Preliminary plan

1. {1-2}, 3-7, {7-16}, 25-30, 49-57, (app. A)
   - o Course overview
   - o Analog filters: Applications
   - o The Butterworth approximation
   - o Passive filter realisation (ladder structure)
   - o Design procedure, frequency and impedance scaling

2. **7-20, 30-36, 58-62, {App. A}**
   - o **The Chebyshev approximation**
   - o **Other filter types**
   - o **Impact of group delay variations**

3. 37-38, {67-71}, 77-88, 171-184, 187-189, {190-196}, 197-208
   - o Frequency transformations, LP-HP, LP-BP & LP-BS
   - o Sensitivity analysis
     - o How sensitive is a given circuit to component variations?
     - o Used as a tool to evaluate filter circuits

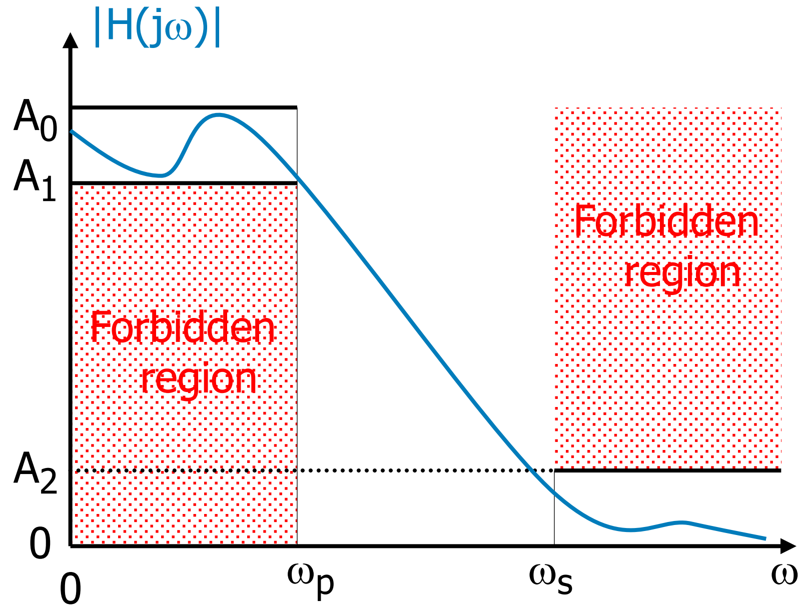
4. 217-238, 253-260, 263-264
   - o OpAmps applied as building blocks in active RC-filters
   - o 2nd order Sallen-Key
   - o 2nd order multiple feed-back
   - o Higher order filters

5. Design/lab. exercise

Lecture 1: Recap

Important parameters are:

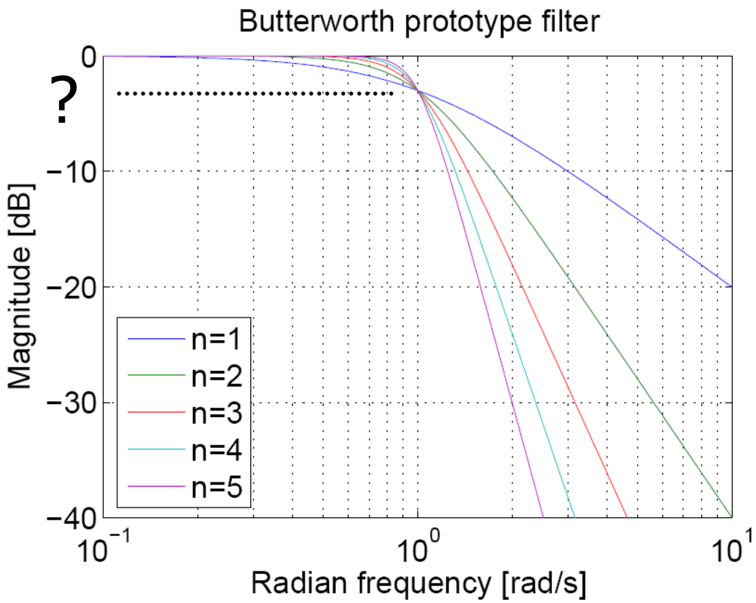
- o Transition band ratio = shape factor, ω_s/ω_p
- o Stopband attenuation A_2/A_0
- o Passband variation, A_1/A_0



An n[th] order (n = number of poles) normalized Butterworth filter is defined by ☺:

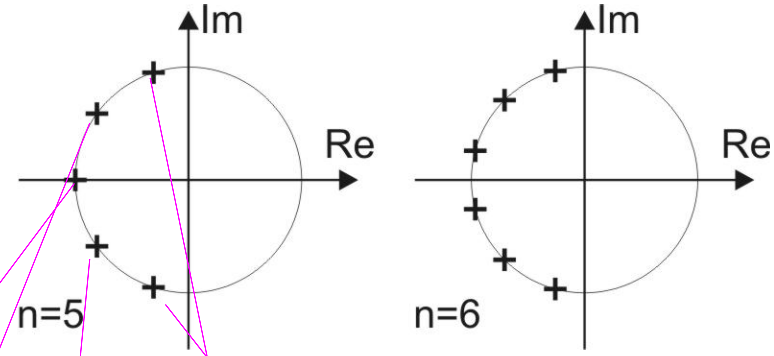
$$|H(j\omega)|^2 = \frac{1}{1+\omega^{2n}} \quad (A_0 = 1)$$

$$|H(j1)| = \frac{1}{\sqrt{2}} \quad \sim -3\ dB$$



Lecture 1: Recap

Poles of H(s) on the unit circle ☺:



$$\boxed{p_k = e^{j\left(\frac{2k-1}{2n}\pi + \frac{\pi}{2}\right)} \qquad k = 1,2,3\ldots n}$$

Note:

- o Only complex conjugated pole pairs when n is even
- o Complex conjugated pole pairs and one real pole when n is odd

Transfer function:

$$H(s) = \frac{K}{(s-p_r)\underline{(s-p_{c1})(s-p_{c1}^*)}(s-p_{c2})(s-p_{c2}^*)\ldots\ldots}$$

n odd:

$$H(s) = \frac{1}{(s+1)\underline{(s^2-2\,\mathrm{Re}\{p_{c1}\}s+1)}(s^2-2\,\mathrm{Re}\{p_{c2}\}s+1)\ldots}$$

?

n even:

$$H(s) = \frac{1}{(s^2-2\,\mathrm{Re}\{p_{c1}\}s+1)(s^2-2\,\mathrm{Re}\{p_{c2}\}s+1)\ldots}$$

Frequency scaling:

The prototype has a normalized transfer function

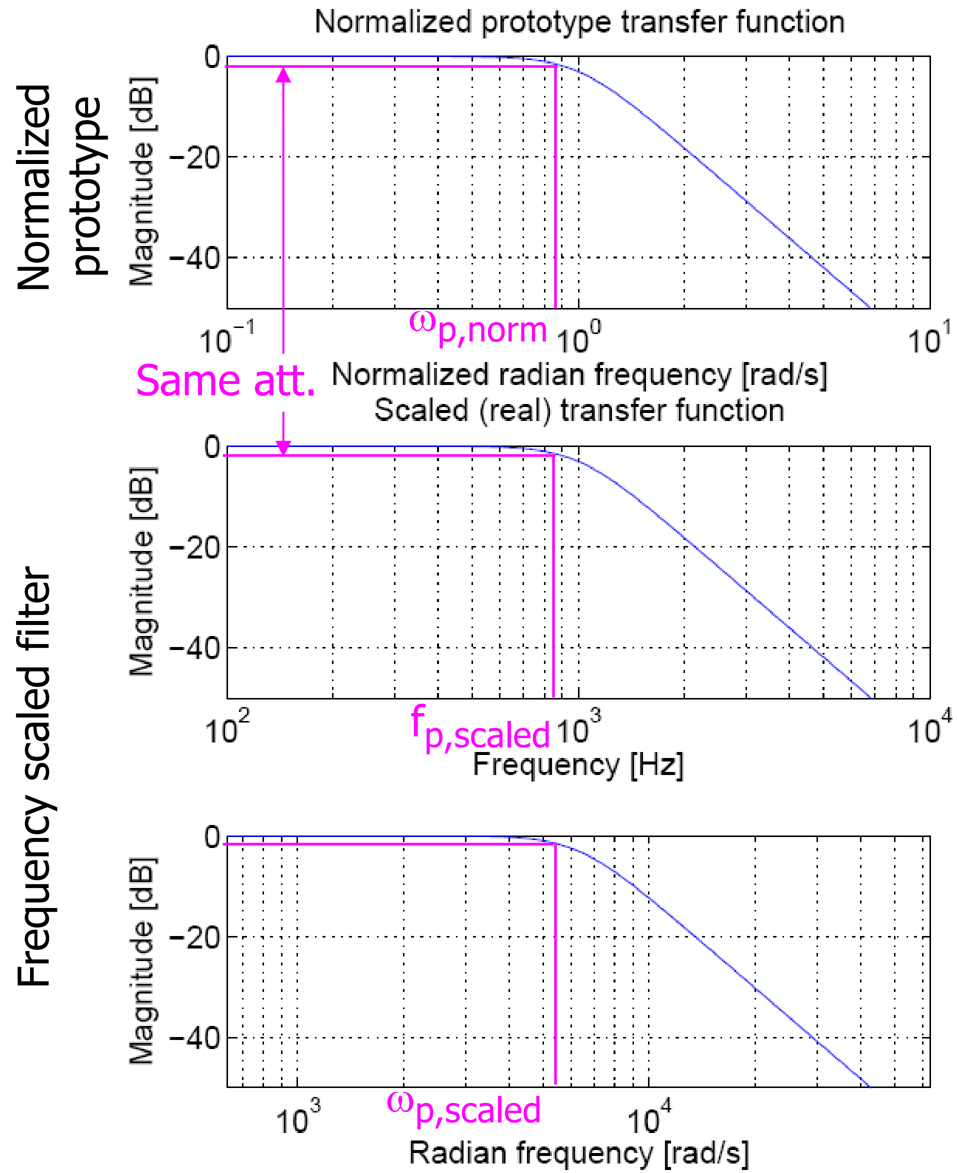
It's wanted to have some other bandedge frequency

The frequency scaling factor is ☺:

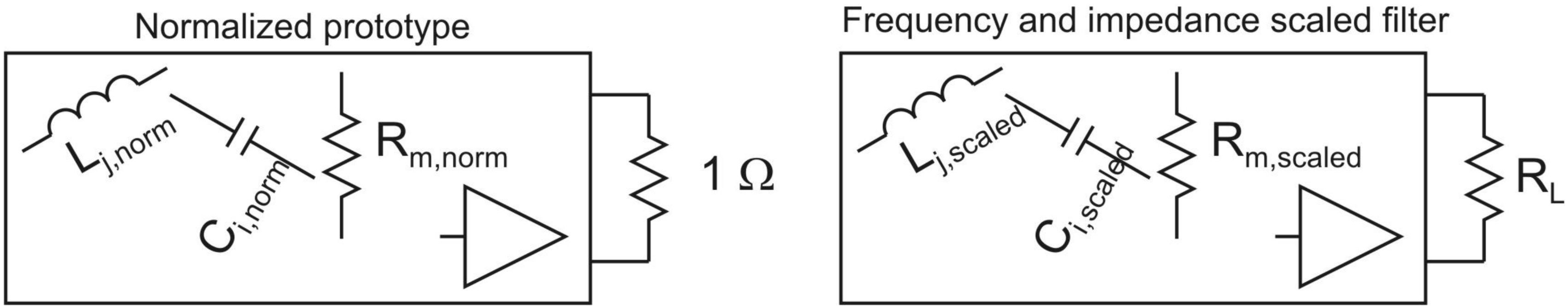
$$k_f = \frac{\omega_{p,scaled}}{\omega_{p,norm}} = \frac{2\pi \cdot f_{p,scaled}}{\omega_{p,norm}}$$

And the transfer functions ☺:

$$\boxed{H_{scaled}(j\omega) = H_{norm}\left(\frac{j\omega}{k_f}\right)}$$



Frequency and impedance scaling ☺:



$$k_z = \frac{R_L}{1\,\Omega}$$

$$k_f = \frac{\omega_{p,scaled}}{\omega_{p,norm}}$$

$$L_{j,scaled} = \frac{k_z}{k_f} L_{j,norm}$$

$$C_{i,scaled} = \frac{1}{k_f k_z} C_{i,norm}$$

$$R_{m,scaled} = k_z R_{m,norm}$$

Exercise 1.2

2nd order Butterworth filter ☺

- -0.5 dB @ 20 kHz
- -30 dB @ f_{SW}.

