MAT157: Analysis I — Tutorial 2

Topics: Bijections and Cardinality.

Bijective Functions

Question 1. Let $f : A \to B$ be a function. Argue why f is a bijection if and only if for every $b \in B$, there is a *unique* $a \in A$ for which $b = f(a)$.

Question 2. Let $f : A \to B$ be a function, and suppose $C \subseteq A$ and $D \subseteq B$.

(a) Prove that $C \subseteq f^{-1}(f(C))$, and that equality holds if f is injective. Can equality sometimes hold even if f is not injective?

(b) Prove that $f(f^{-1}(D)) \subseteq D$, and that equality holds if f is surjective. Can equality sometimes hold even if f is not surjective?

Question 3. Let A be a set. Prove that there is a bijection $F : \{f : \{1, 2\} \to A\} \to A \times A$.

Bonus Problem. Let $g : A \to B$ and $f : B \to C$ be functions. If neither f nor g are bijections, is it possible for $f \circ g$ to be a bijection?

Cardinality

Question 4. Suppose A and B are disjoint sets, and there exist bijections $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$. Using f and g, construct a bijection $h : \mathbb{N} \to A \cup B$. *Hint:* Think about the even and odd natural numbers.

Question 5. Suppose A and B are countable sets. Is it necessarily true that $A \times B$ is countable? Prove the result or provide a counterexample. *Hint:* First consider $\mathbb{N} \times \mathbb{N}$.

Bonus Problem. Prove that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$. *Hint:* Consider decimal representations of real numbers.