Demography and Mortality

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Part 1

Let the setup and notation be as stated in the assignment.

1.1)

Problem 1.1

From the definition of the expected residual life, $e_i(0) = \int_0^\infty S_i(u|0)du = \int_0^\infty \frac{S_i(u)}{S_i(0)}du = \int_0^\infty S_i(u)du$, since $S_i(0) = 1$. For ease of notation write $M_i = M_i(u) = \int_0^u \mu_i(z)dz, \quad i = 1, 2$ we have

$$
\begin{aligned}
e_2(0) - e_1(0) &= \int_0^\infty e^{-M_2}du - \int_0^\infty e^{-M_1}du \\
&= \int_0^\infty e^{-M_2} - e^{-M_1}du \\
&= \int_0^\infty \left[e^{M_1-M_2} - 1\right]e^{-M_1}du
\end{aligned}
$$

Now, since $e^{-M_1} = S_1(u)$ and noting that

$$
\frac{d}{du} - S_1(u)e_i(u) = \frac{d}{du} - S_1(u)\int_u^\infty \frac{S_1(z)}{S_1(u)}du = \frac{d}{du} - \int_u^\infty S_1(z)dz = S_1(u),
$$

by the fundamental theorem of calculus and switching limits of the integral. Then using integration by parts we get

$$
\begin{aligned}
\int_0^\infty \left[e^{M_1-M_2} - 1\right]e^{-M_1}du &= \left[(e^{M_1-M_2} - 1)(-S_1(u)e_1(u))\right]_0^\infty - \int_0^\infty (M_1 - M_2)'e^{M_1-M_2}(-S_1(u)e_1(u))du \\
&= \int_0^\infty (\mu_1(u) - \mu_2(u))e^{\int_0^u \mu_1(v)-\mu_2(v)dv}S_1(u)e_1(u)du
\end{aligned}
$$

1.2)

Problem 1.2

For this subproblem we will mainly be using Appendix 3.1 on page 69, in wich we find several expressions for the terms used in Arriagas decomposition.

First we will look at the Direct effect term. We start off by inserting expressions from p.69 for nL_x in the decomposition. And we get:

$$\frac{l_x^1}{l_0^1}\left(\frac{l_x^2\int_x^y e^{-\int_x^a \mu_2(u)du}da}{l_x^2} - \frac{l_x^1\int_x^y e^{-\int_x^a \mu_1(u)du}da}{l_x^1}\right)$$

We see that l_x^2 and l_x^1 terms cancel out in the fractions. As they cancel out and as we use a expression from p.69 for l_x^1 in insert in the fraction outside of the pharentesis, we get:

$$\frac{l_0^1 e^{-\int_0^x \mu_1(u)du}}{l_0^1}\left(\int_x^y e^{-\int_x^a \mu_2(u)du} - e^{-\int_x^a \mu_1(u)du}da\right)$$

And as 1_{0}^{1} cancel out in the fraction, we see that the exponential term in the nominator is actually a survival function and my using some basic rules of integrations we then get the following equations:

$$S_1(x)\cdot\left(\int_x^y\left(\frac{e^{-\int_0^x \mu_2(u)du-\int_x^a \mu_2(u)du}}{e^{-\int_0^x \mu_2(u)du}} - \frac{e^{-\int_0^x \mu_1(u)du-\int_x^a \mu_1(u)du}}{e^{-\int_0^x \mu_2(u)du}}\right)da\right)$$

.

$$= S_1(x)\cdot\left(\int_x^y \frac{e^{-\int_0^a \mu_2(u)du}}{e^{-\int_0^x \mu_2(u)du}} - \frac{e^{-\int_0^a \mu_1(u)du}}{e^{-\int_0^x \mu_1(u)du}}da\right)$$

.

$$S_1(x)\left(\int_x^y \frac{S_2(a)}{S_2(x)} - \frac{S_1(a)}{S_1(x)}\right)da = S_1(x)\left(\int_x^y S_2(a|x) - S_1(a|x)da\right) = S_1(x)\cdot(e_2(x,y)-e_1(x,y))$$

\ We have now expressed the direct effect in terms of our usual notation.

Next we will be looking at the expression for the indirect effect

First we will once again use the expressions from page 69. We see that $\frac{l_x}{l_y} = nP_x$ and after rearranging our fractions and then inserting that we get the following:

$$\frac{T_y^1}{l_0^1}\cdot\left(e^{\int_x^y \mu_1(u)du}\cdot e^{-\int_x^y \mu_2(u)du} - 1\right)$$

Then by applying some integral rules we obtain the following:

$$\frac{T_y^1}{l_0^1}\cdot\left(\frac{e^{\int_x^y \mu_1(u)du}\cdot e^{-\int_x^y \mu_1(u)du}}{e^{-\int_x^y \mu_1(u)du}}\cdot\frac{e^{-\int_x^y \mu_2(u)du}\cdot e^{\int_0^x \mu_2(u)du}}{e^{\int_0^x \mu_2(u)du}} - 1\right)$$

.

$$=$$

$$\frac{T_y^1}{l_0^1}\cdot\left(\frac{1}{e^{-\int_x^y \mu_1(u)du}}\cdot\frac{e^{-\int_0^y \mu_2(u)du}}{e^{\int_0^x \mu_2(u)du}} - 1\right) = \frac{T_y^1}{l_0^1}\cdot\left(\frac{e^{-\int_0^x \mu_1(u)du}}{e^{-\int_0^y \mu_1(u)du}}\cdot S_2(y|x) - 1\right)$$

.

$$= \frac{T_y^1}{l_0^1}\cdot\left(\frac{S_1(x)}{S_1(y)}\cdot S_2(y|x) - 1\right)$$

We only need to think about the fraction outside of the pharentesis now . We have that $T_y^1 = e_y^1\cdot l_y^1$ and that $l_y^1 = l^1(0)\cdot e^{-\int_0^y \mu_1(u)du}$ So by that we get:

2

$$\frac{e_y l^1(0) \cdot e^{-\int_0^y \mu_1(u)du}}{l_0^1} \cdot \left(\frac{S_1(x)}{S_1(y)} \cdot S_2(y|x) - 1\right) = e_y \cdot S_1(y)\left(\frac{S_1(x)}{S_1(y)} \cdot S_2(y|x) - 1\right) == e_y^1 \cdot S_1(x) \cdot S_2(y|x) - e_y^1 \cdot S_1(y)$$

.

We finally just need to write the last part of the decomposition. The interaction effect. We have looked at T_y and for l_y before and we will use the same again and get following:

$$X(x,y) = \left(\frac{e_y^2 \cdot l_y^2 - e_y^1 \cdot l_y^1 \frac{l_y^2}{l_y^1}}{l_0^1}\right) \cdot \left(\frac{l_x^1}{l_x^2} - \frac{l_y^1}{l_y^2}\right) = l^2(0) e^{-\int_0^y \mu_2(u)du} \cdot \left(\frac{e_y^2 - e_y^1}{l_0^1}\right) \cdot \left(\frac{l_x^1}{l_x^2} - \frac{l_y^1}{l_y^2}\right)$$

.

$$\frac{l_0^2}{l_0^1} \cdot S_2(y) \cdot \left(e_y^2 - e_y^1\right) \cdot \left(\frac{l_x^1}{l_x^2} - \frac{l_y^1}{l_y^2}\right) = S_2(y) \cdot \left(e_y^2 - e_y^1\right) \cdot \left(\frac{l_0^2}{l_0^1} \cdot \frac{l_x^1}{l_x^2} - \frac{l_0^2}{l_0^1} \cdot \frac{l_y^1}{l_y^2}\right)$$

.

$$S_2(y) \cdot \left(e_y^2 - e_y^1\right) \left(\frac{l_0^2}{l_0^1} \cdot \frac{l^1(0)e^{-\int_0^x \mu_1(u)du}}{l^2(0)e^{-\int_0^x \mu_2(u)du}} - \frac{l_0^2}{l_0^1} \cdot \frac{l^1(0)e^{-\int_0^y \mu_1(u)du}}{l^2(0)e^{-\int_0^y \mu_2(u)du}}\right)$$

.

$$S_2(y) \cdot \left(e_y^2 - e_y^1\right) \left(\frac{e^{-\int_0^x \mu_1(u)du}}{e^{-\int_0^x \mu_2(u)du}} - \frac{e^{-\int_0^y \mu_1(u)du}}{e^{-\int_0^y \mu_2(u)du}}\right) = S_2(y) \cdot \left(e_y^2 - e_y^1\right) \left(\frac{S_1(x)}{S_2(x)} - \frac{S_1(y)}{S_2(y)}\right)$$

1.3)

Problem 1.3

We want to show that for any x and y with $x < y$

$$D(x,y) + I(x,y) = A(x,y)$$

We have from question 1.2 that

$$D(x,y) = S_1(x)(e_2(x,y) - e_1(x,y))$$
$$I(x,y) = e_1(y)S_1(x)S_2(y \mid x) - e_1(y)S_1(y)$$

We have from question 1.2 that

$$D(x,y) = S_1(x)(e_2(x,y) - e_1(x,y))$$
$$I(x,y) = e_1(y)S_1(x)S_2(y \mid x) - e_1(y)S_1(y)$$

Having a look at $D(x,y)$ we get by rewriting:

$$D(x,y) = S_1(x)(e_2(x,y) - e_1(x,y))$$
$$= S_1(x)\left(\int_x^y S_2(z \mid x)dz - \int_x^y S_1(z \mid x)dz\right)$$
$$= S_1(x)\left(\int_x^y \frac{S_2(z)}{S_2(x)}dz - \int_x^y \frac{S_1(z)}{S_1(x)}dz\right)$$