Prototype:

- -0.5 dB @ $\omega_{0.5dB}$:
- -30 dB @ ω_{SW}:

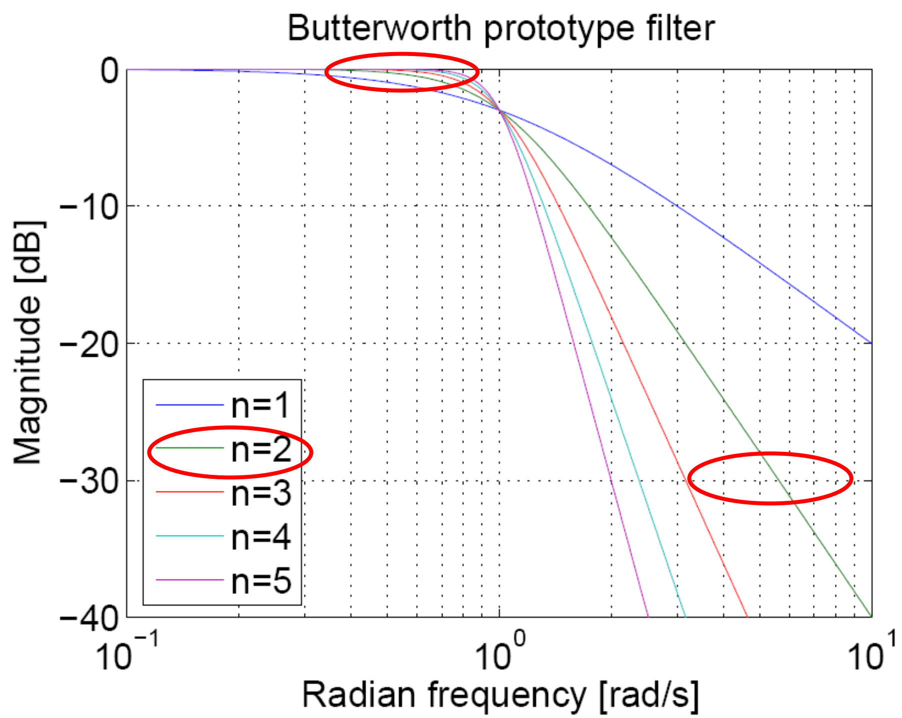
$$\left|H_{norm}(j\omega_{0.5dB})\right|^2 = \frac{1}{1+(\omega_{0.5dB})^{2n}} = 10^{-0.5/10}$$

$$(\omega_{0.5dB})^{2\cdot 2} = 10^{0.5/10} - 1$$

$$\omega_{0.5dB} = \sqrt[4]{10^{0.5/10} - 1} = 0.591\,rad/s$$

$$\omega_{30dB} = \sqrt[4]{10^{30/10} - 1} = 5.62\,rad/s$$

$$\frac{\omega_{30dB}}{\omega_{0.5dB}} = 9.51$$



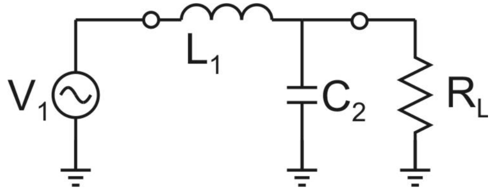
$$k_f = \frac{2\pi 20e3}{0.591} = 212.6e3$$

$$f_{SW} = 9.51 \cdot 20\,kHz = 190\,kHz$$

Exercise 1.2

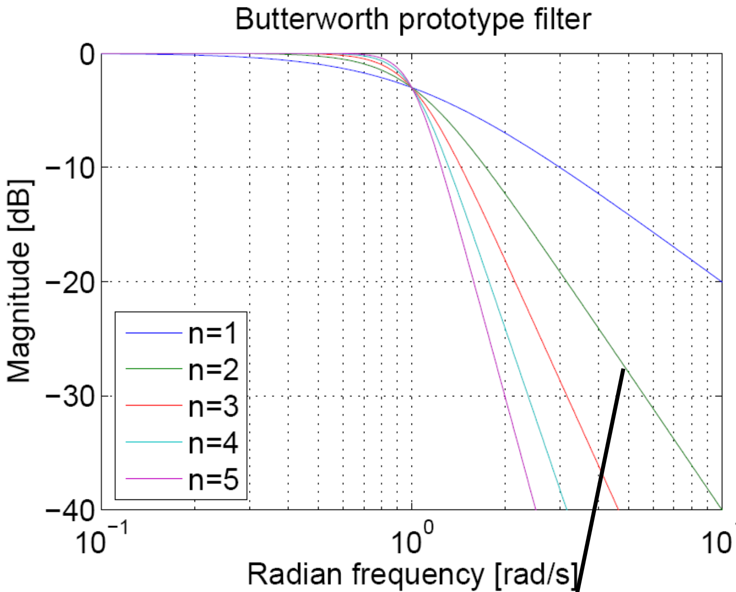
2nd order Butterworth prototype filter

- $L_1 = 1.4142$ H
- $C_2 = 0.7071$ F
- $R_L = 1\ \Omega$.



$1/k_f$

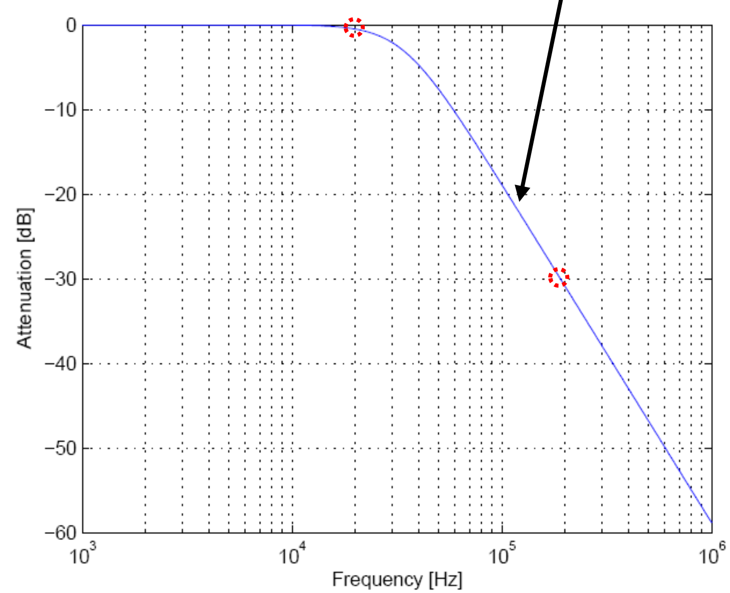
$$k_f = \frac{2\pi 20e3}{0.591} = 212.6e3$$



Frequency scaled filter

- $L_1 = 6.65\ \mu$H
- $C_1 = 3.33\ \mu$F

$1/k_z$ $*k_z$



Frequency & impedance scaled filter

- $L_1 = 26.6\ \mu$H
- $C_1 = 831$ nF
- $R_L = 4\ \Omega$.

Invent a filter by your name ☺

$$|H(j\omega)|^2 = \frac{A_0}{1+F(\omega^2)} \quad where$$

$$0 < F(\omega^2) << 1 \quad for \quad \omega < \omega_p$$

$$F(\omega^2) >> 1 \quad for \quad \omega > \omega_s$$

$$F(\omega^2)?$$

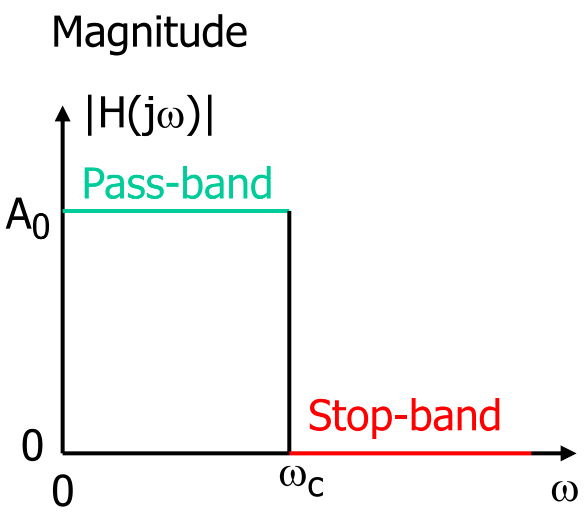
~~5 min. break~~

Break over

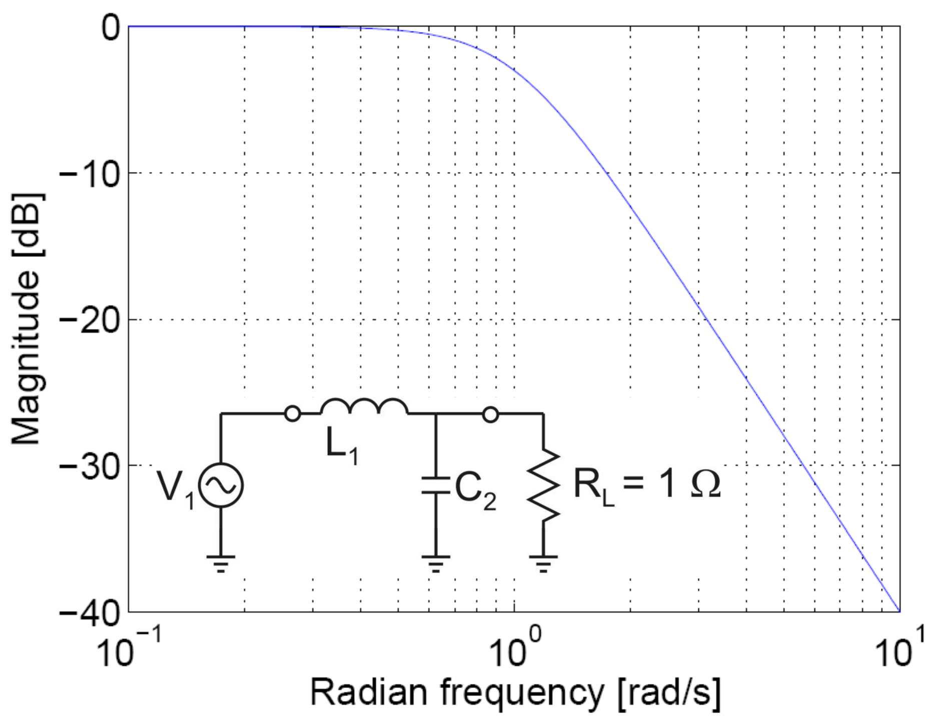
The Chebyshev approximation

Motivation:

- The Butterworth approximation suffers from a ”soft” transition near the bandage
- A sharper cut-off is wanted
- The price to be paid is going to be the ripple in the passband and increased phase distortion



Idea low pass filter



Butterworth low pass filter

The Chebyshev approximation

A transfer function of the form:

$$|H(j\omega)|^2 = \frac{A_0}{1+F(\omega^2)} \quad where$$

$$0 < F(\omega^2) << 1 \quad for \quad \omega < \omega_p$$

$$F(\omega^2) >> 1 \quad for \quad \omega > \omega_s$$

1821 - 1894

will make a low-pass function. Here the Chebyshev functions are given by:

$$0 \le C_n^2(\omega) \le 1 \quad for \quad |\omega| \le 1$$

$$C_n^2(\omega) > 1 \quad for \quad |\omega| > 1$$

$$|H(j\omega)|^2 = \frac{1}{1+\varepsilon^2 C_n^2(\omega)}$$

$$C_n(\omega) = \begin{cases} \cos(n \cdot \cos^{-1}\omega) & |\omega| \le 1 \\ \cosh(n \cdot \cosh^{-1}\omega) & |\omega| \ge 1 \\ \tfrac{1}{2}\left[(\omega+\sqrt{\omega^2-1})^n + (\omega+\sqrt{\omega^2-1})^{-n}\right] & |\omega| \ge 1 \end{cases}$$

Notation:

- $\cos^{-1}x = \mathrm{acos}(x)$
- $\cosh^{-1}x = \mathrm{acosh}(x)$

Chebyshev low pass filter

The transfer function is given by:

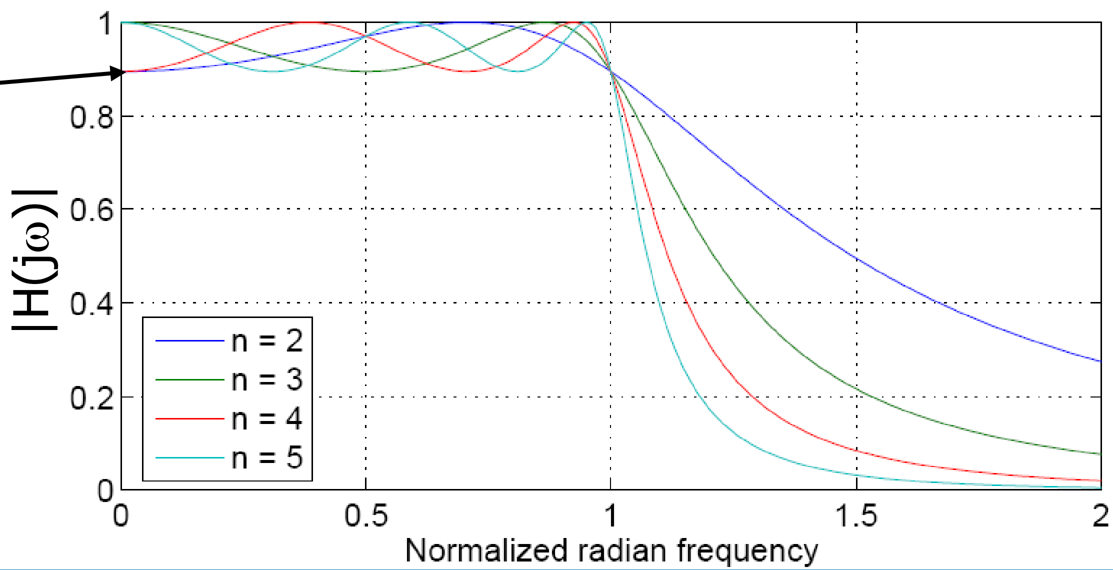
$$0 \le C_n^2(\omega) \le 1 \quad for \quad |\omega| \le 1$$
$$C_n^2(\omega) > 1 \quad for \quad |\omega| > 1$$
$$|H(j\omega)|^2 = \frac{1}{1+\varepsilon^2 C_n^2(\omega)}$$

$$C_n(\omega) = \begin{cases} \cos(n \cdot \cos^{-1}\omega) & |\omega| \le 1 \\ \cosh(n \cdot \cosh^{-1}\omega) & |\omega| \ge 1 \\ \tfrac{1}{2}\left[(\omega+\sqrt{\omega^2-1})^n + (\omega+\sqrt{\omega^2-1})^{-n}\right] & |\omega| \ge 1 \end{cases}$$

