HW 1

① (1.1-1c)

let $f(x) = 2x\cos(2x) - (x-2)^2$, intervals: $[2,3]$ and $[3,4]$

$2x\cos(2x)$ is continuous

$(x-2)^2$ is continuous.

Since sum of cont. functions is continuous, $f(x)$ is continuous.

Intermediate Value Theorem:

If $f \in C[a,b]$ and k is any number between $f(a)$ and $f(b)$, then there exists a number c in (a,b) for which $f(c) = k$.

Looking at the function values at the bounds

(i) $[2,3]$

$f(2) = 2 \cdot 2\cos(2 \cdot 2) - (2-2)^2 = 4 \cdot \cos(4) - 0 = 4\cos(4) = -2.6146$

$f(3) = 2 \cdot 3\cos(2 \cdot 3) - (3-2)^2 = 6\cos(6) - 1 = 4.7610$

Since $f(3) > 0$ and $f(2) < 0$, there must a point $x = c$ s.t f intercepts the x-axis at that point, that is $f(c) = 0 \Rightarrow c$ is a solution of $f(x)$.

(ii) $[3,4]$

$f(3) = 4.7610$ [from (i)]

$f(4) = 2 \cdot 4\cos(2 \cdot 4) - (4-2)^2 = 8\cos(8) - 2^2 = 8\cos(8) - 4 = -5.1640$

Since $f(3) \geq 0$ and $f(4) < 0$, there must be a point $x = c$ s.t f intercepts the x-axis at that point, that is $f(c) = 0 \Rightarrow c$ is a solution of $f(x)$.

2) (1.1-5a)

(a) $\max_{a \leq x \leq b} |f(x)|$

$f(x) = (2 - e^x + 2x)/3$, $[0,1]$.

① find points of inflection.

$$f'(x) = \frac{-e^x + 2}{3} = 0 \Rightarrow e^x = 2 \Rightarrow x = \ln 2.$$

$\therefore$ minima or maxima at $x = \ln 2 = 0.6931$

The max value of $|f(x)|$ in $[0,1]$ could be at $x=0$, $x=1$ or $x = \ln 2$.

at $x = 0$, $\dfrac{2 - e^0 + 2(0)}{3} = \dfrac{2-1}{3} = \dfrac{1}{3} = 0.3333$

at $x = 1$, $\dfrac{2 - e^1 + 2}{3} = \dfrac{4-e}{3} = 0.4272$

at $x = \ln 2$, $\dfrac{2 - e^{\ln 2} + 2\ln 2}{3} = \dfrac{2 - 2 + 2\ln 2}{3} = \dfrac{2\ln 2}{3} = 0.4621$

$\therefore \max_{a \leq x \leq b} |f(x)| = \underline{0.4621}$

3) (1.1-16) $\sin x \approx x$

let: $f(x) = \sin x$, $x_0 = 0$

1^0 in radians is $\frac{2\pi}{360} = \frac{\pi}{180} = x$

Since $\sin x \approx x$, compute

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

Assume 2nd Taylor polynomial to minimize error, then remainder term associated with $P_2(x)$ is $R_2(x) = \frac{f'''(\varepsilon(x))}{3!} x^3$

$$\Rightarrow \frac{-\cos(\varepsilon(x))}{3!} \cdot x^3 \le \frac{|\cos(\varepsilon(x))|\,|x^3|}{3!} = \frac{|x^3|}{3!} = \left(\frac{\pi}{180}\right)^3 \cdot \frac{1}{6}$$

$$= 8.8610 \cdot 10^{-7}$$

$\therefore$ error $= (8.8610) \cdot 10^{-7}$.

4) (1.1-19) $f(x) = e^x$, $x_0 = 0$.

(i) find $P_n(x)$ for $f(x)$ about x_0.