Inserting the expressions for $S_1(x)$ and $S_2(x)$ we get:

$$= S_1(x)\left(\int_x^y \frac{e^{-\int_0^z \mu_2(u)du}}{e^{-\int_0^x \mu_2(u)du}}dz - \int_x^y \frac{e^{-\int_0^z \mu_1(u)du}}{e^{-\int_0^x \mu_1(u)du}}dz\right)$$

$$= S_1(x)\left(\int_x^y e^{-\int_0^z \mu_2(u)du+\int_0^x \mu_2(u)du}dz - \int_x^y e^{-\int_0^z \mu_1(u)du+\int_0^x \mu_1(u)du}dz\right)$$

Using that $-\int_0^z f(z)dz + \int_0^x f(z)dz = -\int_0^z f(z)dz - \int_x^0 f(z)dz = -\int_x^z f(z)dz$ and collecting the integral over z we get

$$= S_1(x)\left(\int_x^y e^{-\int_x^z \mu_2(u)du} - e^{-\int_x^z \mu_1(u)du}dz\right)$$

Writing out $S_1(x)$ and rearranging the terms we get:

$$= S_1(x)\left(\int_x^y \left(e^{-\int_x^z (\mu_2(u)-\mu_1(u))du} - 1\right)e^{-\int_x^z \mu_1(u)du}dz\right)$$

$$= e^{-\int_0^x \mu_1(u)du}\left(\int_x^y \left(e^{\int_x^z (\mu_1(u)-\mu_2(u))du} - 1\right)e^{-\int_x^z \mu_1(u)du}dz\right)$$

$$= \left(\int_x^y \left(e^{\int_x^z (\mu_1(u)-\mu_2(u))du} - 1\right)e^{-\int_0^z \mu_1(u)du}dz\right)$$

Using partial integration with $u(z) = e^{\int_x^z (\mu_1(u)-\mu_2(u))du} - 1$ and $v'(z) = e^{-\int_0^z \mu_1(u)du}$ we recognize $\int v'(z)dz = -S_1(z)e_1(z)$ since

$$v'(z) = S_1(z)$$
$$= \frac{d}{dz} - \int_z^\infty S_1(u)du$$
$$= \frac{d}{dz} - S_1(z)\int_z^\infty \frac{S_1(u)}{S_1(z)}du$$
$$= \frac{d}{dz} - S_1(z)e_1(z)$$

and

$$\frac{d}{dz}u(z) = \frac{d}{dz}e^{\int_x^z (\mu_1(u)-\mu_2(u))du} - 1$$
$$= (\mu_1(z) - \mu_2(z))e^{\int_x^z (\mu_1(u)-\mu_2(u))du}$$

Collecting we thus get

$$\int_x^y u(z)v'(z)dz = [u(z)v(z)]_x^z - \int_x^y u'(z)v(z)dz$$

$$= \left[\left(e^{\int_x^z (\mu_1(u)-\mu_2(u))du} - 1\right)(-S_1(z)e_1(z))\right]_x^y$$

$$- \int_x^y (\mu_1(z) - \mu_2(z))e^{\int_x^z (\mu_1(u)-\mu_2(u))du}(-S_1(z)e_1(z))$$

$$= \left[\left(e^{\int_x^z (\mu_1(u)-\mu_2(u))du} - 1\right)(-S_1(z)e_1(z))\right]_x^y$$

$$+ \int_x^y (\mu_1(z) - \mu_2(z))e^{\int_x^z (\mu_1(u)-\mu_2(u))du}S_1(z)e_1(z)$$

We recognize the second term as $A(x,y) = \int_x^y (\mu_1(z) - \mu_2(z))e^{\int_x^z (\mu_1(u)-\mu_2(u))du}S_1(z)e_1(z)$. So in order to show that $D(x,y) + I(x,y) = A(x,y)$ holds, we now need to show that $A(x,y) - D(x,y) = I(x,y)$ i.e. we need to show that

$$-\left[\left(e^{\int_x^z (\mu_1(u)-\mu_2(u))du} - 1\right)(-S_1(z)e_1(z))\right]_x^y = I(x,y)$$

Rewriting the left hand side we get

$$-\left[\left(e^{\int_x^z (\mu_1(u)-\mu_2(u))du} - 1\right)(-S_1(z)e_1(z))\right]_x^y$$

$$= -\left(e^{\int_x^y \mu_1(u)-\mu_2(u)du} - 1\right)(-S_1(y)e_1(y)) + 0$$

$$= \left(e^{\int_x^y \mu_1(u)-\mu_2(u)du} - 1\right)(S_1(y)e_1(y))$$

$$= e^{\int_x^y \mu_1(u)-\mu_2(u)du}e^{-\int_0^y \mu_1(u)du}e_1(y) + S_1(y)e_1(y)$$

$$= e^{-\int_x^y \mu_2(u)du + \int_x^y \mu_1(u)du - \int_0^y \mu_1(u)du}e_1(y) + S_1(y)e_1(y)$$

Using that $-\int_x^y f(z)dz = -\int_0^y f(z)dz - \int_x^0 f(z)dz = -\int_0^y f(z)dz + \int_0^x f(z)dz$ and that $\int_x^y f(z)dz - \int_0^y f(z)dz = \int_x^y f(z)dz + \int_y^0 f(z)dz = \int_x^0 f(z)dz = -\int_0^x f(z)dz$ we get:

$$= e^{-\int_0^y \mu_2(u)du + \int_0^x \mu_2(u)du}e^{-\int_0^x \mu_1(u)du}e_1(y) + S_1(y)e_1(y)$$

$$= \frac{e^{-\int_0^y \mu_2(u)du}}{e^{-\int_0^x \mu_2(u)du}}S_1(x)e_1(y) + S_1(y)e_1(y)$$

$$= \frac{S_2(y)}{S_2(x)}S_1(x)e_1(y) + S_1(y)e_1(y)$$

$$= S_2(y \mid x)S_1(x)e_1(y) + S_1(y)e_1(y)$$

$$= I(x,y)$$

as wanted.

1.4)

Problem 1.4

We have from ealier that:

$$D(x,y) = S_1(x)(e_2(x,y) - e_1(x,y))$$

$$I(x,y) = S_1(y)e_1(y)(\frac{S_2(y|x)}{S_1(y|x)} - 1)$$

$$X(x,y) = S_2(y)(e_2(y) - e_1(y))(\frac{S_1(x)}{S_2(x)} - \frac{S_1(y)}{S_2(x)})$$

We want to show that:

$$\sum_{i=0}^{n-1} [D(x_i, x_{i+1}) + I(x_i, x_{i+1}) + X(x_i, x_{i+1})] = e_2(0) - e_1(0)$$

.

We have that:

$$D(x_i, x_{i+1}) = S_1(x_i)(e_2(x_i, x_{i+1}) - e_1(x_i, x_{i+1}))$$

$$= e^{-\int_0^{x_i} \mu_1(u)du}(\int_{x_i}^{x_{i+1}} S_2(z|x_i) - S_1(z|x_i)dz)$$

$$= e^{-\int_0^{x_i} \mu_1(u)du}(\int_{x_i}^{x_{i+1}} \frac{e^{-\int_0^z \mu_2(u)du}}{e^{-\int_0^{x_i} \mu_2(u)du}} - \frac{e^{-\int_0^z \mu_1(u)du}}{e^{-\int_0^{x_i} \mu_1(u)du}}dz)$$

$$= e^{\int_0^{x_i} \mu_2(u) - \mu_1(u)du}(\int_{x_i}^{x_{i+1}} e^{-\int_0^z \mu_2(u)du} - e^{-\int_0^z \mu_1(u)du}dz)$$

$$= \frac{S_1(x_i)}{S_2(x_i)} \int_{x_i}^{x_{i+1}} S_2(u)du - \int_{x_i}^{x_{i+1}} S_1(u)du$$

By taking the the sum of that we get:\

$$\sum_{i=0}^{n-1}[D(x_i, x_{i+1})] =$$

$$e_2(0) - e_1(0) - S_2(x_1)e_2(x_1)$$

$$+ S_1(x_1)e_2(x_1) - \frac{S_1(x_1)}{S_2(x_1)}S_2(x_2)e_2(x_2)$$

$$+ S_1(x_2)e_2(x_2) - \frac{S_1(x_2)}{S_2(x_2)}S_2(x_3)e_2(x_3)$$

$$...$$

$$+ S_1(x_{n-1})e_2(x_{n-1}) - \frac{S_1(x_{n-1})}{S_2(x_{n-1})}S_2(x_n)e_2(x_n)$$

From that we observe that we have $e_2(0) - e_1(0)$, which must mean, that the sum over X+I and the other terms above must be equal to zero. So lets look at the sum over X+I:

$$\sum_{i=0}^{n-1}[I(x_i, x_{i+1}) + X(x_i, x_{i+1})] =$$

$$- e_1(x_1)S_1(x_1) + e_2(x_1)(\frac{S_2(x_1)S_1(x_0)}{S_2(x_0)} - \frac{S_2(x_1)S_1(x_1)}{S_2(x_0)}) + e_1(x_1)(\frac{S_1(x_1)S_2(x_1)}{S_2(x_0)})$$

$$- e_1(x_2)S_1(x_2) + e_2(x_2)(\frac{S_2(x_2)S_1(x_1)}{S_2(x_1)} - \frac{S_2(x_2)S_1(x_2)}{S_2(x_1)}) + e_1(x_2)(\frac{S_1(x_2)S_2(x_2)}{S_2(x_1)})$$

...

$$- e_1(x_n)S_1(x_n) + e_2(x_n)(\frac{S_2(x_n)S_1(x_{n-1})}{S_2(x_{n-1})} - \frac{S_2(x_n)S_1(x_n)}{S_2(x_{n-1})}) + e_1(x_n)(\frac{S_1(x_n)S_2(x_n)}{S_2(x_{n-1})})$$

So what we wanna show now is that:\

$$- S_2(x_1)e_2(x_1)$$

$$+ S_1(x_1)e_2(x_1) - \frac{S_1(x_1)}{S_2(x_1)}S_2(x_2)e_2(x_2)$$

$$+ S_1(x_2)e_2(x_2) - \frac{S_1(x_2)}{S_2(x_2)}S_2(x_3)e_2(x_3)$$

...

