Signalbehandling for computer-ingeniører
COMTEK-5, E20

&

Signalbehandling
EIT-5, E20

9. Digital FIR Filters – LP/HP/BP/BS – Analysis of Transfer Functions

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Signal Processing
9th Lecture

Last time we talked about systems with generalized Linear Phase ;

$$① \quad H(e^{j\omega}) = A(e^{j\omega}) \cdot e^{-j(\alpha\omega - \beta)}$$

Real Amplitude Function

Complex Phase Function Equ. for straight line
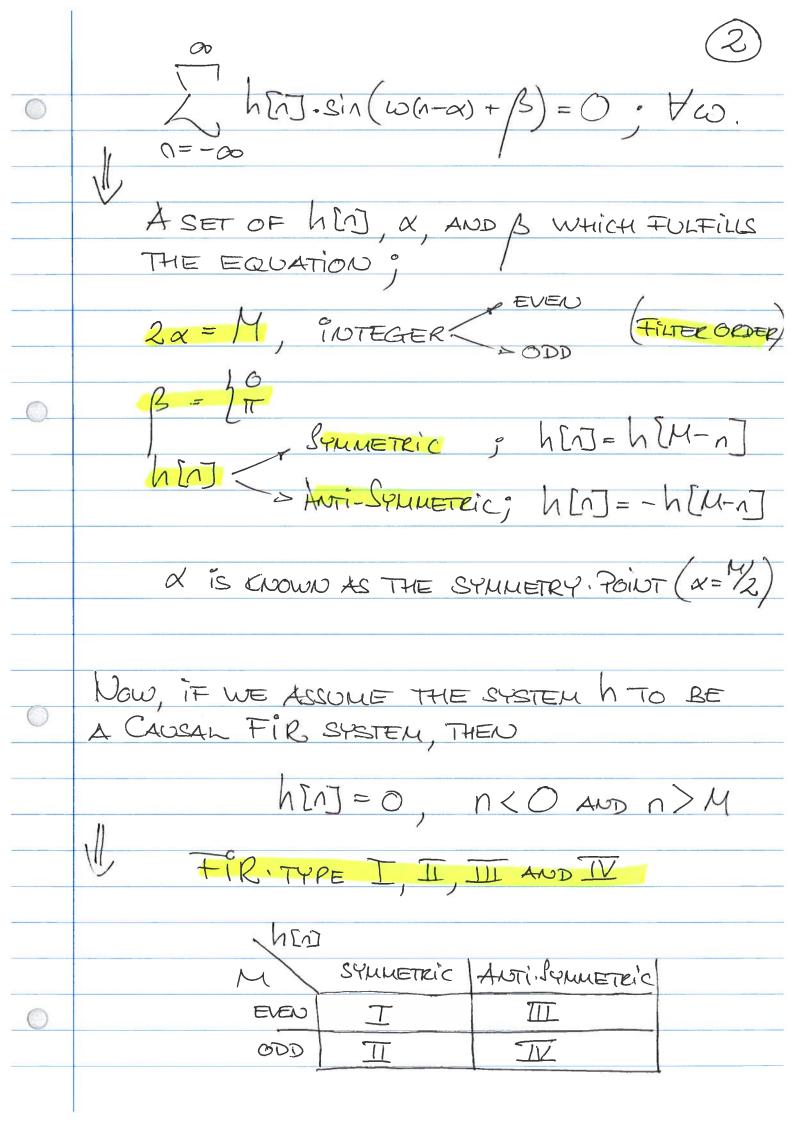
Frequency response for general system ;

$$② \quad H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] \cdot e^{-j\omega n}$$

Deriving expressions for $\tan\underline{H(e^{j\omega})}$, both for ① and ②, and equalizing these two, we got;

$$\sum_{n=-\infty}^{\infty} h[n] \cdot \sin(\omega(n-\alpha)+\beta) = 0, \quad \forall \omega$$
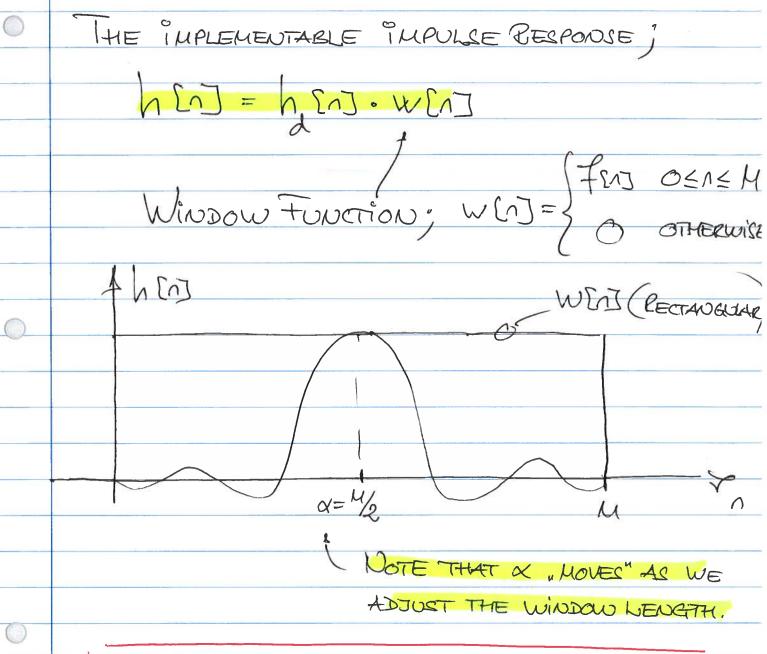
Which is a necessary condition for Linear Phase.

$$\sum_{n=-\infty}^{\infty} h[n] \cdot \sin\left(\omega(n-\alpha) + \beta\right) = 0 \; ; \; \forall \omega.$$

$\Downarrow$

A SET OF $h[n]$, α, AND β WHICH FULFILLS THE EQUATION ;

$2\alpha = M$, INTEGER $\begin{cases} \text{EVEN} \\ \text{ODD} \end{cases}$ (FILTER ORDER)

$\beta = \begin{cases} 0 \\ \pi \end{cases}$

$h[n] \begin{cases} \text{SYMMETRIC} \; ; \; h[n] = h[M-n] \\ \text{ANTI-SYMMETRIC} ; \; h[n] = -h[M-n] \end{cases}$

