Q1. Find the laplace transformation of cosat.

Using Euler's identity:

$$\cos at = \frac{e^{iat} + e^{-iat}}{2}$$

Using Laplace transform on both sides.

$$L\{\cos at\} = L\left\{\frac{e^{iat} + e^{-iat}}{2}\right\}$$

$$= L\left\{\frac{e^{iat}}{2}\right\} + L\left\{\frac{e^{-iat}}{2}\right\}$$ Using Linearity property

$$= \frac{1}{2}\left\{\frac{1}{s-ia} + \frac{1}{s+ia}\right\}$$ $L\{e^{at}\} = \frac{1}{s-a}$.

$$= \frac{1}{2}\left\{\frac{s+ia+s-ia}{s^2+a^2}\right\}$$

$$L\{\cos at\} = \frac{s}{s^2+a^2}.$$

Q2. Find the laplace transformation of sinhat.

Using Euler's identity:

$$\sinh at = \frac{e^{at} - e^{-at}}{2}$$

Using Laplace transform on both sides.

$$L\{\sinh at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\}$$

$$= L\left\{\frac{e^{at}}{2}\right\} - L\left\{\frac{e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\}$$

$$= \frac{1}{2}\left\{\frac{\cancel{s}+a-\cancel{s}+a}{s^2 \quad a^2}\right\}$$

$$L\{\sinh t\} = \frac{a}{s^2-a^2}$$

Q. State and prove the laplace transformation of integral.

Laplace transformation of an integral can be defined as;

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}F(s), \text{ where } F(s) = L(f(t))$$

Proof: Laplace transformation of derivative is given as;

$$L\{g'(t)\} = sL\{g(t)\} - g(0)$$

$\therefore$ Putting $g(t) = \int_0^t f(t)\,dt$, $f(0) = 0$

$$\therefore L\left\{\left(\int_0^t f(t)\,dt\right)'\right\} = sL\left\{\int_0^t f(t)\,dt\right\} - \left(\int_0^t f(0)\,dt\right)$$

$$L\{f(t)\} = sL\left\{\int_0^t f(t)\,dt\right\}$$

$$\Rightarrow L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}F(s) \qquad L\{f(t)\} = F(s).$$

Q. State and prove Laplace transformation of second shifting theorem by using rule of definite integral and improper integral.

Statement: If $L\{f(t)\} = F(s)$, then;

$$f(t) = \begin{cases} f(t-a), & t > 0 \\ 0, & t < 0 \end{cases}$$

$$\therefore L\{f(t)\} = e^{-as}F(s).$$

Proof: By the definition of the Laplace Transformation;

$$L\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$$

$$= \int_0^a e^{-st}\cancel{f(t)}^{\,0}\,dt + \int_a^\infty e^{-st}f(t)\,dt$$

$$L\{f(t)\} = \int_a^\infty e^{-st}f(t)\,dt$$

Let, $t - a = u$

$\therefore t = u + a$

$dt = du$

$\therefore t = a,\ u = 0$

$t = \infty,\ u = \infty$

$$\Rightarrow L\{f(t)\} = \int_0^\infty e^{-s(u+a)} f(u)\,du = e^{-sa}\int_0^\infty e^{-su} f(u)\,du$$

$$L\{f(t)\} = e^{-as} L\{f(u)\}$$

$$L\{f(t)\} = e^{-as} F(s).$$ Proved.

Q. State and prove the convolution theorem.

Statement: If $L^{-1}[F(s)] = f(t)$ and

$L^{-1}[G(s)] = g(t)$ then

$$L^{-1}[F(s).G(s)] = \int_0^t f(u)\,g(t-u)\,du = (f*g)(t)$$

Proof: If $L^{-1}[F(s).G(s)] = \int_0^t f(u)\,g(t-u)\,du$

$$\Rightarrow F(s).G(s) = L\left\{\int_0^t f(u)\,g(t-u)\,du\right\}$$

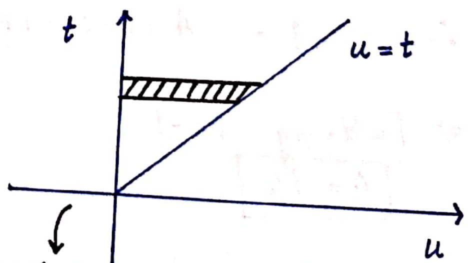
Using definition of Laplace transformation:

$$\therefore L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\Rightarrow L\left\{\int_0^t f(u)\,g(t-u)\,du\right\} = \int_{t=0}^{t=\infty} e^{-st}\left(\int_{u=0}^{u=t} f(u)\,g(t-u)\,du\right)dt.$$

Now, evaluating the integral with the help of change of order of an integration.

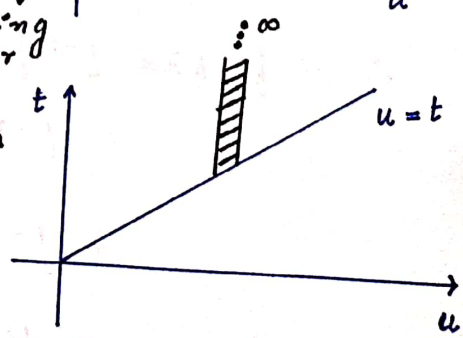
$$= \int_{t=0}^{t=\infty}\int_{u=0}^{u=t} e^{-st} f(u)\,g(t-u)\,du\,dt$$