$$+ S_1(x_{n-1})e_2(x_{n-1}) - \frac{S_1(x_{n-1})}{S_2(x_{n-1})}S_2(x_n)e_2(x_n) - e_1(x_1)S_1(x_1) + e_2(x_1)(\frac{S_2(x_1)S_1(x_0)}{S_2(x_0)} - \frac{S_2(x_1)S_1(x_1)}{S_2(x_0)}) + e_1(x_1)(\frac{S_1(x_1)S_2(x}{S_2(x_0)}$$

$$- e_1(x_2)S_1(x_2) + e_2(x_2)(\frac{S_2(x_2)S_1(x_1)}{S_2(x_1)} - \frac{S_2(x_2)S_1(x_2)}{S_2(x_1)}) + e_1(x_2)(\frac{S_1(x_2)S_2(x_2)}{S_2(x_1)})$$

...

$$- e_1(x_n)S_1(x_n) + e_2(x_n)(\frac{S_2(x_n)S_1(x_{n-1})}{S_2(x_{n-1})} - \frac{S_2(x_n)S_1(x_n)}{S_2(x_{n-1})}) + e_1(x_n)(\frac{S_1(x_n)S_2(x_n)}{S_2(x_{n-1})}) = 0$$

The relationship between Arriaga's decomposition and Pollard's decomposition is very similar. The age intervals must be finer and finer, since broad intervals will result in discrepancies between the 2 decomposition. Arriaga's approach is discreet and it is analytically easier to work with. However, there is the downside that it is not as accurate as Pollard's decomposition.

1.5)

Problem 1.5

We want to make a Taylor expansion of $f(x) = e^{\int_0^x \mu_1(v) - \mu_2(v)dv}$ around $a = 0$, i.e we want to calculate

$$f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + ...$$

We get:

$$f(0) = e^{\int_0^0 \mu_1(v) - \mu_2(v)dv} = 1$$

and

$$f'(x) = e^{\int_0^x \mu_1(v) - \mu_2(v)dv}(\mu_1(x) - \mu_2(x))$$

So the second term of the Taylor expansion is

$$\frac{f'(0)}{1}(x - 0) = e^{\int_0^0 \mu_1(v) - \mu_2(v)dv}(\mu_1(0) - \mu_((0)))(x) = (\mu_1(0) - \mu_((0)))x$$

We get the Taylor expansion

$$f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + ...$$
$$= 1 + (\mu_1(0) - \mu_((0)))x + ... \qquad\qquad = f(x)$$

Taking only the lead term we get that $f(x) \approx 1$ and thus we have our answer. This approximation will only be good if the two mortality rates are similar to each other, and would we not recommend this approximation unless it is assumed that the mortalities behave the same at age 0.

1.6)

Problem 1.6

Direct effect: This is the effect that changes life expectancy within an age group due to mortality changes in the group.

Indirect effect: when we have mortality change in an age group and that results in a change in the number of survivors by the end of the subgroup, This is then the number of life years we ADD to the life expectancy within the.

Interaction effect: This is the rest of the effect that can not be appointed to the changes in mortality or life expectancy. It is a result of the years of life that are added because improved life expectancy results in additional survivors that will continue to live under the new mortality after the change in mortality.

1.7)

Problem 1.7

Let's start by running the **structure.R** script, so the relevant classes and aux-functions are available

```
source("Code/structure.R")
```

We start by creating an HMD object, with Danish data, and using the corresponding class function "getSmoothMu" to get the smoothed mortality surface.

```
HMDobj <- HMDdatClass$new(
  Otxtfile = 'Data/DNK_Deaths_1x1.txt',
  Etxtfile = 'Data/DNK_Exposures_1x1.txt'
  )

surf_obj <- HMDobj$getSmoothMu()
```

From the surf object, we can use the aux-function "surf2lifeexp" to calculate Period life expectancies for all groups, ages = 0, and all times. Then plot using the aux-function **life.exp.base**
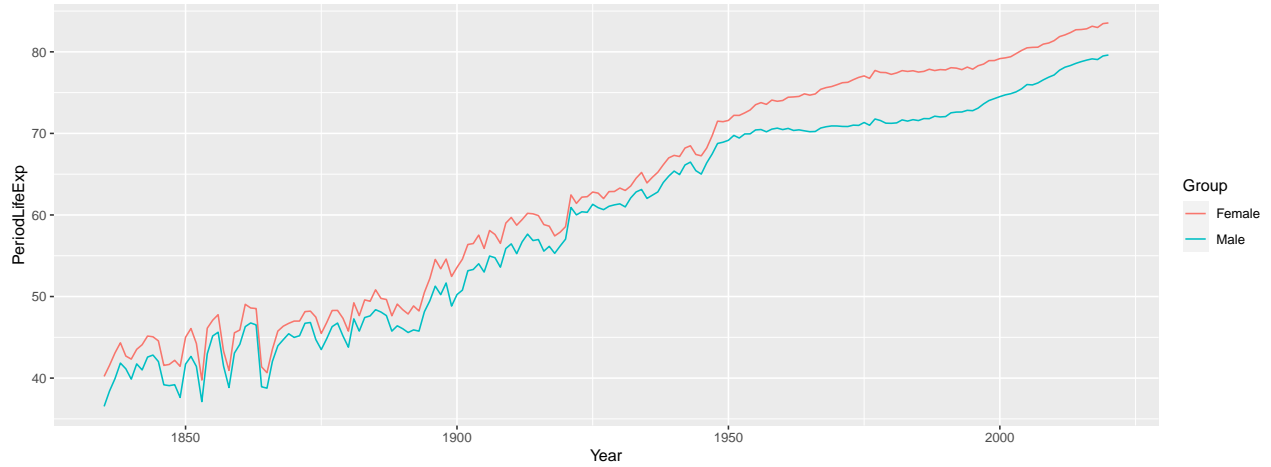
```
lifeexp_obj <- surf2lifeexp(surf = surf_obj, agelim = 0, type = "period")

plot_dat_lifeexp <- reshape2::melt(lifeexp_obj$lifeexp , varnames = c("Group", "Age", "Year"), value.nam

ggplot(data = plot_dat_lifeexp, aes(x = Year, y = PeriodLifeExp, fill = Group)) + geom_bar(stat = "ident
```
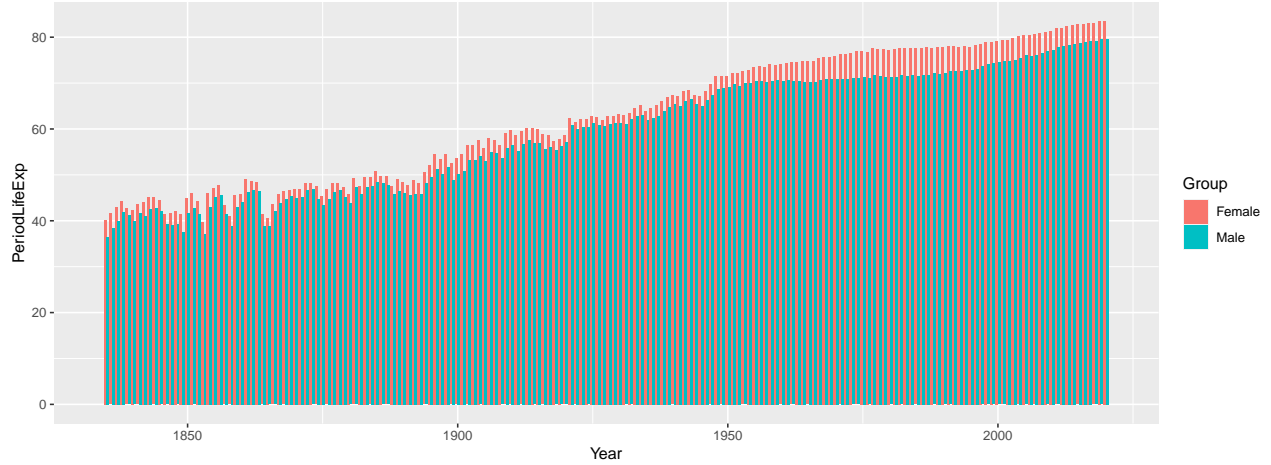


```
ggplot(data = plot_dat_lifeexp, aes(x = Year, y = PeriodLifeExp, col = Group)) + geom_line()
```


1.8)

Problem 1.8

Mortality improvement rates is defined in Vaupel & Romo (2003) eq. (4) as

$$\rho(a,t) = -\acute{\mu}(a,t) = -\frac{\frac{\partial}{\partial t}\mu(a,t)}{\mu(a,t)} = -\frac{d}{dt}log\mu(a,t)$$

.

While the average pace of mortality improvement is defined in eq. (6):

$$\bar{\rho}(t) = \int_0^\infty \rho(a,t)\mu(a,t)S(a,t)du = \int_0^\infty \rho(a,t)f(a,t)da$$

9

Where $f(a, t)$ is the distribution of deaths among the period t cohort.
Since we get data by 1x1 format, we are not able to get analytic expressions for the derivative wrt time or age for that matter.

