Pension Systems / Demography & Mortality

Lecture notes: Demography – part III

University of Copenhagen, Autumn 2021

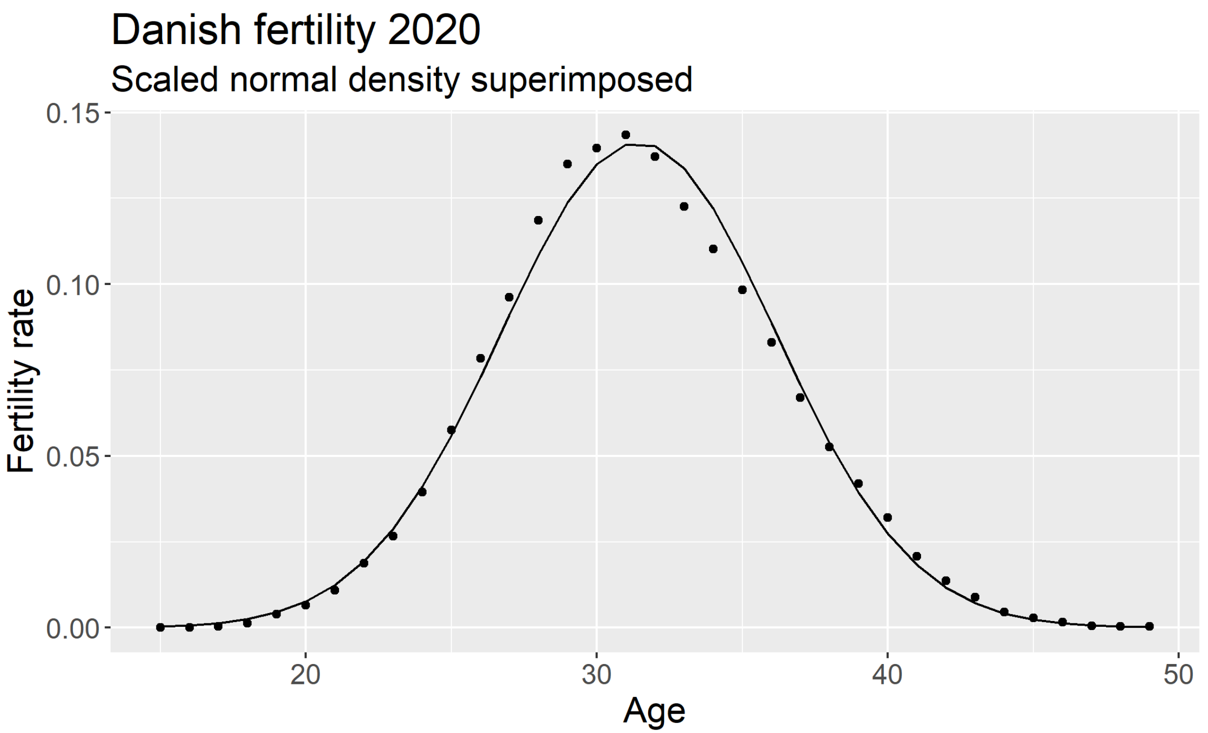
Snorre Jallbjørn & Søren F. Jarner

Recap – Danish fertility

- Total fertility rates from Statistics Denmark
  - Data: $\{\bar{F}(x,t)\}_{x=15,\dots,49,t=1973,\dots,2017}$
  - Figure shows data for $t = 2017$

- Summary statistics used for modelling
  - $TFR(t) = \sum_{x=15}^{49} \bar{F}(x,t)$
  - Mean age of mother$(t) = \bar{x}(t) = \sum_x x \frac{\bar{F}(x,t)}{TFR(t)}$
  - SD age of mother$(t) = \sqrt{\sum_x [x - \bar{x}(t)]^2 \frac{\bar{F}(x,t)}{TFR(t)}}$

