Three Hard Theorems

Each of these theorems will involve continuous functions on closed and bounded intervals, namely sets of the form $[a,b]$. Before we state and prove each of the theorems, let us recalls some preliminary definitions.

Definition 1. A function $f:[a,b]\to\mathbb{R}$ is continuous if for all $c\in[a,b]$ and for all $\varepsilon>0$, there is $\delta>0$ such that for all $x\in[a,b], |x-a|<\delta\Rightarrow|f(x)-f(a)|<\varepsilon$.

One of our concerns will be with a slightly stronger version of continuity, namely uniform continuity. Let's recall the definition, and the distinction from regular continuity.

Definition 2. A function $f:[a,b]\to\mathbb{R}$ is uniformly continuous if for all $\varepsilon>0$, there is $\delta>0$ such that for all $x,y\in[a,b], |x-y|<\delta\Rightarrow|f(x)-f(y)|<\varepsilon$.

The main difference is that with regular continuity, the $\delta>0$ that we choose will depend on the point of continuity, namely for each $c\in[a,b]$, we have $\delta>0$ that depends on c. With uniform continuity, this is not the case, given $\varepsilon>0$ we have one $\delta>0$ that works for all points in $[a,b]$. We're now ready to state the first hard theorem, which has the most complex proof of the three.

Theorem 1. If $f:[a,b]\to\mathbb{R}$ is continuous, then f is uniformly continuous.

Note that this is saying continuous functions on closed and bounded intervals are uniformly continuous. The fact that the domain is a closed and bounded interval is crucial, as the result fails otherwise, which we will see after the proof. But for now, let's prove this.

Proof. Let $\varepsilon>0$, and define the following set

$$U=\{c\in[a,b]:\exists\delta>0\text{ such that }\forall x,y\in[a,c],|x-y|<\delta\Rightarrow|f(x)-f(y)|<\varepsilon\}$$

U is the collection of points $c\in[a,b]$ for which f is uniformly continuous on $[a,c]$ given $\varepsilon>0$. Our goal will be to show $U=[a,b]$. Note that $a\in U$ as $[a,a]=\{a\}$, and so any $\delta>0$ will work. Also, $U\subseteq[a,b]$ by construction, so $c\leq b$ for all $c\in U$. Thus, $\alpha:=\sup(U)$ exists by completeness, and $\alpha\leq b$. We want to show $U=[a,b]$, so we need to show $b\in U$. Suppose $\alpha<b$, as f is continuous at α, there is $\delta_1>0$ such that for all $x\in[a,b],|x-\alpha|<\delta_1\Rightarrow|f(x)-f(\alpha)|<\frac{\varepsilon}{2}$. By the criterion for supremum, there is $c\in U$ such that $\alpha-\delta_1<c\leq\alpha$, and as $c\in U$, there is $\delta_2>0$ such that for all $x,y\in[a,c],|x-y|<\delta_2\Rightarrow|f(x)-f(y)|<\varepsilon$. Set $\delta=\min\left\{\frac{\delta_1}{2},\delta_2\right\}$, to arrive at a contradiction, we show $\alpha+\frac{\delta}{2}\in U$. Fix $x,y\in\left[a,\alpha+\frac{\delta}{2}\right]$, and suppose $|x-y|<\delta$. If $x,y\in\left[a,\alpha-\frac{\delta_1}{2}\right]$, then $x,y\in[a,c]$, and so as $c\in U,|x-y|<\delta\leq\delta_2\Rightarrow|f(x)-f(y)|<\varepsilon$. Conversely, without loss of generality if $x\in\left(\alpha-\frac{\delta_1}{2},\alpha+\frac{\delta}{2}\right]$, then $|x-\alpha|<\frac{\delta_1}{2}$, and as $|x-y|<\delta$, it follows by the triangle inequality that $|y-\alpha|\leq|x-y|+|x-\alpha|<\delta+\frac{\delta_1}{2}\leq\frac{\delta_1}{2}+\frac{\delta_1}{2}=\delta_1$, and so

$$|f(x)-f(y)|\leq|f(x)-f(\alpha)|+|f(y)-f(\alpha)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

