Exercises in Pension Systems

University of Copenhagen, Autumn 2021

20 September – 26 September (Week 3)

Exercise 1: Fertility and population projections

This exercise deals with the three classes `mortClass`, `fertClass`, and `popClass`, and how you can use them to support different kinds of calculations. *Important: Start by sourcing the file Code/structure.R, which contains demographic objects and data to be used below.*

Question 1.1: Using the fertility class, `fertClass`, reproduce the three plots on p. 15 of Wednesday's presentation (20210915 Demography II (W2/L2)) but with the following specification

- Mean age of mother increasing from current age to 40 in 2080; initial slope of 3 years per 30 years.
- Standard deviation of mother kept constant at current level.
- Total fertility increasing to a replacement level of 2.1 in 2080 with an initial slope of 0.005.

The `fertClass` contains a method, `$plotdatEstimates`, which you can use to extract the relevant estimates and their projection.

Question 1.2: Using the population class, `popClass`, calculate and plot the old-age dependency ratio

$$\text{Old-age Dependency Ratio} = \frac{\text{Senior population (ages +65)}}{\text{Working population (ages 15-65)}}, \tag{1}$$

for the Danish population for the calendar period 1900–2250. The population projection should be based on the fertility projection produced in Question 1.1 and a mortality projection based on the Lee-Carter method estimated on the calendar period 1995–2020. What does the plot tell us about the past and future Danish population? (*Hint:* To extract the population data you need, you can use the `$plotdat`-method of `popClass`.)

Exercise 2: Stable population theory

Consider a stable population with force of mortality μ, highest attainable age ω, and intrinsic growth rate r. For $0 \leq x \leq \omega$, we define

$$T^r(x) = \int_x^\omega \exp\{-r(y-x)\} \frac{S(y)}{S(x)} dy, \tag{2}$$

where $S(x) = \exp\{-\int_0^x \mu(y)dy\}$ is the probability of surviving to age x for a newborn. Let c denote the age-composition of the (stable) population.

Question 2.1: Prove that $\int_0^\omega T^r(x)c(x)dx = \int_0^\omega xc(x)dx$. How can we interpret the quantity $\int_0^\omega xc(x)dx$?

Let $f(a|x)$ denote the density of dying in a years conditional on being alive at age x. For $0 \leq x \leq \omega$, we define the expected remaining lifetime at age x

$$e_x = \int_0^{\omega - x} af(a|x)da = \int_0^{\omega - x} a\mu(x+a)\frac{S(x+a)}{S(x)}da. \tag{3}$$

Question 2.2: Prove that $e_x = T^0(x)$ by rewriting the right-hand side of (3). Conclude that in a stationary population, $\int_0^\omega e_x c(x)dx = \int_0^\omega xc(x)dx$. How can we interpret this relation?

Assume now that the population is **stationary**. Let $g(a)$ denote the (unconditional) density of dying in a years from now for a person chosen at random from the population.

Question 2.3: Show that $g(a) = c(a)$ for $0 \leq a \leq \omega$. How can we interpret this relation?

Question 2.4: Imagine that we have a large sample from a stationary population of, say, ants at a given point in time. We do not know the ages of the ants in the sample, but we record the remaining lifetime for each of them. How can we use the data on remaining lifetimes to estimate the age-composition? Express the survival function, S, and then the force of mortality, μ, in terms of g. Which expression is easiest to estimate in terms of precision and why?

Exercise 3: The stable population equivalent

In this exercise, we will compute the age composition of the Danish female and male population on 1 January 2020 and projected to 1 January 2120 with the age composition of the stable equivalent population superimposed.

Question 3.1: Plot the age composition of the Danish population in 2020.

Question 3.2: Use the cohort component method, cf. Preston et al. pp. 119–120, to project the population to 2100 using 2020 mortality and fertility rates assuming a closed population:

- You should decide on a discretization scheme for the projection. Given the structure of the data it is most natural to consider 1-by-1 age-time cells, but in principle other discretizations could be used.
- In each (one year) time step you must reduce each cohort with the expected number of deaths using the age- and gender-specific mortality rates, and calculate the expected number of newborns of each sex. Note, of course, that only females give birth.
- The total fertility rates give the total number of newborns, and these should be split between boys and girls. We can assume that 48.7% of newborns are girls and 51.3% are boys, i.e. approximately 5% more boys than girls.

Plot the age composition of the resulting population in 2100.

Question 3.3: Compute the intrinsic growth rate r by finding the solution to the equation

$$1 = \int_0^\infty e^{-rx} S(x) F(x) \, \mathrm{d}x, \tag{4}$$

where S is the survival function and F is the rate of bearing female children, in each of the two cases above. Superimpose the stable equivalent population on the two plots from the previous question.

Hopefully, you get a nice fit between the projected and asymptotic age compositions. If the fit is not as precise as you would like, you should reconsider how best to approximate the integral defining r given the discrete nature of S and F.

Question 3.4: What happens to the intrinsic growth rate, and, by extension, the stable equivalent population if all age-specific fertility rates are increased by, say, 40%?

Exercise 4 (BONUS): Population momentum

In this the exercise you should reproduce the plot of population momentum below. Since you already have the projections, all that remains is to calculate the asymptotics (and the stationary variant).

You can compute the momentum in two ways, either as $M = C/N$ with C computed via the general formula

$$C = \frac{\int_0^\infty e^{-rt} G(t)dt}{\int_0^\infty x e^{-rx} S(x)F(x)dx}, \tag{5}$$

or using the expression (which rest on the asymptotic stationarity)

$$M = \frac{N_S}{N} = \int_0^\infty \frac{c(x)}{c_0(x)} \frac{\int_x^\infty S(y)F(y)dy}{A_0} dx = \int_0^\infty \frac{c(x)}{c_0(x)} w(x)dx. \tag{6}$$

You will probably find that the results are rather sensitive to how you approximate these integrals, so you have to be careful.