α IS KNOWN AS THE SYMMETRY POINT $\left(\alpha = \frac{M}{2}\right)$

Now, IF WE ASSUME THE SYSTEM h TO BE A CAUSAL FIR SYSTEM, THEN

$$h[n] = 0, \quad n < 0 \text{ AND } n > M$$

$\Downarrow$

FIR TYPE $\underline{I}$, $\underline{II}$, $\underline{III}$ AND $\underline{IV}$

M $\diagdown$ $h[n]$	SYMMETRIC	ANTI-SYMMETRIC
EVEN	$\underline{I}$	$\underline{III}$
ODD	$\underline{II}$	$\underline{IV}$
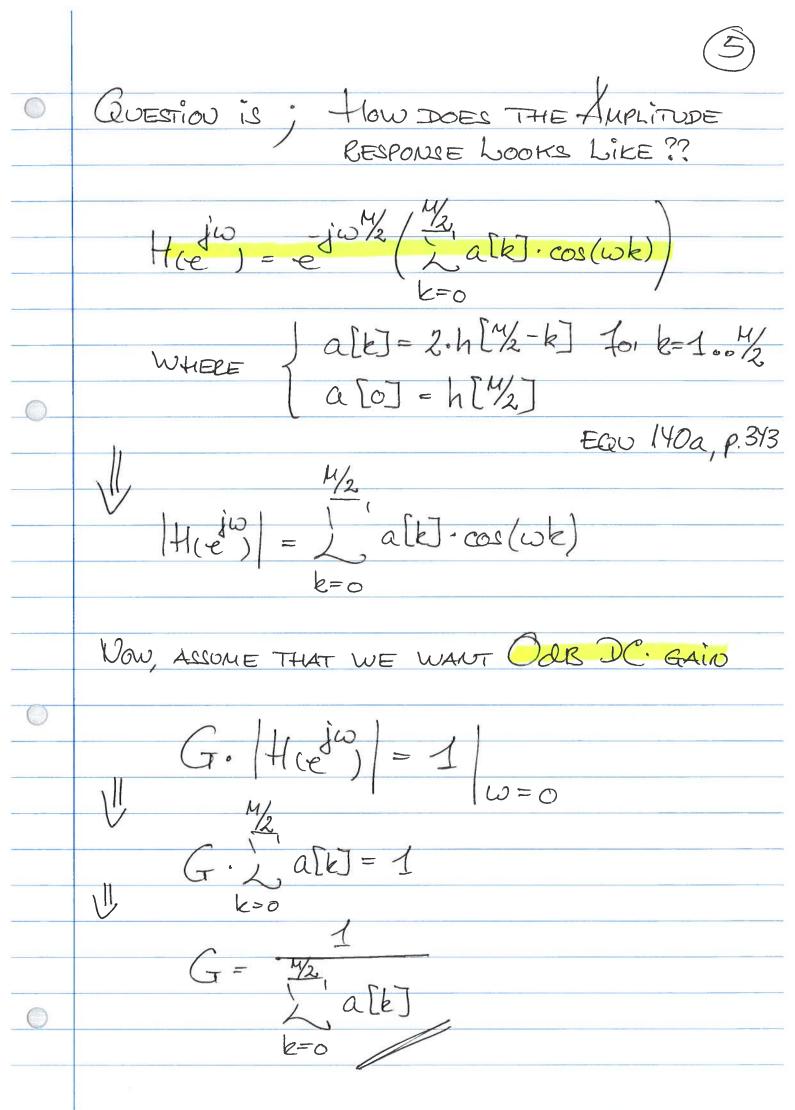
DESIGNING AN F.I.R. FILTER.

- ==THE DESIRED AMPLITUDE RESPONSE==
- ==LINEAR PHASE RESPONSE==

THE DESIRED BRICKWALL FILTER

==THE DESIRED FREQUENCY RESPONSE==

$$H_d(e^{j\omega}) = \begin{cases} 1 \cdot e^{-j\omega \frac{M}{2}} & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

$$\alpha = \frac{M}{2} \quad \text{AND} \quad \beta = 0$$

USING THE ==INVERSE== DTFT, WE CAN FIND

$$h_d[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega \frac{M}{2}} \cdot e^{j\omega n} \, d\omega$$

$$\Downarrow$$

$$h_d[n] = \frac{\sin\left(\omega_c\left(n - \frac{M}{2}\right)\right)}{\pi\left(n - \frac{M}{2}\right)}$$

==NON. CAUSAL==
==INFINITE==

↳ SYMMETRIC FUNCTION

THE IMPLEMENTABLE IMPULSE RESPONSE;

$$h[n] = h_d[n] \cdot w[n]$$

Window Function; $w[n] = \begin{cases} f[n] & 0 \le n \le M \\ 0 & \text{OTHERWISE} \end{cases}$

$\alpha = M/2$

NOTE THAT α „MOVES" AS WE ADJUST THE WINDOW LENGTH.

ALSO NOTE THAT $w[n]$ IS SYMMETRIC, THUS

$$h[n] = h_d[n] \cdot w[n]$$ IS ALSO SYMMETRIC

AND THEREFORE $H(z)$ HAS LINEAR PHASE

QUESTION IS ; HOW DOES THE AMPLITUDE
RESPONSE LOOKS LIKE ??

$$H(e^{j\omega}) = e^{-j\omega M/2} \left(\sum_{k=0}^{M/2} a[k] \cdot \cos(\omega k) \right)$$

WHERE
$$\begin{cases} a[k] = 2 \cdot h[M/2 - k] & \text{for } k = 1 .. M/2 \\ a[0] = h[M/2] \end{cases}$$

EQU 140a, p.343

$$|H(e^{j\omega})| = \sum_{k=0}^{M/2} a[k] \cdot \cos(\omega k)$$

Now, ASSUME THAT WE WANT $0 dB$ DC. GAIN

$$G \cdot |H(e^{j\omega})| = 1 \Big|_{\omega = 0}$$

$$G \cdot \sum_{k=0}^{M/2} a[k] = 1$$

$$G = \frac{1}{\displaystyle\sum_{k=0}^{M/2} a[k]}$$

FOR GIVEN WINDOW FUNCTION W[n], OUR TASK NOW IS TO FIND THE SMALLEST POSSIBLE VALUE FOR THE FILTER ORDER M WHICH SATISFIES THE SPECIFICATIONS ;

THE PROBLEM WE ARE FACING HERE IS

$$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega})$$
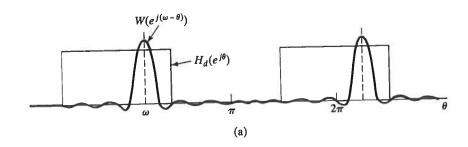
IDEALY WE WOULD LIKE $W(e^{j\omega})$ TO BE AN IMPULSE TRAIN, WHICH HOWEVER WOULD REQUIRE $W[n] = 1, \forall n$ ⟵ non·causal infinite.