Hence we will use the observed data for the distribution of deaths, and the estimates from the smoothed μ-surface, one could use the approximation

$$\rho(a, t) \approx \frac{\mu(a, t+1) - \mu(a, t)}{\mu(a, t)}$$

Although we will primarily use the approximation:

$$\rho(a, t) \approx \log\left\{\mu(a, t+1) / \mu(a, t)\right\}$$

But still keep the previous one, to see how it performs.
We will use the smoothed $\mu(a, t)$ from the 'surf' object, and get the deaths from the HMD objects.

```
#get deaths
deaths <- HMDobj$getOEdata()


#reshape to long format, add groups
arr_long <-
  reshape2::melt(deaths$O) %>% rename(Deaths = value) %>%
  filter(Year %in% 1835:2020) %>%
  mutate(Age_grp = cut(Age,
                       breaks=c(-Inf,0,10*(1:9),Inf),
                       include.lowest=TRUE,
                       labels=c("0",paste(10*(0:8)+1, 10*(1:9), sep = "-"),"91+")),
         Year_grp = case_when(Year %in% 1835:1900 ~ "1835-1900",
                              Year %in% 1900:1950 ~ "1900-1950",
                              Year %in% 1850:2020 ~ "1950-2020")) %>%
    arrange(Year, Group, Age)

#calculate normalized death-distribution across time, gender, and age-groups
arr_long_2 <-
  arr_long %>% group_by(Year, Group, Age_grp) %>%
  mutate(normalised = Deaths/sum(Deaths))
```
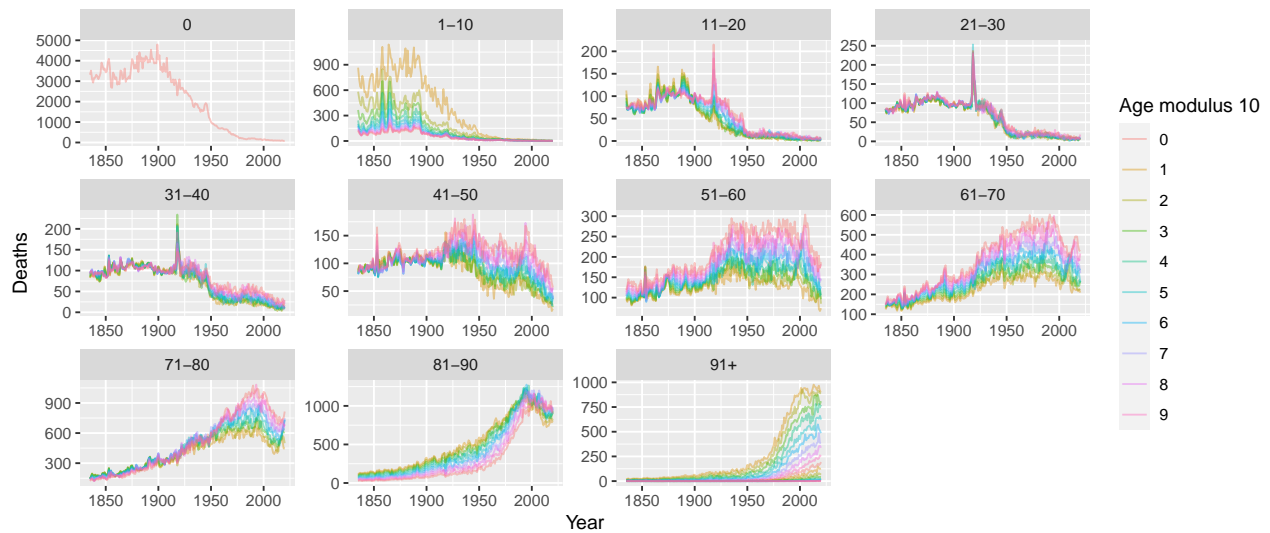
Let's plot to get overview of data

```
ggplot(arr_long %>% filter(Group == "Female") %>% mutate(Age_mod = Age %% 10),
       aes(x = Year, y = Deaths)) +
  geom_line(aes(group = Age, col = factor(Age_mod)), alpha = 0.4) +
  labs(title = "Females: Deaths, facet'd by age groups, coloured by Age",
       col = "Age modulus 10") +
  scale_color_discrete() +
  facet_wrap(vars(Age_grp), scales = "free")
```

Females: Deaths, facet'd by age groups, coloured by Age


```
ggplot(arr_long %>% filter(Group == "Male") %>% mutate(Age_mod = Age %% 10),
       aes(x = Year, y = Deaths)) +
  geom_line(aes(group = Age, col = factor(Age_mod)), alpha = 0.4) +
  labs(title = "Males: Deaths, facet'd by age groups, coloured by Age",
       col = "Age modulus 10") +
  scale_color_discrete() +
  facet_wrap(vars(Age_grp), scales = "free")
```

Males: Deaths, facet'd by age groups, coloured by Age
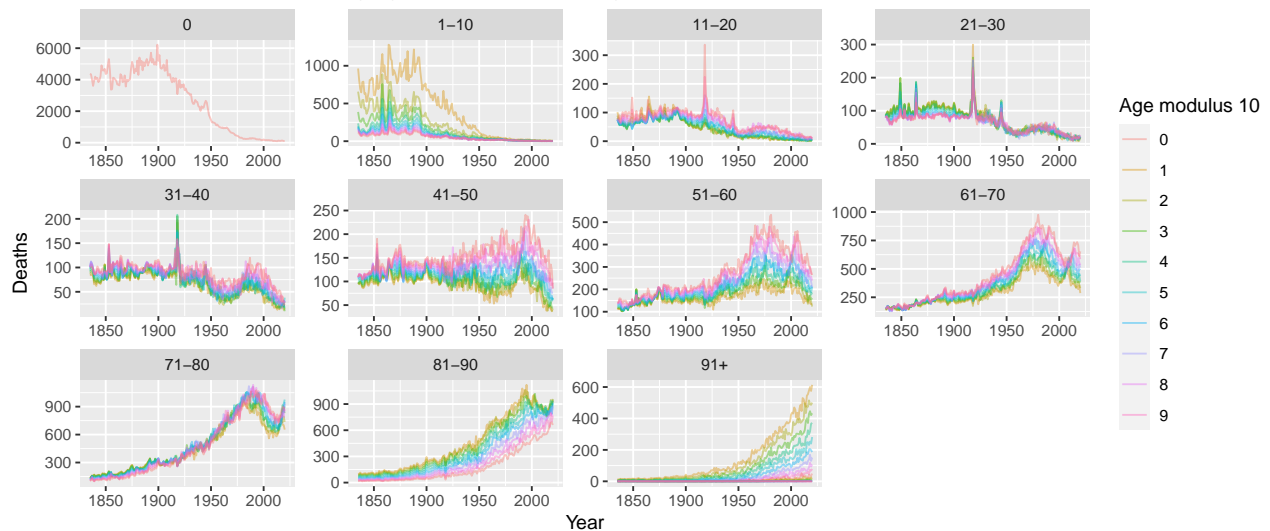
A few notes: -hello -hello2

```
#calculate change from year t -> t+1 in mortality, using surf object
rel_diff_surf_log <-
  -log( surf_obj$mu[,,dimnames(surf_obj$mu)$Year %in% as.character(1836:2020)] /
  surf_obj$mu[,,dimnames(surf_obj$mu)$Year %in% as.character(1835:2019)])

#other approximation, not used for plots
rel_diff_surf <-
  -(surf_obj$mu[,,dimnames(surf_obj$mu)$Year %in% as.character(1836:2020)] -
  surf_obj$mu[,,dimnames(surf_obj$mu)$Year %in% as.character(1835:2019)]) /
```

```
  surf_obj$mu[,,dimnames(surf_obj$mu)$Year %in% as.character(1836:2020)]


#Make into long format
rel_diff_surf_long <-
  reshape2::melt(rel_diff_surf) %>%
  arrange(Year, Group, Age) %>%
  rename(rel_diff_mu = value)

rel_diff_surf_long_log <-
  reshape2::melt(rel_diff_surf_log) %>%
  arrange(Year, Group, Age) %>%
  rename(rel_diff_mu_log = value)

#combine Deaths data set with discrete rho
comb <-
  left_join(x = arr_long_2, y = rel_diff_surf_long, by = c("Year", "Group", "Age")) %>%
  left_join(rel_diff_surf_long_log, by = c("Year", "Group", "Age") )


comb_2 <-
  comb %>% group_by(Year,Year_grp, Group, Age_grp) %>%
  #Get weighted average of rho, death distributed
  summarise(rho_bar = sum(rel_diff_mu * normalised),
            rho_bar_log = sum(rel_diff_mu_log*normalised))


#plot
ggplot(comb_2, aes(x = Year, y = rho_bar_log, col = Age_grp)) +
  geom_line(alpha = 0.6) +
  facet_wrap(vars(Group, Year_grp), nrow = 3, scales = "free_x") +
  geom_abline(slope = 0, intercept = 0, linetype = "dashed")
```
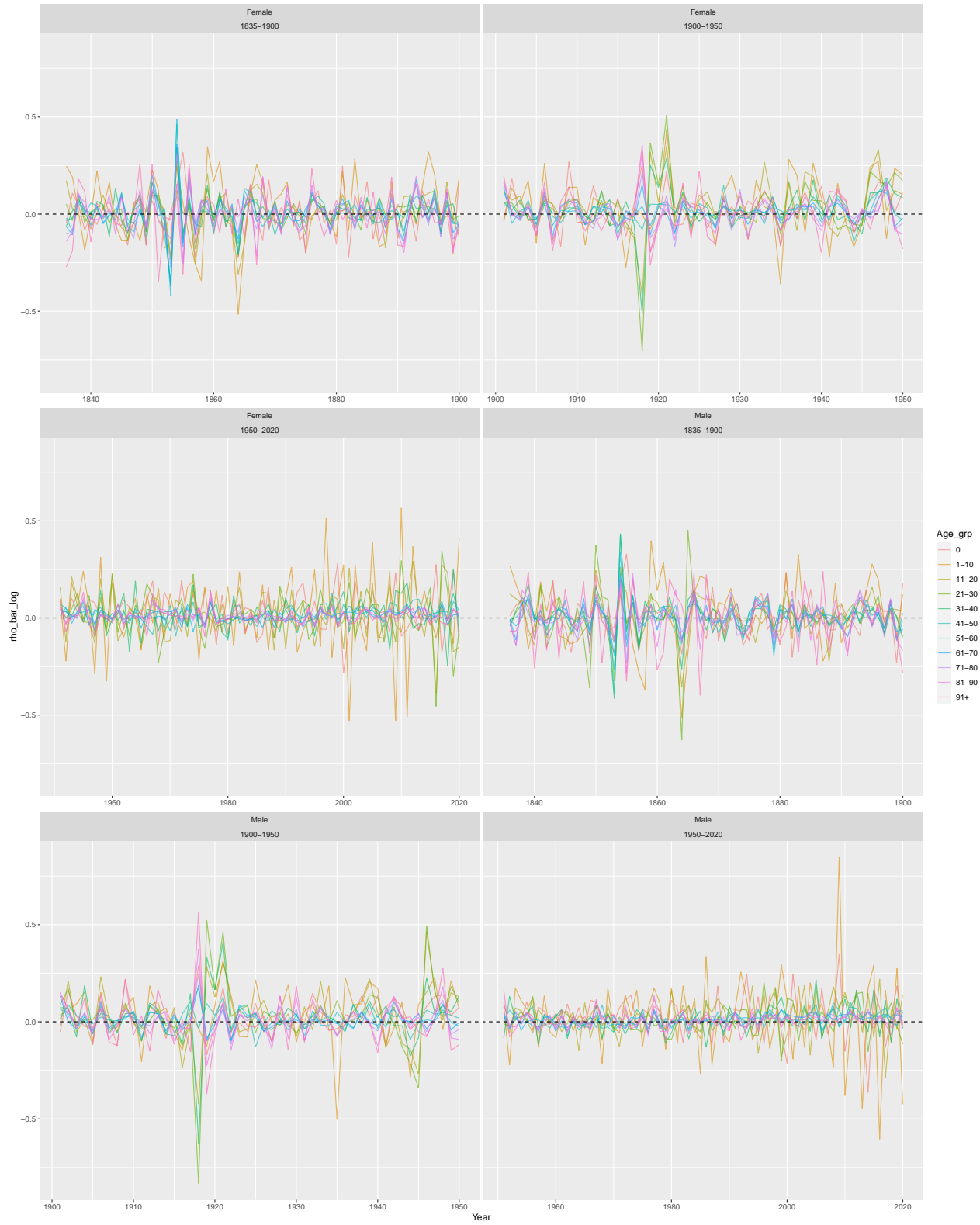
```
## Warning: Removed 11 row(s) containing missing values (geom_path).
```