Thus, $\alpha+\frac{\delta}{2}\in U$, contradicting that $\alpha=\sup(U)$, and so $\alpha=b$. Lastly, we show $b=\alpha\in U$. By the left continuity of f at b, there is $\delta_1>0$ such that $x\in(b-\delta_1,b]\Rightarrow|f(x)-f(y)|<\frac{\varepsilon}{2}$. As $b=\sup(U)$, there is $c\in U$ such that $c\in(b-\delta_1,b]$, and so there is $\delta_2>0$ such that for all $x,y\in[a,c],|x-y|<\delta_2\Rightarrow|f(x)-f(y)|<\varepsilon$. Set $\delta=\min\left\{\frac{\delta_1}{2},\delta_2\right\}$ and suppose $x,y\in[a,b]$ such that $|x-y|<\delta$. If $x,y\in\left[a,b-\frac{\delta_1}{2}\right]\subseteq[a,c]$, then $|x-y|<\delta\leq\delta_2\Rightarrow|f(x)-f(y)|<\varepsilon$.

If, without loss of generality, $x \in \left(b - \frac{\delta_1}{2}, b\right]$, then $|x-b| < \frac{\delta_1}{2}$, and by the triangle inequality, $|y-b| \leq |x-y| + |x-b| < \frac{\delta_1}{2} + \delta \leq 2\frac{\delta_1}{2} = \delta_1$, and so it follows that

$$|f(x) - f(y)| \leq |f(x) - f(b)| + |f(y) - f(b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, $b \in U$ and hence f is uniformly continuous. $\square$

Now let's see where this result can go wrong if we change the domain of the continuous function.

Example 1. The function $f : (0,1) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is continuous and not uniformly continuous.

Proof. Fix $c \in (0,1)$, it follows from the limit laws that f is continuous at c, and so f is continuous on $(0,1)$. We claim f is *not* uniformly continuous. Indeed, let $\varepsilon_0 = \frac{1}{2}$ and $\delta > 0$ be given. Choose some $x_0 \in (0,1)$ such that $0 < x_0 < \min\{1, \delta\}$ and set $y_0 = \frac{x_0}{2}$. Note that $|x_0 - y_0| = \frac{x_0}{2} < \delta$ and

$$|f(x_0) - f(y_0)| = \left|\frac{1}{x_0} - \frac{1}{y_0}\right| = \frac{|x_0 - y_0|}{x_0 y_0} = \frac{x_0}{2}\left(\frac{1}{x_0 y_0}\right) = \frac{1}{2y_0} \geq \frac{1}{2} = \varepsilon_0$$ $\square$

This emphasizes the importance that the interval be closed. Conversely, we show why the interval needs to be bounded

Example 2. The function $f : [0,\infty) \to \mathbb{R}$ given by $f(x) = x^2$ is continuous and not uniformly continuous.

Proof. Let $c \in [0,\infty)$, once again it follows from the limit laws that f is continuous at c at hence continuous on $[0,\infty)$, we show f is not uniformly continuous. Pick $\varepsilon_0 = 1$ and let $\delta > 0$ be given. Set $x_0 = \frac{1}{\delta}$ and $y_0 = x_0 + \frac{\delta}{2}$. Then $|x_0 - y_0| = \frac{\delta}{2} < \delta$, and

$$|f(x_0) - f(y_0)| = x_0\delta + \frac{\delta^2}{4} > x_0\delta = 1 = \varepsilon_0$$

Thus, f is not uniformly continuous. $\square$

Moving on to the second of the hard theorems, which is likely the most intuitive, and is the easiest to prove of the three.

Theorem 2. If $f : [a,b] \to \mathbb{R}$ is continuous and $f(a)f(b) < 0$, there is $c \in (a,b)$ such that $f(c) = 0$.

