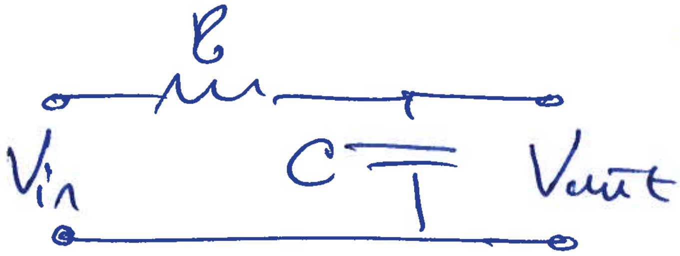
# Signal Processing, 6th lecture
## Suggested Solutions



Given




$R = 1\,k\Omega$
$C = 1\,\mu F$

1) Discrete time impulse response

$$\begin{cases} h(t) = \frac{1}{RC} \cdot e^{-t/RC} \\ h[n] = T \cdot h(t)\Big|_{t=nT} = \frac{T}{RC} \cdot e^{-\frac{nT}{RC}} \\ T = \frac{1}{8000} = 125 \cdot 10^{-6}\,s. \end{cases}$$

$\Downarrow$

$$h[n] = 0.125 \cdot e^{-n \cdot 0.125}$$

Calculate $h[n]$ for $n=0,1,2,3,4$ ;

2

$h[0] = 0.1250$

$h[1] = 0.1103$

$h[2] = 0.0974$

$h[3] = 0.0859$

$h[4] = 0.0758$

2) Convolution Sum

$$y[n] = h[n] * x[n]$$

$$\Downarrow$$

$$y[n] = \sum_{m=-\infty}^{\infty} h[m] \cdot x[n-m]$$

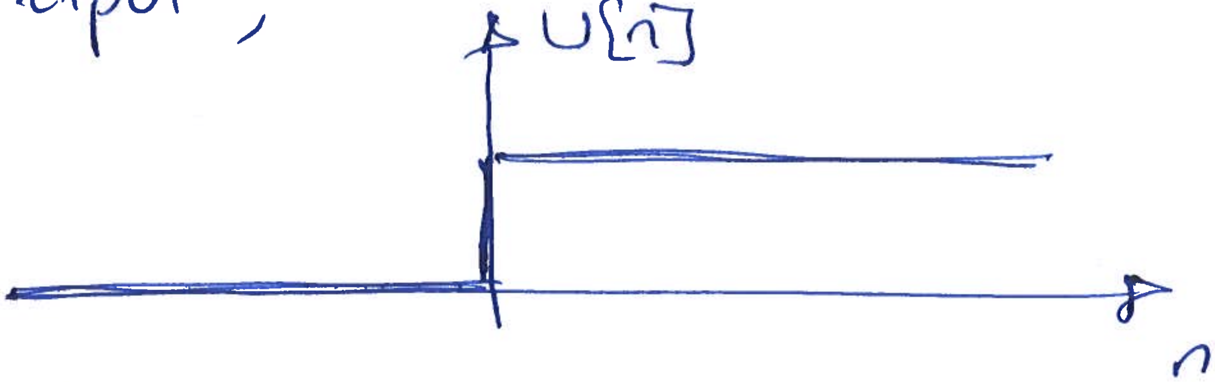
$$\Downarrow$$

$$y[n] = \sum_{m=-\infty}^{\infty} 0.125 \cdot e^{-m \cdot 0.125} \cdot x[n-m]$$

$$y[n] = 0.125 \sum_{m=0}^{\infty} e^{-m \cdot 0.125} \cdot x[n-m]$$

for Causal LTI system

Step-response is the output when $x[n] = u[n]$ is applied to the input;

