

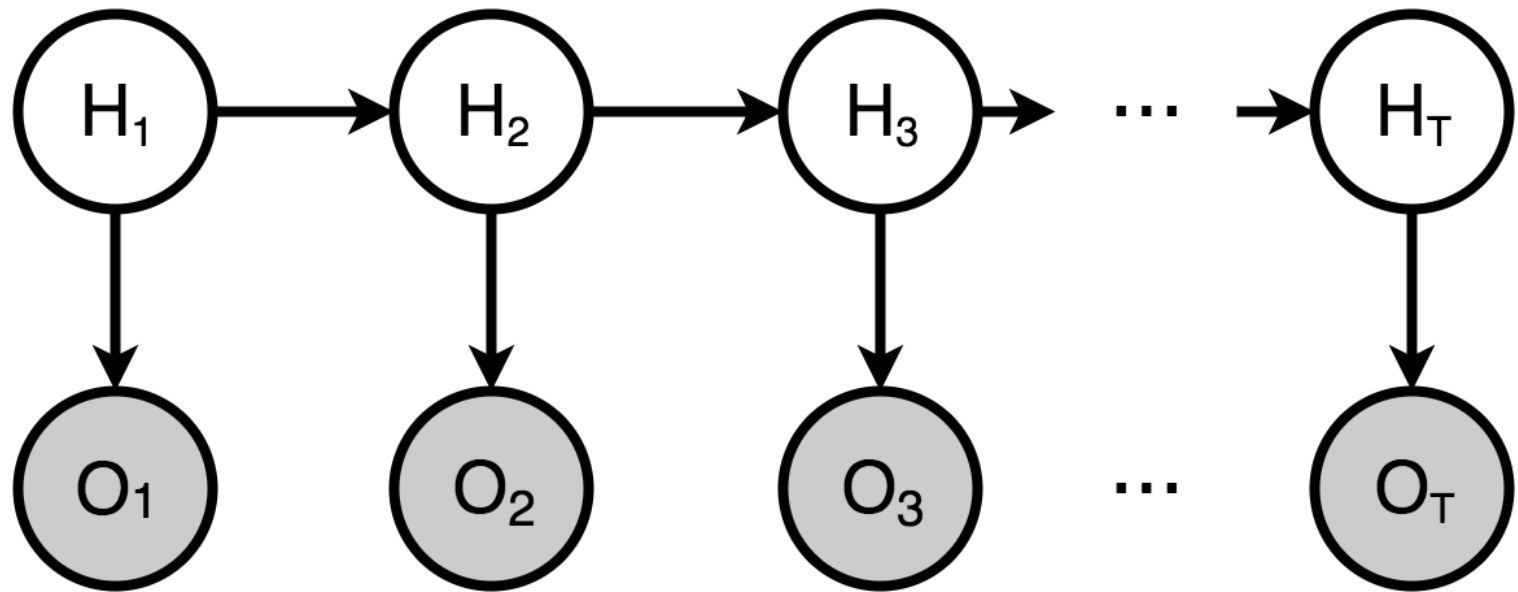
INF 552, Machine Learning for Data Science

University of Southern California

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Lesson 11

Hidden Markov Models



Hidden Markov Models

What is a hidden Markov model (HMM)?

A machine learning technique and...

...a discrete hill climb technique

Two for the price of one!

Where are HMMs used?

Speech recognition, information security,
and too many other things to list

Q: Why are HMMs so useful?

A: Widely applicable and ***efficient algorithms***

Markov Chain

Markov chain

“Memoryless random process”

Transitions depend only on current state (Markov chain of order 1)...

...and transition probability matrix

Markov Chain

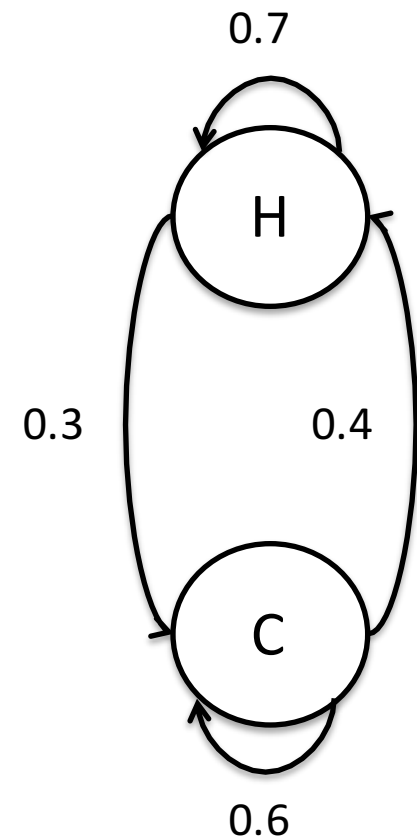
Suppose we're interested in
average annual temperature

Only consider Hot and Cold

From recorded history, obtain
probabilities for...

Year-to-year transitions

Based on thermometer
readings for “recent” years



Markov Chain

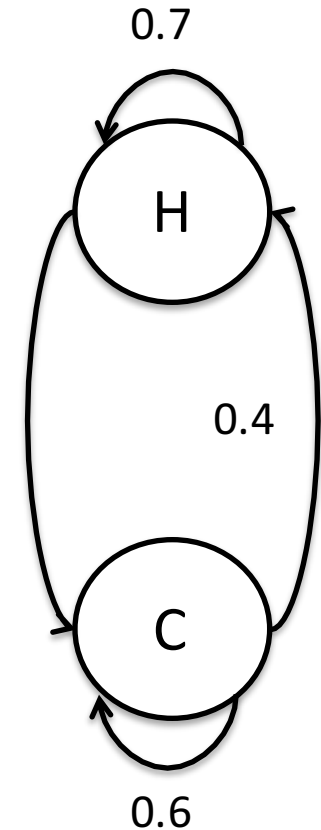
Transition probability matrix

Matrix is denoted as A

$$\begin{matrix} & \begin{matrix} H & C \end{matrix} \\ \begin{matrix} H \\ C \end{matrix} & \left[\begin{array}{cc} 0.7 & 0.3 \\ 0.4 & 0.6 \end{array} \right] \end{matrix}$$

Note, A is “row stochastic”

$$A = \left[\begin{array}{cc} 0.7 & 0.3 \\ 0.4 & 0.6 \end{array} \right]$$



So, **each element of A is between 0 and 1** and each row satisfies the definition of a discrete probability distribution, thus the elements of any given row sum to 1.

