# Preliminary plan (analog filters)

1. {1-2}, 3-7, {7-16}, 25-30, 49-57, (app. A)
   - o Course overview
   - o Analog filters: Applications
   - o The Butterworth approximation
   - o Passive filter realisation (ladder structure)
   - o Design procedure, frequency and impedance scaling

2. 7-20, 30-36, 58-62, {App. A}
   - o The Chebyshev approximation
   - o Other filter types
   - o Impact of group delay variations

3. **37-38, {67-71}, 77-88, 171-184, 187-189, {190-196}, 197-208**
   - o **Frequency transformations, LP-HP, LP-BP & LP-BS**
   - o **Sensitivity analysis**
     - o **How sensitive is a given circuit to component variations?**
     - o **Used as a tool to evaluate filter circuits**

4. 217-238, 253-260, 263-264
   - o OpAmps applied as building blocks in active RC-filters
   - o 2nd order Sallen-Key
   - o 2nd order multiple feed-back
   - o Higher order filters

5. Design/lab. exercise

# Lecture 2: Recap

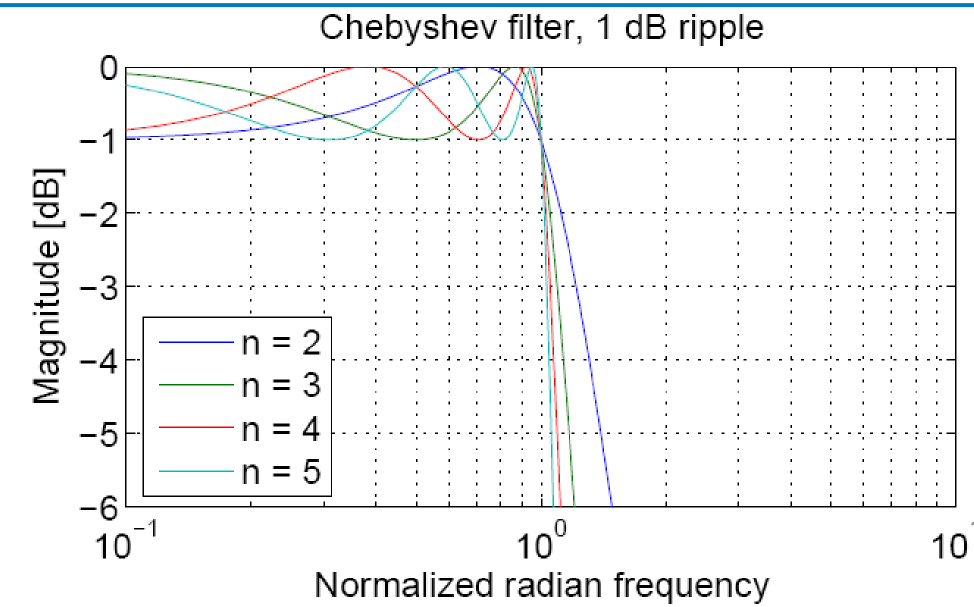
Chebyshev lowpass filter:

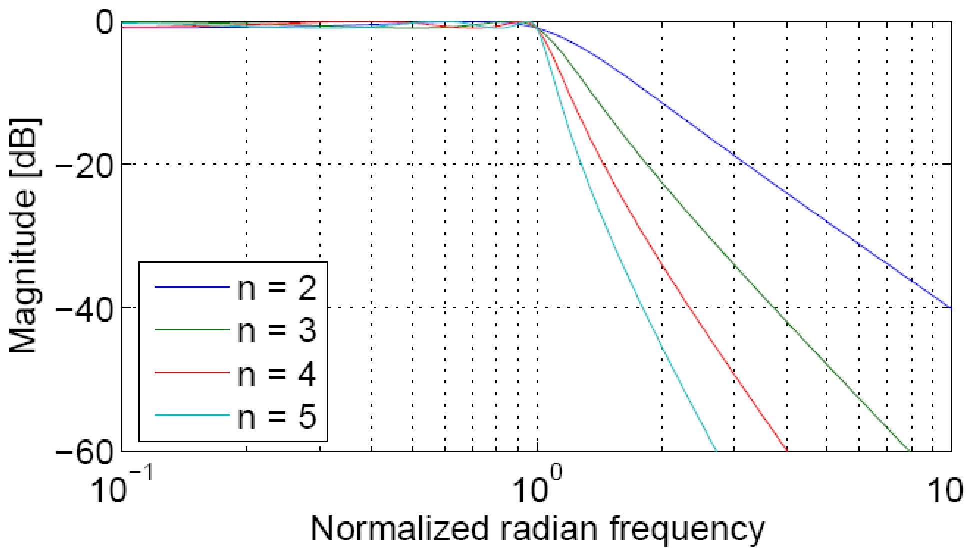
$$|H(j\omega)|^2 = \frac{1}{1+\varepsilon^2 C_n^2(\omega)}$$

$$C_n(\omega) = \begin{cases} \cos(n \cdot \cos^{-1}\omega) & |\omega| \le 1 \\ \cosh(n \cdot \cosh^{-1}\omega) & |\omega| \ge 1 \end{cases}$$

$$Ripple_{dB} = 10\log(1+\varepsilon^2)$$

$$\varepsilon = \sqrt{10^{Ripple_{dB}/10} - 1}$$

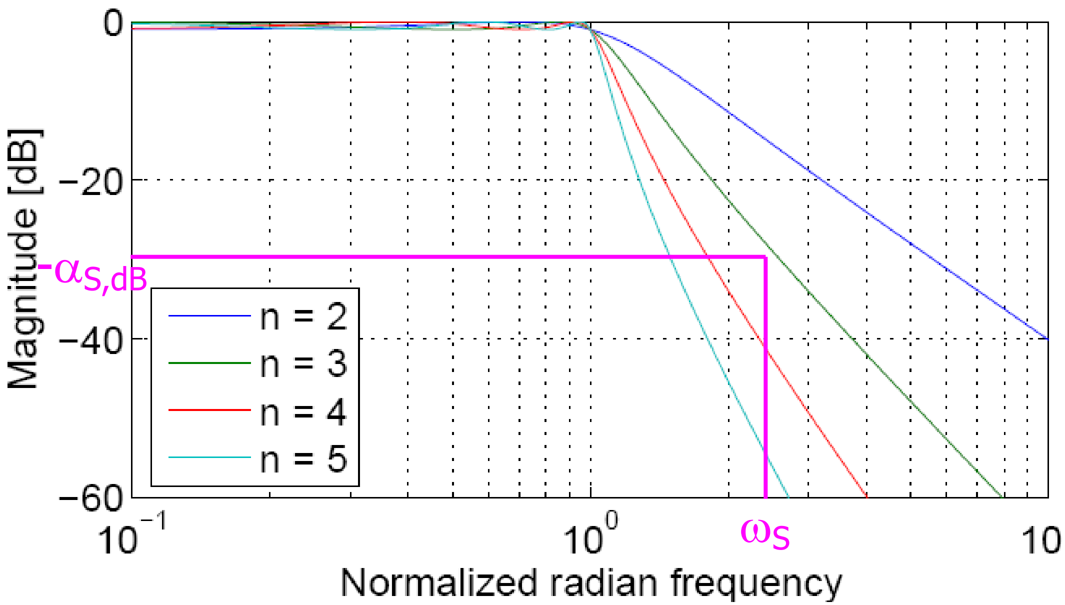



Compared to Butterworth:

- o Higher stopband attenuation
- o Ripple in the passband
- o Higher phase nonlinearity
- o Longer impulse response

If the filter requirements are:
- o Passband ripple (dB)
- o Stopband attenuation (= $-H(j\omega)_{dB}$) at the normalized stopband radian frequency $\omega_S$.




Then the necessary filter order is found from:

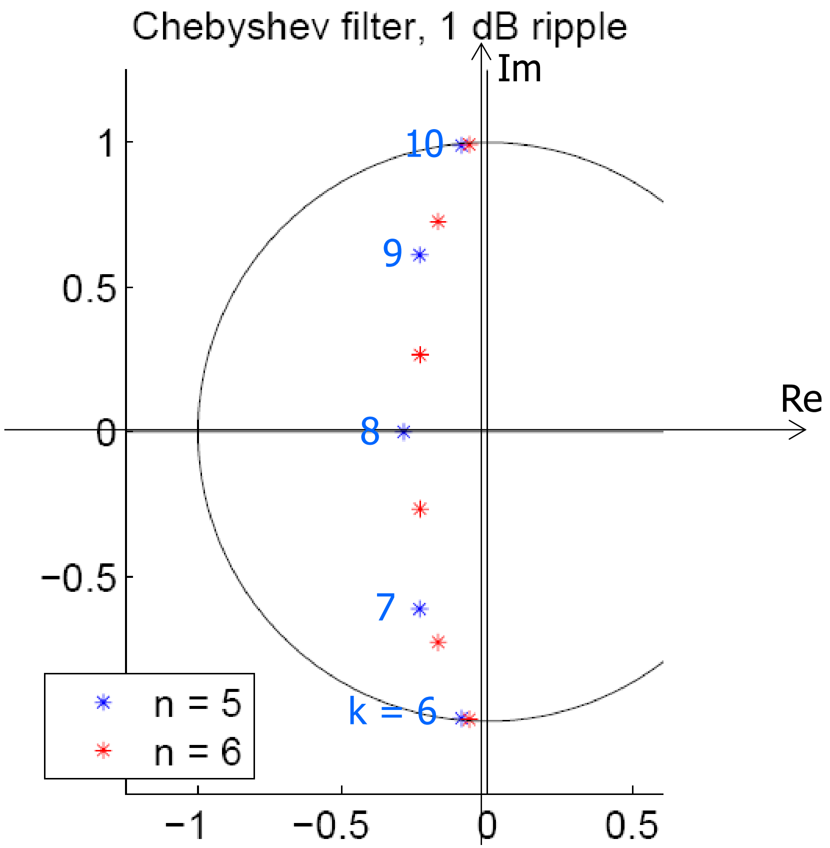
$$\boxed{n \geq \frac{1}{\cosh^{-1}\omega_S}\cosh^{-1}\sqrt{\frac{10^{\alpha_{S,dB}/10}-1}{10^{Ripple_{dB}/10}-1}}}$$

-rounded up to the nearest integer. Note the sign definition: $\alpha_{S,dB} > 0$, $Ripple_{dB} > 0$

$\omega_S$ is the stopband frequency for the normalized low-pass-filter: $\omega_S = \omega_{S,scaled}/k_f$

Chebyshev poles on an ellipse:
(Butterworth poles on the unit circle)

No zeros




$$p_k = \sin\frac{(2k-1)\pi}{2n}\cdot\sinh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right) + j\cos\frac{(2k-1)\pi}{2n}\cdot\cosh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right)$$

k = n+1, n+2...... 2n

# Exercise

The requirements for a Chebyshev low-pass filter are:

- Passband ripple: 1 dB
- Ripple bandwidth: 30 kHz
- The attenuation at 120 kHz shall be at least 61 dB

a. Make a sketch of the magnitude of the filter
b. Calculate $\varepsilon$
c. Calculate the necessary order

Hint:

$$|H(j\omega)|^2 = \frac{1}{1+\varepsilon^2 C_n^2(\omega)}$$

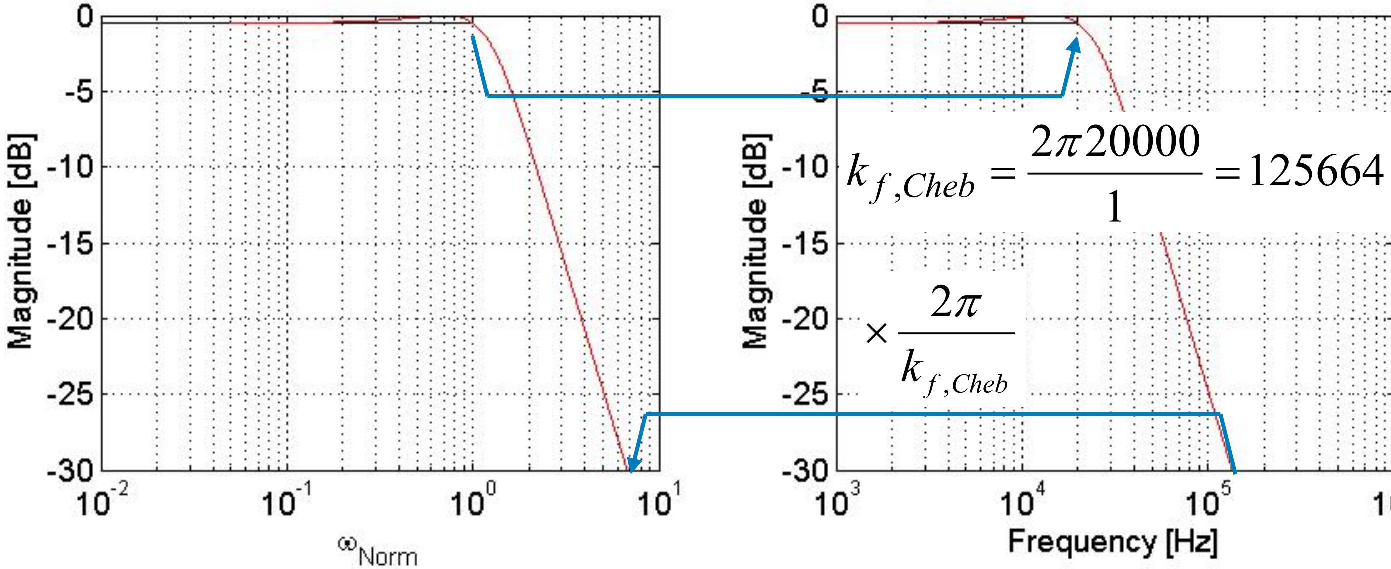
$$Ripple_{dB} = 10\log(1+\varepsilon^2)$$

