Suggested Solution, DFT

1) Find $X[k]$ given $x(t)$

$$X[k] = \frac{1}{T}\int_{-1}^{1} e^{-t} \cdot e^{-jk\Omega_f t}\,dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-t} \cdot e^{-jk\pi t}\,dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+jk\pi)t}\,dt$$

$$= \frac{-1}{2(1+jk\pi)} \cdot \left[e^{-(1+jk\pi)t}\right]_{-1}^{1}$$

$$= \frac{e^{(1+jk\pi)} - e^{-(1+jk\pi)}}{2(1+jk\pi)} \qquad k \in]-\infty;\infty[$$

It is possible though to „polish" this expression a little bit;

$$X[k] = \frac{e^{(1+jk\pi)} - e^{-(1+jk\pi)}}{2(1+jk\pi)}$$

$$= \frac{e \cdot e^{jk\pi} - e^{-1} \cdot e^{-jk\pi}}{2(1+jk\pi)}$$

$$= \frac{e(\cos k\pi + j\sin k\pi) - \frac{1}{e}(\cos k\pi - j\sin k\pi)}{2(1+jk\pi)}$$

$$= \frac{(e-\frac{1}{e})\cos(k\pi)}{2(1+jk\pi)}$$

$$= \frac{(1-jk\pi)(e-\frac{1}{e})\cos(k\pi)}{2(1+k^2\pi^2)}$$

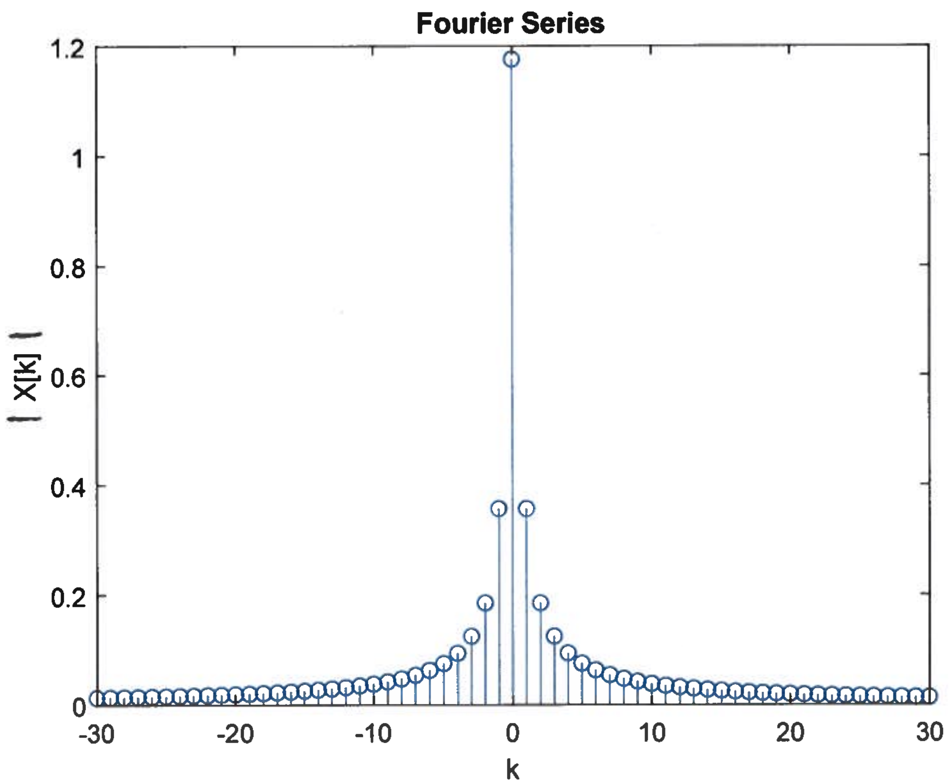
Plot $|X[k]|$ for $k \in [-30;30]$

$$|X[k]| = \left| \frac{(e-\frac{1}{e})\cos(k\pi)}{2(1+k^2\pi^2)} \right| \cdot \sqrt{1+(k\pi)^2}$$

$k \in [-30;30]$

The plot is easily made in term of a small Matlab program.

Amplitude of the complex FS coefficients.



If you look at x(t) over one period from $t=-1$ to $t=1$ you will realize that the signal actually is very slowly varying. This is consistent with the plot above which shows a significant DC-component and very little „high" frequencies.

Prob. 4 p. 713.

$x[n] = a^n u[n] \quad \alpha < 1$

A periodic sequence is constructed from $x[n]$;

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n + rN]$$

a) Fourier Transform $X(e^{j\omega})$ of $x[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n$$

$$= \frac{1}{1 - \alpha e^{-j\omega}} \quad |\alpha| < 1$$

Geometric Series.

b) The DFS of $\tilde{x}[n]$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] . W_N^{kn}$$

Twiddle factor.

periodic sequence which is now inserted.

