MATH 12002 - CALCULUS I
§1.3: Introduction to Limits

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Average and Instantaneous Velocity

Suppose I drive in a straight line 150 miles in 3 hours.
What is my *average* velocity?
Average velocity is distance divided by time, so in this case is

$$\frac{150 \text{ miles}}{3 \text{ hours}} = 50 \text{ miles per hour.}$$

Velocity at time $t = 1$ hour?

We can Compute the average velocity on the time interval

$$t = 1 \text{ to } t = 1 + h$$

for smaller and smaller values of h.

The number the average velocity approaches as the length of the time interval, h, approaches 0 is the *instantaneous* velocity at time $t = 1$.

Average and Instantaneous Velocity

This is the idea of a *limit*:
The number the average velocity is approaching (the instantaneous velocity) is the LIMIT of the average velocity as h approaches 0.
In symbols, if $s(t)$ is the position at time t, then the average velocity on the time interval from $t = a$ to $t = a + h$ is the distance

$$s(a + h) - s(a)$$

divided by the length of the time interval

$$(a + h) - a = h.$$

That is,

$$v_{avg} = \frac{s(a + h) - s(a)}{h}.$$

Instantaneous velocity is expressed as

$$v_{inst} = \lim_{h \to 0} \frac{s(a + h) - s(a)}{h}.$$

Limit of a Function

More generally, we are interested in the behavior of the y values of a function $y = f(x)$ when the value of x is near some number a.

Example

Let $y = f(x) = \frac{x^2-4}{x-2}$ *and let* $a = 2$. *Note that* $f(2)$ *is undefined.*
Values of $y = f(x)$ *for* x *near* 2*:*

x	y
1	3
1.5	3.5
1.9	3.9
1.99	3.99
1.999	3.999

x	y
3	5
2.5	4.5
2.1	4.1
2.01	4.01
2.001	4.001

As x *gets close to* 2 *from either side, the* y *values approach* 4.
We say the limit of $f(x)$ *as* x *approaches* 2 *is* 4*, that is,* $\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$.

Limit of a Function

Definition

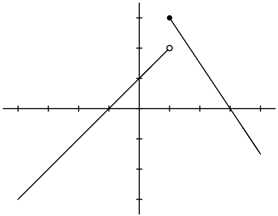
Let $y = f(x)$ be a function and let a and L be numbers.
We say that the limit of f as x approaches a is L
if y can be made arbitrarily close to L
by taking x close enough to a, but not equal to a.
We write $\lim_{x \to a} f(x) = L$.

Notes:

- What happens when $x = a$ is not relevant.
  We are interested only in the value of y when x is *near* a.
- y must be close to L when x is close to a on *both* sides of a, that is, whether $x < a$ or $x > a$.

One-Sided Limits

Let $y = f(x)$ be the function whose graph is shown below:



As x approaches 1 from the left, y approaches 2.
We say the left-hand limit of $f(x)$ as x approaches 1 (or the limit as x approaches 1 from the left) is 2, and write $\lim_{x \to 1^-} f(x) = 2$.
As x approaches 1 from the right, y approaches 3.
We say the right-hand limit of $f(x)$ as x approaches 1 (or the limit as x approaches 1 from the right) is 3, and write $\lim_{x \to 1^+} f(x) = 3$.
Since the two one-sided limits are not equal, $\lim_{x \to 1} f(x)$ *does not exist*.

One-Sided Limits

In general, we have

Theorem

Let $y = f(x)$ be a function and let a and L be numbers. Then

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L$$