CHAPTER 26

Case Analysis

Chapter Guide. This chapter presents 15 OR applications. Each case starts with a description of the situation, followed by a detailed analysis that includes collection of data, development of the mathematical model, solution of the model using AMPL, Excel, or TORA, and interpretation of the results. The unpredictable computational behavior of ILPs occurs in a number of cases where an excessively long execution time fails to produce a solution and where it is necessary to modify the original model to circumvent the computational difficulty. The table below lists the cases discussed in this chapter. The AMPL/Excel/Solver/TORA programs are in folder ch26Files.

	Case	Application area	Analytic tools	Software
1	Airline fuel allocation using optimum tankering	Airlines	LP, heuristic	Excel, AMPL
2	Optimization of heart valves production	Production planning	LP	AMPL
3	Scheduling appointments at Australian tourist commission trade events	Tourism	Assignment model, heuristic	Excel, AMPL
4	Saving federal travel dollars	Business travel	Shortest route	TORA, Excel
5	Optimal ship routing and personnel assignments for naval recruitment in Thailand	Transportation, routing	Transportation model, ILP	AMPL
6	Allocation of operating room time in Mount Sinai hospital	Health care	ILP, GP	AMPL
7	Optimizing trailer payloads at PFG Building Glass	Distribution	Free-body diagram, ILP	AMPL
8	Optimization of crosscutting and log allocation at Weyerhauser	Mill operation	DP	Excel
9	Layout planning for a computer integrated manufacturing (CIM) facility	Layout planning	AHP, GP	Excel, AMPL
10	Booking limits in hotel reservations	Hotels	Decision tree	Excel
11	Casey's problem: Interpreting and evaluating a new test	Medical tests	Bayes probabilities, decision trees	Excel

(continued)

	Case	Application area	Analytic tools	Software
12	Ordering golfers on the final day of Ryder Cup matches	Sports	Game theory	Excel, TORA
13	Inventory decisions in Dell's supply chain	Inventory control	Inventory models	Excel
14	Analysis of internal transport system in a manufacturing plant	Materials handling	Queuing theory, simulation	Excel, TORA
15	Telephone sales manpower planning at Qantas Airways	Airlines	ILP, queuing theory	Excel, AMPL, TORA

Case 1: Airline Fuel Allocation Using Optimum Tankering[1]

Tools: Heuristics, LP

Area of application: Airlines

Description of the situation:

A typical commercial flight route usually forms a loop that starts and ends at the airline operation hub with stopovers at intermediate cities. An example of a route is a departure from Los Angeles (LAX) for Tampa (TPA), Miami (MIA), Fort Lauderdale (FLL), New York (LGA), Miami (MIA), and Houston (IAH), before returning to Los Angeles. The fueling of the aircraft can take place anywhere along the flight route. However, because fuel cost varies among the stopovers, potential savings in the cost of fuel can be realized through *tankering*. Tankering means loading extra fuel at a stopover to take advantage of lower prices and then using the additional fuel to cover subsequent flight legs. The disadvantage of tankering is the excess burn of gasoline resulting from the increase in the weight of the plane. Thus, tankering can be recommended only if the savings in the fuel cost is larger than the cost of excess burn. The problem reduces to determining the optimum amount of tankering, if any, that should take place along the flight route, taking into account the weight limit of the plane.

A heuristic for solving the tankering problem:

Figure 26.1 shows an n-city flight route with city 1 representing the start and end of the route. Tankering for flight leg j between stopovers j and $j + 1$ can occur in any of the cities $1, 2, \ldots,$ and $j - 1$. Define

$$x_{ij} = \text{Fraction of the amount of fuel tankered for leg } j \text{ in stopover } i < j$$

By definition, $\sum_{i=1}^{j-1} x_{ij} \leq 1$, $x_{ij} \geq 0$. If $\sum_{i=1}^{j-1} x_{ij} < 1$, then tankering does not provide all the fuel requirement for leg j, and the balance, $1 - \sum_{i=1}^{j-1} x_{ij}$, must be loaded at stopover j.

The idea of the heuristic is to compare the weighted price of tankering for leg j at stopovers $i = 1, 2, \ldots, j - 1$, with the base price of purchasing the fuel for leg j at stopover j. The weighted price of tankering takes into account the fact that loading fuel at a stopover for a succeeding flight leg results in additional fuel burn-off (excess burn penalty) because of weight increase. Let

[1] *Source:* B. Nash, "A Simplified Alternative to Current Airline Fuel Allocation Models," *Interfaces*, Vol. 11, No. 1, pp. 1–8, 1981.

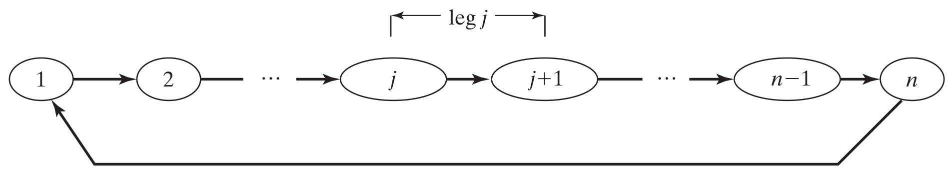


FIGURE 26.1

An n-stopover flight route

q = fraction of additional burn-off per gallon per hour for extra fuel on board

t_{ij} = Number of block hours from tankering station i to the start of leg j, $i < j$

$\hat{p}_j$ = Base price per gallon at stopover j

The weighted price, p_{ij}, can be computed from the formula

$$p_{ij} = \frac{\hat{p}_i}{(1 - q)^{t_{ij}}}, i < j$$

Let

$$p_{i^*j} = \min_{i<j}\{p_{ij}\}$$

The heuristic rule recommends tankering at station i^* for leg j if

$$p_{i^*j} < \hat{p}_j, i^* < j$$

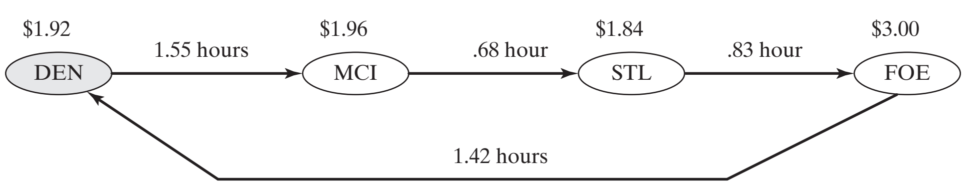
To demonstrate the use of the heuristic, Figure 26.2 gives a flight route departing from Denver (DEN) for Kansas City (MCI), St. Louis (STL), and Topeka (FOE), before returning to Denver. Flight times in hours are shown on the respective arcs. The base gasoline prices per gallon were indicated on the respective nodes.

Figure 26.3 provides the tankering opportunities for the example route. The respective numeric codes 1, 2, 3, and 4 represent DEN, MCI, STL, and FOE. Also, flightlegs (DEN, MCI), (MCI, STL), (STL, FOE), (FOE, DEN) are represented by the numeric codes 1, 2, 3, and 4, respectively. The figure shows that stopover 1 (DEN) can tanker for legs 2, 3, and 4, stopover 2 (MCI) for legs 3 and 4, and stopover 3 (STL) for leg 4 only. Because DEN is the start and end of the flight loop, tankering is not an option for leg 1 (DEN, MCI). The corresponding tankering ratios, x_{ij}, are defined on the associated arcs.

The implementation of the heuristic can be done conveniently with a spreadsheet (file excelCase1.xls). Figure 26.4 divides the spreadsheet into two sections: input (rows 3–9) and output (rows 12–27). The input data include the value of q (= .05), the base price per gallon at each

FIGURE 26.2

Gasoline prices and flight times on route DEN-MCI-STL-FOE-DEN



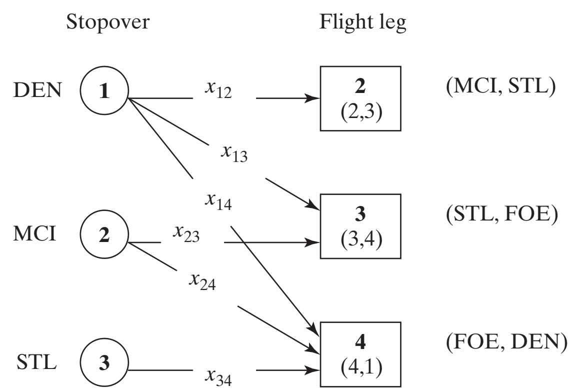


FIGURE 26.3
Representation of the elements of the flight route in Figure 26.2

FIGURE 26.4
Spreadsheet computations for the tankering heuristic (file excelCase1.xls)

	A	B	C	D	E
1		**Tankering Heuristic**			
2	***Input data:***				
3	q	0.05			
4		**DEN**	**MCI**	**STL**	**FOE**
5	**Stopover**	1	2	3	4
6	**Price per gallon($)**	1.92	1.96	1.84	3
7		**DEN-MCI**	**MCI-STL**	**STL-FOE**	**FOE-DEN**
8	**Leg**	1	2	3	4
9	**Flight hours**	1.55	0.68	0.83	1.42
10					
11	***Output:***				
12		**Block hours**			
13		Leg 2	Leg 3	Leg 4	
14	Stopover 1	1.55	2.23	3.06	
15	Stopover 2		0.68	1.51	
16	Stopover 3			0.83	
17	Stopover 4				
18					
19		**Weighted price**			
20		Leg 2	Leg 3	Leg 4	
21	stopover 1	2.0788811	2.15267	2.246295	
22	stopover 2		2.02957	2.117841	
23	stopover 3			1.920027	
24	stopover 4				
25	**Min weighted price**	2.0788811	2.02957	1.920027	
26	**Tankering(yes/no)?**	no	no	yes	
27	**Tankering stopover**			STL	

stopover, and flight leg hours. The output section starts with the computation of block hours representing the tankering time from a stopover to the start of a flight leg. For example, fuel tankered in DEN (stopover 1) will take 1.55 hours to reach MCI, the start of leg 2, 2.23

(= 1.55 + .68) hours to reach STL, the start of leg 3, and 3.06 (= 1.55 + .68 + .83) hours to reach FOE, the start of leg 4. Block hours are then used to compute the weighted prices (rows 19-24). For example, the weighted price from DEN (stopover 1) for leg 2 (MCI, STL) is computed as

$$p_{12} = \frac{\hat{p}_1}{(1 - q)^{t_{12}}} = \frac{1.92}{(1 - .05)^{1.55}} = 2.079$$

The weighted prices indicate that tankering is recommended only at stopover 3 (STL) for leg 4 (FOE, DEN) because it offers a weighted price of $1.92 which is less than the base price of $3.00 at FOE.

The proposed heuristic solution makes two basic assumptions:

1. There is no limit on fuel supply at any station.
2. The plane capacity can accept any tankered amount of fuel.

Under such conditions, the heuristic solution is always optimal. For other situations, it is necessary to modify the heuristic rules to deliver a feasible solution in the following manner:

1. Rank candidate tankering cities in ascending order of their weighted prices. (A tankering stopover must have a weighted price less than the base price at the start of a target leg.)
2. Allocate as much fuel as possible at the stopover with the smallest weighted price (subject to stopover supply and plane capacity limits).
3. Repeat step 2 for the next-ranked stopover until either a leg requirement is satisfied or that tankering cannot be used without violating stopover supply and/or capacity limits.
4. If a leg is still short of its fuel requirement, additional fuel is purchased at base price at the start stopover of the leg.

To demonstrate the use of the capacitated heuristic, consider the set of data in Table 26.1 that specify the amount of fuel needed for leg j, the tankering capacity of the plane, A_j, at stopover j, and the fuel supply limit, B_j, also at stopover j.

If the tankering capacity A_j and the supply limit B_j at stopover j are not taken into account, the solution in Figure 26.4 calls for tankering $\frac{920}{(1 - .05)^{.83}} = 961$ gallons in STL for leg 4. The total cost for the round trip is $1200 \times \$1.92 + 645 \times \$1.96 + (880 + 961) \times \$1.84 = \$6,955.64$. On the other hand, if the limits are considered, the given solution will not be feasible because fueling $880 + 961 = 1841$ gallons at STL exceeds both the tankering capacity ($A_3 = 900$ gallons) and the supply limit ($B_3 = 1400$ gallons). This means that the most that can be tankered at STL is $900 - 880 = 20$ gallons, which is equivalent to a net of $20(1 - .05)^{.83} \approx 19$ gallons for leg 4, leaving leg 4 short of $920 - 19 = 901$ gallons. This amount must be purchased at FOE, because tankering at DEN and/or MCI is not possible as the plane is already at its tankering capacity (= 900 gallons)

TABLE 26.1 Example Data for Case 1

Flight leg $j \equiv (j, j+1)$	Required gallons of fuel for leg j, f_j	Available tankering capacity at stopover j, A_j	Supply limit in gallons at stopover j, B_j	Base price per gallon at stopover j, $\hat{p}_j$
1 (DEN, MCI)	1200	860	1500	$1.92
2 (MCI, STL)	645	960	900	1.96
3 (STL, FOE)	880	900	1400	1.84
4 (FOE, DEN)	920	900	1500	3.00

in STL. The associated fuel cost for the tour thus equals $1200 \times \$1.92 + 645 \times \$1.96 + (20 + 880) \times \$1.84 + 901 \times \$3.00 = \$7927.20$, an increase of $7927.20 - 6955.64 = \$971.56$ (or about 14%) over the cost of uncapacitated solution.

Optimum solution using linear programming:

The linear programming model minimizes the total expenditures for fuel subject to two types of restrictions: (1) plane tankering capacity, and (2) stopover supply limit. The decision variables x_{ij} represent the proportion of fuel tankered in stopover i for leg j. Let

f_j = Net gallons of fuel needed for leg j

F_{ij} = Gross gallons of fuel (including excess burn-off) tankered in stopover i for leg j, $i < j$

The quantity F_{ij} is computed from f_j as

$$F_{ij} = \frac{f_j}{(1 - q)^{t_{ij}}}, i < j$$

As defined earlier, q is the fraction of excess burn-off per gallon per hour and t_{ij} is the number of block hours from tankering stopover i to the start of leg j.

The amount of *tankered* fuel on board the plane at stopover k is computed as

$$T_k = \sum_{j=k+1}^{n} F_{kj}\left(\sum_{i \le k} x_{ij}\right), k = 1, 2, \ldots, n - 1$$

The formula recognizes that amounts tankered in stopovers up to and including stopover k for all succeeding legs $k + 1, k + 2, \ldots,$ and $n - 1$ pass through stopover k (see also Figure 26.3).

Next, the amount of fuel purchased at stopover k is computed as

$$Q_k = f_k\left(1 - \sum_{i<k} x_{ik}\right) + \sum_{j>k} F_{kj} x_{kj}, k = 1, 2, \ldots, n$$

The amount purchased at stopover k includes two portions: (1) The additional fuel needed for leg k and (2) the tankered fuel used by subsequent legs $k + 1, k + 2,$ and n. Note that only $f_k\left(1 - \sum_{i<k} x_{ik}\right)$ gallons, instead of f_k gallons, need to be purchased from stopover k to cover leg k because the amount $f_k \sum_{i<k} x_{ik}$ has already been tankered in preceding stopovers $i < k$.

Using A_i and B_i as defined earlier to represent the tankering capacity of the plane and the supply limit at stopover i, the linear program is given as

$$\text{Minimize } z = \sum_{i=1}^{n} \hat{p}_i Q_i$$

subject to

$$T_i \le A_i, i = 1, 2, \ldots, n - 1$$
$$Q_i \le B_i, i = 1, 2, \ldots, n$$
$$\sum_{i=1}^{j} x_{ij} \le 1, j = 2, 3, \ldots, n$$
$$x_{ij} \ge 0, i < j$$

TABLE 26.2 Computation of F_{ij} using f_j and t_{ij}

	F_{ij}		
i	$j = 2$	$j = 3$	$j = 4$
1	699	987	1077
2		912	995
3			961

The model is solved by substituting for T_i and Q_i in terms of x_{ij} as given previously. Note that in the third set of constraints the use of inequality ($\leq$) instead of strict equality allows for the possibility of no tankering for leg j.

The data of the heuristic example in Table 26.1 will be used to demonstrate the use of the linear programming model. The first step is to compute F_{ij} using f_j and t_{ij} as shown in Table 26.2. All values have been raised to the next integer value for convenience. To illustrate the computations, we have

$$F_{12} = \frac{f_2}{(1 - q)^{t_{12}}} = \frac{645}{(1 - .05)^{1.55}} = 698.37 \approx 699 \text{ gallons}$$

Thus, we get

$$\begin{aligned}
T_1 &= 699x_{12} + 987x_{13} + 1077x_{14} \\
T_2 &= 912(x_{13} + x_{23}) + 995(x_{14} + x_{24}) \\
T_3 &= 961(x_{14} + x_{24} + x_{34}) \\
Q_1 &= 1200 + 699x_{12} + 987x_{13} + 1077x_{14} \\
Q_2 &= 645(1 - x_{12}) + 912x_{23} + 995x_{24} \\
Q_3 &= 880(1 - x_{13} - x_{23}) + 961x_{34} \\
Q_4 &= 920(1 - x_{14} - x_{24} - x_{34})
\end{aligned}$$

The complete linear program is thus given as

$$\text{Minimize } z = 1.92Q_1 + 1.96Q_2 + 1.84Q_3 + 3.00Q_4$$

subject to

$$\begin{aligned}
&Q_1 \leq 1500, Q_2 \leq 900, Q_3 \leq 1400, Q_4 \leq 1500 \\
&699x_{12} + 987x_{13} + 1077x_{14} \leq 860 \\
&912x_{13} + 912x_{23} + 995x_{14} + 995x_{24} \leq 960 \\
&961x_{14} + 961x_{24} + 961x_{34} \leq 900 \\
&Q_1 - 699x_{12} - 987x_{13} - 1077x_{14} = 1200 \\
&Q_2 + 645x_{12} - 912x_{23} - 995x_{24} = 645 \\
&Q_3 + 880x_{13} + 880x_{23} - 961x_{34} = 880 \\
&Q_4 + 920x_{14} + 920x_{24} + 920x_{34} = 920
\end{aligned}$$

$$x_{12} \leq 1$$
$$x_{13} + x_{23} \leq 1$$
$$x_{14} + x_{24} + x_{34} \leq 1$$
$$x_{ij} \geq 0, i = 1, 2, 3; j = 2, 3, 4$$

AMPL model

File amplCase1.txt provides the AMPL model for the tankering linear program. Input includes the raw data of the problem (n, q, $\hat{p}_i$, f_i, A_i, and B_i, as defined in the mathematical model). Figure 26.5 provides a summary output for the data in Table 26.1. Tankered fuel is purchased for leg 4 (STL-FOE) at stopovers 1, 2, and 3 (DEN, MCI, and STL) in the amounts 150 ($= 1350 - 1200$), 255 ($= 900 - 645$), and 520 ($= 1400 - 880$) gallons, respectively. Removing the excess burn-off fuel, tankered purchases reduce to $150(1 - .05)^{(1.55+.68+.83)} + 255(1 - .05)^{(.68+.83)} + 520(1 - .05)^{.83} \approx 862$ gallons. This amount together with the 58 gallons purchased in city 4 (FOE) provides $862 + 58 = 920$ gallons, the fuel requirement for leg 4. As should be expected, the optimal LP cost is lower than that of the heuristic by $\$7927.20 - \$7104.44 = \$822.76$.

It may appear "odd" that the solution in Figure 26.5 calls for the purchase of 58 gallons in stopover 4 (FOE) at \$3.00 per gallon instead of simply boosting the tankered purchase in stopover 1 (DEN) at \$1.92 where there is a surplus supply of 150 gallons. The reason no additional fuel can be purchased in stopover 1 (DEN) is that the given optimum solution uses the entire tankering capacity ($= 900$ gallons) at stopover 3 (STL). In fact, if the tankering capacity at STL is increased, say, to 1000 gallons, no purchase will be made in FOE.

Questions:

1. In Figure 26.5, verify the values under `Tankered amt`.
2. Given that the basic prices for DEN, MCI, STL, and FOE are \$2.40, \$2.70, \$2.80, and \$3.00, respectively, use the heuristic to determine the tankering policy.
3. In Question 1, if the plane fuel capacity is 1000 gallons and each stopover can supply a maximum of 600 gallons, determine the tankering policy, first using the heuristic and then using the LP.