$$\varepsilon = \sqrt{10^{Ripple_{dB}/10} - 1}$$

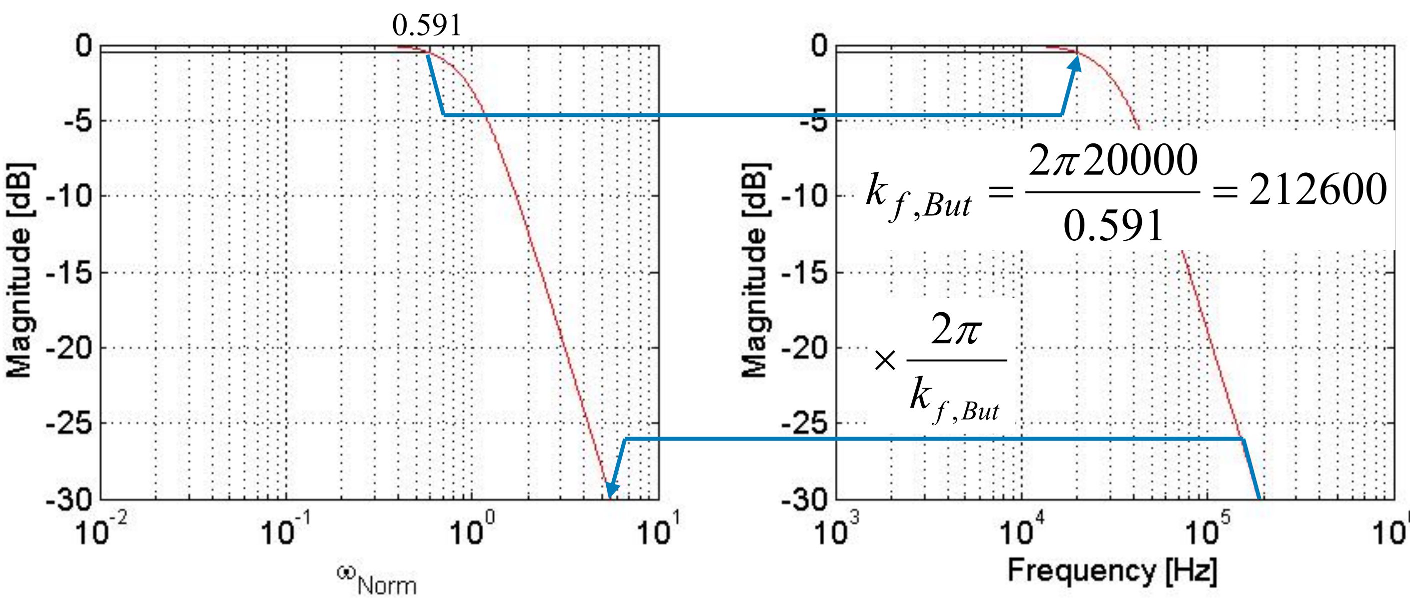
$$n \geq \frac{1}{\cosh^{-1}\omega_S}\cosh^{-1}\sqrt{\frac{10^{\alpha_{S,dB}/10}-1}{10^{Ripple_{dB}/10}-1}}$$

# Frequency scaling, exercise 2.2 & 1.2 (2nd order filters)

$$\left|H_{Scaled}(j\omega)\right| = \left|H_{Norm}\left(\frac{j\omega}{k_{f,Cheb}}\right)\right| = \frac{1}{\sqrt{1+\varepsilon^2 C_2^2\left(\frac{\omega}{k_{f,Cheb}}\right)}}$$




$$\left|H_{Scaled}(j\omega)\right| = \left|H_{Norm}\left(\frac{j\omega}{k_{f,But}}\right)\right| = \frac{1}{\sqrt{1+\left(\frac{\omega}{k_{f,But}}\right)^4}}$$

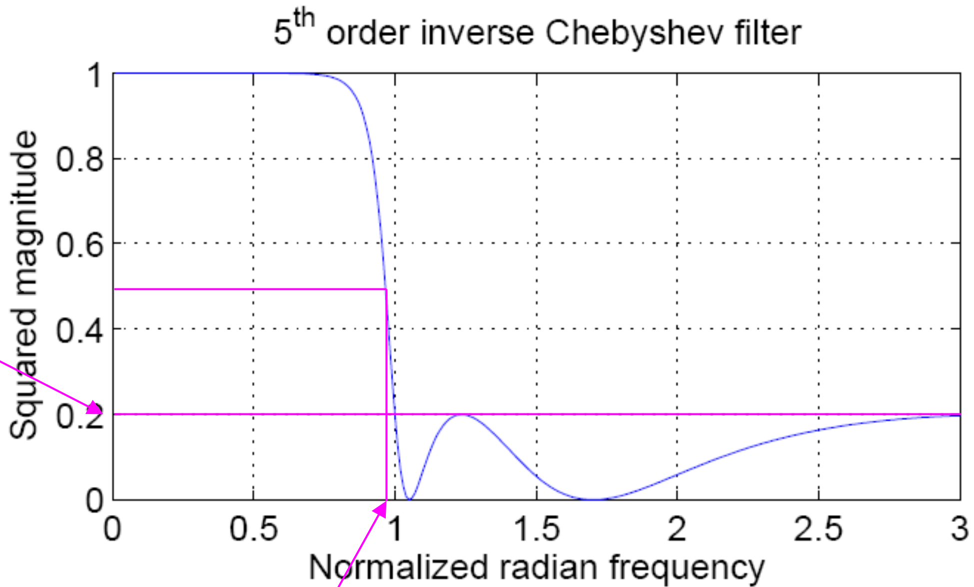


# Inverse Chebyshev filter

$$|H_{InvCheb}(j\omega)|^2 = 1 - |H_b(j\omega)|^2$$

$$= \frac{\varepsilon^2 C_n^2\left(\frac{1}{\omega}\right)}{1+\varepsilon^2 C_n^2\left(\frac{1}{\omega}\right)}$$

$$\frac{\varepsilon^2}{1+\varepsilon^2}$$




$$-\alpha_{SdB} = 10 \cdot \log \frac{\varepsilon^2}{1+\varepsilon^2}$$

$$\frac{1+\varepsilon^2}{\varepsilon^2} = 10^{\alpha_{SdB}/10}$$

$$1+\frac{1}{\varepsilon^2} = 10^{\alpha_{SdB}/10}$$

$$\boxed{\varepsilon = \frac{1}{\sqrt{10^{\alpha_{SdB}/10}-1}} \qquad (\alpha_{SdB} > 0)}$$

The 3-dB bandwidth is determined from:

$$1 - \frac{1}{1+\varepsilon^2 C_n^2\left(\frac{1}{\omega_{-3dB}}\right)} = \frac{1}{2}$$

"It is easily seen that":

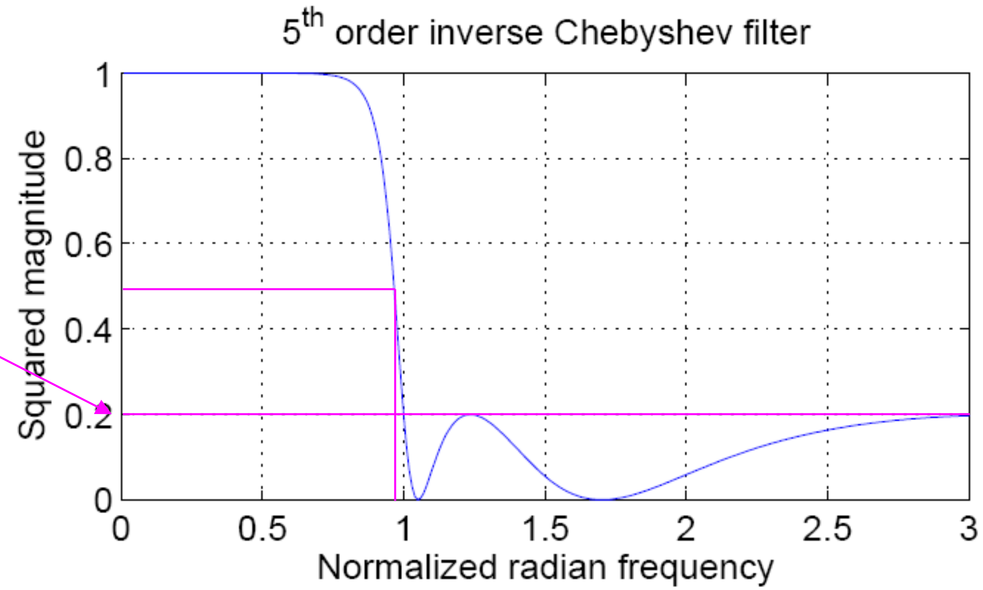
$$\boxed{\omega_{-3dB} = \frac{1}{\cosh\left(\frac{1}{n}\cosh^{-1}\frac{1}{\varepsilon}\right)}}$$

# Exercise

$$|H_{InvCheb}(j\omega)|^2 = 1 - |H_b(j\omega)|^2$$

$$= \frac{\varepsilon^2 C_n^2(\frac{1}{\omega})}{1+\varepsilon^2 C_n^2(\frac{1}{\omega})}$$

$$\frac{\varepsilon^2}{1+\varepsilon^2}$$




$$\boxed{\varepsilon = \frac{1}{\sqrt{10^{\alpha_{SdB}/10} - 1}} \quad (\alpha_{SdB} > 0)}$$

Inverse Chebyshev filter with requirements:

- Min. 40 dB attenuation in stop band

Calculate:

- Epsilon
- Magnitude at 1 rad/s

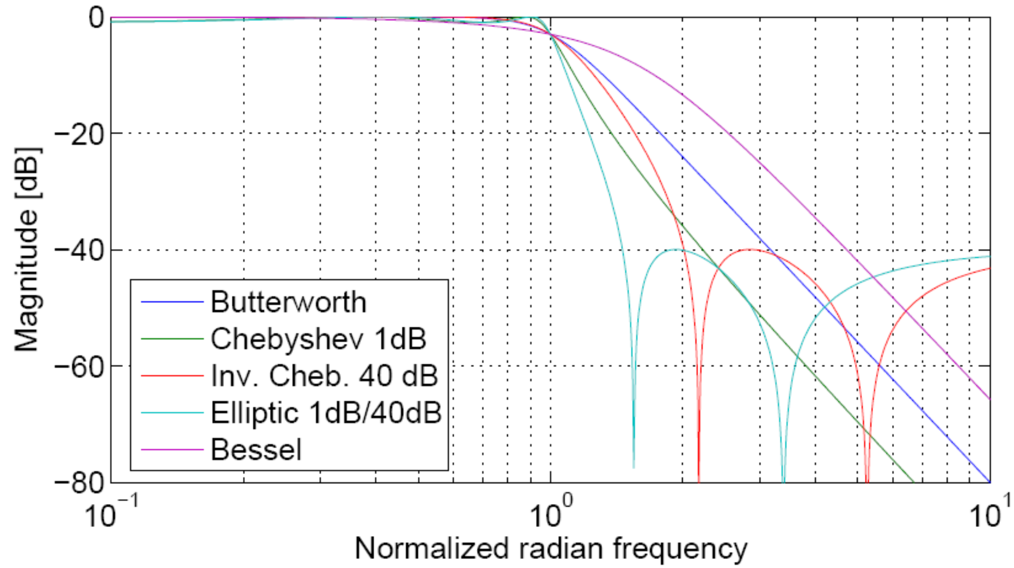
# Other filter types

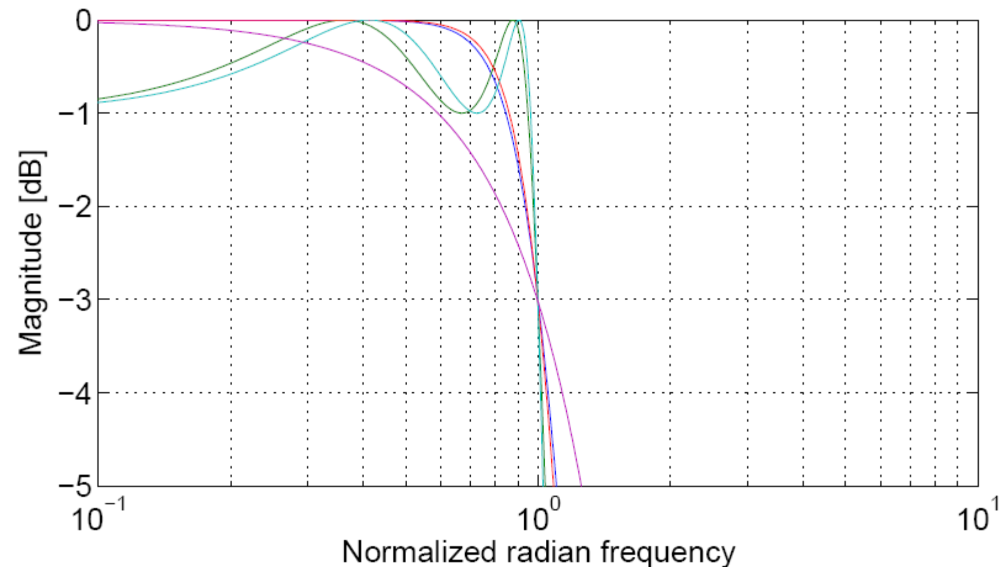
Elliptic function filters:
- o Ripple in both passband and stopband ☹
- o Very sharp cut-off ☺
- o High group delay distortion ☹
- o Matlab: ”ellip”

Bessel/Thomson filters:
- o Maximally flat group delay ☺
- o ”Soft” magnitude cut-off ☹
- o Matlab: ”besself”

Gaussian filters:
- o No ripple in the impulse response ☺
- o ”Very soft” magnitude cut-off ☹