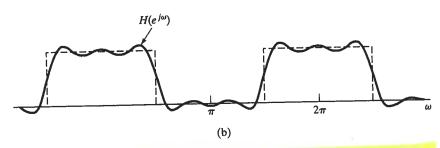SO, WE ARE LOOKING FOR A COMPROMISE.

Filter Design Techniques

(a)

(b)

Figure 27 (a) Convolution process implied by truncation of the ideal impulse response. (b) Typical approximation resulting from windowing the ideal impulse response.
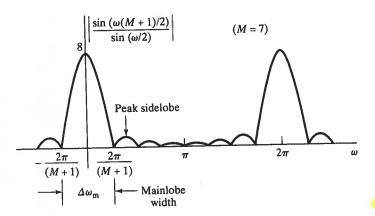
Figure 28 Magnitude of the Fourier transform of a rectangular window ($M = 7$).

5.1 Properties of Commonly Used Windows

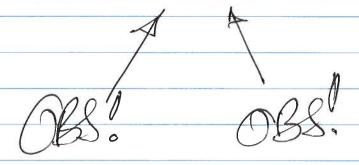Some commonly used windows are shown in Figure 29. These windows are defined by the following equations:

Rectangular

$$w[n] = \begin{cases} 1, & 0 \leq n \leq M, \\ 0, & \text{otherwise} \end{cases} \tag{60a}$$

TABLE 2   COMPARISON OF COMMONLY USED WINDOWS

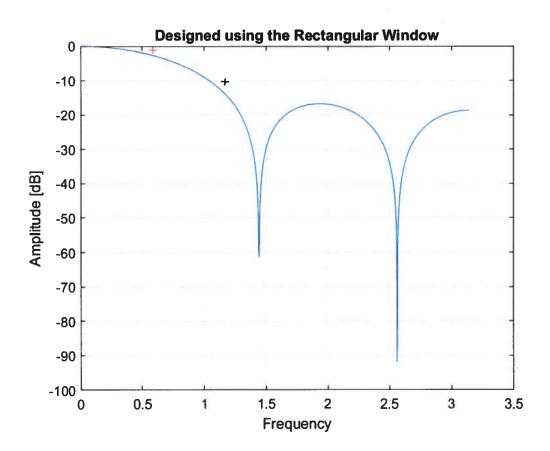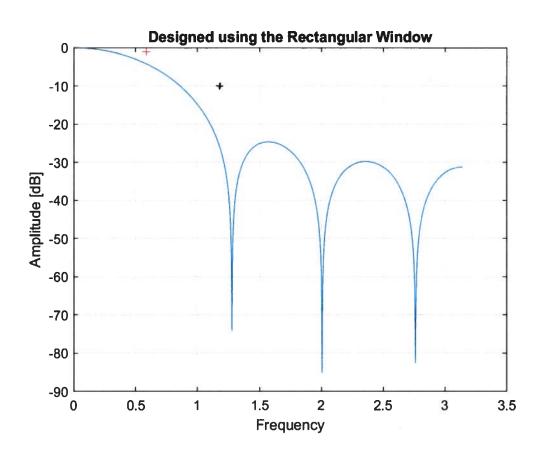| Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hann	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

OBS!   OBS!

⊙ PEAK OF SIDE·LOBE IS CONSTANT
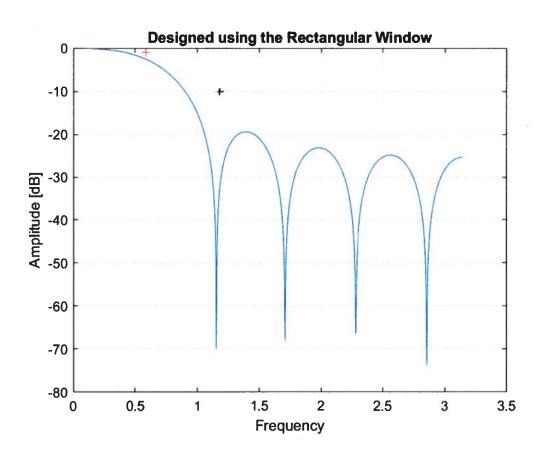
⊙ WIDTH OF MAIN·LOBE IS INVERSE PROPORTIONAL TO M

So, NO MATTER HOW HIGH A FILTER ORDER WE CHOOSE, WE CANNOT (USING THE SAME TYPE OF WINDOW) REDUCE THE PASS- AND STOP·BAND RIPPLE.
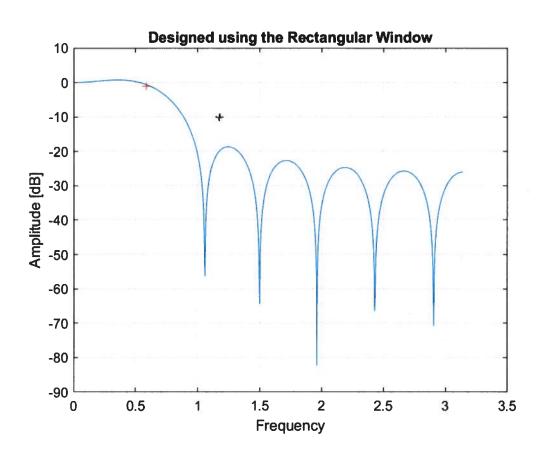
WE CAN HOWEVER, BY INCREASING M, OBTAIN A BETTER APPROXIMATION TO THE TRANSITION FROM PASS- TO STOP·BAND ~ A NARROWER TRANSITION BAND.

Designed using the Rectangular Window

Amplitude [dB] vs Frequency

$\mu = 4$

Designed using the Rectangular Window

$\mu = 8$

Designed using the Rectangular Window

$\mu = 10$

Designed using the Rectangular Window

M=12

Designed using the Rectangular Window

SIDE LOBE DOWN APPROX. 20dB.