Markov Chain

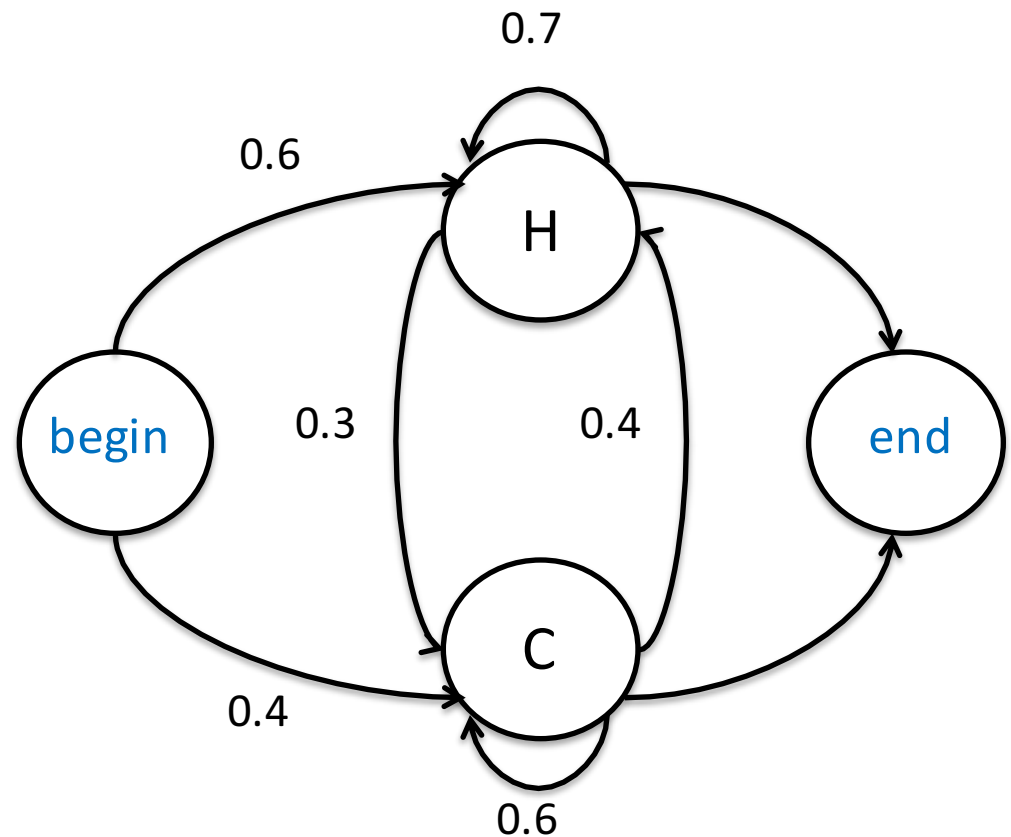
Can also include
begin, **end** states

Matrix for begin
state denoted π

In this example,

$$\pi = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix}$$

Note that π is also
row stochastic



Hidden Markov Model

HMM includes a Markov chain/ process

- But the Markov process is “hidden”, i.e., we can't directly observe the Markov process
- Instead, observe things that are probabilistically related to hidden states
- It's as if there is a “curtain” between Markov chain and the observations

HMM Example

Consider H/C annual temp example

Suppose we want to know H or C annual temperature in distant past

- Before thermometers were invented
- Note, we only distinguish between H and C

We assume transition between Hot and Cold years is same as today

Then the A matrix is **known**

HMM Example

Temps in past follow a Markov process

But, we cannot observe temperature in past

We find evidence that **tree ring size** is related to temperature

Looking at historical data, we find this holds

We only consider 3 tree ring sizes

Small, Medium, Large (S, M, L, respectively)

Measure tree ring sizes and recorded temperatures to determine relationship

HMM Example

We find that tree ring sizes and temperature related by

$$\begin{array}{c} S \quad M \quad L \\ H \quad \left[\begin{array}{ccc} 0.1 & 0.4 & 0.5 \end{array} \right] \\ C \quad \left[\begin{array}{ccc} 0.7 & 0.2 & 0.1 \end{array} \right] \end{array}$$

This is known as the B matrix

The matrix B is also row stochastic

$$B = \left[\begin{array}{ccc} 0.1 & 0.4 & 0.5 \\ 0.7 & 0.2 & 0.1 \end{array} \right]$$

HMM Example

Can we now find H/C temps in past?

We cannot measure (observe) temps

But we can measure tree ring sizes...

...and tree ring sizes related to temps

By probabilities in the B matrix

Can we say something intelligent about
temps over some interval in the past?

HMM Notation

A lot of notation is required

Notation may be the most difficult part...

T = the length of the observation sequence

N = the number of states in the model

M = the number of observation symbols

Q = $\{q_0, q_1, \dots, q_{N-1}\}$ = the states of the Markov process

V = $\{0, 1, \dots, M - 1\}$ = set of possible observations

A = the state transition probabilities

B = the observation probability matrix

π = the initial state distribution

\mathcal{O} = $(\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_{T-1})$ = observation sequence.

