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# DECOMPOSING CHANGE IN LIFE EXPECTANCY: A BOUQUET OF FORMULAS IN HONOR OF NATHAN KEYFITZ'S 90th BIRTHDAY\*

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*We extend Nathan Keyfitz's research on continuous change in life expectancy over time by presenting and proving a new formula for decomposing such change. The formula separates change in life expectancy over time into two terms. The first term captures the general effect of reduction in death rates at all ages, and the second term captures the effect of heterogeneity in the pace of improvement in mortality at different ages. We extend the formula to decompose change in life expectancy into age-specific and cause-specific components, and apply the methods to analyze changes in life expectancy in Sweden and Japan.*

**D**emographers frequently apply methods for decomposing change in a demographic variable into components of change. Kitagawa (1955), Das Gupta (1978, 1993), and Kim and Strobino (1984) developed widely used approaches; and we recently suggested a new approach (Vaupel and Canudas Romo 2002). Specific methods to analyze change in life expectancy over time have been developed by various demographers. A United Nations report (1982), Pollard (1982, 1988), Arriaga (1984), Pressat (1985), and Andreev (1982; Andreev et al. 2002) focused on the discrete difference in life expectancy between two periods. Keyfitz (1977, 1985) considered continuous change and derived a formula that relates the time derivative of life expectancy to the entropy of life table survivorship. Mitra (1978), Demetrius (1979), Goldman and Lord (1986), Vaupel (1986), Hakkert (1987), and Hill (1993) further developed this approach.

In this article we present and prove a new decomposition of change in life expectancy over time that generalizes Keyfitz's results.<sup>1</sup> In addition, we extend this new method to analyze age-specific and cause-specific effects. We begin with some notation and the proof of the decomposition formula. We provide a number of illustrative examples using data for Sweden and Japan.

## PRELIMINARIES

The new decomposition relates the time derivative of life expectancy to the average pace of improvement in mortality, the average number of life-years lost as a result of death, and the covariance between age-specific rates of mortality improvement and age-specific remaining life expectancies. Before presenting the decomposition, we will briefly discuss each of these quantities.

Life expectancy at birth at time  $t$  can be expressed as

$$e^o(0, t) = \int_0^{\infty} \ell(a, t) da, \quad (1)$$

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1. This article is in honor of Nathan Keyfitz's 90th birthday. He was born on June 29, 1913 in Montreal, Canada.

where  $\ell(a, t)$  is the life table probability at time  $t$  of surviving from birth to age  $a$ , and  $\omega$  is the highest age attained.

It is convenient to use a **dot** over a variable to denote the derivative with respect to time:

$$\dot{v} \equiv \dot{v}(a, t) \equiv \frac{\partial}{\partial t} v(a, t), \quad (2)$$

where  $v(a, t)$  is some demographic function. Hence, the time derivative of life expectancy at birth is expressed as  $\dot{e}^o(0, t)$ .

The force of mortality at age  $a$  and at time  $t$  is denoted by  $\mu(a, t)$ . Using an **acute accent** over a variable to represent the relative derivative or intensity with respect to  $t$ ,

$$\dot{v} \equiv \dot{v}(a, t) \equiv \frac{\dot{v}(a, t)}{v(a, t)} = \frac{\frac{\partial}{\partial t} v(a, t)}{v(a, t)}, \quad (3)$$

we define  $\rho(a, t)$  as the rate of progress in reducing death rates:

$$\rho(a, t) = -\dot{\mu}(a, t). \quad (4)$$

The acute accent notation, which reduces the clutter in many demographic formulas, was originated by Vaupel (1992) and was used in Vaupel and Canudas Romo (2000, 2002).

Let  $\bar{v}(t)$  denote the average of  $v(a, t)$  over  $a$ :

$$\bar{v}(t) = \frac{\int_0^\omega v(a, t) w(a, t) da}{\int_0^\omega w(a, t) da}, \quad (5)$$

where  $w(a, t)$  is some weighting function. We define the **average improvement in mortality** as

$$\bar{\rho}(t) = \int_0^\omega \rho(a, t) f(a, t) da, \quad (6)$$

where  $f(a, t)$  is the probability density function describing the distribution of deaths (i.e., life spans) in the life table population at age  $a$  and time  $t$ . In this average, which is a weighted average, the denominator is 1 because  $\int_0^\omega f(a, t) da = 1$ . Throughout this article, *average* refers to the mean value of a variable weighted by the distribution of life spans  $f(a, t)$ .

Let  $e^o(a, t)$  denote remaining life expectancy at age  $a$  and time  $t$ :

$$e^o(a, t) = \frac{\int_a^\omega \ell(x, t) dx}{\ell(a, t)}. \quad (7)$$

Then the average number of life-years lost as a result of death is given by

$$e^\dagger(t) = \int_0^\omega e^o(a, t) f(a, t) da. \quad (8)$$

The covariance between functions  $u(a, t)$  and  $v(a, t)$ , with weighting function  $w(a, t)$ , is

$$\begin{aligned} \text{Cov}_w(u, v) &= \frac{\int_0^\omega [u(a, t) - \bar{u}(t)][v(a, t) - \bar{v}(t)] w(a, t) da}{\int_0^\omega w(a, t) da} \\ &= \frac{\int_0^\omega u(a, t) v(a, t) w(a, t) da}{\int_0^\omega w(a, t) da} - \frac{\int_0^\omega u(a, t) w(a, t) da}{\int_0^\omega w(a, t) da} \frac{\int_0^\omega v(a, t) w(a, t) da}{\int_0^\omega w(a, t) da} \\ &\equiv \overline{uv} - \bar{u}\bar{v}. \end{aligned} \quad (9)$$

Eq. (9) implies that the expectation of a product can be decomposed as

$$\overline{uv} = \overline{u}\overline{v} + \text{Cov}_w(u, v), \quad (10)$$

a result central to our derivation and useful in other contexts.

