LINEAR ALGEBRA

Matrix is an arrangement of a mn elements in rectangular array form where 'm' represent rows and 'n' represent coloumns.

→ Upper Triangular Matrix

A matrix $[a_{ij}]_{m\times n}$ is said to be upper triangular if $a_{ij}=0$ for $i>j$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

→ Lower Triangular Matrix

A matrix $[a_{ij}]_{m\times n}$ is said to be lower triangular if $a_{ij}=0$ for $i<j$.

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

→ Nilpotent Matrix

A matrix $[a_{ij}]_{m\times n}$ is said to be nilpotent if $A^k=0$, where k is a positive integer.

$$\therefore \quad A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

$$\Rightarrow \quad A^2 = AA = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} a^2b^2-a^2b^2 & ab^3-ab^3 \\ -a^3b+a^3b & -a^2b^2+a^2b^2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad K = 2.$$

order is 2

→ Idompotent Matrix

A matrix is said to be idompotent matrix if $A^2 = A$

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 2 \\ 1 & -2 & -3 \end{bmatrix}$$

→ Complex Matrix

A matrix $[a_{ij}]_{m \times n}$ is said to be complex matrix if elements of the matrix (at least one) is in complex form.

$$A = \begin{bmatrix} 2 & c+id \\ a+ib & 3 \end{bmatrix}$$

→ Complex Congugate Matrix

A matrix $[a_{ij}]_{m \times n}$ is said to be complex congugate matrix if we occur congugate of the complex element, i.e

$$a_{ij} = \overline{a_{ij}}$$

→ Hermitian Matrix

A complex matrix is said to be Hermitian matrix if

$$a_{ij} = \overline{a_{ij}}^{T}.$$

Ex :

$$A = \begin{bmatrix} 2 & a+ib & c+id \\ a-ib & 3 & e+if \\ c-id & e-if & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 2 & a-ib & c-id \\ a+ib & 3 & e-if \\ c+id & e+if & 0 \end{bmatrix}$$

$$\bar{A}^{T} = \begin{bmatrix} 2 & a+ib & c+id \\ a-ib & 3 & e+if \\ c-id & e-if & 0 \end{bmatrix}$$

$\Rightarrow \quad \bar{A}^T = A \quad ; \quad A \rightarrow$ Hermitian Matrix

→ Skew Hermitian Matrix

A complex matrix is said to be skew hermitian matrix, if $a_{ij} = -\overline{a_{ij}}^{T}$

NOTE: The principal diagonal of the skew hermitian matrix either 0 or pure complex

Ex:
$$A = \begin{bmatrix} 0 & a+ib & c+id \\ -(a-ib) & 0 & e+if \\ -(c-id) & -(e-if) & 0 \end{bmatrix}$$

$$\therefore \bar{A} = \begin{bmatrix} 0 & a-ib & c-id \\ -(a+ib) & 0 & e-if \\ -(c+id) & -(e+if) & 0 \end{bmatrix} = -\bar{A} = \begin{bmatrix} 0 & -(a-ib) & -(c-id) \\ a+ib & 0 & -(e-if) \\ c+id & e+if & 0 \end{bmatrix}$$

$$\therefore -\bar{A}^{T} = \begin{bmatrix} 0 & a+ib & c+id \\ -(a-ib) & 0 & e+if \\ -(c-id) & -(e-if) & 0 \end{bmatrix} = A.$$

↳ Skew hermitian matrix

→ Rank

Matrix

Singular Matrix

If the determinant of given matrix $A = 0$

$|A| = 0$

Non-singular matrix

If the determinant of given matrix $A \neq 0$

$|A| \neq 0$

Rank: If we calculate the determinant of the matrix and find the value of determinant equal to zero.

$$AX = B$$
$$X = A^{-1}B, \quad A \neq 0$$

If $|A| = 0$ (Rank of Matrix)

Definition : Highest order of the ~~matrix~~ non-zero minor of the given matrix is called rank of matrix.

$$A = \begin{vmatrix} 1 & 1 & 3 \\ 4 & 2 & 6 \\ 2 & 1 & 3 \end{vmatrix}_{3\times 3} = 1(6-6) - 12(1-1) + 3(4-4) = 0$$

$$M = \begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix}_{2\times 2} = 2-4 = -2 \neq 0.$$

Rank = 2

→ Echelon Form

AB → post multiplication (column) (N)

pre multiplication (row)

1. Entry should be equal to 1. (First element of matrix)
2. First row deal with second and third row } same for column
3. Second row deal with third row

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Step ① : $R_1 \to R_2$ and $R_3 \to \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$

$R_2 \to R_3$

⇓

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$ → May or may not be equal zero.

⇒ No. of non-zero rows = Rank of Matrix

Q. Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 2 & 1 \\ 3 & 2 & 1 & 2 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

Sol. By using elementary transformation

$$\therefore \quad R_2 \rightarrow R_2 - 2R_1$$
$$R_3 \rightarrow R_3 - 3R_1$$
$$R_4 \rightarrow R_4 - 6R_1$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & -4 & -8 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & -4 & 0 & 0 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$$R_4 \rightarrow R_4 \quad R_3$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -11 & 5 \end{bmatrix}$$

Rank of Matrix = No. of non-zero raw = 4.

Q. Find the ~~mo~~ rank of matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

Sol.