HMM Notation

Note that for simplicity, observations taken from $V = \{0, 1, \dots, M-1\}$

That is, $O_i \in V$ for $i = 0, 1, \dots, T-1$

The matrix $A = \{a_{ij}\}$ is $N \times N$, where

$$a_{ij} = P(\text{state } q_j \text{ at } t+1 \mid \text{state } q_i \text{ at } t)$$

The matrix $B = \{b_j(k)\}$ is $N \times M$, where

$$b_j(k) = P(\text{observation } k \text{ at } t \mid \text{state } q_j \text{ at } t).$$

HMM Example

Consider our temperature example...

What are the possible observations?

$V = \{0, 1, 2\}$, corresponding to S, M, L

What are states of Markov process?

$Q = \{H, C\}$

What are A, B, π , and T ?

A, B, π on previous slides

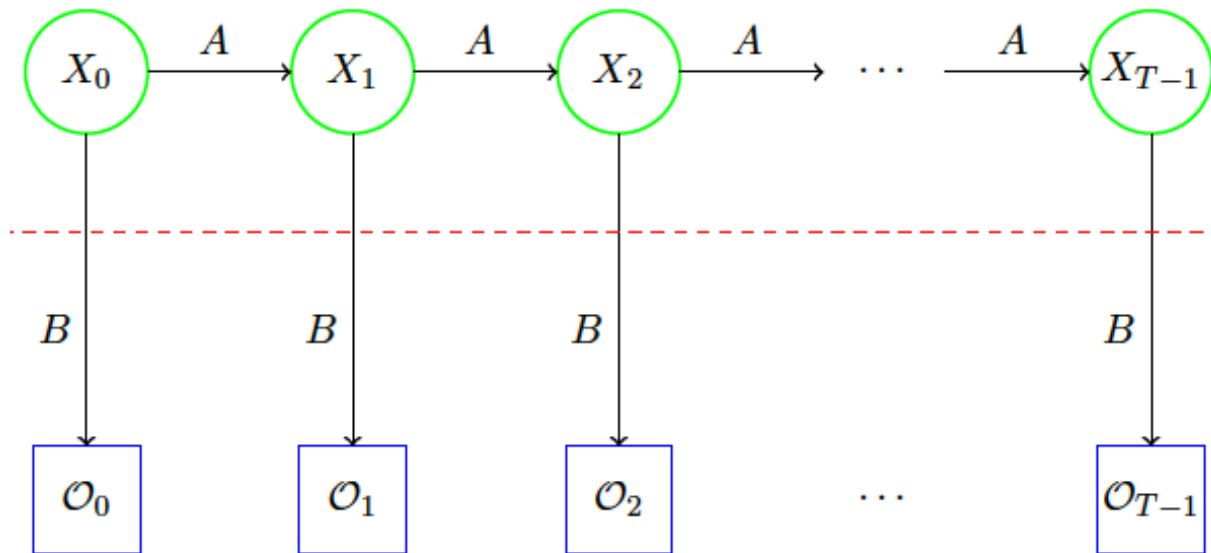
T is number of tree rings measured

What are N and M ?

$N = 2$ and $M = 3$

Generic HMM

Generic view of HMM



HMM defined by A , B , and π

We denote HMM “model” as $\lambda = (A, B, \pi)$

HMM Example

Suppose that we observe tree ring sizes

For a 4 year period of interest: S,M,S,L

Then $\mathcal{O} = (\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3) = (0, 1, 0, 2)$

Most likely (hidden) state sequence?

That is, most likely $X = (X_0, X_1, X_2, X_3)$

Let π_{x_0} be prob. of starting in state x_0

Note $b_{x_0}(\mathcal{O}_0)$ prob. of initial observation

And a_{x_0, x_1} is prob. of transition X_0 to X_1

And so on...

HMM Example

$$\pi = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix}$$

Bottom line?

$$\begin{array}{c} S \quad M \quad L \\ H \quad \begin{bmatrix} 0.1 & 0.4 & 0.5 \end{bmatrix} \quad H \quad \begin{bmatrix} 0.7 & 0.3 \end{bmatrix} \\ C \quad \begin{bmatrix} 0.7 & 0.2 & 0.1 \end{bmatrix} \quad C \quad \begin{bmatrix} 0.4 & 0.6 \end{bmatrix} \end{array}$$

We can compute $P(X, O)$ for any X

For $X = (X_0, X_1, X_2, X_3)$ we have $P(X, O)$

$$= \pi_{x_0} b_{x_0}(\mathcal{O}_0) a_{x_0, x_1} b_{x_1}(\mathcal{O}_1) a_{x_1, x_2} b_{x_2}(\mathcal{O}_2) a_{x_2, x_3} b_{x_3}(\mathcal{O}_3)$$

Suppose we observe (0,1,0,2), then what is probability of, say, HHCC?

S,M,S,L

Plug into formula above to find $P(HHCC, 0, 1, 0, 2)$

$$= 0.6(0.1)(0.7)(0.4)(0.3)(0.7)(0.6)(0.1) = 0.000212.$$

HMM Example

Do same for all 4-
state sequences

We find that the
winner is...

CCCH

Not so fast!

state	probability	normalized probability
<i>HHHH</i>	.000412	.042787
<i>HHHC</i>	.000035	.003635
<i>HHCH</i>	.000706	.073320
<i>HHCC</i>	.000212	.022017
<i>HCHH</i>	.000050	.005193
<i>HCHC</i>	.000004	.000415
<i>HCCH</i>	.000302	.031364
<i>HCCC</i>	.000091	.009451
<i>CHHH</i>	.001098	.114031
<i>CHHC</i>	.000094	.009762
<i>CHCH</i>	.001882	.195451
<i>CHCC</i>	.000564	.058573
<i>CCHH</i>	.000470	.048811
<i>CCHC</i>	.000040	.004154
<i>CCCH</i>	.002822	.293073
<i>CCCC</i>	.000847	.087963

$$\text{Sum} = P(O) \quad P(X|O) = P(X,O)/P(O)$$

HMM Example

The *path* CCCH scores the highest
In dynamic programming (DP), we find
highest scoring path

But, in HMM we maximize *expected
number of correct states*

Sometimes called “EM algorithm”

For “Expectation Maximization”

How does HMM work in this example?

