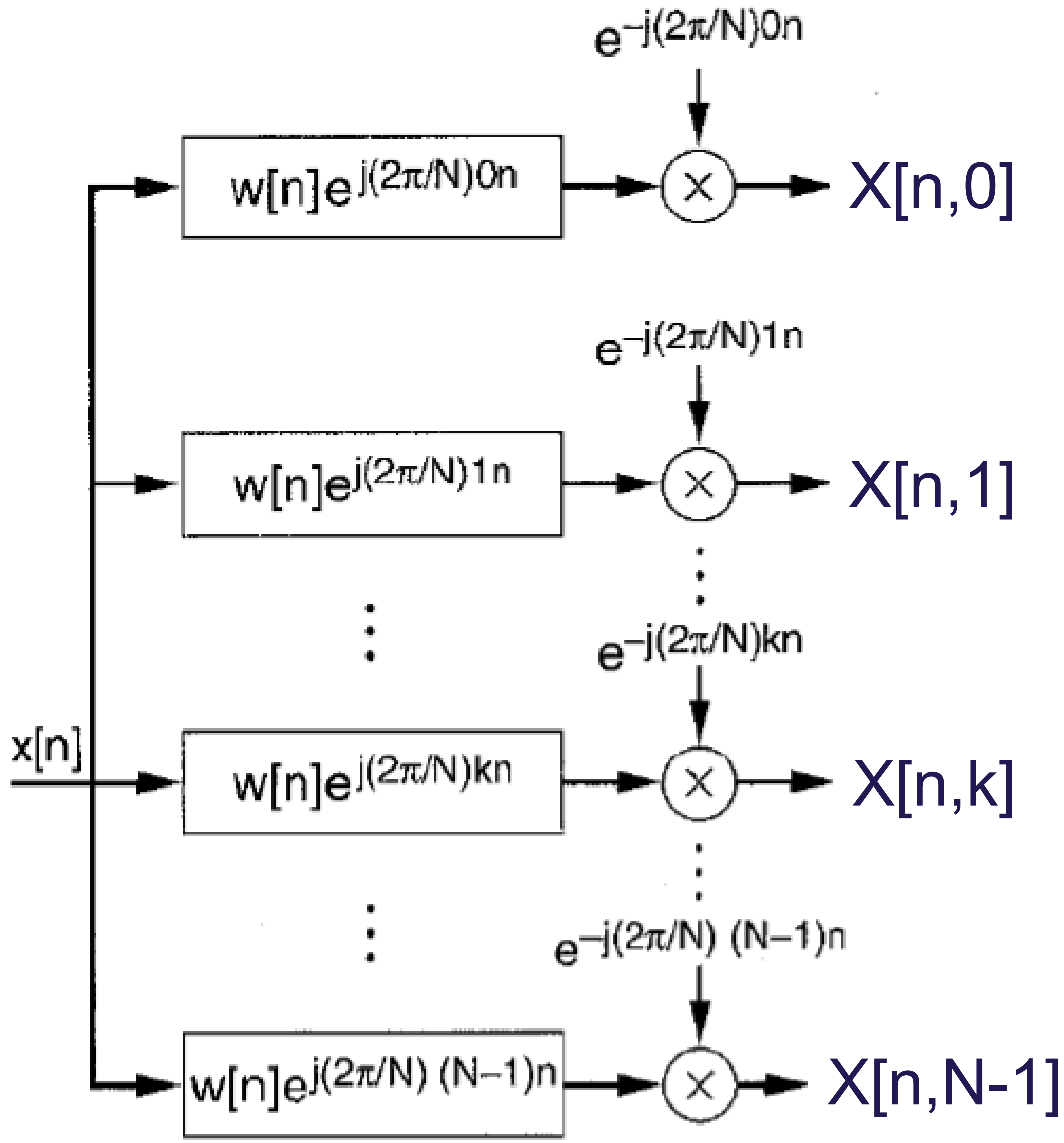
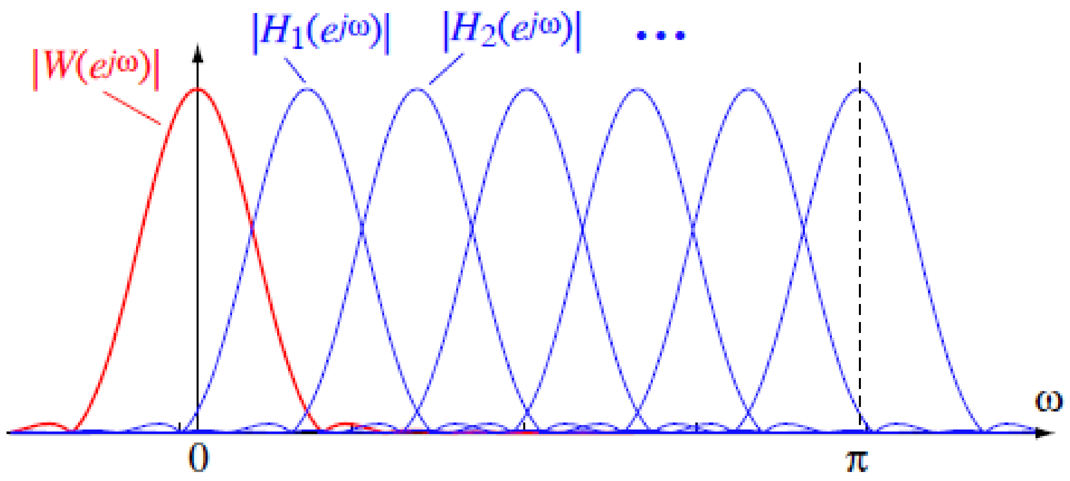
The STFT – the Filter Bank interpretation