The covariance between age-specific rates of mortality improvement and age-specific remaining life expectancies, weighted by the distribution of deaths, is

$$\text{Cov}_f(\rho, e^o) = \int_0^\infty [\rho(a, t) - \bar{\rho}(t)] [e^o(a, t) - e^*(t)] f(a, t) da. \quad (11)$$

## THE TIME DERIVATIVE OF LIFE EXPECTANCY

### A New Decomposition

Change in life expectancy at birth can be decomposed as follows:

$$\dot{e}^o(0, t) = \bar{\rho}(t)e^*(t) + \text{Cov}_f(\rho, e^o). \quad (12)$$

*Proof*

From the definition of life expectancy at birth in Eq. (1) and the fact that  $\ell(a, t) = e^{-\int_0^a \mu(x, t) dx}$ , it follows that the time derivative of life expectancy is

$$\dot{e}^o(0, t) = \int_0^\infty \dot{\ell}(a, t) da = \int_0^\infty \ell(a, t) \dot{\ell}(a, t) da = - \int_0^\infty \ell(a, t) \int_0^a \dot{\mu}(x, t) dx da. \quad (13)$$

In terms of the rate of progress in reducing death rates,  $\rho(a, t) = -\dot{\mu}(a, t)$ , the derivative of life expectancy can be expressed as

$$\dot{e}^o(0, t) = \int_0^\infty \ell(a, t) \int_0^a \mu(x, t) \rho(x, t) dx da = \int_0^\infty \mu(a, t) \rho(a, t) \int_a^\infty \ell(x, t) dx da. \quad (14)$$

See Goldman and Lord (1986) and Vaupel (1986) for further discussion of the reversal of the integration used to derive Eq. (14). Given Eq. (7) for the remaining expectation of life at age  $a$  and time  $t$ ,  $e^o(a, t)$ , and the probability density function describing the distribution of deaths,  $f(a, t) = \mu(a, t) \ell(a, t)$ , Eq. (14) implies that

$$\dot{e}^o(0, t) = \int_0^\infty \mu(a, t) \rho(a, t) \ell(a, t) e^o(a, t) da = \int_0^\infty \rho(a, t) e^o(a, t) f(a, t) da. \quad (15)$$

The right-hand side of Eq. (15) can be decomposed using (10), the formula for the expectation of a product:

$$\dot{e}^o(0, t) = \bar{\rho}(t)e^*(t) + \text{Cov}_f(\rho, e^o).$$

Q.E.D.

The decomposition in Eq. (12) expresses the change in life expectancy at birth as the sum of two terms. The first term is the product of the average rate of mortality improvement and the average number of life-years lost. This term captures the general effect of a reduction in death rates and in this article will be called the “level-1 change.” Note that  $\bar{\rho}$  can be interpreted as the proportion of deaths averted (or lives saved), and  $e^*$  can be interpreted as the average number of life-years gained per life saved.

The second term, the covariance between rates of mortality improvement and remaining life expectancies, increases or decreases the general effect, depending on whether the covariance is positive or negative. If  $\rho(a, t)$  is constant at all ages, then the covariance is zero. Hence, the covariance captures the effect of heterogeneity in  $\rho(a, t)$  at different ages. The covariance term will be called the “level-2 change” in this article.

Whether the covariance is positive or negative is determined by Eq. (11). It is difficult to capture the formula in a simple sentence. The basic idea is that the covariance will be positive if the age-specific pace of mortality improvement tends to be higher (or lower) than average at those ages when remaining life expectancy tends to be higher (or lower) than average; the heaviest weights are given to ages at which death is most common.

**Table 1.** Life Expectancy at Birth,  $e^o(0,t)$ , and the Decomposition of its Annual Change Around January 1, 1903, 1953 and 1998, in Sweden

$t$	1903	1953	1998
$e^o(0,t)$	53.383	71.858	79.262
$e^o(0,t-2.5)$	52.239	71.130	78.784
$e^o(0,t+2.5)$	54.527	72.586	79.740
$\dot{e}^o(0,t)$	0.458	0.291	0.191
$\bar{\rho}$ (%)	1.852	2.083	1.587
$e^\dagger$	22.362	11.988	10.053
$\bar{\rho}e^\dagger$	0.414	0.249	0.159
$\text{Cov}_f(\rho, e^o)$	0.044	0.042	0.032
$\dot{e}^o(0) = \bar{\rho}e^\dagger + \text{Cov}_f(\rho, e^o)$	0.458	0.291	0.191

Sources: Authors' calculations described in Appendix B. Life table data are derived from the Human Mortality Database (2002). The data used pertain to single years of age. Life table values for the years 1900 and 1905, 1950 and 1955, and 1995 and 2000, were used to obtain results for the midpoints around January 1, 1903, 1953, and 1998, respectively.

Remaining life expectancy generally declines with age. At some age  $a^*$ ,  $e^o(a^*,t) = e^\dagger(t)$ . The covariance will be positive if before age  $a^*$  the age-specific pace of mortality improvement tends to be higher than average and if after age  $a^*$  the age-specific pace of mortality improvement tends to be lower than average. Ages are weighted according to the distribution of deaths.

We do not consider the effects of measurement error. We assume that the data are highly reliable. This is certainly the case for the Swedish and Japanese data used in the illustrative applications. Measurement errors, however, may distort the covariance term by creating heterogeneity in the age-specific rates of mortality improvement. Because  $\bar{\rho}$ ,  $e^\dagger(t)$ , and  $e^o(a,t)$  are averages, they will be less affected than  $\rho(a,t)$  by random error, but they could be distorted by systematic biases. A decomposition without the covariance term is provided in Appendix A.

Eq. (12) is analogous to the decomposition we presented in a previously published article (Vaupel and Canudas Romo 2002). That formula separates the change in an average into a level-1 term involving the average of age-specific changes and a level-2 covariance term that captures the effect of heterogeneity in age-specific or subpopulation-specific changes.

### An Illustration: Change in Swedish Life Expectancy

Table 1 shows the application of Eq. (12) to the annual change in life expectancy at birth for the Swedish population in 1903, 1953 and 1998. Over the course of the twentieth century, Swedish life expectancy increased substantially. The average pace of mortality improvement,  $\bar{\rho}$ , fluctuated from about 1.9% at the turn of the century to 2.1% at mid-century and 1.6% at the end of the century. The average number of life-years lost as a result of death,  $e^\dagger$ , dropped from 22 years in 1903 to around 12 years in 1953 and 10 years in 1998. The product  $\bar{\rho}e^\dagger$  describes the increase in life expectancy owing to the general advance in survivorship. This level-1 component is positive and is the main contributor to the increase in life expectancy.

The level-2 component is the covariance between age-specific improvements in mortality and remaining life expectancies. This term is positive but relatively small.