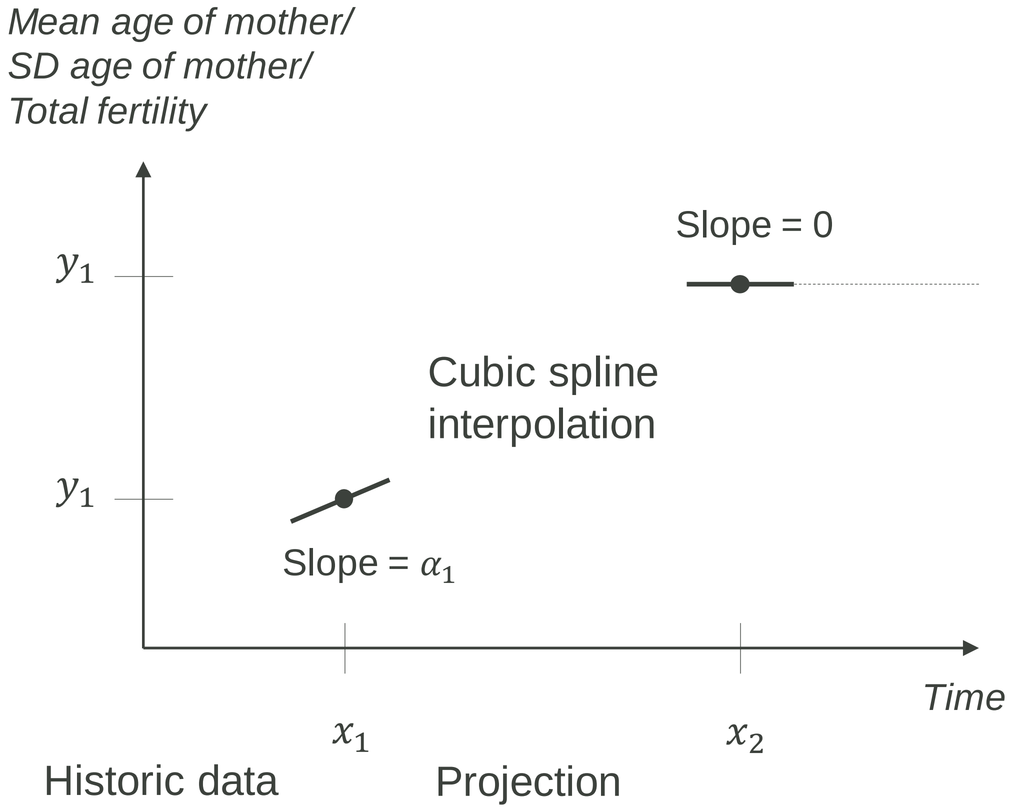
- Approximation by scaled normal density
  - $\bar{F}(x,t) \approx \frac{1}{\sqrt{2\pi}} \frac{TFR(t)}{SD(t)} \exp\left\{-\frac{1}{2}\left(\frac{x-\bar{x}(t)}{SD(t)}\right)^2\right\}$



Recap – Fertility projection methodology

- Smooth transition from current level to asymptotic level
  - Cubic spline interpolation over given horizon
  - $P(x) = a + b(x - x_1) + c(x - x_1)^2 + d(x - x_1)^3$
  - $P'(x) = b + 2c(x - x_1) + 3d(x - x_1)^2$
  - Initial conditions: $P(x_1) = y_1, \;\; P'(x_1) = \alpha_1$
  - Terminal conditions: $P(x_2) = y_2, \;\; P'(x_2) = 0$

- Solution
  - $a = y_1$ $\qquad b = \alpha_1$
  - $c = 3\frac{y_2 - y_1}{(x_2 - x_1)^2} - 2\frac{\alpha_1}{x_2 - x_1}$ $\qquad d = \frac{\alpha_1}{(x_2 - x_1)^2} - 2\frac{y_2 - y_1}{(x_2 - x_1)^3}$