FIGURE 26.5

Output of the AMPL model

```
Total cost =    7104.44

City    Supply limit    Purchased amount   Min requirement     Cost
  1         1500              1350              1200          2591.90
  2          900               900               645          1764.00
  3         1400              1400               880          2576.00
  4         1500                58               920           172.54

City  Tankered amt  Tankering capacity
  1       150            860
  2       393            960
  3       900            900
```

4. Show that if the tankering capacity at stopover 3 (STL) is increased from 900 to 1000 gallons, no purchase is made in stopover 4 (FOE) and the purchase in stopover 1 (MCI) is increased to meet the fuel requirement for leg 4.

Case 2: Optimization of Heart Valves Production[2]

Tool: LP

Area of application: Bioprostheses (production planning)

Description of the situation:

Biological heart valves are bioprostheses manufactured from porcine hearts for human implantation. Replacement valves needed by the human population come in different sizes. On the supply side, porcine hearts cannot be "produced" to specific sizes. Moreover, the exact size of a manufactured valve cannot be determined until the biological component of pig heart has been processed. As a result, some needed sizes may be overstocked and others may be understocked.

Raw hearts are provided by several suppliers in six to eight sizes, usually in different proportions depending on how the animals are raised. The distribution of sizes in each shipment is expressed in the form of a histogram. Porcine specialists work with suppliers to ensure distribution stability as much as possible. In this manner, the manufacturer can have a reasonably reliable estimate of the number of units of each size in each shipment. The selection of the mix of suppliers and the size of their shipments is thus crucial in reducing mismatches between supply and demand.

LP model:

Let

m = Number of valve sizes

n = Number of suppliers

p_{ij} = Proportion of raw valves of size i supplied by vendor j, $0 < p_{ij} < 1$,

$$i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n, \; \sum_{i=1}^{m} p_{ij} = 1, \; j = 1, 2, \ldots, n$$

c_i = Purchasing and processing cost of a raw heart of size i, $i = 1, 2, \ldots, m$

$\bar{c}_j$ = Average cost from supplier j

$$= \sum_{i=1}^{m} c_i p_{ij}, \; j = 1, 2, \ldots, n$$

D_i = Average monthly demand for valves of size i

H_j = Maximum monthly supply vendor j can provide, $j = 1, 2, \ldots, n$

L_j = Minimum monthly supply vendor j is willing to provide, $j = 1, 2, \ldots, n$

[2]*Source:* S.S. Hilal and W. Erikson, "Matching Supplies to Save Lives: Linear Programming the Production of Heart Valves," *Interfaces*, Vol. 11, No. 6, pp. 48–55, 1981.

The variables of the problem can be defined as

$$x_j = \text{Monthly supply amount (number of raw hearts) by vendor } j, j = 1, 2, \ldots, n$$

The LP model seeks to determine the amount from each supplier that will minimize the total cost of purchasing and processing subject to demand and supply restrictions.

$$\text{Minimize } z = \sum_{j=1}^{n} \bar{c}_j x_j$$

subject to

$$\sum_{j=1}^{n} p_{ij} x_j \geq D_i, i = 1, 2, \ldots, m$$

$$L_j \leq x_j \leq H_j, j = 1, 2, \ldots, n$$

To be completely correct, the variables x_j must be restricted to integer values. However, the parameters p_{ij} and D_i are mere estimates and, hence, rounding the continuous solution to the closest integer may not be a bad approximation in this case. This is particularly desirable for large problems, where imposing the integer restriction may result in unpredictable computational experiences.

AMPL Implementation:

Although the LP is quite simple as an AMPL application, the nature of the input data is somewhat cumbersome. A convenient way to supply the data to this model is through a spreadsheet. File excelCase2.xls gives all the tables for the model and AMPL file amplCase2.txt shows how the data involving 8 valve sizes and 12 suppliers are read from Excel tables.[3]

Analysis of the results:

The output of the AMPL model for the data in excelCase2.xls is given in Figure 26.6. In the strict sense, the solution results cannot be used for scheduling purposes because the demand D_i for heart valve i is based on *expected* value calculations. Thus, the solution x_j, $j = 1, 2, \ldots, n$, will result in some months showing surplus and others exhibiting shortage.

How useful then is the model? Actually, the results can be used effectively for planning purposes. Specifically, the solution suggests grouping the vendors into three categories:

1. Vendors 1, 2, and 3 must be deleted from the list of suppliers because $x_1 = x_2 = x_3 = 0$.
2. Vendors 5, 6, 8, and 9 are crucial for satisfying demand because the solution requires these vendors to supply all the hearts they can produce.
3. The remaining vendors (4, 7, 10, 11, and 12) exhibit "moderate" importance from the standpoint of satisfying demand because their maximum production capacity is not fully utilized.

The given recommendations are further supported by the values of the *reduced costs* in Figure 26.6. Vendor 9 can raise its average unit prices by as much as \$4.00 and still remain viable

[3]There is one requirement about reading the data in array format from spreadsheet excelCase2.xls as used in file amplCase2.txt. The ODBC handler requires column headings in an Excel read table to be strings, which means that a pure numeric heading is not acceptable. To get around this restriction, all column headings are converted to strings using the Excel TEXT function. Thus, the heading 1 can be replaced with the formula =TEXT(COLUMN(A1), "0"). Copying this formula into succeeding columns will automatically convert the numeric code into the desired strings.

```
Cost = $  42210.82
solution:
  j     L[j]       x[j]       H[j]      reduced cost   Av. unit price
  1       0         0.0        500        2.39             14.22
  2       0         0.0        500        0.12             15.88
  3       0         0.0        400        5.22             15.12
  4       0       116.4        500        0.00             14.70
  5       0       300.0        300       -0.49             16.68
  6       0       500.0        500       -2.13             14.89
  7       0       250.5        600        0.00             18.12
  8       0       400.0        400       -6.22             16.61
  9       0       300.0        300       -4.20             17.19
 10       0       357.4        500       -0.00             14.47
 11       0       112.9        400        0.00             15.62
 12       0       293.1        500        0.00             16.31

  i      D[i]  Surplus[i]     Dual value
  1      275        0.0        29.28
  2      310       28.9         0.00
  3      400        0.0        19.18
  4      320       88.1         0.00
  5      400        0.0        24.33
  6      350        0.0         8.55
  7      300        0.0        62.41
  8      130       28.2         0.00
```

FIGURE 26.6

Output of the valve production model

in the optimum solution, whereas vendor 3 will continue to be unattractive even if it reduces the average unit cost by as much as \$5.00. This result is true despite the fact that the average unit prices for excluded vendor 3 is among the lowest (= \$15.12) and that for "star" vendor 9 is among the highest (= \$17.19). The reason for this apparently unintuitive conclusion is that the model is primarily demand driven, in the sense that vendors 5, 6, 8, and 9 provide relatively more of the sizes needed than the remaining vendors. The opposite is true for vendors 1, 2, and 3. This means that a change in levels of demand could result in a different mix of vendors. This is the reason that, under reasonably steady projected demand, the manufacturer works closely with its "star" vendors, providing them with nutrition and animal care recommendations that ensure that their distributions of valve sizes will remain reasonably stable.

Valve size 7 appears to be the most critical among all sizes because it has the highest dual price (= \$62.41), which is more than twice the dual prices of other sizes. This means that size 7 stock should be monitored closely to keep its surplus inventory at the lowest level possible. On the other hand, sizes 2, 4, and 8 exhibit surplus, and efforts must be made to reduce their inventory.

Comments on the implementation of the model:

The proposed LP model is "rudimentary," in the sense that its results produce general planning guidelines rather than definitive production schedules. Yet, the monetary savings from the proposed plan, as reported in the original article, are impressive. The elimination of a number of vendors from the pool of suppliers and the identification of "star" vendors have resulted in reduction in inventory with significant cost savings. The same plan is responsible for reducing chances of shortage that were prevalent before the model results were used. Also, by identifying the most favored vendors, it was possible for porcine specialists in the production facility to train the workers in the slaughterhouses of these vendors to provide well-isolated and well-trimmed hearts. This, in turn, has lead to streamlining production at the production facility.

Questions:

1. If the demands for the respective 8 sizes are 300, 200, 350, 450, 500, 200, 250, and 180 valves, find the new solution and compare it with the one given in the case analysis.
2. Suppose that vendor 3 is already under contract to supply 400 hearts for next year and that through special diet it can alter the proportion of heart sizes it provides. What should be the ideal proportion of sizes supplier 3 should provide?

Case 3: Scheduling Appointments at Australian Tourist Commission Trade Events[4]

Tools: Assignment model, heuristics

Area of application: Tourism

Description of the situation:

The Australian Tourist Commission (ATC) organizes trade events around the world to provide a forum for Australian sellers to meet international buyers of tourism products that include accommodation, tours, transport, and others. During these events, sellers are stationed in booths and are visited by buyers according to pre-scheduled appointments. Because of the limited time slots available in each event and the fact that the number of buyers and sellers can be quite large (one such event held in Melbourne in 1997 attracted 620 sellers and 700 buyers), ATC attempts to schedule the seller-buyer appointments in advance of the event in a manner that maximizes the preferences. The idea is to match mutual interests to produce the most effective use of the limited time slots available during the event.

Analysis:

The problem is viewed as a three-dimensional assignment model representing the buyers, the sellers, and the scheduled time slots. For an event with m buyers, n sellers, and T time slots, define

$$x_{ijt} = \begin{cases} 1, \text{ if buyer } i \text{ meets with seller } j \text{ in period } t \\ 0, \text{ otherwise} \end{cases}$$

$$c_{ij} = \text{A score representing the mutual preferences of buyer } i \text{ and seller } j$$

The associated assignment model can be expressed as

$$\text{Maximize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\left(\sum_{t=1}^{T} x_{ijt}\right)$$

subject to

$$\sum_{i=1}^{m} x_{ijt} \leq 1, j = 1, 2, \ldots, n, t = 1, 2, \ldots, T$$

$$\sum_{j=1}^{n} x_{ijt} \leq 1, i = 1, 2, \ldots, m, t = 1, 2, \ldots, T$$

[4]*Source:* A.T. Ernst, R.G.J. Mills, and P. Welgama, "Scheduling Appointments at Trade Events for the Australian Tourist Commission," *Interfaces*, Vol. 33, No. 3, pp. 12–23, 2003.

$$\sum_{t=1}^{T} x_{ijt} \leq 1, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$$
$$x_{ijt} = (0, 1) \text{ for all } i, j, \text{ and } t$$

The model expresses the basic restrictions of an assignment model: Each buyer or seller can meet at most one person per session and a specific buyer-seller meeting can take place in at most one session. In the objective function, the coefficients c_{ij} representing the buyer-seller preferences for meetings are not session dependent, because it is assumed that buyers and sellers are indifferent to the session time.

How are the coefficients c_{ij} determined? Following the registration of all buyers and sellers, each seller provides ATC with a prioritized list of buyers whom the seller wants to see. A similar list is demanded of each buyer with respect to sellers. The list assigns the value 1 to the top choice, with higher values implying lower preferences. These lists need not be exhaustive, in the sense that sellers and buyers are free to express interest in meeting with some but not all registered counterparts. For example, in a list with 100 sellers, a buyer may seek meetings with 10 sellers only, in which case the expressed preferences will be $1, 2, \ldots, 10$ for the selected sellers.

The raw data gathered from the buyers/sellers list may then be expressed algebraically as

b_{ij} = ranking assigned by buyer i to a meeting with seller j

s_{ji} = ranking assigned by seller j to a meeting with buyer i

B = maximum number of preferences elected by all buyers

S = maximum number of preferences elected by all sellers

α = relative weight of buyer preferences (in calculating scores c_{ij}), $0 < \alpha < 1$

$1 - \alpha$ = relative weight of seller preferences.

From these definitions, the objective coefficients c_{ij} can be calculated as

$$c_{ij} = \begin{cases} 1 + \alpha\left(\dfrac{B - b_{ij}}{B}\right) + (1 - \alpha)\left(\dfrac{S - s_{ji}}{S}\right), & \text{if } b_{ij} \neq 0 \text{ and } s_{ji} \neq 0 \\ 1 + \alpha\left(\dfrac{B - b_{ij}}{B}\right), & \text{if } b_{ij} \neq 0 \text{ and } s_{ji} = 0 \\ 1 + (1 - \alpha)\left(\dfrac{S - s_{ji}}{S}\right), & \text{if } b_{ij} = 0 \text{ and } s_{ji} \neq 0 \\ 0, & \text{if } b_{ij} = s_{ji} = 0 \end{cases}$$

The logic behind these formulas is that a smaller value of b_{ij} means a higher value of $(B - b_{ij})$ and, hence, a higher score assigned to a requested meeting between buyer i and seller j. A similar interpretation is given to the score $S - s_{ji}$ for seller j's requested meeting with buyer i. Both scores are normalized to values between 0 and 1 by dividing them by B and S, respectively, and then are weighted by α and $1 - \alpha$ to reflect the relative importance of the buyer and seller preferences, $0 < \alpha < 1$. Thus, values of α less than .5 favor sellers' preferences. Note that $b_{ij} = 0$ and $s_{ji} = 0$ indicate that no meetings are requested between buyer i and seller j. The quantity 1 appears in the top three formulas of c_{ij} to give it a relatively larger preference than the case where no meetings are requested (i.e., $b_{ij} = s_{ji} = 0$). The normalization of the raw scores ensures that $0 \leq c_{ij} < 2$.

Reliability of input data:

A crucial issue in the present situation is the reliability of the preference data provided by buyers and sellers. A preference collection tool is devised to guarantee that the following restrictions are observed:

1. Lists of buyers and sellers are made available only after the registration deadline has passed.
2. Only registered buyers and sellers can participate in the process.
3. Participants' preferences are kept confidential by ATC. They may not be seen or altered by other participants.

Under these restrictions, an interactive internet site is created to allow participants to enter their preferences conveniently. More importantly, the design of the site ensures valid input data. For example, the system prevents a buyer from seeking more than one meeting with the same seller, and vice versa.

Solution of the problem:

The given assignment model is straightforward and can be solved by available commercial packages. File amplCase3a.txt and file amplCase3b.txt provide two AMPL models for this situation. The data for the two models are given in a spreadsheet format (file excelCase3.xls). In the first model, the spreadsheet is used to calculate the coefficients c_{ij}, which are then used as input data. In the second model, the raw preference scores, b_{ij} and s_{ji}, are the input data and the model itself calculates the coefficients c_{ij}. The advantage of the second is that it allows computing the percentages of buyer and seller satisfaction regarding their expressed preferences.

The output of model amplCase3b.txt for the data in file excelCase3.xls (6 buyers, 7 sellers, and 6 sessions) is given in Figure 26.7. It provides the assignment of buyers to sellers within each session as well as the percent satisfaction for each buyer and seller for a weight factor $\alpha = .5$. The results show high buyer and seller satisfactions (92% and 86%, respectively). If $\alpha < .5$, seller satisfaction will increase.

Practical considerations:

For the solution of the assignment model to be realistic, it must take into consideration the delays between successive appointments. Essentially, a buyer, once through with an appointment, will most likely have to move to another cubicle for the next appointment. A feasible schedule must thus account for the transition time between successive appointments. The following *walking* constraints achieve this result:

$$x_{ijt} + \sum_{k \in J_i} x_{i,k,t+1} = 1, i = 1, \ldots, m, j = 1, \ldots, n, t = 1, \ldots, T$$

The set J_i represents the sellers buyer i cannot reach in period $t + 1$ without experiencing undue delay. The logic is that if buyer i has an appointment with seller j in period $t(x_{ijt} = 1)$, then the same buyer may not schedule a next-period $(t + 1)$ appointment with seller k who cannot be reached without delay (that is, $x_{i,k,t+1} = 0$). We can reduce the number of such constraints by eliminating period t that occurs at the end of a session block (e.g., coffee breaks, lunch break, and end of day).

The additional constraints increase the computational difficulty of the model considerably. In fact, the model may not be solvable as an integer linear program considering the computational limitations of present-day IP algorithms. This is the reason a heuristic is needed to determine a "good" solution for the problem.

The heuristic used to solve the new restricted model is summarized as follows:

For each period t do

1. *Set* $x_{ijt} = 0$ *if the location of buyer i's last meeting in period* $t - 1$ *does not allow reaching seller j in period t.*
2. *Set* $x_{ijt} = 0$ *if a meeting between i and j has been prescheduled.*
3. *Solve the resulting two-dimensional assignment model.*

Next t

```
Optimal score =   50.87
Optimal assignments:

Session 1:
  Assign buyer 1 to seller 1
  Assign buyer 2 to seller 5
  Assign buyer 3 to seller 4
  Assign buyer 4 to seller 6
  Assign buyer 5 to seller 2
  Assign buyer 6 to seller 7

Session 2:
  Assign buyer 1 to seller 3
  Assign buyer 2 to seller 6
  Assign buyer 3 to seller 5
  Assign buyer 4 to seller 2
  Assign buyer 5 to seller 1
  Assign buyer 6 to seller 4

Session 3:
  Assign buyer 1 to seller 2
  Assign buyer 2 to seller 4
  Assign buyer 3 to seller 6
  Assign buyer 4 to seller 5
  Assign buyer 5 to seller 3
  Assign buyer 6 to seller 1

Session 4:
  Assign buyer 1 to seller 5
  Assign buyer 2 to seller 3
  Assign buyer 3 to seller 1
  Assign buyer 4 to seller 7
  Assign buyer 5 to seller 4
  Assign buyer 6 to seller 2

Session 5:
  Assign buyer 2 to seller 2
  Assign buyer 3 to seller 3
  Assign buyer 4 to seller 4
  Assign buyer 5 to seller 5
  Assign buyer 6 to seller 6

Session 6:
  Assign buyer 1 to seller 4
  Assign buyer 2 to seller 7
  Assign buyer 3 to seller 2
  Assign buyer 4 to seller 1
  Assign buyer 5 to seller 6
  Assign buyer 6 to seller 5

Buyers satisfaction: Average =  92
Buyer:      1    2    3    4    5    6
Percent:  100   86  100   80   86  100

Sellers satisfaction: Average =  86
Seller:     1    2    3    4    5    6    7
Percent:   83  100   60  100  100  100   60
```

FIGURE 26.7
AMPL output of the assignment model

The quality of the heuristic solution can be measured by comparing its objective value (preference measure) with that of the original assignment model (with no walking constraints). Reported results show that for five separate events the gap between the two solutions was less than 10%, indicating that the heuristic provides reliable solutions.

Of course, the devised solution does not guarantee that all preferences will be met because of the limit on the available number of time periods. Interestingly, the results recommended by the heuristic show that at least 80% of the highest-priority meetings (with preference 1) are selected by the solution. This percentage declines almost linearly with the increase in expressed scores (higher score indicates lower preference).

Questions:

1. Suppose that the locations of booths preclude scheduling a buyer from holding meetings in two successive sessions with sellers 1 and 3. Modify the AMPL model adding the walking constraints and find the optimum solution.
2. Apply the heuristic to the situation in Question 1 and compare it with the optimum solution.

Case 4: Saving Federal Travel Dollars[5]

Tools: Shortest-route algorithm

Area of application: Business travel

Description of the situation:

U.S. Federal Government employees are required to attend development conferences and training courses. Currently, the selection of the city hosting conferences and training events is done without consideration of incurred travel cost. Because federal employees are located in offices scattered around the United States, the location of the host city can impact travel cost, depending on the number of participants and the locations from which they originate.

The General Services Administration (GSA) issues a yearly schedule of airfares that the Government contracts with different U.S. air carriers. This schedule provides fares for approximately 5000 city-pair combinations in the contiguous 48 states. It also issues per-diem rates for all major cities and a flat daily rate for cities not included in the list. Participants using personal vehicles for travel receive a flat rate per mile. All rates are updated annually to reflect cost-of-living increase. The travel cost from a location to the host city is a direct function of the number of participants, the cost of travel to the host city, and the per-diem allowed for the host city.

The problem is concerned with the optimal location of host city for an event, given a specified number of applicants from participating locations around the country.