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 + R_1$

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 5 & -3 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, Rank = 2.

Rank → Echelon Form
- ↳ Row ET → upper Δ mat
- ↳ Column ET → lower Δ mat

diagonal matrix ← $D = I_1 A I_2$

$I_1 \rightarrow$ row ET , $I_2 \rightarrow$ col ET

Always use equivalent sign ~

$a_{33} \longrightarrow$ zero or non-zero

→ Linear system of equation

(i) Homogenous $b_1 = b_2 = b_3 = 0$ (c always)

(ii) non-Homogenous $b_1 = b_2 = b_3 \neq 0$ (c or ic)

→ Augmented Matrix

Role: Solution for L.S.E

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times1} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times1}$$

$$A.X = B$$

→ If 'A' is invertible (inverse)

then $X = A^{-1}B$, $|A| \neq 0$

→ If det $|A| = 0$, then how to cal^n? → Augmented matrix

Notation : $[A|B]$

$$= \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

→ Rank of augmented matrix = c = $R[A|B]$

Note :→ Notation of rank of mat; $r(A)$ or $\rho(A)$

→ Consistent and Inconsistent

Consistent ↓ Solution

You can calculate value of unknown

Inconsistent ↓ No solution

You can't calculate value of unknown

$$\left.\begin{array}{l} \text{Rank of augmented matrix } [A|B] = \text{ROM } A \\ R[A|B] = r(A) \\ c = r(A) = \rho(A) \end{array}\right\} C$$

$$\left.\begin{array}{l} \text{Rank of ag. mat } \neq \text{ROM } A \\ c \neq r(A) \text{ or } \rho(A) \end{array}\right\} IC$$

→ Nature of solution of LSE

Solution → unique
Solution → infinite number of solution

Unique → no. of unknown = ROM

$n = r(A)$

if no. of sol → $n > r(A)$
infinite solution

Q. Solve the system of equations

$$x + y + z = -3$$
$$3x + y - 2z = -2$$
$$2x + 4y + 7z = -7$$

Sol.

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ -7 \end{bmatrix}$$

augmented matrix →
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & -7 \end{array}\right]$$

By using echelon form and applying ET

$$R_2 \to R_2 - 3R_1$$
$$R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 6 & 26 \end{array}\right]$$

Rank of mat = 3

$C = R[A|B] = 3$

$r(A) = 2$ IC system

Solve the linear equations

$$4x - 2y + 6z = 8$$
$$x + y - 3z = -1$$
$$15x - 3y + 9z = 21.$$

Given:

System of equations can be expressed as:

$$AX = B$$

$$\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

Using echelon form to calculate rank of augmented matrix

$$\therefore \quad [A|B] = \left[\begin{array}{ccc|c} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

By using elementary row transformation:

$$R_2 \to R_2 - 4R_1$$
$$R_3 \to R_3 - 15R_1$$

$$[A|B] \sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{array}\right]$$

$$R_3 \to R_3 - 3R_2$$

$$[A|B] \sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank of augmented matrix $[A|B]$

= No. of non-zerow rows

= 2.

And,

Rank of matrix A = No. of non-zero rows

= 2

$R[A|B] = r(A)$

Hence, ~~the~~ system is consistent

Rank of matrix $A = 2$

No. of unknown = 3

$\therefore \quad n > r$ (Infinite no. of solution)

→ Eigenvalues and Eigenvectors
(Characteristic values and characteristic vectors)

1. Characteristic Matrix (Eigen Matrix)
2. Characteristic polynomial
3. Characteristic equation
4. Characteristic roots.

$$\text{Vector} \to \text{Column vector}, \quad \text{Row vector}$$

Eigenvalues

If $A = [a_{ij}]_{m \times n}$ be a square matrix of order 'n', λ is in indeterminant form with identity matrix, then $a - \lambda i$ is called characteristic matrix.

$$\therefore \quad A - \lambda I = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If we take determinant of the characteristic matrix we arrive at characteristics polynomial.

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \det(A - \lambda I)$$

If we equate characteristic polynomial with 0, we obtain characteristic equation.

$$\det(A - \lambda I) = \text{characteristic polynomial} = 0$$

$$\text{Characteristic equation} = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

If we calculate characteristic equation $|A - \lambda I| = 0$, we get the roots of the characteristic equation, obtained roots are called Eigenvalues or characteristic roots or latent roots.

NOTE: The set of eigenvalues is called spectrum.

Eigenvector (Characteristic vector)

Let A be a square matrix,

$$A - \lambda I = 0,$$

if X is any vector (column vector), then the eigen-vector is defined as $(A - \lambda I)X = 0$, where O is null matrix.

Q. Find the eigenvalue or the characteristic value of the matrix :

$$A = \begin{bmatrix} 5 & 2 \\ 0 & 2 \end{bmatrix}$$

∴ To determine eigenvalue of the given matrix, 'A'; We will construct characteristic matrix, which is given as;

$$A - \lambda I = \begin{bmatrix} 5-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix}$$

Now, characteristic polynomial;

∴ $|A - \lambda I| = (5-\lambda)(2-\lambda) = \det(A - \lambda I)$

Characteristic equation is given as;

$|A - \lambda I| = 0$

∴ $(5-\lambda)(2-\lambda) = 0$

$\Rightarrow \lambda = 5, 2 \rightarrow$ These are the eigenvalues of the given matrix.