So:

$$\frac{1}{1+\varepsilon^2} \le |H(j\omega)|^2 \le 1 \quad for \quad |\omega| \le 1$$

$$\frac{1}{\sqrt{1+\varepsilon^2}}$$



Exercise

The transfer function is given by:

$$0 \le C_n^2(\omega) \le 1 \quad for \quad |\omega| \le 1$$

$$C_n^2(\omega) > 1 \quad for \quad |\omega| > 1$$

$$|H(j\omega)|^2 = \frac{1}{1+\varepsilon^2 C_n^2(\omega)}$$

$$C_n(\omega) = \begin{cases} \cos(n \cdot \cos^{-1}\omega) & |\omega| \le 1 \\ \cosh(n \cdot \cosh^{-1}\omega) & |\omega| \ge 1 \\ ½\left[(\omega+\sqrt{\omega^2-1})^n + (\omega+\sqrt{\omega^2-1})^{-n}\right] & |\omega| \ge 1 \end{cases}$$

So:

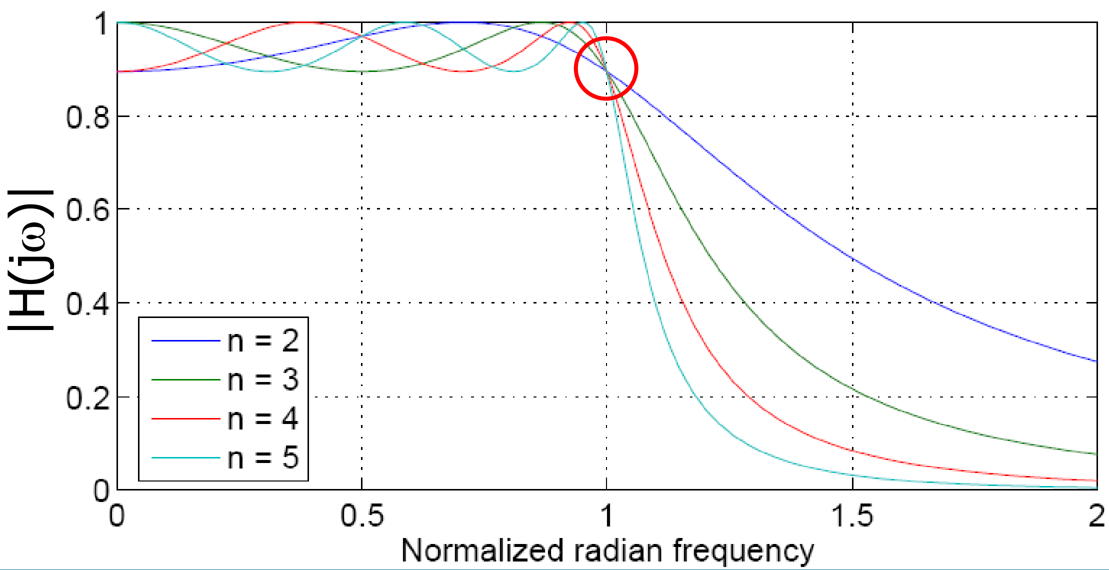
$$\frac{1}{1+\varepsilon^2} \le |H(j\omega)|^2 \le 1 \quad for \quad |\omega| \le 1$$

Find $|H(j1)|^2$ when $\varepsilon = 0.5$

$$C_n(1) = \cos\left(n \cdot cos^{-1}(1)\right) = \cos(n \cdot 0) = 1$$

So:

$$|H(j1)|^2 = \frac{1}{1+\varepsilon^2} = 0.8 \; when \; \varepsilon = 0.5$$



Chebyshev low pass filter - Closer to the idea filter?

$$C_n(\omega) = \begin{cases} \cos(n \cdot \cos^{-1} \omega) & |\omega| \le 1 \\ \cosh(n \cdot \cosh^{-1} \omega) & |\omega| \ge 1 \end{cases}$$

It is seen that:

$$C_0(\omega) = 1 \quad and \quad C_1(\omega) = \omega$$

"It may be shown" that:

$$C_{n+1}(\omega) = 2\omega \cdot C_n(\omega) - C_{n-1}(\omega)$$

By recursion, the Chebyshev functions are found as polynomials:

$$C_1(\omega) = \boxed{\omega}$$

$2^{n-1}\omega^n$

$$C_2(\omega) = \boxed{2\omega^2} - 1$$

$$C_3(\omega) = \boxed{4\omega^3} - 3\omega$$

$$C_4(\omega) = \boxed{8\omega^4} - 8\omega^2 + 1$$

$$C_5(\omega) = \boxed{16\omega^5} - 20\omega^3 + 5\omega$$

So

$$|H(j\omega)|^2 \rightarrow \frac{1}{1 + \varepsilon^2 (2^{n-1} \omega^n)^2}$$

$$\omega \rightarrow \infty$$

Chebyshev low pass filter – Yes and No

Asymptotes:

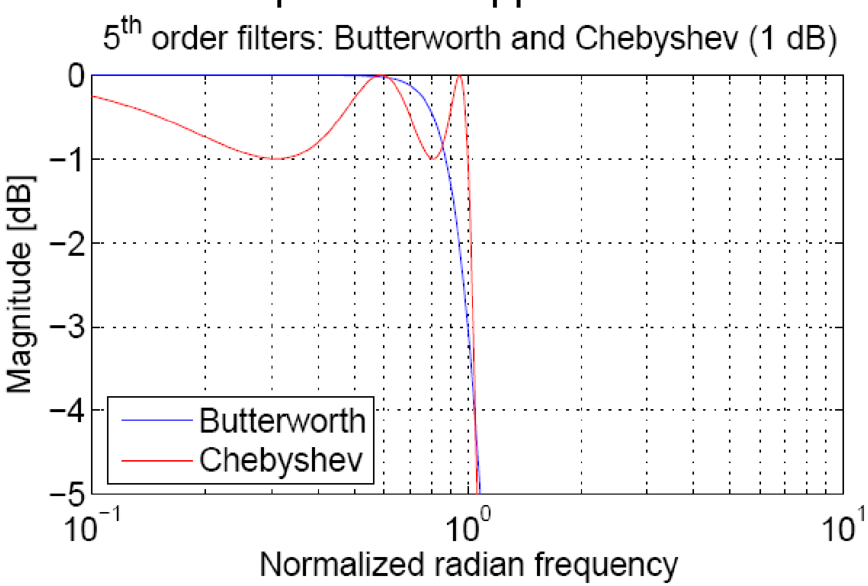
Using the polynomial representation, it may be shown that:

$$|H(j\omega)|^2 \to \frac{1}{1+\varepsilon^2(2^{n-1}\omega^n)^2}$$
$$\omega \to \infty$$

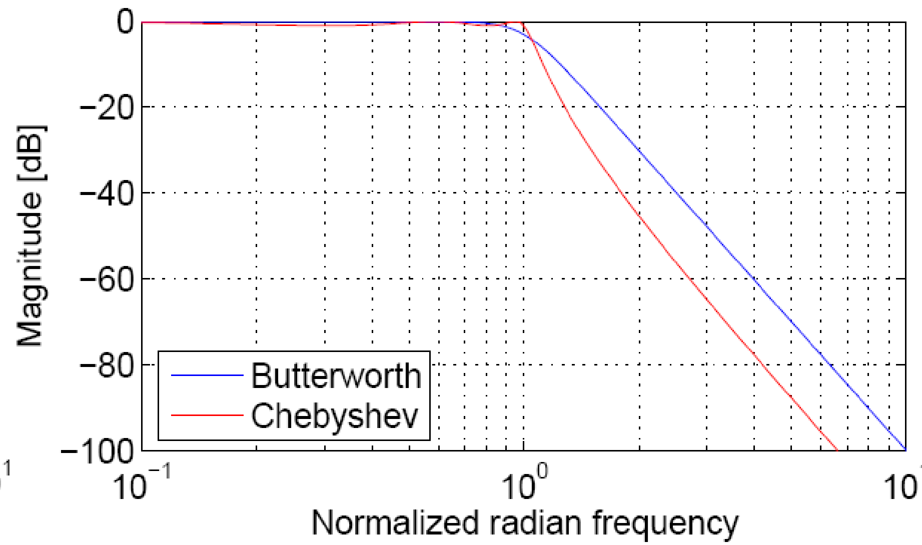
or:

$$\underset{\omega \to \infty}{H(j\omega)_{dB}} \to \underbrace{-20\log(2^{n-1}\varepsilon)}_{\text{"Offset" giving extra attenuation}} - \underbrace{n \cdot 20\log(\omega)}_{\text{Same as Butterworth}}$$

In passband ripple



Higher stopband attenuation



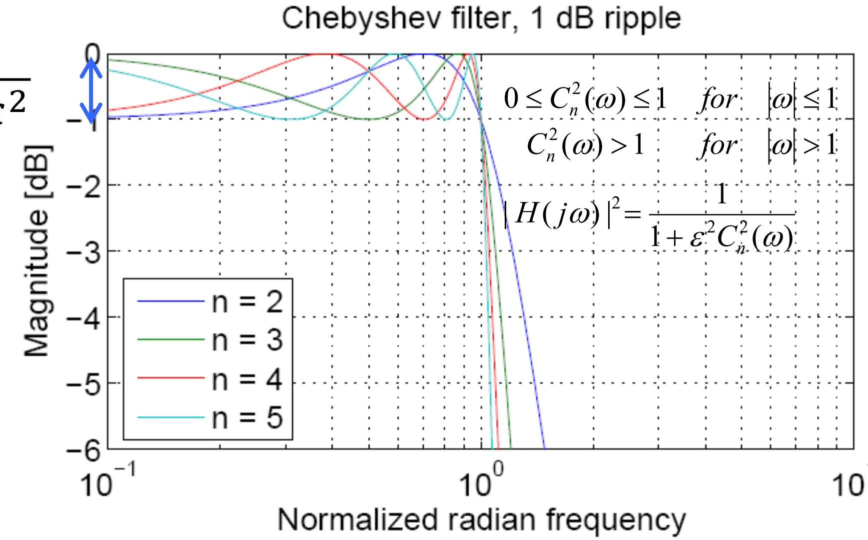
Chebyshev low pass filter – ripple

For a required ripple (the smaller the better), ε is found:

$$Ripple_{dB} = 10\log(1+\varepsilon^2)$$

$$\varepsilon = \sqrt{10^{Ripple_{dB}/10} - 1}$$

$$\frac{1}{1+\varepsilon^2}$$



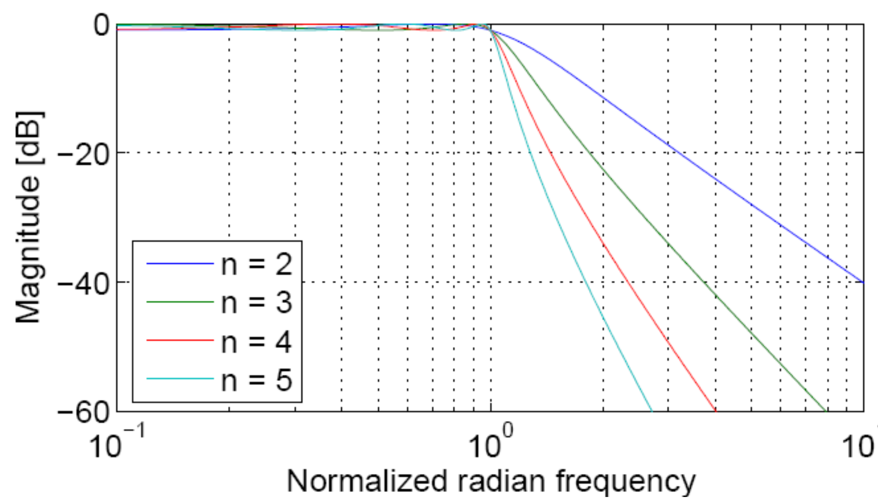
The stopband attenuation (= -H(jω)$_{dB}$) is found from:

$$H(j\omega)_{dB} = -10\log(1+\varepsilon^2 C_n^2(\omega))$$

Where:

$$C_n(\omega) = \cosh(n \cdot \cosh^{-1}\omega)$$

and ω is the normalized radian frequency at the stopband



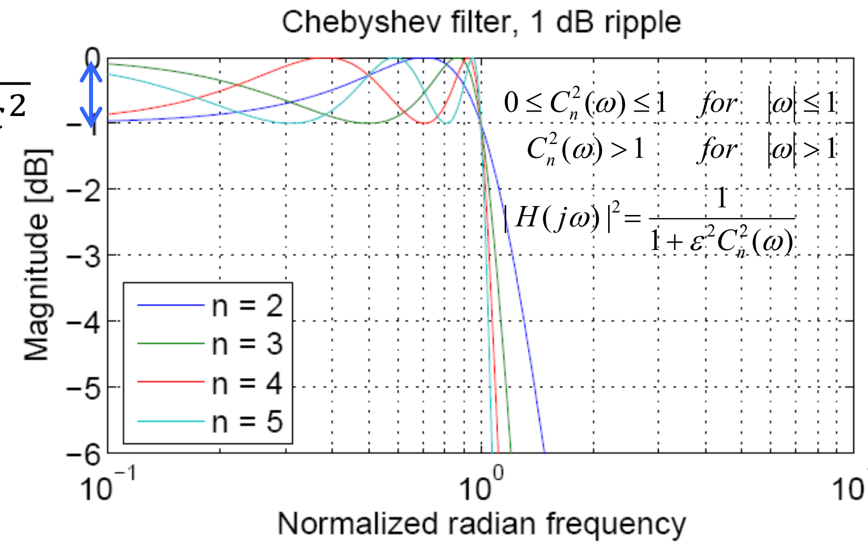
Exercise

For a required ripple (the smaller the better), ε is found:

$$Ripple_{dB} = 10\log(1+\varepsilon^2)$$

$$\varepsilon = \sqrt{10^{Ripple_{dB}/10} - 1}$$

$$\frac{1}{1+\varepsilon^2}$$



For a second order Chebyshev low pass filter, find ε when ripple is 1 dB

$$\varepsilon = \sqrt{10^{1/10} - 1} = \sqrt{1.2589 - 1} = 0.5088$$

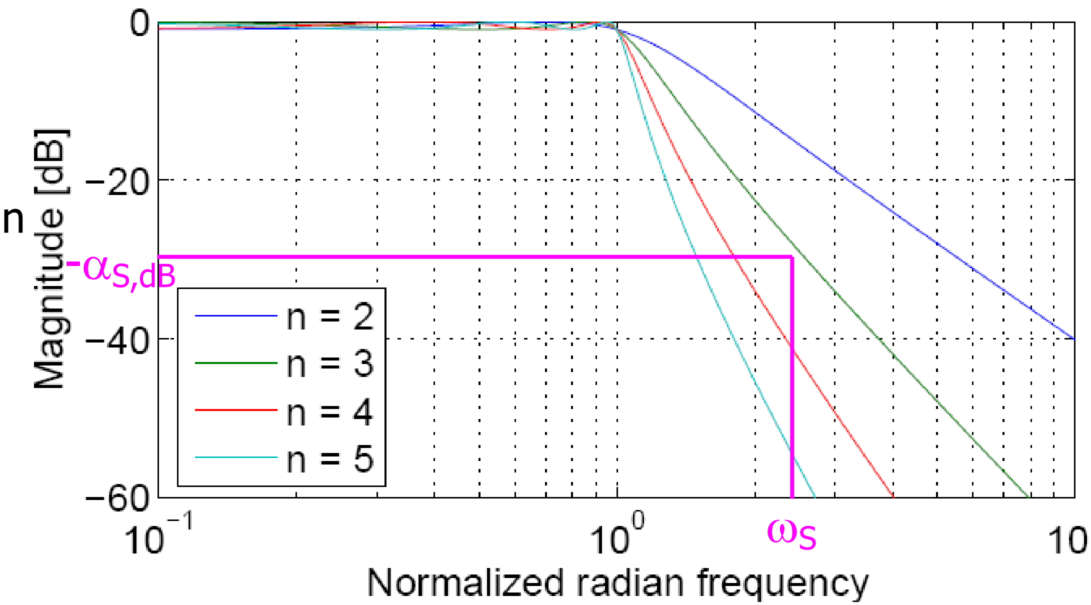
5 min. break

Break over

Chebyshev low pass filter: necessary order

If the filter requirements are:

- o Passband ripple (dB)
- o Stopband attenuation α_S [dB] at the normalized stopband radian frequency ω_S.



Then the necessary filter order is found from:

$$\boxed{n \geq \frac{1}{\cosh^{-1}\omega_S}\cosh^{-1}\sqrt{\frac{10^{\alpha_{S,dB}/10}-1}{10^{Ripple_{dB}/10}-1}}}$$

rounded up to the nearest integer. Note the sign definition: $\alpha_{S,dB} > 0$, $Ripple_{dB} > 0$

ω_S is the stopband frequency for the normalized low-pass-filter: $\omega_S = \omega_{S,scaled}/k_f$

Chebyshev low pass filter

Ripple bandwidth and 3-dB bandwidth

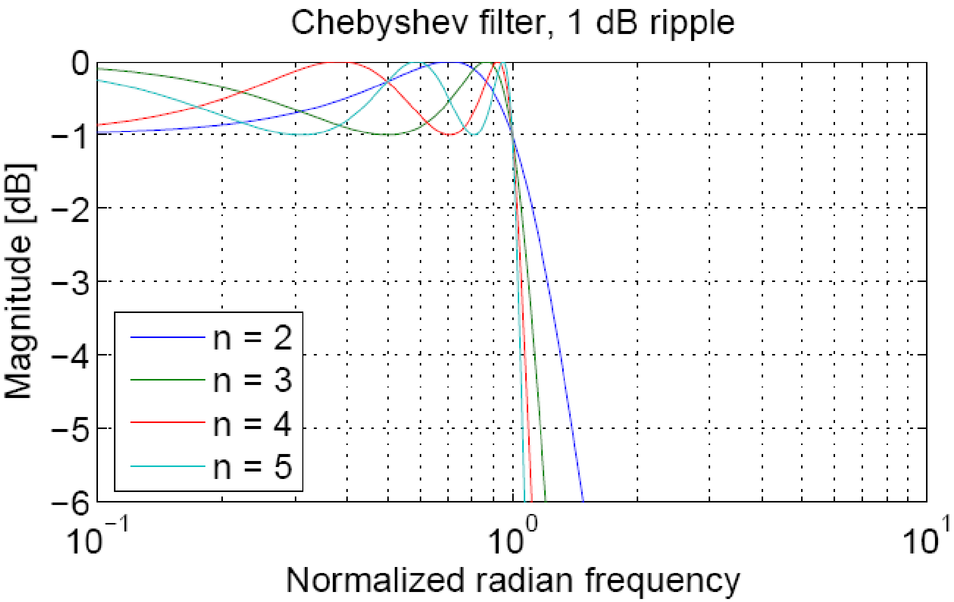
The ratio of ripple bandwidth and 3-dB bandwidth depends on:

- Filter order and
- Ripple level

It may be shown (Exercise 2.1) that:

$$\omega_{3dB} = \cosh\left(\frac{1}{n}\cosh^{-1}\frac{1}{\varepsilon}\right)$$

(for a ripple bandwidth of 1 rad/s)



Different ways of describing the filter response

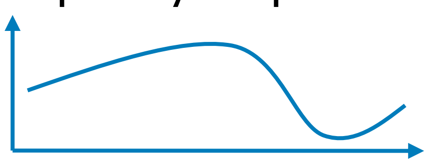
Zeros & Poles:

$$K\frac{(s-z_1)(s-z_2).....}{(s-p_1)(s-p_2)....}$$

Polynomials:

$$\frac{b_1 s^n + \ldots + b_n s + b_{n+1}}{a_1 s^n + \ldots + a_n s + a_{n+1}}$$

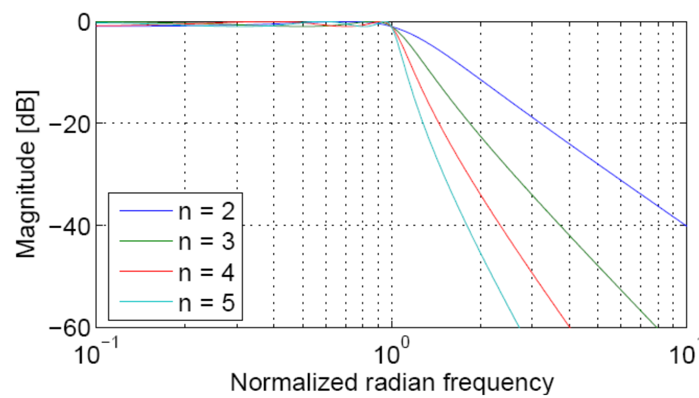
Frequency response:



$p_k = ?\ z_k = ?$

$$0 \le C_n^2(\omega) \le 1 \quad for \quad |\omega| \le 1$$
$$C_n^2(\omega) > 1 \quad for \quad |\omega| > 1$$
$$|H(j\omega)|^2 = \frac{1}{1+\varepsilon^2 C_n^2(\omega)}$$

but H(s)=?



Chebyshev transfer function and poles

Given:

$$|H(j\omega)|^2 = \frac{1}{1+\varepsilon^2 C_n^2(\omega)}$$

Find H(s)

Same procedure as for Butterworth, but a ”bit” more complicated:

A transfer function of the form:

$$H(s) = \frac{K}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_2 s^2 + a_1 s + a_0} \quad \textit{where } a_i\textit{'s are real}$$

has the property:

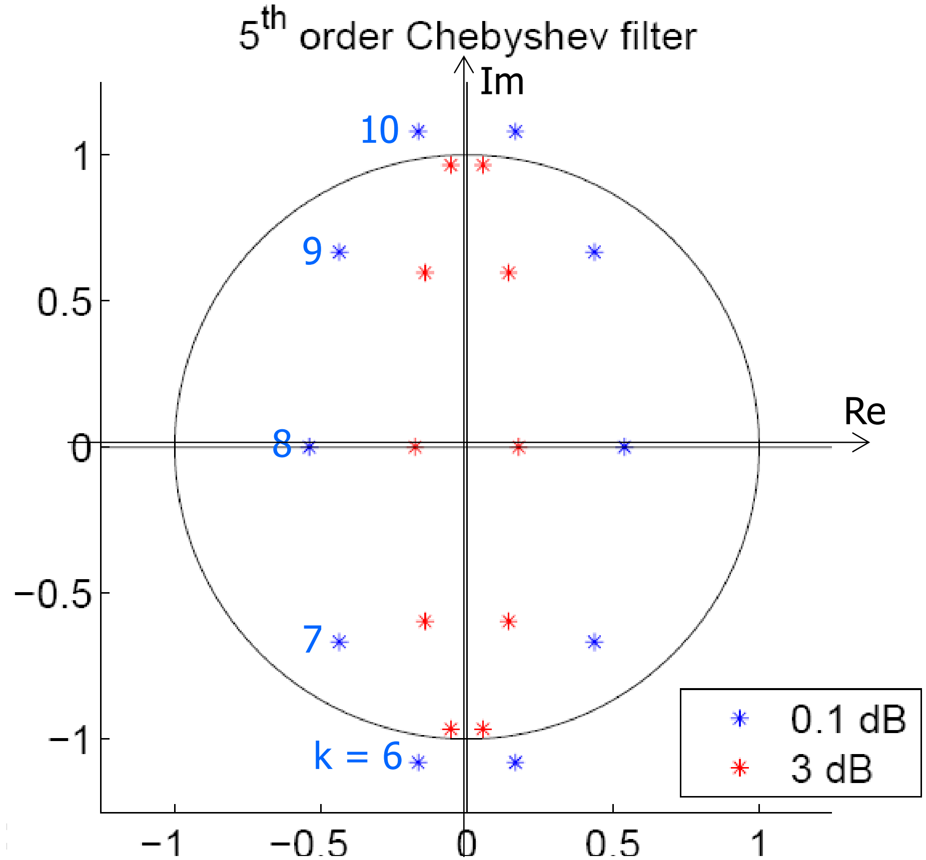
so: $$H(-j\omega) = H^*(j\omega)$$

$$\left|H(j\omega)\right|^2 = H(j\omega)\cdot H^*(j\omega) = H(s)\cdot H(-s)\Big|_{\omega=-js \Leftrightarrow s=j\omega}$$

$$H(s)\cdot H(-s) = \frac{1}{1+\varepsilon^2 C_n^2(-js)}$$

Chebyshev transfer function and poles

It may be shown that the poles are located on an ellipse:
(Butterworth: On the unit circle)



Left hand poles are assigned to H(s)
Right hand poles are assigned to H(-s)

Poles for H(s) (see appendix for the math)

