Derivation of the Cylindrical Mirror Anamorphosis

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OUTLINE

A cylindrical mirror σ is to be centered at the origin. The vantage point V is to be somewhere on the xz-plane outside of the cylinder and the picture plane π is to be on the yz-plane inside the cylinder.

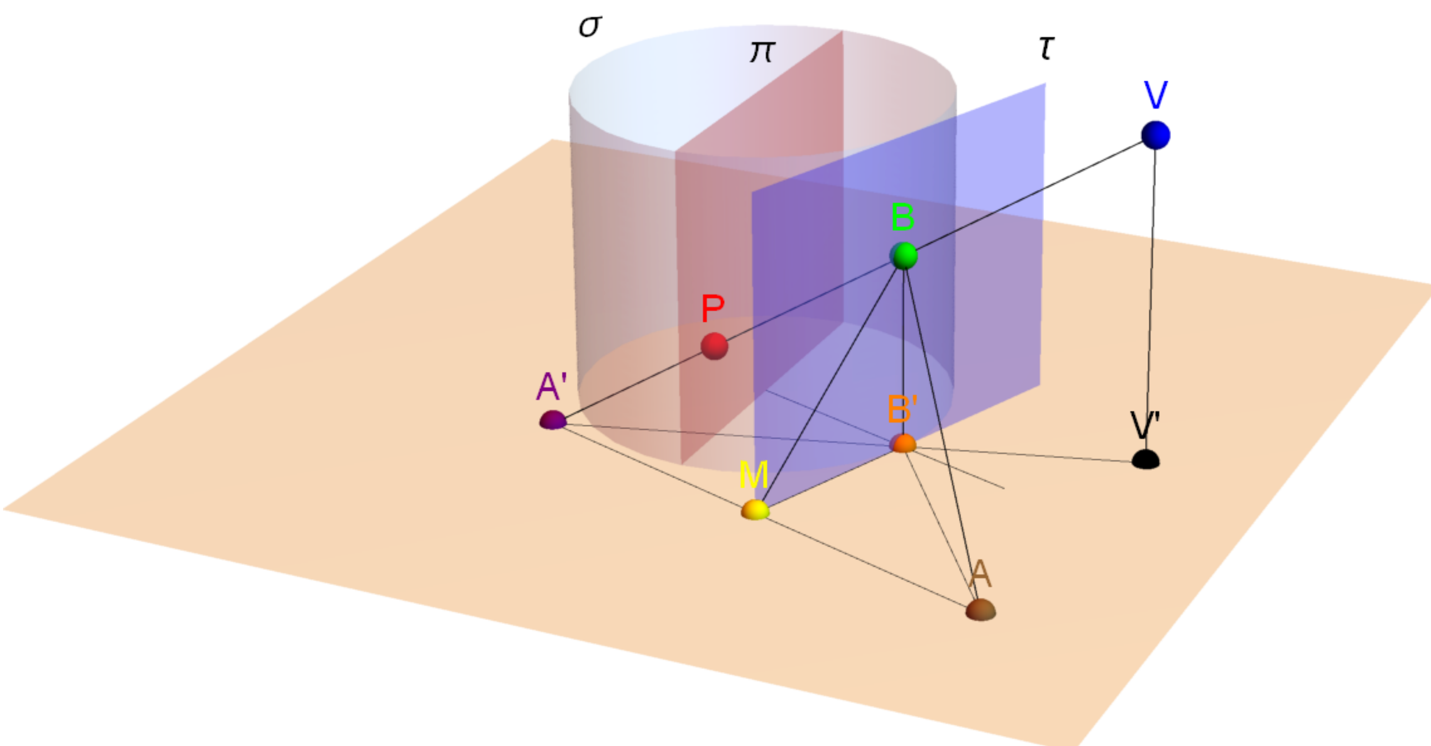

Figure 1: 3D View of Geometric Argument

1) Construct a line from V to some point P on π, call this $\vec{r}(t)$.
2) Find the intersection of $\vec{r}(t)$ with the xy-plane, call this point A'.

