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Addendum to "Analytic and bootstrap estimates of prediction errors in claims reserving"

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Abstract

In England and Verrall [Insur. Math. Econ. 25 (1999) 281], an appropriate residual definition was considered for use in a bootstrap exercise to provide a computationally simple method of obtaining reserve prediction errors for the chain ladder model. However, calculation of the first two moments of the predictive distribution only was considered. In this paper, the method is extended by using a two-stage process: bootstrapping to obtain the estimation error and simulation to obtain the process error. This has the advantage of providing realisations from the whole predictive distribution, rather than just the first two moments.

Keywords: Bootstrap; Prediction errors; Claims reserving

1. Introduction

To date, the primary focus of stochastic claims reserving methods has been obtaining the reserve root mean square error of prediction (prediction error) in addition to a measure of location, such as the mean. Essentially, this provides the first two moments of the predictive distribution only. Further assumptions are usually necessary if other statistics are required. For example, in Mack (1993), the distribution of the underlying data is not fully specified, only the first two moments. Although formulae are developed for the reserve prediction errors, further assumptions that the predictive distribution of the reserves is approximately *normal* or *lognormal* are required in the calculation of confidence intervals.

This paper augments England and Verrall (1999) and extends the methods to obtain a full predictive distribution of reserve estimates. This is achieved by simulating the process error in addition to using bootstrapping to obtain the estimation error. The procedure is simple to implement, and all calculations can be performed within a spreadsheet; there is no need for sophisticated software.

2. Methodology

In England and Verrall (1999), we focused on the model described by Renshaw and Verrall (1998), who proposed modelling the incremental claims using an "over-dispersed" Poisson distribution. If the incremental claims for origin year i in development year j are denoted C_{ij}, then

$$E[C_{ij}] = m_{ij} \quad \text{and} \quad \text{Var}[C_{ij}] = \phi E[C_{ij}] = \phi m_{ij}, \tag{2.1}$$

$$\log(m_{ij}) = \eta_{ij}, \tag{2.2}$$

$$\eta_{ij} = c + \alpha_i + \beta_j, \quad \alpha_1 = \beta_1 = 0. \tag{2.3}$$

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Eqs. (2.1)–(2.3) define a generalised linear model in which the response is modelled with a logarithmic link function and the variance is proportional to the mean (hence "over-dispersed" Poisson). The parameter ϕ is an unknown scale parameter estimated as part of the fitting procedure. With certain positivity constraints, predicted values and reserve estimates from this model are exactly the same as those from the chain ladder model.

In England and Verrall (1999), bootstrapping was used to obtain reserve standard errors (the estimation error component of the prediction error). The procedure involves sampling with replacement from an appropriate set of residuals to obtain a large number of sets of pseudo-data. The chain ladder model can be fitted to each set of pseudo-data, and reserve estimates obtained. The standard deviation of the set of reserve estimates obtained in this way provides a bootstrap estimate of the standard error (estimation error). The particular residual definition used in England and Verrall (1999) is the unscaled Pearson residual, r_{P}, defined as

$$r_{\mathrm{P}} = \frac{C - m}{\sqrt{m}}. \tag{2.4}$$

It is unscaled in the sense that it does not include the scale parameter ϕ which is not needed when performing the bootstrap calculations, but is needed when considering the process error. An estimate of the scale parameter consistent with the definition of residuals is the Pearson scale parameter, given by

$$\phi_{\mathrm{P}} = \frac{\sum r_{\mathrm{P}}^2}{n - p}, \tag{2.5}$$

where n is the number of data points in the sample, p the number of parameters estimated and the summation is over the number (n) of residuals. It can be seen that an increased number of parameters used in fitting the model introduces a penalty (ceteris paribus).

