Laplace Transformation

Transformation

Plane to plane (co-ordinate) (x, y, z)

space to space (parameter) (t)

Notation: $L\{f(t)\}$

Laplace transformation of function of f.

→ Formula:

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt, \text{ where } t > 0.$$

$f(t)$ is known as kernal,

$f(t)$: $e^{at}, \sin at, \cos at, t^n$.

Example: Find $L\{1\}$.

$\because L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$L\{1\} = \int_0^\infty e^{-st} \cdot 1 \cdot dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}\left[\lim_{t\to\infty} e^{-st} - 1\right]$$

$$= -\frac{1}{s}[0-1]$$

$$L\{1\} = \frac{1}{s}. \quad \text{Ans.}$$

=> $L\{t^n\} = ?$

Using Laplace transformation:

Let $st = z$, $s\,dt = dz$

$L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\,dt$

$$L\{t^n\} = \int_0^\infty e^{-st} \cdot t^n \cdot dt = \int_0^\infty e^{-z} \cdot \left(\frac{z}{s}\right)^n \cdot s \cdot dt = \frac{\Gamma(n+1)}{s^{n+1}}.$$

$$= \frac{n!}{s^{n+1}}.$$

$$L\{t^2\} = \int_0^\infty \underset{II}{e^{-st}} \cdot \underset{I}{t^2} \cdot dt$$

$$= \left[t^2 \cdot \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty 2 \underset{I}{t} \cdot \frac{\overset{II}{e^{-st}}}{-s} dt$$

$$= \left[\frac{t^2}{-s}\left(\left[\lim_{t\to\infty} e^{-st}\right] - 1\right)\right] + \frac{2}{s}\left[\left[t \cdot \frac{e^{-st}}{s}\right]_0^\infty - \left[\frac{1}{s^2} \cdot e^{-st}\right]_0^\infty\right]$$

$$= \cancelto{0}{\frac{t^2}{s}} + \frac{2}{s}\left[\left(\cancelto{0}{\lim_{t\to\infty} \frac{t \cdot e^{-st}}{s}} - \cancelto{0}{\frac{t}{s}}\right) - \frac{1}{s^2}[0-1]\right]$$

$$= \cancelto{0}{\frac{t^2}{s}} + \frac{2}{s}\left[\left(-\cancelto{0}{\frac{t}{s}}\right) + \frac{1}{s^2} \right]$$

$$L\{t^2\} = \frac{2}{s^3}$$ Ans

$\therefore$ If $n \rightarrow$ fractional;

$$L\{t^n\} = \frac{\sqrt{(n+1)}}{s^{n+1}}$$ where $n > 0$

If $n \rightarrow$ integer

$$L\{t^n\} = \frac{n!}{s^{n+1}}.$$

Q. Find Laplace Transformation of e^{at}.

$$L\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at} dt = \int_0^\infty e^{-t(s-a)} dt$$

$$= \left[-\frac{e^{-t(s-a)}}{s-a} \right]_0^\infty = \lim_{t\to\infty}\left(-\frac{e^{-\infty(s-a)}}{s-a}\right) + \left(\frac{1}{s-a}\right)$$

$$L\{e^{at}\} = \frac{1}{s-a}.$$

→ Laplace transformation of sin at.

$$\therefore L\{\sin at\} = L\left\{\frac{e^{iat}-e^{-iat}}{2i}\right\} \qquad \sin at = \frac{e^{iat}-e^{-iat}}{2i}$$

$$= \frac{1}{2i}\left[L\{e^{iat}\} - L\{e^{-iat}\}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right]$$

$$= \frac{1}{2i}\left[\frac{\cancel{s}+ia-\cancel{s}+ia}{s^2-(ia)^2}\right]$$

$$= \frac{1}{2i} \times \frac{2ia}{s^2+a^2} = \frac{a}{s^2+a^2}$$

$$\Rightarrow L\{\sin at\} = \frac{a}{s^2+a^2}.$$

$$\cos at = \frac{e^{iat}+e^{-iat}}{2}$$

$$L\{\cos at\} = L\left\{\frac{e^{iat}+e^{-iat}}{2}\right\}$$

$$\sinh at = \frac{e^{at}-e^{-at}}{2}$$

$$\cosh at = \frac{e^{at}+e^{-at}}{2}.$$

$$= \frac{1}{2}\left\{\frac{1}{s-ia} + \frac{1}{s+ia}\right\}$$

$$= \frac{1}{2}\left\{\frac{s+\cancel{i}a+s-\cancel{i}a}{s^2+a^2}\right\}$$

$$\Rightarrow L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$$

$$= \frac{1}{2} L\{e^{at}+e^{-at}\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$L\{\cosh at\} = \frac{s}{s^2-a^2} \qquad \text{Ans.}$$

Change of Scale

$L\{f(at)\} = ?$

By using Laplace transformation

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt$$

Let $at = u \Rightarrow t = \frac{u}{a}$

$a\,dt = du$

$dt = \frac{1}{a}\,du$

$\therefore$ When $t = 0$, then $u = 0$,
$t = \infty$, then $u = \infty$.

$$L\{f(u)\} = \frac{1}{a}\int_0^\infty e^{-\frac{su}{a}} \cdot f(u)\,du$$

$$L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right).$$

→ First shifting Theorem

If Laplace $L\{f(t)\} = F(s)$, $s > 0$

then $L\{e^{at} f(t)\} = F(s-a)$, $s-a > 0$

Proof: By using Laplace transformation

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) = F(s)$$

$$L\{e^{at} \cdot f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} \cdot f(t)\,dt = \int_0^\infty e^{-t(s-a)} f(t)\,dt$$

$$= F(s-a)$$

(i) $L\{e^{at}\cos bt\} = ?$

$$L\{\cos bt\} = \frac{s}{s^2+b^2}$$

By using shifting theorem

$$L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$$

(ii) $L\{e^{at}\sin bt\} = ?$

$$L\{\sin bt\} = \frac{b}{s^2+b^2}$$

By using first shifting

$$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$$

(iii) $L\{e^{at}\cosh bt\} = ?$

$$L\{\cosh bt\} = \frac{s}{s^2-b^2}$$

$$\therefore L\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2-b^2}.$$

(iv) $L\{e^{at}\sinh bt\} = \dfrac{b}{(s-a)^2-b^2}.$

→ Laplace Transformation of Derivative

If $L\{f(t)\} = F(s)$

then $L\{f'(t)\} = ?$

Proof: By using Laplace transformation

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{f'(t)\} = \int_0^\infty \underset{I}{e^{-st}} \underset{II}{f'(t)}\,dt$$

$$= \left[e^{-st}.f(t)\right]_0^\infty + s\int_0^\infty e^{-st}.f(t)\,dt$$

$$= f(0) + sF(s)$$

$$\therefore L\{f'(t)\} = sL\{f(t)\} + f(0)$$

$$L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - f''(0).$$

→ Multiplication by 't'