   A' is the projection of P onto xy-plane.
3) Find the intersection of $\vec{r}(t)$ and σ, call this point B.
4) Find a vector normal to the cylinder at B, call this $\vec{n}_B$.
5) Construct a line parallel to $\vec{n}_B$ that leaves point A', call this $\vec{N}(s)$.
6) Construct a plane tangent to the cylinder at B, call this $\vec{\tau}_B(u,v)$.
7) Find the intersection of $\vec{\tau}_B$ with the xy-plane, call this line $\vec{T}(u)$.
8) Find the intersection of $\vec{T}(u)$ and $\vec{N}(s)$, call this point M.
9) Translate A' by two times $\overrightarrow{A'M}$: $A = A' + 2\overrightarrow{A'M}$.

   A is the reflection of P onto xy-plane.

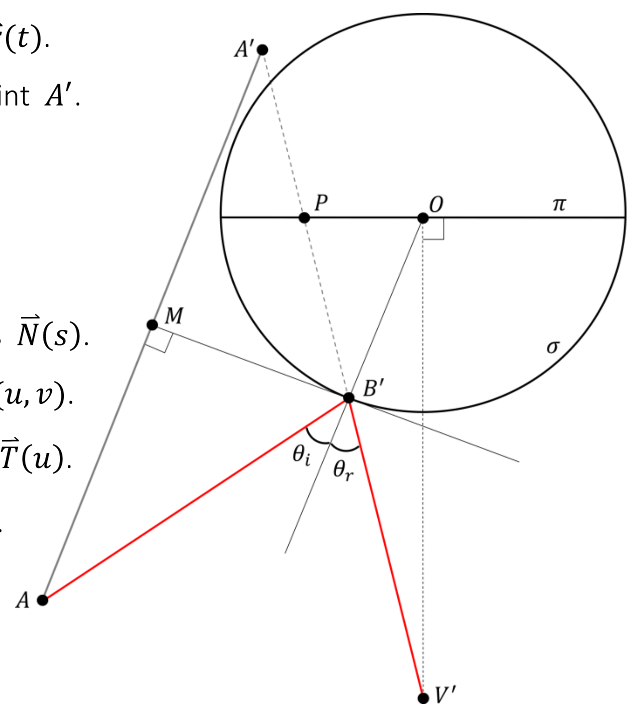

Figure 2: 2D View of Geometric Argument

DERIVATION

We proceed to derive the cylindrical mirror anamorphosis using an extension of plane mirror virtual images.

Let

$$\sigma = \{(x, y, z) : x^2 + y^2 = r^2,\ 0 \leq z \leq h\}$$

be a cylinder centered at the origin with height h and radius r.

We now assign $V(v_x, v_y, v_z)$ to be a fixed point on the xz-plane. To ensure all rays leaving V which intersect σ are to reflect onto the xy-plane, we must require v_z to be greater than the height of the cylinder and v_x to be greater than its radius. Thus, we have that

$$V \in \{(x, y, z) : x > r,\ y = 0,\ z > h\}.$$

For simplicity, let the picture plane

$$\pi = \{(x, y, z) : x = 0,\ -r \leq y \leq r,\ 0 \leq z \leq h\}$$

be the yz-plane contained inside the cylinder.[1]

We first construct a line from V to a point P on the picture plane.

Let $P(p_x, p_y, p_z) \in \pi$ and $t \geq 0$. Then we have

$$\vec{r}(t) = \langle v_x, v_y, v_z \rangle + t\langle p_x - v_x, p_y - v_y, p_z - v_z \rangle.$$

Or in parametric form,

$$(1) \qquad \begin{cases} x = v_x + t(p_x - v_x) \\ y = v_y + t(p_y - v_y) \\ z = v_z + t(p_z - v_z) \end{cases}.$$

We now find where $\vec{r}(t)$ intersects the plane $z = 0$.

Setting $t(p_z - v_z) + v_z = 0$, we have that $t = \frac{v_z}{v_z - p_z}$. Substituting this for t in (1), we obtain the point

$$(2) \qquad \begin{aligned} A' &= \left(\frac{v_z p_x - v_x p_z}{v_z - p_z}, \frac{v_z p_y - v_y p_z}{v_z - p_z}, 0\right) \\ &= (a'_x, a'_y, 0), \end{aligned}$$

which is the projection of P onto the xy-plane from the vantage point V.[2]

Next, we find where $\vec{r}(t)$ intersects the cylinder σ.

