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Series solution of Differential equations and Special Functions.

POWER SERIES METHOD

Consider a homogenous linear second order D.E. with variable coefficients

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\cdot\frac{dy}{dx} + P_2(x)y = 0 \quad \text{—(1)}$$

$$\frac{d^2y}{dx^2} + \frac{P_1(x)}{P_0(x)}\frac{dy}{dx} + \frac{P_2(x)}{P_0(x)}y = 0$$

Let $\frac{P_1(x)}{P_0(x)} = P(x)$ and $\frac{P_2(x)}{P_0(x)} = Q(x)$

then, $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \quad \text{—(2)}$

Eq (2) is called normal form or canonical form. ~~of~~ or standard form of eq^n (1)

The power series solution of eq. (2) about a point $x = x_0$ depends on the following definitions:

1. Ordinary point / Regular point:
A point $x = x_0$ is called an ordinary point of eq (2) if $P(x)$ and $Q(x)$ are both analytic (i.e., differentiable) at $x = x_0$.

If $P_0(x) \neq 0$ at $x = x_0$, then x_0 is an ordinary point.

2. Singular point

A point $x = x_0$ is called a singular point of eq. (2) if either $P(x)$ or $Q(x)$ or both are not analytic at x_0.

If $P_0(x) = 0$ at $x = x_0$, then x_0 is a singular point.

Regular singular point.

A singular point is called a regular singular point of eq^n (2) if $(x - x_0) P(x)$ and $(x - x_0)^2 Q(x)$ both are analytic (or differentiable) at $x = x_0$

$$\lim_{x \to x_0} (x - x_0) P(x) = \text{finite value}$$

$$\lim_{x \to x_0} (x - x_0)^2 Q(x) = \text{finite value}$$

Irregular Singular Point

If either $(x - x_0) P(x)$ or $(x - x_0)^2 Q(x)$ or both are not analytic at $x = x_0$

Series solution about an Ordinary Point

Let the power series solution of eqⁿ

$$P_0(x)\, y'' + P_1(x)\, y' + P_2(x)\, y = 0 \quad \text{--- (1)}$$

about an ordinary point x_0 be given as

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots \quad \text{--- (2)}$$

The coefficients $a_1, a_2, a_3, \cdots$ are obtained as follows:

1) Let $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ be series solution of given equation.
2) Differentiate 'y' w.r.t x twice to get y, y', y'' then substitute y, y', y'' in eqⁿ (1).
3) Shift the summation index to obtain a common power of x in each term.
4) Equate the coefficients of various powers of x to zero to obtain $a_1, a_2, \ldots$ in terms of a_0.
5) Substitute $a_1, a_2, \cdots$, in eqⁿ (2) to obtain the reqd. solution of given equation.

FROBENIUS METHOD

Series solution when $x=0$ is a regular singular point of the equation

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \quad \text{—} (1)$$

(1) Assume the solution of eq^n (1) to be

$$y = \sum_{n=0}^{\infty} a_n \cdot x^{m+n} = x^m \cdot \sum_{n=0}^{\infty} a_n \cdot x^n$$

$$y = x^m (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) \quad \text{—} (2)$$

(2) Differentiate w.r.t. x twice and substitute y, y' and y'' in eq^n (1)

(3) Equate to zero the coefficient of lowest degree term in x. It gives a quadratic equation (known as Indicial equation).

(4) Equate to zero the coefficients of other powers of x, find the values of a_1, a_2, ... in terms of a_0.

The complete solution depends on the nature of roots of indical equation.

Case I : When roots are distinct and do not differ by an integer, then complete solution is

$$y = c_1 (y)_{m=m_1} + c_2 (y)_{m=m_2}$$

m_1 and m_2 are the roots.

Case II: When roots are repeated, then the complete solution is

$$y = c_1 (y)_{m=m_1} + c_2 \left(\frac{\partial y}{\partial m}\right)_{m=m_1}$$

m_1 is the root.

case III: When roots are distinct and differ by an integer, making a coefficient of y infinite.

Let m_1 & m_2 be the roots (such that $m_1 < m_2$). If some of co-efficients of y- series become infinite when $m = m_1$, we modify the form of y by replacing a_0 by $b_0 (m - m_1)$. Then the complete solution is

$$y = c_1 (y)_{m=m_2} + c_2 \left(\frac{\partial y}{\partial m}\right)_{m=m_1}$$