Signalbehandling for computer-ingeniører

COMTEK-5, E20

&

Signalbehandling

EIT-5, E20

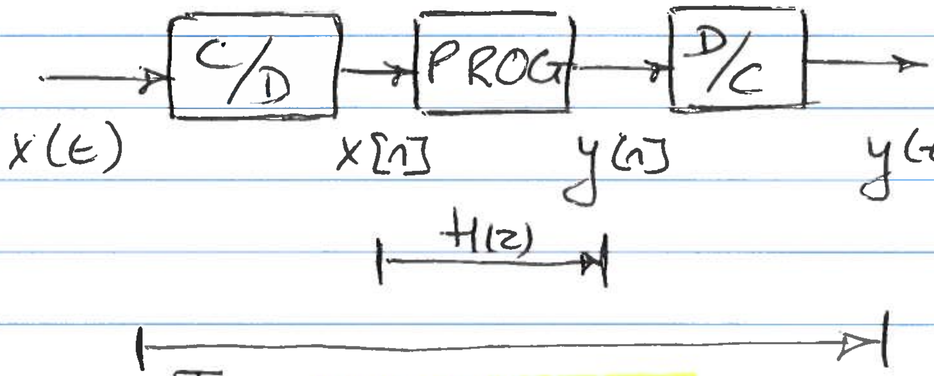
8. Digital FIR Filters – Linear Phase and The Window Method

Assoc. Prof. Peter Koch, AAU

8^{th} LECTURE

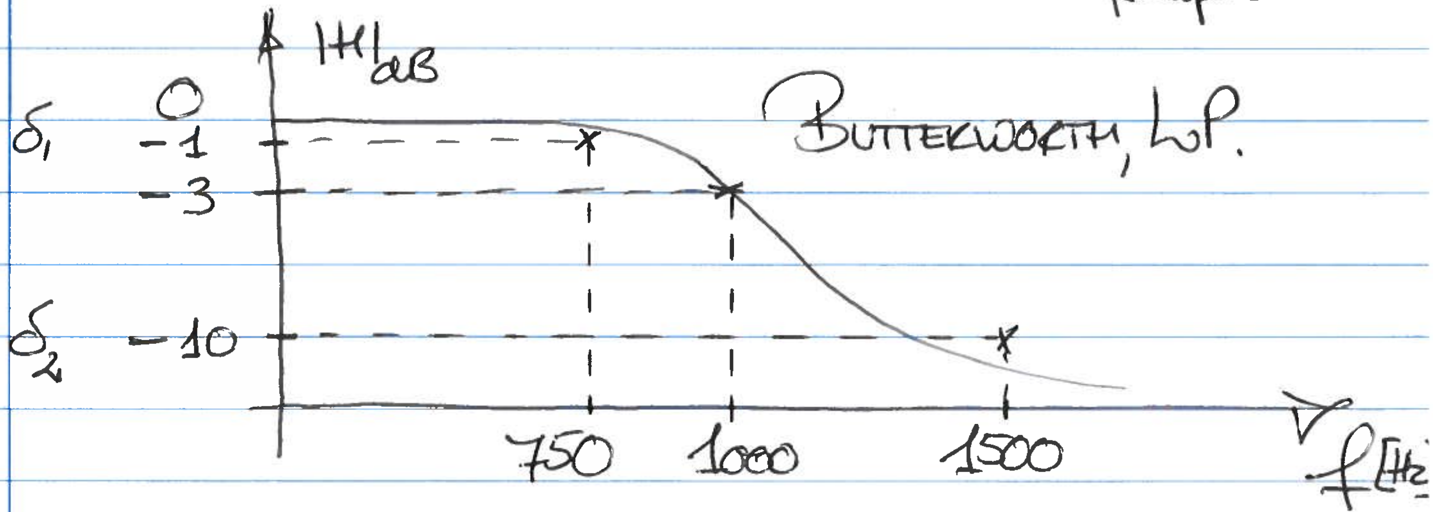
LET'S START WITH A SHORT REVIEW OF LECTURE 7.

IN THE EXERCISE WE WERE ASKED TO APPLY THE BILINEAR TRANSFORMATION ON A SET OF SPECIFICATIONS FOR A DIGITAL FILTER



THE EFFECTIVE FILTER FOR WHICH THE SPEC'S APPLY

THUS, WHEN WE "LOOK INTO" THE C/D WE SHOULD SEE A FILTER RUNNING WITH $f_{sample} = 8kHz$



$2\pi 750 \quad 2\pi\cdot 1000 \quad 2\pi 1500 \qquad \Omega \ [rad/s]$

$\frac{3\cdot\pi}{16} \quad \frac{\pi}{4} \quad \frac{3\pi}{8} \qquad \omega \ [rad]$

$$\boxed{\omega = T\cdot\Omega}$$

FIRST WE NEED TO FIND THE NECESSARY ORDER FOR THE ANALOG PROTOTYPE FILTER.

$$\delta_1 = -1\,dB \Rightarrow \delta_1 = 0.89125$$
$$\delta_2 = -10\,dB \Rightarrow \delta_2 = 0.31623$$

$$\text{Given } |H_c(j\Omega)|^2 = \frac{1}{1+(j\Omega/j\Omega_c)^{2N}}$$

WE CAN NOW DETERMINE „THE STRONGEST" REQUIREMENT;

$$\delta_1^2 \leq \frac{1}{1+(\Omega_1/\Omega_c)^{2N_1}} \quad \text{and} \quad \delta_2^2 \geq \frac{1}{1+(\Omega_2/\Omega_c)^{2N_2}}$$

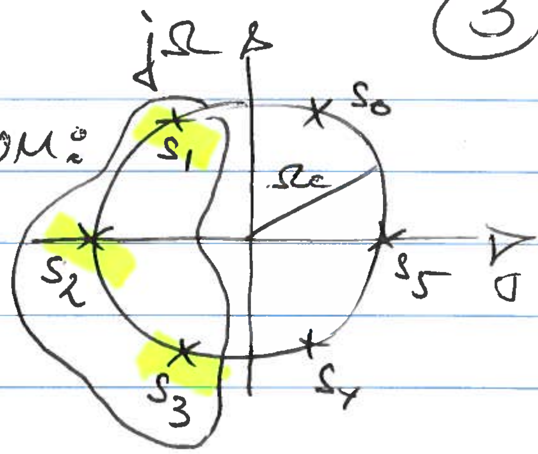
SINCE WE AFTERWARD HAS TO DO $H(s) \rightarrow H(z)$ USING THE BILINEAR TRANSFORMATION, WE MUST APPLY THE PRE-WARPED VERSIONS OF Ω_1, Ω_2 AND Ω_c