Substituting $x = v_x + t(p_x - v_x)$ and $y = v_y + t(p_y - v_y)$ into $0 = x^2 + y^2 - r^2$, we have

$$(3) \qquad \begin{aligned} 0 &= \big(v_x + t(p_x - v_x)\big)^2 + \big(v_y + t(p_y - v_y)\big)^2 - r^2 \\ &= \big(p_x^2 + p_y^2 - 2(v_x p_x + v_y p_y) + v_x^2 + v_y^2\big)t^2 + 2\big(v_x p_x + v_y p_y - v_x^2 - v_y^2\big)t + v_x^2 + v_y^2 - r^2. \end{aligned}$$

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[1] Note that the plane need not necessarily be on the yz-axis, though this simplifies the equations. In fact, any surface which is bounded by the cylinder such that a ray from V can simultaneously intersect both the cylinder and surface may be used.

[2] The set of all these points forms a perspective or oblique anamorphosis of the picture plane.

Set

$$(4)\qquad \begin{aligned} a &= p_x^2 + p_y^2 - 2\left(v_x p_x + v_y p_y\right) + v_x^2 + v_y^2 \\ b &= 2\left(v_x p_x + v_y p_y - v_x^2 - v_y^2\right) \\ c &= v_x^2 + v_y^2 - r^2 \end{aligned},$$

then (3) becomes

$$0 = at^2 + bt + c.$$

We can solve for t using the quadratic equation, where

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

To determine which solution we require, we first simplify (4). Note that $v_y = 0$ and $p_x = 0$, then (4) reduces to:

$$\begin{aligned} a &= p_y^2 + v_x^2 \\ b &= -2v_x^2 \\ c &= v_x^2 - r^2 \end{aligned}$$

This implies that $a > 0$ and $-b > 0$, and since $\vec{r}(t)$ approaches σ for $t > 0$, the first intersection of $\vec{r}(t)$ with σ is given by the negative root (the smaller of the two solutions). Hence,

$$(5)\qquad \begin{aligned} t_\sigma &= \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{v_x^2 - \sqrt{r^2\left(p_y^2 + v_x^2\right) - v_x^2 p_y^2}}{p_y^2 + v_x^2}. \end{aligned}$$

Then the point of intersection between $\vec{r}(t)$ and σ is given by

$$(6)\qquad \begin{aligned} B &= \left(v_x + t_\sigma(p_x - v_x), v_y + t_\sigma\left(p_y - v_y\right), v_z + t_\sigma(p_z - v_z)\right) \\ &= \left(b_x, b_y, b_z\right). \end{aligned}$$

To ensure all points on the picture plane can be reflected off σ, $\vec{r}(t)$ must simultaneously intersect the cylinder and picture plane. This becomes a concern for the z-coordinates of the picture plane as $\vec{r}(t)$ traces along the top rim of the cylinder, where we require

$$v_z + t_\sigma(p_z - v_z) \leq h.$$

This suggests we must restrict the allowable z-coordinates of P so that

$$p_z \leq v_z - \frac{v_z - h}{t_\sigma} = v_z - \frac{(v_z - h)\left(p_y^2 + v_x^2\right)}{v_x^2 - \sqrt{r^2\left(p_y^2 + v_x^2\right) - p_y^2 v_x^2}}.$$

Let

$$p_{z\,\max} = v_z - \frac{(v_z - h)\left(p_y^2 + v_x^2\right)}{v_x^2 - \sqrt{r^2\left(p_y^2 + v_x^2\right) - p_y^2 v_x^2}}.$$

To keep the picture plane rectangular, we find the minimum of $p_{z\,\max}$ with respect to p_y. Using some basic calculus, we find that $p_{z\,\max}$ obtains a minimum for $p_y \in [-r, r]$ at $p_y = 0$, where

$$\min\left(p_{z\,\max}(p_y)\right) = p_{z\,\max}(0) = \frac{v_x h - v_z r}{v_x - r}.$$

Then the usable z-coordinates must be less than or equal to this value to be visible:

(7) $$p_z \le \frac{v_x h - v_z r}{v_x - r}.$$

We now proceed to find the midpoint M between the virtual image A' and the anamorphic image A.

To find M we need to construct two additional lines. The first line starts at A' and is parallel to a vector normal to the cylinder at B. The second line is formed by a plane tangent to B intersecting the xy-plane.