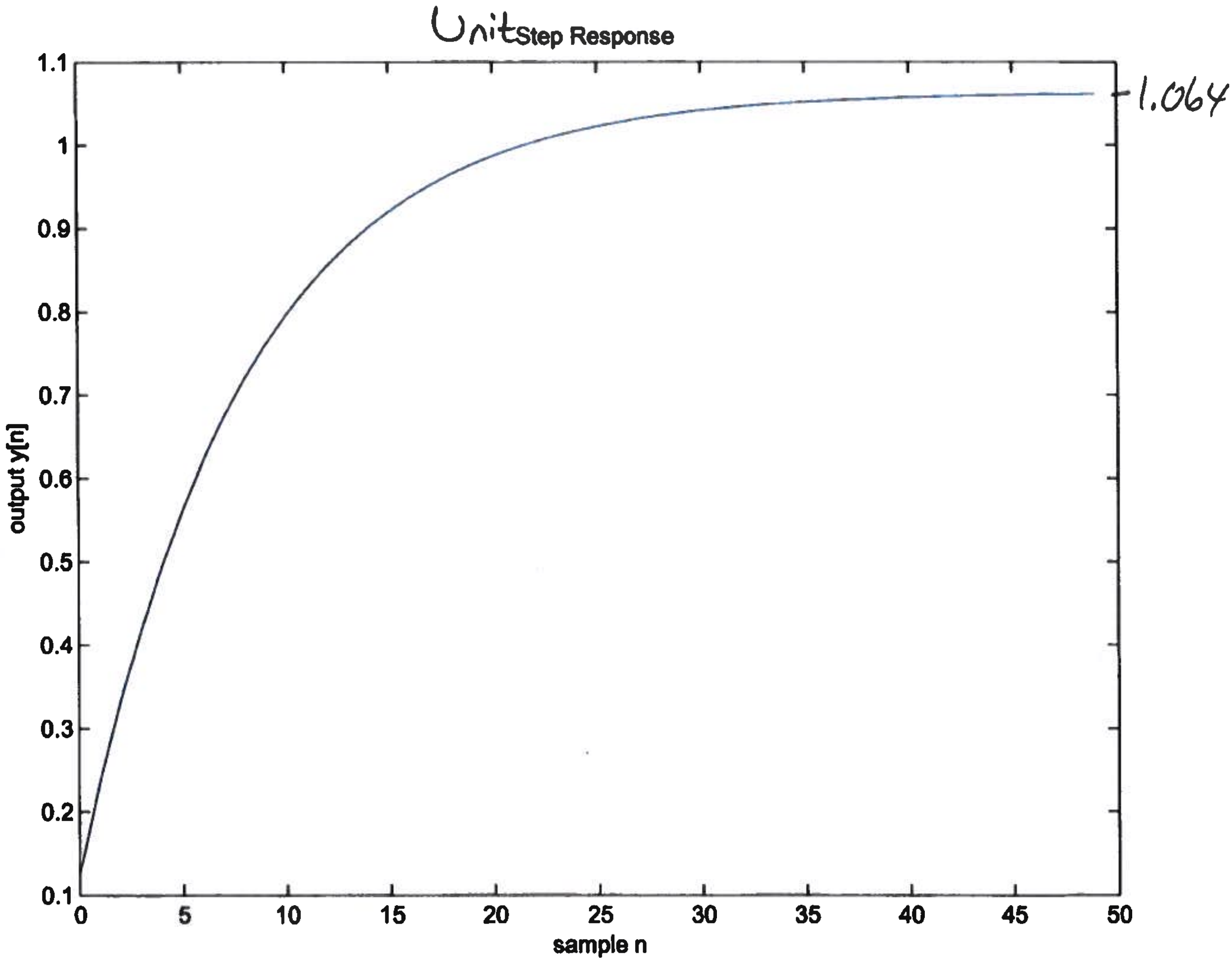



Write a Matlab-program to do the calculation.

```
% Beregning af Step-respons vha. foldnings-summen
% y[n] = SUM[m=0..infinite] 0.125e^(-0.125m) x[n-m]

% Steprespons'en beregnes ved hjælp af to nestede loops
% Den ydre løkke opdaterer sample nummeret mens den indre
% løkke beregner selve produkt-summen.
% Der beregnes 50 samples.



clear;

for n=0:49, % n er sample-nummeret
  sum = 0;  % sum benyttes til beregning af produktsummen for sample n
            % og sættes følgelig lig 0 initialt

  % Den indre løkke beregner selve produktsummen -- baseres på 100 led
  for m=0:99,

    % Bestem værdien af impulsresponsen til tidspunktet, m
    h = 0.125*exp(-0.125*m);

    % Bestem værdien af step-funktionen u[n-m]
    if (n-m) >= 0
      u = 1;
    else
      u = 0;
    end;

    sum = sum + (h * u);
  end;

  % Opdater output-sekvensen
  y(n+1) = sum; % Bemærk at MatLab ikke kan indeksere 0