In England and Verrall (1999), the reserve prediction error was given by

$$\mathrm{PE}_{\mathrm{bs}}(R) = \sqrt{\phi_{\mathrm{P}} R + \frac{n}{n - p}(\mathrm{SE}_{\mathrm{bs}}(R))^2}, \tag{2.6}$$

where R is an origin year or total reserve, and $\mathrm{SE}_{\mathrm{bs}}(R)$ the bootstrap standard error of the reserve estimate. The process variance, $\phi_{\mathrm{P}} R$, is calculated analytically and simply added to the estimation variance (which is suitably scaled to take account of the degrees-of-freedom).

The purpose of this paper is to describe an extended methodology which replaces the analytic calculation of the process error with a simulation approach, thereby providing a way of simulating a complete predictive distribution. The methodology proceeds by interrupting the bootstrapping procedure at each iteration and drawing a random observation from the underlying process distribution, conditional on the bootstrap value. The procedure is performed by completing the following steps:

- Obtain the standard chain ladder development factors from cumulative data.
- Obtain cumulative fitted values for the past triangle by backwards recursion, as described in Appendix A of England and Verrall (1999).
- Obtain incremental fitted values for the past triangle by differencing.
- Calculate the unscaled Pearson residuals for the past triangle using Eq. (2.4).
- Calculate the Pearson scale parameter, ϕ, using Eq. (2.5).
- Adjust the Pearson residuals using Eq. (3.1).
- Begin iterative loop, to be repeated N times ($N = 1000$, say):
  - Resample the adjusted residuals with replacement, creating a new past triangle of residuals.
  - For each cell in the past triangle, solve Eq. (2.4) for C, giving a set of pseudo-incremental data for the past triangle.
  - Create the associated set of pseudo-cumulative data.
  - Fit the standard chain ladder model to the pseudo-cumulative data.
  - Project to form a future triangle of cumulative payments.
  - Obtain the corresponding future triangle of incremental payments in each cell (i, j) by differencing, to be used as the mean ($\tilde{m}_{ij}$) when simulating from the process distribution.
  - For each cell (i, j) in the future triangle, simulate a payment from the process distribution with mean $\tilde{m}_{ij}$ (obtained at the previous step), and variance $\phi\tilde{m}_{ij}$, using Eq. (2.1) and the value of ϕ calculated previously.
  - Sum the simulated payments in the future triangle by origin year and overall to give the origin year and total reserve estimates, respectively.

- Store the results, and return to start of iterative loop.

The set of stored results forms the predictive distribution. The mean of the stored results should be compared to the standard chain ladder reserve estimates to check for bias. The standard deviation of the stored results gives an estimate of the prediction error.

It can be seen that essentially the bootstrap procedure provides a distribution of "means" in the future triangle, and the process error is replicated by sampling from the underlying distribution conditional on those means. The result is a simulated predictive distribution of future payments which when summed appropriately provides a predictive distribution of reserve estimates, from which summary statistics can be obtained.

3. Practical issues

In England and Verrall (1999), an adjustment was made to the bootstrap standard error to take account of the degrees-of-freedom (see Eq. (2.6)). This was to enable a comparison to be made with the results obtained analytically. It is desirable to make a similar adjustment here too, but to enable the adjustment to follow through to the predictive distribution automatically, it is suggested that the residuals are adjusted prior to implementing the procedure. That is, replace r_{P} by r'_{P}, where

$$r'_{\mathrm{P}} = \sqrt{\frac{n}{n-p}}\frac{C-m}{\sqrt{m}}. \tag{3.1}$$

Since the mean of the residuals should be close to zero, this has the effect of inflating the variance while leaving the mean largely unchanged.

Due to the required constraints on the parameterisation (commonly known as "corner constraints"), it will be noticed that the residuals at either end of the latest diagonal are zero. It can be argued that those residuals should not be used when resampling in the bootstrap procedure. However, they are included to ensure that the prediction errors given by the method are analogous to those calculated analytically. Excluding them would have a minimal effect on the mean of the bootstrap reserves, but would inflate the standard deviation.