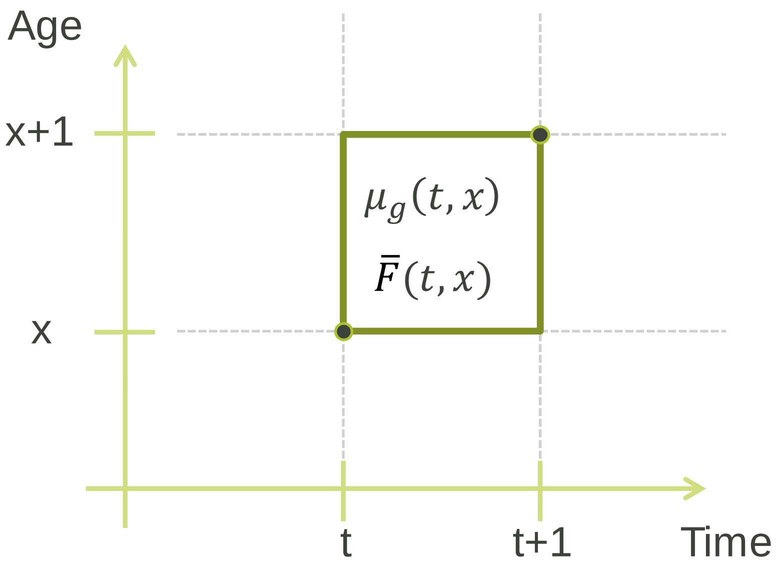
Recap – Population projection

- We use the following (crude) population projection
  - Projection of "1 January"-population at integer ages
  - No immigration or emigration

- Assume given vital rates
  - Gender-specific, mortality surface: $\mu_g(t,x)$, $g \in \{F,M\}$
  - Fertility rate at age x: $\bar{F}(t,x)$
    - Proportion of female births: ρ (≈ 0.487 in DK)
    - $F(t,x) = \rho\bar{F}(t,x)$

- Dynamics (for integer values of x and t)
  - $N_g(t+1,x+1) = N_g(t,x)\exp\left(-\mu_g(t,x)\right)$
  - $N_F(t+1,0) = \rho\sum_x N_F(t,x)\,\bar{F}(t,x)$
  - $N_M(t+1,0) = (1-\rho)\sum_x N_F(t,x)\,\bar{F}(t,x)$

Cellwise constant vital rates indexed by lower-left corner



#1

Stable population theory

1. Stable population theory

- Definitions
  - A population with an age distribution which remains constant over time is referred to as **stable**.
  - A stable population of constant size is said to be **stationary**.

- Sharpe and Lotka (1911) showed that
  - *Assuming constant age-specific fertility and mortality rates, and zero net migration in all ages*
  - *A stable population will arise over time, and it is independent of the initial age distribution*