$$\boxed{\Omega' = \frac{2}{T} \cdot \tan\left(\frac{\omega}{2}\right)}$$

$\Downarrow$

$$N = 3$$

NEXT WE FIND $H(s)$ FROM:



$$s_1 = \Omega_c\left(-\frac{1}{2} + j\frac{1}{2}\sqrt{3}\right)$$

$$s_2 = -\Omega_c$$

$$s_3 = \Omega_c\left(-\frac{1}{2} - j\frac{1}{2}\sqrt{3}\right)$$

$$H(s) = \frac{G_T}{\prod_{k=1}^{3}(s - s_k)}$$

0dB DC GAIN

$$|H(s)| = 1\Big|_{s=0} \Rightarrow G = \Omega_c^3$$

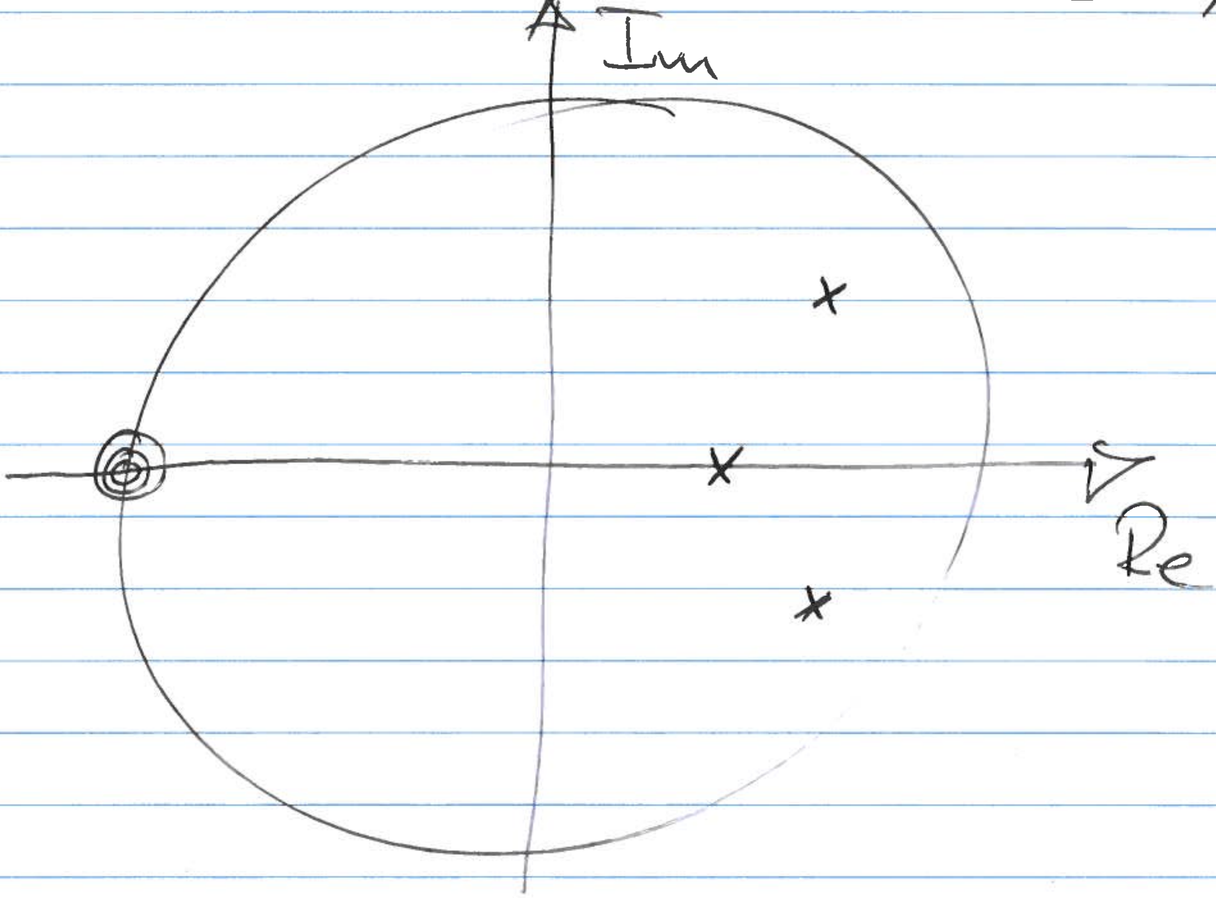
$$\left\{\begin{array}{l} H(s) = \dfrac{\Omega_c^3}{(s+\Omega_c)(s^2+\Omega_c s+\Omega_c^2)} \\[2em] H(z) = H(s)\Big|_{s=\frac{2}{T}\cdot\frac{z-1}{z+1}} \quad \text{BILINEAR TRASF.} \\[2em] \Omega_c' = \dfrac{2}{T}\cdot\tan\left(\dfrac{\omega_c}{2}\right) \quad \text{PREWARP} \end{array}\right.$$

$\Downarrow$

3dB FREQUENCY

$$H(z) = 0.0317 \cdot \frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-1{,}4590z^{-1}+0.9104z^{-2}-0{,}1978z^{-3}}$$

3rd ORDER IIR-FILTER.
- 3 POLES
- 3 ZEROS (ALL IN $z=-1$)



