Non-life — Assignment NL1

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Generating multinormal and multi-student r.v.'s

Q1

First we run the following script:

```
> set.seed(1)
> sum(duplicated(runif(1e6))) ## = 120
[1] 120
> sum(duplicated(rnorm(1e8))) ## = 0
[1] 0
```

The function `duplicated` returns a logical array where unique numbers are marked with 0 and duplicates are marked with 1 (the first occurence of the number is marked with a 0). Summming this array thus gives the total number of duplicates. The uniform distribution gives 120 duplicates in a much smaller sample size than the normal distribution, which gives 0 duplicates.

In the assignment, the expected number of different numbers is derived to be

$$\mathbb{E}[N_n] = \frac{1 - f^n}{1 - f}$$

The number of duplicates is then given by $n - \mathbb{E}[N_n]$ We run the following script:

```
> m <- 2^32;n <- 1e6
> f <- 1 - 1/m
> num_dup_unif <- n - (1-f^n)/(1-f)
> num_dup_unif
[1] 116.3988
```

The expected result of 116.4 is quite close to the generated result. The outcome of 120 is therefore consistent with the assumption that `runif` produces different values uniformly.

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The resolution of `rnorm` is somewhere in the 2^{50}'s. Directly calculating $1-f^n$ will give 1, because f is so close to 1. First, we use to approximation given in the assignment.

$$f^n = \left(1-\frac{1}{m}\right)^n \approx 1-\frac{n}{m}+\frac{n^2}{2m^2}$$

Inserting this into our equation for the number of different numbers gives

$$\frac{1-f^n}{1-f} \approx \frac{1-1+\frac{n}{m}-\frac{n^2}{2m^2}}{1-\left(1-\frac{1}{m}\right)} = \frac{\frac{n}{m}-\frac{n^2}{2m^2}}{\frac{1}{m}} = n-\frac{n^2}{2m}$$

This results in the following equation for the expected number of duplicates.

$$n-\left(n-\frac{n^2}{2m}\right) = \frac{n^2}{2m}$$

Next we check in `R` if the number of duplicates is consistent with values for m of 10^{15}, 10^{16}, 10^{17} or 10^{18}, when n is 10^8

```
> n_norm <- 1e8
> m_norm <- c(1e15,1e16,1e17,1e18)
> num_dup_norm <- n_norm^2/(2*m_norm)
> num_dup_norm
[1] 5.000 0.500 0.050 0.005
```

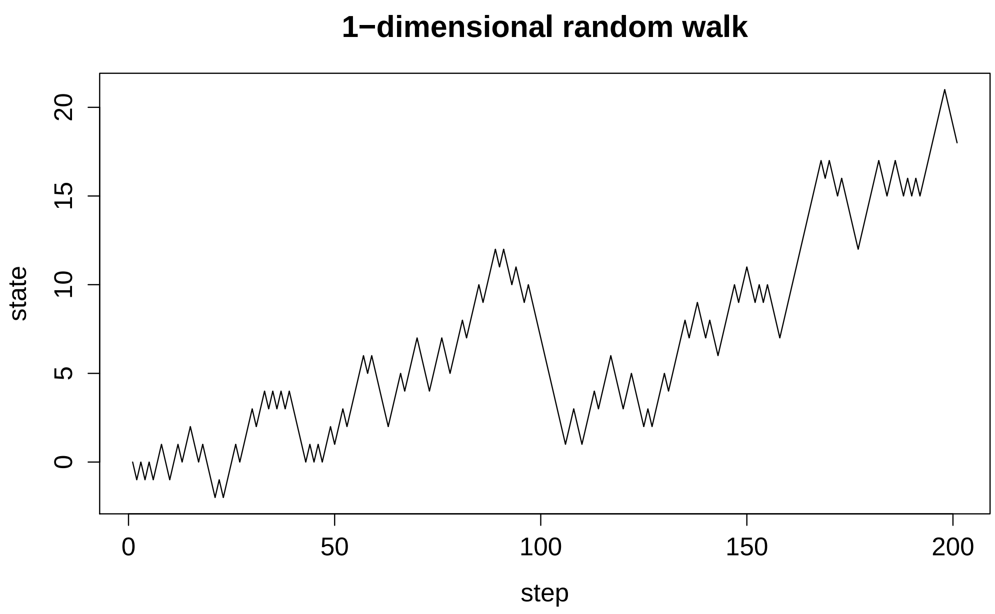
The obtained result seems to be consistent with a resolution of 10^{16} or higher.

Q2

The following code is executed in `R`.

```
> n <- 200; p <- 0.52
> x <- c(0,cumsum(2*rbinom(n,1,p)-1))
> plot(x, type="l", lwd=1, ylab="state", xlab="step", main="1-dimensional random walk")
```

This gives the following biased random walk:



Q3

Given that X, Y $\sim$N(0,1), $wewanttotransform$(X, Y)$into$(X, Y*), with $Y^* = aX + bY$, with a, b chosen in such a way that $\mathrm{Var}[Y^*] = 1$ and $r(X, Y^*) = 0.8$.

$$\begin{aligned} r(X, Y^*) &= \frac{\mathbb{E}[XY^*] - \mathbb{E}[X]\,\mathbb{E}[Y*]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y^*]}} = \frac{\mathbb{E}[aX^2 + bXY] - 0 * \mathbb{E}[Y*]}{\sqrt{1 * 1}} \\ &= a * \mathbb{E}[X^2] + b * \mathbb{E}[XY] = a(\mathrm{Var}[X] - \mathbb{E}[X]^2) = a \end{aligned}$$

Here we use that $\mathbb{E}[X] = 0$, $\mathrm{Var}[X] = \mathrm{Var}[Y] = \mathrm{Var}[Y^*] = 1$. Also $\mathbb{E}[XY] = 0$, because X and Y are independent. Next we use the condition that the variance of Y^* must also be 1.

$$\mathrm{Var}[Y^*] = \mathrm{Var}[aX + bY] = a^2\,\mathrm{Var}[X] + b^2\,\mathrm{Var}[Y] + 2ab\,\mathrm{Cov}[X, Y] = a^2 + b^2 = 1$$

We can conclude from this that $a = 0.8$ and $b = \sqrt{1 - a^2} = 0.6$.

In `R` code:

```
a <- .8; b <- sqrt(1 - a^2); Y <- a*X + b*Y
```

Q4

The variance-covariance matrix Σ of the random vector (X, Y^*) is equal to the correlation matrix because $\mathrm{Var}[X] = \mathrm{Var}[Y^*] = 1$. It is given by the following expression.

$$\Sigma = \begin{pmatrix} \mathrm{Cov}[X, X] & \mathrm{Cov}[X, Y^*] \\ \mathrm{Cov}[Y^*, X] & \mathrm{Cov}[Y^*, Y^*] \end{pmatrix} = \begin{pmatrix} \mathrm{Var}[X] & r(X, Y^*) \\ r(X, Y^*) & \mathrm{Var}[Y^*] \end{pmatrix} = \begin{pmatrix} 1 & 0.8 \\ 0.8 & 1 \end{pmatrix}$$

Q5

If A is to be the Cholesky decomposition of Σ, it should be a lower triangular matrix with real and positive entries and $AA^* = \Sigma$, where A^* is the conjugate transpose of A. Checking this gives

$$\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1 & a \\ a & a^2 + b^2 \end{pmatrix} = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.8 \\ 0.8 & 1 \end{pmatrix} = \Sigma$$