HMM Example

For first position...

Sum probabilities for all paths that have H in 1st position, compare to sum of probabilities for paths with C in 1st position: biggest wins

Repeat for each position and we find

	element			
	0	1	2	3
$P(H O)$	0.188182	0.519576	0.228788	0.804029
$P(C O)$	0.811818	0.480424	0.771212	0.195971

HMM Example

Note: We could use both $P(X,O)$ and $P(X|O)$ to decide the best. Especially in paper and pencil problems, it is easier to use the un-normalized $P(X,O)$

	element			
	0	1	2	3
$P(H O)$	0.188182	0.519576	0.228788	0.804029
$P(C O)$	0.811818	0.480424	0.771212	0.195971

HMM Example

So, HMM solution gives us CHCH

While DP solution is CCCH

Which solution is better?

Neither solution is better!

Just using different definitions of “best”

	element			
	0	1	2	3
$P(H O)$	0.188182	0.519576	0.228788	0.804029
$P(C O)$	0.811818	0.480424	0.771212	0.195971

HMM Example

	element			
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So, HMM solution gives us CHCH

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Just using different definitions of “best”

HMM Paradox?

HMM maximizes expected number of correct states

Whereas DP chooses “best” overall path
Possible for HMM to choose a “path” that is impossible

Could be a transition probability of 0
Cannot get impossible path with DP
Is this a flaw with HMM?

No, it's a feature

Probability of Observations

Table computed for

$$\mathcal{O} = (\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3) \\ = (0, 1, 0, 2)$$

For this sequence,

$$P(\mathcal{O}) = .000412 + .000035 \\ + .000706 + \dots + \\ .000847 \\ = \text{left to the reader}$$

Similarly for other
observations \mathcal{O}

state	probability	normalized probability
<i>HHHH</i>	.000412	.042787
<i>HHHC</i>	.000035	.003635
<i>HHCH</i>	.000706	.073320
<i>HHCC</i>	.000212	.022017
<i>HCHH</i>	.000050	.005193
<i>HCHC</i>	.000004	.000415
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<i>CCHC</i>	.000040	.004154
<i>CCCH</i>	.002822	.293073
<i>CCCC</i>	.000847	.087963

Probability of Observations

If this calculation is made for all possible 4-observation sequences then the sum of the resulting probabilities (not the normalized probabilities) will be 1.

state	probability	normalized probability
<i>HHHH</i>	.000412	.042787
<i>HHHC</i>	.000035	.003635
<i>HHCH</i>	.000706	.073320
<i>HHCC</i>	.000212	.022017
<i>HCHH</i>	.000050	.005193
<i>HCHC</i>	.000004	.000415
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<i>CCCH</i>	.002822	.293073
<i>CCCC</i>	.000847	.087963

HMM Model

An HMM is defined by the three matrices, A , B , and π

Note that M and N are implied, since they are the dimensions of matrices

So, we denote an HMM “model” as

$$\lambda = (A, B, \pi)$$

The Three Problems

HMMs used to solve 3 problems:

Problem 1: Given a model $\lambda = (A, B, \pi)$ and observation sequence O , find $P(O|\lambda)$

That is, we can **score** an observation sequence to see how well it fits a given model

The Three Problems

HMMs used to solve 3 problems

Problem 2: Given $\lambda = (A, B, \pi)$ and O , find an optimal state sequence (in HMM sense)

Uncover hidden part (like previous example)

In many applications in NLP, the solution to Problem 2 is crucial. Example: Finding the grammatical roles of words in a sentence.

The Three Problems

HMMs used to solve 3 problems

Problem 3: Given O , N , and M , find the model λ that maximizes probability of O

That is, *train* a model to fit observations

HMMs in Practice

Often, HMMs used as follows:

Given an observation sequence...

Assume that (hidden) Markov process exists

Train a model based on observations

That is, solve Problem 3

“**Best**” N can be found by trial and error

Then given a sequence of observations, score it
versus the model we trained

This is Problem 1: high score implies similar to
training data, low score says it's not

HMMs in Practice

In this sense, HMM is a “machine learning” technique

To train a model, we do not need to specify anything except the parameter N
“Best” N often found by trial and error

So, we don’t need to “think” too much

Just train HMM and then use it

Fortunately, there are efficient algorithms for HMMs

The Three Solutions

We give detailed solutions to 3 problems

Note: We must find ***efficient*** solutions

The three problems:

Problem 1: Score an observation sequence versus a given model

Problem 2: Given a model, “uncover” hidden part

Problem 3: Given an observation sequence, train a model

Recall that we considered example for 2 and 1, but direct solutions are **very inefficient**

The Three Solutions

Appendix: Algorithmic Details

Solution 1

Score observations versus a given model

Given model $\lambda = (A, B, \pi)$ and observation sequence $O = (O_0, O_1, \dots, O_{T-1})$, find $P(O|\lambda)$

Denote hidden states as

$$X = (X_0, X_1, \dots, X_{T-1})$$

Then from definition of B,

$$P(O|X, \lambda) = b_{X_0}(O_0) b_{X_1}(O_1) \dots b_{X_{T-1}}(O_{T-1})$$

And from definition of A and π ,

$$P(X|\lambda) = \pi_{X_0} a_{X_0, X_1} a_{X_1, X_2} \dots a_{X_{T-2}, X_{T-1}}$$

Solution 1

Elementary conditional probability fact:

$$P(O, X | \lambda) = P(O | X, \lambda) P(X | \lambda)$$

Sum over all possible state sequences X ,

$$P(O | \lambda) = \sum P(O, X | \lambda) = \sum P(O | X, \lambda) P(X | \lambda)$$

$$= \sum \pi_{x_0} b_{x_0}(O_0) a_{x_0, x_1} b_{x_1}(O_1) \dots a_{x_{T-2}, x_{T-1}} b_{x_{T-1}}(O_{T-1})$$

This “works” but way too costly

Requires about $2TN^T$ multiplications

Why?