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \sum_{r=-\infty}^{\infty} x[n+rN] W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \sum_{r=-\infty}^{\infty} \alpha^{n+rN} u[n+rN] W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \sum_{r=0}^{\infty} \alpha^{n+rN} . W_N^{kn}$$

Rearranging the summations;

$$\tilde{X}[k] = \sum_{r=0}^{\infty} \alpha^{rN} \sum_{n=0}^{N-1} \alpha^{n} W_N^{kn}$$

$\Downarrow$

Again, let's utilize that there are geometric series:

$$W_N^{kn} = e^{-j\frac{2\pi}{N}kn}$$

$$\Downarrow$$

$$\alpha^n W_N^{kn} = \alpha^n e^{-j\frac{2\pi}{N}kn} = \left(\alpha \cdot e^{-j\frac{2\pi}{N}k}\right)^n$$

$$= \beta^n$$

$$\sum_{n=0}^{N-1} \beta^n = \frac{1-\beta^{N}}{1-\beta}$$

$(N-1)+1$

$$= \frac{1-\alpha^N e^{-j2\pi k}}{1-\alpha \cdot e^{-j\frac{2\pi}{N}k}}$$

$$\Downarrow$$

$$\tilde{X}[k] = \sum_{r=0}^{\infty} \alpha^{r \cdot N} \cdot \underbrace{\frac{1-\alpha^N e^{-j2\pi k}}{1-\alpha e^{-j\frac{2\pi}{N}\cdot k}}}_{} \qquad |\alpha|<1$$

As related to r, this can now be considered a constant.

$$\Downarrow$$

$(\alpha^N)^r$

$$\tilde{X}[k] = \frac{1}{1-\alpha^N} \cdot \frac{1-\alpha^N \cdot \underbrace{e^{-j2\pi k}}_{=1}}{1-\alpha \cdot e^{-j\frac{2\pi}{N}\cdot k}} \qquad |\alpha|<1$$

$\Downarrow$

$$\tilde{X}[k] = \frac{1}{1-\alpha \cdot e^{-j\frac{2\pi}{N}\cdot k}} \qquad |\alpha|<1$$

c) Compare a) and b)

$$\begin{cases} X(e^{j\omega}) = \dfrac{1}{1-\alpha e^{-j\omega}} & |\alpha|<1 \\ \tilde{X}[k] = \dfrac{1}{1-\alpha e^{-j\frac{2\pi}{N}\cdot k}} & |\alpha|<1 \end{cases}$$

$\Downarrow$

$$\tilde{X}[k] = X(e^{j\omega})\Big|_{\omega = 2\pi k/N}$$

So, the Fourier Series coefficients are the Fourier transform (freq. response) sampled in frequencies $\omega_k = \frac{2\pi}{N}\cdot k$

Prob. 5 p. 713.

Compute DFT of finite length sequences considered to be of length N (even).

a) $x[n] = \delta[n]$

$$X[k] = \sum_{n=0}^{N-1} \delta[n] \cdot W_N^{kn} \qquad 0 \leq k \leq N-1$$

$$= \sum_{n=0}^{N-1} \delta[n] \, e^{-j\frac{2\pi}{N}kn} = \underline{\underline{1}}$$

b) $x[n] = \delta[n-n_0] \qquad 0 \leq n_0 \leq N-1$

$$X[k] = \sum_{n=0}^{N-1} \delta[n-n_0] \cdot W_N^{kn} \qquad 0 \leq k \leq (N-1)$$

$$= \sum_{n=0}^{N-1} \delta[n-n_0] \, e^{-j\frac{2\pi}{N}kn}$$

$$= \underline{\underline{W_N^{kn_0}}}$$

Prob. 7 p. 714.

We have a six-point uniform sequence $x[n]$ (figure P7 p. 714) which is non-zero for $0 \le n \le 5$

$X(z)$ is the Z-transform of $x[n]$

If we sample $X(z)$ @ $z = e^{j\frac{2\pi}{4}k}$ $k = 0,1,2,3$

the we obtain $X_1[k] = X(z)\Big|_{z = e^{j\frac{2\pi}{4}k}}$ $k = 0..3$.

Sketch the inverse DFT of $X_1[k]$.

Now; $X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n} = \sum_{n=0}^{5} z^{-n}$

Next we substitute $z = e^{j\frac{2\pi k}{4}}$

$\Downarrow$

$$X[k] = \sum_{n=0}^{5} e^{-j\frac{2\pi k}{4}\cdot n} = \sum_{n=0}^{5} e^{-j\frac{2\pi}{4}kn}$$

$$= \sum_{n=0}^{5} W_4^{kn} \qquad 0 \le k \le 4$$

The period $N=4$.

This is a 4-point DFT.
The original sequence was of length 6

$\Downarrow$

Going back from $X[k]$ to the time domain we may therefor expect some aliasing

Let's now do inverse DFT

$$X[k] = W_4^{0k} + W_4^{k} + W_4^{2k} + W_4^{3k} + W_4^{4k} + W_4^{5k} \qquad 0 \le k \le 4.$$

Based on the result we found in prob. 5a. we can now do inverse DFT by inspection.

$\Downarrow$

Six impulses (one for each value of n)

However;

$$W_4^{4k} = e^{-j\frac{2\pi}{4}\cdot 4k} = e^{-j\frac{2\pi}{4}0\cdot k} = W_4^{0k}$$

and

$$W_4^{5k} = e^{-j\frac{2\pi}{4}5k} = e^{-j\frac{2\pi}{4}\cdot k} = W_4^{k}$$

Therefor two points are aliased;