$\mu = 20$

Designed using the Rectangular Window

NOTE THAT THE SIDE LOBE IS DOWN ONLY ~20dB
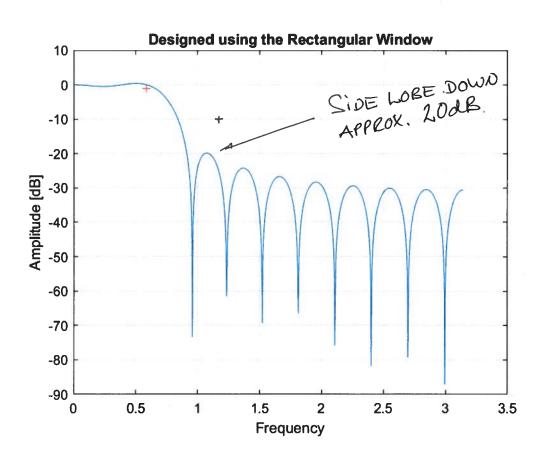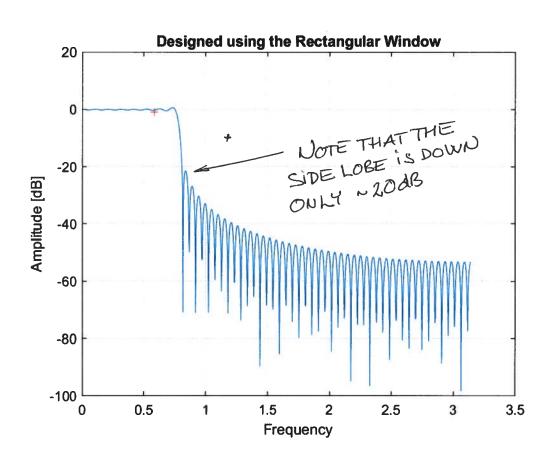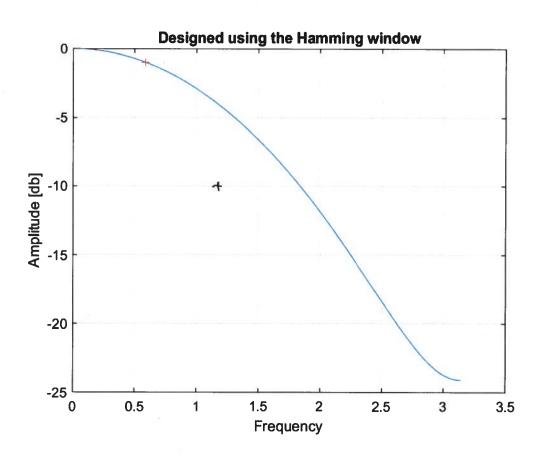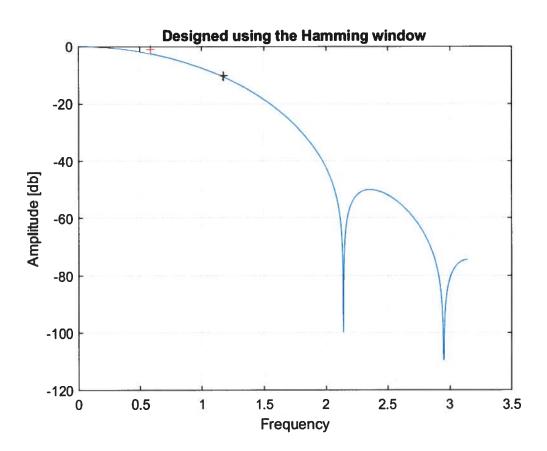
$M = 120$

LET'S TRY WITH THE Hamming Window

Designed using the Hamming window

Amplitude [db] vs Frequency

$u=4$

Designed using the Hamming window

$\mu = 8$

Designed using the Hamming window

$M = 12$

Designed using the Hamming window

$M = 24$

Designed using the Hamming window

Side lobe is down approx 55 dB

$\mu = 28$

Designed using the Hamming window

SIDE LOBE STILL DOWN APPROX. 55dB.

$M = 120$

Other Filter Types - High·Pass

$$H_d(e^{j\omega}) = \begin{cases} 0 & |\omega| < \omega_c \\ e^{-j\omega M/2} & \omega_c \leq |\omega| \leq \pi \end{cases}$$

THE DESIRED FREQUENCY RESPONSE — WHICH WE CAN NOW REWRITE;

$$H_{d_{HP}}(e^{j\omega}) = e^{-j\omega M/2} - H_{d_{LP}}(e^{j\omega})$$

USING THIS EXPRESSION, WE CAN NOW FIND THE DESIRED IMPULSE RESPONSE;

$$h_{d_{HP}}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{d_{HP}} \cdot e^{j\omega n} \, d\omega$$

$\Downarrow$

$$h_{d_{HP}}[n] = \frac{\sin\left(\pi\left(n - M/2\right)\right)}{\pi\left(n - M/2\right)} - \frac{\sin\left(\omega_c\left(n - M/2\right)\right)}{\pi\left(n - M/2\right)}$$

$$-\infty < n < \infty$$

EQU. 80, p. 571.

$$h_{d_{HP}}[n] = \frac{\sin(\pi(n - M/2))}{\pi(n - M/2)} - \frac{\sin(\omega_c(n - M/2))}{\pi(n - M/2)}$$

$$= \begin{cases} 1 & \text{for } n = M/2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{IDENTICAL TO LP}$$

⇓

THE IMPULSE RESPONSE (AND THUS THE FILTER COEFFICIENT) FOR THE HP·FILTER CAN BE EXPRESSED IN TERMS OF THE L·P·FILTER.

$$\circledcirc \quad b_{i_{HP}} = -b_{i_{LP}} \qquad i = 0, .., M/2 - 1, M/2 + 1, .., M$$

