Survival analysis 2020/2021

Exercises week 2

Exercise 1 (JL 1.17) : Competing risks

In a competing risk setting, individuals can die from $K > 1$ different causes. Each individual has a lifetime T^* and a mode of failure ϵ. The cause-specific hazard function is defined as

$$\alpha_k(t) = \lim_{dt \to 0} \frac{pr(t \leq T^* < t + dt, \epsilon = k | T^* \geq t)}{dt}$$

(a) Show that the hazard function for T^* is $\sum_{k=1}^{K} \alpha_k(t)$ and obtain the marginal survival function $S(t) = pr(T^* > t)$

(b) Find $F_k(t) = pr(T^* \leq t, \epsilon = k)$ and thereby also $pr(\epsilon = k)$ and $pr(\epsilon = k | T \leq t)$

Exercise 2 (JL 3.8a) : Piecewise constant hazards

Assume that the hazard of T^* is constant on the intervals defined by the fixed numbers $0 = a_0 < a_1 < \ldots < a_m = \infty$,

$$\alpha(t) = \alpha_j, \text{ for } a_{j-1} \leq t < a_j, j = 1, \ldots, m.$$

Define $D_j(t) = \int_{a_{j-1}}^{a_j} I(u \leq t) du$, $j = 1, \ldots, m$. The survival function is given by

$$S(t) = \exp\left(- \sum_{j=1}^{m} \alpha_j D_j(t) \right).$$

Assume that the event times subject to independent and noninformative right-censoring, such that we observe $T_i = T_i^* \wedge C_i$ and $\Delta_I = I(T^* \leq C)$, $i = 1, \ldots, n$, where the censoring time C_i is independent of the event time T_i^*.

(a) Write up the likelihood for α_j, $j = 1, \ldots, m$ in terms of occurrence and exposure within the m intervals

(b) Find the maximum likelihood estimate of the hazard $\alpha(t)$ and use this to find an estimate of the survival $S(t)$

Exercise 3

(a) Let $T_1, \ldots, T_n$ be a random sample from a distribution with survival function $S(t)$ such that for t near 0,

$$S(t) = 1 - \lambda t + o(t)$$

for some $\lambda > 0$ and where a $o(t)$ is a function such that $\lim_{t \to 0} o(t)/t = 0$. Show that the limiting distribution of $X_n = n \min(T_1, \ldots, T_n)$ is exponential with failure rate λ.

A function $g(s)$ is $o(s^r)$ if $\lim_{s \to 0} g(s)/s^r = 0$.

(b) Suppose that for t near 0,

$$S(t) = 1 - (\lambda t)^\gamma + o(t^\gamma), \lambda > 0, \gamma > 0.$$

Show that the limiting distribution of $Y_n = n^{1/\gamma} \min(T_1, \ldots, T_n)$ is Weibull with shape γ and scale λ, i.e. has hazard $\lambda\gamma(\lambda t)^{\gamma-1}$ and cumulated hazard $(\lambda t)^\gamma$.

(b) As seen in Exercise C from week 1, in the Weibull regression model covariates act multiplicatively both on the hazard and the log-survival time. That is, it can be expressed both as a proportional hazards model

$$\alpha(t|X) = \alpha_0(t) \exp(\beta^T X)$$

and an accelerated failure time model

$$\log T = \theta^T X + \epsilon.$$

Show that the Weibull model is the only model where the proportional hazards and linear log-survival time models intersect.

Exercise 4 (MS 3.9a)

Assume that X_1 and X_2 are two covariates that take the values $\{0, 1\}$ and have joint distribution given by $pr(X_1 = 0|X_2 = 0) = 2/3$, $pr(X_1 = 0|X_2 = 1) = 1/3$ and $pr(X_2 = 1) = 1/2$. Let $\alpha(t)$ be a locally integrable non-negative function and assume that the survival time T^* (no censoring) given X_1 and X_2 has hazard function

$$\alpha(t) \exp(0.1X_1 + 0.3X_2). \tag{1}$$

Assume that only X_1 is observed. What is the intensity function of T^* given X_1?

Hint: Calculate $pr(X_2 = x_2|X_1 = x_1, T^* > t)$ for $x_1, x_2 \in \{0, 1\}$ and use the innovation theorem (MS p. 27).