## THE ENTROPY OF THE SURVIVAL FUNCTION

Following Keyfitz (1985), let  $\mathcal{H}(t)$  denote the entropy of the survival function

$$\mathcal{H}(t) = - \frac{\int_0^\infty \ell(a, t) \ln[\ell(a, t)] da}{\int_0^\infty \ell(a, t) da}. \quad (16)$$

Goldman and Lord (1986) and Vaupel (1986) showed that this entropy can also be expressed as

$$\begin{aligned} \mathcal{H}(t) &= \frac{\int_0^\infty \ell(a, t) \int_0^a \mu(x, t) dx da}{\int_0^\infty \ell(a, t) da} \\ &= \frac{\int_0^\infty \int_0^a \ell(a, t) \mu(x, t) dx da}{\int_0^\infty \ell(a, t) da} = \frac{\int_0^\infty \int_x^\infty \ell(a, t) \mu(x, t) da dx}{e^o(0, t)} \\ &= \frac{\int_0^\infty \mu(x, t) \int_x^\infty \ell(a, t) da dx}{e^o(0, t)} = \frac{\int_0^\infty \mu(x, t) \ell(x, t) e^o(x, t) dx}{e^o(0, t)}. \end{aligned} \quad (17)$$

Given Eq. (8) for  $e^\dagger$ , it follows that

$$e^\dagger(t) = \mathcal{H}(t) e^o(0, t). \quad (18)$$

If  $\rho(a, t)$  is constant over age,  $\rho(a, t) = \rho(t)$  for all  $a$ , then Eq. (12) reduces to

$$\dot{e}^o(0, t) = \rho(t) e^\dagger(t). \quad (19)$$

Substituting Eq. (18) yields

$$\dot{e}^o(0, t) = \rho(t) \mathcal{H}(t) e^o(0, t), \quad (20)$$

or

$$\dot{e}^o(0, t) = \frac{\dot{e}^o(0, t)}{e^o(0, t)} = \rho(t) \mathcal{H}(t), \quad (21)$$

which was Keyfitz's (1985) main result. In Eq. (20), the change depends not only on mortality progress and entropy  $\mathcal{H}(t)$  but also on the level of life expectancy at birth,  $e^o(0, t)$ .

If  $\rho(a, t)$  varies with age, then Eq. (18) implies that our main result, Eq. (12), can be reexpressed as

$$\dot{e}^o(0, t) = \bar{\rho}(t) \mathcal{H}(t) e^o(0, t) + \text{Cov}_f(\rho, e^o). \quad (22)$$

Alternatively, the relative change in life expectancy at birth can be decomposed as

$$\dot{e}^o(0, t) = \frac{\dot{e}^o(0, t)}{e^o(0, t)} = \bar{\rho}(t) \mathcal{H}(t) + \frac{\text{Cov}_f(\rho, e^o)}{e^o(0, t)}. \quad (23)$$

This result generalizes Keyfitz's result in Eq. (21).

If mortality follows a shifting Gompertz trajectory with changing level but constant rate of increase,

$$\mu(a, t) = \mu(0, t) e^{ba}, \quad (24)$$

then as Vaupel (1986; Vaupel and Canudas Romo 2000) proved,

$$\mathcal{H}(t) e^o(0, t) \approx \frac{1}{b} \quad (25)$$

and the accuracy of this approximation improves as life expectancy increases. From Eq. (18) it follows that the average life expectancy lost because of death can be approximated by

$$e^{\dagger}(t) \approx \frac{1}{b}. \quad (26)$$

The value of  $b$  can be estimated from the slope of a regression line fitted to the logarithm of age-specific death rates from ages 30–95, the span of life when mortality approximately follows a Gompertz trajectory. For Sweden in 1900, 1950, and 2000, the values of  $1/b$  were around 13.803, 10.480, and 10.127, respectively. For the same years, the average number of life-years lost as a result of death after age 30,  $e_{30}^{\dagger}(t)$ , were 12.382, 10.012, and 10.010. (Although all the formulas presented here are for life expectancy at birth, they can also be used at any other age, by using a life table starting at that age.)

### AGE DECOMPOSITION

Eq. (15) can be decomposed by age category as follows:

$$\begin{aligned} \dot{e}^o(0, t) &= \int_0^{\omega} \rho(a, t) e^o(a, t) f(a, t) da \\ &= \int_0^{x_1} \rho(a, t) e^o(a, t) f(a, t) da + \dots + \int_{x_n}^{\omega} \rho(a, t) e^o(a, t) f(a, t) da \\ &= \frac{\int_0^{x_1} \rho(a, t) e^o(a, t) f(a, t) da}{\int_0^{x_1} f(a, t) da} \int_0^{x_1} f(a, t) da \\ &\quad + \dots + \frac{\int_{x_n}^{\omega} \rho(a, t) e^o(a, t) f(a, t) da}{\int_{x_n}^{\omega} f(a, t) da} \int_{x_n}^{\omega} f(a, t) da. \end{aligned} \quad (27)$$

The averages in this formula can be decomposed using Eq. (10), the formula for the expectation of a product. For the age group  $x_i$  to  $x_{i+1}$ ,

$$\frac{\int_{x_i}^{x_{i+1}} \rho(a, t) e^o(a, t) f(a, t) da}{\int_{x_i}^{x_{i+1}} f(a, t) da} = [\bar{\rho}]_{x_i}^{x_{i+1}} [e^{\dagger}]_{x_i}^{x_{i+1}} + [\text{Cov}_f(\rho, e^o)]_{x_i}^{x_{i+1}}, \quad (28)$$

where  $[\bar{\rho}]_{x_i}^{x_{i+1}}$  is average improvement in mortality in the age group  $x_i$  to  $x_{i+1}$ ,

$$[\bar{\rho}]_{x_i}^{x_{i+1}} = \frac{\int_{x_i}^{x_{i+1}} \rho(a, t) f(a, t) da}{\int_{x_i}^{x_{i+1}} f(a, t) da}, \quad (29)$$

where the number of life-years lost as a result of death  $[e^{\dagger}]_{x_i}^{x_{i+1}}$  in the age group  $x_i$  to  $x_{i+1}$  is defined as

$$[e^{\dagger}]_{x_i}^{x_{i+1}} = \frac{\int_{x_i}^{x_{i+1}} e^o(a, t) f(a, t) da}{\int_{x_i}^{x_{i+1}} f(a, t) da}, \quad (30)$$

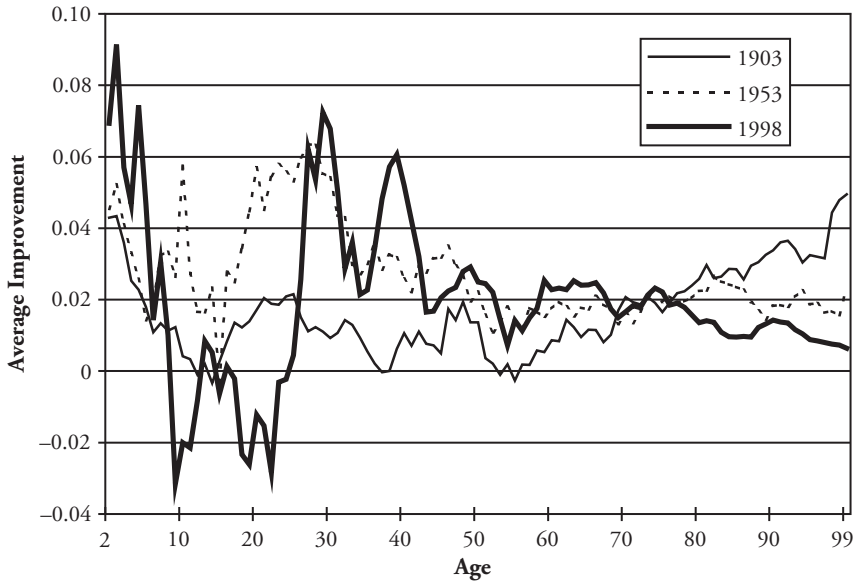
and where the component of the covariance in the age group  $x_i$  to  $x_{i+1}$  is defined as

$$[\text{Cov}_f(\rho, e^o)]_{x_i}^{x_{i+1}} = \frac{\int_{x_i}^{x_{i+1}} [\rho(a, t) - \bar{\rho}(t)][e^o(a, t) - e^{\dagger}(t)] f(a, t) da}{\int_{x_i}^{x_{i+1}} f(a, t) da}, \quad (31)$$

where  $\bar{\rho}(t)$  and  $e^{\dagger}(t)$  are as defined in Eqs. (29) and (30).

