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Stability analysis of a two-channel feedback networked control system

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***Abstract*— In this paper we propose a novel stabilization analysis of networked control of a controllable linear time invariant plant by a non co-located controller communicating with the actuator and the sensors of the plant via wireless channels of an existing wireless LAN (WLAN). We consider a lossless forward communication channel connecting the controller with the actuator and a lossy feedback channel connecting the sensors with the controller, where data transfer through both these channels suffer from significantly variable channel access delays. Sufficient conditions, using parameter dependent Lyapunov functions and perturbation theorem, are proposed for guaranteeing asymptotic stability of the networked control system (NCS) by treating it as a perturbation of a linear discrete-time system with variable sampling intervals. The efficacy of our results for network control synthesis is demonstrated using numerical simulations of a rotary inverted pendulum.**

***Index Terms*— WLAN, NCS, channel access delays, packet drops, variable sampling intervals, parameter dependent Lyapunov functional.**

I. INTRODUCTION

In recent years control of spatially distributed systems, where communication between sensors-controller and controller-actuators is accomplished using wireless channels like bluetooth, ZigBee or WiFi, has gained significant attention. Wireless networks provide a low-cost solution for setting up a low-power, low-maintenance and flexible network for networked control of systems like robotic manipulators, mobile vehicles, etc. using controllers that need not be co-located with either the actuators or the sensors. However, performance of wireless network compared to its counterpart wired network, is seriously degraded due to the phenomena like interference and multipath fading. If nearby devices transmit at the same time, their signals can interfere, resulting in collisions [1] and eventually packet losses. Wireless transmissions in indoor environment are also susceptible to mutlipath fading, an unwanted phenomenon in narrowband communication (most of control applications use narrowband for communicating states and actions[2]).

Research in wireless NCSs focuses on two primary areas: one is the development of protocols for network communications, while the other is the design and analysis of networked controllers [3]. Authors in [4] study two wireless network protocols WirelessHART and ISA100a, designed specifically to handle issues like multipath fading and interferences in industrial NCSs. Considerable work has been done in designing network protocols to manage sharing of communication channels between multiple heterogenous nodes (sensors and controllers) of the NCS. Authors of [5], [6] discuss round-robin (RR) protocol where nodes (sensors and controllers) take turns transmitting data in a periodic fashion and maximum error first- try once discard (MEF-TOD) protocol, wherein priority to utilize the communication channel is decided by the difference between the current value and the last transmitted value. Under this setup, authors in [7], [8] determine an upper bound on the MATI (Maximum Allowable Transfer Interval) for which stability can be guaranteed.

In researches mentioned above, complete control over the network resources is assumed i.e. the focus is on dedicated networks that will only handle communication of the NCS. Whereas, in this paper we investigate the possibility of setting up an indoor NCS using an existing wireless network like WiFi in a building. In this case, the network is shared by the NCS nodes and other devices like mobile phones, laptops, etc. The major advantage of doing so is; a quick, reliable and easy communication setup between controller-actuators and sensors-controllers of the NCS in any indoor environment equipped with WiFi (which is quite pervasive). But like other users connected to the WiFi network, the controller and sensors of the proposed NCS will also be dependent on DCF (Distributed Coordination Function), a IEEE 802.11 MAC protocol for getting access to the wireless channel. The waiting time for gaining channel's access (henceforth, referred to as channel access delay) is often quite variable, significant and can substantially affect the performance of the NCS. Besides, DCF protocol doesn't guarantee packet delivery and packet losses due to issues like hidden terminal problem and cross-technology interferences are highly common.

Authors in [9] propose methods to study the behavior of NCS by modeling packet drops in feedback and forward channels as two independent Bernoulli processes. Using similar stochastic packet drop modeling, optimal control laws are proposed by [10] for both TCP and UDP protocols. A general approach by seeking the LQG optimal control for the mitigating the ill effects of packet dropouts is proposed by [11], for the case when the control and actuator are co-located. In [11], they jointly design the controller, the encoder and the decoder to solve the optimality problem by showing separability of the control and the estimation costs. Similar stochastic approaches to handle random channel access delays are proposed in [12], [13], [14] and [15], by modeling the random channel access delays as i.i.d random

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processes or by Markov chains. The control synthesis and stability analysis presented in these papers rely significantly on the assumed stochastic models of the channel delays and packet drops. However, WiFi channel access delays and packet drops are highly unpredictable due to bursty nature of interference and data traffic [16]. Hence we focus on the robustness of the NCS, by assuming bounds on the channel access delays and consecutive packet drops in the feedback channel, instead of estimating them using stochastic models.

In this paper, we consider the problem of networked control of a linear LTI plant using a non co-located controller (Fig. 1). The forward channel connecting the controller with the actuator is assumed lossless, but the feedback channel connecting the sensors with the controller is assumed lossy. Bounded and varying channel access delays are considered in both the channels. In the proposed problem formulation, the effect of packet drops in the feedback channel and variable channel access delays are mapped to a single entity referred to as network loop delay (shown in Fig. 2). The major contribution of this paper is the novel stability analysis of a linear NCS using concepts of robust stability of discrete-time system with variable sampling intervals [17] and stability of perturbed systems [18]. We show equivalence between the original system and a perturbed nominal system. The nominal system is of the form of a discrete-time system with variable sampling intervals. For asymptotic stability analysis of the nominal system we utilize the concept of parameter dependent Lyapunov functions (PDLF) [19]. Finally, we combine these results to derive sufficient conditions of stability of the considered NCS.

