Ex 1.2:

①

(a) $p = \pi$ , $p^* = 22/7 = 3.\overline{142857}$

$$\begin{array}{r} 3.142857 \\ 3.141592 \\ \hline 0.0012651 \end{array}$$

Abs error $= |p - p^*| = |3.141592 - 3.142857| = |0.0012651|$
$= 0.1265 \times 10^{-2}$

Rel error $= \dfrac{\text{Abs error}}{p} = \dfrac{0.1265 \times 10^{-2}}{3.141592} = 0.40266 \times 10^{-3}$

②

(d) $p = 9!$
$p^* = \sqrt{18\pi}\,(9/e)^9$

$p - p^* = 9! - \sqrt{18\pi}\,(9/e)^9 = 362880 - \sqrt{2 \cdot 9\pi}\,(9/e)^9$
$= 362880 - 3\sqrt{2\pi}\,(9/e)^9$
$= 3.343127 \times 10^3$

abs error $= |p - p^*| = 3.343127 \times 10^3$
relerror $=$ absorror$/p$ $= 0.009212762 = 9.212762 \times 10^{-3}$

⑤

(c) $\left(\frac{1}{3} - \frac{3}{11}\right) + \frac{3}{20}$

(i) exactly: $\left(\frac{1}{3} - \frac{3}{11}\right) + \frac{3}{20} = \left(\frac{11-9}{3 \cdot 11}\right) + \frac{3}{20} = \left(\frac{2}{33}\right) + \frac{3}{20} = \frac{40+99}{33 \cdot 20}$
$= \frac{139}{660}$

(ii) three-digit chopping arithmetic.

$$\begin{array}{r} 0.150 \\ 0.061 \\ \hline 0.211 \end{array}$$

let $x = 1/3 \Rightarrow fl(x) = 0.333$
let $y = 3/11 \Rightarrow fl(y) = 0.272$
let $z = 3/20 \Rightarrow fl(z) = 0.150$

$$\begin{array}{r} 0.333 \\ 0.272 \\ \hline 0.061 \end{array}$$

$m = fl(fl(x) - fl(y)) = fl(0.333 - 0.272) = fl(0.061) = 0.061$

$\Rightarrow fl(m + fl(z)) = fl(0.061 + 0.150) = fl(0.211) = \underline{0.211}$

(iii) three-digit rounding arithmetic.

$$\begin{array}{r} 0.060 \\ 0.150 \\ \hline 0.210 \end{array}$$

let $x = 1/3 \Rightarrow fl(x) = 0.333$
let $y = 3/11 \Rightarrow fl(y) = 0.273$
let $z = 3/20 \Rightarrow fl(z) = 0.150$

$m = fl(fl(x) - fl(y)) = fl(0.333 - 0.273) = fl(0.060) = 0.060$
$\Rightarrow fl(m + fl(z)) = fl(0.060 + 0.150) = fl(0.210) = \underline{0.210}$

(iv) (a) chopping: rel error $= \dfrac{\left| 139/660 - 0.211 \right|}{\frac{139}{660}}$

$= \dfrac{\left| 139 - (0.211)\,660 \right|}{139}$

$= \dfrac{\left| 139 - 139.26 \right|}{139}$

$= \dfrac{0.26}{139} \approx 0.00187$

(b) rounding: rel error $= \dfrac{\left| 139/660 - 0.210 \right|}{139/660}$

$= \dfrac{\left| 139 - (0.210)\,660 \right|}{139}$

$= \dfrac{\left| 139 - 138.6 \right|}{139} = \dfrac{0.4}{139} \approx 0.00287$

(11) $\arctan(x) \approx x - (1/3)x^3 + (1/5)x^5$

(a) $p = 4\left[\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)\right]$

$p^* = 4\left[\left(\left(\frac{1}{2}\right) - \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^3 + \left(\frac{1}{5}\right)\left(\frac{1}{2}\right)^5\right) + \left(\left(\frac{1}{3}\right) - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{1}{5}\right)\left(\frac{1}{3}\right)^5\right)\right]$