Analysis:

The idea of the solution is simple: The host city must yield the lowest travel cost that includes (1) minimum transportation cost and (2) per-diem allowance for the host city. The determination of the transportation cost requires identifying the locations from which participants depart. It is reasonable to assume that for locations within 100 miles from the host city, participants use personal vehicles as the selected mode of transportation. Others travel by air. The cost basis for air

[5]*Source:* J.L. Huisingh, H.M Yamauchi, and R. Zimmerman, "Saving Federal Travel Dollars," *Interfaces,* Vol. 31, No.5, pp. 13–23, 2001.

travelers consists of the sum of contracted airfares along the legs of the *cheapest* route to the host city. To determine such routes, it is necessary to identify the locations around the United States from which participants depart. Each such location is a possible host city candidate provided it offers adequate airport and conference facilities. In the present case, 261 such locations with 4640 contracted airport links are identified.

The determination of the *cheapest* airfare routes among the selected 261 locations with 4640 air links is no simple task because a trip may involve multiple legs. Floyd's algorithm (Section 6.3.2) is ideal for determining such routes. The "distance" between two locations is represented by the contracted airfare provided by the government. Per the contract, the cost of the round trip ticket then equals double the cost of the one-way trip.

To simplify the analysis, the study does not allow the use of car rentals at destinations. The plausible assumption here is that the host hotel is in the vicinity of the airport, usually with free shuttle service.

Per-diems cover lodging, meals, and incidental expenses. Participants arrive the day before the event starts. However, those arriving from locations within 100 miles arrive the morning of the first day of the event. All participants will check out of the hotel on the last day. Days of arrival and departure, government regulations for meals and incidental expenses allow only a 75% reimbursement of the full per-diem rate.

Numerical Example:

For the sake of this illustration, we will assume a 12-host-city situation. Table 26.3 provides the contracted one-way airfares for admissible links among the cities. A blank entry indicates that the associated city pair does not have a direct air link.

TABLE 26.3 One-way Airfare for the 12-City Example

	SF	ORD	STL	LAX	TUL	DEN	DC	ATL	DAL	NY	MIA	SPI
SF				$70		$120			$220			
ORD			$99			$140	$150					
STL		$99			$95	$110						$78[(a)]
LAX	$70					$130						
TUL			$95			$105			$100			
DEN	$120	$140	$110	$130	$105							
DC		$150						$100	$195	$85		
ATL							$100				$125	
DAL	$220				$100		$195					
NY							$85				$130	
MIA								$125		$130		
SPI			$78[(a)]									

[(a)] Air travel cost = $78. Distance <100 miles (= 86 miles). Personal car used for travel between STL and SPI.

TABLE 26.4 Lodging Cost, Per Diem, and Number of Participants in the 12-City Example

City	Lodging per night ($)	Per-diem ($)	Number of participants
SF	115.00	50.00	15
ORD	115.00	50.00	10
STL	85.00	48.00	8
LAX	120.00	55.00	18
TUL	70.00	35.00	5
DEN	90.00	40.00	9
DC	150.00	60.00	10
ATL	90.00	50.00	12
DAL	90.00	50.00	11
NY	190.00	60.00	12
MIA	120.00	50.00	8
SPI	60.00	35.00	2

Maximum lodging and per-diem allowances for the 12 cities together with their associated number of participants for an upcoming event are listed in Table 26.4. The duration of the event is 4 days. The standard mileage allowance for personal vehicles is $.325 per mile (per the year 2000).

The first step in the solution is to determine the cheapest airfare among all city pairs. This step is carried out by TORA using Floyd's shortest-route algorithm (Section 6.3.2). The results of the algorithm are summarized in Table 26.5. Blank entries symmetrically equal those above the main diagonal. Recall that these values represent the cost of one-way tickets and that the cost of round trip tickets is double that amount. Floyd's algorithm automatically specifies the trip legs associated with each city pair.

The final step in the solution is to determine the total cost of the event for all the participants, given that the event is held at one of the listed cities. The city that provides the smallest total cost is then selected as the host city.

TABLE 26.5 Cheapest Airfare in the 12-City Example

	ORD	STL	LAX	TUL	DEN	DC	ATL	DAL	NY	MIA	SPI
SF	$260	$230	$70	$225	$120	$410	$510	$220	$495	$625	$308
ORD		$99	$270	$194	$140	$150	$250	$294	$235	$365	$177
STL			$240	$95	$110	$249	$349	$195	$334	$464	$28*
LAX				$235	$130	$420	$520	$290	$505	$635	$318
TUL					$105	$295	$395	$100	$380	$510	$173
DEN						$290	$390	$205	$375	$505	$188
DC							$100	$195	$85	$215	$327
ATL								$295	$185	$125	$427
DAL									$280	$410	$273
NY										$130	$412
MIA											$542

*Personal vehicle cost based on 86 miles (32.5 cents per mile).

To demonstrate the computations, suppose that STL is the candidate host city. The associated total cost is then computed as:

$$\begin{aligned}\text{Travel cost} &= 2 \times (15 \times 230 + 10 \times 99 + 18 \times 240 + 5 \times 95 + 9 \times 110 + 10 \times 249 \\ &\quad + 12 \times 349 + 11 \times 195 + 12 \times 334 + 8 \times 464) + 2 \times (2 \times 86) \times .325 \\ &= \$53{,}647.80\end{aligned}$$

$$\begin{aligned}\text{Lodging cost} &= \$85 \times [(15 + 10 + 18 + 5 + 9 + 10 + 12 + 11 + 12 + 8) \times 4 + 2 \times 3] \\ &= \$37{,}910\end{aligned}$$

$$\begin{aligned}\text{Per-diem cost} &= \$48 \times [(15 + 10 + 18 + 5 + 9 + 10 + 12 + 11 + 12 + 8) \\ &\quad \times 4.5 + (2 + 8) \times 3.5] \\ &= \$25{,}440\end{aligned}$$

Note that because SPI is located 86 miles (<100 miles) from STL, its participants drive personal vehicles and arrive at STL the morning of the first day of the event. Thus, their per-diem is based on $3\frac{1}{2}$ days and their lodging is based on 3 nights only. Participants from STL receive per diem for $3\frac{1}{2}$ days and no lodging. All other participants arrive at STL a day earlier, and their per-diem is based on $4\frac{1}{2}$ days and 4 nights of lodging.

The computations for all host cities can be done conveniently with a spreadsheet as shown in Figure 26.8 (file excelCase4.xls—all the formulas are appended as cell comments). The results show that TUL offers the lowest total cost ($108,365), followed by DEN ($111,332) and then STL ($115,750).

Questions:

1. Given that TUL is the chosen host city, how should the reservations be made for SF participants?

FIGURE 26.8

Cost comparisons of host cities (file excelCase4.xls)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
1	Lodging	115	115	85	120	70	90	150	90	90	190	120	60					
2	Per-diem	50	50	48	55	35	40	60	50	50	60	50	35					
3	# participants	15	10	8	18	5	9	10	12	11	12	8	2	120				
4	Event days	4																
5																		
6	Air travel cost matrix(determined by Floyd's algorithm):																	
7		SF	ORD	STL	LAX	TUL	DEN	DC	ATL	DA	NY	MIA	SPI	Travel	Lodging	Per-diem	Total	
8	SF	0	260	230	70	225	120	410	510	220	495	625	308	$64,202	$48,300	$26,250	$138,752	SF
9	ORD	260	0	99	270	194	140	150	250	294	235	365	177	$51,220	$50,600	$26,500	$128,320	ORD
10	STL	230	99	0	240	95	110	249	349	195	334	464	28	$53,648	$37,910	$25,440	$116,998	STL
11	LAX	70	270	240	0	235	130	420	520	290	505	635	318	$66,842	$48,960	$28,710	$144,512	LAX
12	TUL	225	194	95	235	0	105	295	395	100	380	510	173	$58,052	$32,200	$18,725	$108,977	TUL
13	DEN	120	140	110	130	105	0	290	390	205	375	505	188	$51,392	$39,960	$21,240	$112,592	DEN
14	DC	410	150	249	420	295	290	0	100	195	85	215	273	$55,836	$66,000	$31,800	$153,636	DC
15	ATL	510	250	349	520	395	390	100	0	295	185	125	427	$72,212	$38,880	$26,400	$137,492	ATL
16	DA	220	294	195	290	100	205	195	295	0	280	410	273	$56,082	$39,240	$26,450	$121,772	DA
17	NY	495	235	334	505	380	375	85	185	280	0	130	412	$69,652	$82,080	$31,680	$183,412	NY
18	MIA	625	365	464	535	510	505	215	125	410	130	0	542	$92,132	$53,760	$26,600	$172,492	MIA
19	SPI	308	177	28	318	173	188	273	427	273	412	542	0	$70,064	$28,320	$18,830	$117,214	SPI
20	Note: Travel cost between STL and SPI is based on mileage (=86*.325=$28)																	
21																		

2. Suppose that the following new air links are added to the list of admissible routes: ORD-NY ($150), LAX-MIA ($300), DEN-DAL ($110), ATL-NY ($150).
   Determine the host city.

Case 5: Optimal Ship Routing and Personnel Assignment for Naval Recruitment in Thailand[6]

Tools: Transportation model, ILP

Area of application: Transportation, routing

Description of the situation:

Thailand Navy recruits are drafted four times a year. A draftee reports to one of 34 centers in a home locality and is then transported by bus to one of four navy branch bases. From there, recruits are transported to the main naval base by ship. All trips originate from and terminate at the main base. The docking facilities at the branch bases may restrict the type of ship that can visit a base. Three classes of ship are available as summarized in Table 26.6. Table 26.7 provides the distances between the main base (0) and branch bases (1, 2, 3, and 4). The matrix is symmetrical and all distances are in kilometers.

Each branch base has a limited capacity but, as whole, the four bases have sufficient capacity to accommodate all the draftees. During the summer of 1983, a total of 2929 draftees were transported from the drafting centers to the four branch bases and then to the main base.

TABLE 26.6 Ship Availability for Transporting Recruits

Ship class	Capacity (number of recruits)	Available number of round trips	Allowed bases
1	100	2	1
2	200	10	1, 2, 3
3	600	2	1, 2, 3, 4

TABLE 26.7 Distance matrix (in kilometers) for main (0) and branch (1, 2, 3, and 4) bases

From \ To	0	1	2	3	4
0	0	310	510	540	660
1	310	0	150	190	225
2	510	150	0	25	90
3	540	190	25	0	70
4	660	225	90	70	0

[6]*Source:* P. Choypeng, P. Puakpong, and R. Rosenthal, "Optimal Ship Routing and Personnel Assignment for Naval Recruitment in Thailand," *Interfaces*, Vol. 16, No. 4, pp. 47–52, 1986.

Two problems arise concerning the transportation of the recruits:

1. How should the draftees be transported by bus from drafting centers to branch bases?
2. How should the draftees be transported by ship from branch bases to the main base?

Model development:

The solution of the problem is done in two independent stages:

1. Stage 1 determines the optimal allocation of draftees from the recruiting centers to the branch bases.
2. Stage 2 then uses the result of Stage 1 to determine the optimal transportation schedules from the branches to the main base.

The first problem is a straightforward transportation model with 34 sources (drafting centers) and 4 branch bases (destinations). For the purpose of the model, the "supply" at a source is the number of recruits a center drafts. The "demand" at a destination equals the number of draftees a branch base can receive. In the present situation the total supply does not exceed the total demand. The unit transportation equals the cost of a bus trip divided by the number of seats on the bus. We can represent the transportation unit as a bus load rather than an individual recruit by dividing the number of recruits at a center by the number of seats available on a bus, and then round up the result. For example, 500 recruits from a drafting center on a bus with 52 seats will require 10 bus trips. The unit transportation cost in this case is the cost per bus trip.

Whether we use a recruit or a bus load as the transportation unit, both representations are approximations. The use of a recruit as the transportation unit can deflate the total transportation cost, because a partially full bus incurs the same cost as a full bus. On the other hand, the use of a bus load as a transportation unit can inflate both the number of recruits and the capacities of the branch bases. Either way, the bias in the solution is not pronounced, because it only results from treating one partial bus load as a full load. In this study, a recruit is used as the transportation unit.

Let

x_{ij} = Number of draftees at recruiting center i transported to branch base j

c_{ij} = Transportation cost per recruit from recruiting center i to branch base j

a_i = Number of recruits at center i

b_j = Capacity (in number of recruits) at branch base j

The mathematical model is

$$\text{Minimize } z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} \geq a_i,\ i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} \leq b_j,\ j = 1, 2, \ldots, n$$

$$x_{ij} \geq 0, \text{ for all } i \text{ and } j$$

The model will yield a feasible (integer) solution so long as $\sum_{i=1}^{m} a_i \leq \sum_{j=1}^{n} b_j$, an acceptable assumption in this situation.

TABLE 26.8 Optimum Number of Recruits Transported to Branch Bases by Bus

Branch Base		Number of recruits
1		475
2		659
3		672
4		1123
	Total	2929

Because the model given above is straightforward, it will not be considered any further. Instead, we will assume that the model has been solved using pertinent data, with the optimum solution as given in Table 26.8.

The optimum solution from Stage 1 is used as input data for Stage 2, which is concerned with the determination of the optimum schedule for transporting the recruits from the four branch bases to the main base. The solution must provide the required trips identified by route and ship class. A trip starts at main base (0) with trips legs that reach one or more branch bases (1, 2, 3, and 4), taking into account allowable port visitations by the three ship classes as given in Table 26.6. For example, a Class 2 ship can visit bases 1, 2, and 3 only.

The first task in the development of the model is to identify all possible trips, taking into account the restrictions on port visitations. A ship of class 1 is allowed to visit base 1 only and the associated round trip is thus given as 0-1-0. For class 2 ships, docking is allowed in bases 1, 2, and 3 only. For the purpose of the solution, a round trip for class 2 may include one base (e.g., 0-1-0), two bases (e.g., 0-1-2-0), or three bases (e.g., 0-1-2-3-0). The same idea applies to class 3 routes, which may include from 1 to 4 branch bases. It is important to note that two round trips involving the same bases do not necessarily produce the same total distance. For example, the tours 0-1-3-4-0 and 0-1-4-3-0 involve the same branch bases (1, 3, and 4), yet they result in different total distances—namely,

$$\text{Distance } (0\text{-}1\text{-}3\text{-}4\text{-}0) = 310 + 190 + 70 + 660 = 1230 \text{ km}$$
$$\text{Distance } (0\text{-}1\text{-}4\text{-}3\text{-}0) = 310 + 225 + 70 + 540 = 1145 \text{ km}$$

For this reason, it is necessary to consider all round trip permutations.

Let

J = set of all (round) trips

A_i = Set of trips that includes base i, $i = 1, 2, 3, 4$

B_j = Set of bases visited on trip j, $j \in J$

C_k = Set of trips that use ship class k, $k = 1, 2, 3$

x_j = Number of times trip j is used, $j \in J$

y_{ij} = Number of recruits transported from branch base i on trip j, $i = 1, 2, 3, 4$, $j \in J$

r_i = Number of recruits at branch base i (as determined from Stage 1), $i = 1, 2, 3, 4$

n_k = Number of ships available in class k, $k = 1, 2, 3$

c_k = Capacity of ship class k, $k = 1, 2, 3$

d_j = Total distance (in km) of trip j, $j \in J$

The model for the ship-routing problem is

$$\text{Minimize } z = \sum_{j \in J} d_j x_j$$

subject to

$$\sum_{j \in A_i} y_{ij} \geq r_i, i = 1, 2, 3, 4$$

$$\sum_{i \in B_j} y_{ij} \leq c_k x_j, j \in C_k, k = 1, 2, 3$$

$$\sum_{j \in C_k} x_j \leq n_k, k = 1, 2, 3$$

$$x_j \geq 0 \text{ and integer}, j \in J$$

$$y_{ij} \geq 0 \text{ and integer}, i = 1, 2, 3, 4, j \in J$$

The first constraint ensures that all recruits will be transported. The second constraint recognizes the capacity limitation of each ship class. The third constraint ensures that the number of round trips used in each class does not exceed availability.

AMPL solution:

File amplCase5.txt provides the AMPL model for the Stage 2 problem. The model automatically generates all possible trips for each ship class and their distances. Perhaps the "trickiest" part of the model deals with the determination of the sets A_i and B_j as defined in the mathematical model. These sets are determined by creating the matrix a_{ij} that mirrors the legs of the different trips by letting $a_{ij} = 1$ if base i is in trip j and $a_{ij} = 0$ otherwise. For example, if trip 30 is 0-1-4-3-0, then $a_{1,30} = 1$, $a_{2,30} = 0$, $a_{3,30} = 1$, and $a_{4,30} = 1$. Using the matrix $\|a_{ij}\|$, the following substitutions in the constraint eliminate the need to determine the sets A_i and B_j explicitly:

$$\sum_{j \in A_i} y_{ij} = \sum_{j \in J} a_{ij} y_{ij}, i = 1, 2, 3, 4$$

$$\sum_{i \in B_j} y_{ij} = \sum_{i=1}^{4} a_{ij} y_{ij}, j \in C_k, k = 1, 2, 3$$

Figure 26.9 provides the output of the model using the data presented earlier. It identifies the trip and its associated ship class together with the number of times the trip is used. The number of recruits from each base is also specified for each trip. The row and column totals affirm the feasibility of the solution by showing that (1) ship capacity restrictions are met, and (2) the number of recruits transported at least equals the number specified by the data of the problem (e.g., at base 1, the solution guarantees that 541 recruits can be transported, which exceeds the actual 475 recruits).

The reason the solution "inflates" the number of transported recruits is that we specified the first constraint as an inequality rather than an equation. This, however, does not mean that the solution will use more "resources" than necessary. The inequality constraint simply allows the solution to "inflate" the number of recruits to make use of any excess capacity that may be available on a ship. This is evident from the fact that the row total in Figure 26.9 exactly equals the ship capacity multiplied by the number of trips.

To show that the use of the inequality does not allow the use of more resources than is absolutely necessary, Figure 26.10 provides the output of the same model with the same data except that the inequality in the first constraint is replaced by an equation. In this case, the column

Total Ship kilometers = 10515

Ship class	Capacity	Tour legs	Nbr of trips	Nbr draftees carried from base 1	2	3	4	Totals
2	200	0-1-0	2	400	0	0	0	400
2	200	0-1-2-0	3	141	459	0	0	600
2	200	0-2-1-0	1	0	200	0	0	200
2	200	0-1-2-3-0	3	0	0	600	0	600
3	600	0-1-2-4-3-0	2	0	0	77	1123	1200
			Totals	541	659	677	1123	

Total number of roundtrips = 11

FIGURE 26.9

AMPL output with the constraint as an inequality $\left(\sum_{j \in A_i} y_{ij} \geq r_i\right)$

Total Ship kilometers = 10515

				Nbr draftees carried from base				
Ship class	Capacity	Tour legs	Nbr of trips	1	2	3	4	Totals
2	200	0-1-0	2	334	0	0	0	334
2	200	0-1-2-0	1	0	200	0	0	200
2	200	0-2-1-0	3	141	459	0	0	600
2	200	0-1-2-3-0	3	0	0	595	0	595
3	600	0-1-2-4-3-0	2	0	0	77	1123	1200
			Totals	475	659	672	1123	

Total number of roundtrips = 11

FIGURE 26.10

AMPL output with the first constraint as an equation $\left(\sum_{j \in A_i} y_{ij} = r_i\right)$

sums exactly equal the number of recruits at each base, whereas the row sums indicate that the "used" capacity on the ships can be less the total available capacity. For example, trip 0-1-0 carries 390 recruits from base 1, which is less than the maximum capacity of two trips of a class 2 ship ($= 2 \times 200 = 400$). The fact remains that two trips will be made, exactly as the solution in Figure 26.9 specifies.

Computational considerations

Despite the fact that the two formulations (with inequality and equality constraints) produce the same optimum solution, the computational experience with the two ILPs are different. In the *equality* constraint case, AMPL generated a total of 34,290 branch-and-bound nodes, which is 70% more than the 20,690 nodes generated in the case of the *inequality* constraint.[7] It would seem that the more restricted solution space should have resulted in a faster convergence of the branch-and-bound algorithm because, in a way, it sets tighter limits on the feasible integer space. Unfortunately, no such rule can be recommended, as the (generally bizarre) behavior of the branch-and-bound algorithm is also problem dependent.