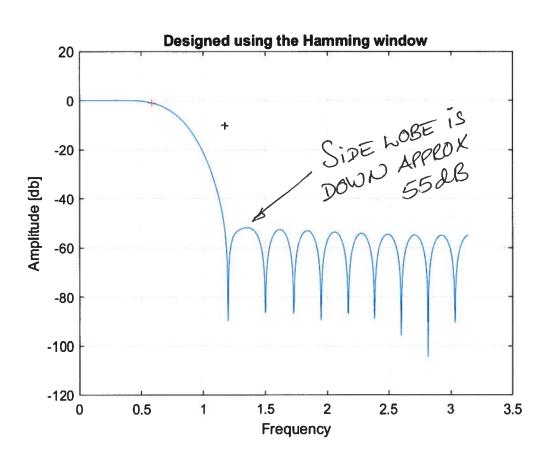$$\circledcirc \quad b_{M/2_{HP}} = 1 - b_{M/2_{LP}}$$
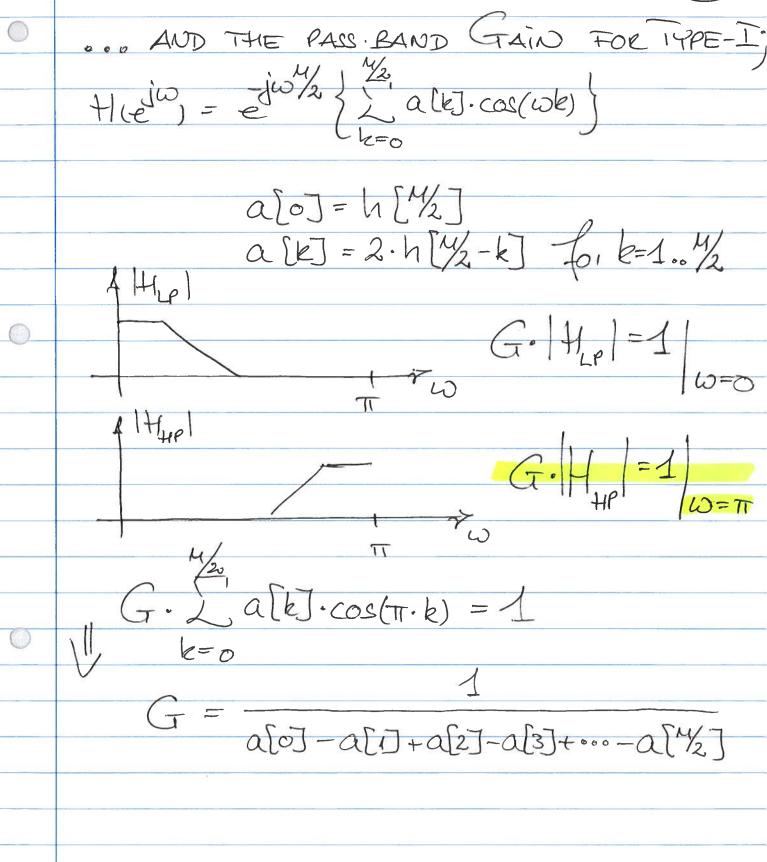
THE IMPULSE RESPONSE IS STILL SYMMETRIC AND THUS STILL LINEAR PHASE.

SO, DESIGNING A HIGH·PASS FILTER CAN BE DONE BY DESIGNING THE „CORRESPONDING" LOW·PASS FILTER, AND THEN SIMPLY JUST NEGATING THE COEFFICIENTS (AND SPECIAL $i = M/2$)
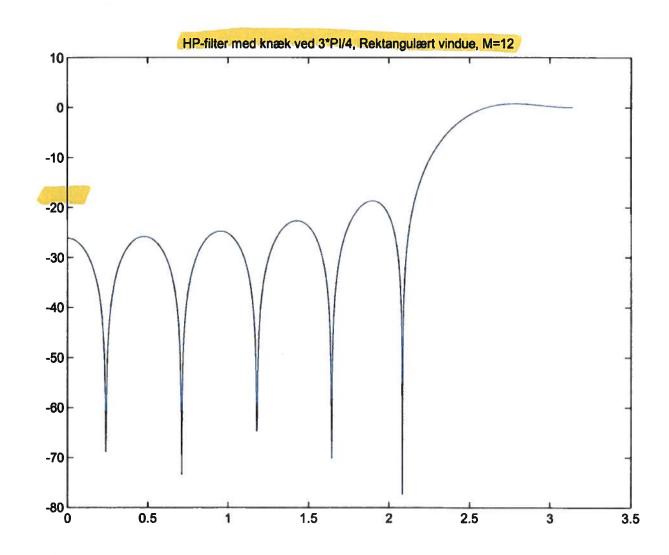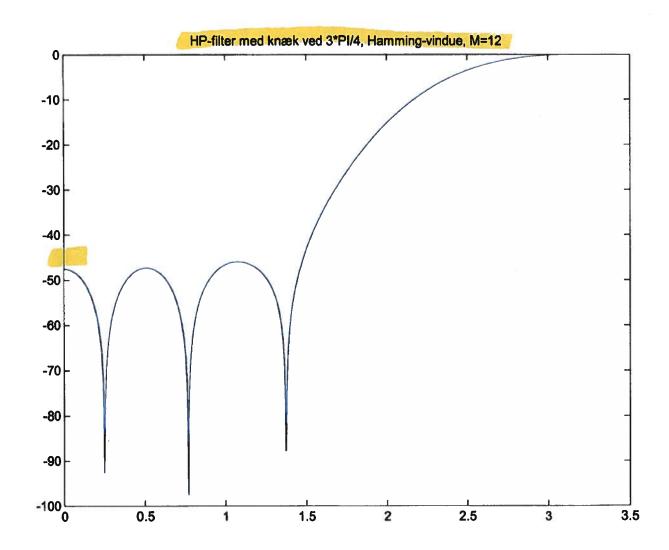
... AND THE PASS·BAND GAIN FOR TYPE-I;

$$H(e^{j\omega}) = e^{-j\omega M/2} \left\{ \sum_{k=0}^{M/2} a[k] \cdot \cos(\omega k) \right\}$$

$$a[0] = h\left[\frac{M}{2}\right]$$

$$a[k] = 2 \cdot h\left[\frac{M}{2} - k\right] \quad \text{for } k = 1 .. \frac{M}{2}$$

$$G \cdot |H_{LP}| = 1 \Big|_{\omega=0}$$

$$G \cdot |H_{HP}| = 1 \Big|_{\omega=\pi}$$

$$G \cdot \sum_{k=0}^{M/2} a[k] \cdot \cos(\pi \cdot k) = 1$$

$$\Downarrow$$

$$G = \frac{1}{a[0] - a[1] + a[2] - a[3] + \cdots - a\left[\frac{M}{2}\right]}$$

HP-filter med knæk ved 3*PI/4, Rektangulært vindue, M=12

HP-filter med knæk ved 3*PI/4, Hamming-vindue, M=12

BAND-PASS FILTERS

$$H_{d_{BP}}(e^{j\omega}) = \begin{cases} e^{-j\omega \frac{M}{2}} & \begin{cases} -\omega_c - \frac{\Delta \omega}{2} \le \omega \le -\omega_c + \frac{\Delta \omega}{2} \\ \omega_c - \frac{\Delta \omega}{2} \le \omega \le \omega_c + \frac{\Delta \omega}{2} \end{cases} \\ \\ 0 & \text{OTHERWISE} \end{cases}$$