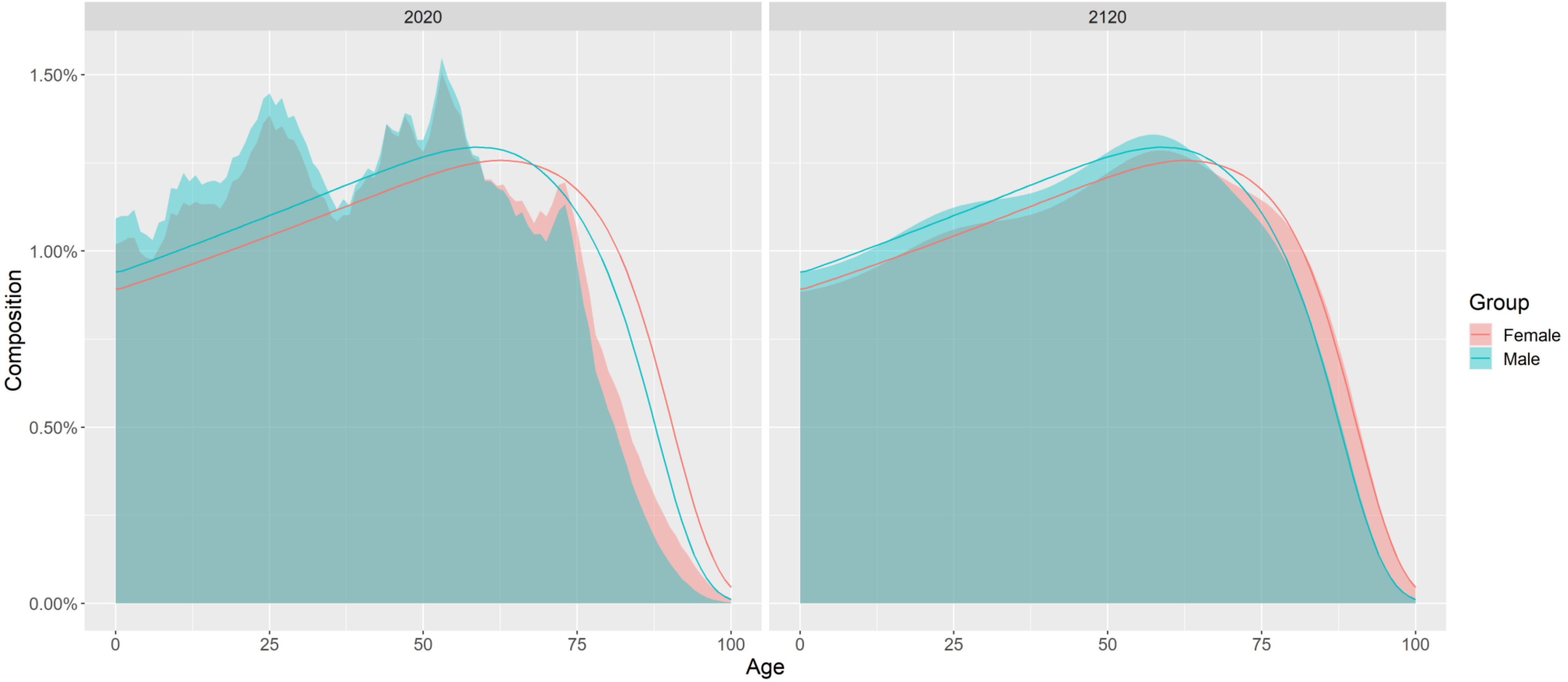
- Stable populations enjoy a number of explicit relations between demographic quantities
  - Useful for examining long-term implications of current demographic patterns
  - Can also be used to estimate demographic parameters in populations assumed to be stable
  - We will use it to gain theoretical insights and to verify our population projections

1. The stable equivalent population

- The stable equivalent population of a given population is the population that would eventually arise, if current (or otherwise specified) mortality and fertility rates were to continue indefinitely

The Danish population 2020 and its stable equivalent ($r \approx -0.65\%$)

Projection to 2120 using 2020 rates



#2

Population momentum

2. Momentum of population growth

Population Momentum

Population momentum refers to the tendency of a population to continue to grow after replacement-level fertility has been achieved. A population that has achieved replacement or below-replacement fertility may still continue to grow for some decades because past high fertility leads to a high concentration of people in the youngest ages. Total births continue to exceed total deaths as these youths become parents. Eventually, this large group becomes elderly and deaths increase to equal the number of births or outnumber them. Thus it may take two or three generations (50 to 70 years) before each new birth is offset by a death in the population. Although replacement-level fertility was reached in Sweden by the late 1960s, there are still about 22,000 more births than deaths each year.