end;

% Plot stepresponsen %
x(1:50)=0:49;
plot(x,y)
title('Step Response');
xlabel('sample n');
ylabel('output y[n]');
```

PK/

Bemærk, at filteret i steady_state har et output, som numerisk er større end input (U[n]).
Se side 12.



3) Transfer function.

$$\begin{cases} H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{b}{1 - a z^{-1}} \\ b = \dfrac{T}{RC} \text{ and } a = e^{-T/RC} \end{cases}$$

$\Downarrow$

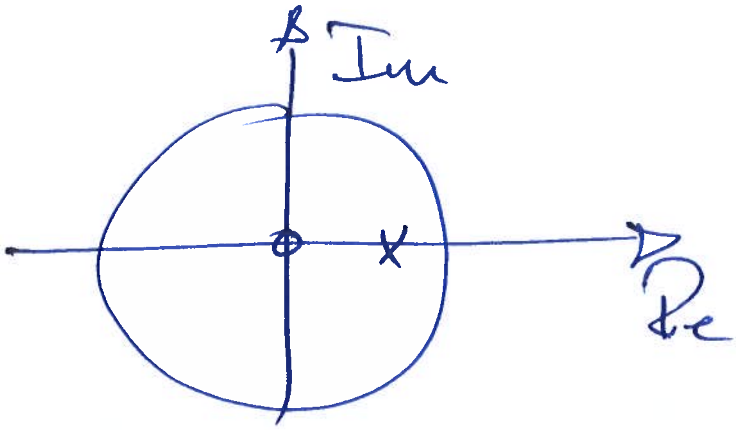
$$H(z) = \frac{0{,}125}{1 - 0{,}8825 z^{-1}}$$

Poles and Zeros:

$$H(z) = \frac{0{,}125}{1 - 0{,}8825 z^{-1}} = \frac{0{,}125 z}{z - 0{,}8825}$$

$\Downarrow$

pole in $z = 0{,}8825$ and zero in $z=0$




Pole is inside the unit circle

$\Longrightarrow$ Stabil filter

4) Difference equation

$$H(z) = \frac{b}{1 - az^{-1}} = \frac{Y(z)}{X(z)}$$

$\Downarrow$

$$\mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\{aY(z)z^{-1} + bX(z)\}$$