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} L\{f(t)\}$$

Example:

$$L\{t\cos t\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= (-1)\left[\frac{(s^2+a^2) - s(2s)}{(s^2+a^2)^2}\right]$$

$$= \frac{-a^2+s^2}{(s^2+a^2)^2} = \frac{(s+a)(s-a)}{(s^2+a^2)^2}$$

$$L\{t\cos t\} = \frac{s^2-1^2}{(s^2+1^2)^2}$$ Ans.

Find the laplace transformation;

$$L\{e^t . t^2 . \cos 2t\}$$

$$\therefore \quad L\{\cos 2t\} = \frac{s}{s^2+4}$$

$$L.\{t^2 . \cos 2t\} = \frac{d^2}{ds^2}\left[\frac{s}{s^2+4}\right]$$

$$= \frac{d}{ds}\left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{4-s^2}{(s^2+4)^2}\right]$$

$$= \frac{-2s(s^2+4)^2 - (4-s^2) \times 2(s^2+4)(2s)}{(s^2+4)^4}$$

$$\therefore \frac{-2s\cancel{(s^2+4)}\left[+(s^2+4) + 2(4-s^2)\right]}{(s^2+4)^4}$$

$$= \frac{-2s\left[12-s^2\right]}{(s^2+4)^3}$$

$$\therefore L\{e^t . t^2 . \cos 2t\} = \frac{-2(s-1)\left[12-(s-1)^2\right]}{((s-1)^2+4)^3}$$

$$= \frac{2(s-1)\left[(s-1)^2-12\right]}{((s-1)^2+4)^3}$$

Ans.

→ Laplace Transformation of Integral.

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}F(s)$$

Ex : If Laplace transformation of

$$L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{p^{3/2}}. \text{ Find } L\left\{\frac{1}{\sqrt{\pi t}}\right\}$$

$$\therefore L\{f'(t)\} = sF(s) + f(0)$$

$$\therefore L\left\{\frac{1}{\sqrt{\pi t}}\right\} = p.\frac{1}{p^{3/2}} + 0 = \frac{1}{p^{1/2}} \text{ Ans.}$$

→ Find $L\{\sin\sqrt{t}\}$

$$\therefore \sin ax = ax - \frac{(ax)^3}{3!} + \frac{(ax)^5}{5!} - \frac{(ax)^7}{7!} + \cdots$$

$$\therefore \sin\sqrt{t} = \sqrt{t} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \cdots$$

and

$$L\{\sin\sqrt{t}\} = L\{\sqrt{t}\} - \frac{1}{3!}L\{t^{3/2}\} + \frac{1}{5!}L\{t^{5/2}\} - \cdots$$

$$= \frac{\sqrt{\frac{1}{2}+1}}{s^{3/2}} - \frac{1}{3!}\left(\frac{\sqrt{\frac{3}{2}+1}}{s^{5/2}}\right) + \frac{1}{5!}\left(\frac{\sqrt{\frac{5}{2}+1}}{s^{7/2}}\right)$$

$$= \frac{\frac{1}{2}\times\sqrt{\pi}}{s^{3/2}} - \frac{1}{3!}\left(\frac{3/2\times\frac{1}{2}\times\sqrt{\pi}}{s^{5/2}}\right) + \frac{1}{5!}\left(\frac{5/2\times 3/2\times 1/2\times\sqrt{\pi}}{s^{7/2}}\right).$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \left(\frac{1}{\cancel{3}\times 2}\times\frac{\cancel{3}}{2s}\right) + \frac{1}{\cancel{5}\times\cancel{3}\times 2}\times\frac{\cancel{5}\times\cancel{3}}{2\times 2\times s^2}\cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{8s^2}\cdots\right]$$

$e^{ax} =$

$2!$ $3!$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\cdot e^{-\frac{1}{4s}} \quad \underline{\underline{\text{Ans.}}}$$

$$e^{-ax} = 1 - ax + \frac{(ax)^2}{2!} - \frac{(ax)^3}{3!} + \cdots$$

$$L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}\, e^{-1/4s} \quad \underline{\underline{\text{Ans.}}}$$

Q. Find Laplace transformation of $\sin^3 t$

$\therefore \sin 3t = 3\sin t - 4\sin^3 t$

$\Rightarrow \sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t$

$$\therefore L\{\sin^3 t\} = \frac{3}{4}L\{\sin t\} - \frac{1}{4}L\{\sin 3t\}$$

$$= \frac{3}{4}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{4}\left\{\frac{3}{s^2+9}\right\} \quad \underline{\underline{\text{Ans.}}}$$

Division by t

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty L\{f(t)\}\,ds = \int_s^\infty F(s)\,ds$$

$\underline{\underline{\text{Q.}}}$ Find $L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\}$

$$\therefore L\left\{\frac{e^{-at}}{t}\right\} - L\left\{\frac{e^{-bt}}{t}\right\}$$

$$= \int_s^\infty L\{e^{-at}\}\,ds - \int_s^\infty L\{e^{-bt}\}\,ds$$

$$\int_s^{\infty} \frac{1}{s+a} \cdot ds - \int_s^{\infty} \frac{1}{s+b} ds$$

$$\therefore \left[\ln|s+a|\right]_s^{\infty} - \left[\ln|s+b|\right]_s^{\infty}$$

$$\left[\ln\left|\frac{s+a}{s+b}\right|\right]_s^{\infty}$$

$$\therefore \lim_{s\to\infty} \ln\left|\frac{1+\cancelto{0}{a/s}}{1+b/s}\right| - \ln\left|\frac{s+a}{s+b}\right|$$

$$= -\ln\left|\frac{s+a}{s+b}\right| = \ln\left|\frac{s+b}{s+a}\right| \text{ Ans.}$$