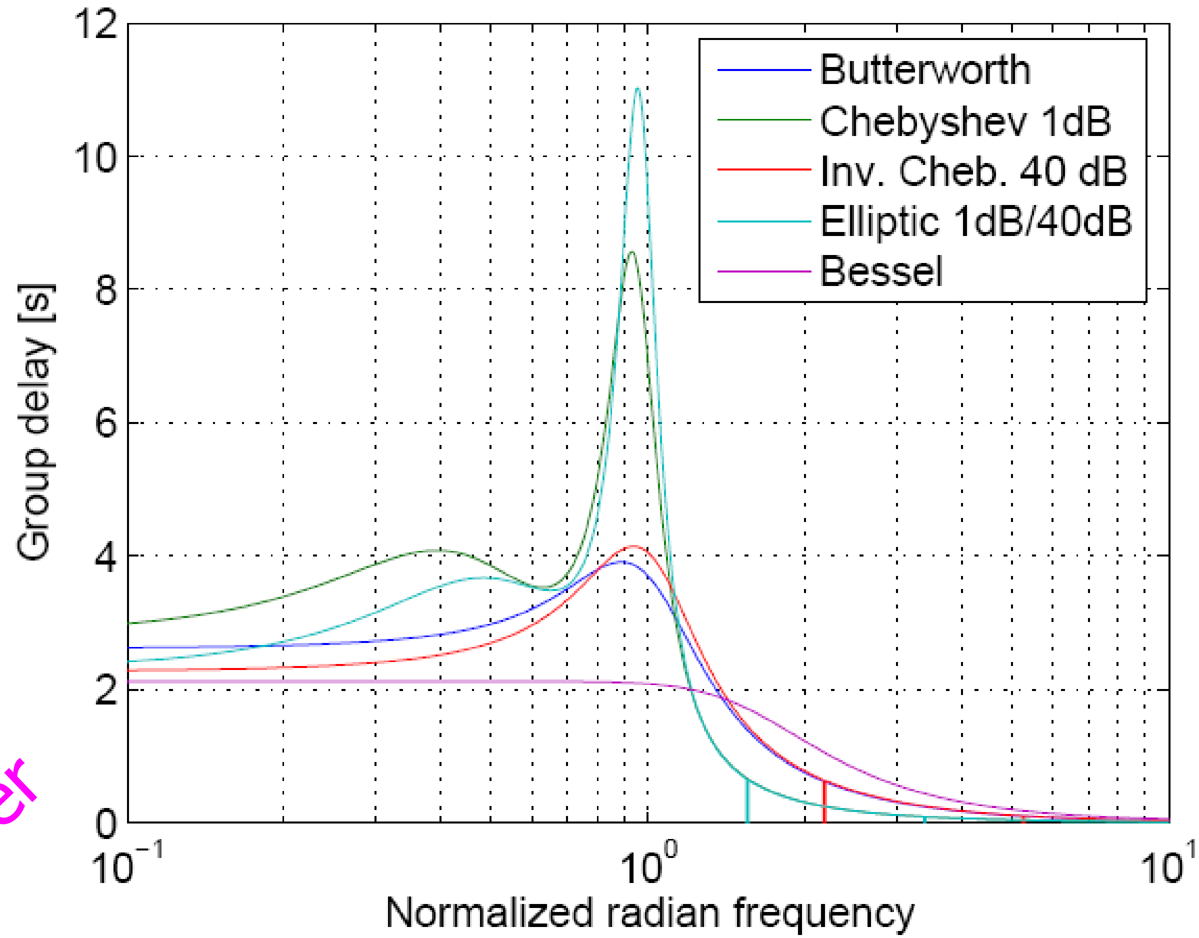
Plots:
- o All filters scaled to have a 3-dB bandwidth of 1 rad/s
- o The comparison depends on choice of parameters (1 dB, 40 dB, n = 4)



# Other filter types

Group delay:




5 min. break

Break over

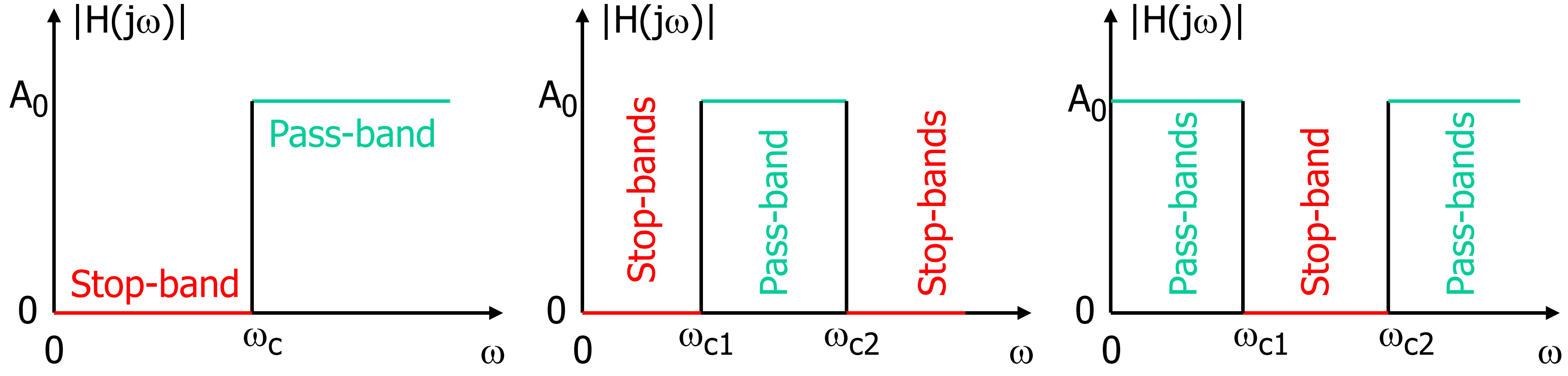
# Frequency transformations

Lowpass filter prototypes can also be used for the design of:

High-pass

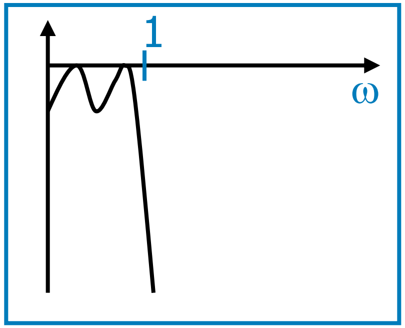
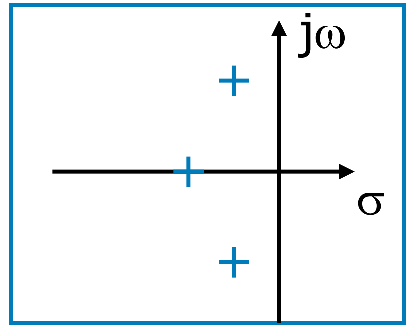
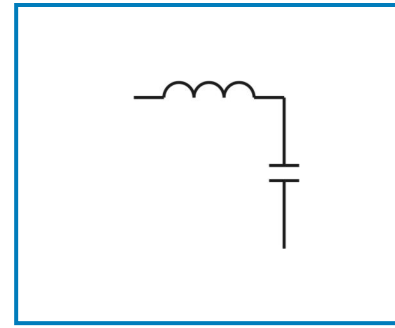
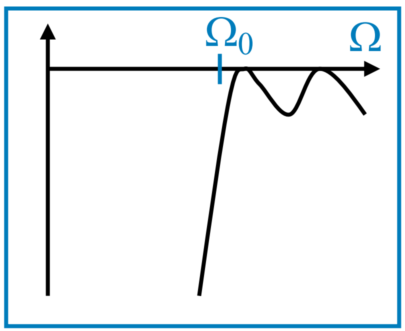
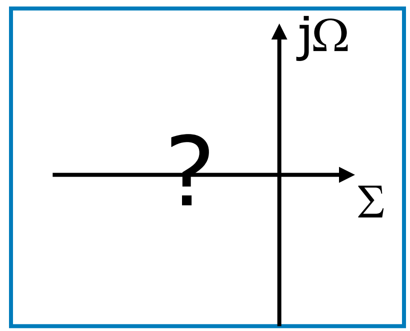
Band-pass

Band-stop (band-rejection)



# Frequency transformations, LP → HP

Low-pass prototype (normalized) $s = \sigma + j\omega$

| Frequency response | Transfer function | Poles / zeros | Circuit |
|---|---|---|---|
|   | $\dfrac{K(s-\quad)}{(s-p)(s-\quad)}$ |   |  |
|   | ? |   | ? |

High-pass $S = \Sigma + j\Omega$

# Frequency transformations, LP → HP

**LP → HP**
**frequency transformation:**

$$\boxed{s = \frac{\Omega_0}{S}}$$

On the imaginary axis:
(setting s = jω
and S = jΩ)

$$\omega = \frac{-\Omega_0}{\Omega}$$

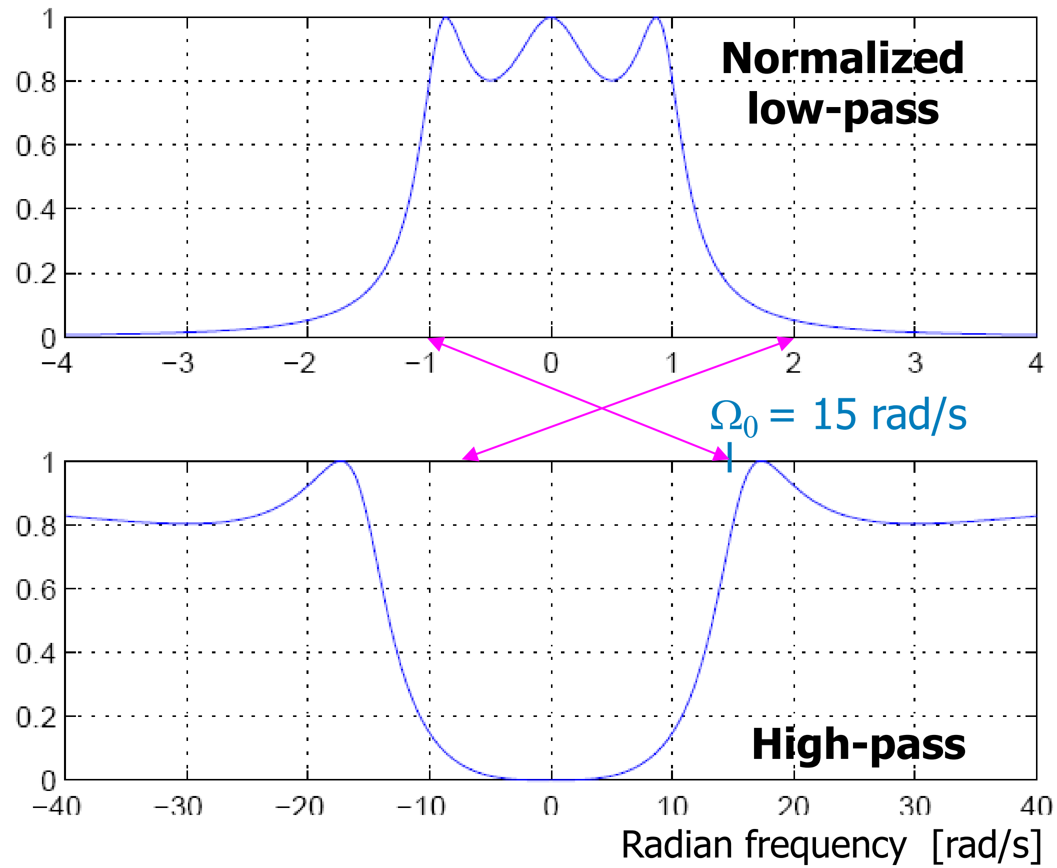
$$|\omega| = \left|\frac{\Omega_0}{\Omega}\right|$$

Transfer functions are linked:

$$H_{HP}(S) = H_{LPP}\left(\frac{\Omega_0}{S}\right)$$

$$H_{HP}(j\Omega) = H_{LPP}\left(j\frac{-\Omega_0}{\Omega}\right)$$

The negative sign only reverses the angle – no effect on the magnitude



# Frequency transformations, LP → HP

$$s = \frac{\Omega_0}{S} \qquad H_{HP}(S) = H_{LPP}\left(\frac{\Omega_0}{S}\right)$$

"All-pole" lowpass filter (Butterworth, Chebyshev, Bessel etc.) transfer function:

$$H_{LPP}(s) = \frac{K}{a_1 s^n + a_2 s^{n-1} + \ldots + a_{n-1} s^2 + a_n s + a_{n+1}}$$

$$H_{HP}(S) = \frac{K}{a_1\left(\frac{\Omega_0}{S}\right)^n + a_2\left(\frac{\Omega_0}{S}\right)^{n-1} + \ldots + a_{n-1}\left(\frac{\Omega_0}{S}\right)^2 + a_n\left(\frac{\Omega_0}{S}\right) + a_{n+1}}$$

$$H_{HP}(S) = \frac{K \cdot S^n}{a_1 \Omega_0{}^n + a_2 \Omega_0{}^{n-1} S + \ldots + a_{n-1} \Omega_0{}^2 S^{n-2} + a_n \Omega_0 S^{n-1} + a_{n+1} S^n}$$

- n-fold zero at S = 0
- $n^{th}$ order denominator polynomial → n poles
- If $p_k$ is a pole in the LP function, then the **HP-poles** are found from: $\boxed{P_k = \frac{\Omega_0}{p_k}}$
- (This rule also applies to zeros, if any)

# Frequency transformations, LP → HP

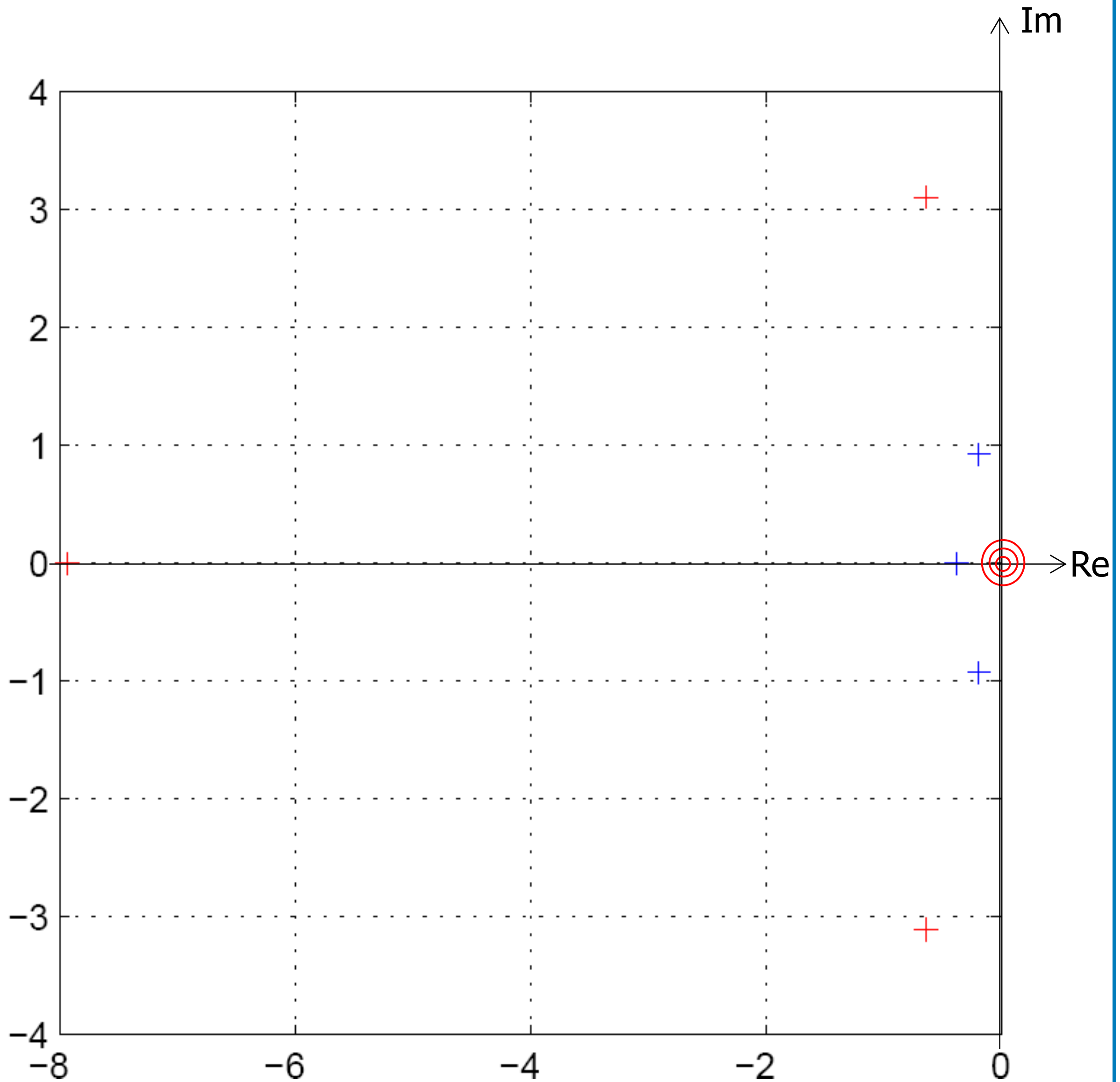
**HP poles and zeros:**

Low-pass prototype poles (+)

$$\boxed{P_k} = \frac{\Omega_0}{\boxed{p_k}}$$

High-pass filter poles and zeros, (+, o)