There is no standard approach for simulating from an over-dispersed Poisson distribution. When fitting the generalised linear model defined by Eqs. (2.1)–(2.3), quasi-likelihood methods are adopted making the assumption that the variance is proportional to the mean. When simulating the process error, it is necessary to choose a suitable method such that the variance is proportional to the mean. There is no restriction that simulated values must be positive integers. As such, there are several candidates which could be justified, with the methods suggested below being ones which have been tried.

A simple way to obtain a realisation from an over-dispersed Poisson distribution with mean m and variance ϕm is to sample from a Poisson distribution with mean m/ϕ, and multiply by ϕ. When $\phi = 1$, this will result in a simulated Poisson distribution. When $\phi \neq 1$, observations will always be multiples of ϕ, which for large ϕ may not be desirable. Furthermore, for non-integer values of ϕ, observations will be non-integer.

If this feature is considered unacceptable, a pragmatic alternative is to sample from a *negative binomial* or *Gamma* distribution parameterised such that the mean is m and the variance is ϕm. The variance remains proportional to the mean, without the simulated values being multiples of ϕ. However, this changes the shape of the predictive distribution, while leaving the first two moments unchanged.

A further approach is to simulate from the quasi-log likelihood using methods such as *adaptive rejection sampling* (Gilks et al., 1995), commonly used in Bayesian statistics for sampling from distributions which cannot be recognised, but where the log density is known.

The bootstrap procedure can occasionally result in a negative mean from which the process error is simulated. Again, this is undesirable and a number of adjustments can be suggested. A practical approach is to simulate an observation from a distribution with mean abs(m), then subtract $2 \times m$ to retain the appropriate scale.

4. Example

Following the example in England and Verrall (1999) the data from Taylor and Ashe (1983) is used, shown here in incremental form:

357,848	766,940	610,542	482,940	527,326	574,398	146,342	139,950	227,229	67,948
352,118	884,021	933,894	1,183,289	445,745	320,996	527,804	266,172	425,046	
290,507	1,001,799	926,219	1,016,654	750,816	146,923	495,992	280,405		
310,608	1,108,250	776,189	1,562,400	272,482	352,053	206,286			
443,160	693,190	991,983	769,488	504,851	470,639				
396,132	937,085	847,498	805,037	705,960					
440,832	847,631	1,131,398	1,063,269						
359,480	1,061,648	1,443,370							
376,686	986,608								
344,014									

The chain ladder reserve estimates together with the means of 1000 simulations of the bootstrap approach of England and Verrall (1999), and the approach adopted here, are shown in Table 1. The results are very close, as expected, with differences due to random variation.

The reserve prediction errors as a percentage of the means are shown in Table 2. The analytic estimates and the bootstrap estimates are taken from England and Verrall (1999). The estimates obtained after simulating the process error as described in this paper are also shown. Again, the results are reassuringly close.

The advantage of the two-stage simulation approach outlined in this paper is the availability of the

Table 1
Estimated reserves (thousands)

i	Chain ladder	Bootstrap mean (England and Verrall, 1999)	Bootstrap/ simulation mean
2	95	96	94
3	470	474	475
4	710	712	719
5	985	996	996
6	1,419	1,425	1,422
7	2,178	2,171	2,164
8	3,920	3,903	3,943
9	4,279	4,268	4,246
10	4,626	4,645	4,629
Total	18,681	18,690	18,688