The rest of the paper is organized as follows. Section II explains the notations and define certain terms used throughout the paper. Section III discusses the data exchange strategy between the plant and the controller, and the problem formulation. Section IV provides the main result on stability analysis of the system formulated in Section III. Finally, the simulation results are presented in Section V and few concluding remarks in VI.

II. Notations and Definitions

Throughout the paper, the sets $\mathbb{N}$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote the sets of nonnegative integers, n-dimensional real-valued vectors and n by n matrices with real-valued elements. The notation $\|\cdot\|$ is used to denote both vector 2-norm and induced 2-norm of a matrix. λ^M_{min} and λ^M_{max} represent the minimum and maximum eigenvalues of a matrix M respectively. The notation $M = M^T > 0$ means the matrix M is symmetric and positive definite.

A square matrix M is called a Schur matrix, if magnitude of all its eigenvalues is less than unity. A continuous function $\alpha : [0, a) \longrightarrow [0, \infty)$ belongs to class $\mathcal{K}$, if it is strictly increasing with $\alpha(0) = 0$. Whereas, a continuous function $\alpha : [0, a) \longrightarrow [0, \infty)$ belongs to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$, $a = \infty$ and $\lim_{t\to\infty} \alpha(t) = \infty$.

III. Problem Formulation

Consider the following linear time-invariant plant system, with (A, B) being controllable and all of the states being measurable

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t), \quad t \geq 0 \\ x(0) &= x_o \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^a$ and $u \in \mathbb{R}^b$ are the state and control input of the plant system respectively.

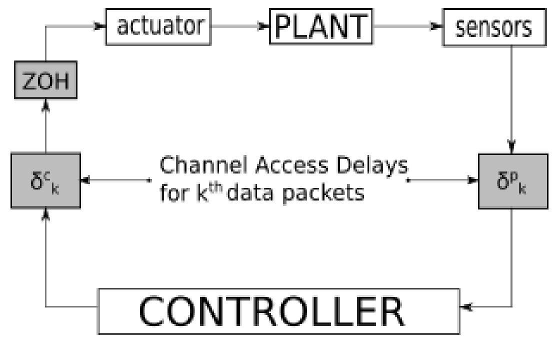

Fig. 1: System design for a single pair of controller and plant communicating through a lossy wireless channel (two-channel feedback NCS [20]).

The controller computes a new control input only when it receives a new plant state, but if the plant does not receive the control input within a time interval of δ_o, it transmits the current state to keep feedback functional even in case of packet drops in the feedback channel. Here, we have assumed that the net channel access delays (channel access delays of the feedback channel + channel access delays of the forward channel) for lossless transmissions (no packet drops in the feedback channel) is less than δ_o. A control input $u(\tau_k)$ that is computed by the controller at time τ_k reaches the plant after a delay of δ^c_k, the channel access delay of the forward channel (shown in Fig 1). Similarly a plant state that was transmitted at $\tau_k - \delta^p_k$ is received by the controller at τ_k. In this paper, transmission delay i.e. the time taken by a packet to travel between any two connected points in the network is assumed to be negligible. This makes sense for systems located in indoor environments where the controller and plant are not located very far from each other and the communication between them is carried out by EM waves in the RF spectrum.

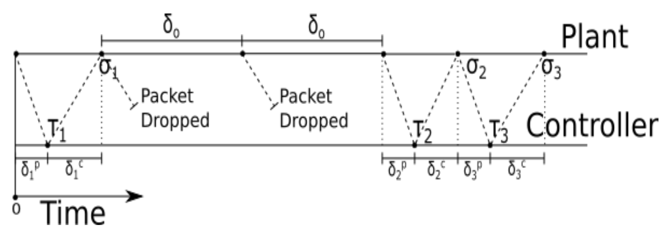

Fig. 2: Example of control and state packets being exchanged by the plant and the controller.

Let the control be of the form

$$u(\tau_k) = -K\hat{x}(\tau_k) \tag{2}$$

where, $\hat{x}(\tau_k)$ is the plant state received by the controller at τ_k.

$$\hat{x}(\tau_k) = x(\sigma_{k-1} + m_{k-1}\delta_o) = x(\sigma_k - \delta_k)$$

The controller immediately transmits this control input to the actuator and so, the plant state finally evolves as

$$\begin{aligned} \dot{x}(t) &= A\,x(t) - BK\,x(\sigma_k - \delta_k), \quad t \in [\sigma_k,\, \sigma_{k+1}) \\ x(\sigma_1) &= e^{A\sigma_1}x_o, \quad \forall k \in \mathbb{N} \end{aligned} \tag{3}$$

where, $\delta_k = \delta_k^c + \delta_k^p$, m_{k-1} is number of state packets dropped after $u(\tau_{k-1})$ was received by the plant until a state packet was successfully transmitted and so in general, $\sigma_{k+1} = \sigma_k + m_k\delta_o + \delta_{k+1}^c + \delta_{k+1}^p$.
To analyze the asymptotic stability of the origin for system (3), few researchers in the past like [21] and [22] have treated the system dynamics as a delayed differential equation, wherein equation (3) can be written as

$$\begin{aligned} \dot{x}(t) &= A\,x(t) - BK\,x(t - \sigma(t)), \quad t \geq 0 \\ x(0) &= e^{A\sigma_1}x_o \end{aligned} \tag{4}$$

where time t is shifted by σ_1 and $\sigma(t)$ is time varying

$$\sigma(t) = t - \sigma_k + \delta_k, \forall t \in [\sigma_k,\, \sigma_{k+1})$$

Subsequently, techniques like Lyapunov-Krasovskii theorem have been utilized to investigate the stability of the DDE (4) by studying the feasibility of a LMI.