If the age category is narrow enough, then  $\rho(a, t)$  and  $e^o(a, t)$  will not vary much within the age category. This implies that the covariance terms will be close to zero. Hence we have the approximation

Figure 1. Five-Year Moving Average of the Improvement in Mortality at Ages 2–99 for Sweden in 1903, 1953, and 1998



$$\begin{aligned}\dot{e}^o(0, t) &\approx [\dot{e}^o]_0^{x_1} + \dots + [\dot{e}^o]_{x_n}^{\omega} \\ &= [\bar{\rho}]_0^{x_1} [e^\dagger]_0^{x_1} [F]_0^{x_1} + \dots + [\bar{\rho}]_{x_n}^{\omega} [e^\dagger]_{x_n}^{\omega} [F]_{x_n}^{\omega},\end{aligned}\quad (32)$$

where  $[F]_{x_i}^{x_{i+1}}$  denotes the proportion of deaths in the age group  $x_i$  to  $x_i + 1$ :

$$[F]_{x_i}^{x_{i+1}} = \int_{x_i}^{x_{i+1}} f(a, t) da. \quad (33)$$

For single years of age this approximation is quite accurate. It can be written as

$$\begin{aligned}\dot{e}^o(0, t) &\approx [\dot{e}^o]_0 + [\dot{e}^o]_1 + \dots + [\dot{e}^o]_{\omega-1} \\ &= \rho(0.5, t) e^o(0.5, t) f(0.5, t) + \rho(1.5, t) e^o(1.5, t) f(1.5, t) \\ &\quad + \dots + \rho(\omega - 0.5, t) e^o(\omega - 0.5, t) f(\omega - 0.5, t),\end{aligned}\quad (34)$$

with the understanding that  $\rho(x + 0.5, t)$  is the rate of progress in reducing mortality,  $e^o(x + 0.5, t)$  is the remaining life expectancy, and  $f(x + 0.5, t)$  is the proportion of deaths, between exact age  $x$  and  $x + 1$ .

The three age-specific components of Eq. (34) are shown in Figures 1, 2, and 3 for Sweden in 1903, 1953, and 1998. Figure 4 shows the value of the resulting age-specific components of the change in life expectancy, the  $[\dot{e}^o]_x$  terms. Because the value of this component was so large at ages 0, 1, and 2 in 1903, Figure 4 is restricted to later ages. In 1903, fully 55% of the change in life expectancy was due to mortality change at ages 0–2.

Figure 2. Remaining Life Expectancy at Ages 0–99 for Sweden in 1903, 1953, and 1998