Questions:

1. Consider the following round trips:
   Ship class 1: 0-1-0
   Ship class 2: 0-1-0, 0-3-2-0, 0-1-2-3-0, 0-2-3-0
   Ship class 3: 0-3-0, 0-1-4-3-2-0, 0-3-1-0, 0-4-2-1-0, 0-3-4-1-0
   (a) Given that these are the only trips allowed, develop the corresponding $\|a_{ij}\|$ matrix.
   (b) Write down the explicit ILP for stage 2.
2. Suppose that there is a fixed cost of $5000 associated with each docking/undocking operation at a port. Further, the operating costs per kilometer are $100, $120, and $150 for ship classes 1, 2, and 3, respectively. Using the data in the case analysis, determine the optimal transportation schedule from the four bases to the main base.

Case 6: Allocation of Operating Room Time in Mount Sinai Hospital[8]

Tools: ILP, GP

Area of application: Health care

Description of the situation:

The situation takes place in Canada, where health care insurance is mandatory and universal for all citizens. Funding, which is based on a combination of premiums and taxes, is controlled by the individual provinces. Under this system, hospitals are advanced a fixed annual budget and each province pays physicians retroactively using a fee-for-service funding mechanism. Local governments control the size of the health-care system by placing strict limits on hospital spending. The result is that the use of health resources, particularly operating rooms, must be controlled effectively.

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[7]This experience is based on CPLEX 9.1.3.
[8]*Source:* J.T. Blake and J. Donald, "Mount Sinai Hospital Uses Integer Programming to Allocate Operating Room Time," *Interfaces,* Vol. 32, No. 2, pp. 63–73, 2002.

TABLE 26.9 Surgery Room Availability in Mt. Sinai Hospital

	Availability hours			
Weekday	*Main "short"*	*Main "long"*	*EOPS "short"*	*EOPS "long"*
Monday	08:00–15:30	08:00–17:00	08:00–15:30	08:00–16:00
Tuesday	08:00–15:30	08:00–17:00	08:00–15:30	08:00–16:00
Wednesday	08:00–15:30	08:00–17:00	08:00–15:30	08:00–16:00
Thursday	08:00–15:30	08:00–17:00	08:00–15:30	08:00–16:00
Friday	09:00–15:30	09:00–17:00	09:00–15:30	09:00–16:00
Number of rooms	4	4	1	1

Mount Sinai Hospital has 10 staffed operating rooms serving 5 departments: surgery, gynecology, ophthalmology, otolaryngology, and oral surgery. There are 8 main surgical rooms and 2 elective outpatient surgery (EOPS) rooms. An operating room is either "short" or "long", depending on the daily number of hours the room is in use. Because of the socialized nature of health care in Canada, all surgeries are scheduled during work days only (Monday through Friday). Table 26.9 summarizes the daily availability of the different types of rooms and Table 26.10 provides the weekly demand for operating room hours. The limit on the underallocated hours in Table 26.10 is the most a department can be denied relative to its weekly request.

The objective of the study is to determine a reasonably equitable daily schedule for the utilization of available operating rooms.

Mathematical model:

The best that we can do in this situation is to devise a daily schedule that most satisfies the weekly target hours for the different departments. In other words, we set the target hours for each department as a goal, and try to satisfy it. The objective of the model is to minimize the total deviation from the weekly target hours.

Let

$$
\begin{aligned}
x_{ijk} &= \text{Number of rooms of type } i \text{ assigned to department } j \text{ on day } k \\
d_{ik} &= \text{Duration (availability in hours) of room type } i \text{ on day } k \\
a_{ik} &= \text{Number of rooms of type } i \text{ available on day } k \\
h_j &= \text{Requested (ideal) target hours for department } j \\
u_j^- &= \text{Maximum underallocated hours allowed in department } j
\end{aligned}
$$

TABLE 26.10 Weekly Demand for Operating Room Hours

Department	Weekly target hours	Allowable limit of underallocated hours
Surgery	189.0	10.0
Gynecology	117.4	10.0
Ophthalmology	39.4	10.0
Oral surgery	19.9	10.0
Otolaryngology	26.3	10.0
Emergency	5.4	3.0

The given situation involves 6 departments and 4 types of rooms. Thus, $i = 1, 2, 3, 4$ and $j = 1, 2, \ldots, 6$. For a 5-day work week, the index k assumes the values 1 through 5.

The following integer-goal programming model represents the Mount Sinai Hospital scheduling situation:

$$\text{Minimize } z = \sum_{j=1}^{6} \left(\frac{1}{h_j}\right) s_j^-$$

subject to

$$\sum_{i=1}^{4} \sum_{k=1}^{5} d_{ik} x_{ijk} + s_j^- - s_j^+ = h_j, \text{ for all } j \tag{1}$$

$$\sum_{j=1}^{6} x_{ijk} \le a_{ik}, \text{ for all } i \text{ and } k \tag{2}$$

$$0 \le s_j^- \le u_j^-, \text{ for all } j \tag{3}$$

$$x_{ijk} \ge 0 \text{ and integer for all } i, j, \text{ and } k \tag{4}$$

$$s_j^-, s_j^+ \ge 0, \text{ for all } j \tag{5}$$

The logic of the model is that it may not be possible to satisfy the target hours h_j for department j, $j = 1, 2, \ldots, 6$. Thus, the objective is to determine a schedule that minimizes possible "underallocation" of rooms to the different departments. To do this, the nonnegative variables s_j^- and s_j^+ in constraint (1) represent the under- and over-allocation of hours relative to the target h_j for department j. The ratio $\frac{s_j^-}{h_j}$ measures the relative amount of underallocation to department j. Constraint (2) recognizes room availability limits. Constraint (3) is used to limit the amount by which a department is underallocated. The limits u_j^- are user-specified.

Model results

File amplCase6.txt gives the AMPL model of the problem. Figure 26.11 gives the solution for the data provided in the statement of the problem. It shows that all goals are satisfied ($z = 0$) and it details the allocation of rooms (by type) to the different departments during the work week (Monday through Friday). Indeed, the departmental summary given at the bottom of the figure shows that the requests for 5 (out of 6) departments are oversatisfied. This happens to be the case because there is abundance of resources for the week and the model does not try to minimize the overallocation of hours to the different departments. Actually, it makes no sense in the present model to try to do away with overallocation of hours, because the rooms are available and might as well be apportioned to the different departments. In essence, the main concern is about underallocation when available resources do not meet demand.

Computational experience:

In the model, the variable x_{ijk} represents the number of allocated rooms. It must assume integer values, and here lies a familiar problem that continues to plague integer programming computations. The AMPL model executed rapidly with the set of data given in the description of the problem. However, when the data representing target hours, h_j, were adjusted slightly (keeping all other data unchanged), the computational experience was totally different. First, the execution time lasted more than one hour (as opposed to a few seconds with the initial set of data) and, after exploring more than 45 million branch-and-bound nodes, failed to produce a feasible solution, let alone the optimum. This experience appears to take place when the demand exceeds the supply. Actually, the behavior of this ILP is unpredictable, because when the objective function

```
z = 0.00
Weekly Time Allocation:
Mon:
  Gynecology: 39.0 hrs
     4 room(s) type Main_L
     1 room(s) type Main_S
  Ophthalmology: 17.0 hrs
     1 room(s) type Main_S
     1 room(s) type EOPS_S
  Oral_surgery: 16.5 hrs
     1 room(s) type Main_S
     1 room(s) type EOPS_L
  Otolaryngology:  9.0 hrs
     1 room(s) type Main_S
Tue:
  Surgery: 17.0 hrs
     1 room(s) type Main_S
     1 room(s) type EOPS_S
  Gynecology: 39.0 hrs
     4 room(s) type Main_L
     1 room(s) type Main_S
  Oral_surgery:  7.5 hrs
     1 room(s) type EOPS_L
  Otolaryngology: 18.0 hrs
     2 room(s) type Main_S
Wed:
  Surgery: 66.5 hrs
     3 room(s) type Main_L
     4 room(s) type Main_S
     1 room(s) type EOPS_S
  Ophthalmology: 15.0 hrs
     1 room(s) type Main_L
     1 room(s) type EOPS_L
Thu:
  Surgery: 72.5 hrs
     4 room(s) type Main_L
     3 room(s) type Main_S
     1 room(s) type EOPS_L
     1 room(s) type EOPS_S
  Ophthalmology:  9.0 hrs
     1 room(s) type Main_S
Fri:
  Surgery: 34.0 hrs
     3 room(s) type Main_S
     1 room(s) type EOPS_S
  Gynecology: 39.0 hrs
     4 room(s) type Main_L
     1 room(s) type Main_S
  Emergency:  6.5 hrs
     1 room(s) type EOPS_L
Departmental summary:
  Surgery allocated 190.0 hrs (101%)
  Gynecology allocated 117.0 hrs (100%)
  Ophthalmology allocated 41.0 hrs (104%)
  Oral_surgery allocated 24.0 hrs (121%)
  Otolaryngology allocated 27.0 hrs (103%)
  Emergency allocated 6.5 hrs (120%)
```

FIGURE 26.11

Output of Mt. Sinai Hospital model

is changed to simply minimize the *unweighted* sum of s_j^-, all previously unsolvable cases are solved instantly. The questions at the end of this case outline these computational experiences.

What courses of action are available for overcoming this problem? At first thought, the temptation may be to drop the integer requirement and then round the resulting linear programming solution. This option will not work in this case because, in all likelihood, it will not produce a feasible solution. Given that a specific number of hospital rooms are available, it is highly unlikely that a trial-and-error rounded solution will meet room availability limits. This means that there is no alternative to imposing the integer condition.

One way to improve the chances for a successful execution of the integer model is to limit the feasible ranges for the variables x_{ijk} by taking into account the availability of other resources. For example, if the hospital has only two dental surgeons on a given day, no more than two rooms (of any type) can be assigned to that department on that day. Setting such bounds may be effective in securing an optimal integer solution. Short of meeting this requirement, the only remaining option is to devise a heuristic for the problem.

Question:

1. In amplCase6.txt,
   (a) Replace the current values of h_j with

   ```
   param h:= Surgery 190  Gynecology 122  Ophthalmology 41.4
             Oral_surgery 20.9 Otolaryngology  25.3 Emergency 6.0;
   ```

   Execute the model with the new data. You will notice that the model will execute for over one hour without finding a feasible solution.

   (b) Using the original data of the model, change parameter a so that only 3 rooms of Main_L are available during all five working days of the week. As in (a), AMPL fails to produce a solution.

   (c) In both cases (a) and (b), replace the objective function with

   ```
   minimize z:  sum{j in departments}(sMinus[j])
   ```

   which call for minimizing the sum of s_j^-. Now, execute the model and you will notice that in both situations, the optimum is found instantly.

Case 7: Optimizing Trailer Payloads at PFG Building Glass[9]

Tools: ILP, static force (free-body diagram) calculations

Area of application: Transportation and distribution

Description of the situation:

PFG, a South African glass manufacturer, uses specially equipped (fifth-wheel) trailers to deliver packs of flat glass sheets to customers. Packs normally vary in both size and weight, and a single trailer load may include different packs depending on received orders. Figure 26.12 illustrates a typical hauler-trailer rig with its axles located at points A, B, and C. Government regulations set

[9] *Source:* H. Taha, "An Alternative Model for Optimizing Payloads of Building Glass at BFG," *South African Journal of Industrial Engineering*, Vol. 15, No. 1, pp. 31–43, 2003.

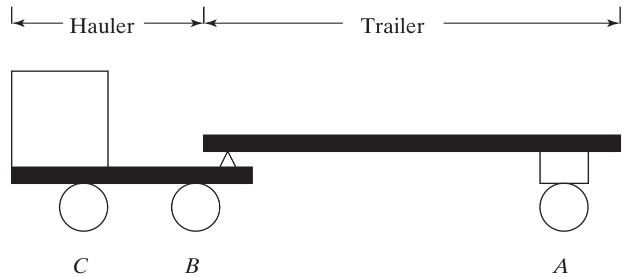


FIGURE 26.12
Points A, B, and C of axle weights in hauler-trailer rig

maximum limits on axle weights. The actual positioning of the packs on the trailer is crucial in determining these weights. Placing heavier packs toward the back of the trailer increases the load on axle A and placing them toward the front of the trailer shifts the load to axles B and C. The objective is to determine the location of a specific order of packs on the trailer bed that satisfies axle weight limits. Of course, the order mix may be such that axle weight limits will be exceeded regardless of the positioning of packs. In such cases, the solution should provide information regarding excess weight on each axle to allow modifying the order mix and, hence, obtaining a new solution. This process is repeated until all weight limits are met.

The nature of the problem precludes minimizing the weight on each of the three axles, because a reduction in the weight on one axle automatically increases the weight on another. For this reason, a realistic goal is to determine a "compromise" distribution of axle weights. This point will be developed further after the details of the problem have been presented.

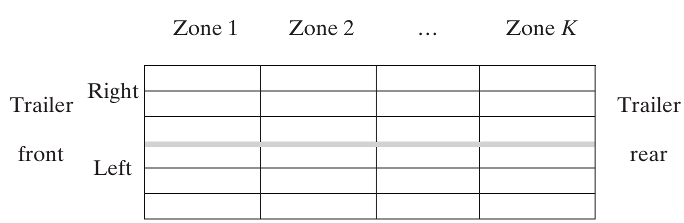
Figure 26.13 provides a schematic top view of a trailer. The length of the trailer is divided into zones whose number and boundaries vary depending on the pack mix. The width of the trailer is divided symmetrically between right and left sides, each side with an equal number of "slots". A slot in a zone is designed to hold one pack.

Calculation of Axle Weights

Axle weights can be determined by applying the following static equilibrium equations, first to the trailer and then to the hauler:

1. Sum of forces = 0.
2. Sum of moments = 0.

FIGURE 26.13
Schematic top view of trailer slots



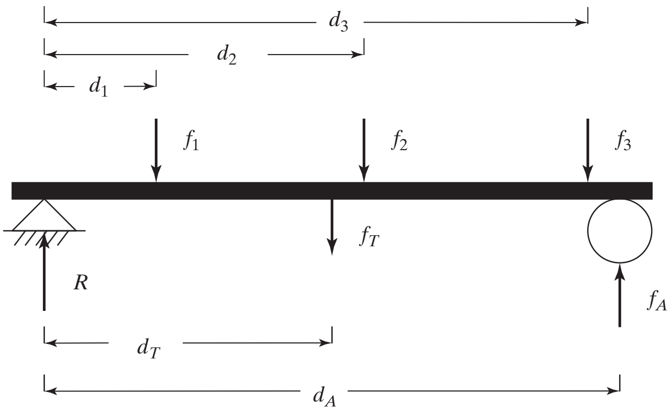


FIGURE 26.14

Trailer free-body diagram

Figure 26.14 provides the free-body diagram for the trailer. For simplicity, the trailer is assumed to have three zones, but the analysis applies readily to any number of zones. The forces (in kilograms) acting on the trailer are

f_k = Weight of glass packs in zone k, $k = 1, 2, 3$
f_T = Trailer weight
R = Reaction force at the fifth wheel (hinge)
f_A = Reaction force at the rear axle

The different distances (in meters) in the figure are known from the geometry of the trailer.

The static equilibrium equations for the trailer are

$$f_1 + f_2 + f_3 + f_T = R + f_A$$
$$f_1 d_1 + f_2 d_2 + f_3 d_3 + f_T d_T - f_A d_A = 0$$

We have two equations in two unknowns, R and f_A, which yield the solution

$$f_A = \tfrac{1}{d_A}(f_1 d_1 + f_2 d_2 + f_3 d_3 + f_T d_T) \quad (1)$$
$$R = f_1 + f_2 + f_3 + f_T - f_A \quad (2)$$

Next, we apply similar analysis to the hauler free-body diagram in Figure 26.15. The weight of the hauler is represented by f_H, and the impact of the trailer forces on the hauler equals the reaction force R computed in equation (2) above. The associated equilibrium equations are

$$f_B + f_C = R + f_H$$
$$f_H d_H + R d_R - f_B d_B = 0$$

Solving for f_B and f_C, we get

$$f_B = \tfrac{1}{d_B}(f_H d_H + R d_R) \quad (3)$$
$$f_C = R + f_H - f_B \quad (4)$$

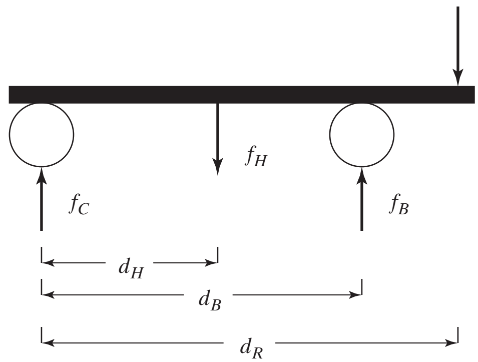


FIGURE 26.15
Hauler free-body diagram

Equations (1), (3), and (4) provide the solution for the reaction forces, f_A, f_B, and f_C, acting on axles A, B, and C of the hauler-trailer rig. Letting w_A, w_B, and w_C represent the dead weights of the axles themselves, the net weights (in kilogram) at the axles are

$$W_A = f_A + w_A \tag{5}$$

$$W_B = f_B + w_B \tag{6}$$

$$W_C = f_C + w_C \tag{7}$$

These are the weights that will be used to enforce government regulations.

Mathematical Model:

We consider the situation in which an order mix of I types of glass packs is placed in K zones with each zone holding J slots. The axle weights given in equations (5), (6), and (7) can be expressed in terms of the original forces, f_H, f_T, and f_k, $k = 1, 2, \ldots, K$, as

$$W_A = w_A + \tfrac{1}{d_A}\sum_{k=1}^{K} f_k d_k + f_T d_T \tag{8}$$

$$W_B = w_B + \tfrac{1}{d_B}(f_H d_H + R d_R) \tag{9}$$

$$W_C = w_C + R\left(1 - \tfrac{d_R}{d_B}\right) + f_H\left(1 - \tfrac{d_H}{d_B}\right) \tag{10}$$

where

$$R = \sum_{k=1}^{K} f_k\left(1 - \tfrac{d_k}{d_A}\right) - \tfrac{1}{d_A}(f_T d_T)$$

Define

$$x_{ijk} = \begin{cases} 1, \text{ if pack type } i \text{ is assigned to slot } j \text{ in zone } k \\ 0, \text{ otherwise} \end{cases}$$

$$w_i = \text{weight per pack of glass type } i$$

Glass weights applied to zone k of the trailer can then be computed in terms of x_{ijk} as

$$f_k = \sum_{i=1}^{I} w_i\left(\sum_{j=1}^{J} x_{ijk}\right), k = 1, 2, \ldots, K$$

All the remaining elements in (8), (9), and (10) are known.

Having expressed axle weights in terms of the variable x_{ijk}, we can now construct the optimization model. The main purpose of the model is to assign packs to slots such that the total axle weights, W_A, W_B, and W_C, remain within the government limits. We cannot just include these limits as simple constraints of the form $W_A \leq L_A$, $W_B \leq L_B$, and $W_C \leq L_C$ because the problem may not have a feasible solution. Ideally, then, we would like to determine the values of x_{ijk} that will reduce the individual axle weights as much as possible. If the resulting minimum weights meet government regulations, then the solution is feasible. Else, the size of the order mix must be reduced and a new solution attempted. The difficulty with this "idealized" solution is that the nature of the problem does not allow the minimization of individual axle weights because, as we stated earlier, they are not independent, and a decrease in one axle load automatically increases another.

A formulation that comes close to minimizing the individual axle weights is to find a solution that *minimizes* the *largest* of the axle weights. This formulation has the advantage of concentrating on the most extreme of all three weights. Mathematically, the objective function is expressed as

$$\text{Minimize } z = \max\{W_A, W_B, W_C\}$$

This function can be linearized readily by using the following standard substitution. Let

$$y = \max\{W_A, W_B, W_C\}$$

Then the objective function can be expressed as

$$\text{Minimize } z = y$$

subject to

$$\begin{aligned} W_A &\leq y \\ W_B &\leq y \\ W_C &\leq y \end{aligned}$$

We now turn our attention to the development of the constraints of the model. These constraints deal with

1. At most one pack per slot is allowed.
2. Zone weight limit cannot be exceeded.
3. Number of packs (by type) to be loaded on trailer bed are limited by order mix.
4. Axle weights should not exceed regulations.
5. Packs are pushed toward the center of each zone to reduce trailer tipping.