$p = \pi$

$\arctan\left(\frac{1}{2}\right) = \frac{1}{2} - \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^3 + \left(\frac{1}{5}\right)\left(\frac{1}{2}\right)^5 = 0.464583$

$\arctan(1/3) = \frac{1}{3} - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{1}{5}\right)\left(\frac{1}{3}\right)^5 = 0.321810$

$p^* = 4.(0.464583 + 0.321810) = 3.145572$

abs error $= |p - p^*| = |\pi - 3.145572| = 0.003979$

rel error $=$ abs error $/\, p = 0.003979/\pi = 0.001266$

(b) $p^* = 16 \arctan\left(\frac{1}{5}\right) - 4 \arctan\left(\frac{1}{239}\right)$

$\arctan(1/5) = 0.197397$

$\arctan(1/239) = 0.00418407 = 0.418407 \times 10^{-2}$

$p^* = 16 \cdot (0.197397) - 4(0.418409 \times 10^{-2}) = 3.14161$
$= 0.314161 \times 10^1$

abs error $= |p - p^*| = |\pi - 0.314161 \times 10^1| = |3.14159 - 3.14161|$
$= |0.00002|$
$= \underline{0.2} \times 10^{-4}$

rel error $=$ abs error $/ \pi = \dfrac{0.2 \times 10^{-4}}{3.14159} = 6.\underline{36620} \times 10^{-6}$

(7)

$$f(x) = \frac{e^x - e^{-x}}{x}$$

(a) $\lim_{x \to 0} \dfrac{e^x - e^{-x}}{x} \Rightarrow \dfrac{0}{0}$ form

Apply L'hopital's rule: $\lim_{x \to 0} \dfrac{e^x + e^{-x}}{1} = 1 + 1 = 2$

(b) $f(0.1) = \dfrac{e^{0.1} - e^{-0.1}}{0.1}$

$fl(e^{0.1}) = fl(0.11051709 \times 10^1) = 0.111 \times 10^1$
$fl(e^{-0.1}) = fl(0.90487) = 0.905$

$f(0.1) = \dfrac{fl(0.111 \times 10 - 0.905)}{0.1} = fl\left(\dfrac{0.205}{0.1}\right) = \underline{2.05}$

(c) $f(x) = \dfrac{e^x - e^{-x}}{x}$

$e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$

$\therefore f(x) = \dfrac{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6}\right)}{x}$

$= \cancel{1} + x + \cancel{x^2/2} + x^3/6 - \cancel{1} + x - \cancel{x^2/2} + x^3/6$

$= \dfrac{2x + x^3/3}{x} = 2 + x^2/3$

$fl(f(0.1)) = fl\left(2 + \dfrac{(0.1)^2}{3}\right) = fl\left(2 + fl\left(\dfrac{0.01}{3}\right)\right)$
$= fl\left(2 + fl\left(\dfrac{0.010}{3}\right)\right) = fl(2.003)$
$= \underline{2.00}$

(d) $f(0.1) = 2.003335000$

rel error $\Rightarrow$ (b) $\Rightarrow$ $\dfrac{|2.003335000 - 2.051|}{2.003335000}$

$= 0.02329$

(c) $\Rightarrow$ $\dfrac{|2.003335000 - 2.001|}{2.003335000}$

$= 0.00166$

(19)

(c) Sign: 0 $\Rightarrow$ +ve

Exp: 01111111111 $\Rightarrow 2^{10}-1 = 1023$

$f$: $2^{-2} + 2^{-4} + 2^{-7} + 2^{-8} = 1/4 + 1/16 + 1/128 + 1/256 = 83/256$

$\Rightarrow (-1)^0 \, 2^{1023-1023} \cdot (1 + 83/256) = 1 + 83/256 = 1.32421875$

(d) Sign: 0 $\Rightarrow$ +ve

Exp: $2^{10}-1 = 1023$

$f$: $2^{-2} + 2^{-4} + 2^{-7} + 2^{-8} + 2^{-52} = 1/4 + 1/16 + 1/256 + 1/256 + 1/2^{52}$

$\Rightarrow (-1)^0 \cdot 2^{1023-1023} (1 + 83/256 + 2^{-52}) = 1.32421875 + 2^{-52}.$

(20)

(c) next smallest:

Exp: 1023

counting in steps of $2^{1023-1023} \cdot 2^{-52} = 2^{-52}$.