There should be a better way...

Forward Algorithm

Instead, use ***forward algorithm***

Or “alpha pass”

For $t = 0, 1, \dots, T-1$ and $i=0, 1, \dots, N-1$, let

$$\alpha_t(i) = P(O_0, O_1, \dots, O_t, X_t = q_i | \lambda)$$

Probability of “partial observation” to t , and

Markov process is in state q_i at step t

Can be computed recursively, efficiently

Forward Algorithm

Let $\alpha_0(i) = \pi_i b_i(O_0)$ for $i = 0, 1, \dots, N-1$

For $t = 1, 2, \dots, T-1$ and $i=0, 1, \dots, N-1$, let

$$\alpha_t(i) = \left(\sum \alpha_{t-1}(j) a_{ji} \right) b_i(O_t)$$

Where the sum is from $j = 0$ to $N-1$

From definition of $\alpha_t(i)$ we see

$$P(O|\lambda) = \sum \alpha_{T-1}(i)$$

Where the sum is from $i = 0$ to $N-1$

This requires only N^2T multiplications

Forward Algorithm

Note: The score $P(O|\lambda)$ is dependent on the length of the observation sequence. Consequently, to compare scores for sequences of different length, we can normalize to a per observation score, that is, $\text{score} = \sum \alpha_{T-1}(i) / T$.

Forward Algorithm

In a nutshell, $\alpha_T(i)$ is the probability that the sequence of observations up to time T correspond to the state being the i^{th} state at time T , given the model λ .

We compute $\alpha_t(i)$ recursively using $\alpha_{t-1}(j)$'s.

Solution 2

Given a model, find hidden states

Given $\lambda = (A, B, \pi)$ and O , find an optimal state sequence

Recall that optimal means “maximize expected number of correct states”

In contrast, DP finds best scoring path

For temp/tree ring example, solved this

But hopelessly inefficient approach

A better way: ***backward algorithm***

Or “beta pass”

Backward Algorithm

For $t = 0, 1, \dots, T-1$ and $i = 0, 1, \dots, N-1$, let $\beta_t(i)$
 $= P(O_{t+1}, O_{t+2}, \dots, O_{T-1} | X_t = q_i, \lambda)$

Probability of partial observation from t to
end and Markov process in state q_i at step t

Analogous to the forward algorithm

As with forward algorithm, this can be
computed recursively and efficiently

$$\beta_t(i) = P(O_{t+1}, O_{t+2}, \dots, O_{T-1} | X_t = q_i, \lambda)$$

Backward Algorithm

Let $\beta_{T-1}(i) = 1$ for $i = 0, 1, \dots, N-1$

For $t = T-2, T-3, \dots, 1$ and $i = 0, 1, \dots, N-1$,
let

$$\beta_t(i) = \sum_j a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)$$

Where the sum is from $j = 0$ to $N-1$

Solution 2

For $t = 1, 2, \dots, T-1$ and $i=0, 1, \dots, N-1$ define

$$\gamma_t(i) = P(X_t=q_i|O, \lambda)$$

(The probability of being in state i at time t)

Most likely state at t is q_i that maximizes $\gamma_t(i)$

$$\tilde{X}_t = \max_i \gamma_t(i).$$

Note that $\gamma_t(i) = \alpha_t(i)\beta_t(i)/P(O|\lambda)$

And recall $P(O|\lambda) = \sum \alpha_{T-1}(i)$

Solution 2

The bottom line?

Forward algorithm solves Problem 1

Forward/backward algorithms solve

Problem 2 by computing $\gamma_t(i)$ for each state at each time step (using both forward and backward paths) and choosing the state that maximizes it.

$$\tilde{X}_t = \max_i \gamma_t(i).$$

Solution 2

Why is it necessary to normalize gamma by dividing by $P(O \mid \lambda)$? Because these probabilities are computed assuming the observation sequence is known (i.e., given O), as opposed to being computed relative to the larger probability space.

Solution 3

Train a model: Given O , N , and M , find λ that maximizes probability of O

We'll iteratively adjust $\lambda = (A, B, \pi)$ to better fit the given observations O

- The size of matrices are fixed (N and M)

- But elements of matrices can change

It is nice that this works...

- ...and amazing that it's efficient!

Solution 3

For $t=0,1,\dots,T-2$ and i,j in $\{0,1,\dots,N-1\}$, define “di-gammas” as

$$\gamma_t(i,j) = P(X_t=q_i, X_{t+1}=q_j|O,\lambda)$$

Note $\gamma_t(i,j)$ is prob of being in state q_i at time t and transiting to state q_j at $t+1$

$$\text{Then } \gamma_t(i,j) = \alpha_t(i)a_{ij}b_j(O_{t+1})\beta_{t+1}(j)/P(O|\lambda)$$

$$\text{And } \gamma_t(i) = \sum \gamma_t(i,j)$$

Where sum is from $j = 0$ to $N - 1$

Baum-Welch Model Re-estimation

Given di-gammas and gammas...

For $i = 0, 1, \dots, N-1$ let $\pi_i = \gamma_0(i)$

For $i = 0, 1, \dots, N-1$ and $j = 0, 1, \dots, N-1$

$$a_{ij} = \sum Y_t(i,j) / \sum Y_t(i)$$

Where both sums are from $t = 0$ to $T-2$

For $j = 0, 1, \dots, N-1$ and $k = 0, 1, \dots, M-1$

$$b_j(k) = \sum Y_t(j) / \sum Y_t(j)$$

Both sums from from $t = 0$ to $T-1$ but only t for which $O_t = k$ are counted in numerator

Baum-Welch Model Re-estimation

Why does this work?