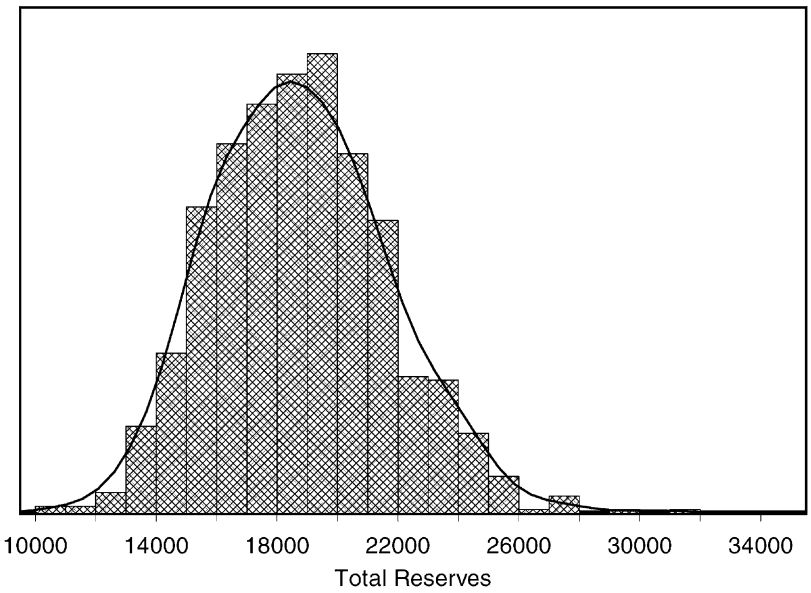

Fig. 1. Predictive aggregate distribution of total reserves.

Table 2
Prediction errors as percentage of reserve estimate

i	Poisson GLM analytic	Bootstrap (England and Verrall, 1999)	Bootstrap/ simulation
2	116	117	117
3	46	46	47
4	37	36	37
5	31	31	31
6	26	26	27
7	23	23	23
8	20	20	21
9	24	24	25
10	43	43	44
Total	16	16	16

Table 3
Sample statistics from predictive aggregate distribution of total reserves

Number of observations	1000
Mean	18,688
Standard deviation	2956
Coefficient of variation	0.158
Skewness	0.350
Kurtosis	0.229
50th percentile	18,532
75th percentile	20,640
90th percentile	22,620
95th percentile	23,827
99th percentile	25,967

full predictive distribution. Summary statistics of the predictive distribution of the total reserves (using 1000 simulations) are shown in Table 3. A histogram of the distribution is also shown in Fig. 1, together with a smoothed density line.

5. Conclusions

Most papers on the topic of stochastic claims reserving consider measures of variability of claims reserves in addition to a "best estimate". This has the advantage of highlighting precision of the estimates, but falls short of providing the full predictive distribution. The predictive distribution of reserve estimates has been considered by very few authors, although some related papers exist, for example Zehnwirth et al. (1998), which considers predictive aggregate claims distributions for the collective risk model used in risk theory.

In this paper, one method of obtaining a predictive distribution for the basic chain ladder model is proposed. The method has the advantage of simplicity, and all calculations can be performed in a spreadsheet. The method involves a two-stage procedure: first simulate a forecast mean using the bootstrap, then simulate an observation conditional on the mean. This has an obvious analogy to a Bayesian Monte Carlo approach, where parameters are sampled from their posterior distribution, and a forecast is then simulated conditional on those parameters. For a further discussion of bootstrapping using a two-stage approach, see Taylor (2000).

Although the bootstrap/simulation procedure provides prediction errors that are consistent with their analytic counterparts, the predictive distribution produced in this way might have some undesirable properties. For example, for some origin year reserves, the minimum values of the predictive distribution could be negative. A number of other practical and theoretical difficulties exist (such as those outlined in Section 3), and alternative adjustments and improvements are likely to be suggested.

This paper only considers the special case of a constant dispersion parameter ϕ. Evidence of heteroskedasticity would require extensions to the approach. Model fitting would proceed using joint modelling (for example, Renshaw, 1994), which would not be amenable to fitting in a spreadsheet. The dispersion parameters would need to be included in the residual definition, and care would need to be exercised in implementing a bootstrap procedure. Unlike the case for a constant dispersion parameter, the expected values under the model may not be identical to those obtained from the traditional deterministic chain ladder method.

It is not suggested that the method proposed here is the only one that is consistent with the chain ladder model. Other approaches are likely to be suggested which provide a different predictive distribution while also having the same first two moments.

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