On changing order

On changing order we obtained:

$$= \int_{u=0}^{u=\infty}\int_{t=u}^{t=\infty} e^{-st} f(u)\,g(t-u)\,dt\,du$$



Now, let

$$t - u = z$$
$$dt = dz$$

$$\int_{u=0}^{u=\infty}\int_{t=u}^{t=\infty} e^{-st} f(u)\, g(t-u)\, dt\, du = \int_{u=0}^{\infty}\int_{z=0}^{\infty} e^{-s(u+z)} f(u)\, g(z)\, dz\, du$$

$$= \int_{u=0}^{u=\infty}\int_{z=0}^{z=\infty} e^{-su} f(u) \cdot e^{-sz} g(z)\, dz\, du$$

$$= \left[\int_{u=0}^{\infty} e^{-su} f(u)\, du\right] \times \left[\int_0^{\infty} e^{-sz} g(z)\, dz\right]$$

$$L\left[\int_0^t f(u)\, g(t-u)\, du\right] = F(s) \cdot G(s) \quad \underline{\underline{\text{Ans.}}}$$

<u>Q.</u> By using convolution theorem, solve;

$$L^{-1}\left\{\frac{1}{(s+1)(s+2)(s+3)(s+4)}\right\}$$

Let $F(s) = \dfrac{1}{(s+1)(s+2)(s+3)}$

Using partial fractions;

$$\therefore \frac{1}{(s+1)(s+2)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{(s+3)}$$

$$\therefore \quad 1 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2)$$

$\Rightarrow$ Putting $s = -1$ $\boxed{A = 1/2}$

$\Rightarrow$ Putting $s = -2$ $\boxed{B = -1}$

$\Rightarrow$ Putting $s = -3$ $\boxed{C = 1/2}$

$$\therefore \quad f(t) = L^{-1}\{F(s)\}$$
$$= L^{-1}\left\{\frac{1}{(s+1)(s+2)(s+3)}\right\}$$
$$= L^{-1}\left\{\frac{1/2}{s+1} - \frac{1}{s+2} + \frac{1/2}{s+3}\right\}$$
$$f(t) = \tfrac{1}{2} e^{-t} - e^{-2t} + \tfrac{1}{2} e^{-3t}.$$

$$\therefore \quad G(s) = \frac{1}{s+4}$$

$$g(t) = L^{-1}\{G(s)\} = L^{-1}\left\{\frac{1}{s+4}\right\}$$

$$g(t) = e^{-4t}$$

$$\therefore \quad g(t-u) = e^{-4(t-u)}.$$

$$\therefore \quad f * g = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \left(\frac{1}{2} e^{-u} - e^{-2u} + \tfrac{1}{2} \cdot e^{-3u}\right) . \, e^{-4t+4u} du$$

$$= \int_0^t \left(\frac{1}{2} e^{-4t+3u} - e^{-4t+2u} + \tfrac{1}{2} e^{u-4t}\right) du$$

$$= \left[\frac{1}{2} \frac{e^{-4t+3u}}{3} - \frac{e^{-4t+2u}}{2} + \frac{1}{2} \frac{e^{u-4t}}{1}\right]_0^t$$

$$L^{-1}\left\{\frac{1}{(s+1)(s+2)(s+3)(s+4)}\right\} = \frac{1}{6} e^{-t} - \frac{e^{-2t}}{2} + \frac{1}{2} e^{-3t} - \frac{1}{6} e^{-4t} \quad \text{Ans.}$$

Q. Solve the differential equation using Laplace Transformation;

$$\frac{d^2y}{dt} + 6\frac{dy}{dt} + 9y(t) = \cos(t), \quad \begin{matrix} y(0) = 0. \\ y'(0) = 1. \end{matrix}$$

$\therefore$ Using Laplace Transformation both sides;

$$L\left\{\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y(t)\right\} = L\{\cos t\}$$

$$L\{y''\} + 6L\{y'\} + 9L\{y\} = \frac{s}{s^2+1}$$

$$\{s^2 L\{y\} - \cancel{y(0)}^{0} - \cancel{y'(0)}^{1}\} + 6\{sL\{y\} - \cancel{y(0)}^{0}\} + 9L\{y\} = \frac{s}{s^2+1}$$

$$s^2 L\{y\} - 1 + 6sL\{y\} + 9L\{y\} = \frac{s}{s^2+1}$$

$$L\{y\}[s^2 + 6s + 9] - 1 = \frac{s}{s^2+1}$$

$$(s^2+6s+9)\, L\{y\} = \frac{s^2+s+1}{s^2+1}$$

$$L\{y\} = \frac{s^2+s+1}{(s^2+1)(s^2+6s+9)} = \frac{s^2+s+1}{(s^2+1)(s+3)^2}$$

$$y = L^{-1}\left\{\frac{s^2+s+1}{(s^2+1)(s+3)^2}\right\}$$

Using partial fractions:

$$\frac{s^2+s+1}{(s^2+1)(s+3)^2} = \frac{As+B}{s^2+1} + \frac{C}{s+3} + \frac{D}{(s+3)^2}$$

$$s^2+s+1 = s^3(A+C) + s^2(6A+B+3C+D) + s(9A+6B+C) + 9B+3C+D$$

On solving, we get:

$A = 2/25$

$B = 3/50$

$C = -2/25$

$D = 7/10$

$$y = L^{-1}\left\{\frac{2/25\, s + 3/50}{s^2+1} + \frac{-2/25}{s+3} + \frac{7/10}{(s+3)^2}\right\}$$

$$= \frac{2}{25} L^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{3}{50} L^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{2}{25} L^{-1}\left\{\frac{1}{s+3}\right\} + \frac{7}{10} L^{-1}\left\{\frac{1}{(s+3)^2}\right\}$$

$$y = \frac{2}{25}\cos t + \frac{3}{50}\sin t - \frac{2}{25}e^{-3t} + \frac{7}{10}te^{-3t}.$$ Ans.