See O&S for a detailed explanation
of the math associated with this figure…

STFT – The Filter Bank interpretation

…and here is a graphical interpretation



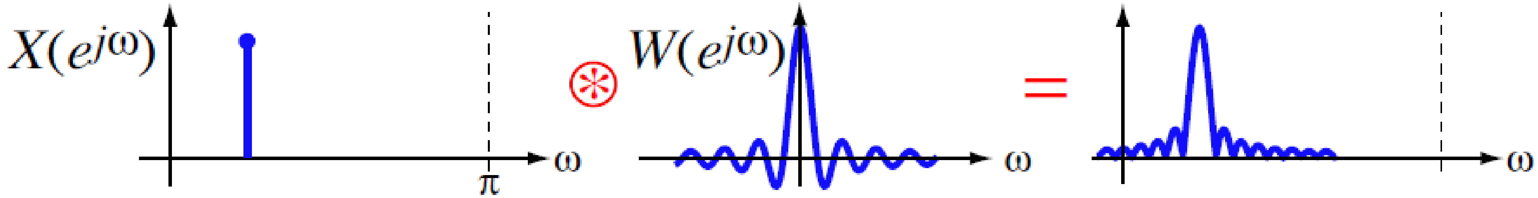
A bank of identical, frequency-shifted bandpass filters: ***"filterbank"***

How to choose an appropriate window - function and length

Choosing an appropriate window is not necessarily an easy and straight forward task…

This is exactly the same problem as we were facing when designing FIR filters using the window method…

- e.g. if $x[n]$ is a pure sinusoid,



Window effect due to width of main lobe -> blurring
Window effect due to non-zero side lobes -> leakage

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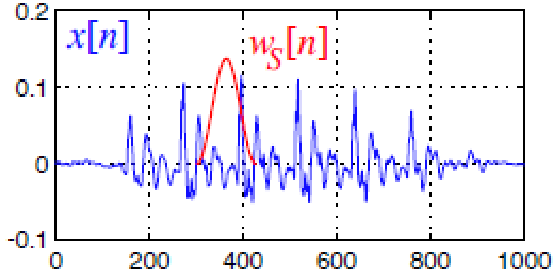
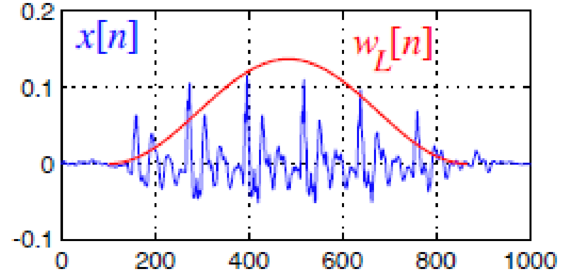
Choosing a window

