Exercises in Pension Systems

University of Copenhagen, Autumn 2021

27 September – 03 October (Week 4)

Exercise 1: Comparing lifetime distributions

Let p_X and p_Y denote the densities of two positive, continuous random variables X and Y, which we interpret as lifetimes. The *survivorship probability* of X over Y is defined as the probability that X outlives Y assuming independence between the two lifetimes, i.e.

$$SV(X,Y) = P(X > Y) = \int_0^\infty \int_0^x p_X(x) p_Y(y) dy dx. \tag{1}$$

The *overlapping coefficient* between X and Y is a measure of similarity and is defined as

$$OL(X,Y) = OL(Y,X) = \int_0^\infty \min\{p_X(z), p_Y(z)\} dz. \tag{2}$$

Note, that if $p_X = p_Y$ then $SV(X,Y) = 1/2$ while $OL(X,Y) = 1$.

We will now compute SV and OL in a special case. Let μ denote a given mortality function, i.e. $\mu(x)$ is the force of mortality at age x. For $a > 0$, we let μ_a denote the mortality function given by

$$\mu_a(x) = a\mu(x) \text{ for } x > 0. \tag{3}$$

Let X^a denote a random variable (lifetime) with force of mortality μ_a. Further, let p_a and S^a denote the corresponding density and survival function, respectively.

Question 1.1: Show $SV(X^a, X^b) = \int_0^\infty \mu_b(y) \exp\{-\int_0^y \mu_a(x) + \mu_b(x) dx\} dy$. Then use the definition of μ_a and μ_b to conclude that $SV(X^a, X^b) = b/(a+b)$. Comment on the result.

Assume for the remaining questions that $\mu(x) > 0$ and consider $a > b > 0$. Assume that the densities p_a and p_b cross each other exactly once, i.e. assume that there exists x_{ab} such that $p_a(x_{ab}) = p_b(x_{ab})$, $p_b(x) < p_a(x)$ for $x < x_{ab}$, and $p_b(x) > p_a(x)$ for $x > x_{ab}$.

Question 1.2: Show that x_{ab} satisfies $\int_0^{x_{ab}} \mu(y) dy = \log\left(\frac{a}{b}\right)/(a-b)$.

Question 1.3: Express $OL(X^a, X^b)$ in terms of the survival functions, and use this expression together with Q1.2 to find a formula, in terms of a and b only, for $OL(X^a, X^b)$.

Question 1.4: Compute $SV(X^a, X^b)$ and $OL(X^a, X^b)$ for $a = 1.25$ and $b = 0.75$. Can you provide an intuitive, approximate relation between the two measures?

Exercise 2: Frailty and selection phenomena

In frailty models, we distinguish between individual and population level mortality. The model for individuals depend on an unobserved quantity Z, called frailty. In this exercise, we will see how an individual-level Gompertz model relates to the population level.

Consider a continuous survival time X. In the multiplicative frailty model, individual mortality is given by the individual's frailty and some baseline rate

$$\mu(x|z) = z\mu_0(x). \tag{4}$$

Conditional on Z, the probability of surviving until age x is then

$$S(x|Z) = \exp(-ZI_0(x)), \quad \text{where} \quad I_0(x) = \int_0^x \mu_0(x)\,\mathrm{d}x. \tag{5}$$

Denote by $\pi_z(x)$ the pdf of the frailty distribution among survivors to age x.

Question 2.1: Show that $S(x) = \int_0^\infty \pi_z(0)S(x|z)\,\mathrm{d}z = \mathbb{E}[S(x|Z)]$. Give an intuitive interpretation of the relation.

A major benefit of assuming that frailty acts multiplicatively on mortality is that we can express the marginal distribution through the Laplace transform of the conditional distribution. Recall that the Laplace transform of a pdf $f(z)$ of Z is defined by

$$\mathcal{L}_z(s) = \int_0^\infty e^{-sz} f(z)\,\mathrm{d}z. \tag{6}$$

Expressed in terms of the above, we have $S(x) = \mathcal{L}_z(I_0(x))$.

Assume that frailty Z follows a Gamma distribution with mean one and variance σ^2 and let

$$\mu(x|z) = z\alpha\exp(\beta x), \tag{7}$$

be a Gompertz mortality curve with parameters $\alpha, \beta \in \mathbb{R}_+$.

Question 2.2: Show that population-level mortality becomes

$$\mu(x) = \frac{\alpha\exp(\beta x)}{1 + \sigma^2\frac{\alpha}{\beta}(\exp(\beta x) - 1)}. \tag{8}$$

Hint: Compute the Laplace transform of the Gamma distribution with mean one and variance σ^2, and exploit the relationship between S and μ.

Question 2.3: What is $\operatorname{sign}(\mu'(x))$? Discuss the limiting behaviour of (8) as x tends to infinity.

Question 2.4: Based on the results from Question 2.2–2.3, argue that the flattening of the mortality curve seen at the highest ages (100+) could be caused by population heterogeneity (i.e., the frailty composition of those who survive to the high ages is different compared to the original frailty distribution.).