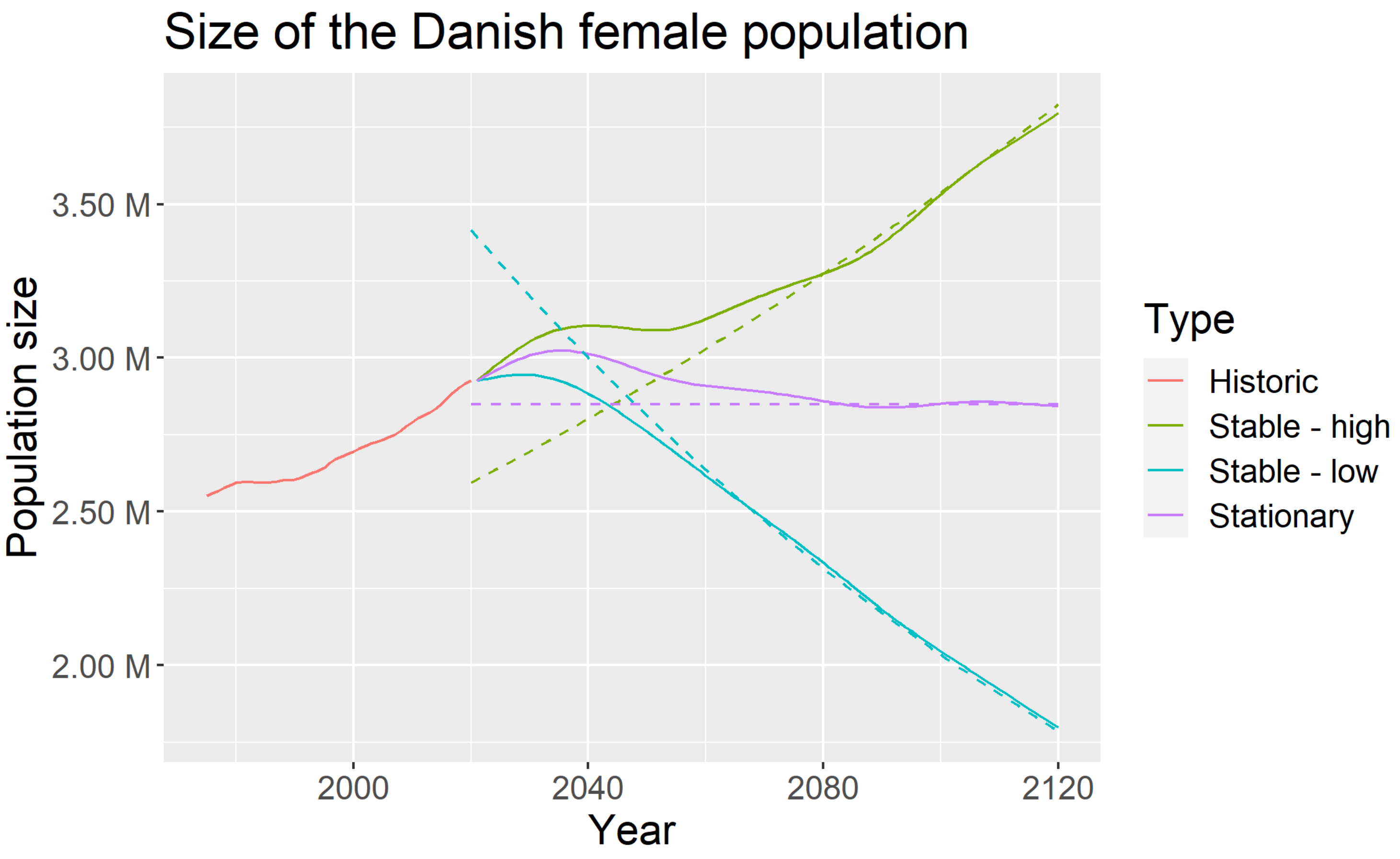
Replacement-level fertility means $NRR = 1$