$$h_{d_{BP}}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{d_{BP}}(e^{j\omega}) \cdot e^{j\omega n} \, d\omega$$

$$h_{d_{BP}}[n] = \frac{1}{2\pi} \left\{ \int_{-\omega_c - \frac{\Delta \omega}{2}}^{-\omega_c + \frac{\Delta \omega}{2}} e^{-j\omega \frac{M}{2}} \cdot e^{j\omega n} \, d\omega + \int_{\omega_c - \frac{\Delta \omega}{2}}^{\omega_c + \frac{\Delta \omega}{2}} e^{-j\omega \frac{M}{2}} \cdot e^{j\omega n} \, d\omega \right\}$$

$$h_{d_{BP}}[n] = \frac{\sin\left(\left(\omega_c + \frac{\Delta \omega}{2}\right)\left(n - \frac{M}{2}\right)\right) - \sin\left(\left(\omega_c - \frac{\Delta \omega}{2}\right)\left(n - \frac{M}{2}\right)\right)}{\pi \left(n - \frac{M}{2}\right)}$$

Center Frequency : $\omega_c = \pi/2$
Bandwidth : $\Delta \omega = \pi/2$

$$h_{d_{BP}}[n] = \frac{\sin\left(\frac{3\pi}{4}(n-M/2)\right) - \sin\left(\frac{\pi}{4}(n-M/2)\right)}{\pi(n-M/2)}$$

$$h[n] = h_{d_{BP}}[n] \cdot w[n]$$

Again, let's experiment with M and the window function.

BP-filter, centerfrek pi/2, båndbredde pi/2, Rekt. vindue, M=12

BP-filter, centerfrek pi/2, båndbredde pi/2, Rekt. vindue, M=28

BP-filter, centerfrek pi/2, båndbredde pi/2, Hamming vindue, M=12

BP-filter, centerfrek pi/2, båndbredde pi/2, Hamming vindue, M=28

BP-filter, centerfrek pi/2, båndbredde pi/2, Hamming vindue, M=128

... AND, OF COURSE, SIMILARLY FOR ==BAND STOP==
==FILTERS== ;

$$|H_{d_{BS}}|$$

$$H_{d_{BS}}(e^{j\omega}) = \begin{cases} e^{-j\omega \cdot 4/2} & \begin{cases} -\pi \le \omega \le -\omega_2 \\ -\omega_1 \le \omega \le \omega_1 \\ \omega_2 \le \omega \le \pi \end{cases} \\ 0 & \text{OTHERWISE} \end{cases}$$

$$\Downarrow$$

$$h_{d_{BS}}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{d_{BS}}(e^{j\omega}) \cdot e^{j\omega n} \, d\omega$$

etc ...

==Now, A FEW WORDS ON DESIGN OF HP==

==IIR FILTER== ;