In this paper, a different perspective to analyze the stability of system (3) is adopted . Instead of working with the continuous-time domain we have dealt with the stability of our system in the discrete-time domain. Applying variation of constants formula [20] to (3), we obtain

$$\begin{aligned} x(\sigma_{k+1}) = &\Phi(\sigma_{k+1} - \sigma_k)x(\sigma_k) \\ &- \Gamma(\sigma_{k+1} - \sigma_k)Kx(\sigma_k - \delta_k) \end{aligned}$$

where, $\Phi(\xi) \triangleq e^{A\xi}$, $\Gamma(\xi) \triangleq \int_0^\xi e^{A(\xi - t)}dtB$. This equation can be written as

$$\begin{aligned} x(\sigma_{k+1}) &= \Psi(\Delta_k)x(\sigma_k) + \Gamma(\Delta_k)Ke(\sigma_k),\, k \in \mathbb{N} \\ e(\sigma_k) &= x(\sigma_k) - x(\sigma_k - \delta_k) \\ x(\sigma_1) &= e^{A\sigma_1}x_o \end{aligned} \tag{5}$$

where, $\Psi(\xi) \triangleq \Phi(\xi) - \Gamma(\xi)K$ and $\Delta_k \triangleq \sigma_{k+1} - \sigma_k$, will be referred to as network loop delay. In our formulation we track the plant state at time instances σ_k, i.e. whenever it receives a new control input. Authors in [23] adopted a similar approach with a different data exchange scheme, but they have analyzed the bound on the performance degradation for a given control gain K using the work of [24], whereas we have given conditions that will help us determine control gain K to improve the system stability if the range of network loop delay Δ_k is known and bounded for every k.

IV. Stability Analysis

We assume that the consecutive feedback channel packet drops are limited by a known upper bound. Using this assumption the network loop delay: $\Delta_k \in [\underline{\Delta}, \overline{\Delta}]$, $\forall k \in \mathbb{N}$. To analyze the stability of system (5), the system (5) can be treated as a perturbed system of the following nominal system

$$\begin{aligned} x(\sigma_{k+1}) &= \Psi(\Delta_k)\,x(\sigma_k),\, k \in \mathbb{N} \\ x(\sigma_1) &= e^{A\sigma_1}x_o \end{aligned} \tag{6}$$

which is a discrete-time linear system with variable sampling intervals.
The perturbation term given by $\Gamma(\Delta_k)Ke(\sigma_k)$ is not vanishing in strict sense i.e. $e(\sigma_k) \neq 0$ if $x(\sigma_k) = 0$, but here we are dealing with discrete-time samples of plant states and it is evident from the way $e(\sigma_k)$ is defined in (5) that if $x(\sigma_k)$ converges to the origin, i.e. if $\lim_{k\to\infty} x(\sigma_k) = 0$ then, $\lim_{k\to\infty} e(\sigma_k) = 0$. Hence, the origin is still the equilibrium point for the system (5).

Lemma 1: [20] A linear discrete-time system with $\chi_k \in \mathbb{R}^a$ and bounded initial state χ_1,

$$\begin{aligned} &\chi_{k+1} = \mathcal{A}(h_k)\chi_k,\, h_k \in [\underline{h},\, \overline{h}] \subset \mathbb{R}^+ \\ &\forall k\, \in \mathbb{N} \end{aligned} \tag{7}$$

is globally asymptotically stable if there exists a symmetric positive definite matrix P such that

$$\mathcal{A}(h_k)^T P \mathcal{A}(h_k) - P < 0, \quad \forall k \in \mathbb{N} \tag{8}$$

where, $\mathcal{A}(h_k) \in \mathbb{R}^{a\times a}, \forall k \in \mathbb{N}$

Proof: Consider a Lyapunov candidate function $V(\chi) = \chi^T P \chi > 0, \forall \chi \in \mathbb{R}^a \neq 0$. Its increment along the solution of (7) is

$$\begin{aligned} &V(\chi_{k+1}) - V(\chi_k) = \chi_k^T[\mathcal{A}(h_k)^T P \mathcal{A}(h_k) - P]\chi_k < 0 \\ &\forall \chi_k > 0 \end{aligned}$$

■

In system (7), if h_k is a constant h for every non-negative integer k, then system (7) is globally asymptotically stable if and only if $\Psi(h)$ is a Schur matrix. Apparently, if h_k is time-varying this does not hold anymore i.e. $\Psi(h_k)$ being Schur is neither necessary nor sufficient condition [25].
Relevant stability analysis based on uncertain representations of $\Psi(h_k)$ have been studied in the past, but extending them for stability analysis of discrete-time systems with variable sampling intervals like the nominal system (6) is not feasible [17]. This is because the state transition matrix $\Psi(\Delta_k)$ in this case does not belong to the specific set of matrices considered in those works (for eg. polytopic set). The following theorem extends Lemma 1 to give conditions for asymptotic stability of the origin of the NCS (5).