Now, Eigenvector X_1 corresponding to eigenvalue $\lambda_1 = 5$

∴ $(A - \lambda I)X = 0$

$$\begin{bmatrix} 5-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

As $\lambda = 5$

$$\Rightarrow \begin{bmatrix} 0 & 2 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

∴ $2x_2 = 0$

$0x_1 - 3x_2 = 0$

$\Rightarrow x_1 = k$

$x_2 = 0$

Hence;

$$X_1 = \begin{bmatrix} k \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Eigenvector X_2 corresponding to eigenvalue $X_2 = 2$.

$\therefore \quad (A - \lambda I)X = 0$

$$\begin{bmatrix} 5-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As, $\lambda = 2$

$$\begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad 3x_1 + 2x_2 = 0$

$\Rightarrow \quad x_1 = 2$

$x_2 = -3$

Hence

$$X_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$ Ans.

□ NOTE: If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A. Then, $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ are eigenvalues of A^{-1}.

Q. Find the eigenvalue and eigenvector of the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

Characteristic Matrix:

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix}$$

Now, characteristic polynomial

$$|A - \lambda I| = (3-\lambda)(2-\lambda) - 2$$

$$= 6 - 3\lambda - 2\lambda + \lambda^2 - 2$$

$$|A - \lambda I| = \lambda^2 - 5\lambda + 4$$

Characteristic equation;

$$|A - \lambda I| = 0$$

$\therefore \quad \lambda^2 - 5\lambda + 4 = 0$

$\lambda^2 - 4\lambda - \lambda + 4 = 0$

$\lambda(\lambda - 4) - 1(\lambda - 4) = 0$

$(\lambda - 1)(\lambda - 4) = 0$

Hence, $\lambda = 1, 4 \rightarrow$ These are the eigenvalues of the given matrix

Now, Eigenvector X_1 corresponding to Eigenvalue $\lambda_1 = 1$.

$$\Rightarrow \quad (A - \lambda I)X = 0$$

$$\begin{bmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad \lambda = 1$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x_1 + x_2 = 0$

$x_1 = 1$

$x_2 = -2$

$$\therefore \quad X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{Ans.}$$

and; Eigenvector X_2 corresponding to Eigenvalue $\lambda_2 = 4$.

$$\Rightarrow \quad (A - \lambda I)X = 0$$

$$\begin{bmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = 4$

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0$

$\Rightarrow x_1 = x_2 = k$

$$\therefore \quad X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ where } k \neq 0. \quad \text{Ans.}$$

Q. Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 9 & 3 \end{bmatrix}$$

Characteristic Matrix ;

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 9 & 3-\lambda \end{bmatrix}$$

Characteristics polynomial ;

$$|A - \lambda I| = (3-\lambda)^2 - 9 = \lambda^2 - 6\lambda$$

Characteristics equation ;

$$|A - \lambda I| = 0$$

$$\lambda^2 - 6\lambda = 0$$

$\lambda = 0, 6 \longrightarrow$ These are the eigenvalues of the given matrix.

$\therefore$ Eigenvector X_1 corresponding to eigenvalue $\lambda_1 = 0$.

$$\therefore \quad (A - \lambda I)X = 0$$

$$\begin{bmatrix} 3-\lambda & 1 \\ 9 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As ; $\lambda = 0$

$$\begin{bmatrix} 3 & 1 \\ 9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x_1 + x_2 = 0$$

$\therefore$ $x_1 = 1$
$x_2 = -3$

Hence,

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

and, eigenvector X_2 corresponding to eigenvalue $\lambda_2 = 6$.

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 3-\lambda & 1 \\ 9 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad \lambda = 6$

$$\begin{bmatrix} -3 & 1 \\ 9 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3x_1 + x_2 = 0$$

$x_1 = 1$
$x_2 = 3$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$ Ans.

Q. Find eigenvalue and eigenvector of the matrix

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$\therefore$ Characteristics Matrix:

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix}$$

Characteristics Polynomial:

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} = (4-\lambda)^2$$

Characteristics equation:

$|A - \lambda I| = 0$

$(4-\lambda)^2 = 0$

$\lambda = 4, 4 \rightarrow$ These are the eigenvalues of the given matrix.

As, eigenvalues are repeated and matrix is symmetrical.
Hence; only ~~=~~ one eigenvector exist for this matrix;
Eigenvector X corresponding to eigenvalue $\lambda = 4$.

$$(A - \lambda I)X = 0$$

$$\therefore \begin{bmatrix} 4-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 0 \cdot x_1 + 0 \cdot x_2 = 0$$

$$\therefore \quad x_1 = x_2 = k, \quad \text{where } k \neq 0.$$

Hence;

$$X = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad \text{Ans.}$$

→ Cayley-Hamilton Theorem

Every square matrix satisfies it's own characteristics equation.