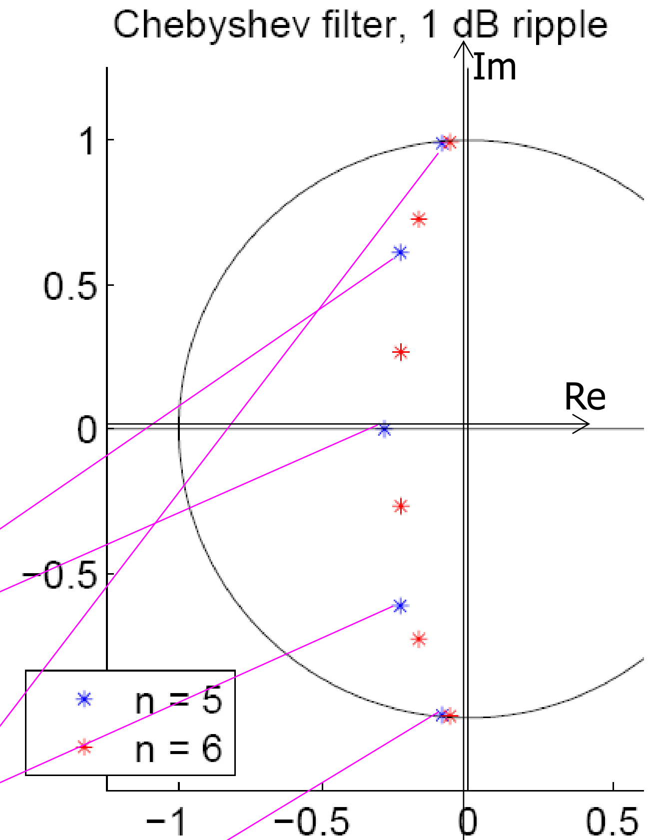
$$p_k = \sin\frac{(2k-1)\pi}{2n}\cdot\sinh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right) + j\cos\frac{(2k-1)\pi}{2n}\cdot\cosh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right)$$

k = n+1, n+2...... 2n

Chebyshev transfer function and poles

Like for Butterworth:

- o Filters of even order have only complex conjugated pole pairs
- o Filters of odd order have one real pole and complex conjugated pole pairs



The transfer function is found from the poles:
n odd:

$$H(s) = \frac{K}{(s-p_r)\underline{(s-p_{c1})(s-p_{c1}^*)}(s-p_{c2})(s-p_{c2}^*)......}$$

$$H(s) = \frac{K}{(s-p_r)\underline{(s^2-2\,\mathrm{Re}\{p_{c1}\}s+|p_{c1}|^2)}(s^2-2\,\mathrm{Re}\{p_{c2}\}s+|p_{c2}|^2)...}$$

Chebyshev functions in Matlab

Matlab:

```
>> % Find the poles from the order and ripple:
Order = 5;
Ripple_dB = 1;
[dummy ChePoles K] = cheb1ap(Order,Ripple_dB);
ChePoles                                      % Another pole numbering
ChePoles =
  -0.0895 + 0.9901i
  -0.2342 + 0.6119i
  -0.2895 + 0.0000i
  -0.2342 - 0.6119i
  -0.0895 - 0.9901i
>> K
K =
    0.1228
>> denom_coeff = real(poly(ChePoles))     % Returns the a_n's. Note the numbering
denom_coeff =
    1.0000    0.9368    1.6888    0.9744    0.5805    0.1228
```

$$H(s) = \frac{K}{(s-p_r)(s-p_{c1})(s-p_{c1}^*)(s-p_{c2})(s-p_{c2}^*)\ldots\ldots}$$

$$H(s) = \frac{K}{a_1 s^n + a_2 s^{n-1} + \ldots + a_{n-1}s^2 + a_n s + a_{n+1}}$$

Chebyshev functions in Matlab

Matlab:

```
>> % Find the Chebyshev polynomial the easiest way:
Order = 5;
Ripple_dB = 1;
Wcut = 1;                                    % Ripple bandwidth [rad/s]
[b a] = cheby1(Order,Ripple_dB,Wcut,'s')     % 's' indicates analog filter
b =
     0      0      0      0      0    0.1228
a =
   1.0000    0.9368    1.6888    0.9744    0.5805    0.1228
% CAUTION: Note the numbering:
```

$$H(s) = \frac{b_1 s^n + b_2 s^{n-1} + \ldots + b_{n-1} s^2 + b_n s + b_{n+1}}{a_1 s^n + a_2 s^{n-1} + \ldots + a_{n-1} s^2 + a_n s + a_{n+1}}$$

Alternatively: Kendall Su: "Analog Filters", App. A

Typical design procedure

The design procedure may vary. It may be like:

1. Calculate ε from the specified ripple
2. Calculate the necessary order, n, from:
   - The required transition band ratio $\omega_{stop}/\omega_{ripple}$.
   - The necessary attenuation at ω_{stop}.
3. Calculate the poles of the normalized filter from ε and n
4. Calculate the transfer function of the normalized filter

10 min. break
Break over

Since both Chebyshev filters like Butterworth filters have a transfer function without zeros, they may be made using the same circuit topology.

Frequency and impedance scaling is also done the same way.

The transfer function of the ”real” (scaled) filter can always be found from the transfer function of the normalized filter:

$$H_{scaled}(j\omega) = H_{norm}\left(\frac{j\omega}{k_f}\right) \qquad k_f = \frac{\omega_{p,scaled}}{\omega_{p,norm}} = \frac{2\pi \cdot f_{p,scaled}}{\omega_{p,norm}}$$

Properties of Butterworth and Chebyshev filters

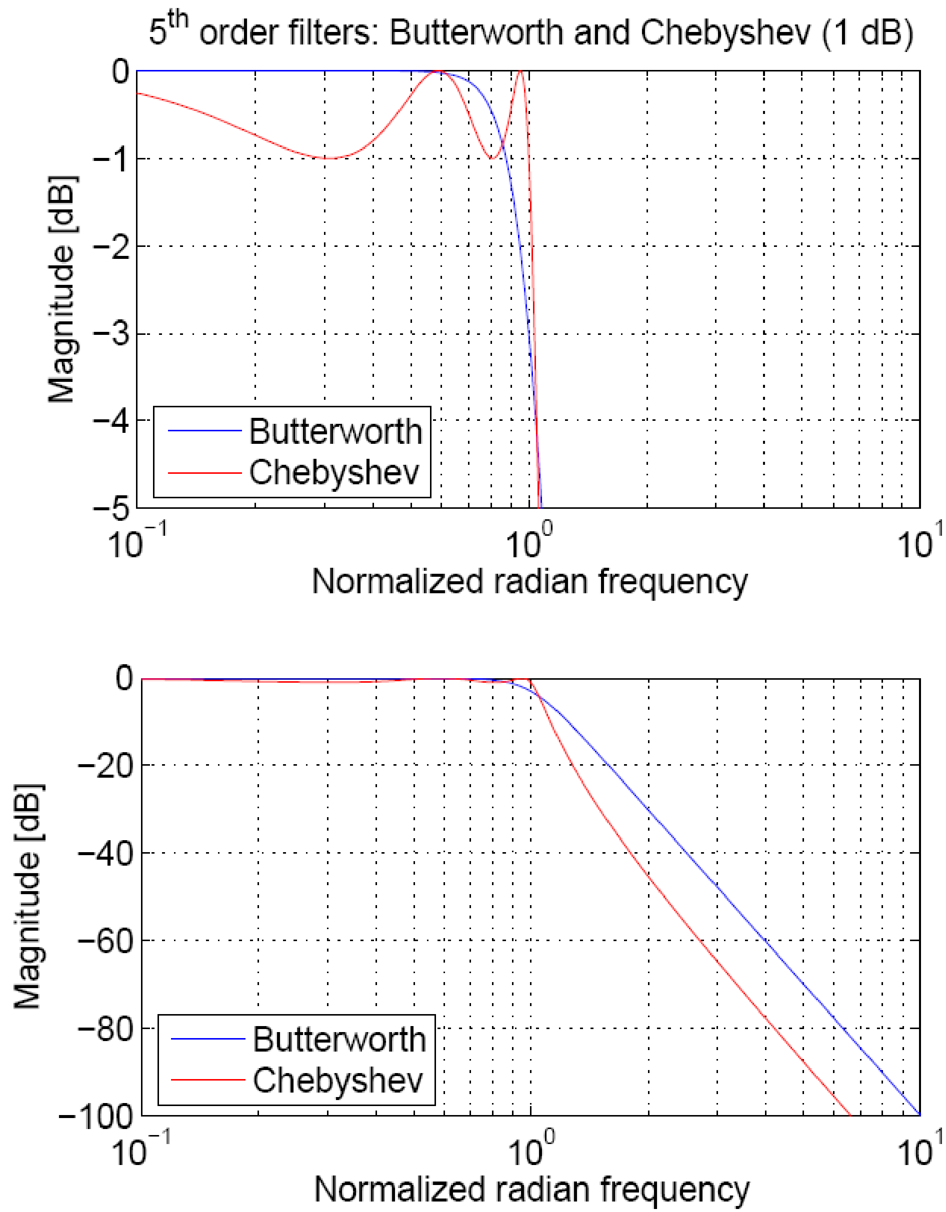
Comparison of 5th order filters. Note:

- o Butterworth: |H(j 1 rad/s)| ~ -3 dB
- o Chebyshev: |H(j 1 rad/s)| ~ -1 dB

(Another normalization might have been used)

Magnitude:

- o The Butterworth filter has a smooth characteristic
- o The Chebyshev filter has ripple in the passband
- o The Chebyshev filter has better stopband attenuation (in this exampe 18 dB more at $\omega \rightarrow \infty$)

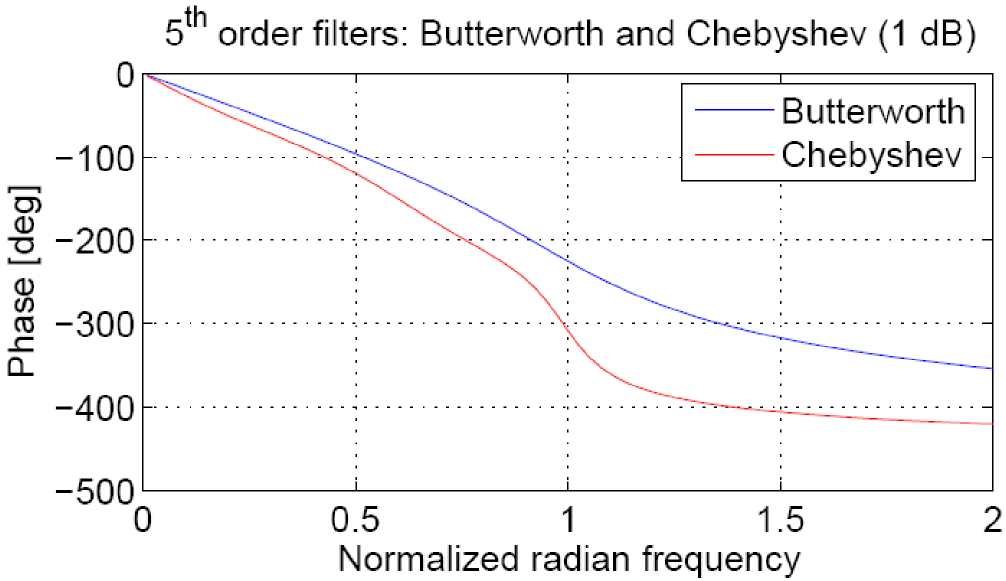


Properties of Butterworth and Chebyshev filters: Group delay

Phase (linear freq. scale):

- The Chebyshev filter has a stronger phase curvature in the passband

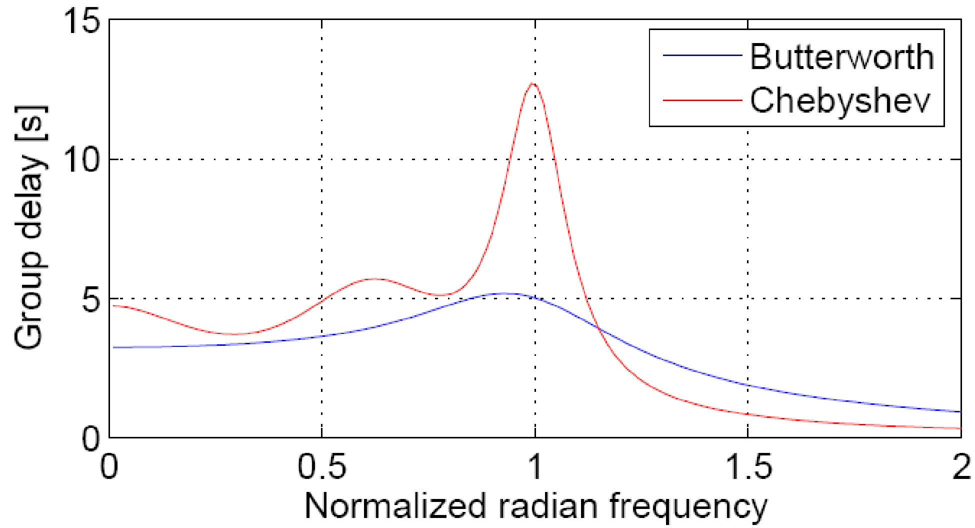
(recall that a linear phase is just a delay and a phase curvature means distortion)



Group delay is related to the phase response.