As can be seen from the plots, the death-weighted improvement rate in mortality is noisy, hence we calculate the average of the $\bar{\rho}$ across `Group` for the specified time limits.

Year_grp	Group	n	mean_rho_bar	mean_rho_bar_log
1835-1900	Female	726	1.05%	0.40%
1835-1900	Male	726	1.11%	0.41%
1900-1950	Female	550	2.76%	2.06%
1900-1950	Male	550	2.70%	1.93%
1950-2020	Female	770	2.46%	1.74%
1950-2020	Male	770	1.94%	1.40%

```
comb_2_means <-
  comb_2 %>%
  group_by(Year_grp, Group) %>%
  summarise(n = n(),
            mean_rho_bar = scales::percent(mean(rho_bar, na.rm = T), accuracy = 0.01),
            mean_rho_bar_log = scales::percent(mean(rho_bar_log, na.rm = T), accuracy = 0.01))

kableExtra::kbl(comb_2_means) %>% kable_styling()
```

1.9)

##Problem 1.9

It is evident from 1.8 that e(0) has been steadily increasing throughout the last 150 years, with some variation, especially prior til 1925.

1.10)

##Problem 1.10 We will now implement the decomposition of changes to the force of mortality by age intervals $[x, y]$, given in equation (2). Since all of the terms can be re-written in terms of $l(0)$ and functions of μ_1, μ_2 we choose to rewrite in terms of $l(0)$ and $S_i(t) = exp(-\int_0^t \mu_i(u)du)$. Formally the r-implementation relies on $x \leq y$ and functions given as the base-r implementation of S_i e.g. *pexp* if constant hazard

```
#function to calculate eq.2
mort_diff <- function(x, y, Smu1, Smu2){
  #first fraction, l_0 cancels
  Smu1(x) * (
  #fraction between person year [x, y] and total remaining person years
(integrate(f = Vectorize(Smu2),
           lower = x, upper = y)$value / Smu2(x)) -
  (integrate(f = Vectorize(Smu1),
           lower = x, upper = y)$value / Smu1(x))
) +
   integrate(f = function(u)Smu2(u),lower = y, upper = Inf)$value *
  (
    Smu1(x)/Smu2(x) - Smu1(y)/Smu2(y)
  )


}
```

Now, we choose to test the function's correctness by looking at differences in means of well known distributions:

```
#Exponential distribution, E(X) = 1/rate
#only settign y = 100 instead of Inf due to numeric instability
mort_diff(x = 0, y = 100,
          Smu1 = function(u) pexp(u, rate = 11, lower.tail = FALSE),
          Smu2 = function(u) pexp(u, rate = 3, lower.tail = FALSE)
          )
```

```
## [1] 0.2424242
```

```
1/3 - 1/11
```

```
## [1] 0.2424242
```

```
#Exponential distribution, E(X) = scale*GammaFunction(1 + 1/shape)
#only settign y = 100 instead of Inf due to numeric instability
mort_diff(x = 0, y = 100,
          Smu1 = function(u) pweibull(u, shape = 11, scale = 3, lower.tail = FALSE),
          Smu2 = function(u) pexp(u, rate = 3, lower.tail = FALSE)
          )
```

```
## [1] -2.531905
```

```
1/3 - 3*gamma(1+1/11)
```

```
## [1] -2.531905
```

1.11)

```
#Problem 1.11
HMDobj$getOEdata(timelim = 1900)
```

2)

2.1)

Problem 2.1

The pros of estimating the model by a poisson distribution instead of using the OLS approach is that we don't have to worry about handling empty cells, for example if there are zero deaths in a given year. Especially in small populations it is a good idea to use the poisson distribution. The cons of using the poisson distribution is that we may miss some information hidden in the data.

2.2)

Problem 2.2

```
HMDobj <- HMDdatClass$new(
  Otxtfile = 'Data/GBRCENW_Deaths_1x1.txt',
  Etxtfile = 'Data/GBRCENW_Exposures_1x1.txt'
  )
```

```
## Imported historic data, (set public initialize) for England and Wales
##    - Groups      : 2
##    - Age range  : 0 - 110
##    - Year range : 1841 - 2018
```

```
surf_obj <- HMDobj$getSmoothMu()
```

Take both sexes for ages 20-100 in the years 1970-2018:

```
OEdata <- HMDobj$getOEdata(agelim = c(20,100), timelim = c(1970, 2018))
```

We want to estimate the models parameters α_t and β_t in the CBD model by using the Poisson assumption:

$$D(x,t) \mid E(x,t) \sim Poisson(E(x,t)\mu(x,t))$$

We already know the exposures from our data, so we get the distribution by estimating $\mu(x,t)$ in terms of α and *beta*. We know that

$$q(x,t) = 1 - exp(-\mu(x,t))$$

and

$$\log\left(\frac{q(x,t)}{1-q(x,t)}\right) = \alpha_t + \beta_t(x-\bar{x})$$

Inserting and isolating $\mu(x,t)$ we get:

$$\log\left(\frac{1-exp(-\mu(x,t))}{1-(1-exp(-\mu(x,t)))}\right) = \alpha_t + \beta_t(x-\bar{x})$$

$$\Leftrightarrow \frac{1-exp(-\mu(x,t))}{exp(-\mu(x,t)))} = exp(\alpha_t + \beta_t(x-\bar{x}))$$

$$\Leftrightarrow \frac{\frac{1}{exp(-\mu(x,t))} - \frac{exp(-\mu(x,t))}{exp(-\mu(x,t))}}{\frac{exp(-\mu(x,t)))}{exp(-\mu(x,t))}} = exp(\alpha_t + \beta_t(x-\bar{x}))$$

$$\Leftrightarrow \frac{\frac{1}{exp(-\mu(x,t))} - 1}{1} = exp(\alpha_t + \beta_t(x-\bar{x}))$$

$$\Leftrightarrow \frac{1}{exp(-\mu(x,t))} - 1 = exp(\alpha_t + \beta_t(x-\bar{x}))$$

$$\Leftrightarrow \frac{1}{exp(-\mu(x,t))} = exp(\alpha_t + \beta_t(x-\bar{x})) + 1$$

$$\Leftrightarrow exp(-\mu(x,t)) = \frac{1}{exp(\alpha_t + \beta_t(x-\bar{x})) + 1}$$

$$\Leftrightarrow -\mu(x,t) = \log\left(\frac{1}{exp(\alpha_t + \beta_t(x-\bar{x})) + 1}\right)$$

$$\Leftrightarrow \mu(x,t) = -\log\left(\frac{1}{exp(\alpha_t + \beta_t(x-\bar{x})) + 1}\right)$$

Inserting this expression for $\mu(x,t)$ into the Poisson distribution, we want to estimate a Poisson distribution with the parameters:

$$D(x,t) \mid E(x,t) \sim Poisson\left(E(x,t)\left(-\log\left(\frac{1}{exp(\alpha_t + \beta_t(x - \bar{x})) + 1}\right)\right)\right)$$

The estimators α_t and β_t have 48 dimensions i.e. we get one parameter for each year.