Q. Verify the Cayley-Hamilton theorem for the given matrix:

$$A = \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix}$$

Constructing characteristic matrix:

$$A - \lambda I = \begin{bmatrix} 5-\lambda & 1 \\ 3 & 2-\lambda \end{bmatrix}$$

Characteristic polynomial:

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda) - 3$$

Characteristic ~~po~~ equation:

$$(5-\lambda)(2-\lambda) - 3 = 0$$

$$10 - 5\lambda - 2\lambda + \lambda^2 - 3 = 0$$

$$\lambda^2 - 7\lambda + 7 = 0$$

Replacing 'λ' with matrix A

$$\therefore \quad A^2 - 7A + 7I = 0$$

$$\Rightarrow \quad A^2 = \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 25+3 & 5+2 \\ 15+6 & 3+4 \end{bmatrix} = \begin{bmatrix} 28 & 7 \\ 21 & 7 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 28 & 7 \\ 21 & 7 \end{bmatrix} - \begin{bmatrix} 35 & 7 \\ 21 & 14 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{Verified.}$$

Hence, matrix A satisfies it's own characteristic equation.

Find A^{-1}.

$\therefore \quad A^2 + (-7)A + 7I = 0$

Pre-multiplying with A^{-1}.

$$A - 7I + 7A^{-1} = 0$$

$$\therefore \quad A^{-1} = \frac{1}{7}(7I - A)$$

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \quad \text{Ans.}$$

Q. Verify the Cayley-Hamilton theorem for the matrix:

$$A = \begin{bmatrix} 6 & 3 \\ 4 & 1 \end{bmatrix}$$

$$\therefore \quad A - \lambda I = \begin{bmatrix} 6-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix}$$

$$\Rightarrow \quad |A - \lambda I| = (6-\lambda)(1-\lambda) - 12$$

$$= 6 - 6\lambda - \lambda + \lambda^2 - 12$$

$$= \lambda^2 - 7\lambda - 6$$

Characteristics equation:

$$|A - \lambda I| = 0$$

$$\lambda^2 - 7\lambda - 6 = 0$$

Replacing λ with matrix 'A'.

$$A^2 - 7A - 6I = 0$$

$$\therefore \quad A^2 = A \cdot A = \begin{bmatrix} 6 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 6 & 3 \\ 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 36+12 & 18+3 \\ 24+4 & 12+1 \end{bmatrix} = \begin{bmatrix} 48 & 21 \\ 28 & 13 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 48 & 21 \\ 28 & 13 \end{bmatrix} - \begin{bmatrix} 42 & 21 \\ 28 & 7 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, matrix 'A' satisfies it's own characteristic equation.

Verified

Find A^{-1}.

$$A^2 - 7A - 6I = 0$$

$\therefore$ Pre-multiplying by A^{-1}.

$$A - 7I - 6A^{-1} = 0$$

$$\therefore A^{-1} = \frac{1}{6}(A - 7I).$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} -1 & 3 \\ 4 & -6 \end{bmatrix}$$ Ans.

→ Diagonilization:

To represent the diagonalization of any matrix A i.e

$$D = P^{-1}AP$$

Condition should be exist, where P is model matrix.

Q. What is model matrix?

A matrix which is construct with the help of the eigenvector is called model matrix.

Ex: Find the diagonalization of matrix A.

$$A = \begin{bmatrix} 5 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\therefore A - \lambda I = \begin{bmatrix} 5-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix}$$

$$\therefore |A - \lambda I| = \begin{vmatrix} 5-\lambda & 2 \\ 0 & 2-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda)$$

$$\Rightarrow |A - \lambda I| = 0$$

$$(5-\lambda)(2-\lambda) = 0$$

$\therefore \lambda = 2, 5 \longrightarrow$ Eigenvalues of the given matrix A.

$\therefore$ ~~Ei~~ Eigenvector X_1 corresponding to eigenvalue $\lambda_1 = 2$

$$\therefore (A - \lambda I)X = 0$$

$$\begin{bmatrix} 5-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x_1 + 2x_2 = 0$$

$$\therefore x_1 = 2$$
$$x_2 = -3$$

$$X_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

~~Eigenvalue~~

Eigenvector X_2 corresponding to eigenvalue $\lambda_2 = 5$

$\therefore \quad (A - \lambda I)X = 0$

$$\begin{bmatrix} 5-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 0 & 2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$0 \cdot x_1 + 2x_2 = 0$$
$$0 \cdot x_1 - 3x_2 = 0$$

$$\therefore \quad x_1 = k$$
$$x_2 = 0$$

$$X_2 = k \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

To construct model matrix (P)

$$P = [X_1 \quad X_2]$$

$$P = \begin{bmatrix} 2 & k \\ -3 & 0 \end{bmatrix}$$

$$\therefore \quad |P| = 3k$$

$$adj(P) = \begin{bmatrix} 0 & -k \\ 3 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{adj(P)}{|P|} = \frac{1}{3k} \begin{bmatrix} 0 & -k \\ 3 & 2 \end{bmatrix}$$

$$D = P^{-1} A P$$

$$= \frac{1}{3k} \begin{bmatrix} 0 & -k \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & k \\ -3 & 0 \end{bmatrix}$$

$$= \frac{1}{3k} \begin{bmatrix} 0 & -2k \\ 15 & 10 \end{bmatrix} \begin{bmatrix} 2 & k \\ -3 & 0 \end{bmatrix} = \frac{1}{3k} \begin{bmatrix} 6k & 0 \\ 0 & 15k \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$ Ans.

Principal matrix of D = Eigenvalue of A.

Find D^8

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} ; \quad D^8 = \begin{bmatrix} 2^8 & 0 \\ 0 & 5^8 \end{bmatrix}$$ Ans.

