# Handwritten Proof #1 - Master Theorem.

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Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative intergers by the recurrence.

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \quad \text{— equation 1.}$$

We interpret $n/b$ to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

Case 1: If $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = O(n^{\log_b a})$

Case 2: If $f(n) = \theta(n^{\log_b a})$, then $T(n) = \theta(n^{\log_b a} \cdot \lg n)$.

Case 3: If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large $n$, the $T(n) = \theta(f(n))$.

Proof.

Solve the recurrence:

$$\begin{aligned} T(n) &= aT\left(\tfrac{n}{b}\right) + f(n) \\ &= a\left[aT\left(\tfrac{n}{b^2}\right) + f\left(\tfrac{n}{b}\right)\right] + f(n) \\ &= a^2 T\left(\tfrac{n}{b^2}\right) + af\left(\tfrac{n}{b}\right) + f(n) \\ &= a^2\left[aT\left(\tfrac{n}{b^3}\right) + f\left(\tfrac{n}{b^2}\right)\right] + af\left(\tfrac{n}{b}\right) + f(n). \end{aligned}$$

$$T(n)=a^3T\left(\frac{n}{b^3}\right)+a^2f\left(\frac{n}{b^2}\right)+af\left(\frac{n}{b}\right)+f(n).$$

$$\vdots \quad T(1)=T\left(\frac{n}{b^k}\right) \Rightarrow \frac{n}{b^k}=1 \Rightarrow n=b^k \Rightarrow k=\log_b n.$$

$$=\sum_{i=0}^{\log_b n} a^i f\left(\frac{n}{b^i}\right)+\underline{O(n^{\log_b a})}.$$

↳ The sum across all leaves.
$(a^{\log_b n}f(1)=n^{\log_b a}f(1))$

$$\Rightarrow T(n)=\sum_{i=0}^{\log_b n} a^i f\left(\frac{n}{b^i}\right)+O(n^{\log_b a}) \quad \text{— equation 2.}$$

**Proof of Case 1:**

$$f(n)=O(n^{\log_b a-\varepsilon}) \Rightarrow f\left(\frac{n}{b^i}\right)=O\left(\left(\frac{n}{b^i}\right)^{\log_b a-\varepsilon}\right)$$

From equation 2, we obtained:

$$T(n)=\sum_{i=0}^{\log_b n} a^i \underline{f\left(\frac{n}{b^i}\right)}+O(n^{\log_b a}) \le \sum_{i=0}^{\log_b n} a^i \underline{\left(\frac{n}{b^i}\right)^{\log_b a-\varepsilon}}+O(n^{\log_b a})$$

Also,

$$\sum_{i=0}^{\log_b n} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\varepsilon} = n^{\log_b a-\varepsilon}\sum_{i=0}^{\log_b n} a^i \cdot b^{-i(\log_b a-\varepsilon)}$$

$\downarrow b^{\log_b a}=a.$

$$=n^{\log_b a-\varepsilon}\sum_{i=0}^{\log_b n} a^i \cdot a^{-i}\cdot b^{i\varepsilon}$$

$$=n^{\log_b a-\varepsilon}\underline{\sum_{i=0}^{\log_b n} b^{i\varepsilon}} \quad 等比級數$$

$$=n^{\log_b a-\varepsilon}\left[\frac{b^{\varepsilon(\log_b n+1)}-1}{b^\varepsilon-1}\right]$$

$$=n^{\log_b a-\varepsilon}\left[\frac{n^\varepsilon\cdot b^\varepsilon-1}{b^\varepsilon-1}\right]$$

$$\Rightarrow \sum_{i=0}^{\log_b n} a^i\left(\frac{n}{b^i}\right)^{\log_b a-\varepsilon} \le n^{\log_b a-\varepsilon}\frac{n^\varepsilon b^\varepsilon}{b^\varepsilon-1}=n^{\log_b a}\cdot\frac{b^\varepsilon}{b^\varepsilon-1}=O(n^{\log_b a})$$

Thus, $T(n) \in O(n^{\log_b a})$ #.

## Proof of Case 2:

$$f(n) = \Theta(n^{\log_b a}) \Rightarrow f\left(\frac{n}{b^i}\right) = \Theta\left(\left(\frac{n}{b^i}\right)^{\log_b a}\right)$$

$$\Rightarrow T(n) = \sum_{i=0}^{\log_b n} a^i f\left(\frac{n}{b^i}\right) + O(n^{\log_b a}) \leq \sum_{i=0}^{\log_b n} a^i \left(\frac{n}{b^i}\right)^{\log_b a} + O(n^{\log_b a})$$

$$\sum_{i=0}^{\log_b n} a^i \left(\frac{n}{b^i}\right)^{\log_b a} = n^{\log_b a} \sum_{i=0}^{\log_b n} a^i \cdot b^{-i \log_b a}$$

$$= n^{\log_b a} \sum_{i=0}^{\log_b n} a^i \cdot a^{-i}$$

$$= n^{\log_b a} \cdot \sum_{i=0}^{\log_b n} 1$$

$$= n^{\log_b a} \cdot (\log_b n + 1)$$

$$= \Theta(n^{\log_b a} \lg n)$$

$$\Rightarrow T(n) = \Theta(n^{\log_b a} \lg n) + O(n^{\log_b a}).$$

Thus, $T(n) \in \Theta(n^{\log_b a} \cdot \lg n)$ #

## Proof of Case 3:

$f(n) = \Omega(n^{\log_b a + \varepsilon})$, for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq c \cdot f(n)$, for some constant $c < 1$, and all sufficient large $n$.

$f(n) = n^{\log_b a + \varepsilon}$ for any $\varepsilon > 0$ with $c = b^{-\varepsilon} < 1$

$$\Rightarrow af\left(\frac{n}{b}\right) = a\left(\frac{n}{b}\right)^{\log_b a + \varepsilon}$$

$$= an^{\log_b a + \varepsilon} \cdot b^{-\log_b a - \varepsilon}$$

$$= an^{\log_b a + \varepsilon} \cdot b^{-\log_b a} \cdot b^{-\varepsilon}$$

$$= an^{\log_b a + \varepsilon} \cdot (a^{-1}) \cdot b^{-\varepsilon}$$

$$= f(n) \cdot b^{-\varepsilon} = c \cdot f(n).$$

easy induction on $i$ using the smoothness condition.

$$\Rightarrow af\left(\frac{n}{b}\right) \le c \cdot f(n). \Rightarrow a^i\left(f\left(\frac{n}{b^i}\right)\right) \le c^i f(n).$$

$$T(n) = \sum_{i=0}^{\log_b n} a^i f\left(\frac{n}{b^i}\right) + O(n^{\log_b a}) \le \sum_{i=0}^{\log_b n} c^i f(n) + O(n^{\log_b a}).$$

$$\le f(n) \sum_{i=0}^{\infty} c^i + O(n^{\log_b a})$$

$$= f(n) \frac{1}{1-c} + O(n^{\log_b a})$$

$$= O(f(n)).$$

In addition, since $f(n)$ is a term of the equation 2, the lower bound is immediate.

Therefore, we obtained $T(n) = \Theta(f(n))$ #.