Q. $L\left\{\frac{\cos at - \cos bt}{t}\right\}$

$$\therefore L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^{\infty} L(\cos at) - L(\cos bt)$$

$$= \int_s^{\infty} \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$= \left[\frac{1}{2}\ln|s^2+a^2| - \frac{1}{2}\ln|s^2+b^2|\right]_s^{\infty}$$

$$= -\frac{1}{2}\ln\left|\frac{s^2+a^2}{s^2+b^2}\right| = \frac{1}{2}\ln\left|\frac{s^2+b^2}{s^2+a^2}\right| \text{ Ans.}$$

Q. Find Laplace transformation of

$$L\{t^2 e^{2t}\cos 4t\}$$

$$= L\{\cos 4t\} = \frac{s}{s^2+16}$$

$$L\{e^{2t}\cos 4t\} = \frac{(s-2)}{(s-2)^2+16}$$

$$L\{t^2 e^{2t}\cos 4t\} = \frac{d^2}{ds^2}\left[\frac{(s-2)}{(s-2)^2+16}\right]$$

$$= \frac{d}{ds}\left[\frac{1((s-2)^2+16) - (s-2)[2(s-2)]}{(s-2)^2+16}\right]$$

$$= \frac{d}{ds}\left[\frac{(s-2)^2+16-2(s-2)^2}{(s-2)^2+16}\right] = \frac{d}{ds}\left[1 - 2\frac{(s-2)^2}{(s-2)^2+16}\right]$$

$$\therefore = -\frac{4(s-2)[(s-2)^2+16] - 2(s(s-2)^2)2(s-2)}{((s-2)^2+16)^2}$$

$$= -\frac{4(s-2)[(s-2)^2+16] - 4(s-2)^3}{((s-2)^2+16)^2}$$

$$= -\frac{4(s-2)(\cancel{(s-2)^2}+16-\cancel{(s-2)^2})}{((s-2)^2+16)^2}$$

$$= \frac{-64(s-2)}{((s-2)^2+16)^2}.$$ Ans.

→ Inverse Laplace Transformation

$$L[f(t)] = F(s)$$

→ F(s) is known as Laplace transformation of f(t).

→ f(t) is known as Inverse Laplace Transformation of F(s).

$$L^{-1}[F(s)] = f(t)$$

→ Laplace Formula

→ $L[t^n] = \dfrac{n!}{s^{n+1}}$, $n \in \mathbb{Z}$

→ $L[t^n] = \dfrac{\Gamma(n+1)}{s^{n+1}}$, $n \in$ fraction

→ $L[1] = \dfrac{1}{s}$.

→ $L[\cos at] = \dfrac{s}{s^2+a^2}$

→ $L[\sin at] = \dfrac{a}{s^2+a^2}$

→ $L[\cosh at] = \dfrac{s}{s^2-a^2}$

→ $L[\sinh at] = \dfrac{a}{s^2-a^2}$

→ First shifting Theorem

$L[e^{at} \cdot f(t)] = F(s-a)$.

→ Inverse Laplace Formula

→ $L^{-1}\left[\dfrac{1}{s^{n+1}}\right] = \dfrac{t^n}{n!}$, $n \in \mathbb{Z}$

→ $L^{-1}\left[\dfrac{1}{s^{n+1}}\right] = \dfrac{t^n}{\Gamma(n+1)}$, $n \in$ fraction

→ $L^{-1}\left[\dfrac{1}{s}\right] = 1$.

→ $L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at$

→ $L^{-1}\left[\dfrac{1}{s^2+a^2}\right] = \dfrac{\sin at}{a}$

→ $\cosh at = L^{-1}\left[\dfrac{s}{s^2-a^2}\right]$

→ $\sinh at = L^{-1}\left[\dfrac{a}{s^2-a^2}\right]$

↳ $L^{-1}\left[\dfrac{1}{s^2-a^2}\right] = \dfrac{\sinh at}{a}$.

→ $L^{-1}(F(s-a)) = e^{at} f(t)$

$f(t) = e^{-at} L^{-1}(F(s-a))$

→ Multiplication by 't'

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$(-1)^n t^n f(t) = L^{-1}\left[\frac{d^n}{ds^n} F(s)\right].$$

→ Division by 't'.

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty L(f(t))\, ds$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$$

$$\to t \cdot L^{-1}\left[\int_s^\infty F(s)\, ds\right] = f(t).$$

→ Laplace of Integral

$$L\left[\int_0^t f(t)\, dt\right] = \frac{F(s)}{s}.$$

$$\to L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\, dt$$

→ Find

$$L^{-1}\left[\frac{s+2}{s^2+2^2}\right] = L^{-1}\left[\frac{s}{s^2+2^2} + \frac{2}{s^2+2^2}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+2^2}\right] + L^{-1}\left[\frac{2}{s^2+2^2}\right]$$

$$= \cos 2t + \sin 2t$$

→ Find

$$L^{-1}\left[\frac{1}{s^2+3^2}\right] = \frac{1}{3} L^{-1}\left[\frac{3}{s^2+3^2}\right] = \frac{1}{3} \cdot \sin 3t.$$

Q. Find

$$\int_0^\infty \frac{e^{-t} \sin\sqrt{3}t\, dt}{t}$$

$$\int_0^\infty e^{-st} f(t)\, dt = L(f(t))$$

$$= \int_0^\infty e^{-t} \frac{\sin\sqrt{3}t}{t}\, dt, \quad \text{let } f(t) = \sin\sqrt{3}t$$

$$\therefore \quad L\left[\frac{f(t)}{t}\right] = L\left[\frac{\sin\sqrt{3}t}{t}\right]$$

$$\Rightarrow \quad L\left[\frac{\sin\sqrt{3}t}{t}\right] = \int_s^{\infty} L(\sin\sqrt{3}t)\, ds$$

$$= \int_s^{\infty} \frac{\sqrt{3}}{s^2+3^2} \cdot ds$$

$$= \frac{\sqrt{3}}{\sqrt{3}} \cdot \left[\tan^{-1}\left(\frac{s}{\sqrt{3}}\right)\right]_s^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{\sqrt{3}}\right)$$

$$= \cot^{-1}\left(\frac{s}{\sqrt{3}}\right)$$

As, $\int_0^{\infty} e^{-st} f(t)\, dt = \int_0^{\infty} e^{-t} \sin\sqrt{3}t\, dt$