Q. Find the diagonalization of matrix :

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$$

$\therefore \quad A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 4 & 3-\lambda \end{bmatrix}$

$$|A - \lambda I| = (3-\lambda)^2 - 4$$
$$= 9 + \lambda^2 - 6\lambda - 4$$
$$= \lambda^2 - 6\lambda + 5$$

$|A - \lambda I| = 0$

$\therefore \quad \lambda^2 - 6\lambda + 5 = 0$

$\lambda^2 - 5\lambda - \lambda + 5 = 0$

$\lambda(\lambda - 5) - 1(\lambda - 5) = 0$

$(\lambda - 1)(\lambda - 5) = 0$

$\lambda = 1, 5 \longrightarrow$ Eigenvalues of the given matrix A.

Eigenvector X_1 corresponding to eigenvalue $\lambda_1 = 1$

$\therefore \quad (A - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & 1 \\ 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x_1 + x_2 = 0$

$x_1 = 1$
$x_2 = -2$

$$X_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Eigenv~~alue~~ector X_2 corresponding to eigenvalue $\lambda_2 = 5$

$$(A - \lambda I)X = 0$$

$$\therefore \begin{bmatrix} 3-\lambda & 1 \\ 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad -2x_1 + x_2 = 0$$

$$x_1 = 1$$

$$x_2 = 2,$$

$$X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\therefore$ Now;

$$P = [X_1 \quad X_2]$$

$$P = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$$

$$|P| = 2 + 2 = 4$$

$$adj(P) = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{4} \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$$

$$\therefore \quad D = P^{-1} A P$$

$$= \frac{1}{4} \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 6-4 & 2-3 \\ 6+4 & 2+3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 2 & -1 \\ 10 & +5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$$

$$D = \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 0 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{Ans.}$$

Solution of Differential Equations using Matrix Method.

Q.

$$y_1' = -2y_1 + y_2$$

$$y_2' = y_1 - 2y_2$$

Let us consider;

$$Y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.

$\therefore$ $Y' = AY$ — ①

$\Rightarrow$ $Y' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} Y$

Let $y = e^{\lambda t} X$ be the solution of ①.

$\therefore$ $\lambda . e^{\lambda t} X = A . e^{\lambda t} X$

$\therefore$ $\lambda X = AX$

Hence,

$$(A - \lambda I) X = 0$$

Calculating eigenvalues ;

$$\det(A - \lambda I) = 0$$

$\therefore$ $\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = 0$

$$(-2-\lambda)^2 - 1 = 0$$
$$4 + \lambda^2 + 4\lambda - 1 = 0$$
$$\lambda^2 + 4\lambda + 3 = 0$$
$$\lambda^2 + 3\lambda + \lambda + 3 = 0$$

$$\lambda(\lambda + 3) + 1(\lambda + 3) = 0$$
$$(\lambda + 1)(\lambda + 3) = 0$$
$$\lambda = -3 \text{ and } -1.$$

$\therefore$ Eigenvector X_1 corresponding to eigenvalue $\lambda_1 = -3$.

$$(A - \lambda I)X = 0$$

$$\Rightarrow \begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} X = 0$$

$$\therefore \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 + x_2 = 0$$

$$\Rightarrow x_1 = 1 \text{ and } x_2 = -1.$$

Hence;
$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

and, Eigenvector X_2 corresponding to eigenvalue $\lambda_2 = -1$.

$$\Rightarrow (A - \lambda I)X = 0$$

$$\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore -x_1 + x_2 = 0$$

$$\Rightarrow x_1 = 1 \text{ and } x_2 = 1$$

Hence,
$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore,

Solution of the given differential equation;

$$y = e^{\lambda_1 t} x^{(1)} + e^{\lambda_2 t} x^{(2)}$$

$$y = e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad y_1 = e^{-3t} + e^{-t}.$$

$$y_2 = -e^{-3t} + e^{-t}. \qquad \text{Ans.}$$

Q.

$$y_1' = 5y_1 + 22y_2$$

$$y_2' = y_1 + 2y_2$$

$\therefore$ Consider

$$Y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 5 & 22 \\ 1 & 2 \end{bmatrix}$$

$$\therefore \quad (A - \lambda I)X = 0$$

$$\Rightarrow \begin{bmatrix} 5-\lambda & 22 \\ 1 & 2-\lambda \end{bmatrix} X = 0$$

$\therefore$ Eigenvalues are given as;

$$\det(A - \lambda I) = 0$$

$$\therefore \begin{vmatrix} 5-\lambda & 22 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 22 = 0$$

$$10 - 5\lambda - 2\lambda + \lambda^2 - 22 = 0$$

$$\lambda^2 - 7\lambda - 12 = 0$$

$$\therefore \quad \lambda = \frac{7 \pm \sqrt{97}}{2}$$

$\therefore$ Eigenvector X_1 corresponding to eigenvalues $\lambda_1 = \frac{7}{2} + \frac{\sqrt{97}}{2}$

$$\therefore \quad (A - \lambda I) X = 0$$

$$\begin{bmatrix} 5-\lambda & 22 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \frac{3}{2} - \frac{\sqrt{97}}{2} & 22 \\ 1 & -\frac{3}{2} - \frac{\sqrt{97}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \left(\frac{3}{2} - \frac{\sqrt{97}}{2}\right) x_1 + 22 x_2 = 0$$

$$\therefore \quad x_1 = 22 \quad \text{and} \quad x_2 = -\frac{3}{2} + \frac{\sqrt{97}}{2}$$