Theorem 1: The origin for system (5) is asymptotically stable if there exists symmetric positive definite matrices P

and Q, along with positive scalars ρ and π s.t. $\forall k \in \mathbb{N}$

$$\Psi(\Delta_k)^T P \Psi(\Delta_k) - P + Q = 0 \tag{9}$$

$$\|P\Psi(\Delta_k)\| \leq \rho,\ \|\Gamma(\Delta_k)K\, e(\sigma_k)\| \leq \pi\, \|x(\sigma_k)\| \tag{10}$$

$$\lambda^P_{max}\pi^2 + 2\rho\pi - \lambda^Q_{min} < 0 \tag{11}$$

where $e(\sigma_k) = x(\sigma_k) - x(\sigma_{k-1})$.

Proof: Consider a Lyapunov candidate function $V(x) = x^T P x$.
For the sake of convenience, let x_k and e_k represent $x(\sigma_k)$ and $e(\sigma_k)$, $\forall k \in \mathbb{N}$ respectively.
The origin for system (5) is asymptotically stable if $V(x_{k+1}) - V(x_k) < 0, \forall k \in \mathbb{N}$, i.e.

$$x^T_{k+1} P x_{k+1} - x^T_k P x_k < 0,\ k \in \mathbb{N}$$

substituting x_{k+1} from equation (5),

$$\begin{aligned} &V(x_{k+1}) - V(x_k) = x^T_k[\Psi(\Delta_k)^T P \Psi(\Delta_k) - P]x_k \\ &+ (\Gamma(\Delta_k)K\, e_k)^T P(\Gamma(\Delta_k)K\, e_k) \\ &+ 2(\Gamma(\Delta_k)K\, e_k)^T P(\Psi(\Delta_k)x_k),\ \forall k \in \mathbb{N} \end{aligned}$$

From equation (9) and the fact that for any positive definite matrix $M \in \mathbb{R}^{n\times n}$,

$$0 < \lambda^M_{min}\|z\|^2 \leq z^T M z \leq \lambda^M_{max}\|z\|^2,\ \forall z \in \mathbb{R}^n$$

we get,

$$x^T_k[\Psi(\Delta_k)^T P \Psi(\Delta_k) - P]x_k = -x^T_k Q x_k \leq -\lambda^Q_{min}\|x_k\|^2$$

and

$$(\Gamma(\Delta_k)K\, e_k)^T P(\Gamma(\Delta_k)K\, e_k) \leq \lambda^P_{max}\|\Gamma(\Delta_k)K\, e_k\|^2$$

Using these inequalities and bounds in (10), we get

$$\begin{aligned} &V(x_{k+1}) - V(x_k) \leq -\lambda^Q_{min}\|x_k\|^2 + \lambda^P_{max}\pi\|x_k\|^2 \\ &+ 2\rho\pi\|x_k\|^2,\ \forall k \in \mathbb{N} \end{aligned}$$

Hence, by the given inequality (11) we have

$$V(x_{k+1}) - V(x_k) < 0\, \forall k \in \mathbb{N}$$

The inequality

$$\lambda^P_{max}\pi^2 + 2\rho\pi - \lambda^Q_{min} < 0$$

has feasible solutions because at $\pi = 0$, LHS $= -\lambda^Q_{min} < 0$ as Q is a positive definite symmetric matrix. So, there will always exist some π and ρ that will satisfy (11).

■

The Theorem above is conceptually important because it informs that if the origin for the nominal system (6) is asymptotically stable, then given the bounds on the perturbation term $\Gamma(\Delta_k)Ke(\sigma_k)$ we can achieve asymptotic stability of the origin for the system (5), by ensuring asymptotic stability of the origin for the nominal system (6).

The crux of Theorem 1 is the asymptotic stability of discrete-time system of the form (7), where h_k is a parameter that maps k to $[\underline{h}, \overline{h}]$. This problem is of great theoretical and practical interest. Lemma 1 requires that there should exist a single quadratic Lyapunov function to ensure system stability of (7) which is quite conservative. In order to reduce this conservatism, the authors in [19] suggest a more general Lyapunov function, known as parameter dependent Lyapunov function (PDLF). Such Lyapunov functions are similar to time-dependent Lyapunov functions but instead of time they depend on the parameter, h_k in case of (7).

Lemma 2: [19] The origin of system (7) is uniformly asymptotically stable if and only if theres exists a parameter dependent Lyapunov function

$$V(h_k,\, \chi_k) = \chi^T_k P(h_k)\chi_k$$

such that

$$\alpha_1(\|\chi\|) \leq \chi^T_k P(h_k)\chi_k \leq \alpha_2(\|\chi\|)$$

and whose difference along the solution of (7) is negative decrescent i.e.

$$V(h_{k+1},\, \chi_{k+1}) - V(h_k,\, \chi_k) \leq -\alpha_3(\|\chi\|)$$

$\forall \chi \in \mathbb{R}^a$, $k \in \mathbb{N}$ and $h_k \in [\underline{h}, \overline{h}]$, where α_1, α_2 and α_3 are $\mathcal{K}_\infty$ functions.