$\Rightarrow \quad s = 1.$

$$\therefore \int_0^{\infty} \frac{e^{-t}\sin\sqrt{3}t\, dt}{t} = \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} \quad \text{Ans.}$$

Q. Find the $L^{-1}\left[\frac{s+9}{s^2-4s+5}\right]$

$$\Rightarrow \quad L^{-1}\left[\frac{s+9}{(s-2)^2+9}\right] = L^{-1}\left[\frac{s}{(s-2)^2+9} + \frac{9}{(s-2)^2+9}\right]$$

$$= L^{-1}\left[\frac{s-2+2}{(s-2)^2+9} + \frac{9}{(s-2)^2+9}\right]$$

$$= L^{-1}\left[\frac{s-2}{(s-2)^2+3^2}\right] + L^{-1}\left[\frac{2}{(s-2)^2+9}\right] + L^{-1}\left[\frac{9}{(s-2)^2+3^2}\right]$$

$e^{2t} \cdot \cos 3t + \frac{2}{3}\left[e^{2t} \cdot \sin 3t\right] + 3\left[e^{2t} \sin 3t\right]$ Ans.

Q. Find

$$L^{-1}\left[\frac{s}{(s-5)^2+3^2}\right]$$

$$\therefore L^{-1}\left[\frac{(s-5)+5}{(s-5)^2+3^2}\right] = L^{-1}\left[\frac{s-5}{(s-5)^2+3^2}\right] + L^{-1}\left[\frac{5}{(s-5)^2+3^2}\right]$$

$$= e^{5t}\cos 3t + \frac{5}{3} e^{5t} \sin 3t.$$

Q. By using partial fraction, solve:

$$L^{-1}\left[\frac{2s+3}{(s-2)(s^2+2s+5)}\right]$$

$$\therefore \frac{2s+3}{(s-2)(s^2+2s+5)} = \frac{A}{(s-2)} + \frac{Bs+C}{s^2+2s+5}$$

$$2s+3 = A(s^2+2s+5) + (Bs+c)(s-2)$$

$$2s+3 = s^2(A+B) + s(2A-2B+C) + 5A-2C$$

$$\Rightarrow A+B=0 \qquad \Rightarrow A = 7/13$$

$$2A-2B+C=2 \qquad B = -7/13$$

$$5A-2C=3 \qquad C = -\frac{2}{13}$$

$$\therefore L^{-1}\left[\frac{2s+3}{(s-2)(s^2+2s+5)}\right] = L^{-1}\left[\frac{7}{13(s-2)} + \frac{-7s+(-2)}{13(s^2+2s+5)}\right]$$

$$= \frac{7}{13} L^{-1}\left[\frac{1}{s-2}\right] + \left(\frac{-1}{13}\right) L^{-1}\left[\frac{7s+2}{s^2+2s+5}\right]$$

$$= \frac{7}{13} e^{2t} - \frac{1}{13} L^{-1}\left[\frac{7s}{(s+1)^2+2^2}\right] - \frac{2}{13} L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= \frac{7}{13} e^{2t} - \frac{7}{13} L^{-1}\left[\frac{s+1}{(s+1)^2+}\right] + \frac{7}{26}\left[\frac{2}{(s+1)^2+2^2}\right] - \frac{2}{13} L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$\frac{7}{13}e^{2t} - \frac{7}{13}e^{-t}\cos 2t + \frac{7}{26}(e^{-t}\sin 2t) - \frac{1}{13}(e^{-t}\sin 2t)$. Ans.

$(s+2)^2 = s^2+4+4s$

Q. Find

$$L^{-1}\left[\frac{3s}{(s+2)(s^2+4s+8)}\right]$$

$$\therefore \frac{3s}{(s+2)(s^2+4s+8)} = \frac{A}{(s+2)} + \frac{Bs+C}{s^2+4s+8}$$

$$3s = A(s^2+4s+8) + (Bs+C)(s+2)$$

$$\Rightarrow 3s = s^2(A+B) + s(4A+2B+C) + 8A+2C$$

$$\Rightarrow A+B=0$$

$$4A+2B+C=3$$

$$8A+2C=0$$

$$\therefore A = -3/2$$

$$B = 3/2$$

$$C = 6$$

$$\therefore L^{-1}\left[\frac{3s}{(s+2)(s^2+4s+8)}\right] = L^{-1}\left[-\frac{3}{2(s+2)} + \frac{(\frac{3}{2})s+6}{(s^2+4s+8)}\right]$$

$$= -\frac{3}{2}L^{-1}\left[\frac{1}{(s+2)}\right] + \frac{3}{2}L^{-1}\left[\frac{s}{(s+2)^2+4}\right] + L^{-1}\left[\frac{6}{(s+2)^2+4}\right]$$

$$= -\frac{3}{2}e^{-2t} + \frac{3}{2}L^{-1}\left[\frac{s+2-2}{(s+2)^2+4}\right] + \frac{6}{2}L^{-1}\left[\frac{2}{(s+2)^2+2^2}\right]$$

$$= -\frac{3}{2}e^{-2t} + \frac{3}{2}e^{-2t}\cos 2t - \frac{3}{2}e^{-2t}\sin 2t + 3e^{-2t}\sin 2t$$

Ans.