Hence, $\quad X_1 = \begin{bmatrix} 22 \\ -\frac{3}{2} + \frac{\sqrt{97}}{2} \end{bmatrix}$

and eigenvector X_2 corresponding to eigenvalues $\lambda_2 = \frac{7}{2} - \frac{\sqrt{97}}{2}$.

$$\therefore \quad (A - \lambda I) X = 0$$

$$\begin{bmatrix} 5-\lambda & 22 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \frac{3}{2} + \frac{\sqrt{97}}{2} & 22 \\ 1 & -\frac{3}{2} + \frac{\sqrt{97}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad \left(\frac{3}{2} + \frac{\sqrt{97}}{2}\right) x_1 + 22 x_2 = 0$$

$$x_1 = 22 \quad \text{and} \quad x_2 = -\frac{3}{2} - \frac{\sqrt{97}}{2}.$$

Hence $\quad X_2 = \begin{bmatrix} 22 \\ -3/2 - \frac{\sqrt{97}}{2} \end{bmatrix}$

There fore,
Solution to the given differential equations ;

$$y = e^{\lambda_1 t} x^{(1)} + e^{\lambda_2 t} x^{(2)}$$

$$\therefore \quad y = e^{\left(\frac{7+\sqrt{97}}{2}\right)t} \begin{bmatrix} 22 \\ -3/2 + \frac{\sqrt{97}}{2} \end{bmatrix} + e^{\left(\frac{7-\sqrt{97}}{2}\right)t} \begin{bmatrix} 22 \\ -3/2 - \sqrt{97}/2 \end{bmatrix}$$

$$y_1 = 22\, e^{\left(\frac{7+\sqrt{97}}{2}\right)t} + 22\, e^{\left(\frac{7-\sqrt{17}}{2}\right)t}$$

$$y_2 = \left(-\frac{3}{2} + \frac{\sqrt{97}}{2}\right) e^{\left(\frac{7+\sqrt{97}}{2}\right)t} + \left(-\frac{3}{2} - \frac{\sqrt{97}}{2}\right) e^{\left(\frac{7-\sqrt{97}}{2}\right)t} \text{. Ans.}$$

Q. Solve the following differential equations using matrix method.

$$y_1' = 4y_1 + 3y_2$$
$$y_2' = 2y_1 + y_2$$

$$\therefore \quad A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

Consider, $Y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\therefore \quad (A - \lambda I)X = 0$$

For eigenvalues ;

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 4-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(1-\lambda) - 6 = 0$$
$$4 - 4\lambda - \lambda + \lambda^2 - 6 = 0$$
$$\lambda^2 - 5\lambda - 2 = 0$$
$$\lambda = \frac{5 \pm \sqrt{25+8}}{2} = \frac{5}{2} \pm \frac{\sqrt{33}}{2}$$

$\therefore$ Eigenvector X_1 corresponding to eigenvalue

$$\lambda_1 = \frac{5}{2} + \frac{\sqrt{33}}{2}.$$

$$\therefore \quad (A - \lambda I)X = 0$$

$$\begin{bmatrix} 4-\lambda & 3 \\ 2 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2} - \frac{\sqrt{33}}{2} & 3 \\ 2 & -\frac{3}{2} - \frac{\sqrt{33}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad \left(\frac{3}{2} - \frac{\sqrt{33}}{2}\right)x_1 + 3x_2 = 0$$

$$x_1 = 3 \quad \text{and} \quad x_2 = -\frac{3}{2} + \frac{\sqrt{33}}{2}$$

Hence, $$X_1 = \begin{bmatrix} 3 \\ \frac{3}{2} + \frac{\sqrt{33}}{2} \end{bmatrix}$$

and eigenvector X_2 corresponding to eigenvalue

$$\lambda_2 = \frac{5}{2} - \frac{\sqrt{33}}{2}.$$

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} \frac{3}{2} + \frac{\sqrt{33}}{2} & 3 \\ 2 & -\frac{3}{2} + \frac{\sqrt{33}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad \left(\frac{3}{2} + \frac{\sqrt{33}}{2}\right)x_1 + 3x_2 = 0$$

$$\therefore \quad x_1 = 3 \quad \text{and} \quad x_2 = -3/2 - \frac{\sqrt{33}}{2}$$

Hence, $$X_1 = \begin{bmatrix} 3 \\ -\frac{3}{2} - \frac{\sqrt{33}}{2} \end{bmatrix}$$

Therefore,

Solution of the given differential equations;

$$\cancel{X_1} \quad y = e^{\lambda_1 t} x^{(1)} + e^{\lambda_2 t} x^{(2)}$$

$$y = e^{\left(\frac{5+\sqrt{33}}{2}\right)t} \begin{bmatrix} 3 \\ -\frac{3}{2} + \frac{\sqrt{33}}{2} \end{bmatrix} + e^{\left(\frac{5-\sqrt{33}}{2}\right)t} \begin{bmatrix} 3 \\ -\frac{3}{2} - \frac{\sqrt{33}}{2} \end{bmatrix}$$

$$y_1 = 3e^{\left(\frac{5+\sqrt{33}}{2}\right)t} + 3e^{\left(\frac{5-\sqrt{33}}{2}\right)t}$$