$= 1.32421875 - 2^{-52}$

next largest:

Exp: 1023

counting in steps of $2^{1023-1023} \cdot 2^{-52} = 2^{-52}$.

$= 1.32421875 + 2^{-52}$.

(d) next smallest:

Exp: 1023

counting in steps of $2^{1023-1023} \cdot 2^{-52} = 2^{-52}$.

$= 1.32421875 + 2^{-52} - 2^{-52} = 1.32421875$

next largest:

Exp: 1023

Downhill in steps of $2^{1023-1023} \cdot 2^{-52} = 2^{-52}$.

$= 1.32421875 + 2^{-52} + 2^{-52} = 1.32421875 + 2^{-51}$

## Ex 1.3

① (a) $\sum_{i=1}^{10} (1/i^2)$

$\Rightarrow 1 + 1/4 + 1/9 + \cdots + 1/100$

$\Rightarrow 1 + 0.250 + 0.111 + 0.062 + 0.040 + 0.027 + 0.020 + 0.015 + 0.012 + 0.010$

$\Rightarrow 1.250 + 0.111 = 1.361 = 1.36 + 0.062 = 1.422 = 1.42 + 0.040$
$= 1.46 + 0.027 = 1.487 = 1.48 + 0.020 = 1.5 + 0.015$
$= 1.51 + 0.012 = 1.522 = 1.52 + 0.010 = \underline{1.53}$

$\Rightarrow 1/100 + 1/81 + \cdots + 1$

$\Rightarrow 0.010 + 0.012 + 0.015 + 0.020 + 0.027 + 0.040 + 0.062 + 0.111 + 0.250 + 1$

$\Rightarrow 0.186 + 0.111 = 0.297 + 0.250 + 1 = 1.547 = \underline{1.54}$

⑥ (a) $n \to \infty$

$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$

Taylor series of $\sin x$ about $x_0 = 0$, $\sin x = 0 + 1(x) - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$\sin\left(\frac{1}{n}\right) = 0 + \frac{1}{n} - \frac{1}{n^3 3!} + \frac{1}{n^5 5!} - \frac{1}{n^7 7!} + \cdots$

$\left|\sin\left(\frac{1}{n}\right) - 0\right| = \left|\frac{1}{n} - \frac{1}{n^3 3!} + \frac{1}{n^5 5!} \cdots - 0\right| \leq \left|\frac{1}{n}\right|$

$\Rightarrow \sin\left(\frac{1}{n}\right) = 0 + O\left(1/n\right)$

(b) $\lim_{n\to\infty} \sin\left(\frac{1}{n^2}\right) = 0$

$\sin\left(\frac{1}{n^2}\right) = 0 + \frac{1}{n^2} - \frac{1}{n^5 3!} + \frac{1}{n^{10} 5!} - \cdots$

$\left|\sin\left(\frac{1}{n^2}\right) - 0\right| = \left|\frac{1}{n^2} - \frac{1}{n^5 3!} + \frac{1}{n^{10} 5!} - \cdots - 0\right| \leq \left|\frac{1}{n^2}\right|$

$\Rightarrow \sin\left(\frac{1}{n^2}\right) = 0 + O\left(1/n^2\right)$

(c) $\lim_{n\to\infty} \left(\sin\left(\frac{1}{n}\right)\right)^2 = 0$

$\sin^2\left(\frac{1}{n}\right) \Rightarrow$ Taylor polynomial of $\sin^2 x$ about $x_0 = x$