Mathematical definition:

$$\boxed{\tau_g = \frac{-d\angle H(j\omega)}{d\omega}}$$



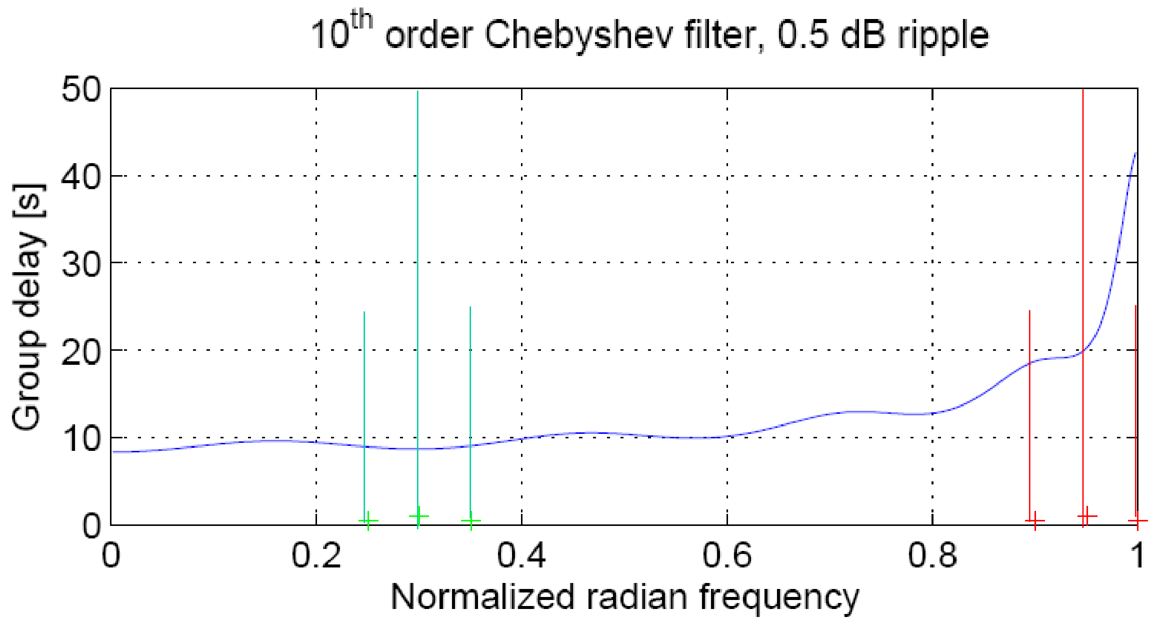
- not "visible" on a sine-signal
- More complicated signals may suffer from group delay variations

Impact of group delay

Example #1:

Input signal =

- o ½cos(0.25t) + cos(0.30t) + ½cos(0.35t)+
- o ½cos(0.90t) + cos(0.95t) + ½cos(1.00t)

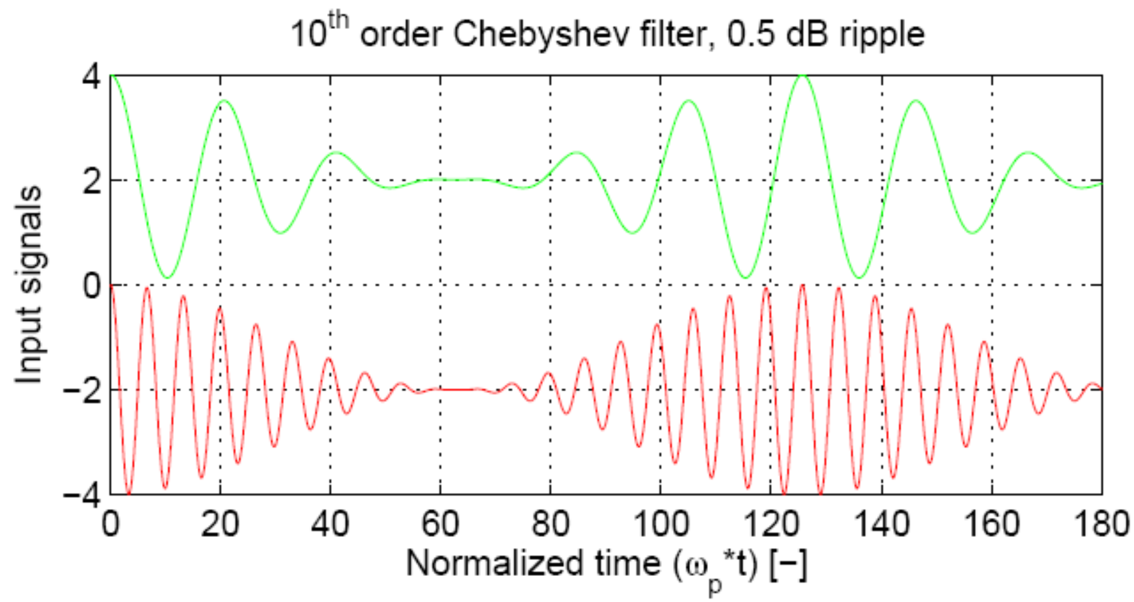


- o The green signal is in a frequency range with low delay
- o The red signal is in a frequency range with higher (varying) delay

Impact of group delay

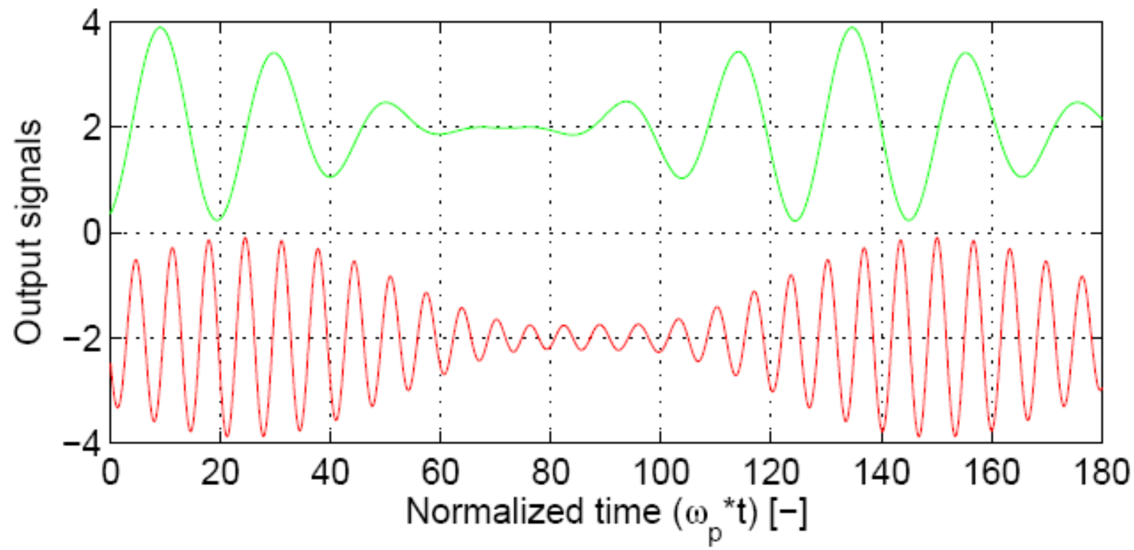
(Signals are shifted vertically +/-2 for better visibility)

Input signals:



Output signals:

- Both signals are delayed
- Different delays for the two (groups of) signals
- The importance depends on the application
- Some distortion



Which filter is best?

If there was a simple answer, I would only have presented one of them.

You must consider:

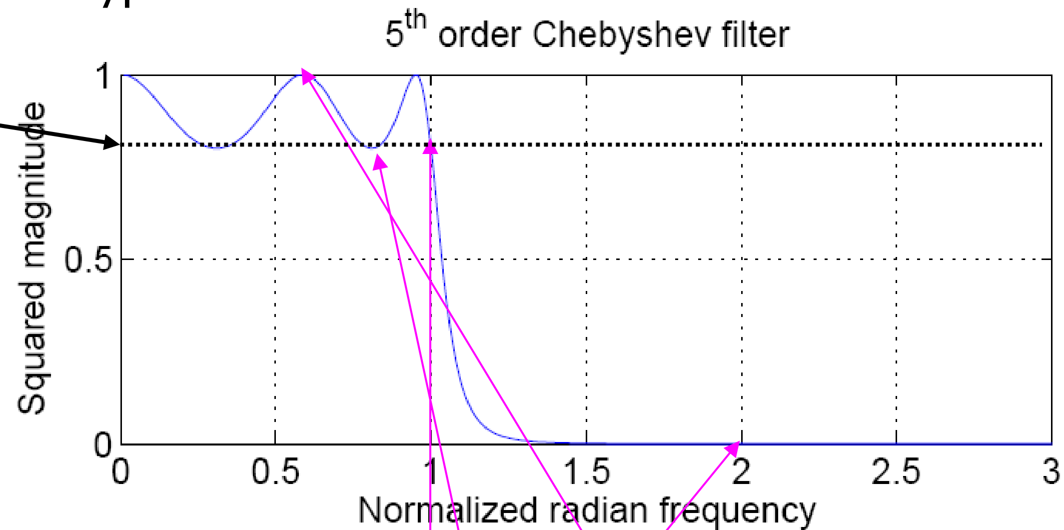
- How sensitive is the wanted signal to passband ripple, group delay variations and impulse response?
- Where are the unwanted signals located, and how much suppression is needed?

Other filter types: Inverse Chebyshev filter

Inverse Chebyshev filter = Chebyshev type 2 filter

$$\frac{1}{1+\varepsilon^2}$$

$$|H_{Cheb}(j\omega)|^2 = \frac{1}{1+\varepsilon^2 C_n^2(\omega)}$$

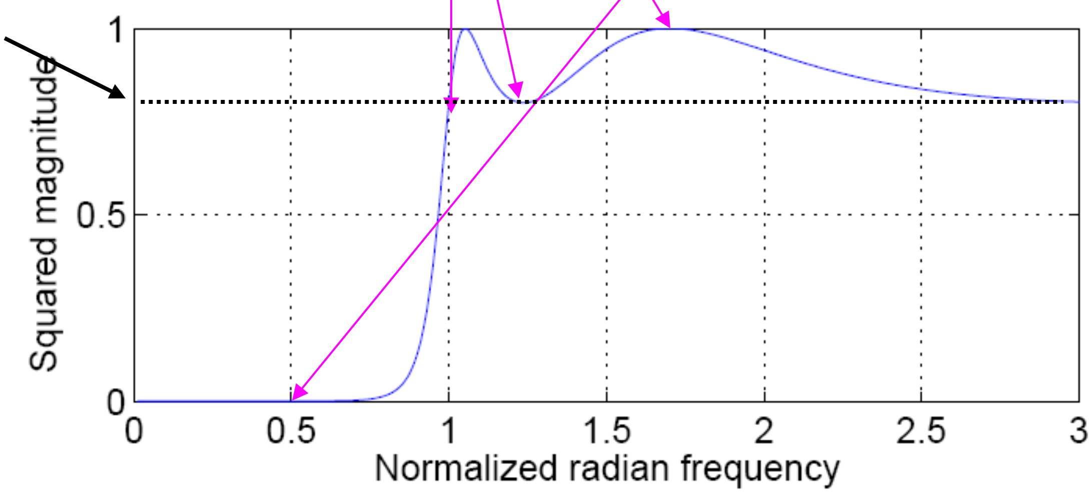


Intermediate transfer function:

$$\frac{1}{1+\varepsilon^2}$$

$$|H_b(j\omega)|^2 = \left|H_{Cheb}\left(j\frac{1}{\omega}\right)\right|^2$$

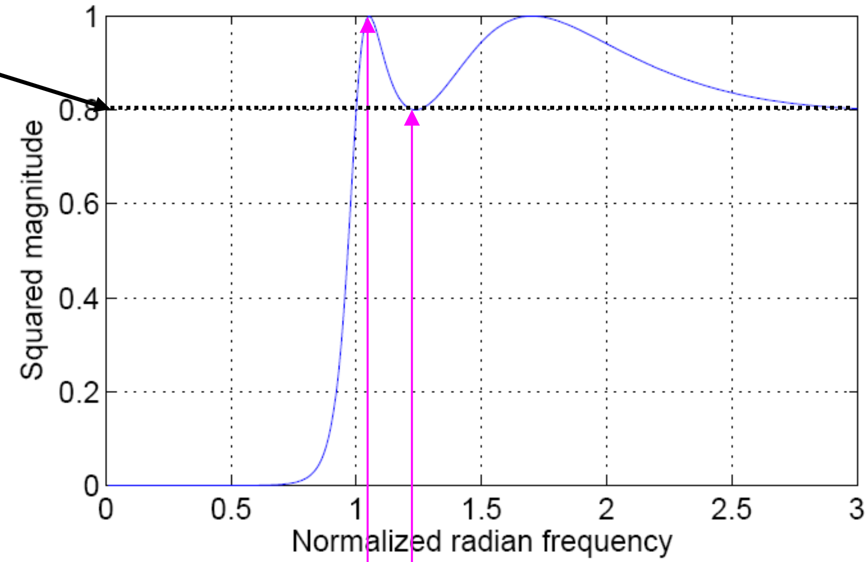
$$|H_b(j\omega)|^2 = \frac{1}{1+\varepsilon^2 C_n^2\left(\dfrac{1}{\omega}\right)}$$