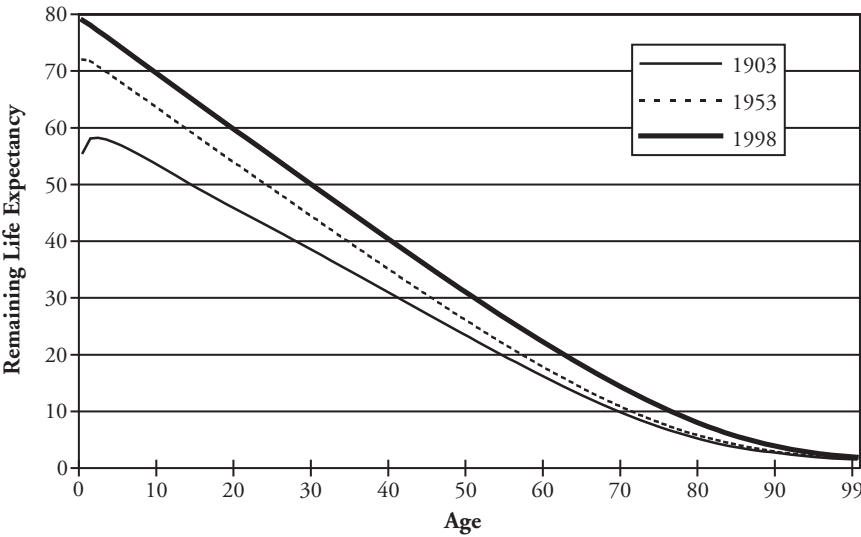


Figure 3. Distribution of Deaths at Ages 0–99 for Sweden in 1903, 1953, and 1998

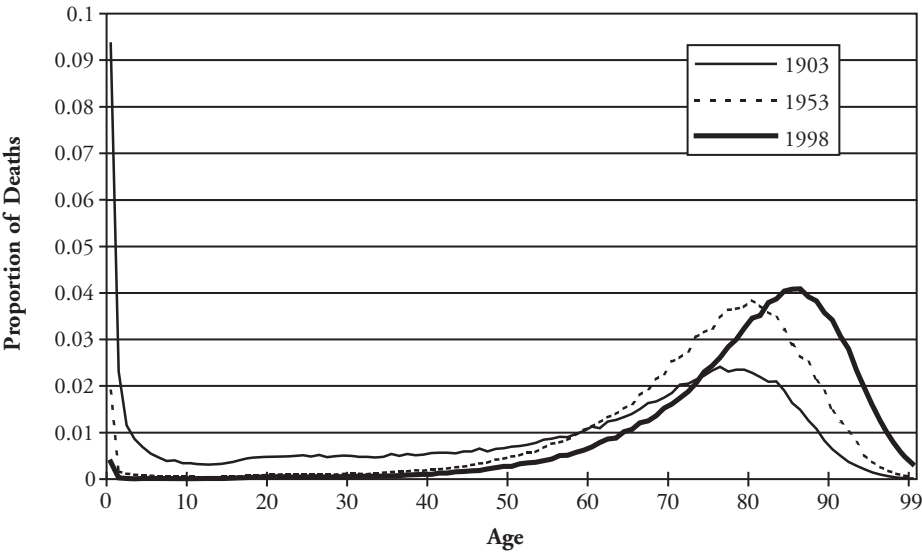
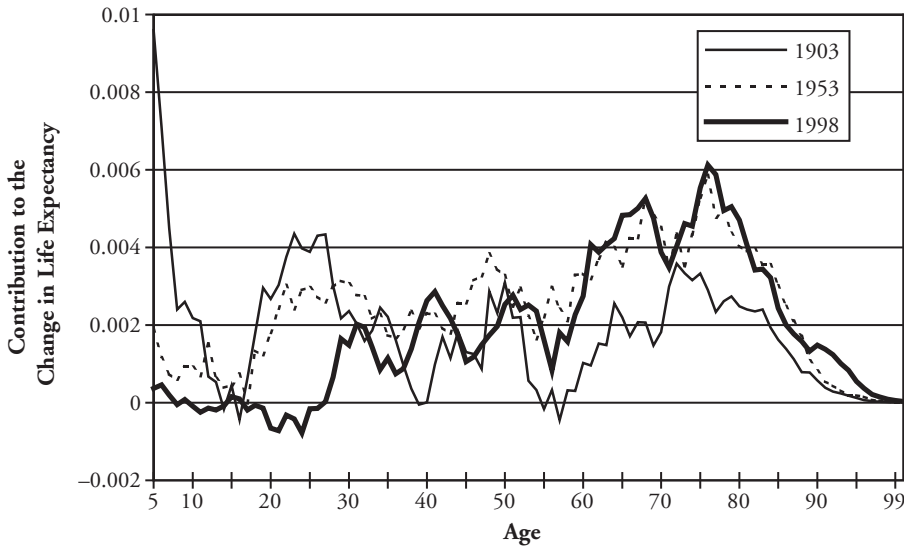




Figure 4. Five-Year Moving Average of the Age Contribution to the Change in Life Expectancy at Birth at Ages 5–99 for Sweden in 1903, 1953, and 1998



## DECOMPOSING LIFE EXPECTANCY BY CAUSE OF DEATH

### A Decomposition Formula

Let  $\mu_i(a, t)$  be the force of mortality from cause of death  $i$  at age  $a$  and time  $t$ . The chance of surviving cause  $i$  (i.e., not dying from cause  $i$ ) is then  $\ell_i(a, t) = e^{-\int_0^a \mu_i(x, t) dx}$ . For competing, independent causes of death  $\ell(a, t) = \ell_1(a, t) \dots \ell_n(a, t)$ . Hence,

$$e^o(0, t) = \int_0^\infty \ell(a, t) da = \int_0^\infty \ell_1(a, t) \dots \ell_n(a, t) da, \quad (35)$$

and

$$\begin{aligned} \dot{e}^o(0, t) &= \int_0^\infty \dot{\ell}_1(a, t) \dots \ell_n(a, t) da + \dots + \int_0^\infty \ell_1(a, t) \dots \dot{\ell}_n(a, t) da \\ &= \int_0^\infty \dot{\ell}_1(a, t) \ell_1(a, t) \dots \ell_n(a, t) da + \dots + \int_0^\infty \dot{\ell}_n(a, t) \ell_1(a, t) \dots \ell_n(a, t) da \\ &= \int_0^\infty \dot{\ell}_1(a, t) \ell(a, t) da + \dots + \int_0^\infty \dot{\ell}_n(a, t) \ell(a, t) da. \end{aligned} \quad (36)$$

Each of the terms in Eq. (36) can be reexpressed using the same logic as explained in the proof of Eq. (12):

$$\begin{aligned} \int_0^\infty \ell(a, t) \dot{\ell}_i(a, t) da &= -\int_0^\infty \ell(a, t) \int_0^a \dot{\mu}_i(x, t) dx da \\ &= -\int_0^\infty \dot{\mu}_i(a, t) \int_a^\infty \ell(x, t) dx da = -\int_0^\infty \dot{\mu}_i(a, t) \ell(a, t) e^o(a, t) da. \end{aligned} \quad (37)$$

Thus

$$\dot{e}^o(0, t) = -\sum_{i=1}^n \int_0^\infty \dot{\mu}_i(a, t) \ell(a, t) e^o(a, t) da. \quad (38)$$

This formula is the continuous version of the discrete-difference formula presented by Pollard (1982, 1988). The change in expectation of life of a population between time 1 and 2 in Pollard's formulation is:

$$e^o(0,2) - e^o(0,1) = \sum_{i=1}^n \int_0^{\omega} [\mu_i(a,1) - \mu_i(a,2)] \frac{\ell(a,2)e^o(a,1) + \ell(a,1)e^o(a,2)}{2} da. \quad (39)$$

Let  $\rho_i(a,t)$  denote the pace of reduction of mortality from cause  $i$ ,  $\rho_i(a,t) = -\dot{\mu}_i(a,t)$ . The proportion of deaths from cause  $i$  at age  $a$  and time  $t$  is  $f_i(a,t) = \mu_i(a,t)\ell(a,t)$ . It then follows from Eq. (38) that

$$\dot{e}^o(0,t) = \sum_{i=1}^n \int_0^{\omega} \rho_i(a,t) e^o(a,t) f_i(a,t) da. \quad (40)$$

Note how concise and elegant Eq. (40) is compared with Eq. (39).

Applying the decomposition in Eq. (10) yields

$$\dot{e}^o(0,t) = \sum_{i=1}^n [\bar{\rho}_i(t) e_i^*(t) + \text{Cov}_{f_i}(\rho_i, e^o)] F_i(t), \quad (41)$$

where

$$F_i(t) = \int_0^{\omega} f_i(a,t) da, \quad (42)$$

$\bar{\rho}_i(t)$  is the average pace of reduction of mortality from cause  $i$ ,

$$\bar{\rho}_i(t) = \frac{\int_0^{\omega} \rho_i(a,t) f_i(a,t) da}{\int_0^{\omega} f_i(a,t) da}, \quad (43)$$

$e_i^*(t)$  is the average number of life-years lost as a result of cause of death  $i$ ,

$$e_i^*(t) = \frac{\int_0^{\omega} e^o(a,t) f_i(a,t) da}{\int_0^{\omega} f_i(a,t) da}, \quad (44)$$

and the covariance is between the rate of improvement in mortality from cause of death  $i$  and the remaining life expectancy at various ages,

$$\text{Cov}_{f_i}(\rho_i, e^o) = \frac{\int_0^{\omega} [\rho_i(a,t) - \bar{\rho}_i(t)] [e^o(a,t) - e_i^*(t)] f_i(a,t) da}{\int_0^{\omega} f_i(a,t) da}. \quad (45)$$

The averages in Eqs. (6), (8), and (11) differ from those in Eqs. (43) to (45), because in the latter equations the denominators do not add to 1; that is,  $F_i(t) = \int_0^{\omega} f_i(a,t) da \neq 1$ .