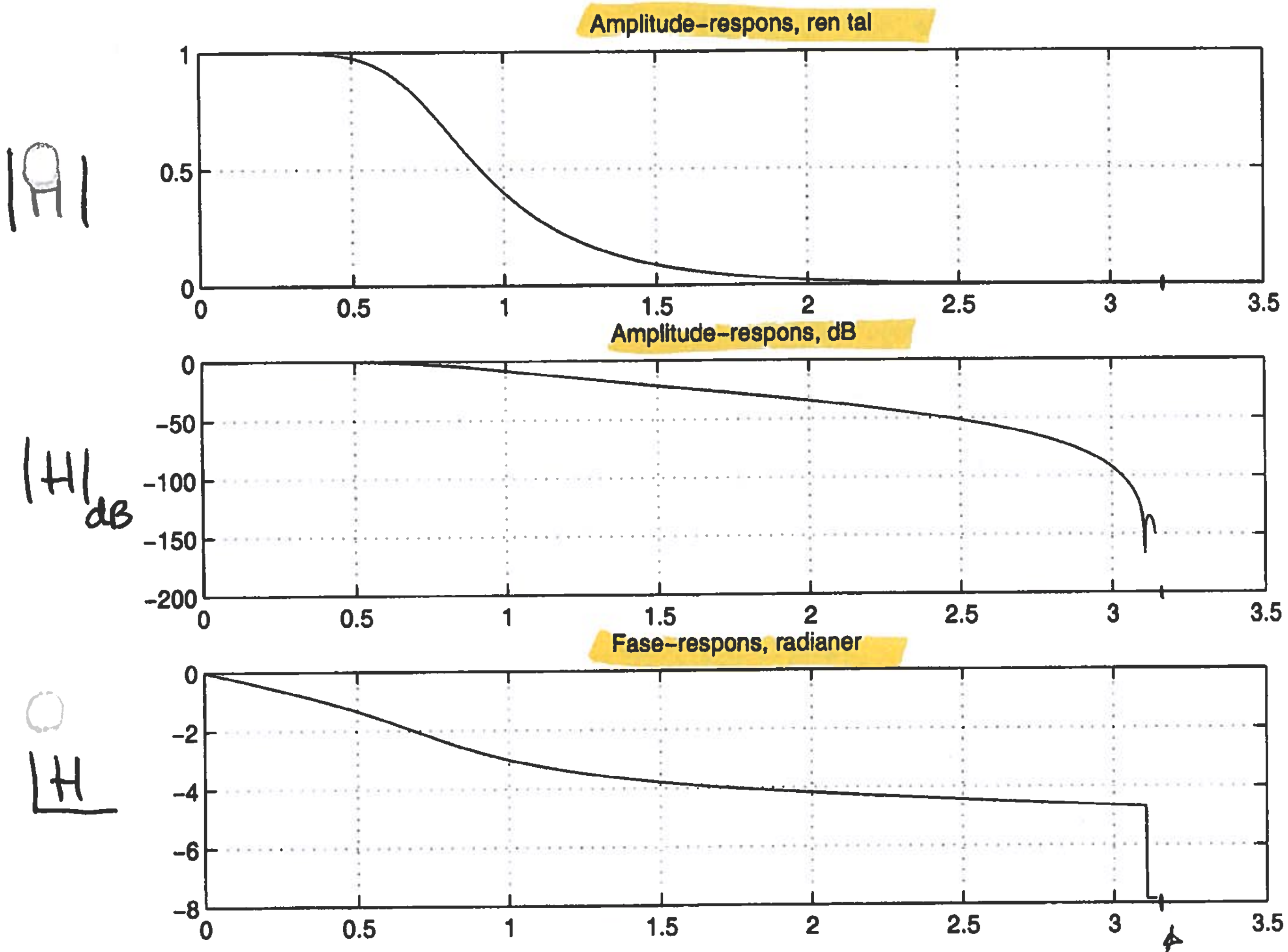
$z_1 = 0.4140$

$(z_2, z_3) = 0{,}5225 \pm j\,0{,}4525$

OVERFØRINGSFUNKTION;

$$H(z) = 0.0317 \cdot \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{1 - 1.4590z^{-1} + 0.9104z^{-2} - 0.1978z^{-3}}$$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$ FREKVENSRESPONS



$\pi \sim f_s/2$

$$H(z) = \frac{Y(z)}{X(z)} = 0.0317 \cdot \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{1 - 1.4590z^{-1} + 0.9104z^{-2} - 0.1978z^{-3}}$$

$$\Downarrow$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{3} b_k z^{-k}}{1 - \sum_{l=1}^{3} a_l \cdot z^{-l}}$$

$$\Downarrow$$

$$y[n] = \sum_{l=1}^{3} a_l y[n-l] + \sum_{k=0}^{3} b_k x[n-k]$$

DIFFERENCE EQUATION

$$\begin{array}{c} x[n] \\ \delta[n] \end{array} \longrightarrow \boxed{\text{PROG}} \longrightarrow \begin{array}{c} y[n] \\ h[n] \end{array}$$

So, FOR EVERY SAMPLE $x[n]$, WE HAVE TO DO;

- 7 MULTIPLICATIONS
- 6 ADDITIONS
- UPDATE OF INTERNAL VARIABLES

IN CONCLUSION WE THEREFOR CAN SAY;

GOOD NEWS:

- FEW NUMBER OF ARITHMETIC OPERATIONS
- APPROX. TO ANALOG TRANSFER-FUNCTION

BAD NEWS:

- POTENTIALLY UNSTABLE
- THE PHASE-RESPONSE HAS TO BE ACCEPTED AS IS.

AN ALTERNATIVE APPROACH.

A FILTER WITH FINITE IMPULSE RESPONSE (FIR)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{l=1}^{N} a_l z^{-l}}$$

$$= \sum_{k=0}^{M} b_k \cdot z^{-k} \Big|_{a_l = 0,\ l = 1..N}$$

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

$$= \frac{\sum_{k=0}^{M} b_k z^{M-k}}{z^M}$$

WE SEE THAT THIS TRANSFER FUNCTION HAS **ALL POLES IN $z=0$**

$\Downarrow$

$$\boxed{\text{STABLE FILTER} - \text{ALWAYS!}}$$

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{Y(z)}{X(z)}$$

$\mathcal{Z}^{-1} \Downarrow$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$\Downarrow$

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = b_k, \quad k=0..M$$

FIR

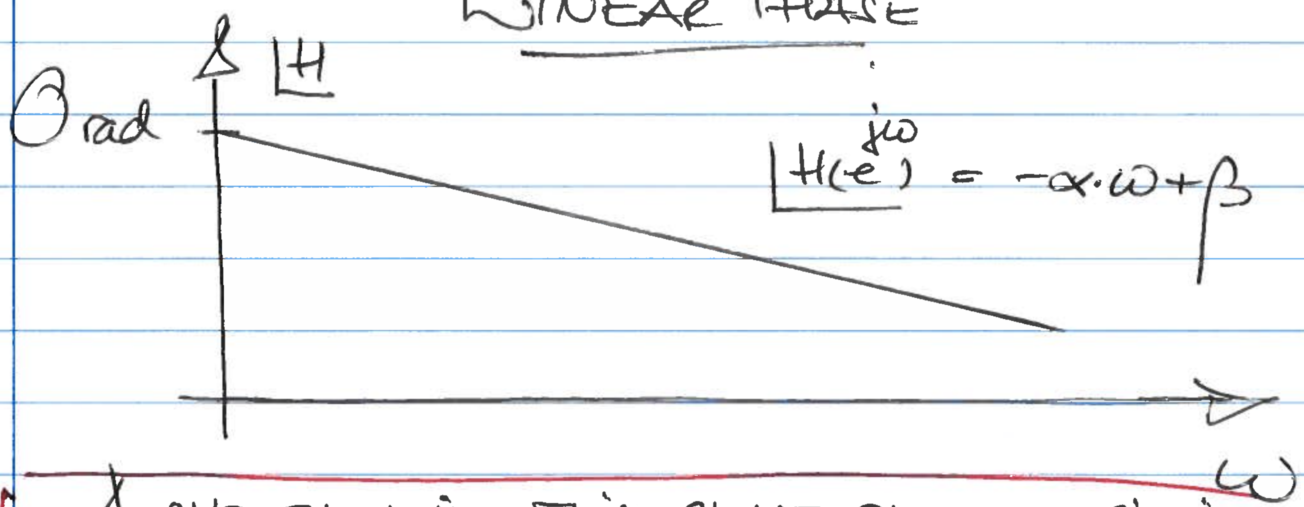
$$\delta[n] \longrightarrow \boxed{\text{FIR}} \longrightarrow \overbrace{\{b_0, b_1, b_2, \dots, b_M\}}^{\text{FIR}}$$

MOREOVER, FIR FILTERS CAN BE DESIGNED HAVING LINEAR PHASE.