A useful vector equation for the cylinder is given by

$$\vec{\sigma}(\theta, z) = \langle r\cos(\theta), r\sin(\theta), z\rangle.$$

Taking the partial derivatives of $\vec{\sigma}$, we have that

$$\begin{aligned}\vec{\sigma}_\theta &= \langle -r\sin(\theta), r\cos(\theta), 0\rangle \\ &= \langle -y, x, 0\rangle\end{aligned}$$

and

$$\vec{\sigma}_z = \langle 0,0,1\rangle.$$

Then the cross product of these is a vector normal to the cylinder:

$$\vec{\sigma}_\theta \times \vec{\sigma}_z = \langle x, y, 0\rangle.$$

Substituting $x = b_x$ and $y = b_y$, a vector normal to the cylinder at the point of intersection is then

$$\vec{n}_B = \langle b_x, b_y, 0\rangle.$$

Thus, the equation of a line parallel to $\vec{n}_B$ which starts at A' is given by

(8) $$\vec{N}(s) = \langle a'_x, a'_y, 0\rangle + s\langle b_x, b_y, 0\rangle.$$

Using the partial derivatives of $\vec{\sigma}$, we can construct a plane tangent to the cylinder.

It follows that an equation of a plane tangent to the cylinder at the point of intersection is given by

$$\begin{aligned}\overrightarrow{\tau_B}(u, v) &= \vec{B} + u\,\vec{\sigma}_\theta|_B + v\,\vec{\sigma}_z|_B \\ &= \langle b_x, b_y, b_z\rangle + u\langle -b_y, b_x, 0\rangle + v\langle 0,0,1\rangle\end{aligned}$$

and the intersection of $\overrightarrow{\tau_B}$ and the plane $z = 0$ forms the line

(9) $$\vec{T}(u) = \langle b_x, b_y, 0\rangle + u\,\langle -b_y, b_x, 0\rangle.$$

We now find the intersection of lines $\vec{N}(s)$ and $\vec{T}(u)$.

Setting the components of (8) and (9) equal to each other, and rearranging, we obtain the linear system of equations

$$\begin{aligned}-b_y u - b_x s &= a'_x - b_x \\ b_x u - b_y s &= a'_y - b_y\end{aligned}.$$

Solving for variables u and s, we have that the intersection occurs at

(10) $$u_m = \frac{a'_y b_x - a'_x b_y}{b_x^2 + b_y^2}$$

and

$$s_m = 1 - \frac{a'_x b_x + a'_y b_y}{b_x^2 + b_y^2}.$$

Substituting[3] (10) for u in (9), we obtain the point

(11) $$\begin{aligned} M &= (b_x - b_y u_m, b_y + b_x u_m, 0) \\ &= (m_x, m_y, 0), \end{aligned}$$

which is the midpoint of the line segment $\overline{A'A}$.

Then translating $\overrightarrow{A'}$ by two times $\overrightarrow{A'M}$ gives the anamorphic point

(12) $$\begin{aligned} A &= (2m_x - a'_x, 2m_y - a'_y, 0) \\ &= (a_x, a_y, 0). \end{aligned}$$

The set of all points A form the desired anamorphic image of the picture plane, where the distortion is removed when the image's reflection is viewed from the vantage point V.

SIMPLIFIED RESULT

Using the simplifying assumptions $v_y = 0$ and $p_x = 0$ and some substitution, we can reduce the number of equations for finding the anamorphic image point to simply:

$$t_\sigma = \frac{v_x^2 - \sqrt{r^2(p_y^2 + v_x^2) - v_x^2 p_y^2}}{p_y^2 + v_x^2}$$

(13) $$u_m = \frac{v_x p_y (t_\sigma (p_z - v_z) + v_z)}{(v_z - p_z)(t_\sigma^2 p_y^2 + v_x^2 (1 - t_\sigma)^2)}$$

(14) $$a_x = 2\left(v_x - t_\sigma\left(u_m p_y + v_x\right)\right) + \frac{v_x p_z}{v_z - p_z}$$

(15) $$a_y = 2\left(t_\sigma p_y + v_x u_m (1 - t_\sigma)\right) - \frac{v_z p_y}{v_z - p_z}$$

where r, h, v_x, and v_z are fixed parameters with p_y and p_z as input variables (specifying a point on the picture plane to be transformed) such that

(16) $$\begin{array}{c} 0 < r < v_x \\ 0 < h < v_z \\ -r \le p_y \le r \\ 0 \le p_z \le \frac{v_x h - v_z r}{v_x - r} \end{array}.$$

[3] Note that it's arbitrary which solution we pick to find the midpoint. We could just as well have substituted s_m into (8) to obtain M; however, the final expressions for a_x and a_y are slightly simpler using u_m.