Unlike Lemma 1, here $P(h_k)$ is not required to be the same for every k. We use Lemma 2 to get the following sufficient conditions, which would help us determine the control gain K for asymptotically stabilizing the origin for the system (5).
However, determining different $P(h_k)$ for different k's that satisfy the conditions in lemma 2 is a tedious task. Hence, in Theorem 2 we have derived sufficient conditions based on one kind of $P(h_k)$.

Theorem 2: If $\mathcal{A}(h_k)$ is Schur with h_k mapping to a *finite* set of values in $[\underline{h},\, \overline{h}] \subset \mathbb{R}^+$ for all $k \in \mathbb{N}$, and $\exists Q = Q^T > 0$, $Q \in \mathbb{R}^a$ such that

$$\mathcal{A}(h_k)^T(P(h_{k+1}) - P(h_k))\mathcal{A}(h_k) - Q < 0,\ \forall k \in \mathbb{N} \tag{12}$$

where,

$$P(h_k) = \sum_{i=0}^{\infty}(\mathcal{A}(h_k)^T)^i\, Q\, (\mathcal{A}(h_k))^i \tag{13}$$

$\forall k \in \mathbb{N}$. Then the origin for the system (7) is uniformly asymptotically stable.

Proof: $P(h_k)$ given by equation (13) is well defined because $\lim_{i\to\infty}\mathcal{A}(h_k)^i = 0$ if $\mathcal{A}(h_k)$ is Schur $\forall k \in \mathbb{N}$.
Observing the definition of $P(h_k)$, one can be readily verify that $P(h_k) = P(h_k)^T > 0$ and

$$\mathcal{A}(h_k)^T P(h_k)\mathcal{A}(h_k) - P(h_k) + Q = 0,\ \forall k \in \mathbb{N}. \tag{14}$$

Now, consider a PDLF

$$V(h_k,\, \chi_k) = \chi^T_k P(h_k)\chi_k$$

then

$$0 < \lambda^P_{min}\|\chi_k\|^2 \leq x_k^T P(h_k)\chi_k \leq \lambda^P_{max}\|\chi_k\|^2,\ \forall \chi_k \in \mathbb{R}^a$$

where, $\lambda^P_{min} = \min_k \lambda^{P(h_k)}_{min}$ and $\lambda^P_{max} = \max_k \lambda^{P(h_k)}_{max}$. Now,

$$\begin{aligned} V(h_{k+1}, \chi_{k+1}) - V(h_k, \chi_k) =& \chi^T_{k+1}P(h_{k+1})\chi_{k+1} \\ &- \chi^T_k P(h_k)\chi_k \end{aligned}$$

which can also be written as,

$$\begin{aligned} & V(h_{k+1}, \chi_{k+1}) - V(h_k, \chi_k) = V(h_{k+1}, \chi_{k+1}) \\ & - V(h_k, \chi_{k+1}) + V(h_k, \chi_{k+1}) - V(h_k, \chi_k) \\ \Longrightarrow\ & V(h_{k+1}, \chi_{k+1}) - V(h_k, \chi_k) = \chi^T_{k+1}P(h_{k+1})\chi_{k+1} \\ & - \chi^T_{k+1}P(h_k)\chi_{k+1} + \chi^T_{k+1}P(h_k)\chi_{k+1} - \chi^T_k P(h_k)\chi_k \\ \Longrightarrow\ & V(h_{k+1}, \chi_{k+1}) - V(h_k, \chi_k) = \\ & \chi^T_{k+1}P(h_{k+1})\chi_{k+1} - \chi^T_{k+1}P(h_k))\chi_{k+1} \\ & - \chi^T_k(\mathcal{A}(h_k)^T P(h_k)\mathcal{A}(h_k) - P(h_k))\chi_k \\ \Longrightarrow\ & V(h_{k+1}, \chi_{k+1}) - V(h_k, \chi_k) = \\ & \chi^T_k(\mathcal{A}(h_k)^T(P(h_{k+1}) - P(h_k))\mathcal{A}(h_k) - Q)\chi_k \end{aligned}$$

Now because h_k takes *finite* number of positive real values, implies there could only be finite number of ordered pairs (h_k, h_{k+1}). That means we can get

$$\eta = \min_{(h_k, h_{k+1})} \lambda^{Q-\mathcal{A}(h_k)^T(P(h_{k+1})-P(h_k))\mathcal{A}(h_k)}_{min} > 0$$

Then,

$$\begin{aligned} & V(h_{k+1}, \chi_{k+1}) - V(h_k, \chi_k) \leq -\eta\|\chi_k\|^2,\ \forall \chi_k \in \mathbb{R}^a \\ & \forall k \in \mathbb{N} \end{aligned}$$

This is a negative definite decrescent quadratic form and according to Lemma 2 we can infer that the origin for system (7) is uniformly asymptotically stable.

■

Theorem 2 provides sufficient conditions that guarantee asymptotic stability of a linear discrete time system with variable sampling intervals like the nominal system (6). The conditions given in Theorem 2 are computationally easier to check, because ensuring if the eigenvalue of maximum magnitude of matrix $\Psi(\Delta_k)$ is less than unity can be done in the very first step of power iteration. Further, substituting $P(\Delta_k)$ given by equation (13) reduces condition (12) it to a feasibility problem affine in Q. The following example illustrates Theorem 2.