SO, OUR PURPOSE TODAY IS TO STUDY;

- THE CONCEPT OF LINEAR PHASE
- ONE METHOD FOR FIR-FILTER DESIGN — THE WINDOW METHOD.

LINEAR PHASE



A SYSTEM WITH THIS PHASE CHARACTERISTIC IS MOST WANTED WHEN WE HAVE TO MODIFY THE SPECTRAL CONTENT OF A SIGNAL AND AT THE SAME TIME MAINTAIN THE WAVEFORM "AS GOOD AS POSSIBLE".

THIS CAN ALSO BE EXPRESSED IN TERMS OF THE GROUP DELAY

$$G(e^{j\omega}) \triangleq -\frac{d}{d\omega} \underline{|H(e^{j\omega})} = \alpha$$

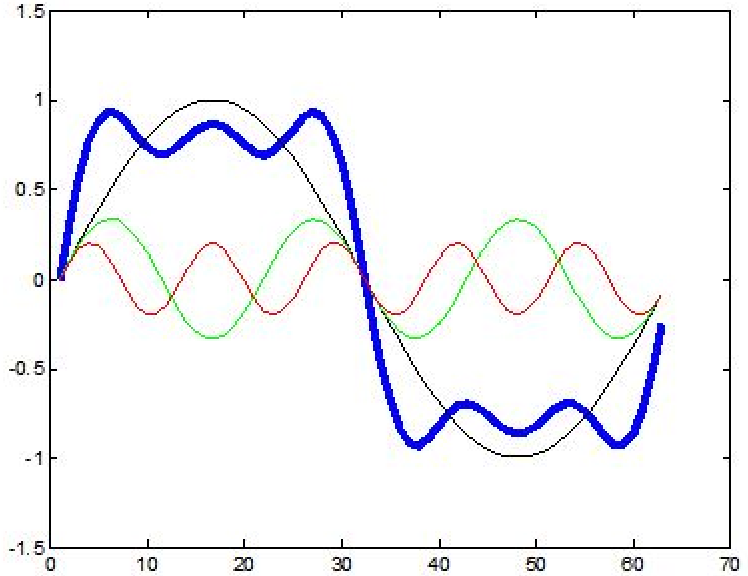
$$\boxed{\text{SO, NO MATTER WHAT FREQUENCY THE SIGNAL HAS } (\omega \in [-\pi;\pi]) \text{, IT WILL BE DELAYED THE SAME AMOUNT OF TIME}}$$

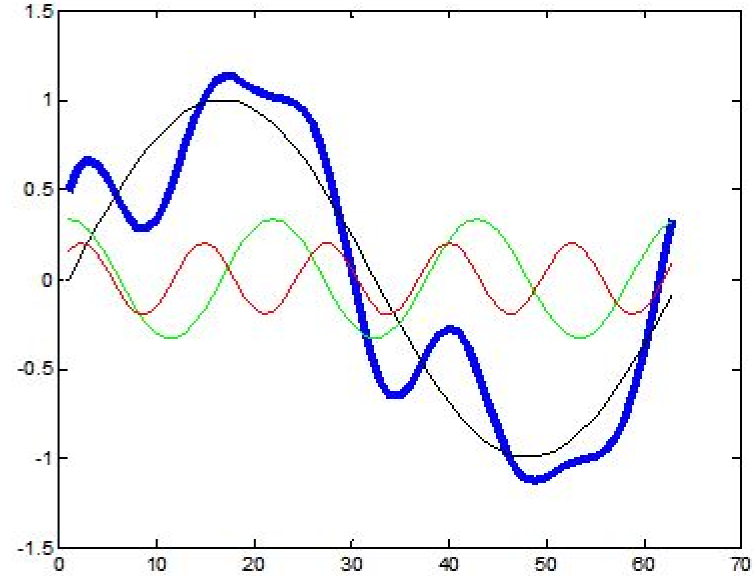
AUDIO / COMMUNICATION / CONTROL / ...

A system with constant Group Delay (gruppeløbs-tid), is a system which maintains the waveform ”as good as possible”

Example: A signal is the sum of three sinusoids having frequencies *f*, 3*f* and 5*f*.

If this signal is the input to an LTI system, then these frequencies will see the same delay, and thus at the output we will also observ a signal which has *approximately* the same waveform as the signal on the input. The three components will potentially be scaled differently in amplitude – thus approximately…





If the same signal is fed into an LTI system with non-linear phase, i.e., frequency dependent group delay, then the components will be delayed differently, and thus the waveform cannot be maintained.

FREQUENCY RESPONSE FOR LINEAR PHASE SYSTEMS.

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \boxed{A(e^{j\omega})\, e^{-j(\alpha\omega-\beta)}}$$

LET'S ASSUME THAT OUR SYSTEM HAS THIS FREQ.-RESPONSE, WHERE $A(e^{j\omega})$ IS A REAL FUNCTION.

- **AMPLITUDE RESPONSE**

$$|H(e^{j\omega})| = |A(e^{j\omega})| \cdot |e^{-j(\alpha\omega-\beta)}| = |A(e^{j\omega})|$$

