(5)

$$p_n = p_{n-1} - \frac{p_{n-1}^3 - 21}{3p_{n-1}^2}$$ converges faster than $$p_n = \left(\frac{21}{p_{n-1}}\right)^{1/2}$$

$\therefore$ (b) faster than (a)

The attached code illustrates this. (b) needs 7 iterations to compute with an accuracy of 1e-5 while (a) requires 19 iterations for the same.

(13) Fixed-Point Theorem: Let $g \in C[a,b]$ be s.t. $g(x) \in [a,b]$, for all x in $[a,b]$. Suppose, in addition, that g' exists on (a,b) and that a constant $0 < k < 1$ exists with

$|g'(x)| \le k$ for all $x \in (a,b)$

Then for any number p_0 in $[a,b]$, the sequence defined by

$p_n = g(p_{n-1})$, $n \geq 1$, converges to the unique fixed point p in $[a,b]$

(a) $x = \frac{2 - e^x + x^2}{3} \Rightarrow g(x) = x$ where $g(x) = \frac{2 - e^x + x^2}{3}$

Let the interval be $[a,b]$

For the interval to be a valid interval for fixed pt iteration, $g(x) \in [a,b]$ for $x \in [a,b]$.

$$g'(x) = \frac{-e^x + 2x}{3}$$

$$g'(b) = \frac{-e^b + 2b}{3} < 1 \Rightarrow -e^b + 2b < 3$$

$$b = 1 \Rightarrow -e + 2 < 3$$

Let $a = 0$, $b = 1$.

$$g(0) = \frac{2 - 1 + 0}{3} = 1/3 \in [0,1]$$

$$g(1) = \frac{2 - e + 1}{3} = 0.0939 \in [0,1]$$

$$g'(1) = \frac{-e + 2}{3} = |-0.239| < 1$$

$$g'(0) = \frac{-1}{3} = |-0.333| < 1$$

$g'(x) < 0$ for $x \in [a,b]$

# of iterations to achieve 10^{-5} accuracy:

$k = 1/3$

$\Rightarrow |p_n - p| \le \frac{k^n}{1-k} |p_1 - p_0| \le 10^{-5}$

$\Rightarrow (1/3)^n |0.200 - 0.5| \cdot \frac{3}{2} \le 10^{-5}$

$\Rightarrow (1/3)^n |(-0.45)| \le 10^{-5}$

$\Rightarrow (1/3)^n \le 2.2 \times 10^{-5}$

$\Rightarrow \frac{1}{2.2} \times 10^5 \le 3^n$

$\Rightarrow \log_3 \left(1/2.2 \times 10^5 \right) \le n.$

$\Rightarrow 9.76 \le n. \Rightarrow \underline{n = 10}.$

Code is attached as well.

(b) $x = \frac{5}{x^2} + 2$

let $g(x) = \frac{5}{x^2} + 2$, interval be $[a, b]$

$g'(x) = 5 \cdot \left(\frac{-2}{x^3} \right) = \left| \frac{-10}{x^3} \right| \le k$ for $k < 1$

$\Rightarrow \frac{10}{x^2} < 1 \Rightarrow 10 < x^3 \Rightarrow |x| > 2.15.$

$\Rightarrow |x| \geq 2.5$

$\Rightarrow x \geq 2.5$ or $x \le -2.5$

let $a = 2.5$, $b = 3$

In $[2.5, 3]$, $g(x)$ is $< 0 \Rightarrow$ strictly decreasing.

To find the # of iterations:

let $k = 10/2.5^3 = 0.64$, $p_0 = 2.75$

$\Rightarrow |p_n - p| \le \frac{k^n}{1-k} |p_1 - p_0|$

$\Rightarrow \frac{k^n}{1-k} |p_1 - p_0| \le 10^{-5}$

$\Rightarrow \frac{0.64^n}{0.36} |2.6611 - 2.75| \le 10^{-5}$

$\Rightarrow (0.64)^n \le 1.1255 \times 10^{-4}$

$\Rightarrow n \geq 20.3 \Rightarrow n = 21$

(f) $x = 0.5(\sin x + \cos x)$

let $g(x) = 0.5(\sin x + \cos x)$ and interval $[a,b]$

$g'(x) = 0.5(\cos x - \sin x) \leq k$

$\Rightarrow |0.5(\cos x - \sin x)| < 1 \Rightarrow |\cos x - \sin x| < 2$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Setting $\beta = \pi/4 \Rightarrow \cos(\alpha+\pi/4) = \cos\alpha \frac{1}{\sqrt{2}} - \sin\alpha\frac{1}{\sqrt{2}}$

$\Rightarrow \cos(x+\pi/4).\sqrt{2} = \cos x - \sin x$

$\Rightarrow \cos(x+\pi/4).\sqrt{2} < 2 \Rightarrow \cos(x+\pi/4) < \sqrt{2}$

$\Rightarrow |\cos(x+\pi/4)| \leq 1$

$\Rightarrow -1 \leq \cos(x+\pi/4) \leq 1 \Rightarrow \pi \leq x+\pi/4 \leq 0$

$\Rightarrow \frac{3\pi}{4} \leq x \leq -\pi/4$

let interval $[-\pi/4, 3\pi/4]$

let $p_0 = 0$.