→ Second Shifting Theorem

If $L[f(t)] = F(s)$

then;

$$f(t) = \begin{cases} f(t-a), & t>0 \\ 0, & t<0 \end{cases}$$

$$\therefore L(f(t)) = e^{-as} F(s)$$

Proof:

$$\therefore L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^a e^{-st} f(t)\,dt + \int_a^\infty e^{-st} \cdot f(t-a)\,dt$$

$$= \int_0^a e^{-st} \cancel{(0)}\,dt + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t)\,dt$$

Let $t-a = u$

$\therefore t = u+a$

$dt = du$

$$\int_0^\infty e^{-s(u+a)} \cdot f(u)\,du = e^{-sa} \int_0^\infty e^{-su} \cdot f(u)\,du$$

$$= e^{-sa} \cdot F(s).$$ Ans.

→ Unit Step Function (Heaviside Function)

$$u(t-a) = \begin{cases} u(t), \text{ if } > a \\ 0, \text{ if } t < a \end{cases}$$

$$u(t) = \begin{cases} 1, \text{ if } t > 0, \\ 0, \text{ if } t < 0 \end{cases}$$

→ Find Laplace transformation of the unit step function.

By the definition of Laplace Transformation ;

$$\Rightarrow L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$L(f(t)) = \int_0^\infty e^{-st} u(t)\,dt$$

$$L(u(t)) = \int_0^a e^{-st} \cancel{u(t)}\,dt + \int_a^\infty e^{-st}(1)\,dt$$

$$L(u(t)) = \int_a^\infty e^{-st}\,dt = \frac{e^{-as}}{}$$

$$\Rightarrow L(u(t)) = \frac{e^{-as}}{}.$$

Q. Find

$$L\,(f(t-a)u(t-a)) = ?$$

By the definition of Laplace Transform;

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\,(f(t-a)u(t-a) = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt$$

$$= \int_0 e^{-st} f(t-a)\underbrace{\cancel{u}(t-a)}_{0} + \int_a^\infty e^{-st} f(t-a)\underbrace{\cancel{u}(t-a)}_{1}\,dt$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)(t-a)\,dt$$

Let $t-a = u$

$\therefore\ t = u + a$

$dt = du$

$$= \int_0^\infty e^{-s(u+a)} \cdot f(u) \cdot \underbrace{\cancel{u}}_{1}\,du$$

$$= e^{-sa} \cdot L[f(u))$$

$$= e^{-sa} \cdot F(s) \quad \text{Ans.}$$

Q. $L[(t-1)^2 u(t-1)] = ?$

$$L[f(t-a)u(t-a)] = e^{-as}F(s)$$

$$\therefore L[(t-1)^2 u(t-1)] = e^{-1s} \cdot L[t]$$

$$= e^{-s} \cdot \frac{2}{s^2} \text{ Ans.}$$

Q. $L(\sin t\, u(t-\pi)) = ?$

$$L(\sin((t-\pi)+\pi) u(t-\pi) = e^{-\pi s} \cdot L[-\sin t]$$

$$= \frac{-e^{-\pi s}}{s^2+1} \text{ Ans.}$$

Q. Express the following in heaviside unit function.

$$f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ \sin 2t, & \pi < t < 2\pi \\ \sin 3t & 2\pi < t < 3\pi \end{cases}$$

$$H(t) = f_1(t)[u_1(t-a_1) - u_2(t-a_2)] +$$
$$f_2(t)[u_2(t-a_2) - u_3(t-a_3)] +$$
$$f_3(t)[u_3(t-a_3) - u_4(t-a_4)] + \dots$$
$$f_n(t)[t u(t-a_n)]$$

$$\therefore\ H(t) = \sin t\,[u_1\, t + u_2(t-\pi)] + \sin 2t\,[u_2(t-\pi) + u_3(t-2\pi)] + \sin 3t\,[u_3(t-3\pi)]$$ Ans.

Q. Express the following function in heaviside unit-step function and find their laplace.

$$f(t) = \begin{cases} t^2 & 1 < t < 2 \\ 4t & 2 < t < 3 \end{cases}$$

$$H(t) = t^2\,[u_1(t-1) + u_2(t-2)] + 4t\,[u_3(t-3)]$$

$$L(H(t)) = L(t^2 u_1(t-1)) + L(t^2 u_2(t-2)) + L(4t\,u_3(t-3))$$

$$= L(((t^2+1)-1)u_1(t-1)) + L(((t^2+2)-2)u_2(t-2)) + 4L(((t+3)-3)u_3(t-3))$$

$$= e^{-s} L(t^2+1) + e^{-2s} L(t^2+2) + 4\cdot e^{-3s} L(t+3)$$

$$= e^{-s}\left[\frac{3!}{s^3} + \frac{1}{s}\right] + e^{-2s}\left[\frac{3!}{s^3} + \frac{2}{s}\right] + 4\cdot e^{-3s}\left[\frac{2!}{s^2} + \frac{3}{s}\right]$$

Ans.

Convolution Theorem

If $L^{-1}[f(t)] = F(s)$

and $L^{-1}[g(t)] = G(s)$

$$L^{-1}[f(t)g(t)] = F(s) * G(s)$$

$$\therefore \quad F * G = \int_0^t f(u)\,g(t-u)\,du.$$

Q. By using convolution theorem. Solve

$$L^{-1}\left[\frac{1}{(s^2+2)(s^2+4)}\right].$$

By convolution theorem;

$$\therefore \quad L^{-1} \qquad = \qquad f(t) * g(t)$$

$$\therefore \quad F(s) = \frac{1}{s^2+2}$$

$$\therefore \quad f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s^2+(\sqrt{2})^2}\right]$$

$$f(t) = \frac{1}{\sqrt{2}}\,L^{-1}\left[\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right]$$

$$f(t) = \frac{\sin\sqrt{2}t}{\sqrt{2}}$$

$$G(s) = \frac{1}{s^2+4}$$

$$\therefore \quad g(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$g(t) = \frac{1}{2}\,L^{-1}\left[\frac{2}{s^2+2^2}\right]$$

$$g(t) = \frac{\sin 2t}{2}$$

$\therefore \quad f*g = \int_0^t f(u)\, g(t-u)\, du$

$\therefore \quad f(u) = \dfrac{\sin\sqrt{2}u}{\sqrt{2}}$

$g(t-u) = \dfrac{\sin 2(t-u)}{2}$

$\therefore \quad f*g = \int_0^t \dfrac{\sin\sqrt{2}u}{\sqrt{2}} \cdot \dfrac{\sin(2t-2u)}{2}\, du$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$\therefore \quad f*g = \dfrac{1}{4\sqrt{2}} \left[\int_0^t \cos(\sqrt{2}u - 2t + 2u) - \cos(\sqrt{2}u + 2t - 2u)\, du \right.$

