The Chain Rule

We start by recalling the definition of differentiability, and a theorem about limits.

Definition 1. A function $f : \mathbb{R} \to \mathbb{R}$ is *differentiable at* $a \in \mathbb{R}$ if the following limit exists

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

In which case the value of the limit is called the *derivative of* f *at* a, and is denoted $f'(a)$.

Recall we have the following *change of variables* theorem for limits

Lemma 2. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = M$. If either

(1) f is continuous at L.

(2) There is $\rho > 0$ such that $g(x) \neq L$ for all x such that $|x - c| < \rho$.

are true, then $\lim_{x \to c} f(g(x)) = \lim_{x \to L} f(x)$.

We now introduce the chain rule, which provides a criterion for the differentiability of the composition of functions, as well as a formula for the derivative.

Theorem 3. If $g : \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $g(c) \in \mathbb{R}$, then $f \circ g : \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$ and $(f \circ g)'(c) = f'(g(c))g'(c)$.

The proof of this theorem is slightly technical, perhaps for a reason that may not be immediately obvious. There is a naive, incorrect approach that we will show prior to the correct proof, as a way of gaining intuition for the correct proof.

Incorrect Proof. We apply Definition 1 to $f \circ g$ at c and manipulate the difference quotient, namely

$$\begin{aligned}
(f \circ g)'(c) = \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} &= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \frac{g(x) - g(c)}{x - c} \\
&= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\
&\overset{(\star)}{=} \lim_{u \to g(c)} \frac{f(u) - f(g(c))}{u - g(c)} \cdot g'(c) \\
&= f'(g(c))g'(c)
\end{aligned}$$

Where in line $(\star)$ we used Lemma 2 with $u = g(x)$ to transform the limit, which gives us the desired result, namely $(f \circ g)'(c) = f(g'(c))g'(c)$.

It may be hard to see why this is an incorrect proof, and rightfully so, this proof will work perfectly fine for almost all of the functions we can think of, but unfortunately will fail for a very select few. The issue arises when we make the substitution $u = g(x)$ on the first limit in line $(\star)$. We did not check the hypothesis of Lemma 2, and so we may not always be able to apply the substitution. Let's explore this more in depth. Let's define $F(x) = \frac{f(x) - f(g(c))}{x - g(c)}$, in line $(\star)$ we claimed that

$$\lim_{x \to c} F(g(x)) = \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} = \lim_{x \to g(c)} \frac{f(x) - f(g(c))}{x - g(c)} = \lim_{x \to g(c)} F(x)$$

Note that Lemma 2 says we can do this if either F is continuous at $g(c)$ or $g(x) \neq g(c)$ in some neighbourhood of c. Note that as is, F is not even defined at $x = g(c)$, so we'd need $g(x) \neq g(c)$ in some neighbourhood of c, which may not always be possible! This is where we run into problems, and need to make changes. The idea here is we can define F at $x = g(c)$ in such a way that makes F continuous. We can do this as $\lim_{x \to g(c)} F(x) = f'(g(c))$ exists.

Proof. We start by properly defining $F : \mathbb{R} \to \mathbb{R}$ so that it is continuous at $g(c)$, namely

$$F(x) = \begin{cases} \frac{f(x) - f(g(c))}{x - g(c)} & x \neq g(c) \\ f'(g(c)) & x = g(c) \end{cases}$$

As f is differentiable at $g(c)$, it follows that $\lim_{x \to g(c)} F(x) = f'(g(c)) = F(g(c))$, namely F is continuous at $g(c)$, allowing us to make the substitution from earlier, namely we have $\lim_{x \to c} F(g(x)) = \lim_{x \to g(c)} F(x)$. The last thing we need is to verify that

$$\frac{f(g(x)) - f(g(c))}{x - c} = F(g(x)) \cdot \frac{g(x) - g(c)}{x - c}$$

which can easily be done by taking cases on whether or not $g(x) = g(c)$. Thus by the limit laws

$$\begin{aligned} (f \circ g)'(c) = \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} &= \lim_{x \to c} F(g(x)) \cdot \frac{g(x) - g(c)}{x - c} \\ &= \lim_{x \to c} F(g(x)) \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\ &= f'(g(c)) g'(c) \end{aligned}$$

□