The first four constraints are essential and must be included in the model. Constraint 5, though included in the model, is really not crucial because the entire glass weight in zone k is represented by f_k, which is not a function of the width of the zone. Whatever the solution, then, the actual zone loading automatically moves the packs toward the center.

Define

$$\begin{aligned} L_k &= \text{Glass load weight limit in zone } k, k = 1, 2, \ldots, K \\ L_A &= \text{Allowable weight limit on axle } A \text{ (trailer rear axle)} \\ L_B &= \text{Allowable weight limit on axle } B \text{ (hauler rear axle)} \\ L_C &= \text{Allowable weight limit on axle } C \text{ (hauler front axle)} \end{aligned}$$

Constraints (1) through (5) are expressed mathematically as

$$\sum_{i=1}^{I} x_{ijk} \le 1, j = 1, 2, \ldots, J; k = 1, 2, \ldots, K \tag{1}$$

$$\sum_{i=1}^{I} w_i \sum_{j=1}^{J} x_{ijk} \le L_k, k = 1, 2, \ldots, K \tag{2}$$

$$\sum_{j=1}^{J} \sum_{k=1}^{K} x_{ijk} \le N_i, i = 1, 2, \ldots, I \tag{3}$$

$$W_A \le L_A \tag{4a}$$

$$W_B \le L_B \tag{4b}$$

$$W_C \le L_C \tag{4c}$$

$$\sum_{i=1}^{I} x_{ijk} \le \sum_{i=1}^{I} x_{i,j+1,k}, j = 1, 2, \ldots, \tfrac{J}{2} - 1; k = 1, 2, \ldots, K \tag{5a}$$

$$\sum_{i=1}^{I} x_{ijk} \ge \sum_{i=1}^{I} x_{i,j+1,k}, j = \tfrac{J}{2}, \tfrac{J}{2} + 1, \ldots, J - 1; k = 1, 2, \ldots, K \tag{5b}$$

I and J, as defined previously, represent the number of glass types and the number of slots per zone.

Constraint (3) is not set as a strict equation to allow the model to load less than the order size, if necessary, to meet the regulations limit specified by constraint (4). However, because the model objective function tries to minimize the maximum of axle weights, y, the obvious optimum solution, given that constraint (3) is an inequality, is not to load any packs on the trailer bed. To remedy the situation, the objective function is modified to allow loading as many packs as possible without violating the regulations.

Define the nonnegative variable s_k, $k = 1, 2, \ldots, K$, such that

$$\sum_{i=1}^{I} w_i \sum_{j=1}^{J} x_{ijk} + s_k = L_k, k = 1, 2, \ldots, K \tag{2a}$$

The idea is to force the slack variables s_k, $k = 1, 2, \ldots, K$, to assume minimum values, which, in turn, requires loading the most packs allowed by constraint (3). The situation now is that we need to minimize y as defined in the objective function and also $s_1, s_2, \ldots,$ and s_K. A reasonable way to achieve this goal is define the objective function as

$$\text{Minimize } z = y + \sum_{k=1}^{K} s_k$$

All the components of the new objective function are in kilogram.

Solution strategy with AMPL

An AMPL model was developed for the situation as given above. Unfortunately, as is typical with many ILPs, the model executed for over 4 hours without producing a feasible solution. This “discouraging” experience prompted modifications in the mathematical model. Essentially, constraint (4) is deleted altogether and the following solution strategy is adopted:

Step 1. Run the AMPL model. If the optimum solution satisfies all the axle weight regulations, stop; else go to step 2.

Step 2. Remove the order with the smallest number of packs. Go to step 1.

The idea here is that by removing orders from the mix, eventually a point will be reached where the regulations are met. It is logical in this case to remove the order with the smallest size.

The proposed strategy assumes that customers must receive their order in one shipment. Alternatively, there is obvious advantage in shipping partial orders to make the most use of the trailer capacity. This can be accommodated in the present strategy simply by reducing the size of an order by one pack at a time (in place of removing the entire order).

File amplCase7.txt gives the AMPL code that implements the proposed strategy. The code is self-explanatory because it uses the notation in the mathematical model. The `solve` segment of the model allows checking the resulting solution for feasibility. If the regulations are not met, it automatically removes the smallest-size order and reoptimizes the model. The process continues until feasibility is realized.

Figure 26.16 provides the output of AMPL for a specific set of input data.[10] The output section shows that the first solution does not satisfy the regulations. By removing order number 3 and reoptimizing, feasibility is achieved and the process ends.

Questions

1. Develop the AMPL model for the original situations in which all the regulations are expressed as explicit constraints and attempt to find a solution.
2. Modify amplCase7.txt to allow partial shipments. How does the solution compare with the one in Figure 26.16?

Case 8: Optimization of Crosscutting and Log Allocation at Weyerhaeuser[11]

Tool: DP

Area of application: Log mill operation

Description of the situation:

Mature trees are harvested and crosscut into logs for use in different mills to manufacture different end products (such as construction lumber, plywood, wafer boards, and paper). Log specifications (e.g., lengths and end diameters) for each mill depend on the end product the mill produces. With harvested trees measuring up to 100 feet in length, the number of crosscut combinations meeting mill requirements can be large. Different revenues can be realized depending on the way logs are cut from a tree. The objective is to determine the crosscut combination that maximizes the total revenue.

Mathematical model:

The basis of the model is that it is not practical to develop an optimum solution that applies to an "average" tree because, in general, harvested trees come in different lengths and end diameters. This means that optimum crosscutting and log-allocation must be based on individual trees.

A simplifying assumption of the model is that the usable length L (feet) of a harvested tree is a multiple of a minimum length K (feet). Additionally, the length of a log cut from the tree is

[10] With CPLEX 9.1.3, the solution is reached in about 20 minutes and generates 4,471,785 B&B nodes.

[11] *Source:* M.R. Lembersky and U.H. Chi, "Decision Simulators Speed Implementation and Improve Operations," *Interfaces*, Vol. 14, No. 4, pp. 1–15, 1984.

```
INPUT DATA:
Nbr of glass types =  4
    Packs of type  1  =  12
    Packs of type  2  =  6
    Packs of type  3  =  4
    Packs of type  4  =  7
Nbr of slots per zone =  10
Nbr of zones =  3

Weight limit for zone  1  =  10000  kg
Weight limit for zone  2  =  10000  kg
Weight limit for zone  3  =  10000  kg

Trailer axle weight = 3000  kg
Hauler rear axle weight =  1500  kg
Hauler front axle weight limit =  2000  kg

Total trailer axle weight limit = 24000  kg
Total hauler rear axle weight limit =  18000  kg
Total hauler front axle weight limit =  8000  kg

OUTPUT RESULTS:
Zone  1  weight =  9982  kg
Zone  2  weight =  9014  kg
Zone  3  weight =  6300  kg
Total weight at trailer axle = 25277.4  kg
Total weight at hauler rear axle =  15915.8  kg
Total weight at hauler front axle =  5902.9  kg
INFEASIBLE SOLUTION: Total trailer axle weight exceeds limit by  1277.4  kg

Packs of type  2  in zone  1  =  4
Packs of type  3  in zone  1  =  1
Packs of type  4  in zone  1  =  5
Packs of type  1  in zone  2  =  3
Packs of type  2  in zone  2  =  2
Packs of type  3  in zone  2  =  3
Packs of type  4  in zone  2  =  2
Packs of type  1  in zone  3  =  9

Searching for a feasible solution...attempt  1:  Order nbr  3 removed
Zone  1  weight =  9930  kg
Zone  2  weight =  7866  kg
Zone  3  weight =  3500  kg
Total weight at trailer axle = 22026.5  kg
Total weight at hauler rear axle =  15098.5  kg
Total weight at hauler front axle =  5971  kg

FEASIBLE SOLUTON: objective = 30730.53
Packs of type  1  in zone  1  =  2
Packs of type  2  in zone  1  =  2
Packs of type  4  in zone  1  =  6
Packs of type  1  in zone  2  =  5
Packs of type  2  in zone  2  =  4
Packs of type  4  in zone  2  =  1
Packs of type  1  in zone  3  =  5
```

FIGURE 26.16

AMPL output for the PFG model

also a multiple of K. This means that logs can only be as small as K feet and as large as NK feet, where, by definition, $N \leq \frac{L}{K}$.

Define

M = Number of mills requesting logs

$I = \frac{L}{K}$

$R_m(i, j)$ = Revenue at mill m from a log of length jK cut from the larger end of a stem of length iK, $m = 1, 2, \ldots, M$; $i = 1, 2, \ldots, I$; $j = 1, 2, \ldots, N$; $j \leq i$

c = Cost of making a crosscut at point i of the tree, $i = 1, 2, \ldots, I - 1$

$$c_{ij} = \begin{cases} c, \text{ if } j < i \\ 0, \text{ if } j = i \end{cases}$$

The definition of c_{ij} recognizes that if the length iK of the stem equals the desired log length jK, then no cuts are made.

To understand the meaning of the notation $R_m(i, j)$, Figure 26.17 provides a representation of a tree with $I = 8$ and $L = 8K$. The crosscuts at points A and B result in one log for mill 1 and two for mill 2. The cutting starts from the larger end of the tree and produces log 1 for mill 2 by making a crosscut at point A. The cut corresponds to $(i = 8, j = 3)$ and produces the revenue $R_2(8, 3)$. The remaining stem now has a length $5K$. The next crosscut at point B produces log 2 for mill 1 with the length $2K$. This log corresponds to $(i = 5, j = 2)$ and generates the revenue $R_1(5, 2)$. The remaining stem of length $3K$ exactly equals the length of log 3 for mill 2. Hence no further cutting is needed. The associated revenue is $R_1(3, 3)$. The crosscutting cost associated with the solution is $c_{83} = c$, $c_{52} = c$, and $c_{33} = 0$.

The problem can be formulated and solved as a dynamic program (DP) model. Let

$f(i)$ = Maximum revenue when the length of the remaining stem is iK, $i = 1, 2, \ldots, I$

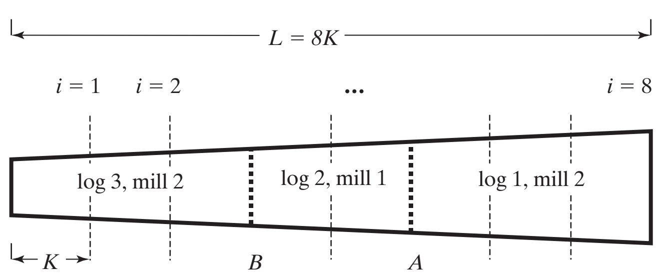
The DP recursive equation is then given as

$$f(0) \equiv 0$$
$$f(i) = \max_{\substack{j=1,2,\ldots,\min(i,N) \\ m=1,2,\ldots,M}} \{R_m(i, j) - c_{ij} + f(i - j)\}, i = 1, 2, \ldots, I$$

The idea is that given a stem of length iK, $f(i)$ is a function of the revenue of cutting a log of length $j(\leq i)$ minus the cost of making a crosscut plus the best cumulative revenue from the remaining stem of length $(i - j)K$.

FIGURE 26.17

Typical solution in a two-mill situation



Example computations:

The recursive equation is computed in the order $f(1), f(2), \ldots, f(I)$. The situation deals with two mills ($M = 2$), a tree of length $L = 12$ feet, and a minimum log length $K = 2$ feet, thus yielding $I = 6$. The cost of a crosscut is $c = \$.15$. Either mill will accept logs of length 2, 4, 6, 8, or 10 feet. This means that $N = 5$. Figure 26.18 provides the spreadsheet solution of the example (File excelCase8.xls). The basic DP calculations (rows 15–20) are partially automated, and will change automatically when $R_m(i, j)$ in rows 6–11 are altered. All boldface entries are entered manually. The spreadsheet is limited to problems with $I = 6, N = 5$, and $M = 2$, in essence allowing changes in the entries of $R_m(i, j)$ only.[12] The values of $R_m(i, j), j \leq i$, are given in rows 5 through 11 in the spreadsheet. Note that for a specific $j = j^*$, the value of $R_m(i, j^*)$ increases with i to reflect increases in end diameters of the log.

To illustrate the DP calculations in rows 15–20, note that each stage consists of one row because the state of the system at stage i consists of one value only—namely, the partial stem

FIGURE 26.18

Spreadsheet solution of the mill example problem

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	Input data:													
2	Note: All italicized boldface data are supplied manually													
3	m=	**2**	K=	**2**	N=	**5**	L=	**12**	I=	**6**	c=	**0.15**		
4	Rm(i,j):			mill m=1						mill m=2				
5	j-->>	1	2	3	4	5		1	2	3	4	5		
6	1	**1**					1	**1.1**						
7	2	**1.1**	**1.15**				2	**1.1**	**2.3**					
8	3	**1.4**	**1.6**	**2.8**			3	**1.33**	**2.4**	**3.4**				
9	4	**1.9**	**1.9**	**3.9**	**4.1**		4	**2.1**	**3.3**	**4.2**	**4.1**			
10	5	**2.1**	**2.9**	**4.4**	**4.8**	**7.2**	5	**2.2**	**3.6**	**4.3**	**4.6**	**6**		
11	6	**2.1**	**3.5**	**4.7**	**6.1**	**8.3**	6	**2.2**	**4.5**	**4.4**	**5**	**6.3**		
12														
13	Calculations:												f(i)	(j*,m*)
14	i	j=1	j=2	j=3	j=4	j=5		j=1	j=2	j=3	j=4	j=5	**$0.00**	
15	1	1						1.1					$ 1.10	**(1,2)**
16	2	2.05	1.15					2.05	2.3				$ 2.30	**(2,2)**
17	3	3.55	2.55	2.8				3.48	3.35	3.4			$ 3.55	**(1,1)**
18	4	5.3	4.05	4.85	4.1			5.5	5.45	5.2	4.1		$ 5.50	**(1,2)**
19	5	7.45	6.3	6.55	5.8	7.2		7.55	7	6.5	5.55	6	$ 7.55	**(1,2)**
20	6	9.5	8.85	8.1	8.3	9.3		9.6	9.85	7.8	7.15	7.25	$ 9.85	**(2,2)**
21													Value=	$9.85
22														
23		m2		m1	m2		m2							
24														
25			i=1	i=2	i=3	i=4	i=5	i=6						

[12]The spreadsheet formulas should provide sufficient information to extend the spreadsheet to other input data. Also, a general spreadsheet solution can be developed using (the more involved) VBA macros to specify the size of the matrices $R_m(i, j)$ and to automate all the calculations.

length. At stage $i = 1$, the (remaining) stem length is $1K$, hence resulting in one log only of length $1K$ (i.e., $j = 1$). Also, $c_{11} = 0$ because no cutting takes place. Thus,

$$\begin{aligned} f(1) &= \max\{R_1(1,1) - c_{11} + f(0), R_2(1,1) - c_{11} + f(0)\} \\ &= \max\{1 - 0 + 0, 1.1 - 0 + 0\} \\ &= 1.1 \end{aligned}$$

The associated optimum decision at $i = 1$ calls for a log of length $1K(j^* = 1)$ for mill $2(m^* = 2)$, or $(j^*, m^*) = (1, 2)$.

For stage $2(i = 2)$, logs can assume a length of $1K$ or $2K$ (i.e., $j = 1$ or 2) for both mills ($m = 1$ or 2). Thus,

$$\begin{aligned} f(2) = \max\ \{&R_1(2,1) - c_{21} + f(1), R_1(2,2) - c_{22} + f(0), R_2(2,1) - c_{21} + f(1), \\ &R_2(2,2) - c_{22} + f(0)\} \\ = \max\ \{&1.1 - .15 + 1.1, 1.15 - 0 + 0, 1.1 - .15 + 1.1, 2.3 - 0 + 0\} \\ = \max&\{2.05, 1.15, 2.05, 2.3\} = 2.3 \end{aligned}$$

The associated optimum decision is $(j^*, m^*) = (2, 2)$, which calls for cutting one log of length $2K$ for mill 2.

The remainder of the calculations are carried out in a similar manner as shown in Figure 26.18, rows 15–20. Note that entries B15:F20, H15:L20, and M15:M20 are automated in the spreadsheet. The entries (j^*, m^*) in N15:N20 are created manually after the automated computations in rows 15–20 are completed. Manually highlighted cells in rows 15–20 define $f(i), i = 1, 2, \ldots, 6$.

The optimum solution is read from cells N15:N20 as follows:

$$\begin{aligned} &(i = 6) \rightarrow (j^*, m^*) = (2, 2) \rightarrow (i = 4) \rightarrow (j^*, m^*) = (1, 2) \rightarrow \\ &(i = 3) \rightarrow (j^*, m^*) = (1, 1) \rightarrow (i = 2) \rightarrow (j^*, m^*) = (2, 2) \end{aligned}$$

The solution translates to making cuts at $i = 2$, 3, and 4 and produces a total value of \$9.85 for the tree.

Practical considerations:

The results of the DP optimization model are used by field operators in the day-to-day operation of the mill. In this regard, the model must be presented in the context of a user-friendly system in which the (intimidating) DP calculations are transparent to the user. This is precisely what Lemberskey and Chi [1] did when they developed the VISION (Video Interactive Stem Inspection and OptimizatioN) computer system. The system is equipped with a database of large representative samples of tree stems from the regions where trees are harvested. The data include the geometry of the stem as well as its quality (e.g., location of knots) and the value (in dollars) for stems with different lengths and diameters. In addition, quality characteristics for the different mills are provided.

A typical user session with VISION includes the following steps:

Step 1. The operator may select a sample stem from the data base or create one using the graphic capabilities of VISION. This will result in a realistic representation of the stem on the computer screen. The mills requesting the logs are also selected from the data base.

Step 2. After inspecting the stem on the screen, the operator can "cut" the stem into logs based on experience. Next, an optimum DP solution is requested. In both cases, graphic displays of the created logs together with their associated values are projected on the screen. The user is then given the chance to compare the two solutions. In particular, the DP solution is examined to make sure that the created logs meet quality specifications. If not, the user may elect to modify the cuts. In each case, the associated value of the stem is displayed for comparison.

In VISION, DP optimization is totally transparent to the user. In addition, the interactive graphic nature of the output makes the system ideal for training operators and improving their decision making skills. The design of the system shows how complex mathematical models can be imbedded within a user-friendly computer system.

Question:

1. Suppose that a third mill that accepts logs of lengths 4, 6, and 8 feet is added to the example given in the case. The new mill has the following $R(i, j)$: $R(2, 2) = \$2.50$, $R(3, 2) = \$2.70$, $R(3, 3) = \$3.10$, $R(4, 2) = \$2.90$, $R(4, 3) = \$3.30$, $R(4, 4) = \$4.20$, $R(5, 2) = \$3.10$, $R(5, 3) = \$3.90$, $R(5, 4) = \$4.60$, $R(6, 2) = \$3.50$, $R(6, 3) = \$4.20$, $R(6, 4) = \$5.10$. Modify the spreadsheet excelCase8.xls to find the optimum solution for trees of length $10'$ and $12'$.

Case 9: Layout Planning for a Computer Integrated Manufacturing (CIM) Facility[13]

Tools: AHP, GP

Area of application: Facility layout planning

Description of the situation:

In an academic institution a vacated 5100-ft^2 building has been made available to the engineering college to establish a CIM laboratory that will serve as a teaching and research facility as well as a center of technical excellence for industry. Recommendations are solicited from the faculty regarding a layout plan for the new laboratory. Table 26.11 lists the individual units comprising the laboratory together with their ideal and absolute minimum areas. Aisles take at least 15% of the total (actual) area. Units 1 through 5 are principally classrooms that must accommodate between 10 and 30 students each.

The ideal areas in Table 26.11 exceed the total area of the vacated building by about 1300 ft^2, which calls for reducing the areas to be allocated to some of the units. Allocation should take into account the fact that the new CIM lab has three missions: teaching, research, and service to industry. Teaching deals with offering instructions in classroom units 1 through 5. Research deals with both consulting contracts and graduate projects, both carried out in the lab. The consensus is that the facility will have at least two contracts and six graduate projects at all times. Service to industry comes in the form of continuing education seminars to be conducted in the reception and presentation area (unit 11).