$\therefore$ Let $\sqrt{2}u - 2t + 2u = v'$ $\qquad u = 0$, $v' = -2t$

$\therefore \quad \sqrt{2}\,du + 2\,du = dv'$

$du = \dfrac{dv'}{\sqrt{2}+2}$ $\qquad u = t$, $v' = \sqrt{2}t$

and

Let $\sqrt{2}u + 2t - 2u = v''$ $\qquad u = 0$, $v'' = 2t$

$du = \dfrac{dv''}{\sqrt{2}-2}$ $\qquad u = t$, $v'' = \sqrt{2}t$

$\therefore \quad f*g = \dfrac{1}{4\sqrt{2}} \left[\int_{-2t}^{\sqrt{2}t} \dfrac{\cos v'\, dv'}{\sqrt{2}+2} - \int_{2t}^{\sqrt{2}t} \dfrac{\cos v''\, dv''}{\sqrt{2}-2} \right]$

$$f*g = \frac{1}{4\sqrt{2}}\left[\frac{[\sin v']_{-2t}^{\sqrt{2}t}}{\sqrt{2}+2} - \frac{[\sin v'']_{2t}^{\sqrt{2}t}}{\sqrt{2}-2}\right]$$

$$f*g = \frac{1}{4\sqrt{2}}\left[\frac{(\sqrt{2}-2)(\sin\sqrt{2}t+\sin 2t)-(\sqrt{2}+2)(\sin\sqrt{2}t-\sin 2t)}{(\sqrt{2})^2-(2)^2}\right]$$

$$f*g = \frac{1}{4\sqrt{2}}\left[\frac{2\sqrt{2}\sin 2t - 4\sin\sqrt{2}t}{-2}\right]$$

$$f*g = -\frac{\sin 2t}{4} + \frac{\sin\sqrt{2}t}{2\sqrt{2}}$$

$$\therefore \quad L^{-1}\left[\frac{1}{(s^2+2)(s^2+4)}\right] = \frac{\sin\sqrt{2}t}{2\sqrt{2}} - \frac{\sin 2t}{4}$$

Ans.

Q. By using convolution theorem solve;

$$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right]$$

$$\therefore \quad F(s) = \frac{1}{s+1}$$

$$\therefore \quad f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s+1}\right]$$

$$f(t) = e^{-t}$$

and

$$G(s) = \frac{1}{s+2}$$

$\therefore \quad g(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{1}{s+2}\right]$

$\Rightarrow \quad g(t) = e^{-2t}$

$\therefore \quad f * g = \int_0^t f(u)\, g(t-u)\, du$

$\therefore \quad f(u) = e^{-u}$

$\quad\quad g(t-u) = e^{-2(t-u)}$

$\therefore \quad f*g = \int_0^t e^{-u} \cdot e^{-2t} \cdot e^{2u} \cdot du$

$= e^{-2t} \cdot \int_0^t e^{u}\, du$

$= e^{-2t} \left[e^{u}\right]_0^t$

$= e^{-2t}\left[e^{t} - 1\right]$

$f * g = e^{-t} - e^{-2t}$. Ans.

$\therefore \quad L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = e^{-t} - e^{-2t}$. Ans.

Q. By using convolution theorem:

$$L^{-1}\left[\frac{1}{s(s^2-a^2}\right].$$

$$\therefore \quad F(s) = \frac{1}{s}.$$

$$\Rightarrow \quad f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s}\right]$$

$$f(t) = 1.$$

$$\therefore \quad G(s) = \frac{1}{s^2-a^2}.$$

$$g(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{1}{s^2-a^2}\right]$$

$$\therefore \quad g(t) = \frac{1}{a} L^{-1}\left[\frac{a}{s^2-a^2}\right]$$

$$g(t) = \frac{1}{a} \sinh at.$$

$$\therefore \quad f * g = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t 1 \cdot \frac{1}{a} \sinh a(t-u)\, du$$

$$= -\frac{1}{a}\left[\cosh(t-u)\right]_0^t$$

$$= -\frac{1}{a}\left[\cosh(0) - \cosh(t)\right]$$

$$= \frac{1}{a}\left[\cosh t - 1\right]. \text{ Ans.}$$

Q. By using convolution theorem Solve :

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$$

$$\therefore \quad F(s) = \frac{1}{s(s+1)}$$

$$\therefore \quad f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s(s+1)}\right]$$

$$f(t) = L^{-1}\left[\frac{(s+1)-s}{s(s+1)}\right]$$

$$f(t) = L^{-1}\left[-\frac{1}{s+1} + \frac{1}{s}\right]$$

$$f(t) = 1 - e^{-t}$$

$$G(s) = \frac{1}{s+2}$$

$$g(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{1}{s+2}\right].$$

$$g(t) = e^{-2t}$$

$$\therefore \quad f * g = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t (1-e^{-t})e^{-2(t-u)}\, du = e^{-2t}\int_0^t e^{2u}\, du - e^{-3t}\int_0^t e^{2u}\, du$$

$$f*g = e^{-2t}\cdot\left[\frac{e^{2u}}{2}\right]_0^t - e^{-3t}\left[\frac{e^{2u}}{2}\right]_0^t$$

$$= e^{-2t}\left[\frac{e^{2t}-1}{2}\right] - e^{-3t}\left[\frac{e^{2t}-1}{2}\right]$$

$$f*g = \frac{1-e^{-2t}}{2} - \frac{e^{-t}-e^{-3t}}{2}$$

$$f*g = \frac{1-e^{-t}-e^{-2t}+e^{-3t}}{2}. \quad \underline{\underline{Ans}}.$$

***Assignment: By using convolution theorem, solve

$$L^{-1}\left[\frac{1}{(s+1)(s+2)(s+3)(s+4)}\right].$$

$\therefore$ $F(s) = \dfrac{1}{(s+1)(s+2)(s+3)}$

$$\therefore \frac{1}{(s+1)(s+2)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{(s+3)}$$