$\Rightarrow |p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|$

$g'(-\pi/4) = 0.7071$

$g'(3\pi/4) = -0.7071$

maxima at $g'(x) = 0 \Rightarrow \cos x = \sin x \Rightarrow x = \pi/4$,

$\Rightarrow g(-\pi/4) = g(3\pi/4) = 5.55 \times 10^{-17}$

$\Rightarrow g(\pi/4) = 0.7071. < 1$

# of iterations $\Rightarrow k = 0.5\sqrt{2}$

$|p_n - p| \leq \frac{(0.5\sqrt{2})^n}{(1-0.5\sqrt{2})}(0.5-0) \leq 10^{-5}$

$\Rightarrow \underline{n \geq 34}$

Ex 2.3

(6)

(a) $e^x + 2^{-x} + 2\cos x - 6 = 0 \qquad x \in [1,2]$

let $f(x) = e^x + 2^{-x} + 2\cos x - 6$

$f'(x) = e^x - 2^{-x}\ln 2 - 2\sin x$

Attached code, $x = 1.8293$

(8)

(a) $e^{x} + 2^{-x} + 2\cos x - 6 = 0 \qquad x \in [1, 2]$

let $p_0 = 1.5$, $p_1 = 1.75$

Attached code, $x = 1.8293$.

(18)

$0 = \frac{1}{2} + \frac{1}{4}x^2 - x\sin x - \frac{1}{2}\cos 2x$, $p_0 = \pi/2$.

let $f(x) = \frac{1}{2} + \frac{1}{4}x^2 - x\sin x - \frac{1}{2}\cos 2x$

$f'(x) = 0 + \frac{1}{2}x - (\sin x + x\cos x) + \sin 2x$

at $p_0 = \pi/2$, $x = 1.892489$

at $p_0 = 5\pi$, $x = 1.892789$

at $p_0 = 10\pi$, $x = 1.897842$.

```
% RUNNING CODE FOR PROBLEM 5

function result = runner(f, act, tol)
    result = 1;
    while abs(f(result) - act) > tol
        result = result + 1;
    end
end


%>> runner(@ex_2_2_p5, 21^(1/3), 1e-5)
%
%ans =
%
%     7
%
%>> runner(@ex_2_2_p5d, 21^(1/3), 1e-5)
%
%ans =
%
%    19


```

```
% PROBLEM 6A FROM EXERCISE 2.3

function r = ex2_3_p6a(f, df, p0, tol)
    if abs(f(p0) - 0) < tol
        r = p0;
    else
        p0 = p0 - f(p0) / df(p0);
        r = ex2_3_p6a(f, df, p0, tol);
    end
end


%>> f = @(x) exp(x) + 2^(-x) + 2*cos(x) - 6
%
%f =
%
%  function_handle with value:
%
%    @(x)exp(x)+2^(-x)+2*cos(x)-6
%
%>> df = @(x) exp(x) - (2^(-x))*log(2) - 2*sin(x)
%
%df =
%
%  function_handle with value:
%
%    @(x)exp(x)-(2^(-x))*log(2)-2*sin(x)
%
%>> ex2_3_p6a(f, df, 1.5, 1e-5)
%
%ans =
%
%   1.829383614494166


% PROBLEM 18 FROM EXERCISE 2.3
%>> f = @(x) 0.5 + (x^2)*0.25 - x*sin(x) - 0.5*cos(2*x)
%
%f =
%
%  function_handle with value:
%
%    @(x)0.5+(x^2)*0.25-x*sin(x)-0.5*cos(2*x)
%
%>> df = @(x) 0.5*x - (sin(x) + x*cos(x)) + sin(2*x)
%
```

```
%df =
%
%  function_handle with value:
%
%    @(x)0.5*x-(sin(x)+x*cos(x))+sin(2*x)
%
%>> ex2_3_p6a(f, df, pi/2, 1e-5)
%
%ans =
%
%   1.892489624534230
%
%>> ex2_3_p6a(f, df, 5*pi, 1e-5)
%
%ans =
%
%   1.892789801826626
%
%>> ex2_3_p6a(f, df, 10*pi, 1e-5)
%
%ans =
%
%   1.897842212555557
```

```
% PROBLEM 8A FROM EXERCISE 2.3

function r = ex2_3_p8a(f, p0, p1, tol)
    if abs(f(p1) - 0) < tol
        r = p1;
    else
        p2 = p1 - (f(p1)*(p1 - p0)) / (f(p1) - f(p0));
        r = ex2_3_p8a(f, p1, p2, tol);
    end
end


%>> f = @(x) exp(x) + 2^(-x) + 2*cos(x) - 6
%
%f =
%
%  function_handle with value:
%
%    @(x)exp(x)+2^(-x)+2*cos(x)-6
%
%>> ex2_3_p8a(f, 1.5, 1.75, 1e-5)
%
%ans =
%
%   1.829383662436248
```

```
% PROBLEM 5B FROM EX 2.2
function p_n = ex_2_2_p5(n)
    % BASE CASE
    if n == 0
        p_n = 1;
    else
        p_n = ex_2_2_p5(n-1) -  ((ex_2_2_p5(n-1)^3 - 21) / (3
 * (ex_2_2_p5(n-1)^2)));
    end
end



```

```matlab
% PROBLEM 13A FROM EXERCISE 2.2
function n = ex_2_2_p13(g, p0, tol)
    n = 1;
    while abs(g(p0) - p0) > tol
        p0 = g(p0);
        n = n + 1;
    end
end
```

```
function p_n = ex_2_2_p5d(n)
    if n == 0
        p_n = 1;
    else
        p_n = (21 / ex_2_2_p5d(n-1))^0.5;
    end
```