We get that the log-likelihood function for the Poisson distribution is:

$$l(\lambda) = \sum_{t=1970}^{2018} \sum_{x=20}^{100} D_{t,x} \log \lambda_{t,x} - \lambda_{t,x} - \log D_{t,x}!$$

$$\Leftrightarrow l((\alpha_{1970}, \beta_{1970}), ..., (\alpha_{2018}, \beta_{2018}))$$

$$= \sum_{t=1970}^{2018} \sum_{x=20}^{100} D_{t,x} \log\left(E(x,t)\left(-\log\left(\frac{1}{exp(\alpha_t + \beta_t(x - \bar{x})) + 1}\right)\right)\right) - \left(E(x,t)\left(-\log\left(\frac{1}{exp(\alpha_t + \beta_t(x - \bar{x})) + 1}\right)\right)\right) - \log D$$

$$= \sum_{t=1970}^{2018} \sum_{x=20}^{100} D_{t,x} \log\left(E(x,t)\log(exp(\alpha_t + \beta_t(x - \bar{x})) + 1)\right) - (E(x,t)\log\left(exp(\alpha_t + \beta_t(x - \bar{x})) + 1)\right) - \log D_{t,x}!$$

Writing the log-likelihood and using optim to maximize minus the log-likelihood(same as minimizing the loglikelihood) we get:

```
# Setup

xmin <- 20
xmax <- 100
xvec <- xmin:xmax
xbar <- sum(xvec)/(xmax-xmin+1) #mean of the agespan considered = 60

tmin <- 1970
tmax <- 2018

# Log-likelihood funktion
loglike <- function(par,Dvec,Evec,xvec) {
    mu <- log(exp(par[1]+par[2]*(xvec-mean(xvec)))+1)
    -sum(Dvec*log(mu)-Evec*mu)
}

Betahat <- matrix(NA, nrow = (tmax-tmin+1),ncol = 1)
Alphahat <- matrix(NA, nrow = (tmax-tmin+1),ncol = 1)
# Optimize minus log-likelihooden for females
for(i in 1:(tmax-tmin+1)){
  Dtest   <- OEdata$O["Female",,i] #    Dødsfald for kvinder i år i aldre xmin til xmax
  Etest   <- OEdata$E["Female",,i] # Eksponering for kvinder i år i aldre xmin til xmax
  opt <- optim(c(-4,0.1), function(par) loglike(par,Dtest,Etest,xvec))
  Betahat[i] <- opt$par[2] # beta parametre under Poisson antagelsen for kvinder i år i
  Alphahat[i] <- opt$par[1]# alpha  parametre under Poisson antagelsen for kvinder i år i
}

Betahat_f <- Betahat
Alphahat_f <- Alphahat

Betahat <- matrix(NA, nrow = (tmax-tmin+1),ncol = 1)
```

```r
Alphahat <- matrix(NA, nrow = (tmax-tmin+1),ncol = 1)
# Optimer minus log-likelihooden
for(i in 1:(tmax-tmin+1)){
  Dtest   <- OEdata$O["Male",,i] #    Dødsfald for Mænd i år i aldre xmin til xmax
  Etest   <- OEdata$E["Male",,i] # Eksponering for Mænd i år i aldre xmin til xmax
  opt <- optim(c(-4,0.1), function(par) loglike(par,Dtest,Etest,xvec))
  Betahat[i] <- opt$par[2] # beta parametre under Poisson antagelsen for kvinder i år i
  Alphahat[i] <- opt$par[1]# alpha  parametre under Poisson antagelsen for kvinder i år i
}


Betahat_m <- Betahat
Alphahat_m <- Alphahat
Par_m <- data.frame(alpha = Alphahat_m, beta = Betahat_m)
Par_f <- data.frame(alpha = Alphahat_f, beta = Betahat_f)
```

We have that number of deaths is Poisson distributed:

$$D(x,t) \mid E(x,t) \sim Poisson\left(E(x,t)\left(-\log\left(\frac{1}{exp(\alpha_t + \beta_t(x - \bar{x})) + 1}\right)\right)\right)$$

Calculating $\lambda = \log\left(exp(\alpha_t + \beta_t(x - \bar{x})) + 1\right)$ we get the value for each year:
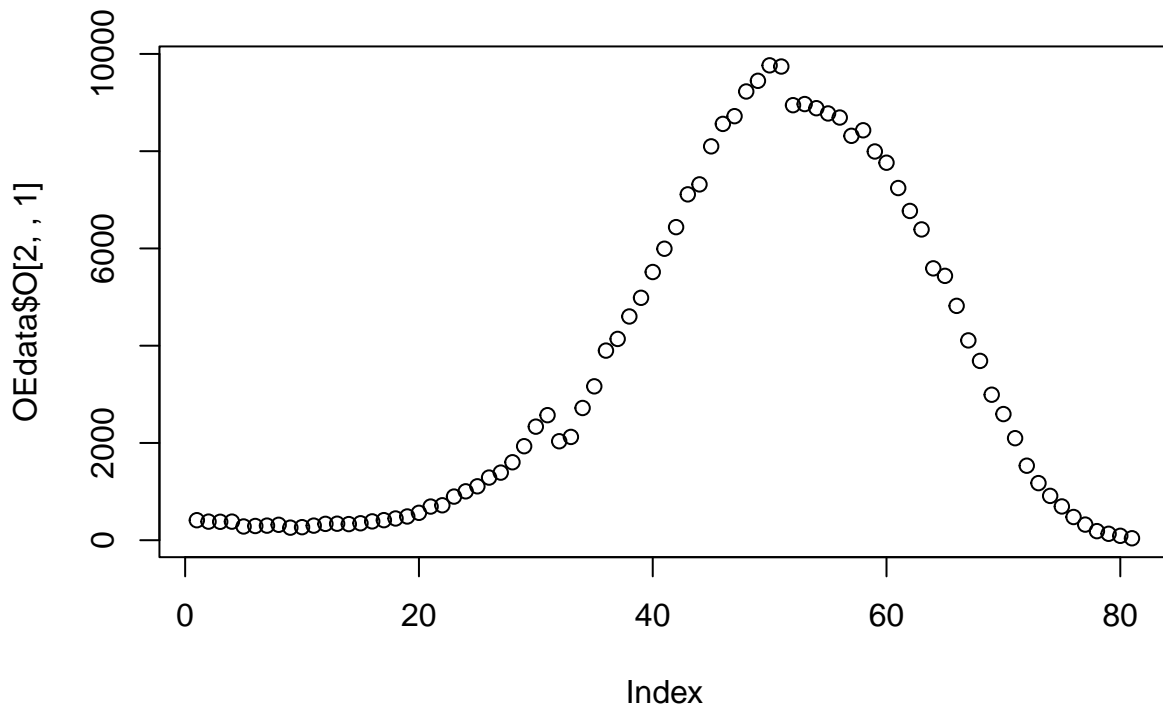
```r
lambda_mat_m <- matrix(NA, nrow = (xmax-xmin +1), ncol = (tmax-tmin +1))
lambda_mat_f <- matrix(NA, nrow = (xmax-xmin +1), ncol = (tmax-tmin +1))


#females + males

for(age in seq_along(xmin:xmax)){
  for(time in seq_along(tmin:tmax)){
    lambda_mat_m[age, time] <- (Par_m[time, "alpha"] + Par_m[time,"beta"]*((xmin:xmax)[age] - xbar))*OE
    lambda_mat_f[age, time] <- (Par_f[time, "alpha"] + Par_f[time,"beta"]*((xmin:xmax)[age] - xbar))*OE

  }
}


#plot first year, men
plot(OEdata$O[2,,1])
```
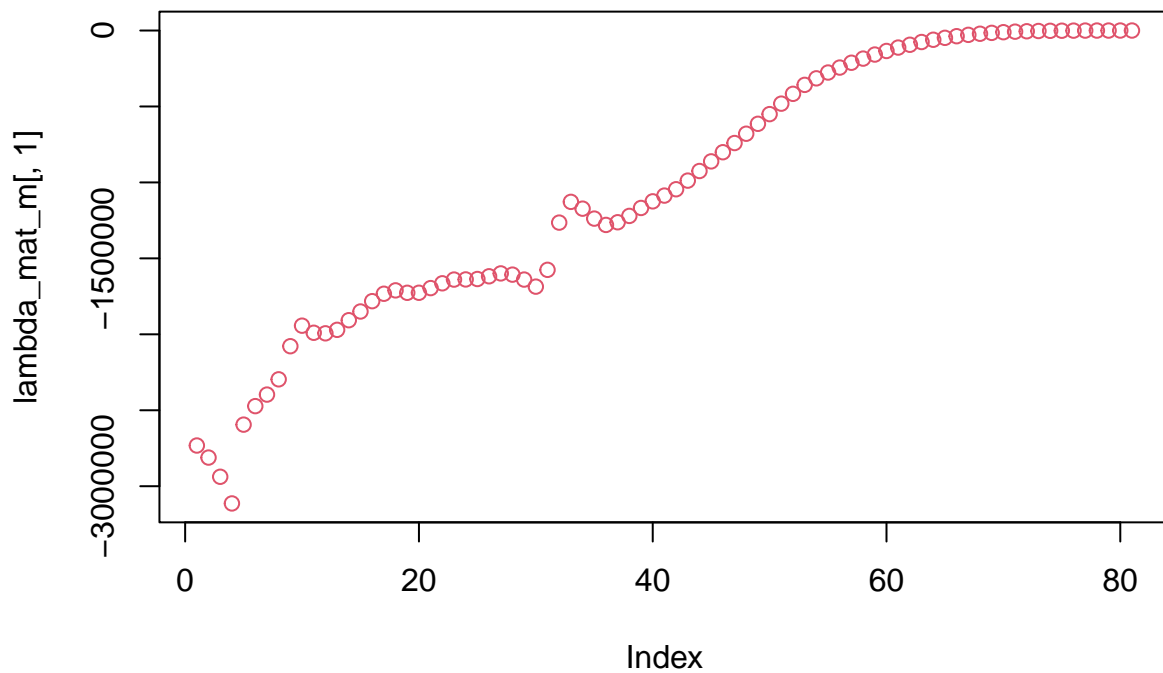
```
plot(lambda_mat_m[,1], col = 2)
```


2.3)

Problem 2.3

As in the original work (Cairns et al. (2006), p. 6), we use a two- dimensional random walk with drift to forecast the model. Estimate the model for each sex and state the parameters. In your opinion, is the use of

a random walk structure justifiable?