Proof. The condition $f(a)f(b) < 0$ says that one of the endpoints is positive and the other negative, let's assume $f(a) < 0 < f(b)$ (otherwise apply the following to $g := -f$). Define the set

$$U = \{c \in [a,b] : f(x) < 0 \quad \forall x \in [a,c]\}$$

As $f(a) < 0$ by assumption and $[a,a] = \{a\}$, it follows that $a \in U$, moreover $U \subseteq [a,b]$ by construction, and so $\alpha := \sup(U)$ exists by completeness. We show $\alpha \in (a,b)$ such that $f(\alpha) = 0$. Suppose $f(\alpha) > 0$, the left continuity of f at α (we can't guarantee two-sided continuity at α since $\alpha = b$ is still technically possible), there is $\delta_1 > 0$ such that $f(x) > 0$ for all $x \in (\alpha - \delta_1, \alpha]$. By the criterion for suprema, there is $c_1 \in U$ such that $\alpha - \delta_1 < c_1 \leq \alpha$, and as $c_1 \in U, f(x) < 0$ for all $x \in [a, c_1]$, and hence $f(c_1) < 0$. But $c_1 \in (\alpha - \delta_1, \alpha]$, and so $f(c_1) > 0$, a contradiction. Thus $f(\alpha) \leq 0$ and hence $\alpha < b$. Suppose now that $f(\alpha) < 0$, by the continuity of f at $\alpha \in (a,b)$, there is

$\delta_2 > 0$ such that $f(x) < 0$ for all $x \in (\alpha - \delta_2, \alpha + \delta_2)$, and hence for all $x \in \left[\alpha - \frac{\delta_2}{2}, \alpha + \frac{\delta_2}{2}\right]$. Again by there criterion for suprema, there is $c_2 \in U$ such that $\alpha - \delta_2 < c_2 \leq \alpha$. As $c_2 \in U$, $f(x) < 0$ for $x \in [a, c_2]$ and hence $f(x) < 0$ for $x \in \left[a, \alpha + \frac{\delta_2}{2}\right]$. Thus $f(x) < 0$ for all $x \in \left[a, \alpha + \frac{\delta_2}{2}\right]$, and hence $a + \frac{\delta_2}{2} \in U$, contradicting $\alpha = \sup(U)$. Thus $\alpha \in (a, b)$ and $f(\alpha) = 0$. $\square$

Corollary 3. If $f : [a, b] \to \mathbb{R}$ is continuous, and d is any value between $f(a)$ and $f(b)$, there is $c \in [a, b]$ such that $f(c) = d$.

Proof. Fix some d between $f(a)$ and $f(b)$. If $d = f(a)$ or $d = f(b)$, we're done, so suppose d lies strictly between them. Define $g := f - d$, so that g is continuous on $[a, b]$ and $g(a)g(b) < 0$. By Theorem 2, there is $c \in (a, b)$ such that $g(c) = 0$, namely $f(c) = d$. $\square$

Moving on to the last and final hard theorem, recall that a function $f : D \to \mathbb{R}$ is bounded on D if there is $M > 0$ for which $|f(x)| \leq M$ for all $x \in D$.

Theorem 3. If $f : [a, b] \to \mathbb{R}$ is continuous then f is bounded.

Proof. We will take a similar approach in the proof, namely define the set

$$U = \{c \in [a, b] : \exists M > 0 \text{ such that } |f(x)| \leq M \ \forall x \in [a, c]\}$$