$$1 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2)$$

$\therefore$ Putting $s=-1$, Putting $s=-2$, Putting $s=-3$

$\therefore$ $1 = A\times 1\times 2$, $1 = -B$, $1 = 2C$

$\boxed{A = 1/2}$ $\boxed{B = -1}$ $\boxed{C = 1/2}$

$$\therefore f(t) = L^{-1}[F(s)]$$

$$= L^{-1}\left[\frac{1}{(s+1)(s+2)(s+3)}\right]$$

$$= L^{-1}\left[\frac{1/2}{(s+1)} - \frac{1}{s+2} + \frac{1/2}{s+3}\right]$$

$$f(t) = \tfrac{1}{2}e^{-t} - e^{-2t} + \tfrac{1}{2}e^{-3t}$$

$$\therefore G(s) = \frac{1}{s+4}$$

$$g(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{1}{s+4}\right]$$

$$g(t) = e^{-4t}.$$

$$\therefore f * g = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \left(\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}\right) . e^{-4(t-u)} du$$

$$= \frac{1}{2}e^{-5t}\int_0^t e^{4u} du - e^{-6t}\int_0^t e^{4u} du + \frac{1}{2}e^{-7t}\int_0^t e^{4u} du$$

$$= \frac{1}{2}e^{-5t}\left[\frac{e^{4t}-1}{4}\right] - e^{-6t}\left[\frac{e^{4t}-1}{4}\right] + \frac{1}{2}e^{-7t}\left[\frac{e^{4t-1}}{4}\right]$$

$$f*g = \left[\frac{e^{4t}-1}{4}\right]\left[\frac{1}{2}e^{-5t} - e^{-6t} + \frac{1}{2}e^{-7t}\right] \quad \underline{\text{Ans}}$$

Assignment: State and prove Convolution Theorem:

Statement: If $L^{-1}[F(s)] = f(t)$ and $L^{-1}[G(\ )] = g(t)$, ~~then~~ then

$$L^{-1}[F(s).G(s)] = \int_0^t f(u)\,g(t-u)\,du = (f*g)(t)$$

Proof:

If $L^{-1}[F(s).G(s)] = \int_0^t f(u)\,g(t-u)\,du$

$$\Rightarrow F(s).G(s) = L\left[\int_0^t f(u)\,g(t-u)\,du\right]$$

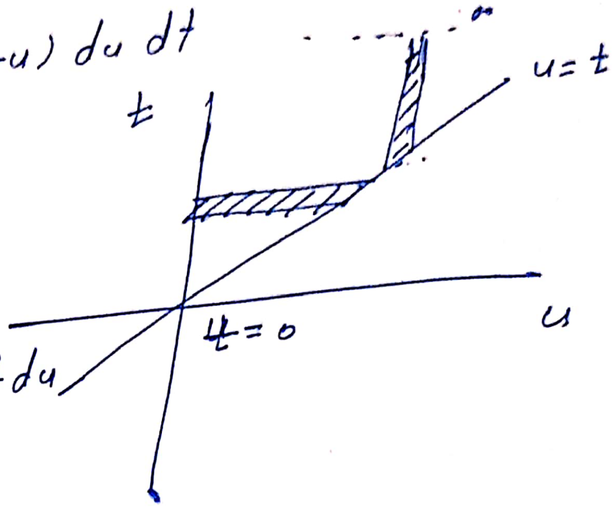
Using definition of Laplace transform

$$\therefore L(f(t)) = \int_0^\infty e^{-st}.f(t)\,dt$$

$$\Rightarrow \quad = \int_{t=0}^{t=\infty} e^{-st}.\left(\int_{u=0}^{u=t} f(u)\,g(t-u)\,du\right)dt$$

Now, evaluating the integral with the help of change of order of integration.

$$\int_{t=0}^{t=\infty}\int_{u=0}^{u=t} e^{-st}\cdot f(u)g(t-u)\,du\,dt$$



$$\int_{u=0}^{u=\infty}\int_{t=u}^{t=\infty} e^{-st} f(u)g(t-u)\,dt\,du$$

Now, let

$$t-u=z$$
$$dt=dz$$

$$\int_{u=0}^{\infty}\int_{t=u}^{\infty} e^{-st} f(u)g(t-u)\,dt\,du = \int_{u=0}^{\infty}\int_{z=0}^{\infty} e^{-s(u+z)} f(u)g(z)\,dz\,du$$

$$= \int_{u=0}^{\infty}\int_{z=0}^{\infty} e^{-su} f(u).\ e^{-sz} g(z)\,dz\,du$$

$$= \left[\int_{u=0}^{\infty} e^{-su} f(u)\,du\right] \times \left[\int_{0}^{\infty} e^{-sz} g(z)\,dz\right]$$

$$= F(s).\ G(s).$$ Hence proved.

Application of Laplace Transformation

Ex : By using Laplace Transformation, solve the differential equation :

$$\frac{d^2y}{dx^2} + 4y = \sin 2t \quad , \quad \begin{array}{l} y(0) = 0 \\ y'(0) = 0 \end{array}$$

$\therefore$ Using Laplace Transformation on both sides ;

$$L[\, y'' - 4y \,] = L[\sin 2t]$$

$$\therefore \quad L[y''] - 4L[y] = L[\sin 2t] \quad (\text{By using linear property})$$

and $\quad L[y''] = s^2 L[y(t)] - s\,y(0) - y'(0)$

$$\therefore \quad s^2 L[y(t)] - \cancelto{0}{s\,y(0)} - \cancelto{0}{y'(0)} - 4L[y] = \frac{2}{s^2 + 2^2}$$

$$\Rightarrow \quad (s^2 - 4)\, L[y(t)] = \frac{2}{s^2+4}$$

$$\therefore \quad L[y(t)] = \frac{2}{(s^2+4)(s^2-4)}$$

$$y(t) = L^{-1}\left[\frac{2}{(s^2+4)(s^2-4)}\right]$$

$$y(t) = \frac{2}{8} L^{-1}\left[\frac{(s^2+4) - (s^2-4)}{(s^2+4)(s^2-4)}\right]$$

$$y(t) = \frac{1}{4} L^{-1}\left[\frac{1}{s^2-4} - \frac{1}{s^2+4}\right]$$

$$= \frac{1}{8} L^{-1}\left[\frac{2}{s^2-4} - \frac{2}{s^2+4}\right]$$

$$= \frac{1}{8}\left[\sinh 2t - \sin 2t\right] . \quad \text{Ans.}$$

Q. Solve the D.E by using Laplace Transformation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y(t) = \cos t \quad , \quad \begin{array}{l} y(0) = 0. \\ y'(0) = 1. \end{array}$$