- **PHASE RESPONSE**

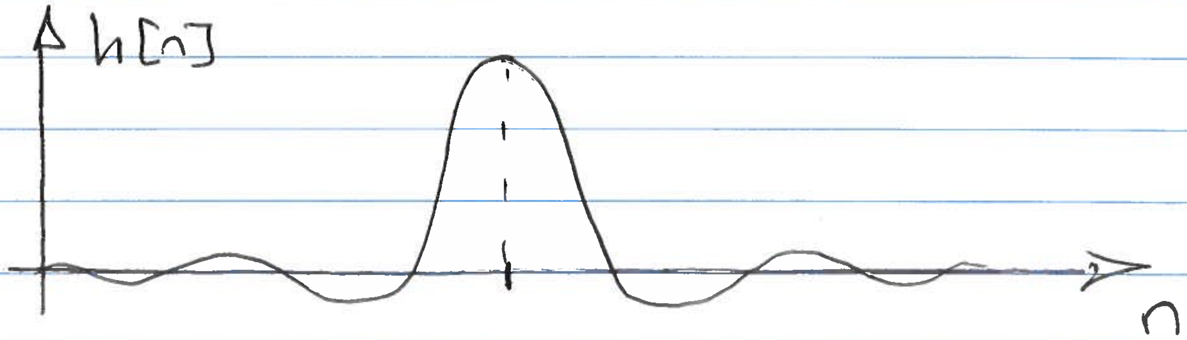
$$\angle H(e^{j\omega}) = \arg\{A(e^{j\omega})[\cos(\alpha\omega-\beta) - j\sin(\alpha\omega-\beta)]\}$$

$$= \arg\{A(e^{j\omega})\} + \operatorname{Arctan}\left(\frac{-\sin(\alpha\omega-\beta)}{\cos(\alpha\omega-\beta)}\right)$$

$$= \begin{cases} -(\alpha\omega-\beta) & \text{for } A(e^{j\omega}) \geq 0 \\ -(\alpha\omega-\beta)-\pi & \text{for } A(e^{j\omega}) < 0 \end{cases}$$

THUS FOR A SYSTEM WITH GENERALIZED LINEAR PHASE, THE ARGUMENT OF $H(e^{j\omega})$ IS THE EQUATION FOR A STRAIGHT LINE, I.E., LINEAR PHASE

NOW, WE WILL SHOW THAT SYSTEMS WITH SYMMETRICAL IMPULSE RESPONSE HAS LINEAR PHASE



OUR OUTSET IS THE FREQUENCY RESPONSE

1) FOR A SYSTEM WITH LINEAR PHASE

$$H(e^{j\omega}) = A(e^{j\omega}) \cdot \cos(\beta - \alpha\omega) + jA(e^{j\omega}) \cdot \sin(\beta - \alpha\omega)$$

EQU 128, p.341.

2) FOR A SYSTEM WITH GENERAL TRANSFER FUNCTION $H(z)$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h[n] \cdot e^{-j\omega n}$$

$\Downarrow$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] \cdot \cos(\omega n) - j \sum_{n=-\infty}^{\infty} h[n] \cdot \sin(\omega n)$$

EQU. 129, p.341

WHERE $h[n]$ IS A REAL SEQUENCE.

FOR 1) AND 2) WE NOW CALCULATE THE ARGUMENT ;

$$1)\ \tan\{\angle H(e^{j\omega})\} = \tan\left\{\mathrm{Arctan}\left(\frac{\sin(\beta-\alpha\omega)}{\cos(\beta-\alpha\omega)}\right)\right\} = \frac{\sin(\beta-\alpha\omega)}{\cos(\beta-\alpha\omega)}$$

$$2)\ \tan\{\angle H(e^{j\omega})\} = \tan\left\{\mathrm{Arctan}\left(\frac{-\sum h[n]\cdot\sin(\omega n)}{\sum h[n]\cdot\sin(\omega n)}\right)\right\}$$

IDENTICAL LEFT-HAND SIDE

$$\Downarrow$$

$$\frac{\sin(\beta-\alpha\omega)}{\cos(\beta-\alpha\omega)} = \frac{-\sum_{n=-\infty}^{\infty} h[n]\cdot\sin(\omega n)}{\sum_{n=-\infty}^{\infty} h[n]\cdot\cos(\omega n)}$$

$$\Downarrow$$

$$\sum_{n=-\infty}^{\infty} h[n]\sin(\beta-\alpha\omega)\cdot\cos(\omega n) = -\sum_{n=-\infty}^{\infty} h[n]\sin(\omega n)\cdot\cos(\beta-\alpha\omega)$$

WE NOW APPLY THE TRIGONOMETRIC IDENTITY

$$\sin(x)\cdot\cos(y) = \frac{1}{2}\{\sin(x-y)+\sin(x+y)\}$$

$$\boxed{\sum_{n=-\infty}^{\infty} h[n]\cdot\sin(\beta+(n-\alpha)\omega) = 0}$$

EQU 130 p. 341

WHICH SHOULD BE VALID FOR ALL ω.

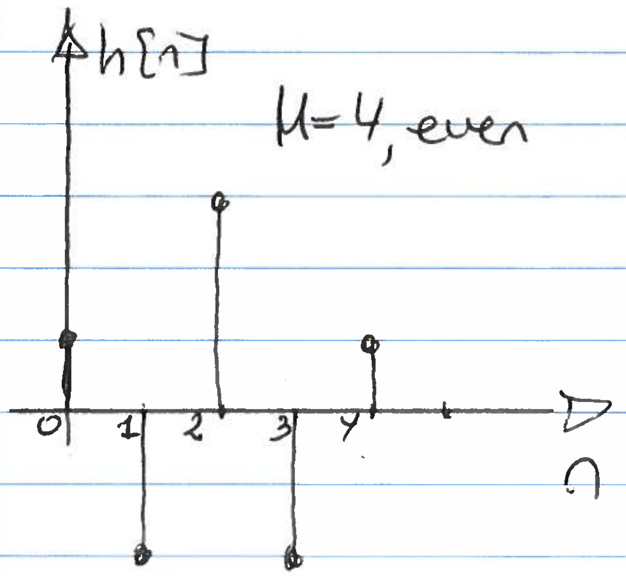
AGAIN $$\sum_{n=-\infty}^{\infty} h[n] \cdot \sin\big(\beta + (n-\alpha)\omega\big) = 0 \quad \forall \omega.$$

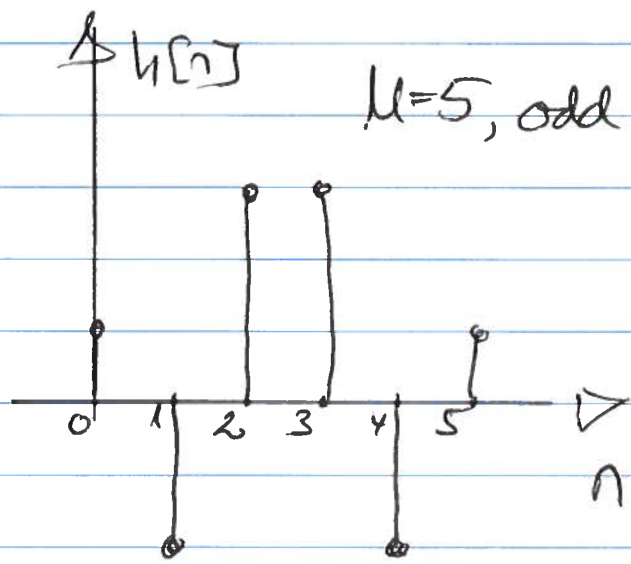
THIS IS A NECESSARY CONDITION FOR LINEAR PHASE.