- Length of $w[n]$ sets temporal resolution

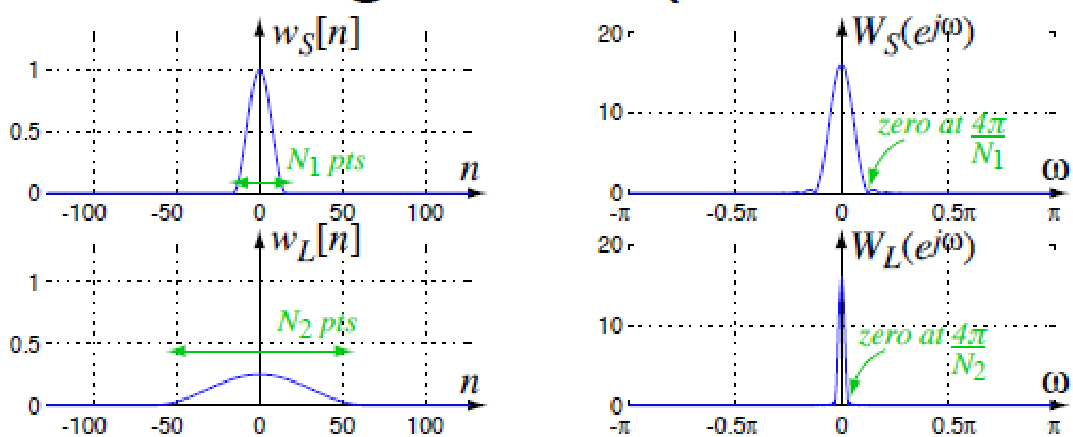




short window measures only local properties

longer window averages spectral character

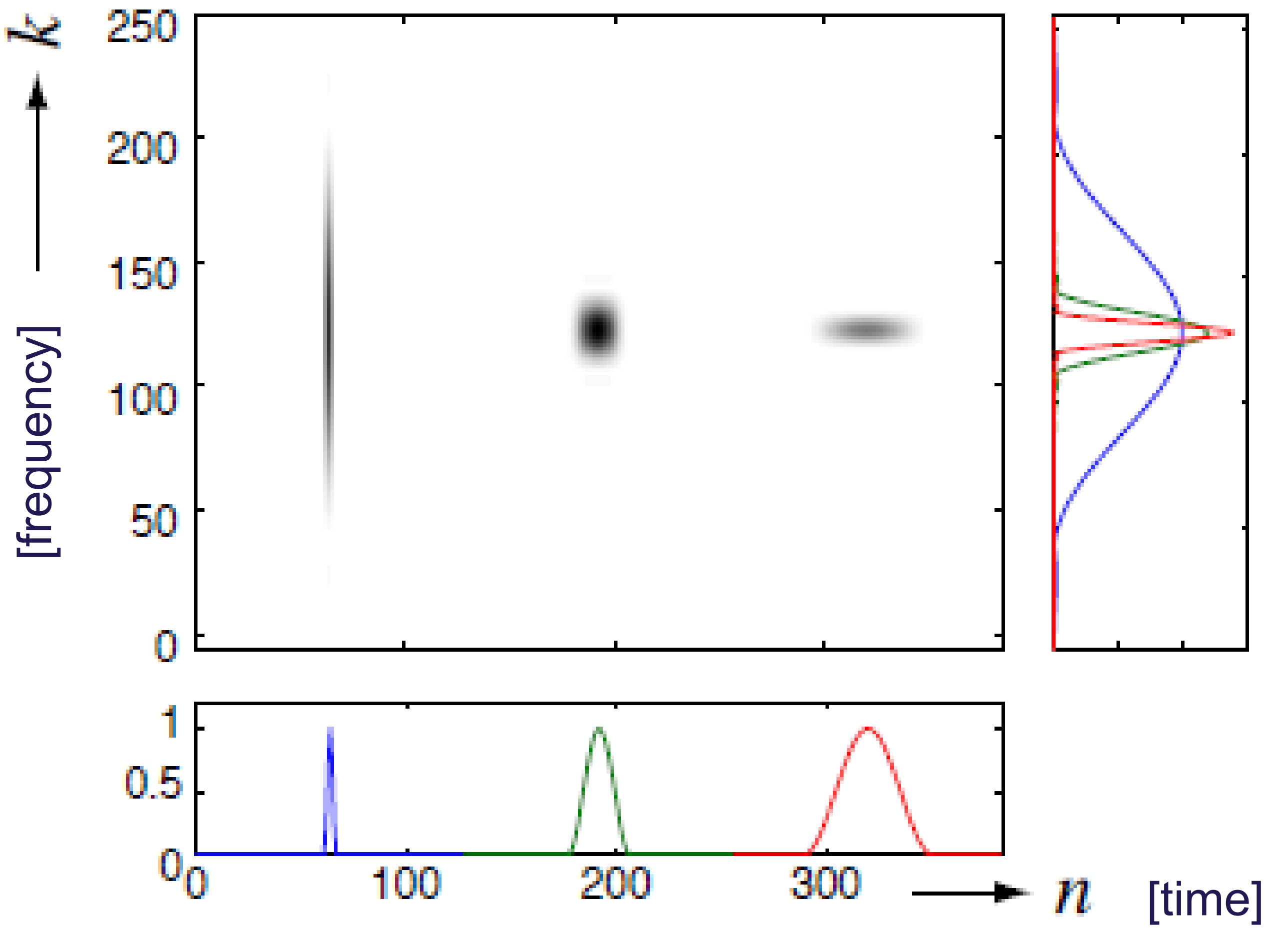
- Window length $\propto$ 1/(Mainlobe width)



shorter window → more blurred spectrum

- *more* time detail ↔ *less* frequency detail

Choosing a window
- here we have three windows with different length



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Choosing a window

- Window should be narrow enough to ensure that the portion of the signal falling within the window is time-invariant.

- But … very narrow windows do not offer good localization in the frequency domain.