Example: Consider a problem of analyzing asymptotic stability of the origin for the system (7), where h_k maps to $\{h_1, h_2\}$ for all values of $k \in \mathbb{N}$, such that

$$\mathcal{A}(h_1) = \begin{bmatrix} 0.9 & 0 \\ 1 & 0.4 \end{bmatrix}, \quad \mathcal{A}(h_2) = \begin{bmatrix} 0.85 & 0 \\ 1.2 & 0.5 \end{bmatrix}$$

Obviously, both $\mathcal{A}(h_1)$ and $\mathcal{A}(h_2)$ are Schur with eigenvalues $(0.9, 0.4)$ and $(0.85, 0.5)$ respectively. To check the second condition, let

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}$$

which gives us

$$P(h_1) = \begin{bmatrix} 10.599 & 0.297 \\ 0.297 & 0.476 \end{bmatrix},\ P(h_2) = \begin{bmatrix} 10.462 & 0.556 \\ 0.556 & 0.533 \end{bmatrix}$$

The set of ordered pairs (h_k, h_{k+1}) is $\{(h_1, h_1), (h_2, h_2), (h_1, h_2), (h_2, h_1)\}$. To check if $\mathcal{A}(h_k)^T(P(h_{k+1}) - P(h_k))\mathcal{A}(h_k) - Q$ is negative definite for all $k \in \mathbb{N}$, we only have to check it for 4 cases in this problem.

1) $(h_k, h_{k+1}) \in \{(h_1, h_1), (h_2, h_2)\}$: For this case $P(h_{k+1}) = P(h_k)$, which implies $\mathcal{A}(h_k)^T(P(h_{k+1}) - P(h_k))\mathcal{A}(h_k) - Q = -Q$.
2) $(h_k, h_{k+1}) = (h_1, h_2)$: The eigenvalues of $\mathcal{A}(h_k)^T(P(h_{k+1}) - P(h_k))\mathcal{A}(h_k) - Q$ are -0.634 and -0.335.
3) $(h_k, h_{k+1}) = (h_2, h_1)$: The eigenvalues of $\mathcal{A}(h_k)^T(P(h_{k+1}) - P(h_k))\mathcal{A}(h_k) - Q$ are -1.537 and -0.395.

According to Theorem 2, the origin for the problem should be asymptotically stable. This is verified through the simulation (Fig. 3), by taking initial state as $\chi_1 = [1, 2]^T$ and h_k taking random values in (h_1, h_2) for all $k \in \mathbb{N}$.

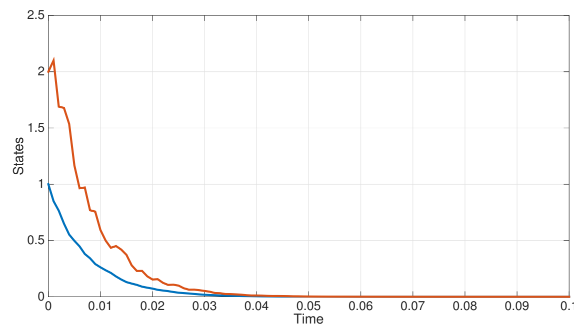

Fig. 3: Simulated state evolution for the example

The nominal system given by (6) is similar to the discrete time system (7). Hence, Theorem 2 can be utilized to study the asymptotic stability of the origin for the perturbation of the nominal system (our formulated NCS) given in (5). This leads us to a modification of Theorem 1 to derive a less conservative result where $P(\Delta_k)$ is not required to be the same $\forall k \in \mathbb{N}$, but could depend on Δ_k.

Theorem 3: If the transition matrix given by $\Psi(\Delta_k)$ is Schur, where Δ_k maps to a *finite* set of values in $[\underline{\Delta}, \overline{\Delta}]$ for all $k \in \mathbb{N}$ and $\exists Q = Q^T > 0$ and a positive scalar γ s.t.

$$\Psi(\Delta_k)^T(P(\Delta_{k+1}) - P(\Delta_k))\Psi(\Delta_k) - Q < 0 \quad (15)$$

$$\|P(\Delta_k)\Psi(\Delta_k)\| \leq \rho,\ \|\Gamma(\Delta_k)K\, e(\sigma_k)\| \leq \pi\, \|x(\sigma_k)\| \quad (16)$$

$$\lambda^P_{max}\pi^2 + 2\rho\pi - \eta \leq -\gamma \quad (17)$$

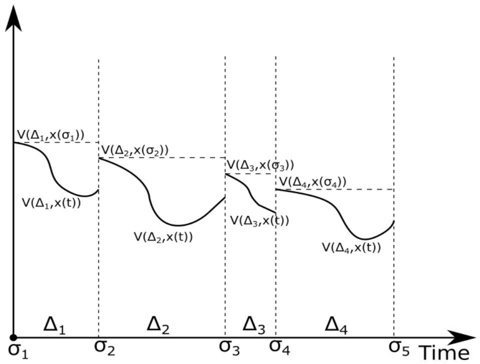

Fig. 4: An illustration of Theorem 2 for the nominal system (6). PDLF vs time

where,

$$P(\Delta_k) = \sum_{i=0}^{\infty} (\Psi(\Delta_k)^T)^i \, Q \, (\Psi(\Delta_k))^i$$

$$\eta = \min_{(\Delta_k,\, \Delta_{k+1})} \lambda_{min}^{Q-\Psi(\Delta_k)^T(P(\Delta_{k+1})-P(\Delta_k))\Psi(\Delta_k)} > 0$$

$\forall k \in \mathbb{N}$, and $\lambda_{max}^{P} = \max_k \lambda_{max}^{P(\Delta_k)}$. Then the origin for system (5) is asymptotically stable.