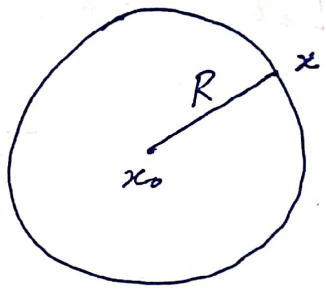
$$y_2 = \left(-\frac{3}{2} + \frac{\sqrt{33}}{2}\right)e^{\left(\frac{5+\sqrt{33}}{2}\right)t} + \left(-\frac{3}{2} - \frac{\sqrt{33}}{2}\right)e^{\left(\frac{5-\sqrt{33}}{2}\right)t}$$ Ans.

Power Series

An expression;

$$\sum_{n=0}^{\infty} (x - x_0)^n z^n$$ is known as Power Series,

where x is the center of of the circle. $(z \in C)$



→ $|x - x_0| = R$
equation of circle.

→ $|x - x_0| < R$
Convergence

→ $|x - x_0| > R$
Divergence.

To calculate radius of convergence;
We use;
① Cauchy n^{th} root test
② Ratio Test.

R is called radius of convergence.

→ Application of the power series in differential equations.
To find solution of differential equation in series form.

Q. To find the series solution by using Fiboni method of the differential equation of;

$$2x^2y'' + xy' - (x^2+1)y = 0 \quad \text{— ①}$$

Given;

Let $y = \sum_{m=0}^{\infty} C_m x^{m+r}$ is series solution of the given differential equation.

Regular Singular Point

$$2(x-x_0)^2y'' + (x-x_0)y' - (x^2+1)y = 0$$

$$y = \sum_{m=0}^{\infty} a_m (x-x_0)^{m+r}, \quad x_0 = 0 \rightarrow \text{Regular singular point.}$$

$$A_0(x)y'' + A_1(x)y' + A_2(x)y = 0$$

Let $x = x_0$ is regular singular point.

$$\frac{p_1}{x-x_0} = \frac{A_1}{A_0}$$

$$\therefore \quad A_1 = \frac{p_1 A_0}{x-x_0}, \quad x \neq x_0.$$

$$y = \sum_{m=0}^{\infty} C_m x^{m+r} \quad \text{— (2)}$$

Differentiate eq (2) w.r.t x

$$y' = \sum_{m=0}^{\infty} C_m (m+r)x^{m+r-1} \quad \text{— (3)}$$

Differentiating again

$$y'' = \sum_{m=0}^{\infty} C_m (m+r)(m+r-1)x^{m+r-2} \quad \text{— (4)}$$

From eq ①, ②, ③ and ④

$$2\left(\sum_{m=0}^{\infty} C_m (m+r)(m+r-1)x^{m+r-2}\right)x^2 +$$

$$\left(\sum_{m=0}^{\infty} C_m (m+r)x^{m+r-1}\right)(x) - (x^2+1)\left(\sum_{m=0}^{\infty} C_m x^{m+r}\right) = 0$$

$$\left[2\left(\sum_{m=0}^{\infty} C_m (m+r)(m+r-1)\right) + \left(\sum_{m=0}^{\infty} C_m (m+r)\right) - \sum_{m=0}^{\infty} C_m\right]x^{m+r}$$

$$-\left[\sum_{m=0}^{\infty} C_m x^{m+r+2}\right] = 0$$

Indicial equation $(m=0)$ [Lowest power term]

$$2C_0(0+r)(0+r-1)x^{0+r} + C_0(0+r)x^{0+r} - C_0x^{0+r+2} -$$
$$C_0x^{0+r} = 0$$

$\therefore$ $2C_0(r)(r-1)x^r + C_0(r)x^r - C_0x^{r+2} - C_0x^r = 0$

$\therefore$ $C_0[2r(r-1) + (r) - \quad]x^r - C_0x^{r+2} = 0$

Lowest power term $\Rightarrow x^r$

$\Rightarrow$ $C_0[2r(r-1) + (r) - 1] = 0$

$2r(r-1) + 1(r-1) = 0$

$(r-1)(2r+1) = 0$

$r = 1$ and $-1/2$.

$r_1 - r_2 = 1 - (-1/2) = 3/2 \neq 0$

Roots are different and not differ by an integer.

Now, $m=1$.

$$\left[2(C_1(r+1)(r)x^{r+1}) + C_1(r+1)x^{r+1} - C_1x^{r+1}\right] - C_1x^{r+3} = 0$$

$$\therefore\ C_1\left\{2(r+1)r + 1(r+1) - 1\right\}x^{r+1} - C_1x^{r+3} = 0$$

Lowest power term $\Rightarrow x^{r+1}$

$$\therefore\ C_1\left\{2r(r+1) + (r+1) - 1\right\} = 0$$

$$C_1\left\{2r(r+1) + (r+1) - 1\right\} = 0$$

$$C_1\left\{2r^2 + 3r\right\} = 0$$

As, $2r^2 + 3r \neq 0$ for $r = 1$ and $-1/2$

$$\Rightarrow \boxed{C_1 = 0}$$

Hence; $C_3 = C_5 = C_7 = \dots = C_{2n+1} = 0$.