Description from the Population Handbook p.13

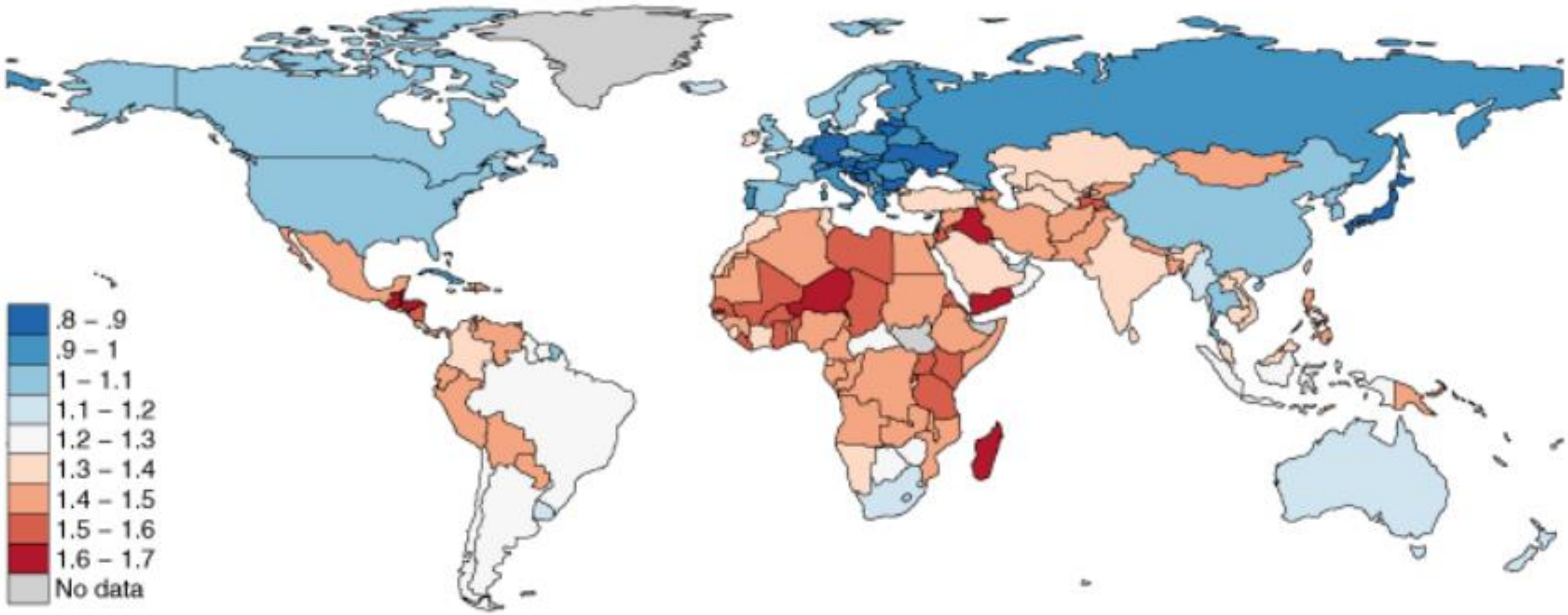
- Population momentum can be studied more formally using stable population theory
  - Keyfitz posed the question: What happens to the size of a previously growing population if its fertility rates are immediately reduced to the replacement level and maintained at this level hereafter?
    - Formally: Consider a population with survival function S and fertility rates F such that $NRR \equiv \int_0^\infty S(x)F(x)dx = 1$. Such a population will converge to a stationary population of size N_S, say.
    - Denote the current size of the population by N, and define the (total) **momentum** as the ratio $M = N_S/N$.

2. Example

- Illustration of projected and asymptotic size of the Danish female population in three cases:
  - Stable equivalent for 2018 (‘Stable - low’: $r \approx -0.6\%$); 40% higher fertility (‘Stable - high’: $r \approx 0.4\%$); Stationary ($r = 0\%$)



2. Population momentum estimates (2010)



Source (data and country classifications): UNPD (2011) World Population Prospects, the 2010 Revision, POP/DBIWPP/Rev.2010/03/F02

2. Population momentum of the world (2010)

- Population momentum coefficients vary from 0.83 (Germany) to 1.69 (Guatemala)