[13] *Source:* C. Benjamin, I. Ehie, and Y. Omurtag, "Planning Facilities at the University of Missouri-Rolla," *Interfaces*, Vol. 22, No. 4, pp. 95–105, 1992.

TABLE 26.11 Ideal and Absolute Minimum Unit Areas

Unit	Description	Ideal area in ft^2	Absolute Minimum area in ft^2
1	Packaging	1450	—
2	Production machines	148	—
3	Autocad/PC area	120	—
4	Physical simulation lab	530	—
5	Intelligent control workstations	130	—
6	Raw material conversion	165	—
7	Assembly	230	172
8	Research projects	400	—
9	Technician work area	160	120
10	Material storage	360	126
11	Reception and presentation area	900	—
12	Robot system	140	60
13	Automatic storage retrieval system	500	200
14	Conveyor systems	200	150
15	Aisle space	1000	—
	Total	6433	

In support of the three missions of the lab, teaching (T), research (R), and service (S), the committee in charge of allocating space has established four principal goals for the facility:

Goal G1: Academic use of the five classrooms is 270 student lab-hours per day, given that the daily use of labs 1 through 5 is 5, 5, 4, 4, and 4 hours, respectively . The enrollment in each lab is between 10 and 30 students.

Goal G2: Space allocation to the physical simulation lab (unit 4) is as close as possible to the ideal area requested ($= 530 \text{ ft}^2$).

Goal G3: The facility supports a total of 15 simultaneous research contracts and graduate projects with at least 2 research contracts and 6 research projects being carried out simultaneously at any one time.

Goal G4: The continuing education facility (unit 11) has a seating capacity of 25 persons with an average of 20 ft^2 per participant.

Prioritizing the goals:

The situation involves multiple goals, and the first step toward the solution of the problem is to prioritize the goals. AHP (Section 13.1) is an appropriate tool for this purpose. Two sets of comparison matrices are compiled by each of the five members of the planning committee: The first set ranks the main areas of teaching, research, and service, and the second set ranks goals $G1$ through $G4$. Tables 26.12 and 26.13 summarize the rankings provided by the five committee members.

The individual comparison matrices for the five members are reduced to a single comparison matrix using the geometric mean because it guarantees that the elements a_{ij} and a_{ji} of the combined matrix will satisfy the required reciprocal relationship $a_{ij} = \frac{1}{a_{ji}}$, for all i and j. For example, from Table 26.12, we have

$$a_{12} = \sqrt[5]{2 \times 3 \times 1 \times 4 \times 3} = 2.352$$

$$a_{21} = \sqrt[5]{\tfrac{1}{2} \times \tfrac{1}{3} \times \tfrac{1}{1} \times \tfrac{1}{4} \times \tfrac{1}{3}} = .425$$

TABLE 26.12 Comparison Matrices for Teaching (T), Research (R), and Service (S)

	Member 1			Member 2			Member 3			Member 4			Member 5		
	T	*R*	*S*	*T*	*R*	*S*	*T*	*R*	*S*	*T*	*R*	*S*	*T*	*R*	*S*
T	1	2	5	1	3	6	1	1	5	1	4	8	1	3	8
R	1/2	1	4	1/3	1	4	1	1	5	1/4	1	5	1/3	1	6
S	1/5	1/4	1	1/6	1/4	1	1/5	1/5	1	1/8	1/5	1	1/8	1/6	1

TABLE 26.13 Comparison Matrices for Goals $G1$, $G2$, $G3$, and $G4$ as a Function of Teaching, Research, and Service

Teaching (T)

	Member 1				Member 2				Member 3				Member 4				Member 5			
	G1	*G2*	*G3*	*G4*	*G1*	*G2*	*G3*	*G4*	*G1*	*G2*	*G3*	*G4*	*G1*	*G2*	*G3*	*G4*	*G1*	*G2*	*G3*	*G4*
*G*1	1	1	2	6	1	2	2	5	1	3	5	7	1	1	1	3	1	2	2	2
*G*2	1	1	3	8	1/2	1	3	6	1/3	1	2	4	1	1	2	4	1/2	1	1	1
*G*3	1/2	1/3	1	6	1/2	1/3	1	4	1/5	1/2	1	5	1	1/2	1	6	1/2	1	1	1
*G*4	1/6	1/8	1/6	1	1/5	1/6	1/4	1	1/7	1/4	1/5	1	1/3	1/4	1/6	1	1/2	1	1	1

Research (R)

	Member 1				Member 2				Member 3				Member 4				Member 5			
	G1	*G2*	*G3*	*G4*	*G1*	*G2*	*G3*	*G4*	*G1*	*G2*	*G3*	*G4*	*G1*	*G2*	*G3*	*G4*	*G1*	*G2*	*G3*	*G4*
G1	1	3	2	4	1	1	2	1	1	2	1/2	3	1	1	1/4	2	1	1	1/3	3
G2	1/3	1	3	5	1	1	1	1	1/2	1	1/2	2	1	1	1/4	2	1	1	1/2	2
G3	1/2	1/3	1	2	1/2	1	1	1	2	2	1	4	4	4	1	3	3	2	1	5
G4	1/4	1/5	1/2	1	1	1	1	1	1/3	1/2	1/4	1	1/2	1/2	1/3	1	1/3	1/2	1/5	1

Service (S)

	Member 1				Member 2				Member 3				Member 4				Member 5			
	G1	*G2*	*G3*	*G4*	*G1*	*G2*	*G3*	*G4*	*G1*	*G2*	*G3*	*G4*	*G1*	*G2*	*G3*	*G4*	*G1*	*G2*	*G3*	*G4*
G1	1	2	2	2	1	1	1	1	1	1	2	2	1	1	1	3	1	1	1	2
G2	1/2	1	3	3	1	1	1	1	1	1	3	2	1	1	1	3	1	1	1	1/2
G3	1/2	1/3	1	3	1	1	1	1	1/2	1/3	1	2	1	1	1	4	1	1	1	2
G4	1/2	1/3	1/3	1	1	1	1	1	1/2	1/2	1/2	1	1/3	1/3	1/4	1	1/2	2	1/2	1

The resulting comparison matrices are listed subsequently (see file excelCase9a.xls):

$$\mathbf{A} = \begin{array}{c} \\ T \\ R \\ S \end{array}\begin{array}{c} \begin{array}{ccc} T & R & S \end{array} \\ \begin{bmatrix} 1.000 & 2.352 & 6.258 \\ 0.425 & 1.000 & 4.743 \\ 0.160 & 0.211 & 1.000 \end{bmatrix} \end{array}$$

$$\mathbf{A}_T = \begin{array}{c} \\ G1 \\ G2 \\ G3 \\ G4 \end{array}\begin{array}{c} \begin{array}{cccc} G1 & G2 & G3 & G4 \end{array} \\ \begin{bmatrix} 1 & 1.644 & 2.091 & 4.169 \\ 0.608 & 1 & 2.048 & 3.776 \\ 0.478 & 0.488 & 1 & 3.728 \\ 0.240 & 0.265 & 0.268 & 1 \end{bmatrix} \end{array}$$

$$\mathbf{A}_R = \begin{array}{c} \\ G1 \\ G2 \\ G3 \\ G4 \end{array}\begin{array}{c} \begin{array}{cccc} G1 & G2 & G3 & G4 \end{array} \\ \begin{bmatrix} 1 & 1.431 & 0.699 & 2.352 \\ 0.699 & 1 & 0.715 & 2.091 \\ 1.431 & 1.398 & 1 & 2.605 \\ 0.425 & .478 & 0.384 & 1 \end{bmatrix} \end{array}$$

$$\mathbf{A}_S = \begin{array}{c} \\ G1 \\ G2 \\ G3 \\ G4 \end{array}\begin{array}{c} \begin{array}{cccc} G1 & G2 & G3 & G4 \end{array} \\ \begin{bmatrix} 1 & 1.149 & 1.320 & 1.888 \\ 0.871 & 1 & 1.933 & 1.431 \\ 0.758 & 0.517 & 1 & 2.169 \\ 0.530 & 0.700 & 0.461 & 1 \end{bmatrix} \end{array}$$

Applying AHP calculations to matrices $\mathbf{A}$, $\mathbf{A}_T$, $\mathbf{A}_R$, and $\mathbf{A}_S$, the ranking of goals $G1, G2, G3$, and $G4$, given in excelCase9b.xls, is as follows:

Goal	Weight
$G1$	0.362
$G2$	0.285
$G3$	0.255
$G4$	0.098

This ranking shows that $G1$ (teaching) receives the highest priority and $G4$ (service) the lowest. This means that in solving the associated goal programs, the order of priority is $G1 > G2 > G3 > G4$.

The AHP calculations in spreadsheet excelCase9b.xls reveal that the consistency ratios (CR) for all the matrices, $\mathbf{A}$, $\mathbf{A}_T$, $\mathbf{A}_R$, and $\mathbf{A}_S$, are within the acceptable range $(<.1)$.

Development of goal programs:

Let

a_i = Ideal area allocated to unit i in ft^2, $i = 1, 2, \ldots, 15$

x_i = Ratio of actual to ideal area allocated to unit i, $0 \leq x_i \leq 1, i = 1, 2, \ldots, 15$

The actual allocation of space to the different units of the CIM facility involves goals $G1, G2, G3$, and $G4$ taken in the order of priority established by the AHP calculations—namely, $G1 > G2 > G3 > G4$. In addition to the flexible goal constraints, a number of strict constraints apply to each goal.

Goal G1 (Target for daily student lab-hours = 270):

Let

$$
\begin{aligned}
y_j &= \text{Student enrollment in section } j, 10 \le y_j \le 30, j = 1, 2, \ldots, 5 \\
h_j &= \text{Daily hours lab } j \text{ is used, } j = 1, 2, \ldots, 5 \ (= 5, 5, 4, 4, 4 \text{ hours, respectively}) \\
c_1 &= \text{Area needed per student in lab units } 1, 2, \ldots, \text{ and } 5 \ (= 10 \text{ ft}^2) \\
H &= \text{Target number of daily hours for all labs } (= 270) \\
p_1 &= \text{Underachievement amount of goal } G1 \\
q_1 &= \text{Overachievement amount of goal } G1
\end{aligned}
$$

Thus the $G1$-problem can be expressed as

$$\text{Minimize } G1 = p_1$$

subject to

$$
\left.
\begin{aligned}
y_j &\le \tfrac{a_j x_j}{c_1} \\
0 \le x_j &\le 1 \\
10 \le y_j &\le 30
\end{aligned}
\right\}, j = 1, 2, \ldots, 5
$$

$$\sum_{j=1}^{5} h_j y_j + p_1 - q_1 = H$$

$$p_1, q_1 \ge 0, \text{ all } y_j \text{ integers}$$

Goal G2 (Ideal space allocation for Physical Simulation Lab, $x_4 = 1$):

The $G2$-problem is given as

$$\text{Minimize G2} = p_2$$

subject to

$$
\begin{aligned}
&x_4 + p_2 - q_2 = 1 \\
&0 \le x_4 \le 1 \\
&p_2, q_2 \ge 0
\end{aligned}
$$

Goal G3 (Simultaneous number of research contracts and projects ≥ 15):

Let

$$
\begin{aligned}
w_1 &= \text{Number of research contracts } (\ge 2) \\
w_2 &= \text{Number of research projects } (\ge 6) \\
b_1 &= \text{Space area required per research contract } (= 36 \text{ ft}^2) \\
b_2 &= \text{Space area required per research project } (= 48 \text{ ft}^2)
\end{aligned}
$$

The $G3$-problem is

$$\text{Minimize } G3 = p_3$$

subject to

$$\begin{aligned}
&w_1 + w_2 + p_3 - q_3 = 15\\
&b_1w_1 + b_2w_2 \leq a_8x_8\\
&w_1 \geq 2\\
&w_2 \geq 6\\
&0 \leq x_8 \leq 1\\
&p_3, q_3 \geq 0
\end{aligned}$$

Goal G4 (*Continuing education classroom capacity* $\geq$25 *persons*):
Let

z = Number of continuing education participants per seminar
d = Average classroom area required per participant (= 20 ft^2)

The $G4$-problem is

$$\text{Minimize } G4 = p_4$$

subject to

$$\begin{aligned}
&z + p_4 - q_4 = 25\\
&dz \leq a_{11}x_{11}\\
&0 \leq x_{11} \leq 1\\
&z \geq 0 \text{ and integer}\\
&p_4, q_4 \geq 0
\end{aligned}$$

Mandatory constraints, applicable to all goals:

1. Available space:

$$\sum_{i=1}^{15} a_ix_i = A$$

2. Minimum required areas:

$$x_7 \geq \frac{172}{230}, x_9 \geq \frac{120}{160}, x_{10} \geq \frac{126}{360}, x_{12} \geq \frac{60}{140}, x_{13} \geq \frac{200}{500}, x_{14} \geq \frac{150}{200}$$

3. Aisles requirement:

$$a_{15}x_{15} \geq .15A$$

Model solution:

The problem is solved by combining all the goals into one objective function using the AHP priority weights .337, .288, .240, and .134 for the respective goals $G1$, $G2$, $G3$, and $G4$; that is,

$$\text{Minimize } G = .362p_1 + .285p_2 + .255p_3 + .098p_4$$

The constraints of the combined problem include all the constraints of goals $G1$ through $G4$ together with the common mandatory constraints.

```
Projects:
Contract  3
Research  6

Space allocation:
Unit   Factor       Actual sq ft
 1     0.094          135
 2     1.000          148
 3     0.833           99
 4     1.000          530
 5     1.000          130
 6     1.000          165
 7     1.000          230
 8     1.000          400
 9     1.000          160
10     1.000          360
11     1.000          900
12     1.000          141
13     1.000          500
14     1.000          200
15     1.000         1000
          Total=     5099

Student enrollments in labs:
Lab 1    11
Lab 2    14
Lab 3    10
Lab 4    17
Lab 5    10

Deviational variables:
     p      q
G1   0      3
G2   0      0
G3   6      0
G4   0      0
```

FIGURE 26.19

AMPL output of space allocation model

File amplCase9.txt provides the AMPL model for the problem. The solution is given in Figure 26.19. The results show that goal $G1$ is overachieved by 3 student-hours per day. Goal $G2$ is achieved exactly. Goal $G3$ is underachieved by 6 projects and goal $G4$ is achieved exactly.

Sensitivity analysis:

The preference matrices providing the original data for the AHP calculations are a subjective assessment by the individuals involved in the ranking process. We can use sensitivity analysis to study the effect of data bias on the solution. Interestingly, experimentation with the AMPL model reveals that the solution remains unchanged regardless of the weights assigned to the goals. Indeed, AMPL sensitivity analysis applied to goal weights provides the following results:

	p.down	p.current	p.up
$G1$	0	0.362	1e + 20
$G2$	0	0.285	1e + 20
$G3$	0	0.255	1e + 20
$G4$	0	0.098	1e + 20

These results show that the optimum solution remains unchanged for any weights between 0 and infinity. This observation suggests a peculiar feasible solution space, perhaps consisting of a single point only, and for this reason the optimum solution is independent of the objective function.

The main lesson from the preceding results is that all the AHP calculations carried out earlier could have been avoided had we formulated the goal programming problem with our best "guesstimates" of the weights for the goals and then studied the impact of changing these weights on the solution. Of course, the manner in which the model has been developed is logical and our observation is made only in hindsight. Nevertheless, this experience shows that one should not resort to the use of sophisticated analytical tools (such as AHP) before investigating the viability of using simpler procedures without impairing the accuracy of the final recommendation.

Question:

1. Suppose that the actual building area is 5600 ft^2. Suppose further that the unit 11 (reception and representation area) ideal request is 1000 ft^2. Find the optimum solution.

Case 10: Booking Limits in Hotel Reservations[14]

Tool: Decision tree analysis

Area of application: Hotels

Description of the situation:

Hotel La Posada has a total of 300 guest rooms. Its clientele includes both business and leisure travelers. Rooms can be sold in advance (usually to leisure travelers) at a discount price. Business travelers, who invariably are late in booking their rooms, pay full price. La Posada must thus establish a *booking limit* on the number of discount rooms sold to leisure travelers to take advantage of the full-price business customers.

Mathematical model:

Let N be the number of available rooms and suppose that the current protection level of rooms sold at full price is $Q + 1, 0 \leq Q < N$. The associated booking limit (rooms sold at a discount) is $N - Q - 1$. Figure 26.20 summarizes the situation.

To determine if the protection level should be lowered from $Q + 1$ to Q, we use the decision tree in Figure 26.21. Let D be the random variable representing historical or forecast demand for full-price (business) rooms. Further, let c be the full price and d be the discount price $(d < c)$. A decision to lower the protection level from $Q + 1$ to Q signifies that room $Q + 1$ will be sold at the discount price d because there will be ample opportunity to do so. Alternatively, not lowering the protection level will result in two probabilistic outcomes: If the demand for business rooms is greater than or equal to $Q + 1$, then room $Q + 1$ will sell at full price, c; else the room will not sell at all. The associated probabilities are $P\{D \geq Q + 1\}$ and $P\{D \leq Q\}$, respectively. It thus follows that the decision to lower the protection level to Q should be adopted if

$$d \geq cP\{D \geq Q + 1\} + 0P\{D \leq Q\}$$

[14]*Source:* S. Netessine and R. Shumsky, "Introduction to the Theory and Practice of Yield Management," *INFORMS Transactions on Education*, Vol. 3, No. 1, pp. 20–28, 2002.

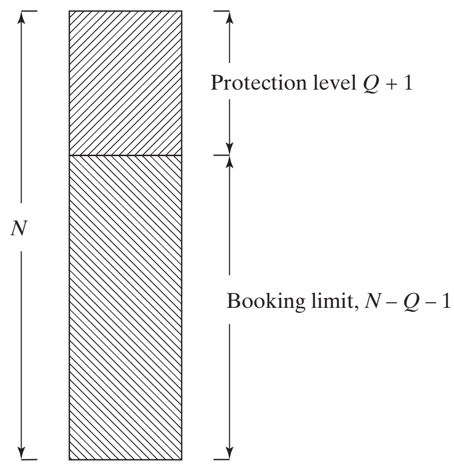


FIGURE 26.20
Booking limit and protection level

or

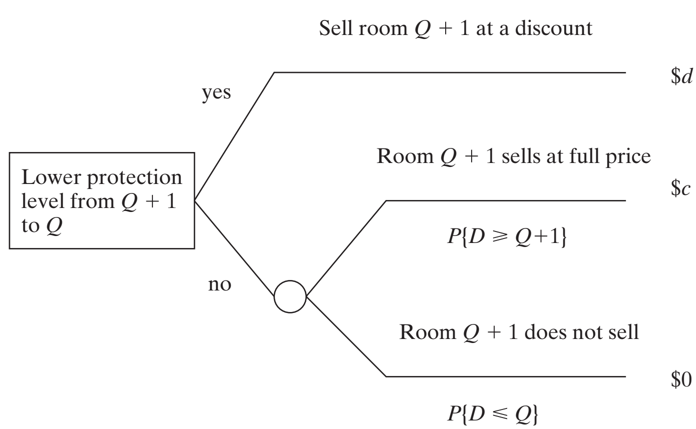
$$P\{D \leq Q\} \geq \frac{c - d}{c}$$

Given the distribution of demand D, together with the unit costs c and d, the protection level Q can be determined readily.

Collection of data:

The most crucial piece of information needed to determine the protection level is the distribution of demand for full price rooms. We can use historical data over a specified time period for this purpose. The number of days a block of rooms Q is reserved at full fare then estimates the

FIGURE 26.21
Decision tree for determining protection level Q



demand probability $P\{D = Q\}$ from which the cumulative probability can be determined. Table 26.14 provides the data for determining the distribution of demand. The first two columns include the raw data.

The use of the information in Table 26.14 can be illustrated by the following situation. Suppose that the full fare is \$159 and the discount fare is \$105. The protection limit is determined such that

$$P\{D \leq Q\} \geq \frac{159 - 105}{159} = .33962$$

The cumulative probability column in Table 26.14 shows the protection level to be Q =79 rooms.

Conclusion:

The ideas presented in this study can be extended similarly to setting booking limits for airline tickets. Additionally, in place of using one booking limit, the analysis can be modified to allow setting several levels of booking limits with the discount price increasing with the nearness of the reservation date. The most important information for the model is a reliable estimate of demand data.