### An Illustration: Change in Japanese Life Expectancy

The life table distribution of deaths in Japan owing to different causes of death in 1980 and 1990 is shown in Table 2. This is a distribution of causes of death for a life table population in which the proportion of people at each age is determined by life table probabilities of survival.

Table 3 and Figure 5 present the results of applying the decomposition formula in Eq. (41) to the data from Japan. Over the decade 1980–1990, Japanese life expectancy rose from 75.91 to 78.80 years, with an estimated annual increase of  $\dot{e}^o(0,1985) = 0.288$ . Three-fifths of this increase in life expectancy at birth can be attributed to a reduction in mortality from cerebrovascular disease and heart disease. This is indicated in Table 3, where  $\dot{e}_i^o(0)$  for heart disease is 0.044 and  $\dot{e}_i^o(0)$  for cerebrovascular disease is 0.129. The sum, 0.173, accounts for 60% of the total change,  $\dot{e}_i^o(0)$ , of 0.288.

**Table 2. Life Table Distribution of Causes of Death for Japan in 1980 and 1990**

Causes of Death	1980 (%)	1990 (%)
Heart Disease	21.4	23.7
Malignant Neoplasm	18.5	21.6
Cerebrovascular Disease	24.3	16.1
Infectious Diseases	8.6	12.8
Violent Deaths	4.6	4.5
Stomach, Liver, and Kidney Disorders	4.3	4.3
Senility Without Psychosis	7.4	5.0
Other Causes	10.9	12.0
All Causes of Death	100.0	100.0

*Source:* Authors' calculations described in Appendix B, based on the Berkeley Mortality Database (2001).

*Notes:* The underlying data pertain to five-years age groups. Heart disease includes hypertensive disease. Other causes of death are those denoted in the Berkeley Mortality Database (2001) as other causes, plus congenital malformations and diabetes mellitus. Infectious diseases include pneumonia and bronchitis.

On average, death rates from malignant neoplasms and infectious diseases increased, yielding negative values of  $\bar{p}_i$  and negative level-1 changes. Conversely, the level-2 changes for these causes of death had positive values because improvements were made at younger ages with high remaining life expectancy. As a result of the balance between level-1 and level-2 change, the final column of Table 3 shows only positive contributions for all the causes of death.

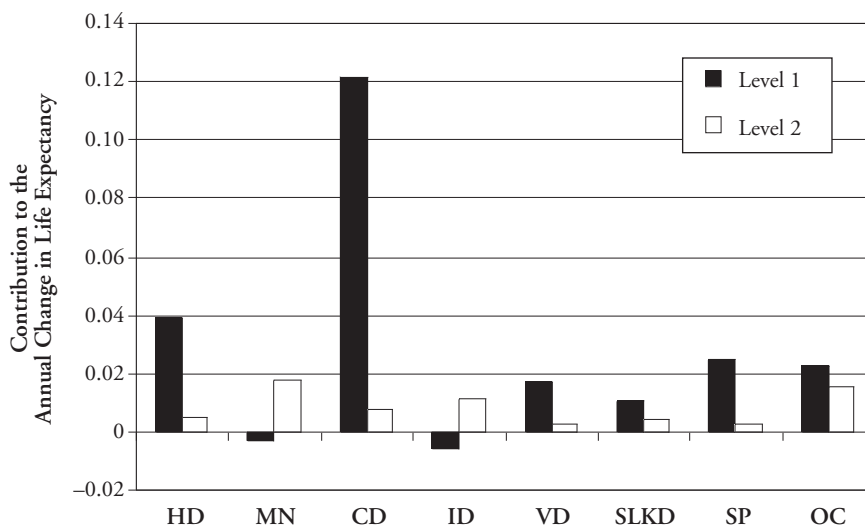
**Table 3. Cause-of-Death Decomposition for the Annual Change Over Time in Life Expectancy for Japan Around January 1, 1985**

Causes of Death	$\bar{p}_i$ (%)	$e_i^\dagger$	$\bar{p}_i e_i^\dagger$	$\text{Cov}_{f_i}(\bar{p}_i, e^\theta)$	$F_i$ (%)	$\dot{e}_i^\theta(0)$
Heart Disease	2.058	8.333	0.172	0.022	22.543	0.044
Malignant Neoplasm	-0.098	13.276	-0.013	0.088	20.042	0.015
Cerebrovascular Disease	6.979	8.594	0.600	0.038	20.226	0.129
Infectious Diseases	-0.703	7.942	-0.056	0.104	10.718	0.005
Violent Deaths	1.608	23.384	0.376	0.058	4.516	0.020
Stomach, Liver, and Kidney Disorders	2.168	11.548	0.250	0.094	4.294	0.015
Senility Without Psychosis	9.379	4.294	0.403	0.040	6.200	0.027
Other Causes	1.432	13.792	0.197	0.137	11.461	0.038
All Causes of Death	2.675	10.527	0.282	0.007	100.000	0.288

*Source:* Authors' calculations described in Appendix B, based on the Berkeley Mortality Database (2001).

*Notes:* The underlying data pertain to five-years age groups. The values of  $\bar{p}_i$ ,  $e_i^\dagger$ ,  $\bar{p}_i e_i^\dagger$ , and  $\text{Cov}_{f_i}(\bar{p}_i, e^\theta)$  for all causes of death are complicated functions and not simple sums of the corresponding cause-specific values. The value of  $\dot{e}^\theta(0)$  for all causes of death is slightly different than the sum of the values of  $\dot{e}_i^\theta$  because of approximation errors.

**Figure 5. Cause-of-Death Decomposition of the Annual Change Over Time in Life Expectancy for Japan Around January 1, 1985**



Note: HD = heart disease; MN = malignant neoplasm; CD = cerebrovascular disease; ID = infectious diseases; VD = violent deaths; SLKD = stomach, liver, and kidney disorders; SP = senility without psychosis; and OC = other causes.

## CAUSE-SPECIFIC ENTROPY

Keyfitz (1977) derived a formula to study the effects that health improvements have on the change of life expectancy over time

$$\dot{e}^o(t) = \sum_{i=1}^n \rho_i(t) \mathcal{H}_i(t), \quad (46)$$

where  $\rho_i(t)$  represents the pace of the improvement, which is assumed to be the same at all ages, at time  $t$  for the  $i$ th cause of death, and  $\mathcal{H}_i(t)$  is the entropy of the  $i$ th cause of death,

$$\mathcal{H}_i(t) = - \frac{\int_0^\omega \ell(a,t) \ln[\ell_i(a,t)] da}{\int_0^\omega \ell(a,t) da}. \quad (47)$$

Keyfitz's describes this entropy as "a measure of the length of time from the nondeath from the  $i$ th cause up to the time when the person dies from the next thing that will hit him" (1977:414). To understand Keyfitz's remark, consider Eq. (18), which indicates that the number of life-years gained when a death is averted is given by the product of life expectancy at birth and the entropy of the survival function. Similarly, the number of life-years gained when a death from cause  $i$  is averted is given by the product of life expectancy at birth and the cause-specific entropy, adjusted by the proportion of deaths from cause  $i$ :

$$e_i^*(t) = \frac{\mathcal{H}_i(t) e^o(0,t)}{F_i(t)}. \quad (48)$$

This result can be derived from Eqs. (44) and (47) using the same strategy, with reversal of the order of integration, used to derive Eq. (18).