- 3rd order Chebyshev
- $\Omega_0$ = 3 rad/s



**HP circuit components**

$$s = \frac{\Omega_0}{S}$$

Inductor impedance:

$$Z_{Li} = sL_i$$

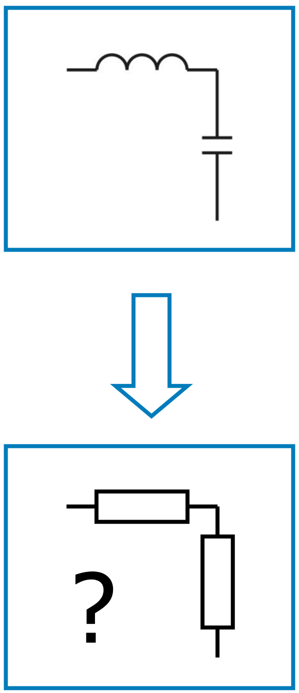
The HP-component must have the impedance:

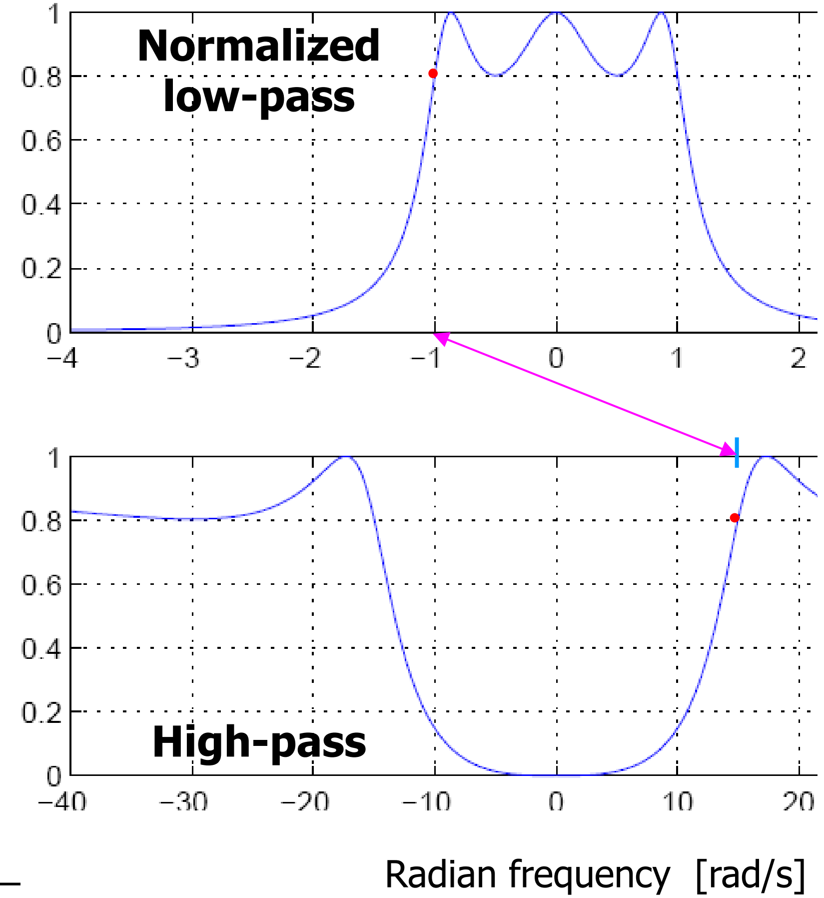
$$Z_{New}(S) = Z_{Li}\left(\frac{\Omega_0}{S}\right)$$

$$= \frac{\Omega_0 L_i}{S} = \frac{1}{\frac{1}{\Omega_0 L_i} S}$$

Corresponding to a capacitor:

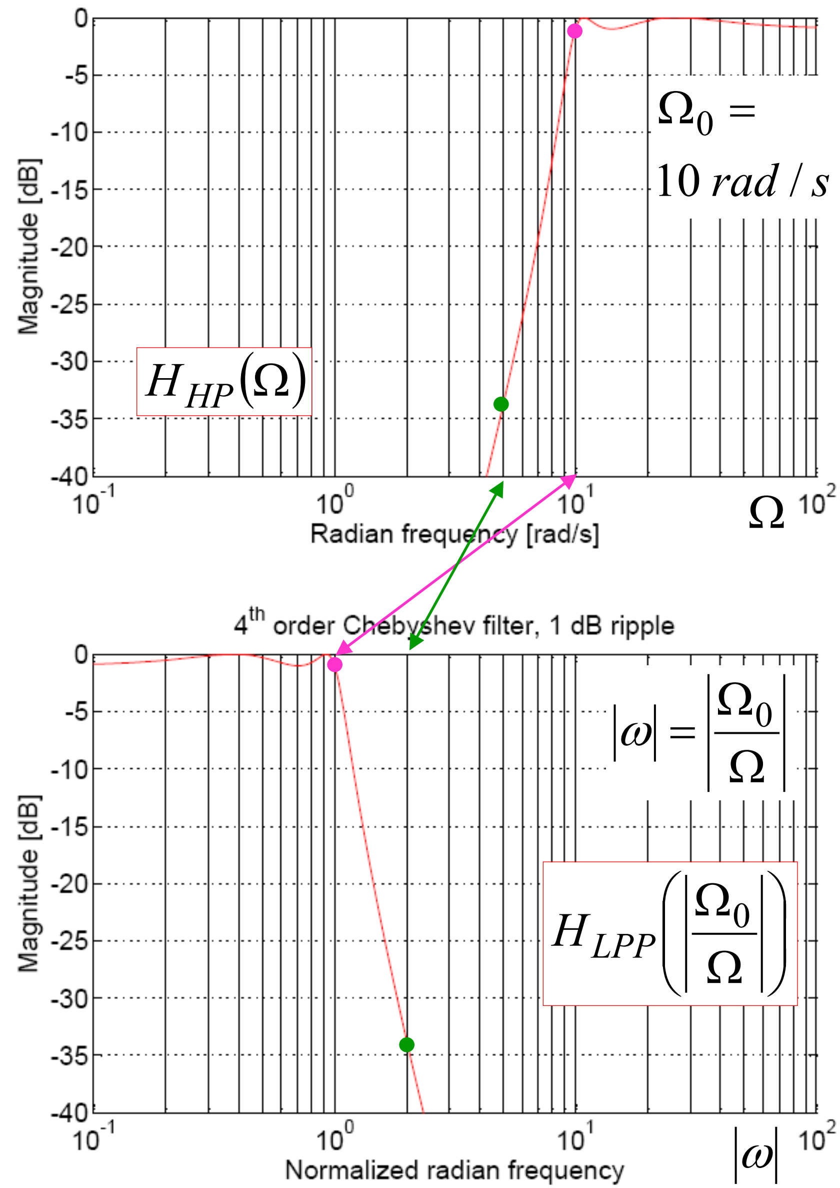
$$C_{HPi} = \frac{1}{\Omega_0 L_i}$$



# Frequency transformations, LP → HP

| Low-pass | High-pass |
|---|---|
| inductor $L_i$ | capacitor $\dfrac{1}{\Omega_0 L_i}$ |
| capacitor $C_i$ | inductor $\dfrac{1}{\Omega_0 C_i}$ |
| | $s = \dfrac{\Omega_0}{S}$ <br> $H_{LPP}\left(\dfrac{\Omega_0}{S}\right)$ |



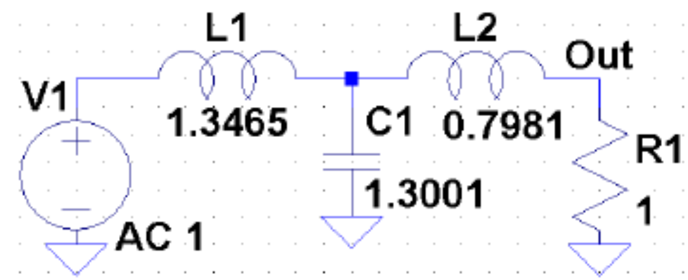
# Exercise

| Low-pass | High-pass |
|---|---|
| $L_i$ | $\dfrac{1}{\Omega_0 L_i}$ |
| $C_i$ | $\dfrac{1}{\Omega_0 C_i}$ |
| | $s = \dfrac{\Omega_0}{S}$ <br> $H_{LPP}\left(\dfrac{\Omega_0}{S}\right)$ |

10 min. break?

Break over

A normalized 3rd order Chebyshev LP-filter can be made using the circuit

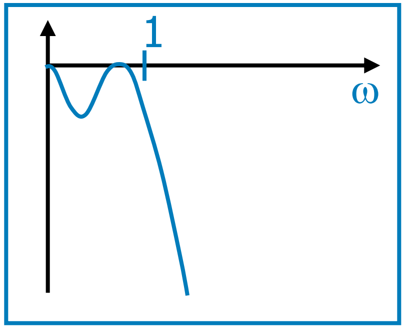


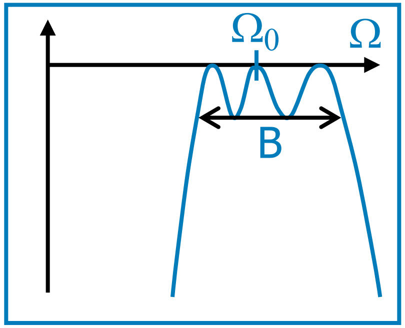

Find the component values in a 10 kHz HP-filter.

# Frequency transformations, LP → BP

**LP → BP transformation**:

$$s = \frac{S^2 + \Omega_0^2}{B \cdot S}$$




On the imaginary axis:

$$j\omega = \frac{-\Omega^2 + \Omega_0^2}{j\Omega \cdot B} = j\frac{\Omega^2 - \Omega_0^2}{\Omega \cdot B} \qquad |\omega| = \left|\frac{\Omega_0^2 - \Omega^2}{\Omega \cdot B}\right|$$

Transfer functions are linked:

$$H_{BP}(S) = H_{LPP}\left(\frac{S^2 + \Omega_0^2}{B \cdot S}\right)$$

$$H_{BP}(j\Omega) = H_{LPP}\left(j\frac{\Omega^2 - \Omega_0^2}{\Omega \cdot B}\right)$$

# Frequency transformations, LP → BP

Low-pass to band-pass

$$H_{BP}(j\Omega) = H_{LPP}\left( j\frac{\Omega^2 - \Omega_0^2}{\Omega \cdot B} \right)$$

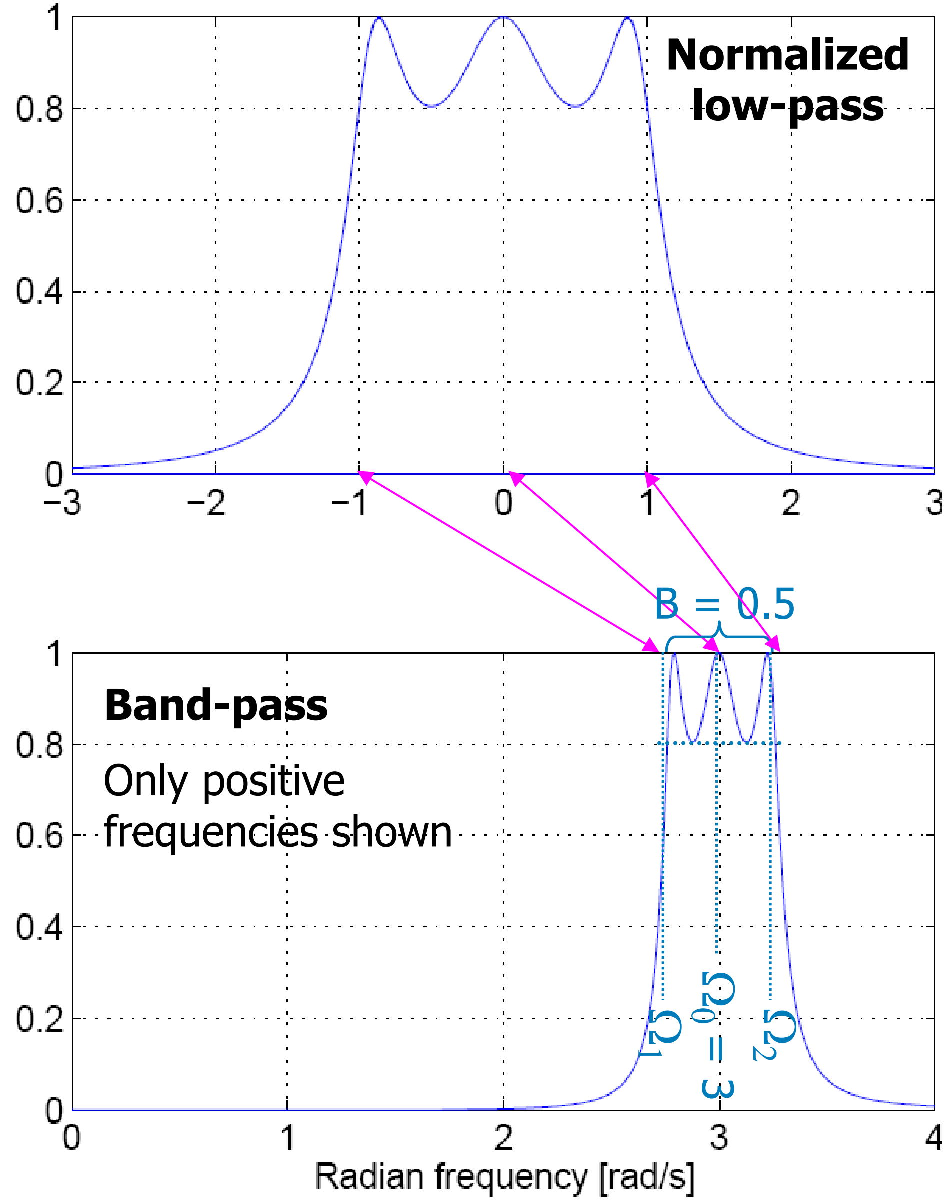
Band-edge frequencies:

$$\pm j = j\frac{\Omega_{edge}^2 - \Omega_0^2}{\Omega_{edge} \cdot B}$$

$$\Omega_{edge}^2 \mp \Omega_{edge} \cdot B - \Omega_0^2 = 0$$

$$\Omega_{edge} = \pm\frac{B}{2} \pm \sqrt{\frac{B^2}{4} + \Omega_0^2}$$

$$\left.\begin{matrix}\Omega_1 \\ \Omega_2\end{matrix}\right\} = \mp\frac{B}{2} + \sqrt{\frac{B^2}{4} + \Omega_0^2}$$