Questions:

1. Suppose that there is a 50-50 chance that an unsold business room will be taken by a leisure traveler at premium price. How does this affect the determination of the booking limit?
2. Specify the basic assumptions on which the devised model are based. How realistic are these assumptions?

TABLE 26.14 Calculation of $P\{D = x\}$ and $P\{D \leq x\}$

Number of rooms, Q	Number of days in demand	$P\{D = Q\}$	$P\{D \leq Q\}$
0–70	12	0.09756	0.097561
71	3	0.02439	0.12195
72	3	0.02439	0.14634
73	2	0.01626	0.16260
74	0	0.00000	0.16260
75	4	0.03252	0.19512
76	4	0.03252	0.22764
77	5	0.04065	0.26829
78	2	0.01626	0.28455
79	7	0.05691	0.34146
80	4	0.03252	0.37398
81	10	0.08130	0.45528
82	13	0.10569	0.56098
83	12	0.09756	0.65854
84	4	0.03252	0.69106
85	9	0.07317	0.76423
86	10	0.08130	0.84553
>86	19	0.15447	1.00000

3. Suppose that two protection levels are used for business travelers: Q_1 rooms are priced at $\$c_1$ per room and Q_2 rooms are priced at $\$c_2$ per room, $c_1 > c_2$. The remaining rooms are discounted at $\$d_1$ per room. Develop a model for determining Q_1 and Q_2. State all assumptions.

Case 11: Casey's Problem: Interpreting and Evaluating a New Test[15]

Tool: Decision tree (Bayes' probabilities)

Area of application: Medical tests

Description of the situation:

A screening test of a newborn baby, named Casey, indicated a C14:1 enzyme deficiency. The enzyme is required to digest a particular form of long-chain fats, and its absence could lead to severe illness or mysterious death (broadly categorized under sudden infant death syndrome or SIDS). The test was administered previously to approximately 13,000 newborns and Casey was the first to test positive. There is no known treatment for this deficiency, but her doctors indicated that the condition could be controlled through dietary restrictions. Though the screening test does not in itself constitute a definitive diagnosis, the extreme rarity of the condition led her doctors to conclude that that there was an 80–90% chance that the baby was suffering from this deficiency.

Because the high probability assessed by the doctors (.8 to .9) is based only on the results of the initial screening test, there is a need for a systematic analysis for interpreting and evaluating the doctors' quantitative assessment of the situation. In particular, given that Casey tested positive, what is the probability that she does have the C14:1 deficiency?

The absence of hard data for the situation complicates the decision-making process. The only solid piece of information is the fact that Casey's positive result was the first in about 13,000 tests. Medical records also show that in all the cases involving sudden infant death syndrome, autopsy showed that none could be attributed to the C14:1 deficiency. In this regard, the deficiency test has no history of (true or false) positive results. Any decision-making in this situation must thus rely on "reasonable" estimates of needed data.

Analysis:

The doctors estimated that, overall, the C14:1 deficiency occurs in about one in 40,000 newborns. However, Casey's case was so rare that its occurrence can probably be lowered to a *prevalence* rate of one in 250,000. Other data that can be estimated relate to false results in which the newborn *incorrectly* tests positive or negative. Starting with false-positive result, other screening tests with known history—such as HIV, drug, or amino fetal protein tests—may be used as guides. These tests have false-positive rates ranging from 1-in-100 to 1-in-1000. However, for Casey's test, this range appears too high, because historical data show only 1 in 13,000. Hence it may be reasonable to assume that this test may have a 1-in-20,000 false-positive rate. For the false-negative rate when the newborn actually has the deficiency, it is estimated that it could not be high based on the strict standards under which the test is administered. On that basis, a rate of 1 in 1000 was used as a best "guesstimate" for this case.

[15] *Source:* J.E. Smith and R.L. Winkler, "Casey's Problem: Interpreting and Evaluating a New Test," *Interfaces*, Vol. 29, No. 3, pp. 63–76, 1999.

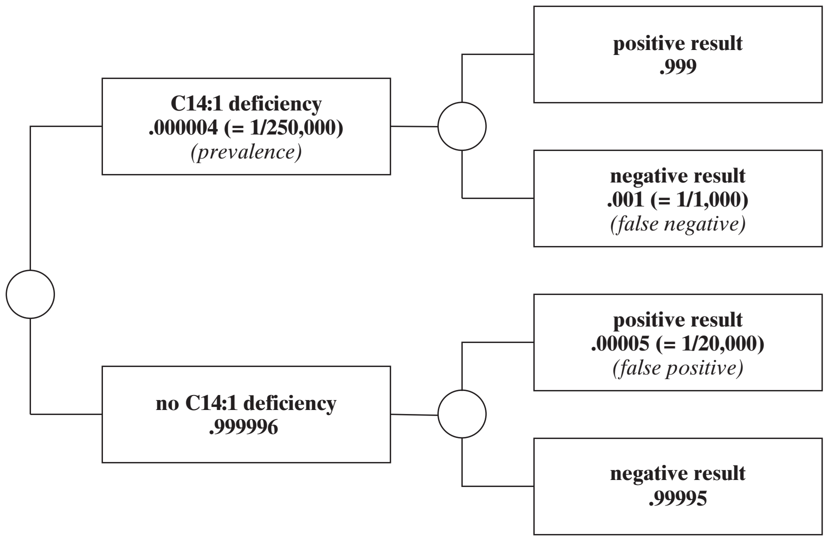


FIGURE 26.22
Decision tree summary of Casey's test probabilities

The decision situation can be analyzed using Bayes' theorem. The objective is to calculate the posterior probability of a C14:1 deficiency given that Casey tested positive. Figure 26.22 summarizes the probabilities of the situation using the estimates given above.

The posterior probability of having C14:1 deficiency given that Casey tested positive is computed using the following Bayes' formula:

$$P\{\text{C14:1}|+ve\text{ test}\} = \frac{P\{+ve|\text{C14:1}\}P\{\text{C14:1}\}}{P\{+ve|\text{C14:1}\}P\{\text{C14:1}\} + P\{+ve|\text{no C14:1}\}P\{\text{no C14:1}\}}$$

$$= \frac{.999 \times .000004}{.999 \times .000004 + .000005 \times .999996} = .074$$

The result shows that the probability that Casey has C14:1 deficiency given the positive test is only .074. Though this probability is considerably less than the .8 to .9 suggested by the doctors, it is a cause for concern because it implies a 1-in-13.5 rate. But how could the doctors have been so wrong? Could it be that the *prevalence, false-negative, and false-positive* probability estimates in Figure 26.23 are highly inaccurate? This is plausible because none of the probabilities are based on reliable data.

One way of testing the reliability of the given result is to carry out sensitivity analysis calculation to study the effect of changes in the prevalence, false-positive, and false-negative rates on the desired posterior probability. Table 26.15 summarizes the proposed ranges suggested for carrying out sensitivity analysis.

Posterior probabilities are computed by varying the range of one item while keeping the remaining two at base rate levels. Using Bayes' formula (file excelBayes.xls), Table 26.16 provides the desired results.

The sensitivity factor in the last column of Table 26.16 is a measure of how sensitive the posterior probability is to changes in the estimates of the prevalence, false-negative, and false-positive rates. It is the ratio of the largest posterior probability to the posterior probability associated

TABLE 26.15 Sensitivity Analysis Ranges

	Rate		
	Maximum	*Base*	*Minimum*
Prevalence	1 in 100,000	1 in 250,0000	1 in 1,000,000
False negative	1 in 100	1 in 1000	1 in 1,000,000
False positive	1 in 5000	1 in 20,000	1 in 1,000,000

TABLE 26.16 Posterior Probabilities Corresponding to Sensitivity Analysis Ranges

	Posterior probability for			
	Maximum rate	*Base rate*	*Minimum rate*	Sensitivity factor
Prevalence	.1665	.0740	.0196	2.25
False negative	.0734	.0740	.0740	1.00
False positive	.0196	.0740	.8000	10.81

with base rates. For example, for the false-positive case, the ratio is $.8/.0740 = 10.81$. The main interest is in the value of the rate that causes an increase in the posterior probability—namely, the minimum rate in the cases of false positive and false negative, and the maximum rate in the case of prevalence.

The sensitivity factor shows that the false-positive rate has the most impact on the posterior probability. The remaining two rates are not as critical. What is alarming, though, is that with a false-positive rate of 1 in 1,000,000, the posterior probability of a C14:1 deficiency reaches the high value suggested by the doctors (= .8 to .9). This means that further analysis is needed regarding the false-positive rate.

In the foregoing analysis, the base false positive rate of 1 in 20,000 is estimated from the knowledge that earlier 13,000 tests produced no false positive. Another way to start is to assume that we had no prior knowledge of the results of the 13,000 screening tests and to make an a priori "guesstimate" of what the rate may be. We can start by hypothesizing that in a sample of 1000 tests there is one false positive, which is equivalent to saying that the *expected* false-positive rate is .001. Now, given the information that 13,000 actual tests have produced no false positives, we can update the corresponding false positive rate to 1 in 14,000 ($= 1000 + 13{,}000$). The associated posterior probability of C14:1 deficiency is thus computed as

$$P\{\text{C14:1 deficiency} \mid +ve \text{ test}\} = \frac{.999 \times .000004}{.999 \times .000004 + (1/14000) \times .999996}$$
$$= .05298 \approx .053$$

If we decrease the sample size from 1,000 to 100 while keeping constant the *expected* false positive rate at .001 (1 in 1,000), the associated false positive rate becomes .1 in 13,100 and the associated posterior probability is

$$P\{\text{C14:1 deficiency} \mid +ve \text{ test}\} = \frac{.999 \times .000004}{.999 \times .000004 + (.1/13100) \times .999996}$$
$$= .3436$$

TABLE 26.17 Comparison of Posterior Probabilities for Three False Positive Rates

	Posterior probability given sample size n		
Expected false positive rate	$n = 100$	$n = 1000$	$n = 10{,}000$
.01	.0497	.00556	.00092
.001	.3436	.05298	.00911
.000001	.9981	.98240	.47890

Table 26.17 compares the values of the posterior probability for different expected false positive rates.

The computations yield an interesting result. For the posterior probability to reach the range of .8 to .9 suggested by the doctors, a combination of small sample size ($n \leq 1000$) and a very low expected false-positive rate ($= .000001$) must occur, a highly unlikely combination. The conclusion here is that the doctors' conclusion that Casey has C14:1 deficiency is overstated.

The given analysis does not conclude that that the doctors are wrong, nor does it show that the C14:1 deficiency does not exist. It simply points to the fact that initial test results cannot be used to make a strong statement about the condition. The only way the issue can be settled is by carrying out follow-up tests, as was actually done in Casey's situation.

Question:

1. Verify the values of the posterior probabilities in the Table 26.17 using excelBayes.xls.

Case 12: Ordering Golfers on the Final Day of Ryder Cup Matches[16]

Tool: Game theory

Area of application: Sports

Description of the situation:

In the final day of a golf tournament, two teams compete for the championship. Each team captain must submit an ordered list of golfers (a *slate*) that automatically determines the matches. Thus, if the slates for teams A and B are (A1, A2, ..., An) and (B1, B2, ..., Bn), then the matches call for A1 to play against B1, A2 against B2, ..., and An against Bn. It is plausible to assume that if two competing players occupy the same order in their respective slates, then there is 50-50 chance that either golfer will win the match. This probability will increase when a golfer of a higher order is matched with a lower order golfer. As an illustration, Table 26.18 provides a set of probabilities for a golfer on team A to win.

The goal of the study is to develop a systematic procedure that will support or refute the idea of using slates. It is assumed that no match can end in a draw and that all players are pressure resistant, in the sense that a golfer's performance remains unchanged throughout the entire competition.

[16]*Source:* W. Hurley, "How Should Team Captains Order Golfers on the Final Day of the Ryder Cup Matches?" *Interfaces,* Vol. 32, No. 2, pp. 74–77, 2002.

TABLE 26.18 Probability of Winning for Team A

	B1	B2	B3
A1	.50	.60	.70
A2	.40	.50	.60
A3	.30	.40	.50

TABLE 26.19 Slates for Teams A and B

	Slates	
Strategy	*Team A*	*Team B*
1	A1, A2, A3	B1, B2, B3
2	A1, A3, A2	B1, B3, B2
3	A2, A1, A3	B2, B1, B3
4	A2, A3, A1	B2, B3, B1
5	A3, A1, A2	B3, B1, B2
6	A3, A2, A1	B3, B2, B1

Analysis:

The situation can be analyzed using a two-person constant-sum game. A slate represents a possible strategy for each player. The payoff for a team is represented by the probability of winning. For a 3-player competition, each team has 6 (= 3!) possible strategies (slates), as shown in Table 26.19.

To illustrate the determination of the payoff matrix (probabilities) for team A, consider strategy 1 for team A and strategy 5 for team B. These strategies match A1 against B3, A2 against B1, and A3 against B2. Team A wins if *at least* two of its players win. The associated probability of winning is computed as:

$$.7 \times .4 \times .4 + (1 - .7) \times .4 \times .4 + .7 \times (1 - .4) \times .4 + .7 \times .4 \times (1 - .4) = .496$$

The complete payoff matrix (probabilities) is computed using excelCase12.xls as given in Table 26.20.

The solution of this game (using toraCase12.txt) calls for each team to mix selected strategies (the problem also has alternative solutions). The value of the game is .5, which favors Team A.

Questions:

1. How should Teams A and B randomize their strategies?
2. Suppose that the probabilities that Team A wins are given as

$$\begin{pmatrix} .4 & .2 & .7 \\ .7 & .5 & .8 \\ .8 & .9 & .6 \end{pmatrix}$$

Determine the optimal strategies for the two players.

TABLE 26.20 Payoff Matrix for Team A

	1	2	3	4	5	6
1	.500	.500	.500	.504	.496	.500
2	.500	.500	.504	.500	.500	.496
3	.500	.496	.500	.500	.500	.504
4	.496	.500	.500	.500	.504	.500
5	.504	.500	.500	.496	.500	.500
6	.500	.504	.496	.500	.500	.500

Case 13: Inventory Decisions in Dell's Supply Chain[17]

Tool: Inventory models

Area of application: Inventory control

Description of the situation:

Dell, Inc., implements a direct-sales business model in which personal computers are sold directly to customers. When an order arrives from a United States customer, the specifications are sent to one of its manufacturing plants in Austin, Texas, where the order is built, tested, and packaged in about eight hours. Dell carries little inventory of its own because its suppliers, normally located in Southeast Asia, are required to keep what is known as "revolving" inventory on hand in *revolvers* (warehouses) in the proximity of the manufacturing plants. These revolvers are owned by Dell and leased to the suppliers. Dell then "pulls" parts as needed from the revolvers, and it is the responsibility of the suppliers to replenish the inventory at the rate that meets Dell's estimated demand. Parts shortage is usually filled through costly overnight shipping. Although Dell does not own the inventory in the revolvers, inventory cost is passed indirectly to Dell through component pricing. Any reduction in inventory benefits Dell's customers by reducing product prices. The objective of the project is to recommend target revolver inventories that minimize inventory costs.

Analysis of the problem:

The analysis starts with the selection of the mathematical model followed by the collection of data needed to drive the model. The last part of the study deals with the analysis of the results for the purpose of making final inventory decisions.

The investigation concentrates on XDX , a principal PC component. The component comes from a number of suppliers and is delivered to the revolvers in batches to offset the fixed cost of

[17] *Source:* I. Millet, D. Parente, J. Fizel, and R. Venkataraman, "Inventory Decisions in Dell's Supply Chain," *Interfaces*, Vol. 34, No. 3, pp. 191–205, 2004.

ordering. The inventory policy followed by each supplier is to deliver a new batch whenever the inventory level drops to a certain level. However, it has been noticed that both the batch size and the reorder level are considerably inconsistent among the suppliers. The first task then was to bring consistency to the suppliers' ordering policies. The model selected to represent the situation is the (Q, R) policy, where Q is the batch size and R is the reorder level. Historical data reveal that the order quantity Q is constant and the inconsistency in the ordering policy stems from the wide variation in the reorder level and the lead time between placing an order and receiving it. Consequently, a decision is made to maintain Q at its current level and to concentrate on the determination of R.

It is reasonable to assume that the demand x during lead time is normal with mean μ and standard deviation σ. The standard normal distribution of the lead time demand is thus defined as

$$z = \frac{x - \mu}{\sigma}$$

Next, let α be the probability of running out of stock during lead time that Dell is willing to tolerate, and define Z such that $P\{z \leq Z] = 1 - \alpha$; then the reorder level R is

$$R = \mu + Z\sigma$$

We can use an approximation borrowed from the single-period newsvendor inventory model to determine Z heuristically (see Section 14.2). The single-period model can be justified by the fact that the component XDX has limited shelf life because of obsolescence. Given that p is the penalty cost per unit of unsatisfied demand and h is the holding cost per unit of surplus inventory, we have

$$\Phi(Z) \equiv P\{z \leq Z\} = \frac{p}{p + h}$$

The function $\Phi(Z)$ is the CDF of the standard normal distribution. Thus,

$$Z = \Phi^{-1}\left(\frac{p}{p + h}\right)$$

The mean μ and standard deviation σ of the demand during lead time are a function of two random variables: lead time and the demand per unit time. Under certain assumptions of independence, it can be proved that[18]

$$\mu = DT$$
$$\sigma = \sqrt{D^2\sigma_t^2 + T\sigma_d^2}$$

where

$$\begin{aligned} D &= \text{Average demand in units per unit time} \\ T &= \text{Average lead time in time units} \\ \sigma_d &= \text{Standard deviation of the demand} \\ \sigma_t &= \text{Standard deviation of the lead time} \end{aligned}$$

[18]G. Hadley and T. Whitin, *Analysis of Inventory Systems*, Prentice Hall, Englewood Cliffs, NJ, 1963, p. 153.

Based on this model and given that the order quantity is fixed at Q units, the inventory policy is "order Q units of XDX whenever the inventory level in the revolvers drops to R units."

Data collection:

Data collection originates from three suppliers, A, B, and C, who replenish Dell's revolvers with different but totally interchangeable XDX parts. The data for the model were collected for the period from December 1, 1998, to May 27, 1999, in two groups: (1) those related to the estimation of the holding and penalty costs, p and h, and (2) those used to estimate the mean and variance of demand during lead time.

The estimation of the holding cost, h, is based on the time period between two consecutive deliveries and includes the cost of capital tied up in the revolvers, price erosion due to obsolescence, and the storage charge assessed by Dell. The penalty cost, p, is estimated based on two factors: lost profit from canceled or partially filled orders and increased shipping cost when the part is not available when needed.

The demand during lead time is estimated based on a number of factors: (1) Dell's daily consumption (called pulls) from the revolvers, (2) daily receipts from the suppliers at the revolvers, (3) daily in-transit inventory from the suppliers to the revolvers, (4) daily PC sales using XDX components, (5) Dell's forecasts to suppliers regarding the expected use of XDX during next month, and (6) suppliers' lead times.

Data analysis:

The first task in the data analysis is to assess whether or not the inventory policies of the three suppliers of XDX have been operating optimally according to the proposed (R, Q) policy. This hypothesis is tested by computing the Z-score for all placed orders during the data collection period (December 1, 1998, to May 27, 1999). Specifically, given the actual reorder level R and the estimated mean and variance of the demand (per day) and the lead time (in days), the formulas presented above yield

$$Z = \frac{R - DT}{\sqrt{D^2\sigma_t^2 + T\sigma_d^2}}$$

The computations reveal that Z varies significantly from .8 to 5.2 over the study period. Given that the estimated holding and penalty costs, h and p, are the same for all three suppliers, the Z-score under optimal conditions should remain steady at about $Z = \Phi^{-1}\left(\frac{p}{p+h}\right)$. The fact that the ordering policy of the suppliers exhibits such erratic variations (between .8 and 5.2) indicates that they are not operating under optimal inventory policy. This discrepancy led the study team to investigate the reasons behind the current practice.