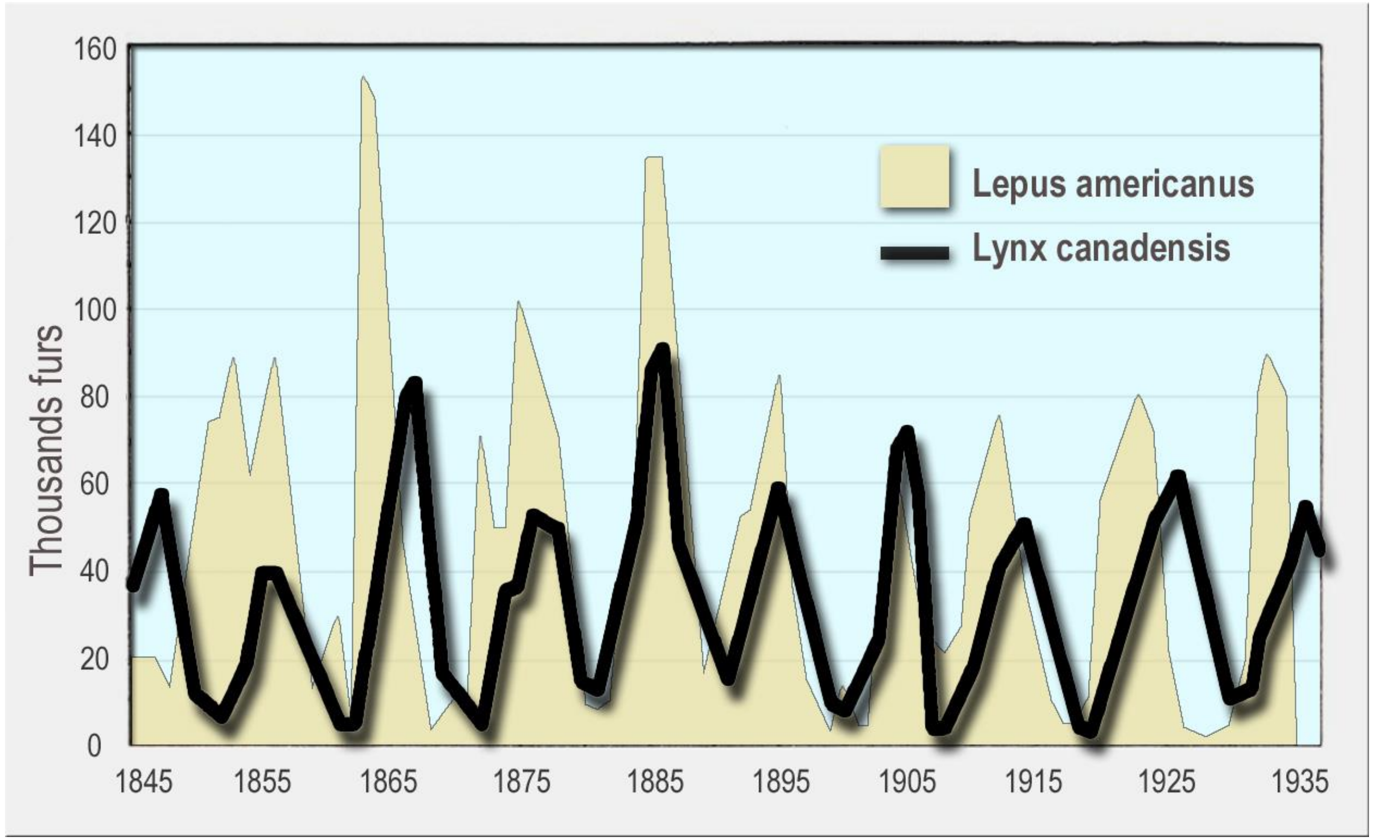
- High momentum
  - Largest momentum where birth rates are high and life expectancy is also (relatively) high
  - This is the case for most countries in Central America; regional momentum is estimated at 1.50
  - Sub-Saharan Africa, Northern Africa and Western Asia all have momentum estimates around 1.45

- Low momentum
  - East, West and Southern Europe are regions with negative population growth (M<1)
  - Even with an immediate rebounding of fertility rates back to replacement level Europe's population as a whole is projected to fall by five percent from current numbers
  - In Germany, the total population would even decline by 17% (M=0.83)
  - Outside Europe, Japan is the most notable country with negative growth (M=0.89)

#3

The Lotka-Volterra equations

Example: Canada lynxes eat snowshoe hares



- Numbers of snowshoe hare (yellow) and Canada lynx (black line) furs sold to the Hudson's Bay Company.

Example: Simulated population

Lotka-Volterra equations

$$\frac{dx}{dt} = \alpha x - \beta xy$$

$$\frac{dy}{dt} = -\gamma y + \delta xy$$

Example

- **Initial population**
  - $x(0) = 10, y(0) = 10$
- **Prey**
  - $\alpha = 1.1, \beta = 0.4$
- **Predators**
  - $\gamma = 0.1, \delta = 0.4$