$f(x) = e^x \qquad f(0) = 1 \qquad f^{(n+1)}$ exists on $[0, 0.5]$

$f'(x) = e^x \qquad f'(0) = 1 \qquad f$ is continuous.

$\vdots \qquad \vdots$

$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = 1$

From Taylor's Theorem,

$$\therefore P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Therefore, $P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$

(ii) find n for $P_n(x)$ to approximate $f(x)$ within 10^{-6} on $[0, 0.5]$

The remainder term, $R_n(x) = \frac{f^{(n+1)}(\varepsilon(x))}{(n+1)!} \cdot (x - x_0)^{n+1}$

$$R_n(x) = \frac{e^{\varepsilon(x)}}{(n+1)!} x^{n+1} \le \frac{e^{\varepsilon(x)}}{(n+1)!} |x|^{n+1} \le \frac{e^{\varepsilon(0.5)}}{(n+1)!} (0.5)^{n+1} \le 10^{-6}$$

Since $\varepsilon(0.5) \in [0, 0.5]$, choosing max $\Rightarrow \varepsilon(0.5) = 0.5$.

$$\frac{e^{0.5}}{(n+1)!}(0.5)^{n+1} \le 10^{-6} \Rightarrow e^{0.5} \cdot 10^{6} \le \frac{(n+1)!}{(0.5)^{n+1}} \Rightarrow 1.648 \cdot 10^{6} \le \frac{(n+1)!}{(0.5)^{n+1}}$$

$\boxed{n = 7}$ is valid for which it is true.

5) (1.1-22)

$f(x) = x^3 + 2x + k$.

f is a continuous function. Let the interval be $(-\infty, \infty)$

f crosses the x-axis exactly once implies.

(i) f must change signs only once in the entire domain of the function.

To show exactly one sign change,

Note $f'(x) = 3x^2 + 2 > 0 \Rightarrow f$ is strictly increasing

$\Rightarrow$ No inflection points.

Since f is strictly increasing, and $\lim_{x \to -\infty} f(x) = -\infty$ and

$\lim_{x \to \infty} f(x) = \infty$, by intermediate value theorem, there must exist $x = c$, where $c \in (-\infty, \infty)$, for which $f(c) = 0$.

6) (1.1-29b)

$f \in C[a,b]$

$x_1, x_2 \in [a,b]$.

(b). $c_1, c_2 > 0$, show $\exists\ \varepsilon$ between x_1 and x_2 with

$$f(\varepsilon) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2}.$$

Without loss of generality assume $f(x_2) \geq f(x_1)$, then.

$$c_2 f(x_1) \leq c_2 f(x_2) \quad \text{since } c_2 > 0.$$

$$c_1 f(x_1) + c_2 f(x_1) \leq c_1 f(x_1) + c_2 f(x_2)$$

$$\therefore \quad f(x_1) \leq \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} \quad \text{since } c_1 + c_2 > 0.$$

Similarly,

$$c_1 f(x_1) \leq c_1 f(x_2) \quad \text{since } c_1 > 0$$

$$c_1 f(x_1) + c_2 f(x_2) \leq c_1 f(x_2) + c_2 f(x_2)$$

$$\frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} \leq f(x_2) \quad \text{since } c_1 + c_2 > 0.$$

$$\therefore \quad f(x_1) \leq f(\varepsilon) \leq f(x_2)$$

Since f is continuous on $[a,b]$ and $x_1, x_2 \in [a,b]$, f is continuous between x_1 and x_2.

Applying Intermediate value theorem, we can conclude for K between $f(x_1)$ and $f(x_2)$, there exists a number ε between x_1 and x_2 such that $f(\varepsilon) = K$, where

$$K = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2}.$$

hw1_4.m

```matlab
% CODE FOR HW1 PROBLEM 4

f = @(n) 1.648*10^6 <= factorial(n+1) / (0.5)^(n+1);

val = 1;

while f(val) ~= 1
    val = val + 1;
end

fprintf('%d\n', val);


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