A supplier bases its replenishment policy on the monthly forecasts provided by Dell. These forecasts can occur at three levels: (1) *aggregate* representing the total number of PCs over the next month, (2) *attach* representing the number of PC units that will be using the component XDX, and (3) *pull* representing the number of XDX units that Dell will be withdrawing from the revolvers of *each* supplier. The effect of errors in aggregate, attach, and pull forecasts is a corresponding increase in the variability in demand, expressed in terms of the variance of the distribution of demand. An investigation of the three types of forecast revealed that Dell's forecast errors are responsible for approximately 50% of the total inventory carried in the revolvers, mainly because the suppliers are trying to compensate for the large variance by maintaining a larger safety stock. This information is important because, aside from the implementation of the optimal (R, Q) system based on the present "inconsistent" forecasts, Dell can play an important role in reducing the inventory in the revolvers by instituting more stringent forecasting techniques.

In particular, the *pull* forecast error is attributed primarily to the fact that Dell withdraws needed XDX units from one supplier at a time instead of in fixed ratios from all suppliers. The effect of "clump" pulling of XDX from the revolvers gives rise to higher replenishment rates and hence larger safety stocks.

Conclusions:

The results of the investigation indicate that eliminating clump pulling and providing more reliable forecasts can potentially reduce the safety stock by as much as 38%. So far, Dell has decreased inventories for one of its lines by about 20% with estimated annual savings of $2.7 million. Perhaps the most remarkable observation about the study is recognizing that the inventory level can be reduced by eliminating the source of variation due to forecast errors. This result is confirmed by the fact that the optimal reorder level depends directly on the standard deviation of demand.

Questions:

1. Evaluate the adverse effect of "clump" pulling from revolvers by comparing its average level of inventory with that when pulling is made uniformly from all the suppliers.
2. Suppose that the average demand per day in one assembly line is $N(300, 10)$ and the demand during lead time is $N(50,4)$. Given that the penalty cost per unit of unsatisfied demand per day is $100 and the holding cost per surplus inventory unit per day is $5, determine the reorder level.
3. For the data in Question 2, study the sensitivity of the reorder level to a $\pm r\%$ error in the estimation of the average demand per day (r varies from 0 to 50). The standard deviation remains unchanged.

Case 14: Analysis of an Internal Transport System in a Manufacturing Plant[19]

Tools: Queuing theory, simulation

Area of application: Materials handling

Description of the situation:

Three trucks are used to transport materials within in a manufacturing plant. The trucks wait in a central parking lot until requested. A requested truck will travel to the customer location, carry load to destination, and then return to the central parking lot. The principal user of the service is production (P) followed by the workshop (W) and maintenance (M). Other departments (O) occasionally may request the use of the trucks. Complaints about the long wait for a free truck have prompted users, especially production, to request adding a fourth truck to the fleet. The study is concerned with the justification of the cost for a fourth truck.

Input data summary:

Information on the operation of the internal transport system was collected over a period of 17 consecutive two-shift work days. Tables 26.21 and 26.22 provide a summary of the collected data.

[19]*Source:* G. P. Cosmetatos, "The Value of Queuing Theory—A Case Study," *Interfaces*, Vol. 9, No. 3, pp. 47–51, 1979.

TABLE 26.21 Summary Data of the Operation of the Internal Transport System

	Truck user				
	P	*W*	*M*	*O*	*Overall*
Average number of truck requests per hour	3.02	.84	.26	.48	4.6
Average in-use truck time per request (min)	18.0	25.0	32.0	20.0	20.3
Standard deviation of truck time per request (min)	8.0	11.0	15.0	14.0	10.6
Average waiting time for a truck request (min)	9.2	9.4	9.2	8.4	9.0

TABLE 26.22 Number of Trucks in Use as a Function of the Number of Requests

	Number of trucks in use at the time a request is made				
	0	*1*	*2*	*3*	Total
Number of requests	862	28	167	115	1172
Percentage of total	73.6	2.4	14.2	9.8	

In Table 26.21, we have the average rate of requests (arrival rate), the average time the truck is in use (service time), and the average waiting time for a request. Table 26.22 gives the number of trucks in use as a function of the number of requests made throughout the observation period.

Analysis of the situation:

Analysis of the raw data used to obtain the information in Table 26.21 yields the following observations:

1. Requests for truck use are random and can be represented by a Poisson distribution.
2. The service time (in-use truck time from the moment it travels to customer until it returns to the parking lot) is unimodal and skewed and does not appear to follow an exponential distribution. Perhaps the triangular distribution can be used to approximate the situation in this case.
3. Although no priority or allocation of trucks to users is in operation, truck drivers tend to show preference to closer customers.

The data in Table 26.22 lead to two observations:

1. In 73.6% of the requests, all three trucks were idle.
2. In only 9.8% of the requests, all three trucks were in use.

Because arrivals are random and can be described by a Poisson distribution and the service time is not exponential distribution, the queuing model that best represents the problem is the $M/G/c/\infty/\infty$. However, computations for the $M/G/c$ model are not convenient. As a result, it is decided that an equivalent $M/M/c$ model may be used to provide an upper-bound estimate on the waiting time in the queue. The justification is that exponential service time is the "most random" of all distributions and hence will result in a worst-case scenario for the present situation. (By the same logic, the $M/D/c$ model provides a lower bound on the average queuing time because the service time is constant and hence represents the "least random" case.)

The following is a summary of the results of the $M/M/c$ model for $c = 3$, $\lambda = \frac{4.6}{60} = .0767$ request per minute and $\mu = \frac{1}{20.3} = .0493$ service per minute:

Probability that the system is empty, $p_0 = .197$

Probability of at least three requests in the system, $p_{n\geq 3} = .133$

Average length of queue, $L_q = .277$ request

Average waiting time in queue, $W_q = 3.6$ minutes

Looking at these results, one notices makes the perplexing observation that the *upper bound* on the average waiting time in the queue (estimated from the $M/M/c$ model) is much lower than what is actually observed ($W_q = 3.6$minutes versus the observed 9.0 minutes given in Table 26.21). This observation leads to one of two conclusions: Either the estimates of λ and μ are highly inaccurate or the estimate of the average waiting is unreliable. A careful study of the data shows that the data are indeed reliable. To reinforce the results of the $M/M/c$ model, simulation is used in which the service time distribution is approximated by a triangular distribution with parameters (15, 20.3, 30). The middle value represents the observed average service time and the lower and upper values are estimated based on the standard deviation of service time (= 10.6 minutes) and the observed minimum and maximum service times. The simulation can be carried out using Excel template excelMultiServer.xls with Poisson arrival rate of .0767 request per minute and triangular service time. With 10 replications that simulate 450 requests each, the average queuing time was found to vary from a minimum of 1.1 minutes to a maximum of 3.62 minutes and an average value of 2.07 minutes. This result gave more credence to the upper bound result of 3.6 minutes obtained from the $M/M/c$ model. Moreover, the high waiting time obtained from the observed data (= 9.0 minutes) seems to contradict the data in Table 26.22, where 73.6% of the time all three trucks were idle when a service request arrived.

How can this inconsistency between observed and estimated results be resolved? Going back to the plant floor to further study the operation of the transport system, an analyst made a fortunate observation: The layout of the parking lot was such that waiting trucks could not be seen by the users, who then assumed that no trucks were available. This in essence was equivalent to operating with less than three trucks, which in turn resulted in an artificial increase in waiting time. Once this problem had been discovered, the solution became obvious: Provide the truck drivers and the users with a two-way communication system. The proposed solution led to immediate improvement in service and a noticeable decrease in the waiting time.

Although the proposed solution was not "propelled" by queuing results in a direct manner, it was the logic inherent in queuing analysis that led to the discovery of data inconsistency and, hence, to pinpointing the source of the problem.

Questions:

1. In Table 26.21, how are the values in the "overall" column computed?
2. If 95% of the requests for a truck come from production and only 5% come from the remaining departments, how would you analyze the situation?
3. Evaluate the robustness of the results by studying the sensitivity of simulation results to changes in the minimum and maximum values of the triangular distribution. Does the sensitivity analysis confirm or refute the conclusion given for the case?

Case 15: Telephone Sales Manpower Planning at Qantas Airways[20]

Tools: ILP, queuing theory

Area of application: Airlines

Description of the situation:

To reduce operating costs, Qantas Airways seeks to staff its main telephone sales reservation office efficiently while providing convenient service to its customers. Reservation office hours are from 7:00 till 22:00. Six shifts start on the hour and half hour during the period 7:00–9:30. An additional shift starts at 15:00. Shifts starting between 7:00 and 8:01 are 8 hours long and shifts starting between 8:30 and 9:31 are $8\frac{1}{2}$ hours long. The seventh shift starting at 15:00 is only 7 hours long. All operators are allowed a half-hour break, except the ones starting between 8:30 and 9:31 who get one-hour breaks. Breaks can be taken after an operator has been at least three hours on the job and are scheduled every half hour over a span of 2 hours. The lengths of shifts and of the breaks are designed to provide a "reasonable" coverage of work hours between 7:00 and 22:00.

Traditionally, staffing needs are estimated by forecasting future telephone calls based on historical increase in business. The increase in staff numbers is then calculated based on the projected average increase in telephone calls divided by the average number of calls an operator can handle. Because the calculations are based on averages, the additional number of hired staff does not take into account the fluctuations in demand during the day. In particular, long waiting time for service during peak business hours has resulted in both customer complaints and lost business.

The problem deals with the determination of a plan that strikes a balance between the number of hired operators and customer needs. The plan must be responsive to the fluctuations in demand during business hours.

Collection of data:

The basic data needed for the study include the distributions of arriving calls and of the time an operator spends with the customer (service time). These data were collected and tallied daily over half-hour time intervals for a period of three months. The analysis yields two main results:

1. Arrivals exhibit substantial variations by time of day but are consistent from one day to another.
2. Service time distribution appears to be stationary over time.
3. The distribution of arrivals over half-hour periods is approximately Poisson albeit with different means (see Table 26.23).
4. The distribution of service time is approximately exponential with mean 3.5 minutes.

Mathematical model:

The problem can be represented as an integer linear program whose objective is to minimize the total number of hired operators while satisfying the half-hourly fluctuations in demand throughout the work day as well as shift durations and breaks. Let

x_j = Number of operators starting in shift j, $j = 1, 2, \ldots, 7$

y_{ij} = Number of operators from shift j taking a break during half-hour period i, $i = 1, 2, 3, 4$

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[20]*Source:* A. Gaballa and W. Pearce, "Telephone Sales Manpower Planning at Qantas," *Interfaces*, Vol. 9, No. 3, pp. 1–9, May 1979.

TABLE 26.23 Average Number of Calls per Minute Computed over 30-Minute Periods from 7:00 till 22:00

Period	Mean	Period	Mean
7:00–7:30	5	14:30–15:00	17
7:30–8:00	8	15:00–15:30	15
8:00–8:30	10	15:30–16:00	12
8:30–9:00	10	16:00–16:30	10
9:00–9:30	12	16:30–17:00	10
9:39–10:00	12	17:00–17:30	10
10:00–10:30	15	17:30–18:00	9
10:30–11:00	17	18:00–18:30	9
11:00–11:30	20	18:30–19:00	9
11:30–12:00	23	19:00–19:30	7
12:00–12:30	23	19:30–20:00	7
12:30–13:00	23	20:00–20:30	5
13:00–13:30	21	20:30–21:00	5
13:30–14:00	17	21:00–21:30	3
14:00–14:30	17	21:30–22:00	3

Mean service time = 3.5 minutes

c_k = Minimum number of operators needed during half-hour k, $k = 1, 2, \ldots, 30$

The model is thus given as

$$\text{Minimize } z = \sum_{j=1}^{7} x_j$$

subject to

$$\sum_{r=1}^{k} x_r \geq c_k,\ k = 1, 2, \ldots, 6$$

$$\sum_{r=1}^{6} x_r - y_{11} \geq c_7$$

$$\sum_{r=1}^{6} x_r - y_{21} - y_{12} \geq c_8$$

$$\sum_{r=1}^{6} x_r - y_{31} - y_{22} - y_{13} \geq c_9$$

$$\sum_{r=1}^{6} x_r - y_{41} - y_{32} - y_{23} - y_{14} \geq c_{10}$$

$$\sum_{r=1}^{6} x_r - y_{42} - y_{33} - y_{14} - y_{24} - y_{15} \geq c_{11}$$

$$\sum_{r=1}^{6} x_r - y_{43} - y_{24} - y_{34} - y_{15} - y_{25} - y_{16} \geq c_{12}$$

$$\sum_{r=1}^{6} x_r - y_{34} - y_{44} - y_{25} - y_{35} - y_{16} - y_{26} \geq c_{13}$$

$$\sum_{r=1}^{6} x_r - y_{44} - y_{35} - y_{45} - y_{26} - y_{36} \geq c_{14}$$

$$\sum_{r=1}^{6} x_r - y_{45} - y_{36} - y_{46} \geq c_{15}$$

$$\sum_{r=1}^{6} x_r - y_{46} \geq c_{16}$$

$$x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \geq c_{17}$$

$$x_3 + x_4 + x_5 + x_6 + x_7 \geq c_{18}$$

$$x_4 + x_5 + x_6 + x_7 \geq c_{19}$$

$$x_4 + x_5 + x_6 + x_7 \geq c_{20}$$

$$x_5 + x_6 + x_7 - y_{17} \geq c_{21}$$

$$x_6 + x_7 - y_{27} \geq c_{22}$$

$$x_7 - y_{37} \geq c_{23}$$

$$x_7 - y_{47} \geq c_{23}$$

$$x_7 \geq c_k, k = 25, 26, \ldots, 30$$

$$\sum_{i=1}^{4} y_{ij} - x_j = 0, j = 1, 2, \ldots, 7$$

all x_j and y_{ij} are nonnegative integers

The logic of the model constraints is as follows: For shifts 1, 2, and 3 starting times 7:00, 7:30, and 8:00, the shift lasts 8 hours or 16 half-hours. This is the reason each of the variables x_1, x_2, and x_3 appears in 16 consecutive constraints. By the same logic, the variables x_4, x_5, and x_6 correspond to the shifts starting at 8:30, 9:00, and 9:30 and lasting $8\frac{1}{2}$ hours each. For this reason, each of these variables appears in 17 consecutive constraints. Finally, x_7 representing the shift that starts at 15:00 (period 17) lasts 7 hours and hence appears in 14 consecutive constraints starting at period 17. As for the variables y_{ij} representing the breaks, the logic can be demonstrated by two cases: For shift 1 (starting at time 7:00), four half-hour breaks start at 10:00 (period 7), 10:30 (period 8), 11:00 (period 9), and 11:30 (period 10). This is the reason for subtracting y_{11}, y_{21}, y_{31}, and y_{41} from the left-hand side of constraints 7 through 10. Similar reasoning applies to y_{ij} for shifts 2 and 3. The other distinct case is shift 4, lasting $8\frac{1}{2}$ hours and starting at 8:30. The four half-hour breaks starting at 11:30 (period 10), 12:00 (period 11), 12:30 (period 12), and 13:00 (period 13) are represented by the respective variables y_{14}, y_{24}, y_{34}, and y_{44}. Because this shift allows for a one-hour break, each of these variables appears in two consecutive constraints. For example, y_{14} appears in constraints 10 and 11. The last constraint guarantees that every operator will have a mid-shift break.

The only data needed to drive the model are the right-hand side of the constraints, $c_k, k = 1, 2, \ldots, 30$. The next section shows how queuing theory is used to determine these values.

Estimation of minimum number of operators, c_k, for period k:

Collected data indicate that the traffic of incoming calls during each half-hour period can be reasonably represented by the multiple-server Poisson queuing model $(M/M/c):(GD/\infty/\infty)$. The arrival rate of calls during each half-hour is determined from data as given in Table 26.23. The same data also show that the service-time distribution is approximately exponential with a stationary mean value of 3.5 minutes. Given that λ_k is the arrival rate per minute during half-hour k and μ is the service rate per minute and defining $[v]$ as the largest integer value $\leq v$, a lower bound on the number of operators needed during period k is

$$c_k = \left[\frac{\lambda_k}{\mu}\right] + 1, \text{k} = 1, 2, \ldots, 30$$

For example, for $\lambda_1 = 5$ calls per minute and $\mu = \frac{1}{3.5} = .286$ service per minute, the minimum number of operators from 7:00 to 7:30 is $\left[\frac{5}{.286}\right] + 1 = 17 + 1 = 18$. The rationale behind these calculations is that c_k is the minimum number needed to maintain steady-state conditions in the queuing model.

The lower-bound estimate of c_k does not provide information about the *quality* of service to calling customers. In particular, to guard against lost calls (as well as loss of goodwill), the waiting time until an operator answers a call should be reasonably small. Management set the goal that at least 90% of the calls should be answered within 20 seconds. This goal translates to the following probability statement:

$$P\{\text{waiting time}, t > T = 20 \text{ sec }\} < .1$$

By varying the number of servers, c, the desired probability can be realized. From the results of queuing theory, given $\rho = \frac{\lambda}{\mu}$, we have[21]

$$P\{t \leq T\} = \begin{cases} 1 - \dfrac{cp^c p_0}{c!\,(c - \rho)}, & T = 0 \\ \dfrac{\rho^c(1 - e^{-\mu(c-\rho)T})p_0}{(c - 1)!\,(c - \rho)} + P\{t = 0\}, & T > 0 \end{cases}$$

The formula for the probability p_0 is given in Section 15.6.3. An estimate of the minimum number of operators needed during period k is the the smallest c that satisfies

$$P\{t > T\} = 1 - P\{t \leq T\} < .1$$

The nature of the probability statement does not allow a closed-form solution of the value of c. Instead, the required solution is determined by substituting successive increasing values of c until the desired condition is satisfied for the first time. The lower bound on c $(= [\rho] + 1)$ provides a starting point for the substitution.

A spreadsheet is ideal for carrying out this type of calculation. File excelCase15.xls is designed to do just that. The results are given in Table 26.24.

AMPL model:

File amplCase15.txt provides the AMPL code for the proposed problem. The solution of the model is translated to the readable format shown in Figure 26.23. It shows that a total of 134 operators are needed. It also details the number of operators for each shift and their assigned break times.

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[21] *Source:* D. Gross and C. Harris, *Fundamentals of Queuing Theory,* Wiley, New York, 1974, Formula 3.20, p. 101. Also, see Problem 16, Set 15.6e.

TABLE 26.24 Estimation of the Minimum Number of Operators

Period, k	λ	c_k	$P\{t > 20\ sec\}$	Period, k	λ	c_k	$P\{t > 20\ sec\}$
1	5	23	0.090	16	17	74	0.082
2	8	35	0.074	17	15	63	0.090
3	10	42	0.093	18	12	50	0.077
4	10	42	0.093	19	10	42	0.093
5	12	50	0.077	20	10	42	0.093
6	12	50	0.077	21	10	42	0.093
7	15	63	0.090	22	9	39	0.068
8	17	74	0.082	23	9	39	0.068
9	18	79	0.090	24	9	39	0.068
10	19	85	0.095	25	7	31	0.080
11	20	90	0.095	26	7	31	0.080
12	20	90	0.095	27	5	23	0.090
13	20	90	0.095	28	5	23	0.090
14	19	85	0.082	29	3	15	0.092
15	18	79	0.082	30	3	15	0.092

HIRING SCHEDULE:

Shift starting time	Shift size
7:00	26
7:30	21
8:00	25
9:00	4
9:30	19
14:00	39
Total =	134

BREAK SCHEDULE:

Shift start time	Number of operators	Break start time	break time (min)
7:00	26	10:00	30
7:30	21	10:30	30
8:00	15	11:00	30
8:00	10	11:30	30
9:00	4	12:00	60
9:30	4	13:00	60
9:30	15	14:00	60
14:00	20	17:00	30
14:00	19	17:30	30

FIGURE 26.23

Workforce size based on $8\frac{1}{2}$ hour durations for shifts 4 through 6

Questions:

1. An interesting question arises regarding the current practice of requiring $8\frac{1}{2}$-hour duration for shifts 4, 5, and 6 (starting at 8:30, 9:00, and 9:30, respectively) together with 1-hour breaks. Preference among most employees is for the shorter shift (8 hours) with a half-hour break rather than the longer shift $\left(8\frac{1}{2}\text{hours}\right)$ with a full-hour break. Study the impact of imposing uniform 8-hour shifts with half-hour breaks for *all* employees.
2. Study the sensitivity of the solution to changing the probability of a 20-second delay in answering a call.