We want to forecast the model by using a two-dimensional random walk:

$$A(t+1) = A(t) + \mu + CZ(t+1)$$

where μ is a constant 2x1 matrix, C is an upper triangular matrix and Z(t) is a two-dimensional standard normal random variable. i.e. We want to estimate:

$$\begin{pmatrix} \alpha_{t+1} \\ \beta_{t+1} \end{pmatrix} = \begin{pmatrix} \alpha_t \\ \beta_t \end{pmatrix} + \begin{pmatrix} \theta_\alpha \\ \theta_\beta \end{pmatrix} + \epsilon_{t+1}$$

where $\epsilon_{t+1} \sim \mathcal{N}(0, \Sigma)$

We choose $\theta = (\theta_\alpha, \theta_\beta)$ to be the difference between the last two observations:

```
theta_f <- matrix(data = c(par_f_data$alpha_f[49]-par_f_data$alpha_f[48], par_f_data$beta_f[49]-par_f_d
theta_m <- matrix(data = c(par_m_data$alpha_m[49]-par_m_data$alpha_m[48], par_m_data$beta_m[49]-par_m_da
```

We calculate Σ as:

$$\Sigma = \begin{pmatrix} \sigma_\alpha^2 & \rho\sigma_{alpha}\sigma_{beta} \\ \rho\sigma_{alpha}\sigma_{beta} & \sigma_{beta}^2 \end{pmatrix}$$

where ρ is the correlation. Calculating we get:

```
sigma_a_f <- var(par_f_data$alpha_f)
sigma_b_f <- var(par_f_data$beta_f)
cor_f <- cor(par_f_data$alpha_f, par_f_data$beta_f)

Sigma_f <- matrix(c(sigma_a_f, cor_f*sqrt(sigma_a_f)*sqrt(sigma_a_f), cor_f*sqrt(sigma_a_f)*sqrt(sigma_a

sigma_a_m <- var(par_m_data$alpha_m)
sigma_b_m <- var(par_m_data$beta_m)
cor_m <- cor(par_m_data$beta_m, par_m_data$alpha_m, method = "kendall")
Sigma_m <- matrix(c(sigma_a_m, cor_m*sqrt(sigma_a_m)*sqrt(sigma_a_m), cor_m*sqrt(sigma_a_m)*sqrt(sigma_a
```

Simulating 100 values of the bivariate normal distribution with mean 0 and the empirical covariancematrix(since the covariance matrices above are not positive definite they cannot be used) for both men and women we get:

```
n <- 100
mean <- c(0,0)
cov <- cov(data.frame(x = rnorm(100), y = rnorm(100)))
library(MASS)

##
## Attaching package: 'MASS'

## The following object is masked _by_ '.GlobalEnv':
##
##     deaths

## The following object is masked from 'package:dplyr':
##
##     select
```

```r
norm <- mvrnorm(n = n, mu = mean, Sigma = cov)
```

We are now ready to create the random walk:

```r
A_0 <- c(par_f_data$alpha_f[49],par_f_data$beta_f[49])
A <- matrix(NA, nrow = 100, ncol = 2)
for(i in (1:100)){
  A[i,1] <- A_0[1] + theta_f[1] + norm[i,1]
  A[i,2] <- A_0[2] + theta_f[2] + norm[i,2]
  A_0 <- c(A[i,1], A[i,2])
}


A_f <- A

A_0 <- c(par_m_data$alpha_m[49],par_m_data$beta_m[49])
A <- matrix(NA, nrow = 100, ncol = 2)
for(i in (1:100)){
  A[i,1] <- A_0[1] + theta_m[1] + norm[i,1]
  A[i,2] <- A_0[2] + theta_m[2] + norm[i,2]
  A_0 <- c(A[i,1], A[i,2])
}
A_m <- A
```
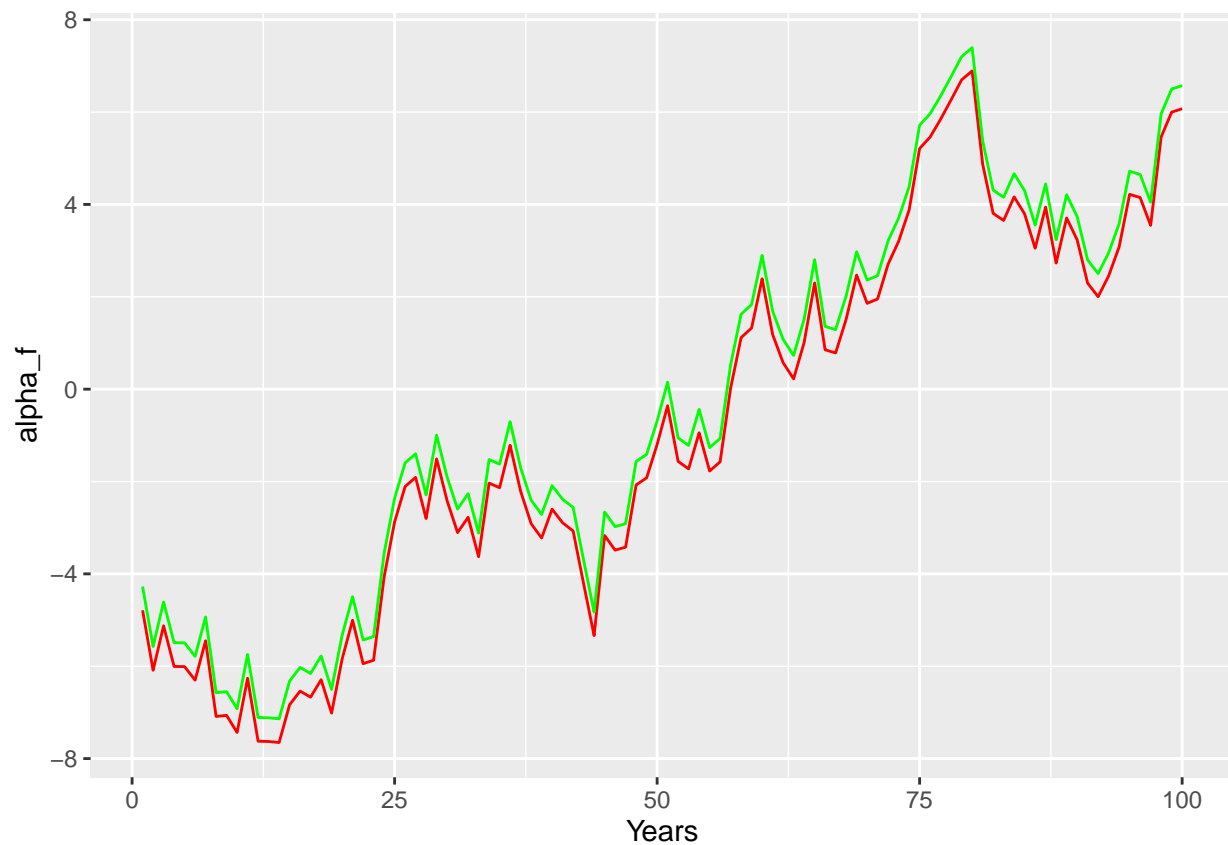
Og plotter:

```r
A_f_d <- as.data.frame(A_f)
A_f_d <- rename(A_f_d, alpha_f = V1)
A_f_d <- rename(A_f_d, beta_f = V2)

A_m_d <- as.data.frame(A_m)
A_m_d <- rename(A_m_d, alpha_m = V1)
A_m_d <- rename(A_m_d, beta_m = V2)

Years = seq(1,100, by = 1)
A_collected <- cbind(A_m_d, A_f_d, Years)
```

Here we see the forecast for α for females(red) and males(green). We see that over time it just gets more negative.

```r
ggplot(A_collected, aes(Years)) +              # basic graphical object
  geom_line(aes(y=alpha_f), colour="red") +    # first layer
  geom_line(aes(y=alpha_m), colour="green")
```