Proof: For the convenience of notation let x_k and e_k represent $x(\sigma_k)$ and $e(\sigma_k)$ respectively. Consider a parameter dependent Lyapunov function candidate $V(\Delta_k, x_k) = x_k^T P(\Delta_k) x_k$, implies

$$0 < \lambda_{min}^{P} \|x_k\|^2 \leq x_k^T P(\Delta_k) x_k \leq \lambda_{max}^{P} \|x_k\|^2, \, \forall k \in \mathbb{N}$$

as $P(\Delta_k) = P(\Delta_k)^T > 0, \, \forall k \in \mathbb{N}$. Further, we have

$$\Psi(\Delta_k)^T P(\Delta_k) \Psi(\Delta_k) - P(\Delta_k) + Q = 0, \, \forall k \in \mathbb{N} \qquad (18)$$

Using equations (5) and (18), the increment of the PDLF i.e.

$$\begin{aligned} V(\Delta_{k+1},\, x_{k+1}) - V(\Delta_k,\, x_k) =& x_{k+1}^T P(\Delta_{k+1}) x_{k+1} \\ & - x_k^T P(\Delta_k) x_k, \, \forall k \in \mathbb{N} \end{aligned}$$

can be written as

$$\begin{aligned} & V(\Delta_{k+1},\, x_{k+1}) - V(\Delta_k,\, x_k) = \\ & x_k^T \{\Psi(\Delta_k)^T (P(\Delta_{k+1}) - P(\Delta_k)) \Psi(\Delta_k) - Q\} x_k \\ & + 2 x_k^T \Psi^T(\Delta_k) P_{k+1} \Gamma(\Delta_k) K e_k \\ & + (\Gamma(\Delta_k) K e_k)^T P_{k+1} (\Gamma(\Delta_k) K e_k), \, \forall k \in \mathbb{N}. \end{aligned}$$

Substituting η, using the inequalities (16) and (17), and the fact that $P(\Delta_k)$ is positive definite symmetric matrix $\forall k \in \mathbb{N}$, we get

$$\begin{aligned} & V(\Delta_{k+1},\, x_{k+1}) - V(\Delta_k,\, x_k) \leq -\eta \|x_k\|^2 + \rho\pi \|x_k\|^2 \\ & \quad + (\Gamma(\Delta_k) K e_k)^T P(\Delta_{k+1}) (\Gamma(\Delta_k) K e_k), \quad \forall k \in \mathbb{N} \\ \implies & V(\Delta_{k+1},\, x_{k+1}) - V(\Delta_k,\, x_k) \leq -\eta \|x_k\|^2 + \rho\pi \|x_k\|^2 \\ & \quad + \lambda_{max}^{P} \pi^2 \|x_k\|^2, \quad \forall k \in \mathbb{N} \\ \implies & V(\Delta_{k+1},\, x_{k+1}) - V(\Delta_k,\, x_k) \leq -\gamma \|x_k\|^2, \quad \forall k \in \mathbb{N} \end{aligned}$$

which is a negative definite decrescent quadratic form and hence, by invoking Lemma 2 we can infer that the origin for system (5) is asymptotically stable. ■

V. Simulation and Remarks

We have simulated control of a rotary inverted pendulum about the unstable equilibrium point in MATLAB. For the simulation we have assumed that both channels i.e. forward and feedback channels are lossless, but data transfer through these channels suffer from time-varying channel access delays. In every simulation run, the inverted pendulum initiated the communication by sending its initial state after waiting for a random amount of time to simulate the access delay in the feedback channel and the controller immediately computed the action value, once the pendulum were received. Similarly, the controller waited for a random amount of time before transmitting it to simulate the access delay in the forward channel.

The pendulum dynamics around the unstable equilibrium point after linearization is given as

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -67 & -25 & 0 \\ 0 & 64 & 12 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 47 \\ -22 \end{bmatrix} u$$

$$x(0) = [0.1,\, 0.1,\, 0.05,\, 0.1]$$

where $x_1 = \phi$ (cart angle), $x_2 = \theta$ (vertical angle of the pendulum), $x_3 = \dot{\phi}$ and $x_4 = \dot{\theta}$. We have compared results of two control gains,

$$K_p = [-1.5,\, -20,\, -2,\, -5]$$
$$K_o = [-2.23,\, -35.20,\, -2.29,\, -5.69]$$

where, K_o was determined using the conventional LQR control method with

$$Q = \begin{bmatrix} 0.15 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; R = 0.05$$

K_p on the other hand was determined by iterating K_o to make $\Psi(\Delta)$ Schur for larger range of Δ (network loop delay), as shown in Figure (5). To demonstrate the dependence

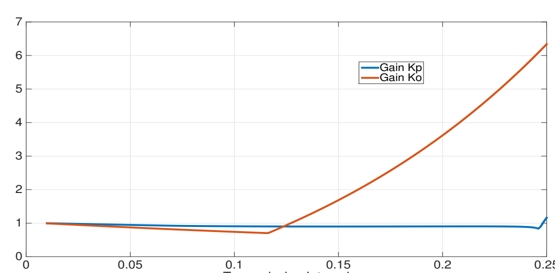

Fig. 5: The maximum magnitude of the eigenvalues of $\Psi(\Delta)$ vs Δ for control gain K_p and K_o.

of system stability on the magnitude of eigenvalues of $\Psi(\Delta)$, we simulated the networked control system for two cases;

1) Δ_k randomly distributed in $[5,\ 185]$ms for every k
2) Δ_k randomly distributed in $[5,\ 255]$ms for every k.