$$|H_b(j\omega)|^2 = \left| H_{Cheb}(j\frac{1}{\omega}) \right|^2$$

$$\frac{1}{1+\varepsilon^2}$$

$$|H_b(j\omega)|^2 = \frac{1}{1+\varepsilon^2 C_n^2(\frac{1}{\omega})}$$

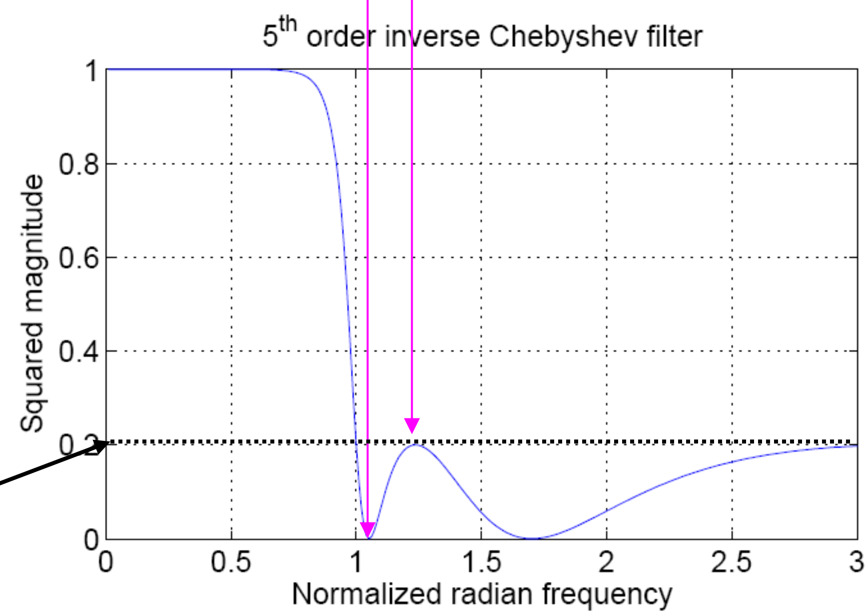


Inverse Chebyshev:

$$|H_{InvCheb}(j\omega)|^2 = 1 - |H_b(j\omega)|^2$$

$$= \frac{\varepsilon^2 C_n^2(\frac{1}{\omega})}{1+\varepsilon^2 C_n^2(\frac{1}{\omega})}$$

$$1 - \frac{1}{1+\varepsilon^2} = \frac{\varepsilon^2}{1+\varepsilon^2}$$

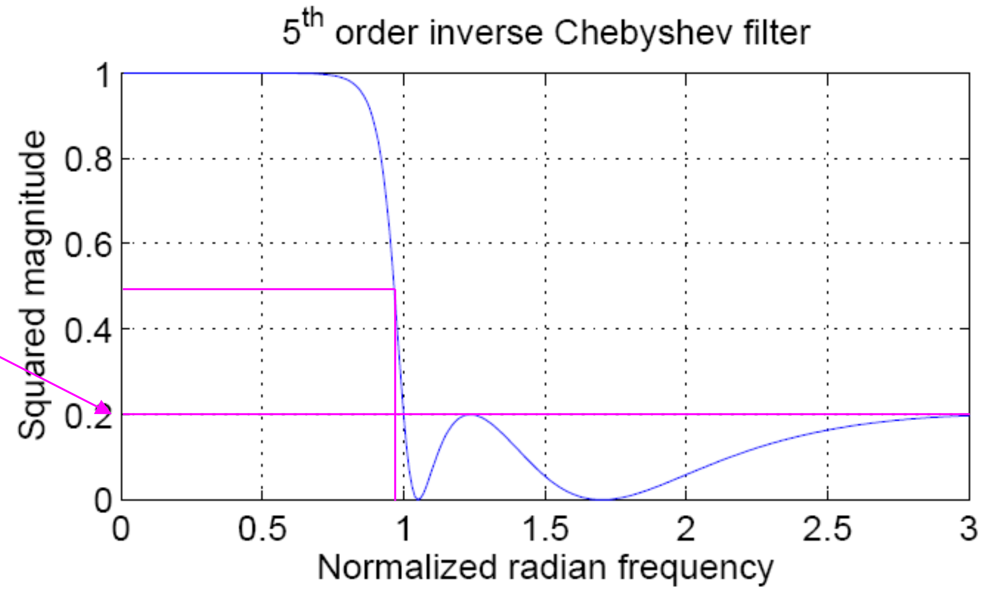


Exercise

$$|H_{InvCheb}(j\omega)|^2 = 1 - |H_b(j\omega)|^2$$

$$= \frac{\varepsilon^2 C_n^2(\frac{1}{\omega})}{1+\varepsilon^2 C_n^2(\frac{1}{\omega})}$$

$$\frac{\varepsilon^2}{1+\varepsilon^2}$$



$$\boxed{\varepsilon = \frac{1}{\sqrt{10^{\alpha_{SdB}/10}-1}} \quad (\alpha_{SdB} > 0)}$$

Inverse Chebyshev filter with requirements:
Min. 40 dB attenuation in stop band

Calculate:
- Epsilon
- Magnitude at 1 rad/s

5 min. break

Break over

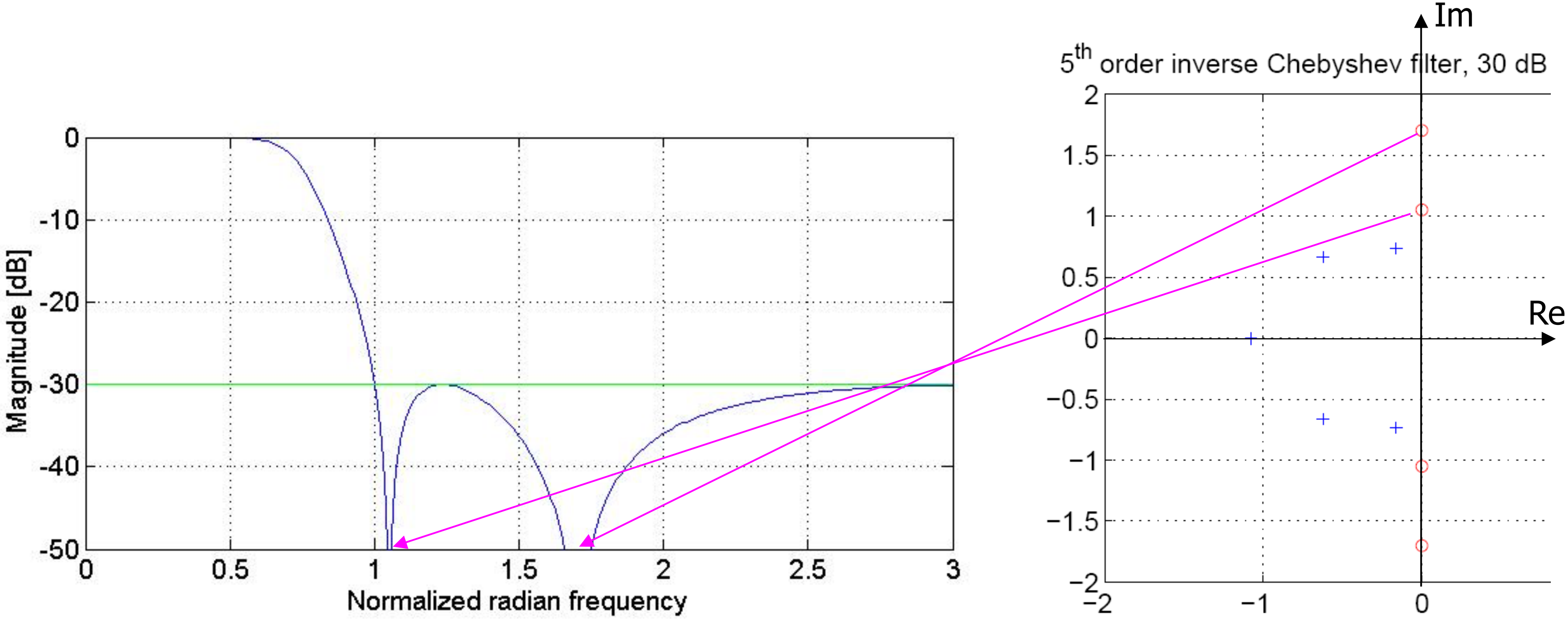
Inverse Chebyshev filter

Transfer function of a Butterworth or Chebyshev filter:

$$H(s) = \frac{K}{(s-p_r)(s-p_{c1})(s-p_{c1}^*)(s-p_{c2})(s-p_{c2}^*)\ldots\ldots}$$

Transfer function of an inverse Chebyshev filter (no proof given here):

$$H(s) = \frac{K(s-j\omega_{z1})(s+j\omega_{z1})(s-j\omega_{z2})(s+j\omega_{z2})\ldots\ldots}{(s-p_r)(s-p_{c1})(s-p_{c1}^*)(s-p_{c2})(s-p_{c2}^*)\ldots\ldots}$$



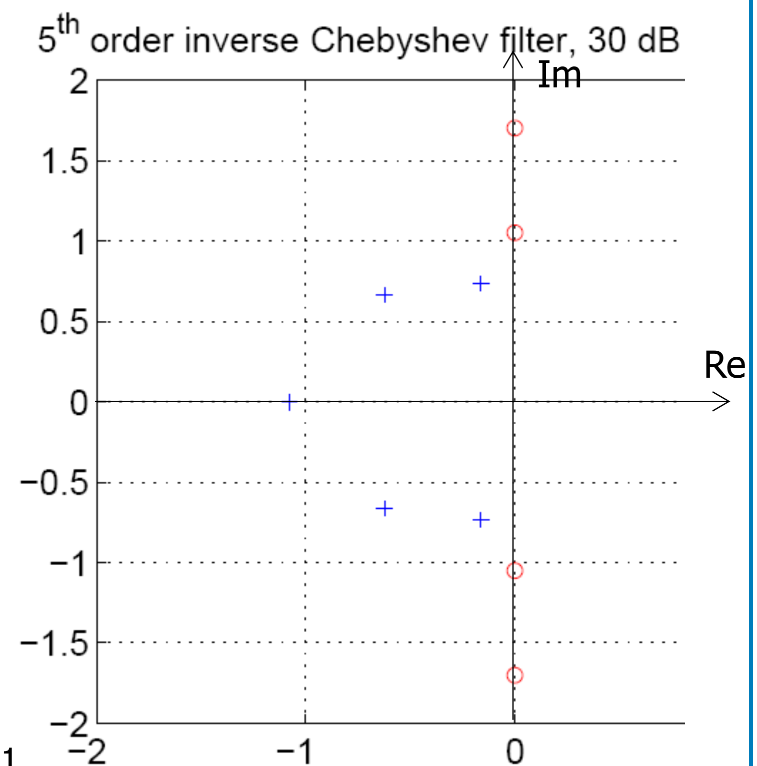
Inverse Chebyshev filter

Poles:

$$p_{k,InvCheb} = \frac{1}{p_{k,Cheb}}$$

$$p_{k,InvCheb} = \frac{1}{\sin\frac{(2k-1)\pi}{2n}\cdot\sinh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right) + j\cos\frac{(2k-1)\pi}{2n}\cdot\cosh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right)}$$

k = n+1, n+2...... 2n



Zeros:

$$z_k = j\omega_{z,k} = \frac{\pm j}{\cos\frac{(2k-1)\pi}{2n}}$$

k = 1, 2....

The number of zeros is rounded down to an even integer, e.g. 5 poles gives 4 zeros (k=1, 2)

Ref.: L.P. Paarmann: "Design and analysis of analog filters", Kluwer 2001

Inverse Chebyshev filter - Matlab

Useful Matlab commands:

```
Order = 5;                                   % Number of poles
dBstopAtt = 30;                              % Stopband attenuation [dB]
[Zeros Poles K] = cheb2ap(Order,dBstopAtt)   % Poles, zeros and gain constant

Zeros =
     0 + 1.0515i
     0 - 1.0515i
     0 + 1.7013i
     0 - 1.7013i
Poles =
  -0.1624 - 0.7349i
  -0.6222 - 0.6647i
  -1.0779
  -0.6222 + 0.6647i
  -0.1624 + 0.7349i
K =
   0.1582
```

Tiny exercise: Give it a try!

$$H(s) = K\frac{(s-j\omega_{z1})(s+j\omega_{z1})(s-j\omega_{z2})(s+j\omega_{z2})......}{(s-p_r)(s-p_{c1})(s-p_{c1}^*)(s-p_{c2})(s-p_{c2}^*)......}$$

Inverse Chebyshev filter - Matlab

- Useful Matlab commands:

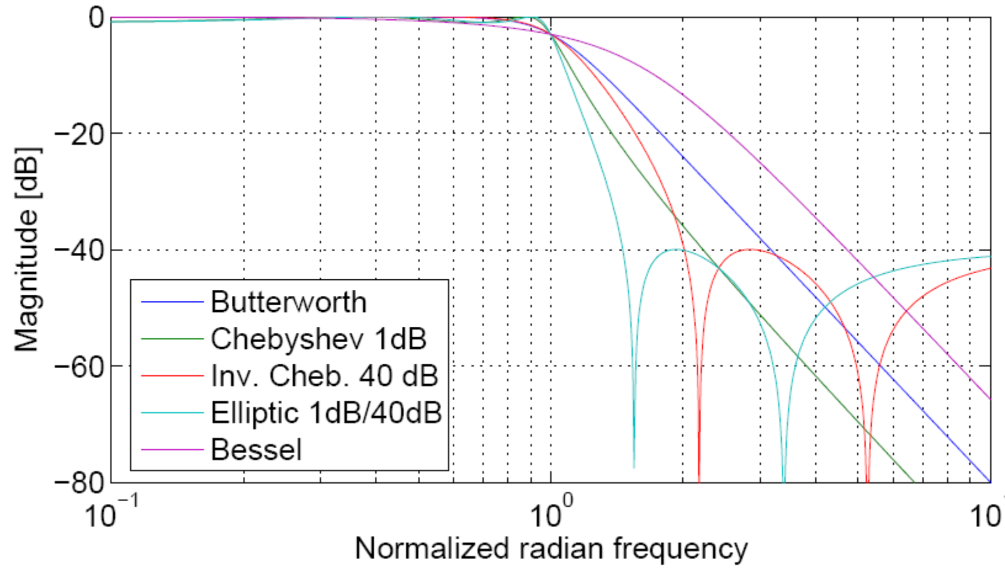
- >> Order = 5; % Number of poles
- dBstopAtt = 30; % Stopband attenuation [dB]
- Wstop = 1; % Stopband edge radian frequency
- [NumPoly DenomPoly] = **cheby2(Order,dBstopAtt,Wstop,'s')**
- % Numerator and denominator polynomials. 's' indicates analog filter
- NumPoly =
- 0 0.1582 -0.0000 0.6328 -0.0000 0.5062
- DenomPoly =
- 1.0000 2.6472 3.4913 2.9142 1.5198 0.5062

$$H(s) = \frac{b_1 s^n + b_2 s^{n-1} + \ldots + b_{n-1} s^2 + b_n s + b_{n+1}}{a_1 s^n + a_2 s^{n-1} + \ldots + a_{n-1} s^2 + a_n s + a_{n+1}}$$