$\Downarrow$

$$\underline{\underline{y[n] = ay[n-1] + bx[n]}}$$

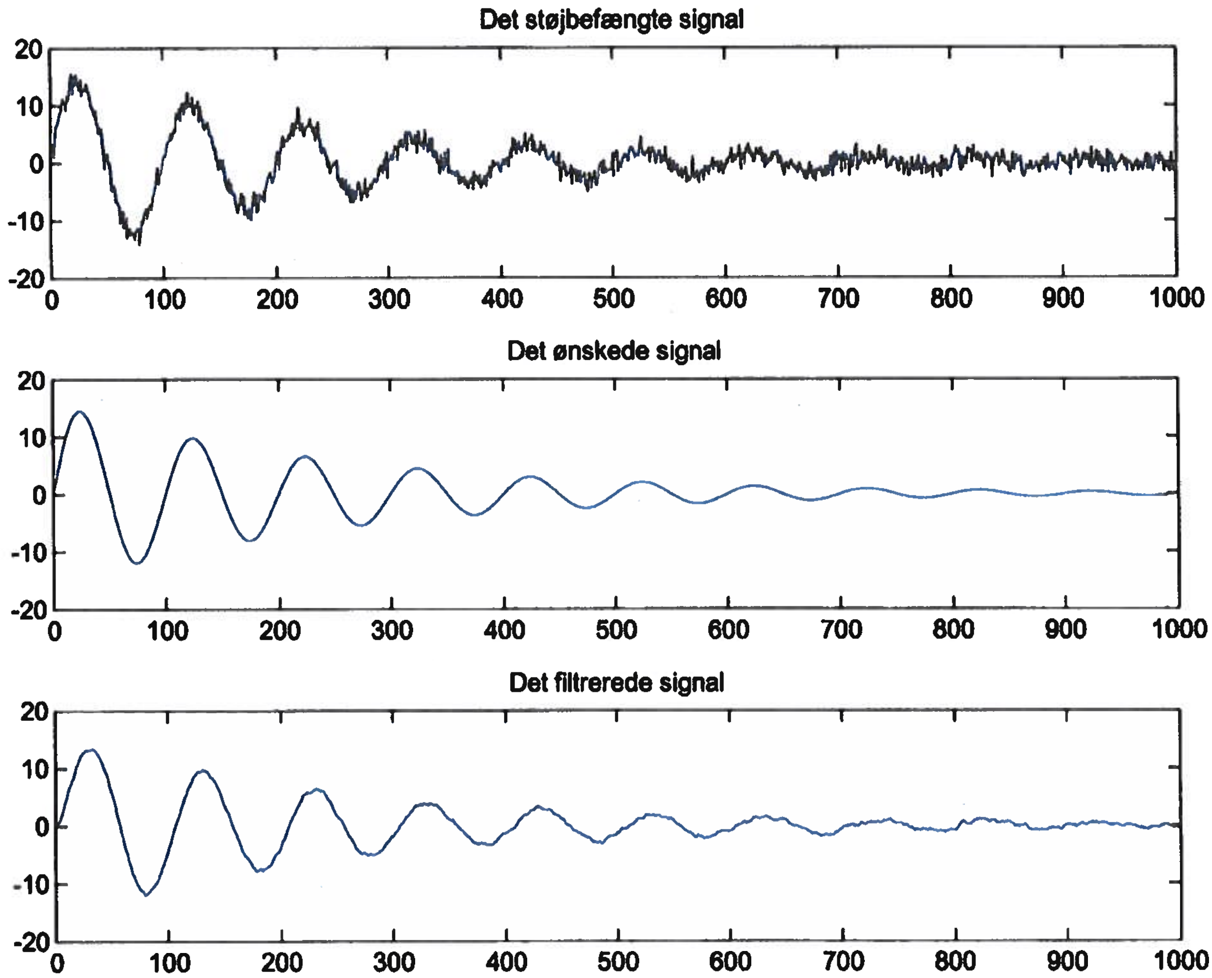
Now, use this diff. equation to do the filtering of the noise contaminated signal.

8

```
% Anvend den styrende differens-ligning til beregning af et filtreret
% output givet et inout-signal, som er overlejret med støj


clear;

load signal_noise.dat  % 1000 samples af støjbefængt signal %
load signal.dat % 1000 samples af støjfrit signal %

y(1) = 0.125 * signal_noise(1);

for n=2:1000,
  y(n) = 0.8825 * y(n-1) + 0.125 * signal_noise(n);
end;

subplot(3,1,1);
plot(signal_noise);
title('Det støjbefængte signal')
subplot(3,1,2);
plot(signal);
title('Det ønskede signal')
subplot(3,1,3);
plot(y);
title('Det filtrerede signal')
```

PK




Det støjbefængte signal
20
10
0
-10
-20
0
100
200
300
400
500
600
700
800
900
1000
Det ønskede signal
20
10
0
-10
-20
0
100
200
300
400
500
600
700
800
900
1000
Det filtrerede signal
20
10
0
-10
-20
0
100
200
300
400
500
600
700
800
900
1000

