Exercise 5.1

a) There are 5 equivalence relations over the set $M = \{ a, b, c \}$.

$$\sim = \{ \langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle \}$$
$$\sim = \{ \langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \langle a,b\rangle, \langle b,a\rangle \}$$
$$\sim = \{ \langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \langle a,c\rangle, \langle c,a\rangle \}$$
$$\sim = \{ \langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \langle c,b\rangle, \langle b,c\rangle \}$$
$$\sim = \{ \langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \langle a,b\rangle, \langle b,a\rangle, \langle a,c\rangle, \langle c,a\rangle, \langle c,b\rangle, \langle b,c\rangle \}$$

b) $[a]_\sim = \{ a \}$, $[b]_\sim = \{ b, d \}$, $[c]_\sim = \{ c, e \}$

Exercise 5.2

a) *Proof.* Let x be an arbitrary element in S. Since an equivalence relation is reflexive, we have $\langle x, x\rangle \in \sim$ which means that $x \in [x]_\sim \subseteq E$.

□

b) *Proof.* We will do a proof by contradiction. Let x, y, z be arbitrary elements in S and assume that $x \in [y]_\sim$ and $x \in [z]_\sim$ but $[y]_\sim \neq [z]_\sim$.

Since $x \in [y]_\sim$ we have that $\langle y, x\rangle \in \sim$. Since $x \in [z]_\sim$ we have that $\langle z, x\rangle \in \sim$ and since an equivalence relation is symmetric we also have that $\langle x, z\rangle \in \sim$.

Because an equivalence relation is also transitive, we have $\langle y, z\rangle \in \sim$.
We now claim that $[y]_\sim = [z]_\sim$ if $\langle y, z\rangle \in \sim$. This would be the needed contradiction to finish the proof.
Proof of the claim: Let $a \in [y]_\sim$ be arbitrary. Then $\langle a, y\rangle \in \sim$ and thus $\langle a, z\rangle \in \sim$ since $\langle y, z\rangle \in \sim$ i.e. $a \in [z]_\sim$. Thus $[y]_\sim \subseteq [z]_\sim$. Analogously $[z]_\sim \subseteq [y]_\sim$. But this means that $[y]_\sim = [z]_\sim$.

□

Exercise 5.3

$R = \{ \langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \langle d,d\rangle, \langle a,b\rangle, \langle c,b\rangle \}$

Our relation R is a partial order because it is reflexive, antisymmetric and transitive. It is reflexive because $\langle x, x\rangle \in R$ for all $x \in S$, antisymmetric because $\langle a, b\rangle \in R$ but $\langle b, a\rangle \notin R$, similarly for $\langle c, b\rangle$ and transitive because $\langle a, b\rangle \in R$ but b does not relate something else except itself and similarly for $\langle c, b\rangle$.

Our minimal elements are a and c.

Exercise 5.4

a) False. It should be [...] *at least* one of xRy and yRx (so possible also both) is true [...].

b) You can always take the relation $R = \emptyset$. This way all $x \in S$ stand in no relation to each other and are therefore each a maximum and minimum.

Exercise 5.5

a) $A^{-1} = \{ \langle 2x, x \rangle \mid x \in \mathbb{N}_0 \}$

b) $B \setminus A^{-1} = \{ \langle i \cdot x, x \rangle \mid i, x \in \mathbb{N}_0, i \neq 2 \}$

c) $C \circ A = \{ \langle 2, 2 \rangle, \langle 2, 7 \rangle, \langle 3, 4 \rangle, \langle 4, 2 \rangle \}$

d) $A \circ (A \circ A) = \{ \langle x, 8x \rangle \mid x \in \mathbb{N}_0 \}$

e) $A^* = \{ \langle x, 2^n \cdot x \rangle \mid x, n \in \mathbb{N}_0 \}$

f) $A \circ B = \{ \langle i \cdot x, 2x \rangle \mid i, x \in \mathbb{N}_0 \}$