Wide window → good frequency resolution, poor time resolution.

Narrow window → good time resolution, poor frequency resolution.

Heisenberg's Uncertainty Principle

$$\Delta t \cdot \Delta f \geq \frac{1}{4\pi}$$

Time resolution:
How well two spikes in time can be separated from each other in the frequency domain.

Frequency resolution:
How well two spectral components can be separated from each other in the time domain

$$\Delta t \; and \; \Delta f \; cannot \; be \; made \; arbitrarily \; small!$$

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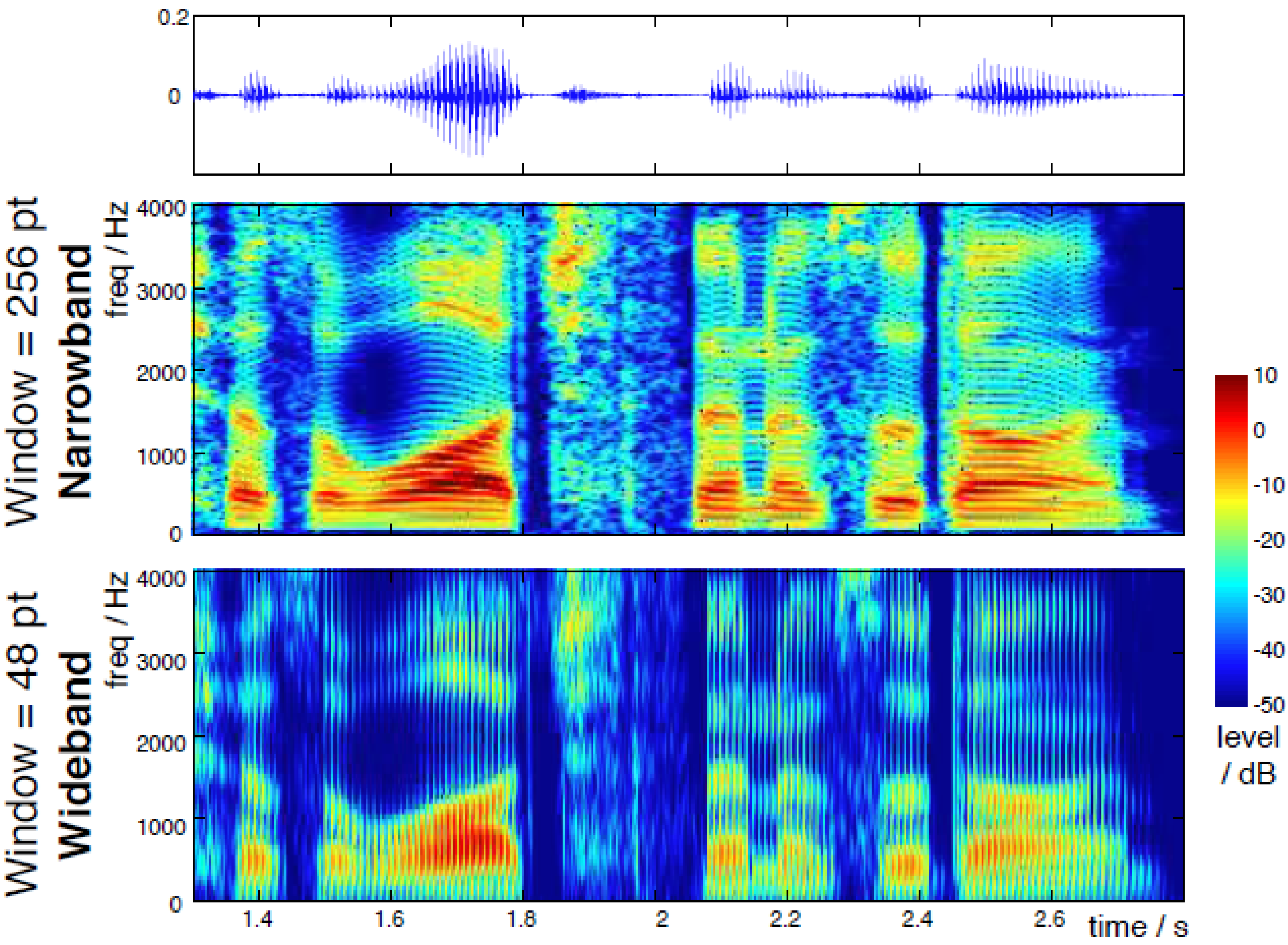
Narrowband vs. Wideband Spectral Analysis



Narrowband vs. Wideband Spectral Analysis

- For a long window $w[n]$, the result is the narrowband spectrogram, which exhibits the harmonic structure in the form of horizontal striations
- For a short window $w[n]$, the result is the wideband spectrogram, which exhibits periodic temporal structure in the form of vertical striations