namely U is the collection of points c for which f is bounded on $[a, c]$. We show $U = [a, b]$. Note that $[a, a] = \{a\}$, and so f is bounded trivially on $[a, a]$, namely $a \in U$, moreover $U \subseteq [a, b]$ by construction, and so $\alpha := \sup(U)$ exists by completeness. Note that $\alpha \leq b$ by construction, suppose $\alpha < b$ so that $\alpha \in (a, b)$. By the continuity of f at α, there is $\delta_1 > 0$ and $M_1 > 0$ such that $|f(x)| \leq M_1$ for all $x \in (\alpha - \delta_1, \alpha + \delta_1)$, and hence for all $x \in \left[\alpha - \frac{\delta_1}{2}, \alpha + \frac{\delta_1}{2}\right]$. By the criterion for suprema, there is $c_1 \in U$ such that $\alpha - \delta_1 < c_1 \leq \alpha$, and so there is $M_2 > 0$ such that $|f(x)| \leq M_2$ for all $x \in [a, c_1]$, and hence for all $x \in \left[a, \alpha - \frac{\delta_1}{2}\right]$. Thus, $|f(x)| \leq \max\{M_1, M_2\}$ for all $x \in \left[a, \alpha + \frac{\delta_1}{2}\right]$, namely $\alpha + \frac{\delta_1}{2} \in U$, contradicting $\alpha = \sup(U)$, and so $\alpha = b$. All that remains to show is that $b \in U$. By the left continuity at b, there is $\delta > 0$ and $M > 0$ such that $|f(x)| \leq M$ for all $x \in (b - \delta, b]$ and hence for all $x \in \left[b - \frac{\delta}{2}, b\right]$. As $b = \alpha$, there is $c \in U$ such that $c \in (b - \delta, b]$, and so there is $N > 0$ such that $|f(x)| \leq N$ for all $x \in [a, c]$, and hence for all $x \in \left[a, b - \frac{\delta}{2}\right]$. Thus, $|f(x)| \leq \max\{M, N\}$ for all $x \in [a, b]$, and so f is bounded. $\square$

Note that this theorem tells us that if $f : [a, b] \to \mathbb{R}$ is continuous, then we can meaningfully write

$$\sup_{x \in [a,b]} f(x) = \sup\{f(x) : x \in [a, b]\} \quad \text{and} \quad \inf_{x \in [a,b]} f(x) = \inf\{f(x) : x \in [a, b]\}$$

A consequence of this result, which will be very useful, is that the supremum and infimum are *witnessed*, namely there are $x_m, x_M \in [a, b]$ such that $f(x_m) = \sup(f)$ and $f(x_m) = \inf(f)$.

Corollary 4. If $f : [a, b] \to \mathbb{R}$ is continuous, then there are points $x_m, x_M \in [a, b]$ such that $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in [a, b]$.

Proof. By Theorem 3, $M = \sup\{f(x) : x \in [a, b]\}$ and $m = \inf\{f(x) : x \in [a, b]\}$ exist as f is bounded. Suppose $M \neq f(x)$ for all $x \in [a, b]$ so that $g(x) := \frac{1}{M - f(x)}$ is well defined and hence continuous on $[a, b]$. Let $N > 0$, by the criterion for supremum, there is $x_0 \in [a, b]$ such that

$$M - \frac{1}{N} < f(x_0) \leq M \quad \iff \quad 0 \leq N < \frac{1}{M - f(x_0)} = g(x_0)$$

Namely g is unbounded, contradicting Theorem 3. Similarly, suppose $m \neq f(x)$ for all $x \in [a, b]$ so that $h(x) := \frac{1}{f(x)-m}$ is well defined and hence continuous on $[a, b]$. Let $N > 0$, by the criterion for infimum, there is $x_1 \in [a, b]$ such that

$$m \leq f(x_1) < m + \frac{1}{N} \quad \iff \quad 0 \leq N < \frac{1}{f(x_1) - m} = h(x_1)$$

Namely h is unbounded, contradicting Theorem 3. Thus, there is $x_m, x_M \in [a, b]$ such that $m = f(x_m)$ and $M = f(x_M)$. □

It's worth noting that like Theorem 1, dropping the conditions of closed and bounded on the domain will result in continuous functions that are not bounded. The same functions from Example 1 and Example 2 will be continuous and not bounded.