Substituting Eq. (48) in Eq. (41), and dividing by the life expectancy yields

$$\dot{e}^o(t) = \sum_{i=1}^n \left( \bar{p}_i(t) \mathcal{H}_i(t) + \frac{\text{Cov}_{f_i}(\rho_i, e^o) F_i(t)}{e^o(0, t)} \right), \quad (49)$$

which generalizes Keyfitz's Eq. (46) to the case in which rates of reduction in cause-specific mortality can vary from age to age.

## DISCUSSION

Over the past two decades, decomposition of change in life expectancy has been a mainstay of demographic analysis. Almost all the many applications have concerned discrete changes in life expectancy, with Arriaga's formulation being particularly popular. Keyfitz's research on time derivatives of life expectancy has been of theoretical interest largely because of the restrictive, unrealistic assumption that the pace of mortality change is constant at all ages.

Time and age are continuous and calculus is elegant, but data are discrete. In this article, we derived decomposition formulas for time derivatives of life expectancy: our formulas involved derivatives and integrals. Pollard (1982, 1988) studied discrete differences in life expectancy between two points in time, using formulas that involve integrals over age. Arriaga (1984) analyzed discrete differences in life expectancy using formulas that take sums over age. All three approaches are closely related when applied to actual data pertaining to time intervals of a few years or less. Depending on the approach, either the formulas or the estimation procedures involve approximations or inelegancies. In implementing our approach we used the approximations described in Appendix B.

Pollard (1988) explained the underlying similarity of his method to Arriaga's; his method can also be shown to be similar, in empirical applications, to ours. Hence it is no surprise that Arriaga's method for decomposing change in life expectancy by age yields the same results as the Vaupel-Canudas method for Sweden around 1998, as shown in Table 4. Similarly, decomposition of change in life expectancy by cause of death using traditional methods will generally produce essentially the same results as our new method. Why, then, should demographers consider the methods developed in this article? There are three main reasons.

First, our method permits further decomposition of age-specific and cause-specific effects into the effects—for each age category or for each cause of death—of the pace of mortality improvement, remaining life expectancy, and the frequency of deaths.

Second, our method permits decomposition of change in life expectancy into the general impact of mortality improvement at all ages (our “level-1” effect) and the additional effect of heterogeneity in the age-specific rates of improvement (our “level-2” effect). The general impact can be further decomposed into the average rate of mortality improvement multiplied by the average number of life-years saved. We conjecture that this kind of decomposition will lead to more interesting demographic insights than Arriaga's distinction between the direct and indirect effects of mortality improvements.

Third, our formulas are both elegant and exact. It is necessary to use approximations when applying them to data, but this is a minor drawback. Because the formulas are elegant, they aid understanding and permit deeper comprehension of the demographic factors that drive change in life expectancy. The formulas are thus in the spirit of Nathan Keyfitz's enduring contribution to demographic research.

**Table 4.** Age Decomposition of the Annual Change Over Time in Life Expectancy Using Arriaga's and Vaupel-Canudas' Decompositions, for Sweden Around January 1, 1998

Age Group	Arriaga	Vaupel-Canudas
0-9	0.013	0.013
10-19	-0.002	-0.002
20-29	-0.001	-0.001
30-39	0.014	0.014
40-49	0.020	0.020
50-59	0.021	0.021
60-69	0.045	0.045
70-79	0.050	0.050
80-89	0.025	0.025
90+	0.007	0.007
All Ages	0.191	0.191

*Source:* Authors' calculations described in Appendix B. Life table data are derived from the Human Mortality Database (2002). Life table values from the years 1995 and 2000 were used to obtain results for the first of January, 1998. The data used pertain to single years of age. The calculation of the Vaupel-Canudas estimates was based on Eq. (32), with the integrals in that formula approximated by sums, as described in Appendix B. Values were calculated for each one-year age category and then summed to obtain the 10-year and open-ended categories in the table. Similar estimates, for both the Arriaga and Vaupel-Canudas methods are obtained if data from abridged life tables, with five-year age groups, are used.

## APPENDIX A: AN ALTERNATIVE DECOMPOSITION

Let  $\tilde{\rho}(t)$  denote the average value of the age-specific rate of progress in reducing death rates,  $\rho(a,t)$ , weighted by the product of the remaining life expectancy  $e^o(a,t)$  and the probability density function  $f(a,t)$ ,

$$\tilde{\rho}(t) = \frac{\int_0^{\infty} \rho(a,t) e^o(a,t) f(a,t) da}{\int_0^{\infty} e^o(a,t) f(a,t) da}. \quad (50)$$

Note that  $\tilde{\rho}(t)$  is the loss-weighted average pace of mortality improvement. Eq. (15) implies that the change in life expectancy can be decomposed as

$$\dot{e}^o(0,t) = \tilde{\rho}(t) e^i(t). \quad (51)$$

We are in the process of applying Eq. (51) to analyze the dynamics of life expectancy.

## APPENDIX B: DISCRETE APPROXIMATIONS

If data are available for time  $t$  and  $t + h$ , then we generally used the following approximations for the value at the midpoint  $t + h/2$ . For the relative derivative of the function  $v(a, t + h/2)$ , we used

$$\dot{v}(a, t + h/2) \approx \frac{\ln \left[ \frac{v(a, t + h)}{v(a, t)} \right]}{h}. \quad (52)$$

