MATH 12002 - CALCULUS I
§1.4: Limit Laws

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Laws

There are a number of basic properties, or laws, satisfied by limits. These can all be proved rigorously using the definition of limit, but we will take a more intuitive approach.

The first two Limit Laws should be fairly obvious.

1. $\lim_{x \to a} x = a$.
2. $\lim_{x \to a} c = c$, if c is a constant.

The statement that the limit of x as x approaches a is a just says that as x gets very close to a, x gets very close to a.

The second law means that if c is a constant, then as x gets close to a, c gets close to c. For example, $\lim_{x \to 3} 5 = 5$ because as x gets close to 3, 5 obviously gets close to (in fact is equal to) 5.

Laws

The next two laws state that limits "respect" addition and multiplication; that is, the limit of a sum is the sum of the limits, and the limit of a product is the product of the limits.

We assume that $\lim_{x \to a} f(x) = M$ and $\lim_{x \to a} g(x) = N$ both exist.

3. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.
4. $\lim_{x \to a} f(x)g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$.

These say that if $f(x)$ is close to M and $g(x)$ is close to N, when x is close to a, then $f(x) + g(x)$ is close to $M + N$ and $f(x)g(x)$ is close to MN, when x is close to a.

Laws

Applying (4) to the case where one of the functions is constant, we obtain

5. $\lim\limits_{x \to a} cf(x) = c \lim\limits_{x \to a} f(x)$.

Applying (4) to repeated multiplication, we obtain the following laws.

6. $\lim\limits_{x \to a} (f(x)^n) = \left(\lim\limits_{x \to a} f(x) \right)^n$, for any positive integer n.
7. $\lim\limits_{x \to a} x^n = a^n$, for any positive integer n.

Finally, we have that limits also respect quotients,
but only under certain conditions.

8. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, **IF** $\lim\limits_{x \to a} g(x) \neq 0$.

A more complete list of Limit Laws is available in the text.

Examples

EXAMPLE 1:

$$\begin{aligned}\lim_{x\to 2}(3x^5+5x^2+7) &= \lim_{x\to 2}3x^5+\lim_{x\to 2}5x^2+\lim_{x\to 2}7,\ \text{by (3)}\\ &= 3\lim_{x\to 2}x^5+5\lim_{x\to 2}x^2+\lim_{x\to 2}7,\ \text{by (5)}\\ &= 3(2^5)+5(2^2)+7,\ \text{by (7) and (2)}\\ &= 3(32)+5(4)+7=123.\end{aligned}$$

Observe that $\lim_{x\to 2}(3x^5+5x^2+7)=3(2^5)+5(2^2)+7$,
that is, if $P(x)=3x^5+5x^2+7$, then $\lim_{x\to 2}P(x)=P(2)$.
Since any **polynomial** $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ involves only addition, multiplication by constants, and positive integer powers, the same reasoning leads to

Theorem

If $P(x)$ is any polynomial and a is a number, then $\lim_{x\to a}P(x)=P(a)$.

Examples

EXAMPLE 2: To evaluate $\lim\limits_{x \to 3} \dfrac{2x^3 + 3x + 5}{(5x^2 + 4)^2}$, first note that by (8),

$$\lim_{x \to 3} \frac{2x^3 + 3x + 5}{(5x^2 + 4)^2} = \frac{\lim\limits_{x \to 3} (2x^3 + 3x + 5)}{\lim\limits_{x \to 3} (5x^2 + 4)^2},$$

provided $\lim\limits_{x \to 3} (5x^2 + 4)^2 \neq 0$. By the previous Theorem and (6),

$$\lim_{x \to 3} (5x^2 + 4)^2 = \left(\lim_{x \to 3} (5x^2 + 4) \right)^2 = (5(3^2) + 4)^2 \neq 0.$$

Therefore, we have

$$\lim_{x \to 3} \frac{2x^3 + 3x + 5}{(5x^2 + 4)^2} = \frac{\lim\limits_{x \to 3} (2x^3 + 3x + 5)}{\lim\limits_{x \to 3} (5x^2 + 4)^2} = \frac{2(3^3) + 3(3) + 5}{(5(3^2) + 4)^2} = \frac{68}{2401}.$$

Examples

As before, if we set $R(x) = \dfrac{2x^3 + 3x + 5}{(5x^2 + 4)^2}$, then we have

$$\lim_{x \to 3} R(x) = \lim_{x \to 3} \frac{2x^3 + 3x + 5}{(5x^2 + 4)^2} = \frac{2(3^3) + 3(3) + 5}{(5(3^2) + 4)^2} = R(3).$$

More generally, if $R(x)$ is any **rational function**,
i.e., $R(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials,
then $\lim_{x \to a} P(x) = P(a)$ and $\lim_{x \to a} Q(x) = Q(a)$ by the previous Theorem,
and by (8),

$$\lim_{x \to a} R(x) = \frac{\lim_{x \to a} P(x)}{\lim_{x \to a} Q(x)} = \frac{P(a)}{Q(a)} = R(a), \textbf{ IF } Q(a) \neq 0.$$

We therefore have

Theorem

If $R(x) = \frac{P(x)}{Q(x)}$ is a rational function and $Q(a) \neq 0$, then $\lim_{x \to a} R(x) = R(a)$.