# Frequency transformations, LP → BP

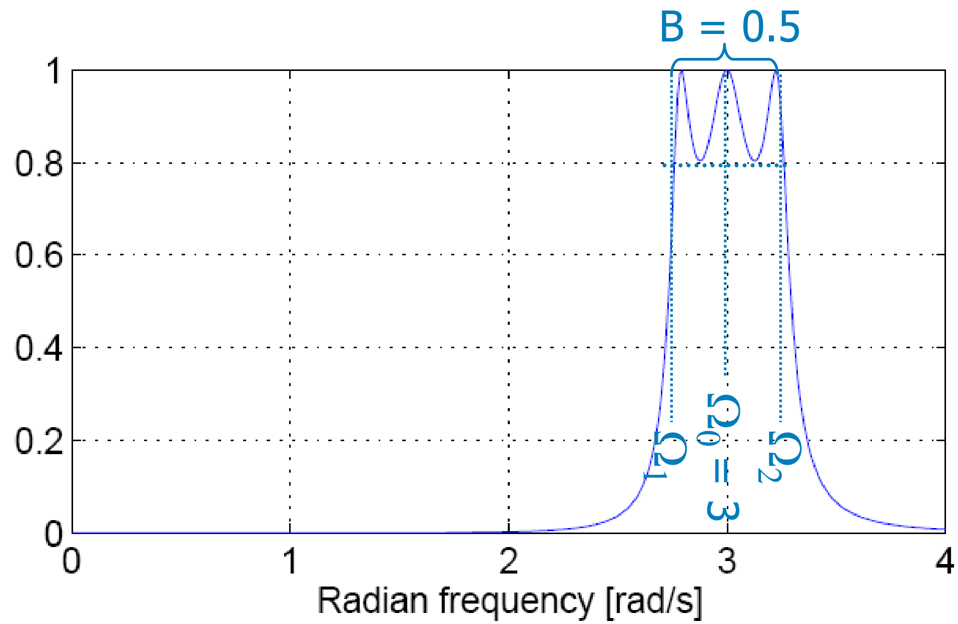
**BP frequency relations:**

$$\left.\begin{matrix}\Omega_1\\ \Omega_2\end{matrix}\right\} = \mp\frac{B}{2} + \sqrt{\frac{B^2}{4} + \Omega_0^2}$$

$$\boxed{\begin{aligned} B &= \Omega_2 - \Omega_1 \\ \Omega_0 &= \sqrt{\Omega_1 \Omega_2} \end{aligned}}$$

$$\Omega_0 \neq \frac{\Omega_1 + \Omega_2}{2}$$

$$\frac{\Omega_0}{\Omega_1} = \frac{\Omega_2}{\Omega_0}$$




Check:

$$\Omega_2\Omega_1 = \left[\frac{B}{2} + \sqrt{\frac{B^2}{4} + \Omega_0^2}\right]\left[-\frac{B}{2} + \sqrt{\frac{B^2}{4} + \Omega_0^2}\right] = -\frac{B^2}{4} - \frac{B}{2}\sqrt{\frac{B^2}{4} + \Omega_0^2} + \frac{B}{2}\sqrt{\frac{B^2}{4} + \Omega_0^2} + \sqrt{\frac{B^2}{4} + \Omega_0^2}^{\,2} = \Omega_0^2$$

**BP poles and zeros**

$$H_{LPP}(s) = \frac{K}{a_1 s^n + a_2 s^{n-1} + \ldots + a_{n-1} s^2 + a_n s + a_{n+1}} \qquad s = \frac{S^2 + \Omega_0^2}{B \cdot S}$$

$$H_{BP}(S) = \frac{K}{a_1 \left( \frac{S^2 + \Omega_0^2}{B \cdot S} \right)^n + a_2 \left( \frac{S^2 + \Omega_0^2}{B \cdot S} \right)^{n-1} + \ldots + a_n \left( \frac{S^2 + \Omega_0^2}{B \cdot S} \right) + a_{n+1}}$$

$$H_{BP}(S) = \frac{K \cdot S^n}{a_1 \left( \frac{S^2 + \Omega_0^2}{B} \right)^n + a_2 \left( \frac{S^2 + \Omega_0^2}{B} \right)^{n-1} S + \ldots + a_n \left( \frac{S^2 + \Omega_0^2}{B} \right) S^{n-1} + a_{n+1} S^n}$$

- **n-fold zero at S = 0**

Let $p_k$ be a pole in the LP function. Then the BP-poles are found from:

$$p_k = \frac{P^2 + \Omega_0^2}{B \cdot P} \Leftrightarrow P^2 - B \cdot P p_k + \Omega_0^2 = 0 \qquad P_{kBP} = ½ B p_k \pm j \sqrt{\Omega_0^2 - \left( \frac{B p_k}{2} \right)^2}$$

resulting in 2 poles, so **the number of poles is doubled**.

# Frequency transformations, LP → BP

**BP poles and zeros:**

Low-pass prototype poles (+)

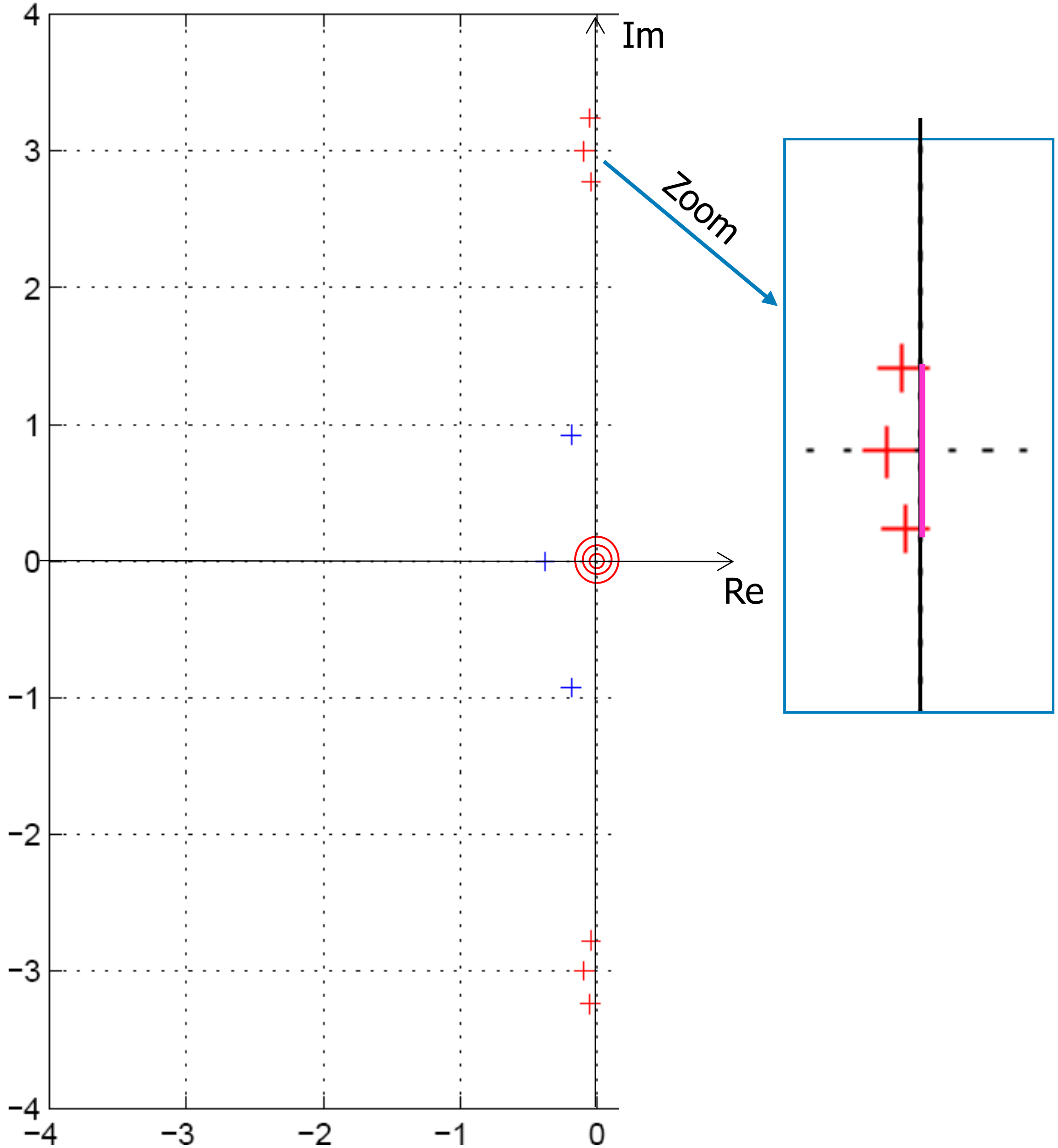
⇩

$$\boxed{P_{kBP}} = \tfrac{1}{2}B\boxed{p_k} \pm j\sqrt{\Omega_0^2 - \left(\frac{B\boxed{p_k}}{2}\right)^2}$$

⇩

Band-pass filter poles and zeros, (+, o)

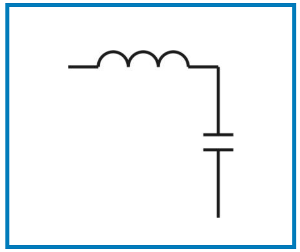
- 3rd order Chebyshev
- $\Omega_0$ = 3 rad/s
- B = 0.5 rad/s



# Frequency transformations, LP → BP

**BP components:**

LP circuit

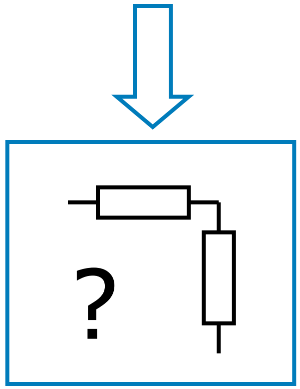


$$H_{BP}(S) = H_{LPP}\left(\frac{S^2+\Omega_0^2}{B\cdot S}\right)$$

LP-inductor impedance:

$$Z_{Li} = sL_i$$

The BP-component must have the impedance:

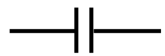
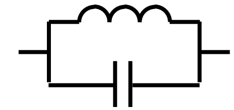
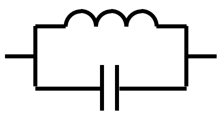



$$Z_{New}(S) = Z_{Li}\left(\frac{S^2+\Omega_0^2}{B\cdot S}\right) = \frac{S^2+\Omega_0^2}{B\cdot S}L_i = \frac{S}{B}L_i + \frac{\Omega_0^2}{B\cdot S}L_i$$

$$Z_{New}(S) = \frac{L_i}{B}S + \frac{1}{\dfrac{B}{\Omega_0^2 L_i}\cdot S}$$

Corresponds to: 

# Frequency transformations

| Low-pass | High-pass | Band-pass | Band-stop |
| --- | --- | --- | --- |
|  $L_i$ |  $\frac{1}{\Omega_0 L_i}$ |  $\frac{L_i}{B}$ $\frac{B}{\Omega_0^2 L_i}$ |  $\frac{BL_{iR}}{\Omega_0^2}$ $\frac{1}{BL_{iR}}$ |
|  $C_i$ |  $\frac{1}{\Omega_0 C_i}$ |  $\frac{B}{\Omega_0^2 C_i}$ $\frac{C_i}{B}$ |  $\frac{1}{BC_{iR}}$ $\frac{BC_{iR}}{\Omega_0^2}$ |
| 5 min. break<br>Break over | $s=\frac{\Omega_0}{S}$<br>$H_{LPP}\left(\frac{\Omega_0}{S}\right)$ | $s=\frac{S^2+\Omega_0^2}{B\cdot S}$<br>$H_{LPP}\left(\frac{S^2+\Omega_0^2}{B\cdot S}\right)$ | $s=\frac{B\cdot S}{S^2+\Omega_0^2}$<br>$H_{LPR}\left(\frac{B\cdot S}{S^2+\Omega_0^2}\right)$ |

