Solutions - Exercises in Pension Systems

University of Copenhagen, Autumn 2021

20 September – 26 September (Week 3)

Exercise 1: Fertility and population projections

See the markdown file for solutions to Exercise 1.

Exercise 2: Stable population theory

We first recall that the age-composition in a stable population is given by

$$c(x) = be^{-rx}S(x), \tag{1}$$

where $b = 1/\int_0^\infty e^{-rx}S(x)dx$ is the birth rate.

Question 2.1: Inserting the expressions for T^r and c we get

$$\begin{aligned}
\int_0^\omega T^r(x)c(x)dx &= \int_0^\omega \left(\int_x^\omega e^{-r(y-x)}\frac{S(y)}{S(x)}dy\right) be^{-rx}S(x)dx \\
&= b\int_0^\omega \left(\int_x^\omega e^{-ry}S(y)dy\right) dx = b\int_0^\omega \left(\int_0^y e^{-ry}S(y)dx\right) dy \\
&= b\int_0^\omega ye^{-ry}S(y)dy = \int_0^\omega yc(y)dy.
\end{aligned}$$

The latter expression can be interpreted as the average age in the population or, equivalently, the average number of years lived for a person chosen at random from the population.

Question 2.2: Start by noting that $(aS(x+a))' = aS'(x+a) + S(x+a) = -a\mu(x+a)S(x+a) + S(x+a)$, where $'$ means differentiation w.r.t. a. We get

$$\begin{aligned}
e_x &= \frac{1}{S(x)}\int_0^{\omega-x} a\mu(x+a)S(x+a)da = \frac{1}{S(x)}\int_0^{\omega-x} S(x+a) - (aS(x+a))'da \\
&= \frac{1}{S(x)}\int_x^\omega S(y)dy - [aS(x+a)]_0^{\omega-x} = T^0(x),
\end{aligned}$$

where the last equality uses $[aS(x+a)]_0^{\omega-x} = 0 - 0 = 0$, since $S(\omega) = 0$.

In a stationary population we have $r = 0$, and it then follows from Q.2.1 and the relation just shown that $\int_0^\omega xc(x)dx = \int_0^\omega T^0(x)c(x)dx = \int_0^\omega e_x c(x)dx$. The relation states that in a stationary population, the average age (= years lived) equals the average remaining life expectancy.

Question 2.3: For a stationary population $c(x) = S(x)/e_0$. Let $f(y) = f(y|0) = \mu(y)S(y)$.

$$\begin{aligned} g(a) &= \int_0^{\omega-a} f(a|x)c(x)dx = \int_0^{\omega-a} \mu(x+a)\frac{S(x+a)}{S(x)}\frac{S(x)}{e_0}dx \\ &= \frac{1}{e_0}\int_0^{\omega-a} \mu(x+a)S(x+a)dx = \frac{1}{e_0}\int_a^{\omega} f(x)dx = \frac{S(a)}{e_0} = c(a). \end{aligned}$$

Question 2.4: The remaining ages is a sample from the distribution with density g. Thus the data can be used to estimate g and thereby c, which is the age-composition. Since the population is assumed to be stationary we have

$$c(0) = \frac{S(0)}{e_0} = \frac{1}{e_0}, \quad S(x) = e_0 c(x) = \frac{c(x)}{c(0)} = \frac{g(x)}{g(0)}, \quad \mu(x) = -\frac{S'(x)}{S(x)} = -\frac{g'(x)}{g(x)}.$$

Regarding precision of estimation, c equals g and is thus the easiest to estimate, then S being a ratio of two g-values, while μ is the hardest to estimate, since it requires an estimate of the derivative of g.

Exercise 3: The stable population equivalent

See the markdown file for solutions to Exercise 3.

Exercise 4 (BONUS): Population momentum

See the markdown file for solutions to Exercise 4.