$\therefore$ Using Laplace Transform both sides:

$$L\left[\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y(t)\right] = L[\cos t]$$

$$L[y''] + 6L[y'] + 9L[y] = \frac{s}{s^2+1}$$

$$\therefore \quad s^2L[y] - \cancelto{0}{y(0)} - \cancelto{1}{y'(0)} + 6\left[sL[y(t)] - \cancelto{0}{y(0)}\right] + 9L[y] = \frac{s}{s^2+1}$$

$$s^2L[y] - 1 + 6sL[y] + 9[L[y]] = \frac{s}{s^2+1}$$

$$L[y]\{s^2+6s+9\} - 1 = \frac{s}{s^2+1}$$

$$L[y] = \frac{s^2+s+1}{s^2+1}$$

$$L[y] = \frac{s^2+s+1}{(s^2+1)(s^2+6s+9)} = \frac{s^2+s+1}{(s^2+1)(s+3)^2}$$

$$\therefore \quad y = L^{-1}\left[\frac{s^2+s+1}{(s^2+1)(s+3)^2}\right]$$

Using partial fraction:

$$\frac{s^2+s+1}{(s^2+1)(s+3)^2} = \frac{As+B}{s^2+1} + \frac{C}{(s+3)} + \frac{D}{(s+3)^2}$$

$$s^2+s+1 = s^3(A+C) + s^2(6A+B+3C+D) + s(9A+6B+C) + 9B+3C+D$$

$\therefore$ On solving, we get;

$A = \frac{2}{25}$

$B = \frac{3}{50}$

$C = -2/25$

$D = 7/10$

$$\therefore \quad y = L^{-1}\left\{\frac{2/25\, s + 3/50}{(s^2+1)}\right\} + L^{-1}\left\{\frac{-2/25}{s+3}\right\} + L^{-1}\left\{\frac{7/10}{(s+3)^2}\right\}$$

E

$$= \frac{2}{25} L^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{3}{50} L^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{2}{25} L^{-1}\left\{\frac{1}{s+3}\right\} + \frac{7}{10} L^{-1}\left\{\frac{1}{(s+3)^2}\right\}$$

$$y = \frac{2}{25}\cos t + \frac{3}{50}\sin t - \frac{2}{25} e^{-3t} + \frac{7}{10} e^{-3t} \cdot t \qquad \text{Ans.}$$

Q. Solve ~~the~~ differential equations :

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = t^2 e^t \; ; \qquad y(0) = 1,\; y'(0) = 0,\; y''(0) = -2.$$

Using Laplace Transforms on both sides

$$L\{y'''\} - 3L\{y''\} + 3L\{y'\} - L\{y\} = L\{t^2 e^t\}$$

$$\therefore \quad L\{y'\} = sL\{y\} - y(0)$$

$$L\{y''\} = s^2 L\{y\} - y'(0) - s(y(0))$$

$$L\{y'''\} = s^3 L\{y\} + y''(0) - s^2(y(0)) - s(y'(0))$$

$$\therefore \quad s^3 L\{y\} + y''(0) - s^2(y(0)) - s(y'(0)) - 3(s^2 L\{y\} - y'(0) - s(y(0)))$$
$$+ 3\{sL\{y\} - y(0)\} - L(y(t)) = \frac{d}{ds}\left[\frac{1}{s-1}\right]$$

$$s^3 L\{y\} - 2 - s^2 - 3s^2 L\{y\} + 3s + 3sL\{y\} - 3 - \underset{L(y(t))}{=} \frac{2}{(s-1)^3}$$

$$L\{y\}\{s^3 - 3s^2 + 3s - 1\} = \frac{2}{(s-1)^3}$$
$$-2 - s^2 + 3s - 3$$

$$L\{y\} = \frac{2}{(s-1)^6} + \frac{(s^2 - 3s + 1)}{(s-1)^3}$$

$$y = L^{-1}\left\{\frac{2}{(s-1)^6}\right\} + L^{-1}\left\{\frac{s^2-3s+1}{(s-1)^3}\right\}$$

$$y = 2\left[\frac{e^t \cdot t^5}{5!}\right] + L^{-1}\left\{\frac{s^2-3s+1}{(s-1)^3}\right\}$$

$$\because \; L^{-1}\left\{\frac{s^2-3s+1}{(s-1)^3}\right\} = L^{-1}\left\{\frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3}\right\}$$

$$\Rightarrow \; y = 2\left[\frac{e^t \cdot t^5}{5!}\right] + e^t - te^t - \frac{t^2}{2}e^t. \quad \text{Ans.}$$

Q. By using Laplace Transforms, solve the following differential equation:

$$\frac{d^2y}{dt^2} + y(t) = t\cos t \quad , \quad t > 0 \qquad \begin{array}{l} y(0) = 0 \\ y'(0) = 0 \end{array}$$

$\because$ Using Laplace transforms on both sides:

$$L\{y'' + y\} = L\{t\cos t\}$$

$$\therefore \; s^2 L\{y\} - y'(0) - s\,y(0) + L\{y\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$s^2 L\{y\} + L\{y\} = \frac{(-1)(2s^2 - s^2 - 1)}{(s^2+1)^2}$$

$$\therefore \; L\{y\} = \frac{1-s^2}{(1+s^2)^{\cancel{2}3}}$$

$$\Rightarrow \; y = L^{-1}\left\{\frac{1-s^2}{(1+s^2)^{\cancel{2}3}}\right\} \quad \ldots$$

These handwritten notes are of MTH-S102 taught to us by Prof. D.K. Singh, compiled and organized chapter-wise to help our juniors. We hope they make your prep a bit easier.

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