# Frequency transformations, LP → BS

**First re-normalize the LP-filter** to have $\omega_{stop}$ = 1 rad/s:

**Next:**
Frequency transformation:

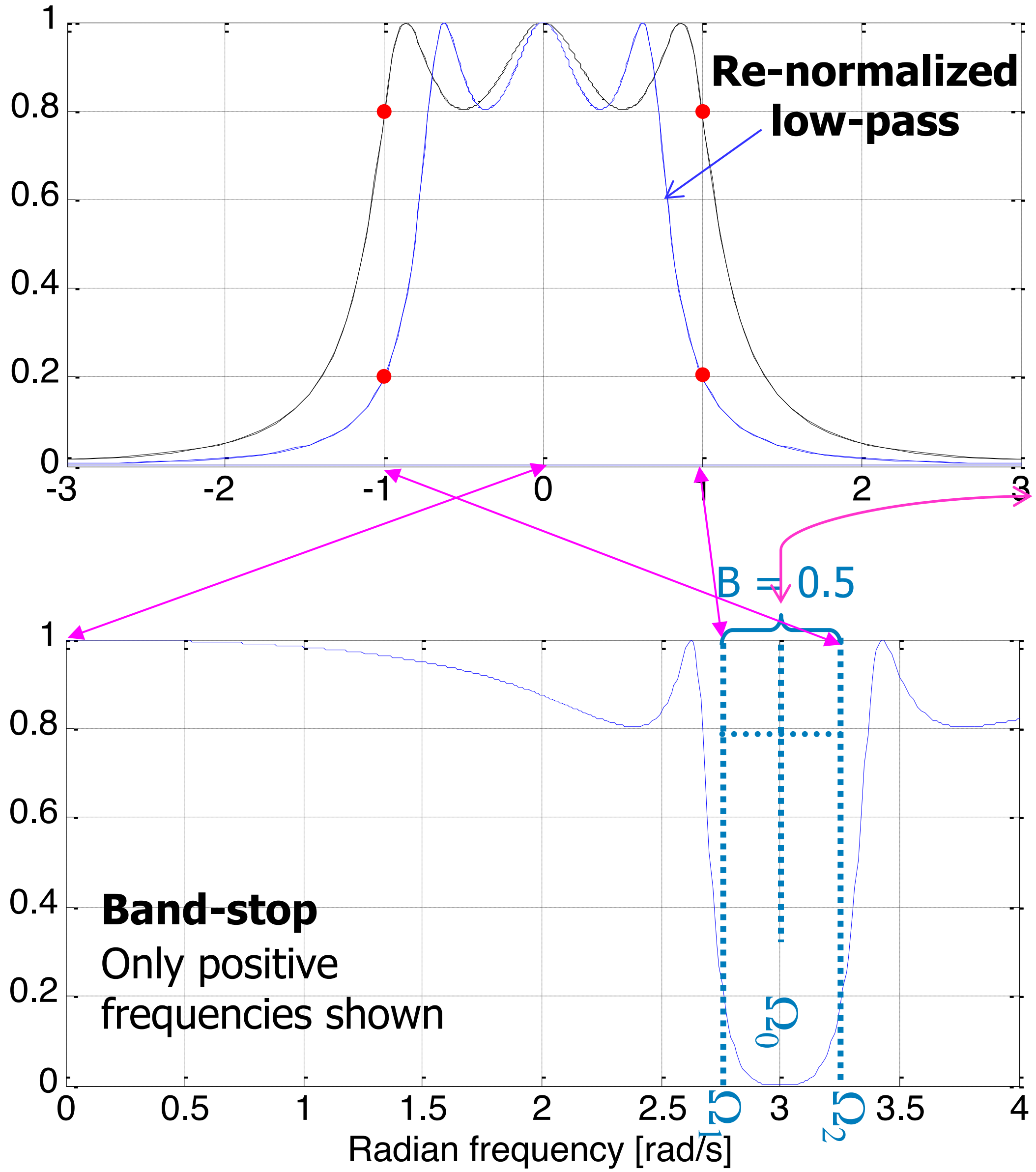
$$s = \frac{B \cdot S}{S^2 + \Omega_0^2}$$

$$j\omega = \frac{j\Omega B}{-\Omega^2 + \Omega_0^2}$$

Transfer function:

$$H_{BS}(S) = H_{LPR}\left(\frac{B \cdot S}{S^2 + \Omega_0^2}\right)$$

$\Omega_1$ and $\Omega_2$ expressions as for BP



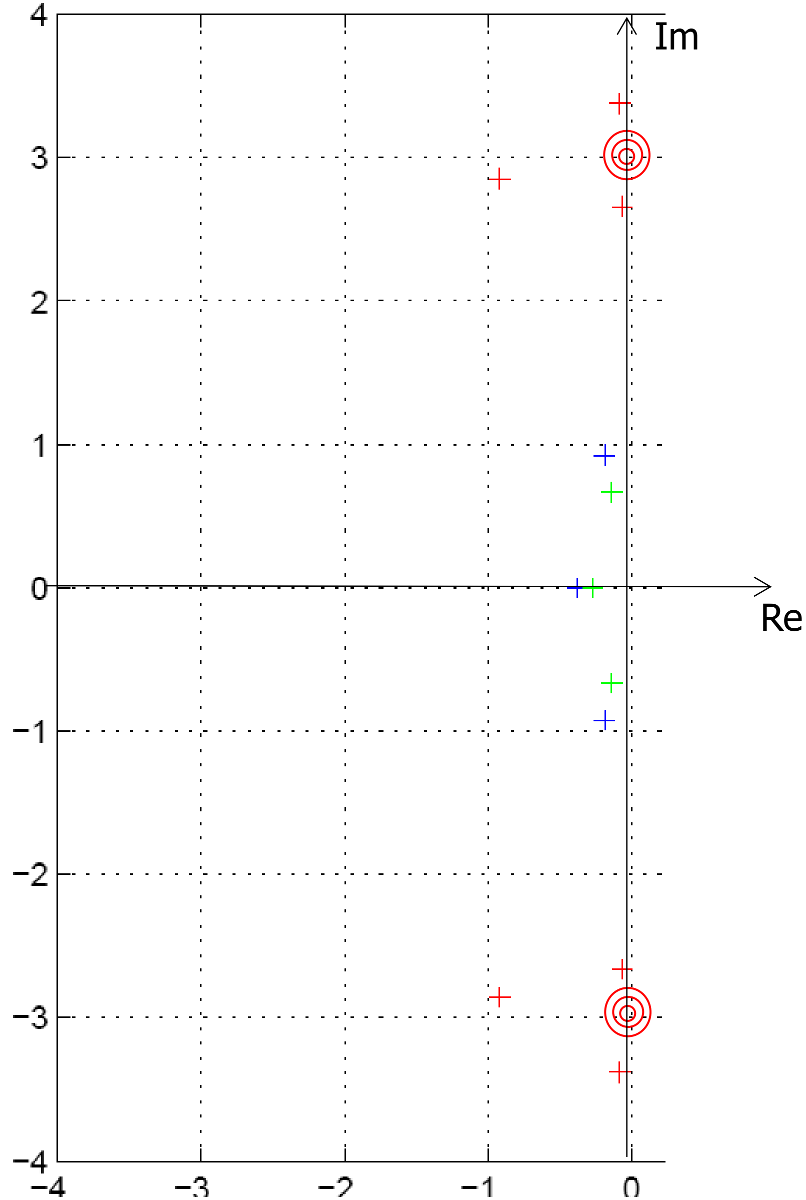
**BS poles and zeros**

Low-pass normalized poles (+)

Low-pass re-normalized poles (+)

$$\boxed{p_R} = \frac{B \cdot \boxed{P}}{\boxed{P}^2 + \Omega_0^2}$$

Solve to find band-stop filter poles and zeros, (+, o)




- 3rd order Chebyshev
- $\Omega_0 = 3$ rad/s
- B = 0.5 rad/s
- (Note n-fold zeros at $\pm j\Omega_0$)

# BP, HP and BS in Matlab

Useful Matlab functions:

Direct HP, BP and BS:

[B,A] = CHEBY1(N,R,Wn,’s’)

If Wn is a two-element vector, Wn = [W1 W2], CHEBY1 returns an order 2N bandpass filter with passband W1 < W < W2 or:

[B,A] = CHEBY1(N,R,Wn,’bandpass’,’s’) is a bandpass filter if Wn = [W1 W2]

[B,A] = CHEBY1(N,R,Wn,'high’,’s’) designs a highpass filter.

[B,A] = CHEBY1(N,R,Wn,'low’,’s’) designs a lowpass filter.

[B,A] = CHEBY1(N,R,Wn,'stop’,’s’) is a bandstop filter if Wn = [W1 W2]

$$H(s)=\frac{b_1 s^n + b_2 s^{n-1} + \ldots + b_{n-1} s^2 + b_n s + b_{n+1}}{a_1 s^n + a_2 s^{n-1} + \ldots + a_{n-1} s^2 + a_n s + a_{n+1}}$$

lp2bp

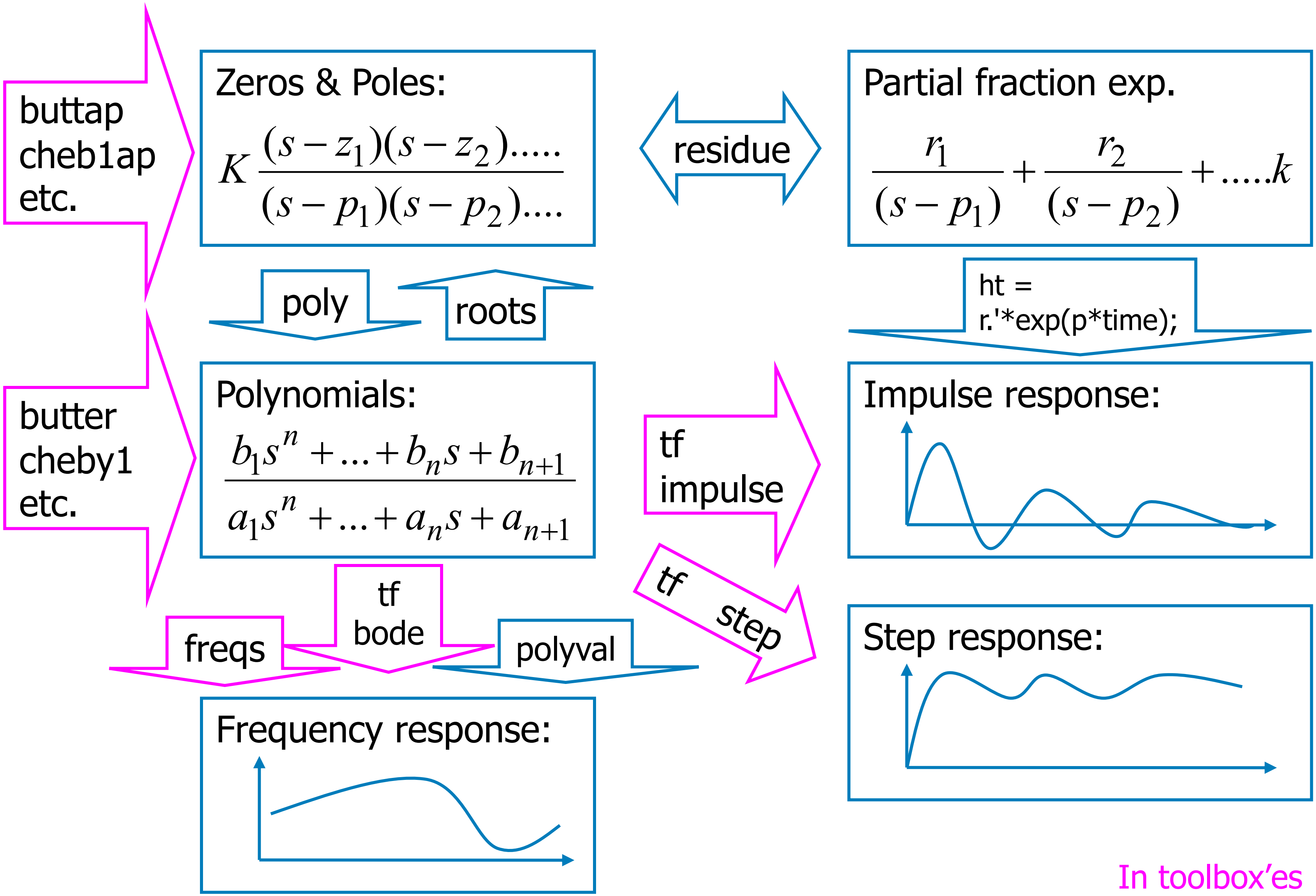
lp2bs

lp2hp

[bt,at] = lp2bp(b,a,Wo,Bw) transforms an analog lowpass filter prototype given by polynomial coefficients into a bandpass filter with center frequency Wo and bandwidth Bw.

Row vectors b and a specify the coefficients of the numerator and denominator of the prototype in descending powers of s.

# Matlab roadmap



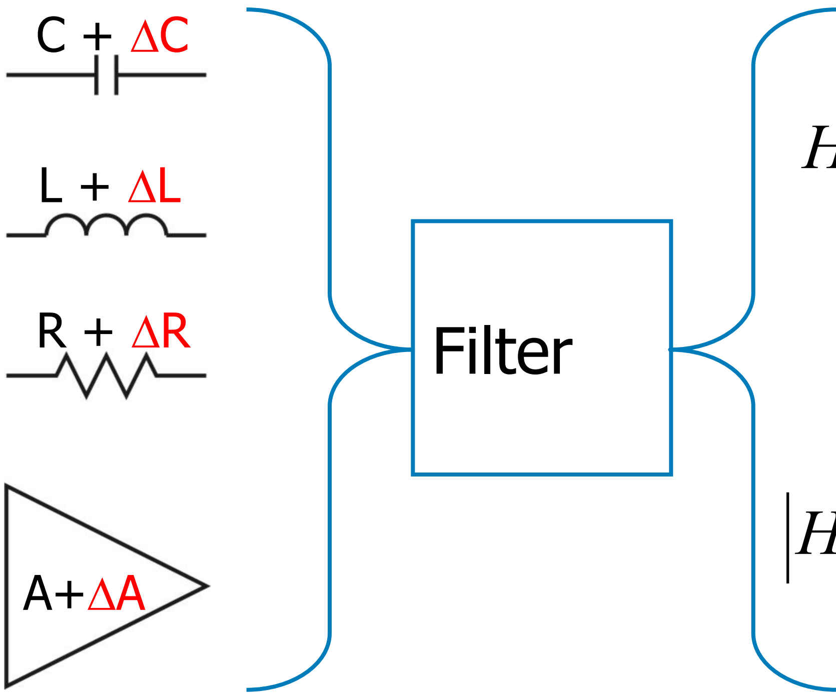
# Sensitivity analysis

- A filter will perform as specified when component values are as calculated
- The parameters of components are not exact due to:
  - Fabrication tolerances, e.g.: $\pm$0.1% . $\pm$1% .... $\pm$20% .
  - Roundoff to standard values: 8.1244 nF → 8.2 nF
  - Temperature drift
  - Ageing

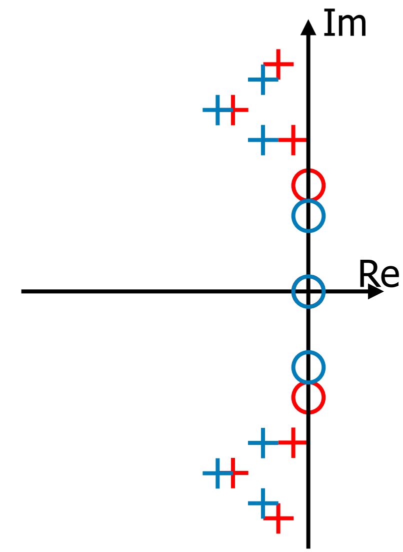
- How sensitive is a filter to parameter variations?
- Are passive and active realisations equally sensitive?
- Are all active realisations equally sensitive?