For the given observation sequence O , the sum $\sum Y_t(i)$ gives us the current best estimate for the total probability of being in state i , while $\sum Y_t(i,j)$ gives us the total probability of transitioning from state i to state j . Hence, the ratio $\sum Y_t(i,j)/\sum Y_t(i)$ enables us to re-estimate a_{ij} based on the current model parameters and observation sequence.

Solution 3

To summarize...

1. Initialize $\lambda = (A, B, \pi)$
2. Compute $\alpha_t(i)$, $\beta_t(i)$, $\gamma_t(i, j)$, $\gamma_t(i)$
3. Re-estimate the model $\lambda = (A, B, \pi)$
4. If $P(O|\lambda)$ increases by more than ε ,
goto 2 (where ε is small)

Solution 3

Some fine points...

Model initialization

If we have a good guess for $\lambda = (A, B, \pi)$ then we can use it for initialization

If not, let $\pi_i \approx 1/N$, $a_{i,j} \approx 1/N$, $b_j(k) \approx 1/M$

Subject to row stochastic conditions

But, do **not** initialize to exactly uniform values

Stopping conditions

Stop after some number of iterations and/or...

Stop if increase in $P(O|\lambda)$ is too small

HMM as Discrete Hill Climb

Algorithm on previous slides shows that HMM is a “discrete hill climb”

HMM consists of discrete parameters

Specifically, the elements of the matrices
And re-estimation process improves model by
modifying parameters

So, process “climbs” toward improved
model

This happens in a high-dimensional space

Dynamic Programming: The Viterbi Algorithm

Brief detour...

For $\lambda = (A, B, \pi)$ as above, it's easy to define a dynamic program (DP)

Executive summary:

DP is forward algorithm, with “sum” replaced by “max”

Dynamic Programming: The Viterbi Algorithm

Let $\delta_0(i) = \pi_i b_i(O_0)$ for $i=0, 1, \dots, N-1$

For $t=1, 2, \dots, T-1$ and $i=0, 1, \dots, N-1$ compute

$$\delta_t(i) = \max \left(\delta_{t-1}(j) a_{ji} \right) b_i(O_t)$$

Where the max is over j in $\{0, 1, \dots, N-1\}$

Note that **at each t , the DP computes best path for each state, up to that point**

So, probability of best path is $\max \delta_{T-1}(j)$

This max gives the highest probability

Not the best path, for that, see next slide

Dynamic Programming: The Viterbi Algorithm

- To determine optimal path

 - While computing deltas, keep track of pointers to previous state

 - When finished, construct optimal path by tracing back points

Dynamic Programming: The Viterbi Algorithm

For example, consider temp example: recall that we observe (0,1,0,2)

$$\pi = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix}$$

$$\begin{array}{c} S \quad M \quad L \\ H \begin{bmatrix} 0.1 & 0.4 & 0.5 \end{bmatrix} \\ C \begin{bmatrix} 0.7 & 0.2 & 0.1 \end{bmatrix} \end{array}$$

$$\begin{array}{c} H \quad C \\ H \begin{bmatrix} 0.7 & 0.3 \end{bmatrix} \\ C \begin{bmatrix} 0.4 & 0.6 \end{bmatrix} \end{array}$$

Probabilities for path of length 1:

$$P(H) = \pi_0 b_0(0) = 0.6(0.1) = 0.06 \text{ and } P(C) = \pi_1 b_1(0) = 0.4(0.7) = 0.28.$$

These are the only “paths” of length 1

Dynamic Programming: The Viterbi Algorithm

Probabilities for each path of length 2

$$P(HH) = 0.06(0.7)(0.4) = 0.0168$$

$$P(HC) = 0.06(0.3)(0.2) = 0.0036$$

$$P(CH) = 0.28(0.4)(0.4) = 0.0448$$

$$P(CC) = 0.28(0.6)(0.2) = 0.0336$$

Best path of length 2 ending with H is
CH

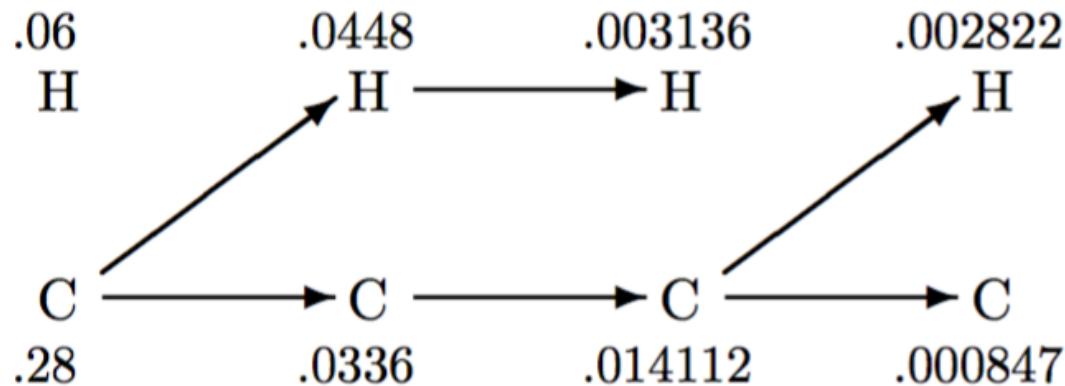
Best path of length 2 ending with C is
CC