OUR NEXT TASK IS TO FIND VALUES FOR α, β AND $h[n]$ SUCH THAT THE CONDITION HOLDS!

WITHOUT PROOF;

- $\alpha = \frac{M}{2} \Rightarrow 2\alpha = M$ WHERE M IS AN INTEGER
- $\beta = \begin{cases} 0 \\ \pi \end{cases}$
- $h[n] = h[2\alpha - n] = h[M-n]$





$h[n]$ IS SYMMETRIC AROUND $\alpha = M/2$

So, LINEAR PHASE IS ACHIEVABLE FOR

- $h[n] = h[M-n]$, i.e., SYMMETRIC IMPULSE RESPONSE

BUT IT IS ALSO ACHIEVABLE FOR

- $h[n] = -h[M-n]$, i.e., ANTI-SYMMETRIC IMPULSE RESPONSE

THUS; FOUR POSSIBILITIES

- M is EVEN/ODD
- $h[n]$ is SYMMETRIC/ANTI-SYMMETRIC

$\Downarrow$ FIR-FILTER TYPE I, II, III, IV

p. 343–344

$h[n]$ \ M	EVEN	ODD
SYMM.	I	II
ANTI-SYMM	III	IV

LET'S LOOK AT THE FREQUENCY RESPONSE FOR AN FIR FILTER TYPE I;

$$H(e^{j\omega}) = A(e^{j\omega}) \cdot e^{-j(\alpha\omega - \beta)}$$

$\Downarrow$ AFTER MANY CALCULATIONS *

$$H(e^{j\omega}) = \left\{ h[M/2] + 2\sum_{k=1}^{M/2} h[M/2+k] \cdot \cos(\omega k) \right\} \cdot e^{-j\omega M/2}$$

Equ 140a, p. 343

$\alpha = M/2$ AND $\beta = 0$

* IF YOU WANT TO EXPLORE THESE CALCULATIONS, YOU MAY FIND THEM ON THE NEXT PAGE

PK

17a

Vis at $H(e^{j\omega}) = \left\{ h[M/2] + 2\sum_{k=1}^{M/2} h[M/2+k]\cdot\cos(\omega k) \right\}\cdot e^{-j\omega M/2}$

Generel frekvensrespons for FIR system $H(e^{j\omega}) = \sum_{n=0}^{M} h[n]\cdot e^{-j\omega n}$

Ny variabel k: $n = M/2 + k \Rightarrow k = n - M/2$

$$H(e^{j\omega}) = \sum_{k=-M/2}^{M/2} h[M/2+k]\cdot e^{-j\omega(M/2+k)}$$

$\Downarrow$

$$H(e^{j\omega}) = e^{-j\omega M/2} \sum_{k=-M/2}^{M/2} h[M/2+k]\, e^{-j\omega k}$$

$\Downarrow$

$$H(e^{j\omega}) = e^{-j\omega M/2} \left\{ \sum_{k=-M/2}^{-1} h[M/2+k]\, e^{-j\omega k} + h[M/2] + \sum_{k=1}^{M/2} h[M/2+k]\cdot e^{-j\omega k} \right\}$$

$\Downarrow$

$$H(e^{j\omega}) = e^{-j\omega M/2} \left\{ \sum_{k=1}^{M/2} h[M/2+k]\cdot e^{j\omega k} + h[M/2] + \sum_{k=1}^{M/2} h[M/2+k]\, e^{-j\omega k} \right\}$$

Lovligt at vende summationsfortegn pga. symmetri i impuls-responsen.

$\Downarrow$

$$H(e^{j\omega}) = e^{-j\omega M/2} \left\{ h[M/2] + \sum_{k=1}^{M/2} h[M/2+k]\cdot \underbrace{\left(e^{j\omega k} + e^{-j\omega k}\right)}_{2\cos(\omega k) \text{ via Eulers identitet}} \right\}$$

$\Downarrow$

$$H(e^{j\omega}) = e^{-j\omega M/2} \left\{ h[M/2] + 2\cdot\sum_{k=1}^{M/2} h[M/2+k]\cdot\cos(\omega k) \right\}$$

Ligen 140a p. 343

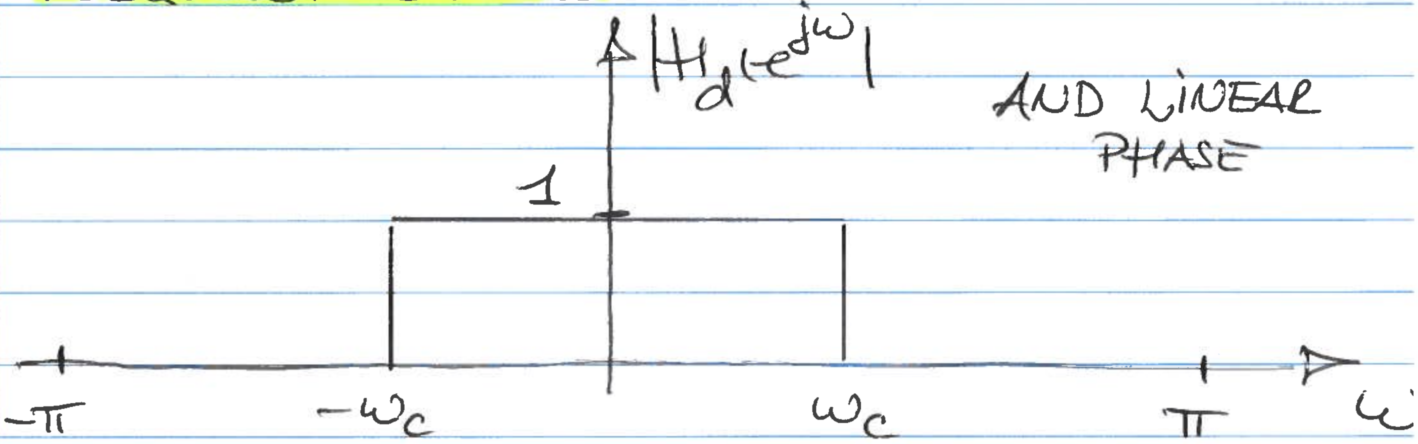
$\Downarrow$

$$H(e^{j\omega}) = A(e^{j\omega})\cdot e^{-j\omega M/2}$$

WE ARE NOW READY TO DESIGN AN FIR-FILTER.