Sensitivity analysis is a tool to investigate this

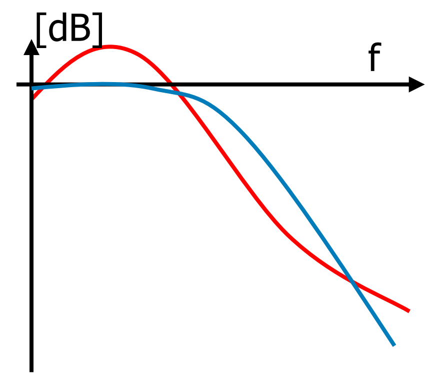
# How is sensitivity defined?




$$H(s) = K\frac{\prod_i (s - z_i)}{\prod_j (s - p_j)}$$



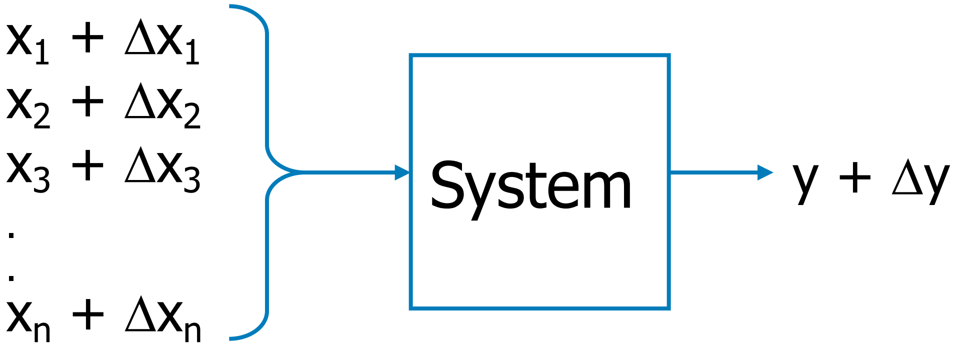

$$\left|H_{But,n}(j\omega)\right| = \frac{1}{\sqrt{1+\omega^{2n}}}$$




The outputs in sensitivity analysis of a system can be defined in a lot of different ways

The outputs are usually not the output signals of the filter

# Sensitivity – multiple input system




Using a Taylor expansion:

$$y + \Delta y \approx y + \frac{\partial y}{\partial x_1}\Delta x_1 + \frac{\partial y}{\partial x_2}\Delta x_2 + \frac{\partial y}{\partial x_3}\Delta x_3 + \ldots \frac{\partial y}{\partial x_n}\Delta x_n$$

+ higher order terms: $\frac{\partial^2 y}{\partial x_i^2}$ + cross-terms

Neglecting higher order terms and "cross-terms":

$$\Delta y \approx \sum_i \frac{\partial y}{\partial x_i}\Delta x_i \Rightarrow \frac{\Delta y}{y} \approx \sum_i \frac{1}{y}\cdot\frac{\partial y}{\partial x_i}\Delta x_i$$

When each perturbation is small the combined effect is found as a sum of the effects from each perturbation

# How is sensitivity defined?

$$x \longrightarrow \boxed{\text{System}} \longrightarrow y \qquad\qquad x + \Delta x \longrightarrow \boxed{\text{System}} \longrightarrow y + \Delta y$$

Mathematical definition of sensitivity:

$$\boxed{S_x^y = \frac{x}{y} \cdot \frac{dy}{dx} \approx \frac{x}{y} \cdot \frac{\Delta y}{\Delta x} = \frac{\Delta y / y}{\Delta x / x}}$$

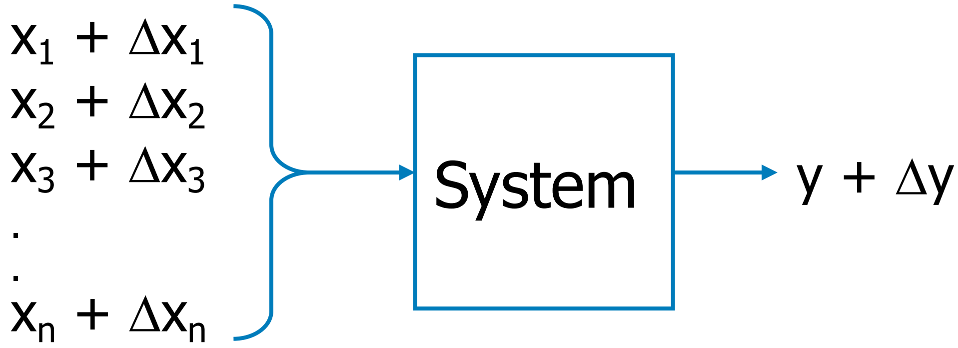
The **relative** change of output divided by the **relative** change of input

E.g.: If $S^y{}_x = 2$, then a 1% change of x will give a 2% change of y

Multiple input system:

$$S_{x_i}^y = \frac{x_i}{y} \cdot \frac{\partial y}{\partial x_i}$$

# Sensitivity – multiple input system




$$\frac{\Delta y}{y} \approx \sum_i \frac{1}{y} \cdot \frac{\partial y}{\partial x_i} \Delta x_i = \sum_i \frac{x_i}{y} \cdot \frac{\partial y}{\partial x_i} \cdot \frac{\Delta x_i}{x_i}$$

$$\boxed{S_{x_i}^{y} = \frac{x_i}{y} \cdot \frac{\partial y}{\partial x_i}}$$

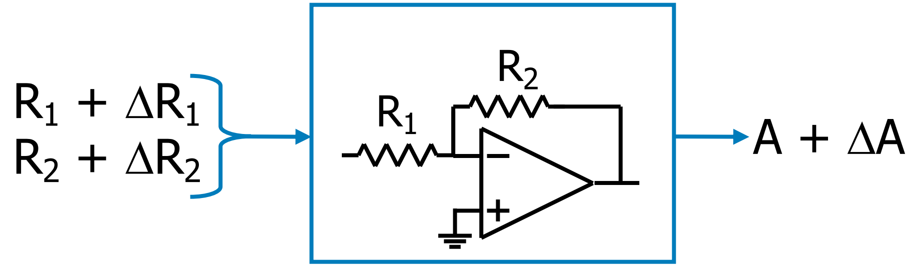
$$\boxed{\frac{\Delta y}{y} \approx \sum_i S_{x_i}^{y} \frac{\Delta x_i}{x_i}}$$

# Sensitivity, example

Inverting amplifer:

Input: Variations on $R_1$ and $R_2$

Output: Amplification, $A = -R_2/R_1$




$$\frac{\Delta A}{A} \approx \sum_{i=1}^{2} S_{R_i}^{A} \frac{\Delta R_i}{R_i}$$

$$\frac{\Delta A}{A} \approx S_{R_1}^{A} \frac{\Delta R_1}{R_1} + S_{R_2}^{A} \frac{\Delta R_2}{R_2}$$

$$S_{R_1}^{A} = \frac{R_1}{A} \cdot \frac{\partial A}{\partial R_1} = \frac{R_1}{A} \cdot \frac{R_2}{R_1^2} = -1$$

$$S_{R_2}^{A} = \frac{R_2}{A} \cdot \frac{\partial A}{\partial R_2} = \frac{R_2}{A} \cdot \frac{-1}{R_1} = 1$$

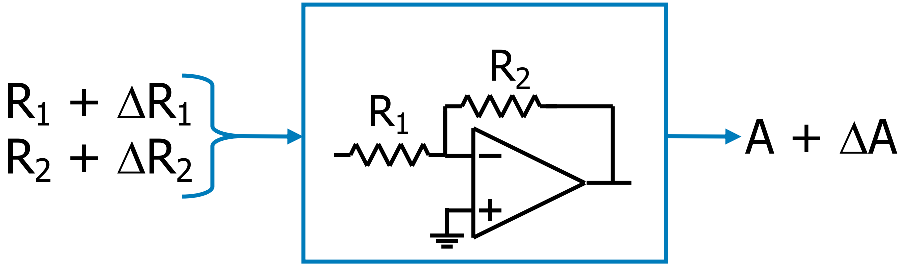
$$\frac{\Delta A}{A} \approx -\frac{\Delta R_1}{R_1} + \frac{\Delta R_2}{R_2}$$

# Exercise

Inverting amplifer:

Input: Variations on $R_1$ and $R_2$

Output: Amplification, $A = -R_2/R_1$




**Prove that the gain of the inverting amplifier $A = -R_2/R_1$**

5 min. break

Break over

# Sensitivity - math

$$x + \Delta x \rightarrow \boxed{\text{System}} \rightarrow \boxed{k} \rightarrow k(y+\Delta y)$$

Scaling y with a constant, k, does not affect the sensitivity

$$S_x^{ky} = \frac{x}{ky} \cdot \frac{\partial (ky)}{\partial x} = \frac{x}{y} \cdot \frac{\partial y}{\partial x} = S_x^y$$

$$x \rightarrow \boxed{\text{System 1}} \rightarrow z_1, z_2, ..z_n \rightarrow \boxed{\text{System 2}} \rightarrow y$$

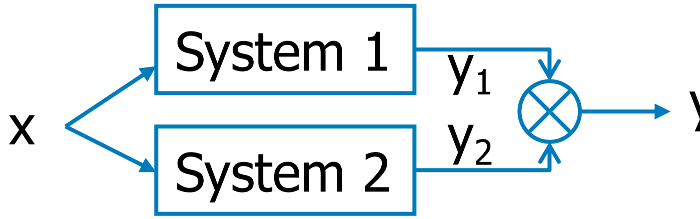
$$\frac{\Delta y}{y} = \sum_{i=1}^{n} S_{z_i}^y \frac{\Delta z_i}{z_i} = \sum_{i=1}^{n} S_{z_i}^y S_x^{z_i} \frac{\Delta x}{x}$$

$$S_x^y = \frac{x}{y} \cdot \frac{\Delta y}{\Delta x} = \sum_{i=1}^{n} S_{z_i}^y S_x^{z_i}$$

$$S_x^{y(z_1, z_2 \ldots z_n)} = \sum_{i=1}^{n} S_{z_i}^y S_x^{z_i}$$

# Sensitivity - math

The sensitivity of the product of 2 parameters wrt. x is the sum of the two individual sensitivities wrt. x




$$S_x^{y_1 \cdot y_2} = \frac{x}{y_1 y_2} \cdot \frac{\partial (y_1 y_2)}{\partial x} = \frac{x}{y_1 y_2} \left( y_2 \frac{\partial y_1}{\partial x} + y_1 \frac{\partial y_2}{\partial x} \right)$$

$$\underline{S_x^{y_1 \cdot y_2}} = \frac{x}{y_1} \cdot \frac{\partial y_1}{\partial x} + \frac{x}{y_2} \cdot \frac{\partial y_2}{\partial x} = \underline{S_x^{y_1} + S_x^{y_2}}$$

Extensions:

$$S_x^{\prod_i y_i} = \sum_i S_x^{y_i} \qquad\qquad S_x^{y^n} = n \cdot S_x^{y}$$

The sensitivity of the ratio between 2 parameters wrt. x is the difference between the two individual sensiti-vities wrt. x

$$S_x^{y_1 / y_2} = \frac{x}{y_1 / y_2} \cdot \frac{\partial (y_1 / y_2)}{\partial x} = \frac{x}{y_1 / y_2} \left( \frac{1}{y_2} \cdot \frac{\partial y_1}{\partial x} - \frac{y_1}{y_2^2} \cdot \frac{\partial y_2}{\partial x} \right)$$

$$\underline{S_x^{y_1 / y_2}} = \frac{x}{y_1} \cdot \frac{\partial y_1}{\partial x} - \frac{x}{y_2} \cdot \frac{\partial y_2}{\partial x} = \underline{S_x^{y_1} - S_x^{y_2}}$$

# Sensitivity – transfer functions

$$S_x^{y_1/y_2} = S_x^{y_1} - S_x^{y_2} \qquad S_x^{y_1 \cdot y_2} = S_x^{y_1} + S_x^{y_2}$$

Transfer function as a ratio of the numerator and denominator polynomials:

$$H(s) = \frac{P(s)}{Q(s)} \qquad S_x^{H(s)} = S_x^{P(s)} - S_x^{Q(s)}$$

Transfer function as magnitude and phase:

$$H(s)\big|_{s=j\omega} = \left|H(j\omega)\right| e^{j\varphi(\omega)}$$

$$S_x^{H(j\omega)} = S_x^{|H(j\omega)|} + S_x^{e^{j\varphi(\omega)}} \qquad \left(S_z^{e^y} = \frac{z}{e^y} \cdot \frac{\partial e^y}{\partial z} = z\frac{\partial y}{\partial z} = y\frac{z}{y} \cdot \frac{\partial y}{\partial z} = yS_z^y\right)$$

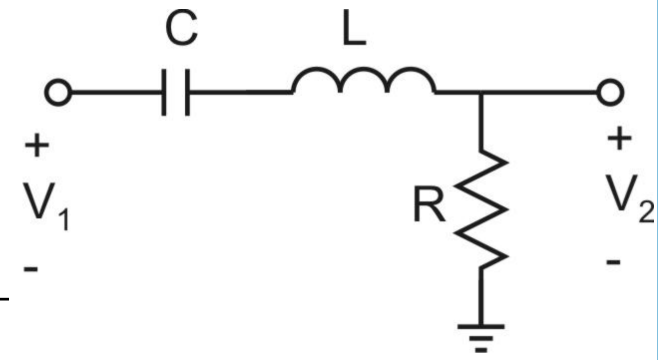
$$S_x^{H(j\omega)} = S_x^{|H(j\omega)|} + j\varphi(\omega) S_x^{\varphi(\omega)} \qquad \left(S_x^{\varphi} = S_x^{j\varphi}\right)$$