Here we see the forecast for β for females(red) and males(green). We see that over time it just gets more negative.

```
ggplot(A_collected, aes(Years)) +                        # basic graphical object
  geom_line(aes(y=beta_f), colour="red") +  # first layer
  geom_line(aes(y=beta_m), colour="green")
```
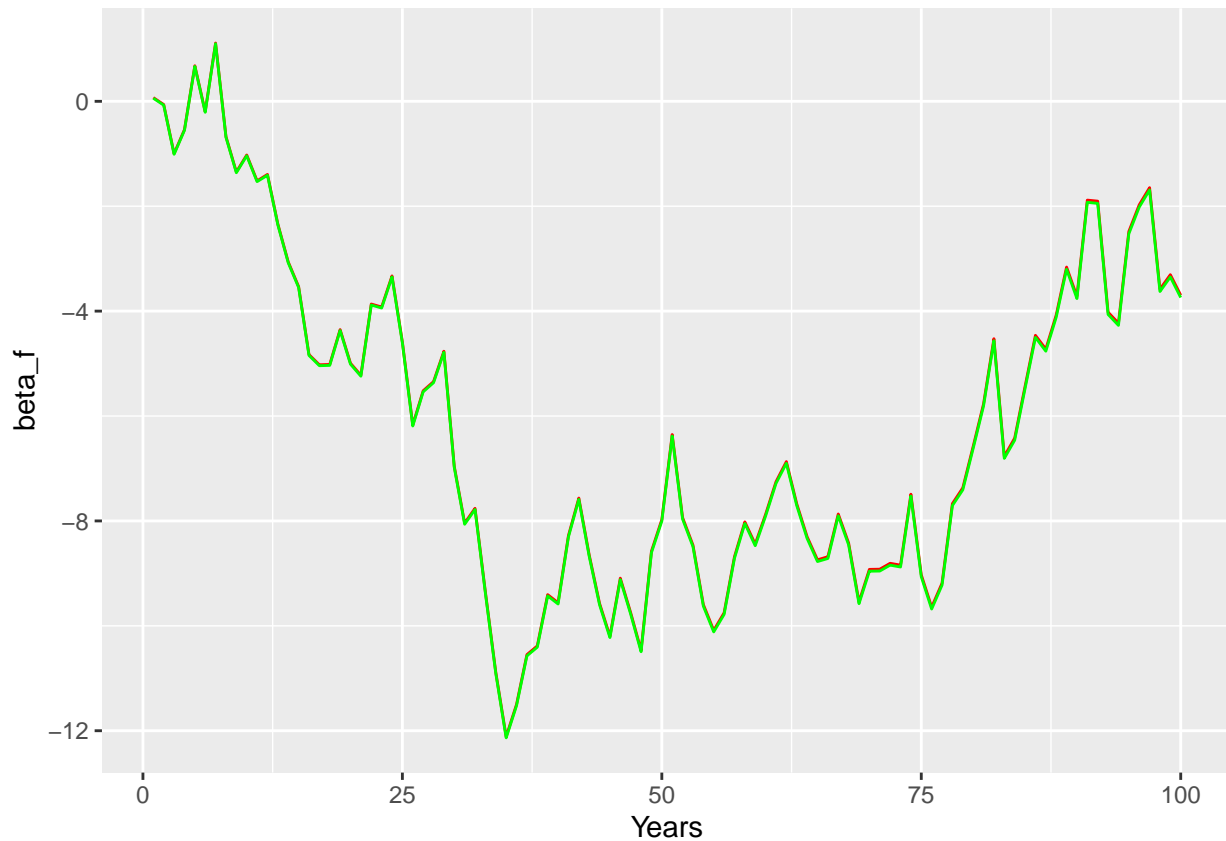
If we look at the historic values of α we get:

```
Years = seq(1970,2018, by = 1)
Par_collected <- cbind(Par_m, Par_f,Years)
Par_collected <- as.data.frame(Par_collected)
Par_collected <- rename(Par_collected, alpha_m = V1)
Par_collected <- rename(Par_collected, beta_m = V2)
Par_collected <- rename(Par_collected, alpha_f = V3)
Par_collected <- rename(Par_collected, beta_f = V4)
```
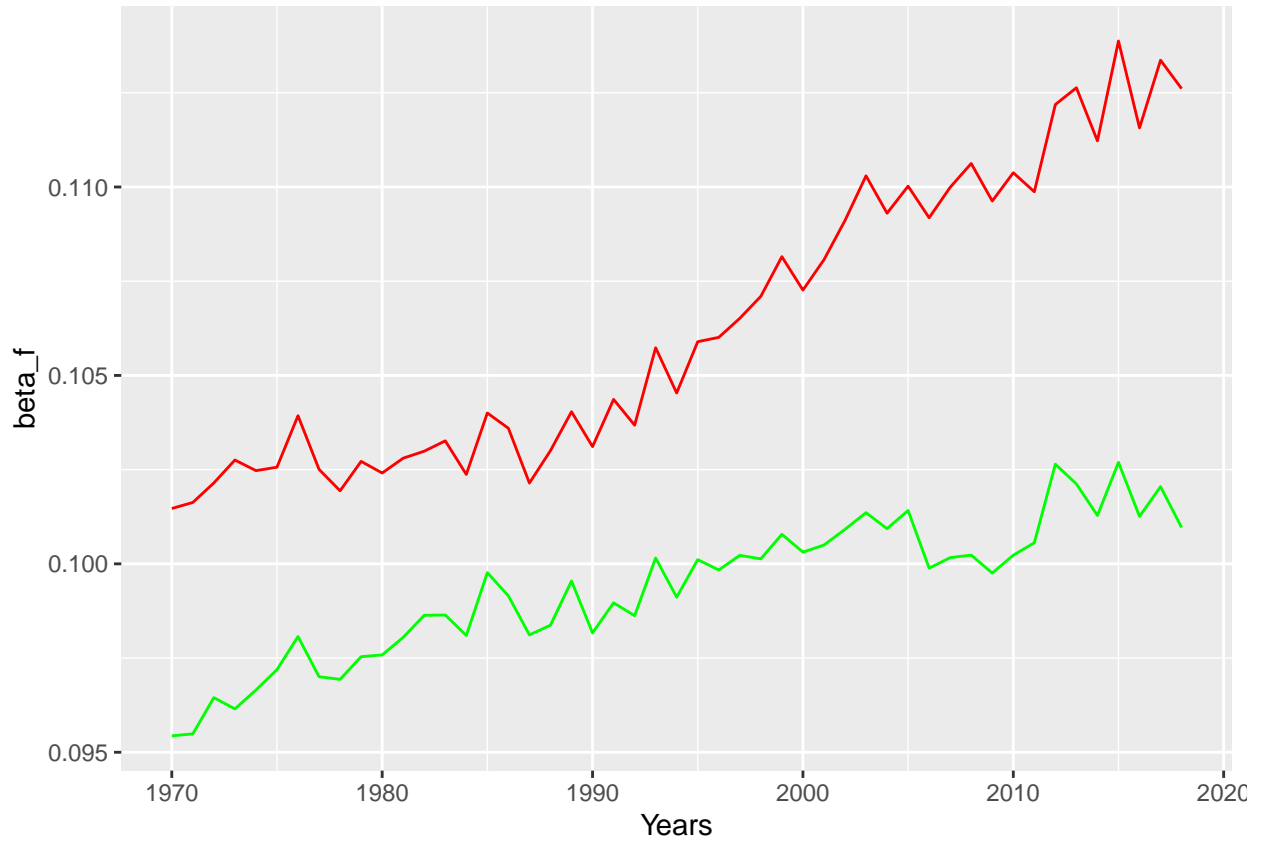
Plotting the historical values of α we get:

```
ggplot(Par_collected, aes(Years)) +                     # basic graphical object
  geom_line(aes(y=alpha_f), colour="red") +  # first layer
  geom_line(aes(y=alpha_m), colour="green")
```
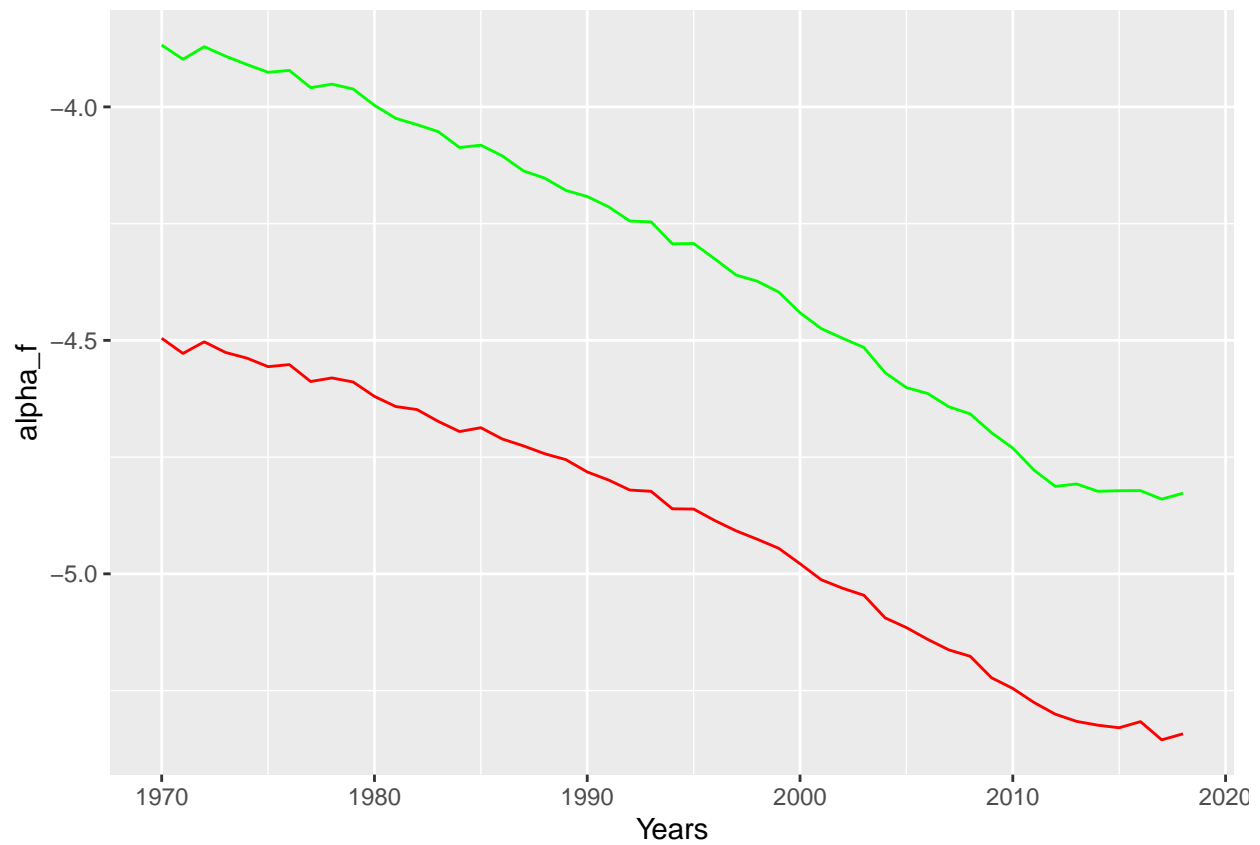
Plotting the historical values for β we get

```
ggplot(Par_collected, aes(Years)) +                        # basic graphical object
  geom_line(aes(y=beta_f), colour="red") +  # first layer
  geom_line(aes(y=beta_m), colour="green")
```

We see that historically α and β are much more stable than they are in the random walk. This is mostly due to the empirical covariance matrix used.

In our opinion it is not justifiable to use a random walk to forecast the future of α and β. One of the main reasons we would not use random walk to forecast the future is, that there is no biological factor preventing the life expectancy to go to infinity in the future.