IIR	FIR
SPEC	SPEC
$H(s)$	IDEAL IMPULSE RESPONSE
$H(s) \rightarrow H(z)$	• FINITE • CAUSAL

IN ORDER TO FIND THE IDEAL IMPULSE RESPONSE, LET'S START OUT WITH THE IDEAL / DESIRED FREQUENCY RESPONSE



AND LINEAR PHASE

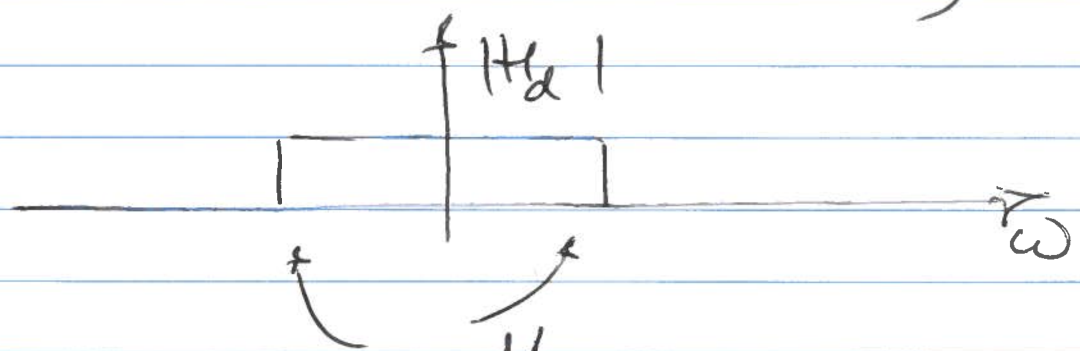
$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n] \cdot e^{-j\omega n}$$ — FOURIER SERIES

THE FOURIER · COEFFICIENTS ARE

$$h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) \cdot e^{j\omega n} \, d\omega$$

EQU 53, 54
p. 537.

WE ARE FACING A PROBLEM HERE ;



OBS!!

AS A CONSEQUENCE, $h_d[n]$ BECOMES NON-CAUSAL AND INFINITE

THUS : ONCE WE HAVE DERIVED $h_d[n]$ IT MUST BE TRUNCATED.

$$h[n] = \begin{cases} h_d[n] & 0 \leq n \leq M \\ 0 & \text{OTHERWISE} \end{cases}$$

THIS IS THE IMPULSE RESPONSE FOR A FILTER THAT WE CAN IMPLEMENT → CAUSAL, FINITE } FIR

THE IDEA IS TO APPLY A WINDOW (IN THE TIME DOMAIN).

So;
$$\boxed{h[n] = h_d[n] \cdot w[n]}$$

WHERE $w[n] = \begin{cases} f[n] & 0 \le n \le M \\ 0 & \text{OTHERWISE} \end{cases}$

NEW PROBLEM?

MULTIPLICATION IN THE TIME DOMAIN IS EQUIVALENT TO CONVOLUTION IN THE FREQUENCY DOMAIN

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})\, W(e^{j(\omega-\theta)})\, d\theta$$

PERIODIC CONVOLUTION. EQU 58, p.558.

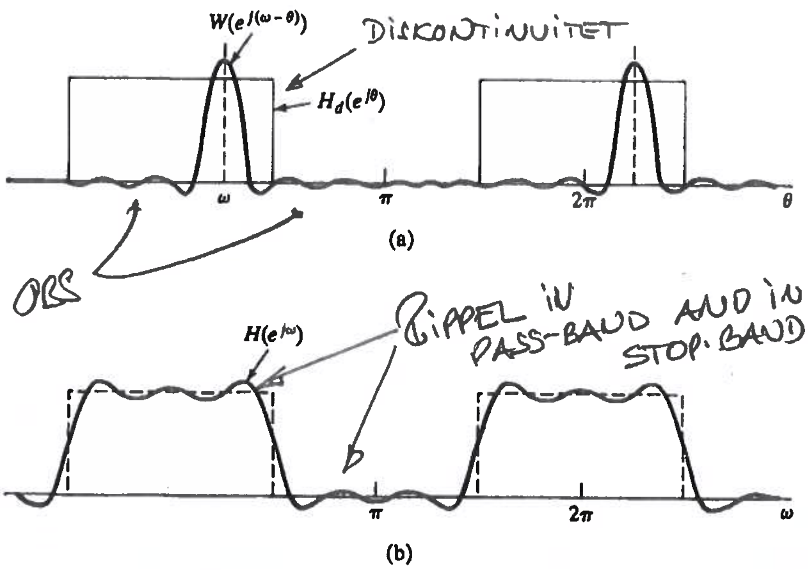


Figure 27 (a) Convolution process implied by truncation of the ideal impulse response. (b) Typical approximation resulting from windowing the ideal impulse response.

GIBBS PHENOMENON

DUE TO GIBB'S PHENOMENON, WE WANT $W(e^{j\omega})$ TO BE AN IMPULSE-TRAIN WITH PERIOD 2π.

$$\Downarrow \quad H(e^{j\omega}) = H_d(e^{j\omega})$$

THE IMPLICATION IN THE TIME DOMAIN IS A CHALLENGE;