Sensitivity of magnitude and phase:

$$S_x^{|H(j\omega)|} = \mathrm{Re}\left(S_x^{H(j\omega)}\right) \qquad S_x^{\varphi(\omega)} = \frac{1}{\varphi(\omega)} \mathrm{Im}\left(S_x^{H(j\omega)}\right)$$

# Sensitivity example: 2nd order BP: Passive LCR

The transfer function is found by voltage division:




$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{R}{\dfrac{1}{sC} + sL + R} = \frac{sCR}{s^2 LC + sCR + 1}$$

This may be written in the standard form:

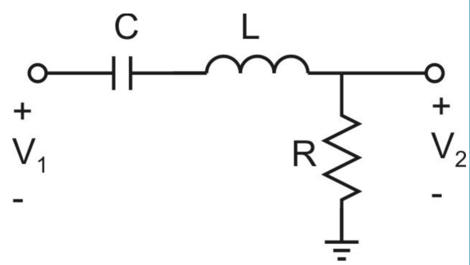
$$H(s) = \frac{s\dfrac{R}{L}}{s^2 + s\dfrac{R}{L} + \dfrac{1}{LC}} = \frac{s\dfrac{\omega_0}{Q}}{s^2 + s\dfrac{\omega_0}{Q} + \omega_0^2} = \frac{s2\zeta\omega_0}{s^2 + s2\zeta\omega_0 + \omega_0^2}$$

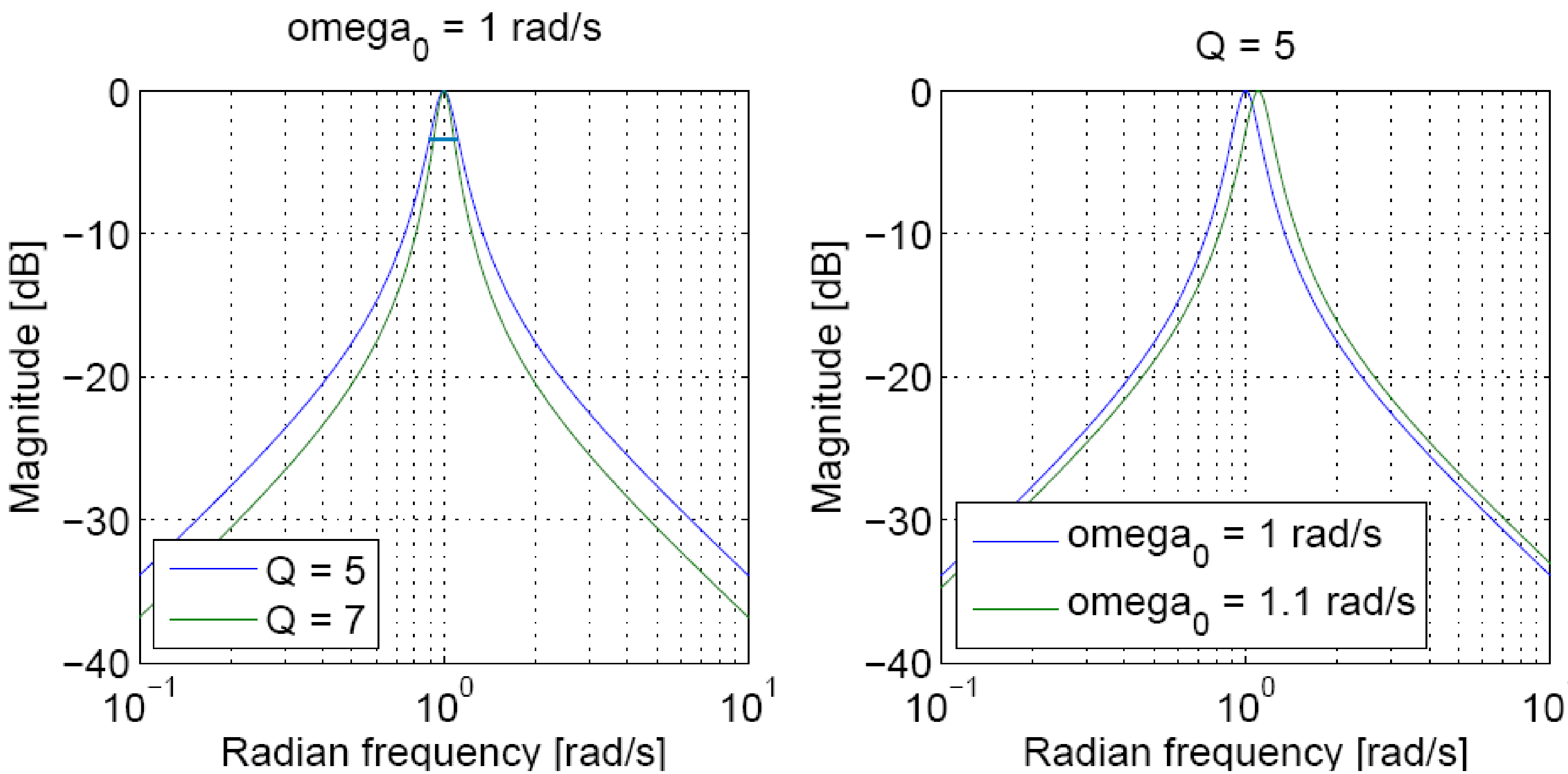
Where:

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} \qquad \zeta = \frac{1}{2Q}$$

# Sensitivity example: 2nd order BP: Passive LCR

$$H(s)=\frac{s\dfrac{\omega_0}{Q}}{s^2+s\dfrac{\omega_0}{Q}+\omega_0^2}\qquad \omega_0=\frac{1}{\sqrt{LC}}\qquad Q=\frac{\omega_0 L}{R}=\frac{1}{\omega_0 CR}$$




Q = Quality factor

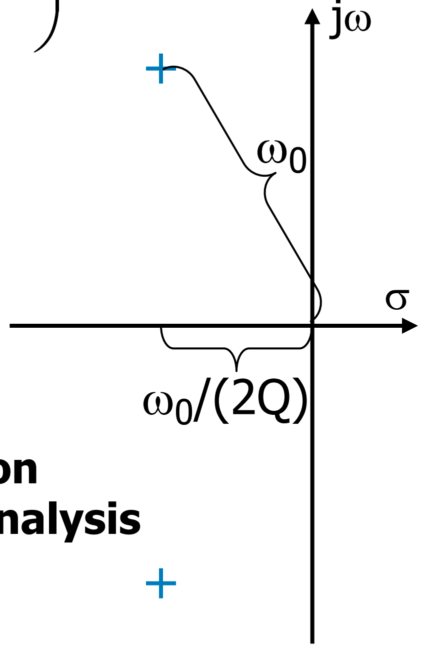
$Q = \omega_0/\Delta\omega_{3dB}$

# Sensitivity example: 2nd order BP

$$H(s)=\frac{s\frac{\omega_0}{Q}}{s^2+s\frac{\omega_0}{Q}+\omega_0^2} \qquad \omega_0=\frac{1}{\sqrt{LC}} \quad Q=\frac{\omega_0 L}{R}=\frac{1}{\omega_0 CR}$$

Poles:

$$p_k=-\frac{\omega_0}{2Q}\pm\frac{1}{2}\sqrt{\left(\frac{\omega_0}{Q}\right)^2-4\omega_0^2}=\omega_0\left(\frac{-1}{2Q}\pm j\sqrt{1-\left(\frac{1}{2Q}\right)^2}\right)$$




A high Q-value means that the poles are close to the imaginary axis

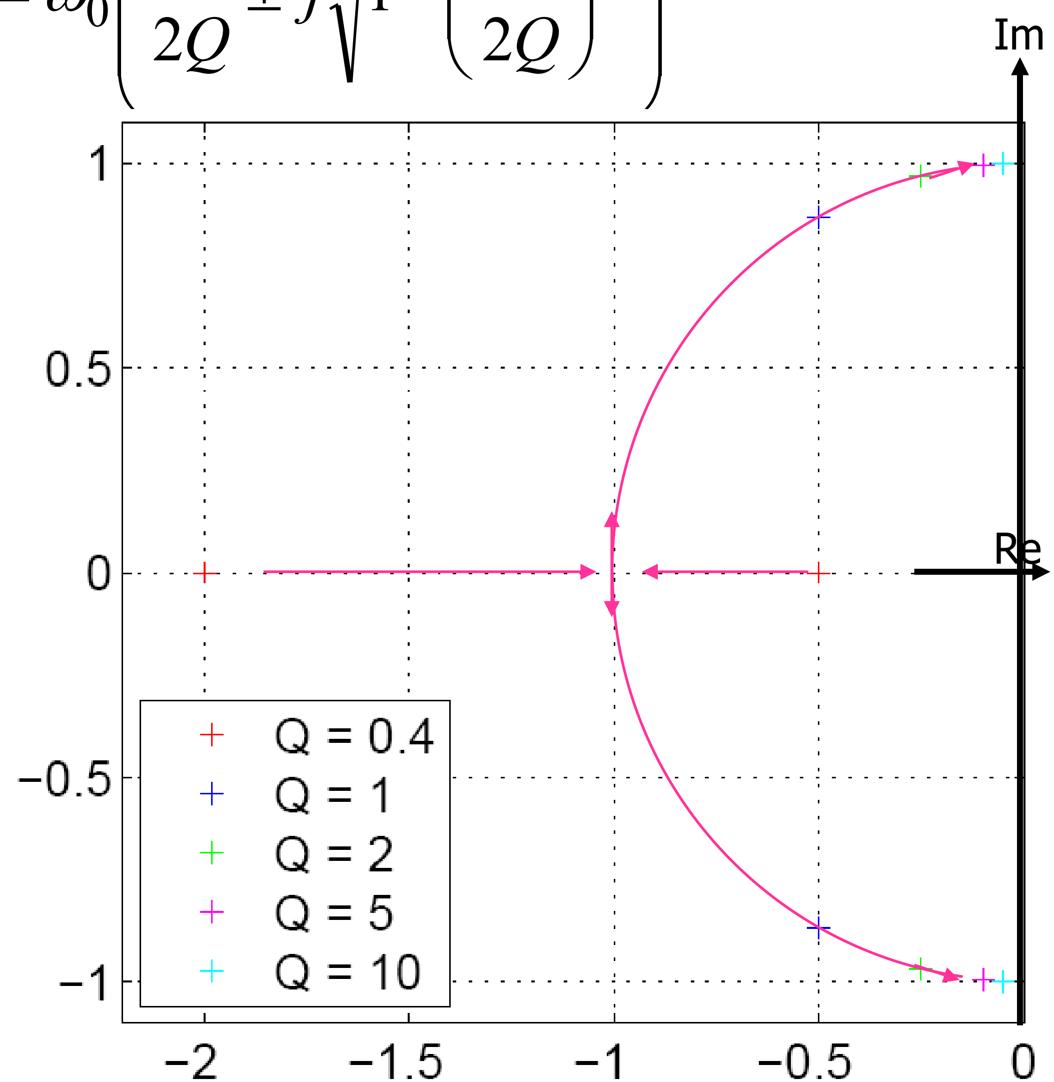
**Since $\omega_0$ and Q are important parameters for the transfer function they are often chosen as output parameters for the sensitivity analysis**

# Sensitivity example: 2nd order BP

$$p_k = -\frac{\omega_0}{2Q} \pm \frac{1}{2}\sqrt{\left(\frac{\omega_0}{Q}\right)^2 - 4\omega_0^2} = \omega_0\left(\frac{-1}{2Q} \pm j\sqrt{1-\left(\frac{1}{2Q}\right)^2}\right)$$

Pole positions as a function of Q:
$\omega_0$ = 1 rad/s

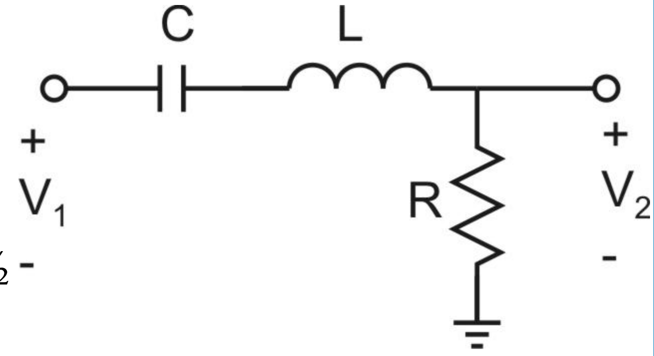
Increasing Q




The pole sensitivity, $S_x^{p_k}$, can be a complex number

# Sensitivity example: 2nd order BP

$$\omega_0 = \frac{1}{\sqrt{LC}} = L^{-1/2} C^{-1/2}$$




$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} = \frac{1}{R}\sqrt{\frac{L}{C}} = R^{-1} L^{1/2} C^{-1/2}$$

Sensitivities:

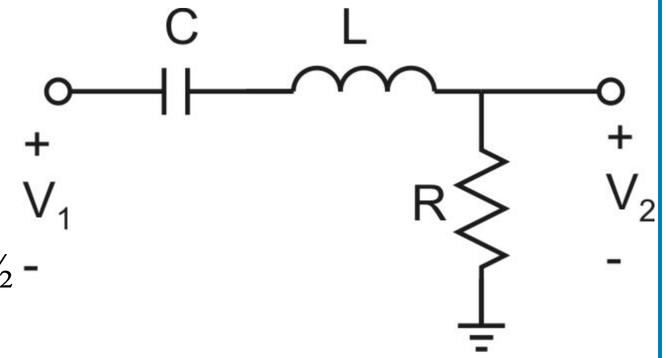
$$S_L^{\omega_0} = \frac{L}{\omega_0} \cdot \frac{\partial \omega_0}{\partial L} = \frac{L}{L^{-1/2} C^{-1/2}} \cdot \frac{\partial (L^{-1/2} C^{-1/2})}{\partial L} = -1/2$$

A 2 % increase of L will cause 1 % decrease in $\omega_0$.

# Sensitivity example: 2nd order BP – small exercise (5 min.)

$$\omega_0 = \frac{1}{\sqrt{LC}} = L^{-1/2}C^{-1/2}$$

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} = \frac{1}{R}\sqrt{\frac{L}{C}} = R^{-1}L^{1/2}C^{-1/2}$$




**Prove that:**

$$S_C^{\omega_0} = -1/2 \quad S_R^{\omega_0} = 0$$

$$S_R^{Q} = -1 \quad S_L^{Q} = 1/2 \quad S_C^{Q} = -1/2$$