Now, $m=2$.

$$\therefore\ \sum_{m=2}^{\infty} C_m\left\{2(m+r)(m+r-1) + (m+r) - 1\right\}x^{m+r} - \sum_{m=0}^{\infty} C_m x^{m+r+2} = 0$$

$\longrightarrow$ Replacing $m \to m-2$

$$\therefore\ \sum_{m=2}^{\infty} C_m\left\{2(m+r)(m+r-1) + (m+r) - 1\right\}x^{m+r} - \sum_{m-2=0}^{\infty} C_{m-2}\, x^{m+r+\cancel{2}-\cancel{2}} = 0$$

$$\therefore C_m\{2(m+r)(m+r-1)+(m+r)-1\}x^{m+r}-C_{m-2}x^{m+r}=0$$

Recurrence relation

$$\therefore C_m = \frac{C_{m-2}}{(2(m+r)(m+r-1)+(m+r)-1)}$$

$3/2-1$

$\therefore \quad m=2$

$$C_2 = \frac{C_0}{2(2+r)(1+r)+(2+r)-1}$$

$$\begin{array}{r} 44 \\ 14 \\ \hline 176 \\ 44\times \\ \hline 616 \end{array}$$

For $r=1$

$$C_2 = \frac{C_0}{2(3)(2)+(3)-1} = \frac{C_0}{12+2} = \frac{C_0}{14}$$

For $r=-1/2$

$$C_2 = \frac{C_0}{2(2-1/2)(1/2)+(2-1/2)-1} = \frac{C_0}{3(1/2)+1/2} = \frac{C_0}{2}.$$

$m=4$

$$C_4 = \frac{C_2}{2(4+r)(3+r)+(4+r)-1}$$

For $r=1$

$$C_4 = \frac{C_2}{2(5)(4)+(5)-1} = \frac{C_0}{14(40+4)} = \frac{C_0}{14\times 44} = \frac{C_0}{616}$$

For $r=-1/2$

$$C_4 = \frac{C_2}{2(4-1/2)(3-1/2)+(4-1/2)-1} = \frac{C_2}{2(7/2)(5/2)+(7/2)-1}$$

$$= \frac{C_2}{7(5/2)+5/2} = \frac{C_2}{40/2} = \frac{C_2}{20} = \frac{C_0}{40}.$$

$\therefore$ For $r = 1$

$$y = C_0 x + C_1 x^2 + C_2 x^3 + C_3 x^4 + C_4 x^5 + \dots$$

$$y = C_0 x + 0x^2 + \frac{C_0}{14} x^3 + 0.x^4 + \frac{C_0}{616} x^5 + \dots$$

$$\Rightarrow y = C_0 \left[x + \frac{x^3}{14} + \frac{x^5}{616} + \dots \right] \text{ Ans.}$$

and For $r = -1/2$

$$y = C_0 x + C_1 x^2 + C_2 x^3 + C_3 x^4 + C_4 x^5 + \dots$$

$$y = C_0 x + 0x^2 + \frac{C_0}{2} x^3 + 0x^4 + \frac{C_0}{40} x^5 + \dots$$

$$y = C_0 \left[x + \frac{x^3}{2} + \frac{x^5}{40} + \dots \right] \text{ Ans.}$$

Vector Space and Linear Transformation

→ Vector space: Let V be a non-empty set and elements of V can be matrix, vector, function, etc.
If v is an element of V such that $v \in V$, then v is called vector.

Properties of vector space for addition

1. Closure property:
$$a+b=c\ ,\quad a,b,c \in V$$
2. Commutative property:
$$a+b=b+a$$
3. Associative property:
$$(a+b)+c = a+(b+c)$$
4. Existence of unique zero, which belong in V.
$$a+0=0+a=a$$
5. Existence of negative
$$a+(-a)=0.$$

Multiplication Property. closure property

1. $\alpha A = b$, where α is scalar and $a \in V$
2. Left distribution law
$$(\alpha+\beta)a = \alpha a+\beta a$$
3. Right distribution law
$$a(\alpha+\beta) = \alpha a+\beta a$$
4. Existence of the identity element
$$a * 1 = a.$$
5. Associative Law
$$(\alpha\beta)a = \alpha(\beta a)$$

→ Linear Transformation

$$T : A \to B$$

$$T : \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \to \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$T(a_{11}) \to b_{11}$$

Let A and B be two non-empty set such that

$$T: A \to B.$$

where, T is called linear transformation from A to B if it follows following properties.

If v in V. Then;

(i) If 'α' is scalar and v in V. Then;

$$T(\alpha v) = \alpha T(v)$$

(ii) $T(v_1 + v_2) = T(v_1) + T(v_2)$

(iii) $T(\alpha_1 v_1 + \alpha_2 v_2) = T(\alpha_1 v_1) + T(\alpha_2 v_2) = \alpha_1 T(v_1) + \alpha_2 T(v_2)$.

Example; If $T\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Find $T\begin{bmatrix} 4 & 5 \\ 3 & 8 \end{bmatrix} = ?$

$$T\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$$T\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$$

$$T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

These handwritten notes are of MTH-S102 taught to us by Prof. D.K. Singh, compiled and organized chapter-wise to help our juniors. We hope they make your prep a bit easier.

— **Saksham Nigam** and **Misbahul Hasan** (B.Tech. CSE(2024-28))