MAT157: Analysis I — Tutorial 4

Topics: Limits

Warm up. Let $D \subseteq \mathbb{R}, f : D \to \mathbb{R}$ be a function, and $a \in D, L \in \mathbb{R}$. Write out the formal definition of the statement "$\lim_{x\to a} f(x) = L$".

Question 1. Using the definition of a limit, prove the following statements.

(a) $\lim_{x\to 1}(3x+4) = 7$.

(b) $\lim_{x\to 0} x^3 = 0$.

(c) $\lim_{x\to 2}(x^2+x-1) = 5$.

(d) $\lim_{x\to 1}\left(\frac{x+5}{x+2}\right) = 2$.

Question 2.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and $a, L \in \mathbb{R}$. Express the statement "$\lim_{x\to a} f(x) \neq L$" mathematically, using logical symbols.

(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is the *characteristic function of* $\mathbb{Q}$, namely

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Prove that $\lim_{x\to 0} f(x) \neq 1$. *Hint:* Both $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense in $\mathbb{R}$.

(c) **Bonus:** Show that $\lim_{x\to a} f(x) \neq L$ for all $a \in \mathbb{R}$ and $L \in \mathbb{R}$.

Question 3. Suppose f is defined on an open interval containing $a \in \mathbb{R}$. If $\lim_{x\to a} f(x) = L$, show that

(a) $\lim_{x\to a}(f(x) - L) = 0$.

(b) $\lim_{h\to 0} f(a+h) = L$.

(c) $\lim_{h\to 1} f(ah) = L$.