$$\delta_t(i) = \max \left(\delta_{t-1}(j) a_{ji} \right) b_i(O_t)$$

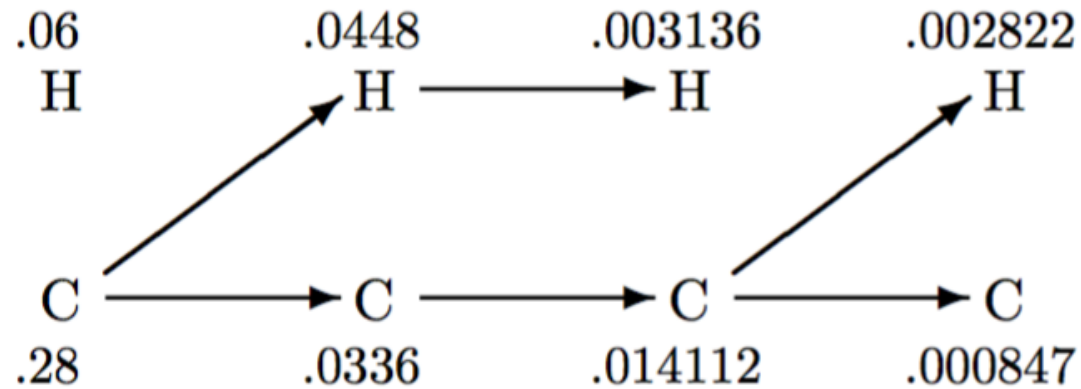
Dynamic Program

Continuing, we compute best path ending at H and C at each step

And save pointers: why?



Dynamic Program



Best final score is .002822

And thanks to pointers, best path is
CCCH

But what about underflow?

A serious problem in bigger cases

Dynamic Program

<https://www.youtube.com/watch?v=6JVqutwtzmo&t=4s>

Underflow Resistant DP

Common trick to prevent underflow:

Instead of multiplying probabilities...

...add logarithms of probabilities

Why does this work?

Because $\log(xy) = \log x + \log y$

Adding logs does not tend to 0

Note that these logs are negative...

...and we must avoid 0 probabilities

Underflow Resistant DP

Underflow resistant DP algorithm:

Let $\delta_0(i) = \log(\pi_i b_i(O_0))$

for $i=0, 1, \dots, N-1$

For $t=1, 2, \dots, T-1$ and $i=0, 1, \dots, N-1$ compute

$$\delta_t(i) = \max \left(\delta_{t-1}(j) + \log(a_{ji}) + \log(b_i(O_t)) \right)$$

Where the max is over j in $\{0, 1, \dots, N-1\}$

And **score** of best path is $\max \delta_{T-1}(j)$

As before, must also keep track of paths

HMM Scaling

Trickier to prevent underflow in HMM

We consider solution 3

Since it includes solutions 1 and 2

Recall for $t = 1, 2, \dots, T-1, i=0, 1, \dots, N-1,$

$$\alpha_t(i) = \left(\sum \alpha_{t-1}(j) a_{j,i} \right) b_i(O_t)$$

The idea is to normalize alphas so that they sum to 1

Algorithm on next slide

HMM Scaling

Given $\alpha_t(i) = \left(\sum \alpha_{t-1}(j) a_{j,i} \right) b_i(O_t)$

Let $a_0(i) = \alpha_0(i)$ for $i=0, 1, \dots, N-1$

Let $c_0 = 1/\sum a_0(j)$

For $i = 0, 1, \dots, N-1$, let $\alpha_0(i) = c_0 a_0(i)$

This takes care of $t = 0$ case

Algorithm continued on next slide...

HMM Scaling

For $t = 1, 2, \dots, T-1$ do the following:

For $i = 0, 1, \dots, N-1$,

$$a_t(i) = \left(\sum \alpha_{t-1}(j) a_{j,i} \right) b_i(O_t)$$

Let $c_t = 1 / \sum a_t(j)$

For $i = 0, 1, \dots, N-1$ let $\alpha_t(i) = c_t a_t(i)$

HMM Scaling

Easy to show $a_t(i) = c_0 c_1 \dots c_t \alpha_t(i)$ (#)

Simple proof by induction

So, $c_0 c_1 \dots c_t$ is scaling factor at step t

Also, easy to show that

$$a_t(i) = \alpha_t(i) / \sum \alpha_t(j)$$

Which implies $\sum a_{T-1}(i) = 1$ (##)

HMM Scaling

By combining (#) and (##), we have

$$\begin{aligned} 1 = \sum \alpha_{T-1}(i) &= c_0 c_1 \dots c_{T-1} \sum \alpha_{T-1}(i) \\ &= c_0 c_1 \dots c_{T-1} P(O|\lambda) \end{aligned}$$

Therefore, $P(O|\lambda) = 1 / c_0 c_1 \dots c_{T-1}$

To avoid underflow, we compute

$$\log P(O|\lambda) = -\sum \log(c_j)$$

Where sum is from $j = 0$ to $T-1$

HMM Scaling

Similarly, scale betas as $c_t \beta_t(i)$

For re-estimation,

Compute $\gamma_t(i,j)$ and $\gamma_t(i)$ using original formulas, but with scaled alphas, betas

This gives us new values for $\lambda = (A, B, \pi)$

“Easy exercise” to show re-estimate is exact when scaled alphas and betas used

Also, $P(O|\lambda)$ cancels from formula

Use $\log P(O|\lambda) = -\sum \log(c_j)$ to decide if iterate improves

All Together Now

Complete pseudo code for Solution 3

Given: $(O_0, O_1, \dots, O_{T-1})$ and N and M

Initialize: $\lambda = (A, B, \pi)$

A is $N \times N$, B is $N \times M$ and π is $1 \times N$

$\pi_i \approx 1/N$, $a_{ij} \approx 1/N$, $b_j(k) \approx 1/M$, each matrix row stochastic, but not uniform