- >> tf(NumPoly,DenomPoly)
-
- Transfer function: 0 0
- 0.1582 ~~s^4 - 1.235e~~-016 s^3 + 0.6328 ~~s^2 - 1.125e~~-016 s + 0.5062
- ---
- s^5 + 2.647 s^4 + 3.491 s^3 + 2.914 s^2 + 1.52 s + 0.5062

Elliptic function filters:

Ripple in both passband and stopband ☹
Very sharp cut-off ☺
High group delay distortion ☹
Matlab: ”ellip”

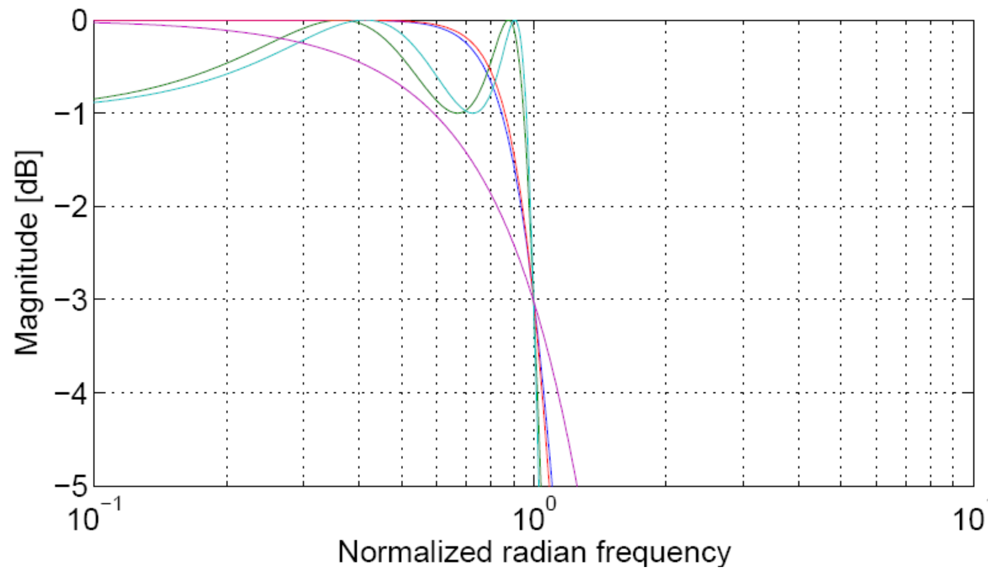


Bessel/Thomson filters:

Maximally flat group delay ☺
”Soft” magnitude cut-off ☹
Matlab: ”besself”

Gaussian filters:

No ripple in the impulse response ☺
”Very soft” magnitude cut-off ☹

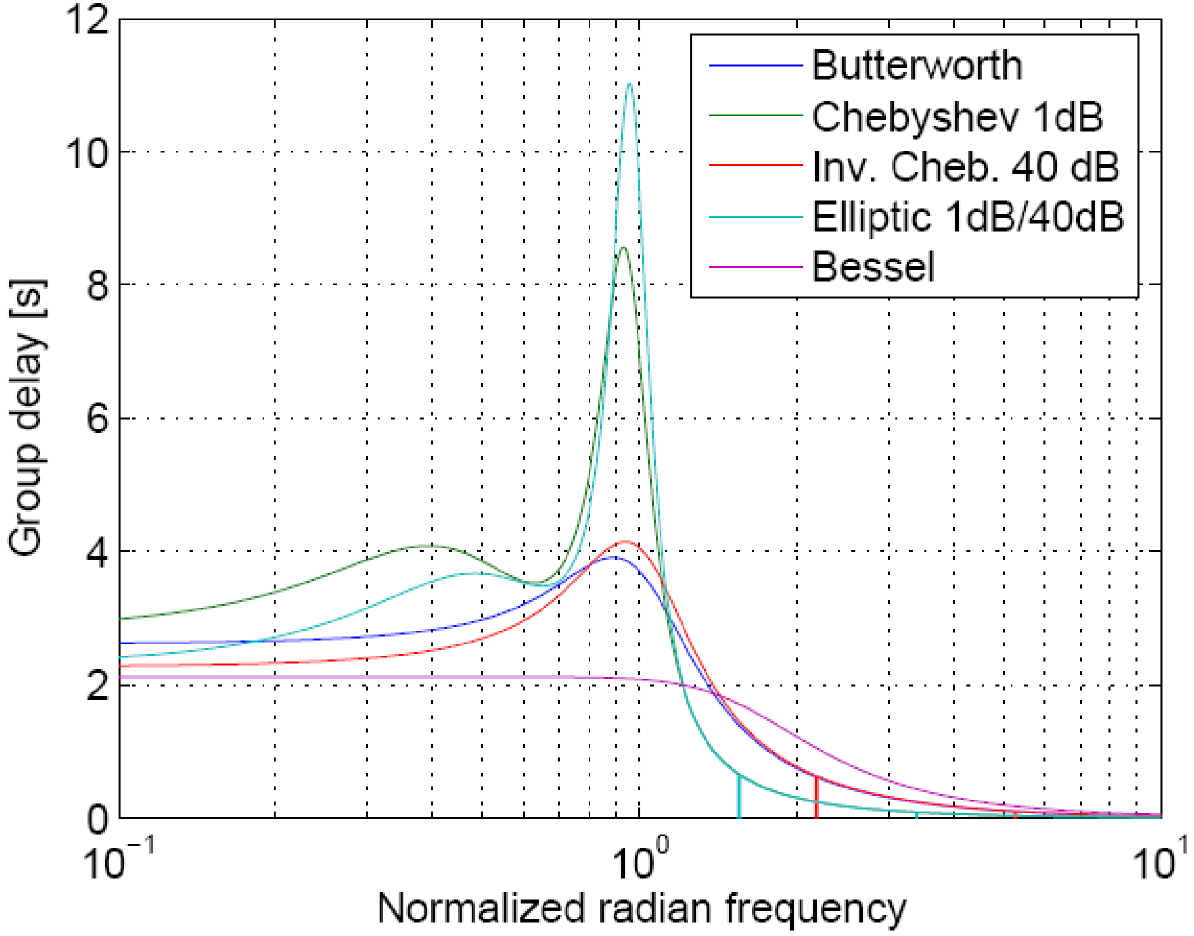


NOTE:

All filters scaled to have a 3-dB bandwidth of 1 rad/s
The comparison depends on choice of parameters (1 dB, 40 dB, n = 4)

Other filter types

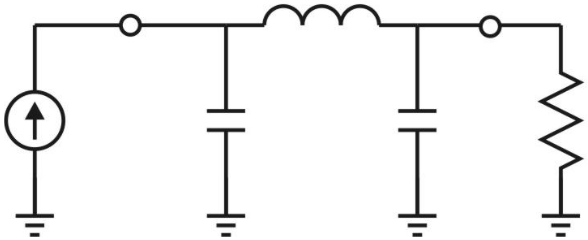
- Group delay:



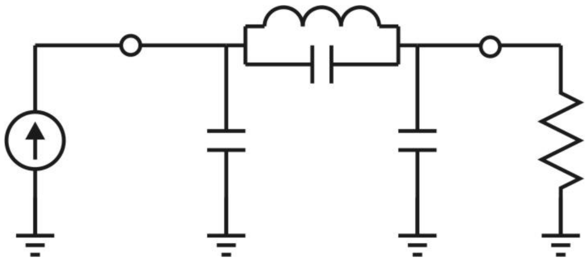
Other filter types, topologies

Transmission zeros require a change of circuit topology:

All-pole:
Butterworth
Chebyshev
Bessel/Thomson

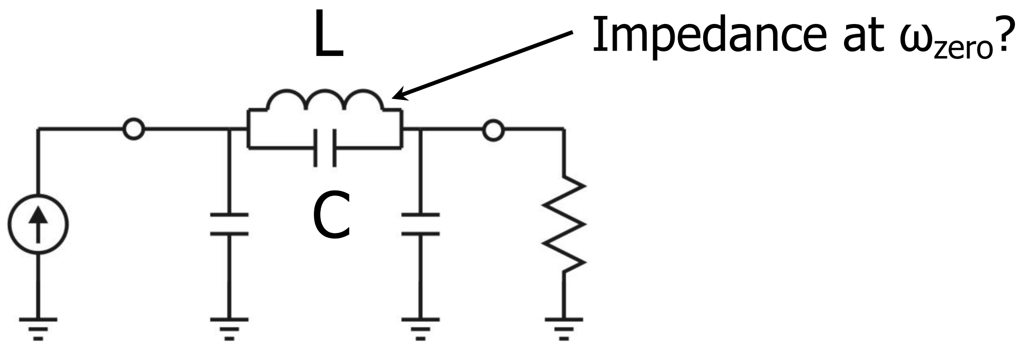


Poles and zeros:
Elliptic
Inverse Chebyshev



Also different topologies for active filters

Exercise



Find the impedance of the shunt resonator consisting of L and C

Appendix: Finding poles for Chebyshev transfer function

$$H(s)\cdot H(-s)=\frac{1}{1+\varepsilon^2 C_n^2(-js)}$$

Find poles ⇔ roots of ☺:

$$1+\varepsilon^2 C_n^2(-js)=0$$

$$C_n(-js)=\pm j\frac{1}{\varepsilon}$$

$$\cos[n\cdot\cos^{-1}(-js)]=\pm j\frac{1}{\varepsilon}$$

Defining u and v (both real):

$$\cos^{-1}(-js)=u+jv$$

$$-js=\cos(u+jv)$$

Using next page:

$$-js=\cos(u)\cdot\cosh(v)-j\sin(u)\cdot\sinh(v)$$

$$s=\sin(u)\cdot\sinh(v)+j\cos(u)\cdot\cosh(v)$$

Math backup

$$\cos(z) = ½(e^{jz} - e^{-jz})$$

$$\underline{\cos(u + jv)} = ½\left(e^{j(u+jv)} + e^{-j(u+jv)}\right)$$

$$= ½\left(e^{-v}(\cos(u) + j\sin(u)) + e^{v}(\cos(-u) + j\sin(-u))\right)$$

$$= ½(e^{v} + e^{-v})\cos(u) - j½(e^{v} - e^{-v})\sin(u)$$

$$\underline{= \cos(u)\cosh(v) - j\sin(u)\sinh(v)}$$

since

$$\cosh(x) = ½(e^{x} + e^{-x})$$

$$\sinh(x) = ½(e^{x} - e^{-x})$$

Chebyshev transfer function

Given:

$$\cos[n\cdot\cos^{-1}(-js)] = \pm j\frac{1}{\varepsilon} \qquad \cos^{-1}(-js) = u + jv$$

$$\cos(u+jv) = \cos(u)\cosh(v) - j\sin(u)\sinh(v)$$

It is found that:

$$\cos[n\cdot(u+jv)] = \pm j\frac{1}{\varepsilon}$$

$$\cos(n\cdot u)\cosh(n\cdot v) - j\sin(n\cdot u)\sinh(n\cdot v) = \pm j\frac{1}{\varepsilon}$$

Taking real and imaginary parts:

$$\cos(n\cdot u)\cosh(n\cdot v) = 0$$

$$\sin(n\cdot u)\sinh(n\cdot v) = \mp\frac{1}{\varepsilon}$$

Since cosh(x)>0 for all x:

$$\cos(n\cdot u) = 0$$

$$n\cdot u = (2k-1)\frac{\pi}{2} \Leftrightarrow u = \frac{(2k-1)\pi}{2n}$$

Chebyshev transfer function

$$u = \frac{(2k-1)\pi}{2n} \quad \Rightarrow \quad \sin(n \cdot u) = \pm 1$$

Given: $$\sin(n \cdot u)\sinh(n \cdot v) = \mp \frac{1}{\varepsilon}$$

Choosing the +sign
it follows: $$v = \frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}$$

Inserting in a previous equation:

$$s = \sin(u) \cdot \sinh(v) + j\cos(u) \cdot \cosh(v)$$

The poles are found:

$$p_k = \sin\frac{(2k-1)\pi}{2n} \cdot \sinh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right) + j\cos\frac{(2k-1)\pi}{2n} \cdot \cosh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right)$$

k = 1,2,.. 2n