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Exercise 2.1

Proof. We want to proof that

$$\sum_{i=0}^{n} i = \frac{n \cdot (n+1)}{2} \text{ for all } n \in \mathbb{N}_0$$

using mathematical induction over n.

Induction basis $n = 0$:

$$\sum_{i=0}^{0} i = 0 = \frac{0 \cdot (0+1)}{2}$$

IH:

$$\sum_{i=0}^{k} i = \frac{k \cdot (k+1)}{2} \text{ for all } 0 \leq k \leq n$$

Induction step $n \to n+1$:

$$\begin{aligned}\sum_{i=0}^{n+1} i &= \left(\sum_{i=0}^{n} i\right) + (n+1) \\ &\overset{IH}{=} \frac{n \cdot (n+1)}{2} + (n+1) \\ &= \frac{n \cdot (n+1)}{2} + \frac{2 \cdot (n+1)}{2} \\ &= \frac{(n+1) \cdot (n+2)}{2}\end{aligned}$$

$\square$

Exercise 2.2

Proof. We want to show that $edges(B) = 2 \cdot leaves(B) - 2$ for all binary trees B with a weak induction.
Induction basis:

$$edges(\square) = 0 = 2 \cdot 1 - 2 = 2 \cdot leaves(\square) - 2$$

IH: To prove that the statement is true for a composite tree $\langle L, \bigcirc, R \rangle$, we may use that it is true for the subtrees L and R.
Inductive step for $B = \langle L, \bigcirc, R \rangle$:

$$\begin{aligned}edges(B) &= edges(\langle L, \bigcirc, R \rangle) \\ &= edges(L) + edges(R) + 2 \\ &\overset{IH}{=} 2 \cdot leaves(L) - 2 + 2 \cdot leaves(R) - 2 + 2 \\ &= 2 \cdot (leaves(L) + leaves(R)) - 2 \\ &= 2 \cdot leaves(\langle L, \bigcirc, R \rangle) - 2 \\ &= 2 \cdot leaves(B) - 2\end{aligned}$$

□

Exercise 2.3

Proof. We want to show that all words in S have odd length with structural induction.
Induction basis: c and baa both have odd length. IH: To prove the statement is true for composite words xyx and bxb, we may use that it is true for the words x and y.
Inductive step: By the definition of the set S, every word in $z \in S$ (that is not c or baa) either has the form xyx or bxb for words $x, y \in S$. We make a case distinction.
Case 1: $z = xyx$
Since by the IH x and y have odd length, xyx also has odd length since the addition of three odd numbers returns an odd number.
Case 2: $z = bxb$
Since by the IH x has odd length, bxb also has odd length since adding two to an odd number returns an odd number. □

Exercise 2.4

(a) $\{2n \mid n \in \mathbb{N}, n < 10\}$

(b) $\mathbb{N} \setminus \{6\}$

Excercise 2.5

(a) $A = \{1, 3\}, B = \{1, 3, 5, 6, 8, 9, 10\}$

(b) $A = \emptyset, B = \{3, 5\}$

(c) $A = \{6, 8, 9\}, B = \{6, 8, 9\}$