The value of the function at the midpoint  $v(a, t + h/2)$  was estimated by

$$v(a, t + h/2) \approx v(a, t) e^{(h/2) \dot{v}(a, t + h/2)}. \quad (53)$$

Substituting the right-hand side of Eq. (52) for  $\dot{v}(a, t + h/2)$  in Eq. (53) yields the equivalent approximation

$$v(a, t + h/2) \approx [v(a, t)v(a, t + h)]^{1/2}. \quad (54)$$

This is a standard approximation in demography (Preston, Heuveline, and Guillot 2001). The derivative of the function  $v(a, t + h/2)$  was estimated by

$$\dot{v}(a, t + h/2) = \dot{v}(a, t + h/2)v(a, t + h/2). \quad (55)$$

We used Eqs. (52), (53), and (55) wherever we thought that the rate of change was more or less constant over the time interval. In some cases it seemed appropriate to assume that change in the interval was linear. This was the case when we estimated the change over time in the survival function  $\ell(a, t)$  and in life expectancy  $e^o(a, t)$ . Then we used

$$v(a, t + h/2) \approx \frac{v(a, t + h) + v(a, t)}{2}. \quad (56)$$

and

$$\dot{v}(a, t + h/2) \approx \frac{v(a, t + h) - v(a, t)}{h}. \quad (57)$$

The period force of mortality in an interval, for all causes of death, was calculated using a formula similar to Eq. (52). If data were available for ages  $a$  and  $a + k$ , we used the following approximation

$$\mu(a + k/2, t) \approx -\frac{\ln \left[ \frac{\ell(a + k, t)}{\ell(a, t)} \right]}{k}. \quad (58)$$

In Tables 2 and 3, we estimated the force of mortality for the  $i$ th cause of death by multiplying the result of Eq. (58) by the proportion of deaths from cause  $i$ ,  $D_i(a + k/2, t)$ , of the total deaths in the age group,  $D(a + k/2, t)$ ,

$$\mu_i(a + k/2, t) \approx \mu(a + k/2, t) \left[ \frac{D_i(a + k/2, t)}{D(a + k/2, t)} \right]. \quad (59)$$

The rate of progress in reducing death rates  $\rho(a + k/2, t + h/2)$  was calculated as

$$\rho(a + k/2, t + h/2) = -\dot{\mu}(a + k/2, t + h/2) \approx -\frac{\ln \left[ \frac{\mu(a + k/2, t + h)}{\mu(a + k/2, t)} \right]}{h}. \quad (60)$$

Because the forces of mortality in Eqs. (58) and (59), and  $\rho(a + k/2, t + h/2)$  in Eq. (60) are at ages  $a + k/2$ , it was necessary to calculate the other functions involved in the decomposition at those ages. We calculated the survivorship function  $\ell(a, t)$  and the remaining life expectancy  $e^o(a, t)$  at age  $a + k/2$  using a formula analogous to Eq. (54). The life table distribution of deaths from cause  $i$  was calculated as

$$f_i(a + k/2, t) \approx \mu_i(a + k/2, t)\ell(a + k/2, t). \quad (61)$$

For the estimation of some formulas, we substituted sums for integrals. Eq. (34) shows how we estimated the right-hand side of Eqs. (15) and (32).

## REFERENCES

Andreev, E.M. 1982. "Method Komponent v Analize Prodoljitelnosty Zjizni" [The Method of Components in the Analysis of Length of Life]. *Vestnik Statistiki* 9:42–47.

- Andreev, E.M., V. Shkolnikov, and A.Z. Begun. 2002. "Algorithm for Decomposition of Differences Between Aggregate Demographic Measures and Its Application to Life Expectancies, Gini Coefficients, Health Expectancies, Parity-Progression Ratios and Total Fertility Rates." *Demographic Research* 7:499–522. Available on-line at <http://www.demographic-research.org/>
- Arriaga, E.E. 1984. "Measuring and Explaining the Change in Life Expectancies." *Demography* 21:83–96.
- Berkeley Mortality Database. University of California, Berkeley (USA). Available on-line at [www.demog.berkeley.edu/wilmoth/mortality/](http://www.demog.berkeley.edu/wilmoth/mortality/) (data downloaded on September 18, 2001).
- Das Gupta, P. 1978. "A General Method of Decomposing a Difference Between Two Rates Into Several Components." *Demography* 15:99–112.
- . 1993. *Standardization and Decomposition of Rates: A User's Manual*. Current Population Reports, Series P-23, No. 186. Washington, DC: U.S. Bureau of the Census.
- Demetrius, L. 1979. "Relations Between Demographic Parameters." *Demography* 16:329–38.
- Goldman, N. and G. Lord. 1986. "A New Look at Entropy and the Lifetable." *Demography* 23:275–82.
- Hill, G. 1993. "The Entropy of the Survival Curve: An Alternative Measure." *Canadian Studies in Population* 20:43–57.
- Hokkert, R. 1987. Lifetable Transformations and Inequality Measures: Some Noteworthy Formal Relations. *Demography* 24:615–22.
- Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available on-line at [www.mortality.org](http://www.mortality.org) or [www.humanmortality.de](http://www.humanmortality.de) (data downloaded on May 23, 2002).
- Keyfitz, N. 1977. "What Difference Does it Make if Cancer Were Eradicated? An Examination of the Taeuber Paradox." *Demography* 14:411–18.
- . 1985. *Applied Mathematical Demography*. 2nd ed. New York: Springer.
- Kim, Y.J. and D.M. Strobino. 1984. "Decomposition of the Difference Between Two Rates With Hierarchical Factors." *Demography* 21:361–72.
- Kitagawa, E.M. 1955. "Components of a Difference Between Two Rates." *American Statistical Association Journal* 50(272):1168–94.
- Mitra, S. 1978. "A Short Note on the Taeuber Paradox." *Demography* 15:621–23.
- Pollard, J.H. 1982. "The Expectation of Life and Its Relationship to Mortality." *Journal of the Institute of Actuaries* 109:225–40.
- . 1988. "On the Decomposition of Changes in Expectation of Life and Differentials in Life Expectancy." *Demography* 25:265–76.
- Pressat, R. 1985. "Contribution des Écarts de Mortalité par Âge à la Différence des Vies Moyennes" [The significance of variations in mortality by age on differences in life expectancy]. *Population* 4–5:765–70.
- Preston, S.H., P. Heuveline, and M. Guillot. 2001. *Demography: Measuring and Modeling Population Processes*. Oxford: Blackwell Publishers.
- United Nations. 1982. *Levels and Trends of Mortality Since 1950, Study 74*. New York: United Nations, Department of International Economic and Social Affairs.
- Vaupel, J.W. 1986. "How Change in Age-Specific Mortality Affects Life Expectancy." *Population Studies* 40:147–57.
- . 1992. "Analysis of Population Changes and Differences." Paper presented at the annual meeting of the Population Association of America, Denver, April 30–May 2.
- Vaupel, J.W. and V. Canudas Romo. 2000. "How Mortality Improvement Increases Population Growth." Pp. 350–57 in *Optimization, Dynamics and Economic Analysis: Essays in Honor of Gustav Feichtinger*, edited by E.J. Dockner, R.F. Hartl, M. Luptacik, and G. Sorger. Heidelberg: Springer. Available on-line at <http://www.demogr.mpg.de/Papers/Working/wp-1999-015.pdf>
- . 2002. "Decomposing Demographic Change into Direct vs. Compositional Components." *Demographic Research* 7:1–14. Available on-line at <http://www.demographic-research.org/>