$\Rightarrow$ $f(x_0) = 0$

$f'(x_0) = 2\sin x \cos x = \sin 2x = 0$

$f''(x_0) = \cos 2x \cdot 2 = 2$

$f^{(3)}(x_0) = -\sin 2x \cdot 2 \cdot 2 = -4\sin 2x = 0$

$f^{(4)}(x_0) = -8\cos 2x = -8$

$f^{(5)}(x_0) = 16\sin 2x = 0$

$f^{(6)}(x_0) = 32\cos 2x = 32.$

$$\therefore \sin^2 x = 0 + \frac{0(x)}{1!} + \frac{2\cdot x^2}{2!} + \frac{0\cdot x^3}{3!} - \frac{8x^4}{4!} + \frac{0\cdot x^5}{5!} + \frac{32\cdot x^6}{6!}$$

$$= \frac{2x^2}{2!} - \frac{8x^4}{4!} + \frac{32x^6}{6!} \cdots$$

$$\sin^2(1/n) = 2\cdot\frac{1}{n^2 2!} - 8\cdot\frac{1}{n^4 \cdot 4!} + 32\cdot\frac{1}{n^6 \cdot 6!} \cdots$$

$$\left|\sin^2(1/n) - 0\right| = \left|\frac{2}{n^2 2!} - \frac{8}{n^4 4!} + \cdots - 0\right| \le \left|\frac{1}{n^2}\right|$$

$$\therefore \sin^2(1/n) = 0 + O\left(\frac{1}{n^2}\right)$$

(d) $\lim_{n\to\infty} [\ln(n+1) - \ln(n)] = 0$

$$\ln(n+1) - \ln(n) = \ln\left(\frac{n+1}{n}\right) = \ln\left(1 + \frac{1}{n}\right)$$

Taylor expansion of $\ln(1+x)$ about $x_0 = 0$

$f(x_0) = \ln(1) = 0$

$f'(x_0) = \frac{1}{1+x} = 1$

$f^{(3)}(x_0) = \frac{-1}{(1+x)^2} = -1$

$f^{(4)}(x_0) = \frac{2}{(1+x)^3} = 2$

$f^{(5)}(x_0) = \frac{-6}{(1+x)^4} = -6$

$$\therefore \ln(1+x) = 0 + 1x - 1\frac{x^2}{2!} + 2\frac{x^3}{3!} - \frac{6x^4}{4!} + \cdots$$

$$\Rightarrow \ln(1 + 1/n) = \frac{1}{n} - \frac{1}{n^2 2!} + \frac{2}{n^3 3!} - \frac{6}{n^4 4!} + \cdots$$

$$\Rightarrow \left| \ln(1+1/n) - 0 \right| = \left| 1/n - 1/n^2 2! + 2/n^3 3! - 6/n^4 4! \cdots - 0 \right| \le |1/n|$$

$$\therefore \ln(1+1/n) = 0 + O(1/n)$$

(7)

(a) $\lim_{h \to 0} \frac{\sin h}{h} = 1$

Taylor series of $\sin x$ about $x_0 = 0$, $\sin x = 0 + 1(x) - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$$\therefore \frac{\sin h}{h} = \frac{\cancel{h} - h^3/3! + h^5/5! + \cdots}{\cancel{h}} = 1 - h^2/3! + h^4/5! - \cdots$$

$$\left| \frac{\sin h}{h} - 1 \right| = \left| -\frac{h^2}{3!} + \frac{h^4}{5!} - \frac{h^6}{7!} + \cdots \right| \le \left| \frac{h^2}{3!} \right|$$

$$\Rightarrow \frac{\sin h}{h} = 1 + O(h^2)$$

(b) $\lim_{h \to 0} \frac{1 - \cos h}{h} = 0$

Taylor series of $\cos x$ about $x_0 = 0$, $\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \cdots$