SPEC^s

```
                    SPEC^s
                  /        \
                 /          \
                ↙            ↘
ANALOG PROTOTYPE          ANALOG PROTOTYPE
   LP FILTER                 HP FILTER
        |                        |
        ↓                        ↓
   BILINEAR ✓               BILINEAR ✓
IMPULSE INVARIANCE ✓    IMPULSE INVARIANCE ∵.
        |                        |
        ↓                        ↓
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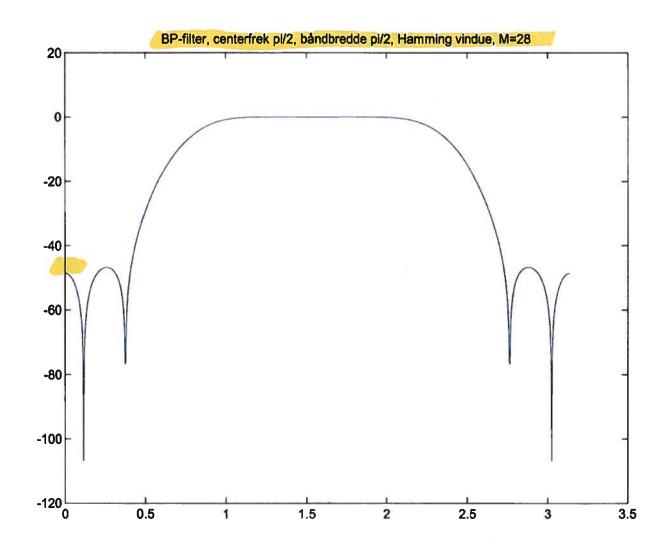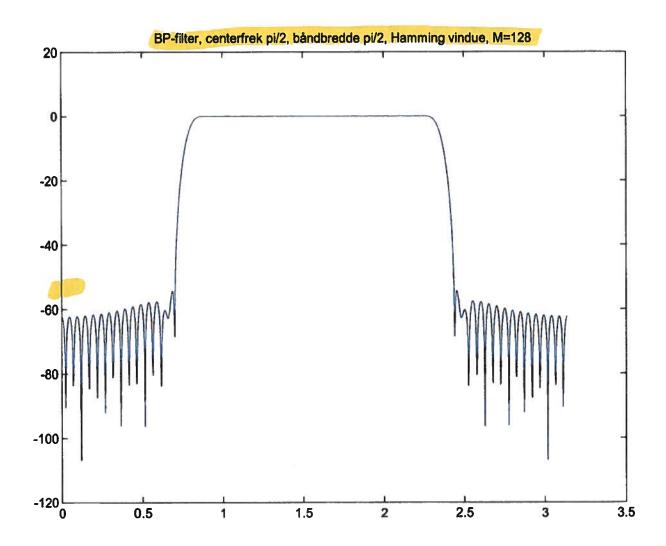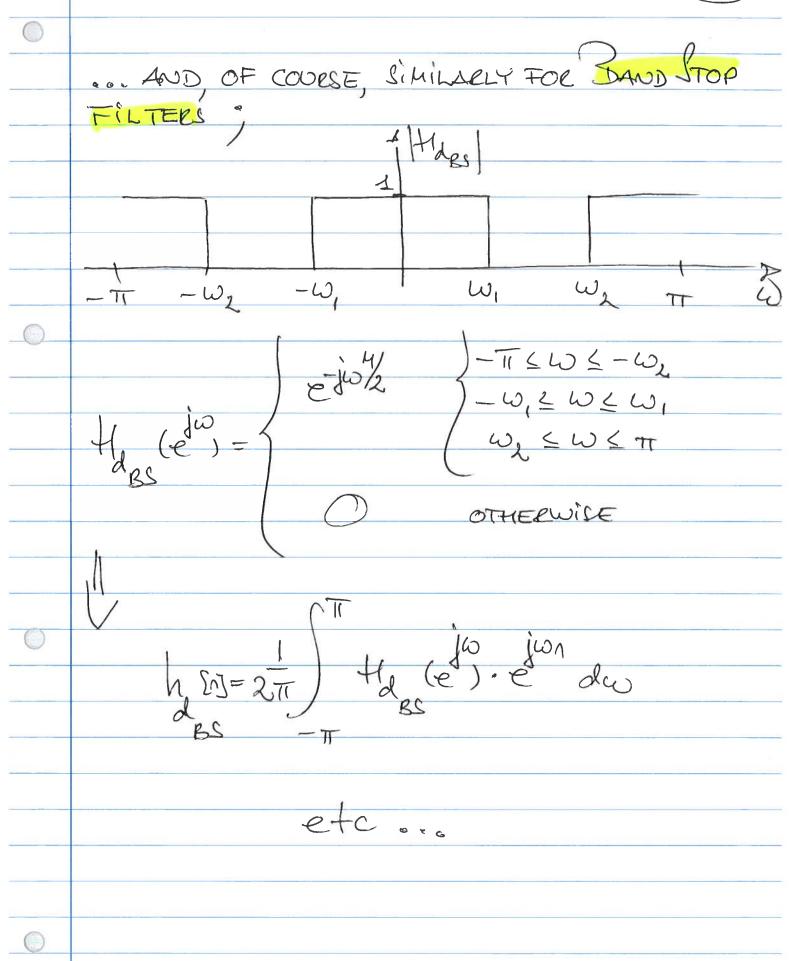
FREQUENCY TRANSFORMATION   DIGITAL HP·FILTER

↓

DIGITAL HP·FILTER.

$|H(\Omega)|$ graph: value 1 from 0 to 1 on the Ω axis, dropping to 0.

$$LP \rightarrow LP$$
$$LP \rightarrow HP$$
$$LP \rightarrow BP$$
$$LP \rightarrow BS$$

==NORMALIZED L.P. FILTER==

FREQUENCY TRANSFORMATION

DIGITAL PROTO-TYPE
LP-FILTER $\longrightarrow H_{wp}(z)$

$\tilde{z}$-plane $z^{-1} = G(\tilde{z}^{-1})$ z-plane

SO, THE IDEA IS TO APPLY AN ALGEBRAIC TRANSFORMATION, WHICH MAPS FROM THE $\tilde{z}$-plane TO THE z-plane.

$$\tilde{z}^{-1} = \pm \prod_{k=1}^{N} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}}$$

$$H_{(z)} = H_{LP}(\tilde{z}) \Big|_{\tilde{z}^{-1} = G(z^{-1})}$$

$$\tilde{z}^{-1} = G(z^{-1})$$

$\tilde{Z}$-plane   z-plane

So, STABLE AND CAUSAL FILTERS ARE MAPPED TO STABLE AND CAUSAL FILTERS.

HOWEVER: THE MAPPING is NON-LINEAR

Ex. $LP \longrightarrow LP$

$$\tilde{z}^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}} \qquad \text{EQU. 44 p.552}$$

WHERE $\alpha = \dfrac{\sin\left\{(\Theta_c - \omega_c)/2\right\}}{\sin\left\{(\Theta_c + \omega_c)/2\right\}}$ EQU 47 p.554.

CRITICAL FREQ. IN THE PROTOTYPE FILTER $(\tilde{z})$

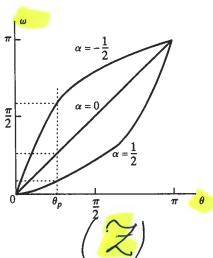CRITICAL FREQUENCY IN THE RESULTING FILTER (z).

Figure 24 Warping of the frequency scale in lowpass-to-lowpass transformation.

TABLE 1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY μ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \dfrac{z^{-1} - \alpha}{1 - az^{-1}}$	$\alpha = \dfrac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Highpass	$Z^{-1} = -\dfrac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\dfrac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Bandpass	$Z^{-1} = -\dfrac{z^{-2} - \frac{2\alpha k}{k+1}z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1}z^{-2} - \frac{2\alpha k}{k+1}z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)\tan\left(\frac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency
Bandstop	$Z^{-1} = \dfrac{z^{-2} - \frac{2\alpha}{1+k}z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k}z^{-2} - \frac{2\alpha}{1+k}z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)\tan\left(\frac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency

ANALYSIS OF DISCRETE-TIME SYSTEMS
(ARE THE SPEC^s FULFILLED ??)

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

FIRST WE NEED TO FIND THE ROOTS IN THE $B(z)$ - AND THE $A(z)$ POLYNOMIAL $\Rightarrow$ THE ZERO^s AND THE POLE^s OF $H(z)$