Initialize:

maxIters = max number of re-estimation steps

iters = 0

oldLogProb = $-\infty$

Forward Algorithm

Forward algorithm With scaling

```
// compute  $\alpha_0(i)$ 
 $c_0 = 0$ 
for  $i = 0$  to  $N - 1$ 
     $\alpha_0(i) = \pi(i)b_i(\mathcal{O}_0)$ 
     $c_0 = c_0 + \alpha_0(i)$ 
next  $i$ 

// scale the  $\alpha_0(i)$ 
 $c_0 = 1/c_0$ 
for  $i = 0$  to  $N - 1$ 
     $\alpha_0(i) = c_0\alpha_0(i)$ 
next  $i$ 

// compute  $\alpha_t(i)$ 
for  $t = 1$  to  $T - 1$ 
     $c_t = 0$ 
    for  $i = 0$  to  $N - 1$ 
         $\alpha_t(i) = 0$ 
        for  $j = 0$  to  $N - 1$ 
             $\alpha_t(i) = \alpha_t(i) + \alpha_{t-1}(j)a_{ji}$ 
        next  $j$ 
         $\alpha_t(i) = \alpha_t(i)b_i(\mathcal{O}_t)$ 
         $c_t = c_t + \alpha_t(i)$ 
    next  $i$ 

    // scale  $\alpha_t(i)$ 
     $c_t = 1/c_t$ 
    for  $i = 0$  to  $N - 1$ 
         $\alpha_t(i) = c_t\alpha_t(i)$ 
    next  $i$ 
next  $t$ 
```


Backward Algorithm

Backward algorithm
or “beta pass”

With scaling

Note: same scaling
factor as alphas

```
// Let  $\beta_{T-1}(i) = 1$  scaled by  $c_{T-1}$ 
for  $i = 0$  to  $N - 1$ 
     $\beta_{T-1}(i) = c_{T-1}$ 
next  $i$ 

//  $\beta$ -pass
for  $t = T - 2$  to  $0$  by  $-1$ 
    for  $i = 0$  to  $N - 1$ 
         $\beta_t(i) = 0$ 
        for  $j = 0$  to  $N - 1$ 
             $\beta_t(i) = \beta_t(i) + a_{ij}b_j(\mathcal{O}_{t+1})\beta_{t+1}(j)$ 
        next  $j$ 
        // scale  $\beta_t(i)$  with same scale factor as  $\alpha_t(i)$ 
         $\beta_t(i) = c_t\beta_t(i)$ 
    next  $i$ 
next  $t$ 
```

Gammas

Using scaled
alphas and
betas

```
for  $t = 0$  to  $T - 2$ 
  denom = 0
  for  $i = 0$  to  $N - 1$ 
    for  $j = 0$  to  $N - 1$ 
      denom = denom +  $\alpha_t(i)a_{ij}b_j(\mathcal{O}_{t+1})\beta_{t+1}(j)$ 
    next  $j$ 
  next  $i$ 
  for  $i = 0$  to  $N - 1$ 
     $\gamma_t(i) = 0$ 
    for  $j = 0$  to  $N - 1$ 
       $\gamma_t(i, j) = (\alpha_t(i)a_{ij}b_j(\mathcal{O}_{t+1})\beta_{t+1}(j))/\text{denom}$ 
       $\gamma_t(i) = \gamma_t(i) + \gamma_t(i, j)$ 
    next  $j$ 
  next  $i$ 
next  $t$ 

// Special case for  $\gamma_{T-1}(i)$ 
denom = 0
for  $i = 0$  to  $N - 1$ 
  denom = denom +  $\alpha_{T-1}(i)$ 
next  $i$ 
for  $i = 0$  to  $N - 1$ 
   $\gamma_{T-1}(i) = \alpha_{T-1}(i)/\text{denom}$ 
next  $i$ 
```

Re-Estimation

Again, using
scaled gammas
So formulas
unchanged

```
// re-estimate  $\pi$ 
for  $i = 0$  to  $N - 1$ 
     $\pi_i = \gamma_0(i)$ 
next  $i$ 

// re-estimate  $A$ 
for  $i = 0$  to  $N - 1$ 
    for  $j = 0$  to  $N - 1$ 
        number = 0
        denom = 0
        for  $t = 0$  to  $T - 2$ 
            number = number +  $\gamma_t(i, j)$ 
            denom = denom +  $\gamma_t(i)$ 
        next  $t$ 
         $a_{ij} = \text{number} / \text{denom}$ 
    next  $j$ 
next  $i$ 

// re-estimate  $B$ 
for  $i = 0$  to  $N - 1$ 
    for  $j = 0$  to  $M - 1$ 
        number = 0
        denom = 0
        for  $t = 0$  to  $T - 2$ 
            if ( $\mathcal{O}_t == j$ ) then
                number = number +  $\gamma_t(i)$ 
            end if
            denom = denom +  $\gamma_t(i)$ 
        next  $t$ 
         $b_i(j) = \text{number} / \text{denom}$ 
    next  $j$ 
next  $i$ 
```

Stopping Criteria

Check that
probability increases

In practice, want

$\log\text{Prob} >$

$\text{oldLogProb} + \varepsilon$

And don't exceed
max iterations

```
// Compute  $\log[P(\mathcal{O} \mid \lambda)]$ 
```

```
logProb = 0
```

```
for  $i = 0$  to  $T - 1$ 
```

```
    logProb = logProb +  $\log(c_i)$ 
```

```
next  $i$ 
```

```
logProb =  $-\log\text{Prob}$ 
```

```
// To iterate or not to iterate, that is the question...
```

```
iters = iters + 1
```

```
if (iters < maxIters and logProb > oldLogProb) then
```

```
    oldLogProb = logProb
```

```
    goto  $\alpha$ -pass
```

```
else
```

```
    output  $\lambda = (\pi, A, B)$ 
```

```
end if
```

References

M. Stamp, [A revealing introduction to hidden Markov models](#)

L.R. Rabiner, [A tutorial on hidden Markov models and selected applications in speech recognition](#)

R.L. Cave & L.P. Neuwirth, [Hidden Markov models for English](#)