5) Frequency response

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

$\Downarrow$

$$H(e^{j\omega}) = \frac{0.125 \cdot e^{j\omega}}{e^{j\omega} - 0.8825}$$

Amplitude response:

$$H(e^{j\omega}) = 0.125 \cdot \frac{\cos\omega + j\sin\omega}{\cos\omega - 0.8825 + j\sin\omega}$$

$\Downarrow$

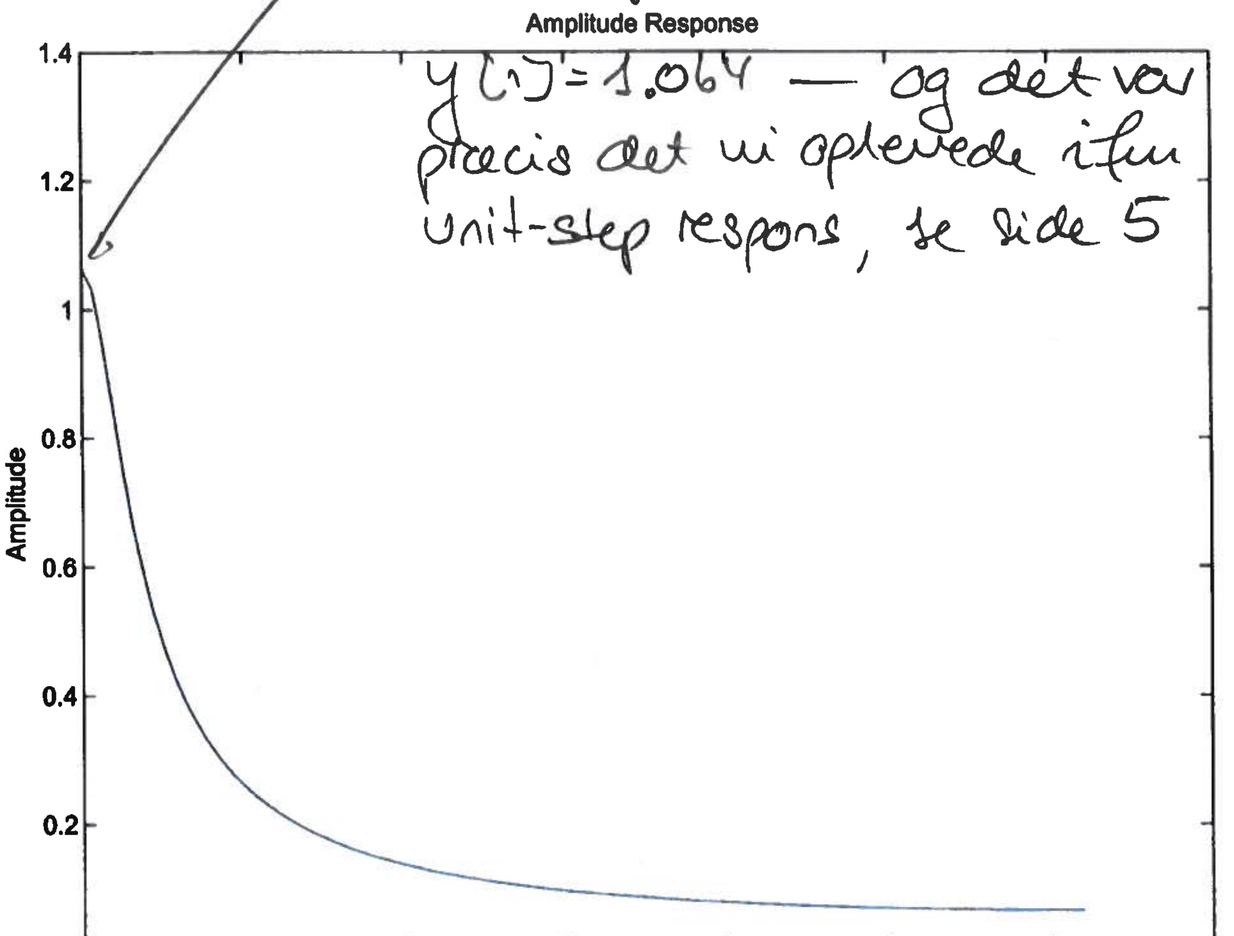
$$|H(e^{j\omega})| = 0.125 \frac{|\cos\omega + j\sin\omega|}{|\cos\omega - 0.8825 + j\sin\omega|}$$