$w[n] = 1 \ \forall n$ AND THUS NO TRUNCATION OF $h_d[n]$

WE NEED TO FIND A COMPROMISE

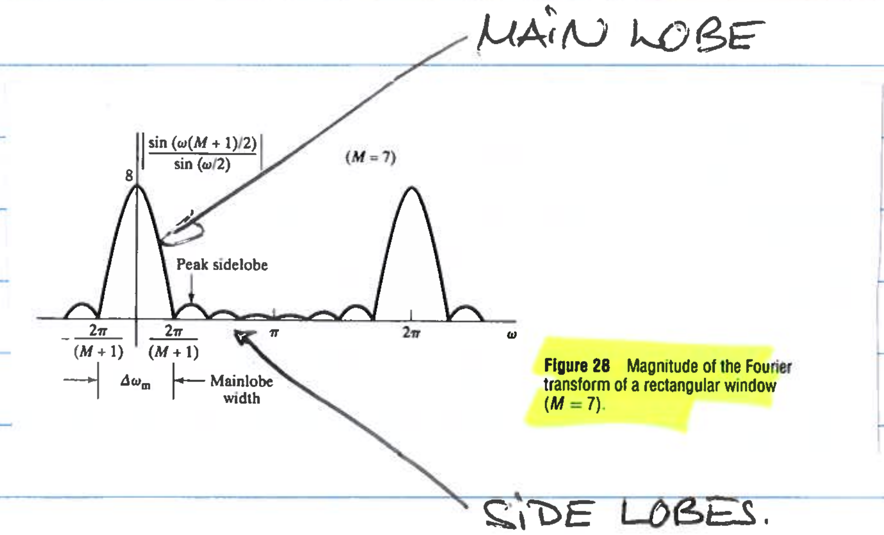


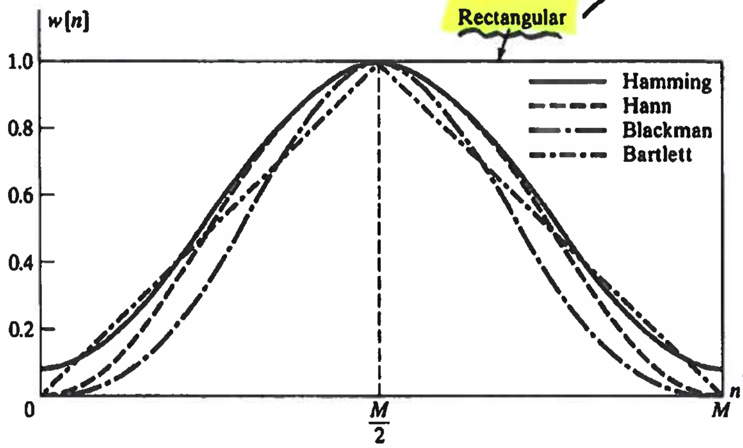
Figure 28 Magnitude of the Fourier transform of a rectangular window ($M = 7$).

WE ARE LOOKING FOR A WINDOW FUNCTION

- NARROW MAIN-LOBE (IMPULSE-TRAIN)
- SMALL SIDE-LOBES (SMALL RIPPLES)

DIFFERENT WINDOW-FUNCTIONS

$$w[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$



OBS!

Figure 29 Commonly used windows.

Bartlett (triangular)

$$w[n] = \begin{cases} 2n/M, & 0 \le n \le M/2, \; M \text{ even} \\ 2 - 2n/M, & M/2 < n \le M, \\ 0, & \text{otherwise} \end{cases} \tag{60b}$$

Hann

$$w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases} \tag{60c}$$

Hamming

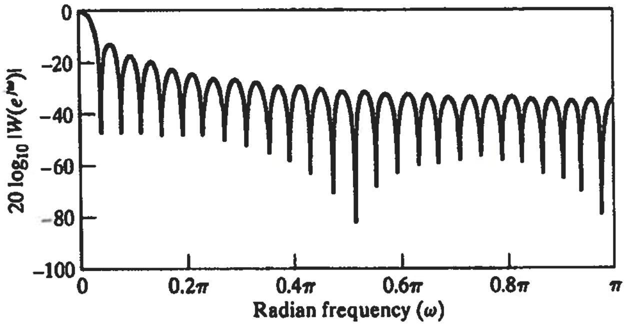
$$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases} \tag{60d}$$

Blackman

$$w[n] = \begin{cases} 0.42 - 0.5\cos(2\pi n/M) + 0.08\cos(4\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases} \tag{60e}$$

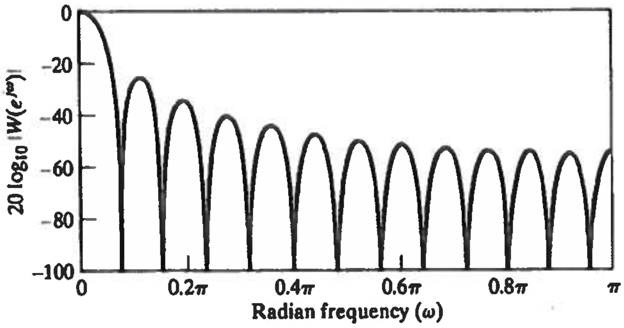
PK

AMPLITUDE RESPONSE



(a)

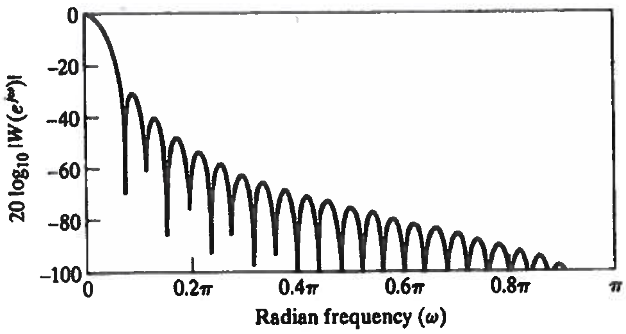
RECTANGULÆR



(b)

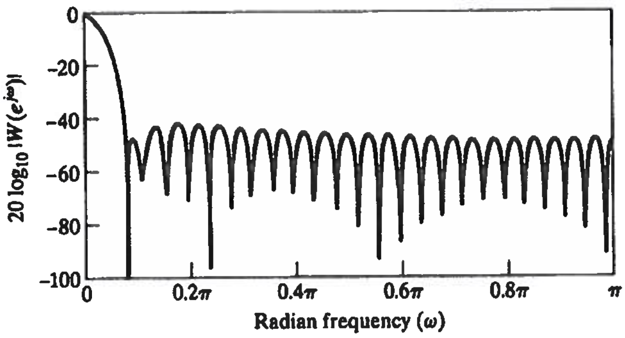
TREKANT

Figure 30 Fourier transforms (log magnitude) of windows of Figure 29 with $M = 50$. (a) Rectangular. (b) Bartlett.



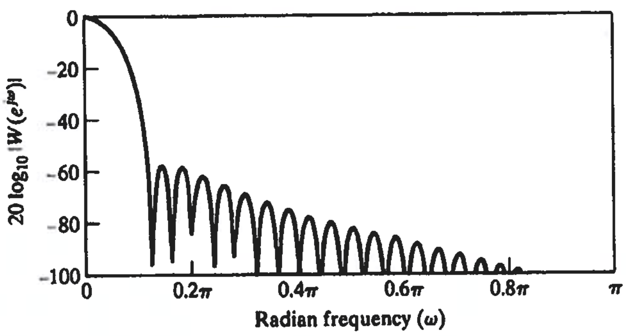
(c)

HANNING



(d)

HAMMING



(e)

BLACKMAN

Figure 30 *(continued)* (c) Hann. (d) Hamming. (e) Blackman.

TABLE 2 COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hann	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

OBS! OBS!

DESIGN PROCEDURE

- $|H_d(e^{j\omega})|$ + LINEAR PHASE
- FIND $h_d[n]$, INVERSE FOURIER TRANSFORM
- CHOOSE WINDOW FUNCTION $w[n]$
- $h[n] = h_d[n] \cdot w[n]$
- EXPRESS $H(e^{j\omega})$ AS A FUNCTION OF M FILTER ORDER
- PLOT $|H(e^{j\omega})|$ AND CHECK SPECIFICATIONS
- OK? YES → DONE, NO → NEW $w[n]$ → TUNE M

ITERATIVE PROCESS