1-D swept equations

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1 Heat

The heat equation without volumetric heat flux:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \tag{1}$$

In 1-D:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{2}$$

Discretizing with forward differencing in time and central differencing in space:

$$\frac{T_i^{m+1} - T_i^m}{\Delta t} = \alpha \frac{T_{i+1}^m + T_{i-1}^m + 2T_i^m}{\Delta x^2} \tag{3}$$

Defining the Fourier number $Fo = \frac{\alpha \Delta t}{\Delta x^2}$, and solving for temperature at the next timestep T_i^{m+1}:

$$T_i^{m+1} = Fo(T_{i+1}^m + T_{i-1}^m) + (1 - 2Fo)T_i^m \tag{4}$$

This is a first-order, explicit, finite-difference method.

With insulated boundary conditions at both ends and n spatial points:

$$T_0 = T_1 \quad and \quad T_{n+1} = T_{n-1} \tag{5}$$

2 Kuramoto-Sivashinsky

The Kuramoto-Sivashinsky equation is a non-linear, fourth-order, one-dimensional partial differential equation that, "generally describes the dynamics near long-wave-length primary instabilities in the presence of appropriate (translational, parity, and Galilean) symmetries) [1]."

$$u_t = -(uu_x + u_{xx} + u_{xxxx}) = -(\frac{1}{2}u_x^2 + u_{xx} + u_{xxxx}) \tag{6}$$

Discretizing the fourth derivative requires neighbors of neighbors in the spatial dimension or an additional step where u_{xx} is calculated for the three points in the domain.

Finite difference discretization with neighbors of neighbors:

$$\frac{u_i^{m+1} - u_i^m}{\Delta t} = -\left(\frac{(u_{i+1}^m)^2 - (u_{i-1}^m)^2}{4\Delta x} + \frac{u_{i+1}^m + u_{i-1}^m + 2u_i^m}{\Delta x^2} + \frac{u_{i+2}^m - 4u_{i+1}^m + 6u_i^m - 4u_{i-1}^m + u_{i-2}^m}{\Delta x^4}\right) \tag{7}$$

Finite difference discretization with flux step:

$$u_{xx} + u_{xxxx} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} = \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial x^2}u_{xx} = \frac{\partial^2}{\partial x^2}\left(u + u_{xx}\right) \tag{8}$$

$$(u_{xx})_i = \frac{u_{i-1} + u_{i+1} - 2u_i}{\Delta x^2} \quad at \;\; i-1,\; i,\; i+1 \tag{9}$$

$$\frac{u_i^{m+1} - u_i^m}{\Delta t} = -\left(\frac{(u_{i+1}^m)^2 - (u_{i-1}^m)^2}{4\Delta x} + \frac{(u + u_{xx})_{i+1}^m + (u + u_{xx})_{i-1}^m + 2(u + u_{xx})_i^m}{\Delta x^2} \right) \tag{10}$$

Using a second-order, explicit Runge-Kutta method the solution at the new timestep is calculated in two steps. First the predictor solution is obtained at $u_i^{m+1/2}$. Then it is evaluated and added to u_i^m to obtain u_i^{m+1}.

The boundary condition is periodic so for example for n spatial points $i+1$ is $mod((i+1), n)$.

3 Euler

References

[1] Encyclopedia of Mathematics. *Kuramoto-Sivashinsky equation*. URL: `http://www.encyclopediaofmath.org/index.php?title=Kuramoto-Sivashinsky_equation&oldid=22687`.