$$= 0.125 \cdot \frac{1}{\sqrt{(\cos\omega - 0.8825)^2 + \sin^2\omega}}$$

```
% Plot filterets amplituderespons

% Først genereres en frekvens-akse med 100 punkter i intervallet [0;PI]

for i=0:99,
  omega(i+1) = (pi*i)/100;
end;


% Dernæst beregnes amplitude-værdierne for hvert enkelt frekvens-værdi

amplitude = zeros(100,1);

for i=1:100,
  naevner = sqrt((cos(omega(i)) - 0.8825)^2 + (sin(omega(i)))^2);
  amplitude(i) = 0.125/naevner;
end;

plot(omega,amplitude)
xlabel('Frequency');
ylabel('Amplitude');
title('Amplitude Response');
```

11

Bemærk, at filteret har en DC-forstærkning som er lig 1.064. Det betyder, at hvis vi påtrykker $x[n] = 1$ på input (altså en DC) så vil systemet svare med værdien




$y[n] = 1.064$ — og det var præcis det vi oplevede ifm unit-step respons, se side 5

3dB frequencies:

The analog filter has H(s):

$$H(s) = \frac{1}{sRC + 1}$$

$\Downarrow$

$$H(j\Omega) = \left.\frac{1}{j\Omega RC + 1}\right|_{s=j\Omega}$$

$\Downarrow$

$$|H(j\Omega)| = \left|\frac{1}{1 + j\Omega RC}\right|$$

$\Downarrow$

$$|H(j\Omega)| = \frac{1}{\sqrt{1 + (\Omega RC)^2}}$$

The 3dB freq. is the freq. where the amplitude has decreased 3dB as related to DC ($\Omega = 0$).

$$-3\,\text{dB} = 20 \log x \Rightarrow x = 0{,}707 \sim \frac{1}{\sqrt{2}}$$

(14)

$$|H(j\Omega)|_{-3dB} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1+(\Omega RC)^2}}$$

$$\Downarrow$$

$$\Omega \cdot RC = 1$$

$$\Downarrow \quad \Omega = \frac{1}{RC}$$

$$\Downarrow \quad 2\pi f = \frac{1}{RC}$$

$$\Downarrow \quad \underline{\underline{f = 159\ Hz}}$$

Similarly, we can calculate the freq. for the discrete-time filter;

$$|H(e^{j\omega})|_{-3dB} = \frac{1}{\sqrt{2}} \cdot |H(e^{j \cdot 0})|$$

$$= \frac{1.06383}{1.41421} = 0{,}7522$$

Thus; $\left|H\left(e^{j\omega_{3dB}}\right)\right| = 0{,}7522$

$$\Downarrow$$

$$\frac{0{,}125}{\sqrt{(\cos\omega_{3dB} - 0{,}8825)^2 + \sin^2\omega_{3dB}}} = 0{,}7522$$

$\Downarrow$ (After several calculations)

$$\begin{cases} \omega_{3dB} = 0{,}125 \text{ rad} \\ \omega_{3dB} = \dfrac{2\pi f_{3dB}}{f_s} \end{cases}$$

$$\Downarrow$$

$$0{,}125 = \frac{2\pi \cdot f_{3dB}}{8000}$$

$$\Downarrow$$

$$f_{3dB} = 159 \text{ Hz}$$

So, the 3dB freq. is identical for the analog and the digital filter.