$$\frac{1 - \cos h}{h} = \frac{1 - (1 - h^2/2! + h^4/4! - h^6/6! + \cdots)}{h}$$

$$= \frac{0 + h^2/2! - h^3/4! + h^5/6! + \cdots}{\cancel{h}}$$

$$= h/2! - h^3/4! + h^5/6! + \cdots$$

$$= O(h)$$

$$\Rightarrow \frac{1 - \cos h}{h} = O(h)$$

(c) $\lim_{h \to 0} \frac{\sin h - h \cos h}{h} = 0$

Apply Taylor expansion

$$\Rightarrow \frac{(h - h^3/3! + h^5/5! - h^7/7! + \cdots) - h(1 - h^2/2! + h^4/4! - h^6/6! + \cdots)}{h}$$

$$\Rightarrow \cancel{h} - h^2/3! + h^4/5! - h^6/7! + \cdots - \cancel{h} + h^2/2! - h^4/4! - h^6/6! + \cdots$$

$$= h^2(1/2! - 1/3!) - h^4(1/4! - 1/5!) - h^6(1/7! + 1/6!) - \cdots$$

$$= O(h^2)$$

$$\therefore \quad \frac{\sin h - h\cos h}{h} = O(h^2)$$

(d) $\lim_{h\to 0} \frac{1-e^h}{h} = -1$

Apply Taylor expansion,

$$\frac{1-e^h}{h} = \frac{1-(1+h+h^2/2!+h^3/3!+h^4/4!+\cdots)}{h}$$

$$= \frac{0-h-h^2/2!-h^3/3!-h^4/4!-\cdots}{h}$$

$$= -1-h/2!-h^2/3!-h^3/4!-\cdots$$

$$= -1-(h/2!+h^2/3!+h^3/4!+\cdots)$$

$$= -1+O(h)$$

$$\Rightarrow \frac{1-e^h}{h} = -1+O(h)$$

(11)

$\lim_{n\to\infty} x_n = x$, $x = (1+\sqrt{5})/2$, $x_n = F_{n+1}/F_n$.

$F_0 = 1$, $F_1 = 1$

$F_{n+2} = F_n + F_{n+1}$ if $n \geq 0$.

$$x_n = \frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} 1+\frac{F_{n-1}}{F_n} = 1+\lim_{n\to\infty}\frac{F_{n-1}}{F_n} = 1+\lim_{n\to\infty}\frac{1}{F_n/F_{n-1}}$$

$$= 1+\lim_{n\to\infty}\frac{1}{x_{n-1}} = 1+\frac{1}{\lim_{n\to\infty} x_{n-1}}$$

Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n-1}$, let $\lim_{n\to\infty} x_n = p$, then

$$p = 1+\frac{1}{p} \Rightarrow p^2 = p+1 \Rightarrow p^2-p-1=0$$

$$p = \frac{1\pm\sqrt{1+4}}{2} = \frac{1\pm\sqrt{5}}{2}$$

Since $\frac{F_{n+1}}{F_n} > 0$, $p = \frac{1+\sqrt{5}}{2}$

$\therefore \lim_{n\to\infty} x_n = \frac{1+\sqrt{5}}{2}$

EX 2.1

① $p_3$ for $f(x) = \sqrt{x} - \cos x = 0$ on $[0,1]$

$p_3 = 0.6250$ using bisection table function and self-defined function (attached code)

⑫ $f(x) = (x+2)(x+1)^2 \times (x-1)^3 (x-2)$

(a) $a = -1.5$
$b = 2.5$

Iteration 1:

$p_n = \frac{a+b}{2} = \frac{-1.5+2.5}{2} = \frac{1}{2} = 0.5$

$f(0.5)\cdot f(-1.5) < 0 \Rightarrow a = -1.5,\ b = 0.5$

Iteration 2:

$p_n = \frac{a+b}{2} = \frac{-1.5+0.5}{2} = -0.5$

$f(-0.5)\cdot f(-1.5) > 0 \Rightarrow a = -0.5,\ b = 0.5$

root $\to 0$ as we iterate forward.