NOW, THREE POSSIBLE SCENARIOS $\int$

① M=N , EXTEND $B(z)$ AND $A(z)$ WITH z^M

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{M-k}}{\sum_{k=0}^{M} a_k z^{M-k}}$$

THE PURPOSE IS TO OBTAIN POSITIVE EXPONENTS ONLY.

(2) M > N, EXTEND B(z) AND A(z) WITH z^M

$$H(z) = \frac{\sum\limits_{k=0}^{M} b_k z^{M-k}}{z^{M-N} \cdot \sum\limits_{k=0}^{N} a_k z^{N-k}}$$

WE GET M-N EXTRA ROOTS IN A(z), i.e.

M-N EXTRA POLES IN z=0 IN H(z)

(3) M < N, EXTEND B(z) AND A(z) WITH z^N

$$H(z) = \frac{z^{N-M} \sum\limits_{k=0}^{M} b_k z^{M-k}}{\sum\limits_{k=0}^{N} a_k z^{N-k}}$$

WE GET N-M EXTRA ROOTS IN B(z), i.e.,

N-M EXTRA ZEROS IN z=0 IN H(z)

a

IT IS NOW POSSIBLE TO FACTORIZED $H(z)$;

$$H(z) = \frac{b_0}{a_0} \cdot \frac{\prod\limits_{k=1}^{L}(z-c_k)}{\prod\limits_{k=1}^{h}(z-d_k)} \qquad L = \max\{0, M\}$$

ZEROS

POLES
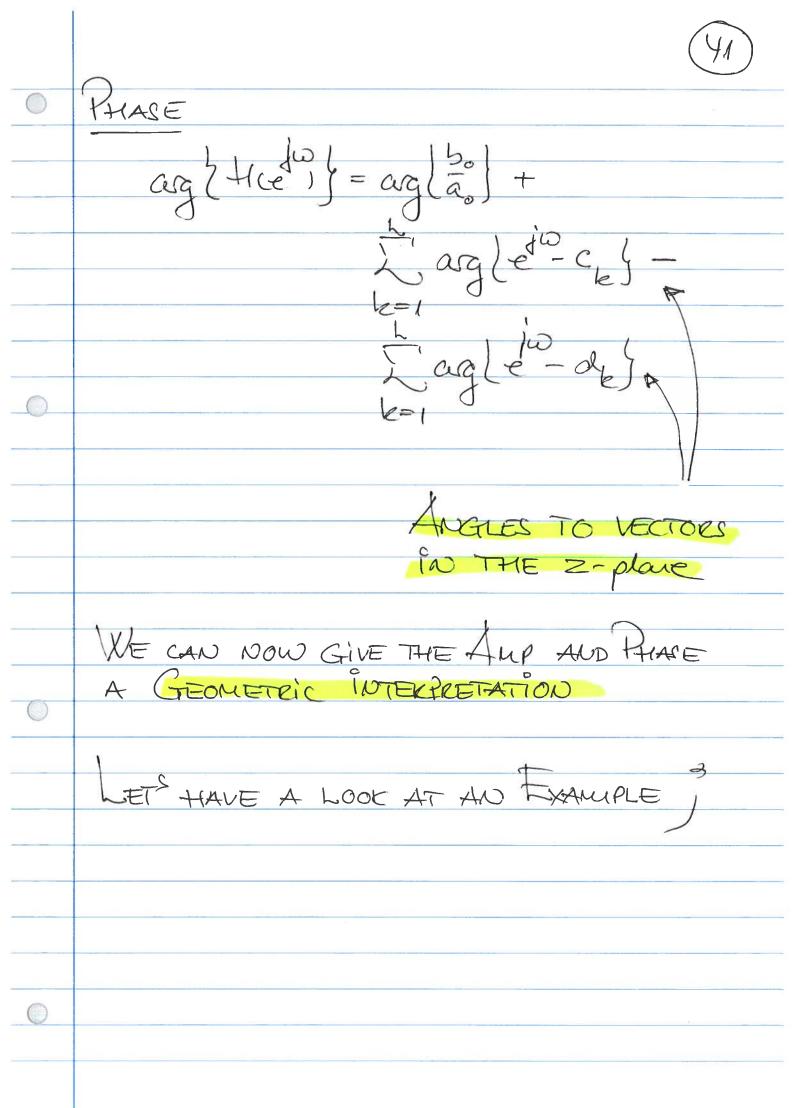
USING THIS EXPRESSION, WE CAN NOW WRITE THE FREQUENCY RESPONSE;

$$H(e^{j\omega}) = \frac{b_0}{a_0} \cdot \frac{\prod\limits_{k=1}^{L}(e^{j\omega}-c_k)}{\prod\limits_{k=1}^{h}(e^{j\omega}-d_k)}$$

WHICH IS NOW USED TO DERIVE THE AMPLITUDE- AND THE PHASE·RESPONSE;

AMPLITUDE

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \cdot \frac{\prod\limits_{k=1}^{h}|e^{j\omega}-c_k|}{\prod\limits_{k=1}^{L}|e^{j\omega}-d_k|}$$

LENGTH OF VECTORS IN THE Z-plane

PHASE

$$\arg\left\{H(e^{j\omega})\right\} = \arg\left[\frac{b_0}{a_0}\right] +$$

$$\sum_{k=1}^{M} \arg\left\{e^{j\omega} - c_k\right\} -$$

$$\sum_{k=1}^{L} \arg\left\{e^{j\omega} - d_k\right\}$$

ANGLES TO VECTORS
in THE z-plane

WE CAN NOW GIVE THE Amp AND PHASE
A GEOMETRIC INTERPRETATION

Let's HAVE A LOOK AT AN EXAMPLE

GIVEN $H(z) = \dfrac{1 + z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$ $\begin{array}{c} M = 1 \\ N = 2 \end{array} \Big\} M < N$

So, EXTEND $B(z)$ AND $A(z)$ WITH z^2

$$H(z) = \frac{z^2 + z}{z^2 - z + \frac{1}{2}}$$

$$H(z) = \frac{1}{1} \cdot \frac{(z - 0)(z - (-1))}{(z - (\frac{1}{2} + j\frac{1}{2}))(z - (\frac{1}{2} - j\frac{1}{2}))}$$

ZEROS: $z = 0$, $z = -1$

POLES: $z = \frac{1}{2} \pm j\frac{1}{2}$

AMPLITUDE RESPONSE

$$\left| H(e^{j\omega}) \right| = \left| \frac{b_0}{a_0} \right| \cdot \frac{|V_1| \cdot |V_2|}{|V_3| \cdot |V_4|}$$

WE NOTE THAT $|V_k|$ ARE ALL FUNCTIONS OF ω.

$V_1:$
$$|V_1| = \sqrt{(\cos\omega + 1)^2 + \sin^2\omega}$$

EVALUATE $\left| H(e^{j\omega}) \right|$ FOR $\omega \in [0; \pi]$

PHASE RESPONSE

$$\arg\left\{ H(e^{j\omega}) \right\} = \arg\{1\} + \sum_{k=1}^{2} \arg\left\{ e^{j\omega} - c_k \right\}$$
$$- \sum_{k=1}^{2} \arg\left\{ e^{j\omega} - d_k \right\}$$

$$= \varphi_1 + \varphi_2 - \varphi_3 - \varphi_4$$

WE NOTE THAT φ_k ARE ALL FUNCTIONS OF ω.

$\varphi_1:$ $\tan(\varphi_1) = \dfrac{\sin\omega}{\cos\omega + 1} \implies \varphi_1 = \text{ARCTAN}\left\{ \dfrac{\sin\omega}{\cos\omega + 1} \right\}$

EVALUATE $\arg\left\{ H(e^{j\omega}) \right\}$ FOR $\omega \in [0; \pi]$

INTERPRETATION OF POLE-ZERO LOCATIONS

A POLE CLOSE TO THE UNIT CIRCLE

A PEAK IN $|H|$ AT ω_0

A ZERO CLOSE TO THE UNIT CIRCLE

A DIP IN $|H|$ AT ω_0