It is clear from Figure 5 that for K_p, $\Psi(\Delta_k)$ is Schur for every k in case (1), but this is not true for case (2). Whereas, for gain K_o, $\Psi(\Delta_k)$ is not Schur for all k in either of the cases. We did multiple simulation runs for both cases. The system was always asymptotically stable for K_p in case (1), but there were results when the system blew up for gain K_o as shown in Figure 6.

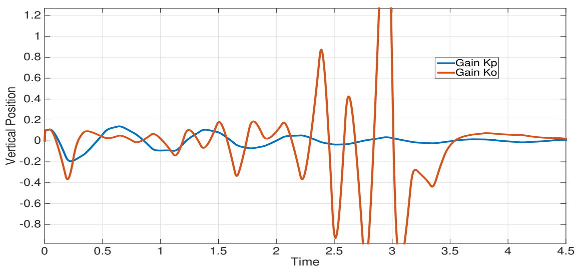

Fig. 6: Vertical angle(θ) of the inverted pendulum vs time, for control gain K_p and K_o in case (1)

The corresponding control inputs to the actuator are shown in Figure 7.

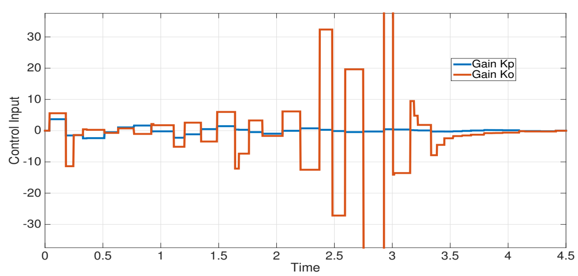

Fig. 7: Control inputs to the actuator vs time, for control gain K_p and K_o in case (1)

Whereas in case (2), neither of the gains guaranteed asymptotic stability of the system. In Figure 8 we have shown the result for gain K_p in case (2). This makes sense as $\Psi(\Delta_k)$ wasn't Schur for all k for neither of the gains K_p and K_o in case (2).

To signify the importance of handling variable channel access delays while determining the control gain, in Figure 9 we have shown the system performance for both the control gains K_p and K_o where the controller and the plant(inverted pendulum) were co-located. This signifies the importance of handling variable channel access delays while tuning the control gain. Also, even though K_o performs better for this case, its performance gets highly degraded (the system even becomes unstable) if the controller and the plant are communicating using channels with variable channel access delays.

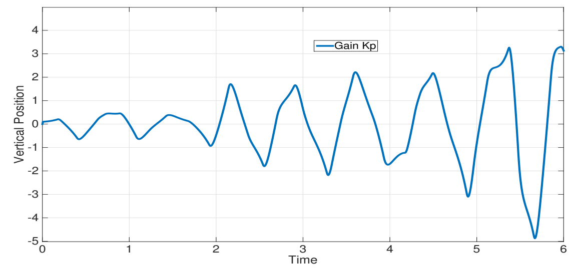

Fig. 8: Vertical angle(θ) for gain K_p when $\Delta \in [5,\ 255]$

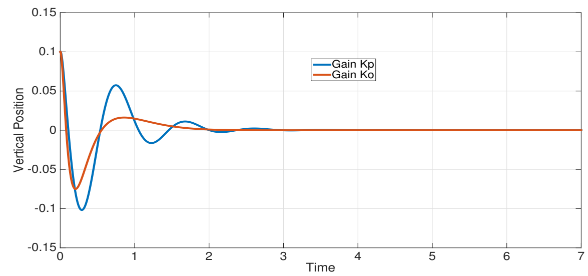

Fig. 9: Vertical angle(θ) for gain K_p and K_o when the controller and plant(inverted pendulum) are co-located

From these results, we can conclude that $\Psi(\Delta_k)$ being Schur for every k is definitely not a necessary condition for stability of systems like (6). But, by imposing additional conditions like in Theorem 2 we can tune our control gain K appropriately such that the conditions of Theorem 3 are satisfied to guarantee asymptotic stability of the given NCS.

VI. Conclusion

In this paper, a linear two-channel feedback networked control system is considered. The NCS is formulated as a perturbation of a linear discrete-time nominal system with variable sampling intervals. To derive sufficient conditions of asymptotic stability of the origin of the nominal system, we extended the work on robust stability analysis of linear discrete-time systems using parameter dependent Lyapunov function. The conditions derived were then used to determine bounds on the perturbation term and state-transition matrix to guarantee asymptotic stability of the origin of the NCS. In these derivations we assumed that the network loop delay took finite set of real values within known bounds. The stability conditions determined in this paper, can be used to compute an appropriate control gain to ensure asymptotic stability of the given NCS with full state feedback control (illustrated through simulation in Section V).

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