(b) $a = -0.5$
$b = 2.4$

Iteration 1:

$p_n = \frac{a+b}{2} = \frac{-0.5+2.4}{2} = \frac{1.9}{2} = 0.95$

$f(0.95)f(-0.5) < 0 \Rightarrow a = -0.5,\ b = 0.95$

root $\to 0$ as we iterate forward.

(c) $a = -0.5$
$b = 3$

Iteration 1:

$p_n = \frac{a+b}{2} = \frac{-0.5+3}{2} = 1.25$

$f(1.25)f(-0.5) > 0 \Rightarrow a = 1.25,\ b = 3$

root $\to 2$ as we iterate forward.

(d) $a = -3$
$b = -0.5$

<u>Iteration 1:</u>

$$p_n = \frac{a+b}{2} = \frac{-3-0.5}{2} = \frac{-3.5}{2} = -1.75$$

$f(-1.75)f(-3) < 0 \Rightarrow a = -3,\ b = -1.75$

root $\rightarrow -2$ as we iterate forward.

(17) $f(x) = x^3 - x - 1$
interval $[1, 2]$, accuracy of $10^{-4}$

<u>Th 2.1</u>: Suppose $f \in C[a,b]$ and $f(a).f(b) < 0$. The bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero $p$ of $f$ with

$$|p_n - p| \leq \frac{b-a}{2^n}, \text{ when } n \geq 1.$$

$\Rightarrow |p_n - p| \leq 10^{-4}$

$\Rightarrow \frac{b-a}{2^n} \leq 10^{-4} \Rightarrow \frac{2-1}{2^n} \leq 10^{-4} \Rightarrow \frac{1}{2^n} \leq 10^{-4}$

$\Rightarrow 2^{-n} \leq 10^{-4}$

$\Rightarrow \log_{10}(2^{-n}) \leq -4 \Rightarrow -n \log_{10}(2) \leq -4 \Rightarrow n \geq 4/\log_{10}(2)$

$\Rightarrow n \geq 13.28 \Rightarrow \underline{n = 14}$

```
% PROBLEM 11 EX 1.2

% FUNCTION DEFN

arc_tan = @(x) x - (1/3) * x^3 + (1/5) * x^5;

p = pi;

% PART A
arc_tan_1 = arc_tan(1/2);
arc_tan_2 = arc_tan(1/3);

p_star = 4 * (arc_tan_1 + arc_tan_2);

% ABS ERROR

abs_error = abs(p - p_star);
rel_error = abs_error / pi;

display(arc_tan_1);
display(arc_tan_2);

display(abs_error);
display(rel_error);

% PART B
arc_tan_1_B = arc_tan(1/5);
arc_tan_2_B = arc_tan(1/239);

p_star_B = 16 * arc_tan_1_B - 4 * arc_tan_2_B;

abs_error_B = abs(p - p_star_B);
rel_error_B = abs_error_B / p;

display(arc_tan_1_B);
display(arc_tan_2_B);

display(abs_error_B);
display(rel_error_B);
```

```
% FUNCTION THAT FINDS THE ROOT OF THE EQUATION IN AN INTERVAL WITH TOLERANCE TOL

function rt = bisection_rec(f, a, b, N, TOL)

    % FIND THE MIDPOINT
    p_n = (a + b) / 2;

    % CHECK TOLERANCE
    if abs(p_n - b) < TOL
        rt = p_n;
        return;
    end

    % STOPPING CONDITION
    if N == 0
        rt = p_n;
        return;
    end

    % CHECK SIGN
    if f(p_n) * f(a) > 0
        a = p_n;
    else
        b = p_n;
    end

    % DO RECURSION ON NEW INPUTS
    rt = bisection_rec(f, a, b, N - 1, TOL);
end
```

```
p = factorial(9);
p_star = 3 * sqrt(2 * pi) * (9^9 / exp(9));

difference = p - p_star;
abs_error = abs(difference);
relative_error = abs_error / p;

display(difference)
display(abs_error)
display(relative_error)

```