MAT157: Analysis I — Lecture 13

Topics: Consequences of continuity

Each of these theorems will involve continuous functions on closed and bounded intervals, namely sets of the form $[a, b]$. Before we state and prove each of the theorems, let us recalls some preliminary definitions.

Definition 1. A function $f : [a, b] \to \mathbb{R}$ is continuous if for all $c \in [a, b]$ and for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $x \in [a, b], |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

One of our concerns will be with a slightly stronger version of continuity, namely uniform continuity. Let's recall the definition, and the distinction from regular continuity

Definition 2. A function $f : [a, b] \to \mathbb{R}$ is uniformly continuous if for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $x, y \in [a, b], |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

The main difference is that with regular continuity, the $\delta > 0$ that we choose will depend on the point of continuity, namely for each $c \in [a, b]$, we have $\delta > 0$ that depends on c. With uniform continuity, this is not the case, given $\varepsilon > 0$ we have one $\delta > 0$ that works for all points in $[a, b]$. We're now ready to state the first hard theorem, which has the most complex proof of the three.

Theorem 1. If $f : [a, b] \to \mathbb{R}$ is continuous, then f is uniformly continuous.

Note that this is saying continuous functions on closed and bounded intervals are uniformly continuous. The fact that the domain is a closed and bounded interval is crucial, as the result fails otherwise, which we will see after the proof. But for now, let's prove this.

Proof. Let $\varepsilon > 0$, and define the following set

$$U = \{c \in [a, b] : \exists \delta > 0 \text{ such that } \forall x, y \in [a, c], |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon\}$$

U is the collection of points $c \in [a, b]$ for which f is uniformly continuous on $[a, c]$ given $\varepsilon > 0$. Our goal will be to show $U = [a, b]$. Note that $a \in U$ as $[a, a] = \{a\}$, and so any $\delta > 0$ will work. Also, $U \subseteq [a, b]$ by construction, so $c \leq b$ for all $c \in U$. Thus, $\alpha := \sup(U)$ exists by completeness, and $\alpha \leq b$. We want to show $U = [a, b]$, so we need to show $b \in U$. Suppose $\alpha < b$, as f is continuous at α, there is $\delta_1 > 0$ such that for all $x \in [a, b], |x - \alpha| < \delta_1 \Rightarrow |f(x) - f(\alpha)| < \frac{\varepsilon}{2}$. By the criterion for supremum, there is $c \in U$ such that $\alpha - \delta_1 < c \leq \alpha$, and as $c \in U$, there is $\delta_2 > 0$ such that for all $x, y \in [a, c], |x - y| < \delta_2 \Rightarrow |f(x) - f(y)| < \varepsilon$. Set $\delta = \min\left\{\frac{\delta_1}{2}, \delta_2\right\}$, to arrive at a contradiction, we show $\alpha + \frac{\delta}{2} \in U$. Fix $x, y \in \left[a, \alpha + \frac{\delta}{2}\right]$, and suppose $|x - y| < \delta$. If $x, y \in \left[a, \alpha - \frac{\delta_1}{2}\right]$, then $x, y \in [a, c]$, and so as $c \in U, |x - y| < \delta \leq \delta_2 \Rightarrow |f(x) - f(y)| < \varepsilon$. Conversely, without loss of generality if $x \in \left(\alpha - \frac{\delta_1}{2}, \alpha + \frac{\delta}{2}\right]$, then $|x - \alpha| < \frac{\delta_1}{2}$, and as $|x - y| < \delta$, it follows by the triangle inequality that $|y - \alpha| \leq |x - y| + |x - \alpha| < \delta + \frac{\delta_1}{2} \leq \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1$, and so

$$|f(x) - f(y)| \leq |f(x) - f(\alpha)| + |f(y) - f(\alpha)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, $\alpha + \frac{\delta}{2} \in U$, contradicting that $\alpha = \sup(U)$, and so $\alpha = b$. Lastly, we show $b = \alpha \in U$. By the left continuity of f at b, there is $\delta_1 > 0$ such that $x \in (b - \delta_1, b] \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}$.

As $b = \sup(U)$, there is $c \in U$ such that $c \in (b - \delta_1, b]$, and so there is $\delta_2 > 0$ such that for all $x, y \in [a, c], |x - y| < \delta_2 \Rightarrow |f(x) - f(y)| < \varepsilon$. Set $\delta = \min\left\{\frac{\delta_1}{2}, \delta_2\right\}$ and suppose $x, y \in [a, b]$ such that $|x - y| < \delta$. If $x, y \in \left[a, b - \frac{\delta_1}{2}\right] \subseteq [a, c]$, then $|x - y| < \delta \leq \delta_2 \Rightarrow |f(x) - f(y)| < \varepsilon$. If, without loss of generality, $x \in \left(b - \frac{\delta_1}{2}, b\right]$, then $|x - b| < \frac{\delta_1}{2}$, and by the triangle inequality, $|y - b| \leq |x - y| + |x - b| < \frac{\delta_1}{2} + \delta \leq 2\frac{\delta_1}{2} = \delta_1$, and so it follows that

$$|f(x) - f(y)| \leq |f(x) - f(b)| + |f(y) - f(b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, $b \in U$ and hence f is uniformly continuous.

Now let's see where this result can go wrong if we change the domain of the continuous function.

Example 1. The function $f : (0, 1) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is continuous and not uniformly continuous.

Proof. Fix $c \in (0, 1)$, it follows from the limit laws that f is continuous at c, and so f is continuous on $(0, 1)$. We claim f is *not* uniformly continuous. Indeed, let $\varepsilon_0 = \frac{1}{2}$ and $\delta > 0$ be given. Choose some $x_0 \in (0, 1)$ such that $0 < x_0 < \min\{1, \delta\}$ and set $y_0 = \frac{x_0}{2}$. Note that $|x_0 - y_0| = \frac{x_0}{2} < \delta$ and

$$|f(x_0) - f(y_0)| = \left|\frac{1}{x_0} - \frac{1}{y_0}\right| = \frac{|x_0 - y_0|}{x_0 y_0} = \frac{x_0}{2}\left(\frac{1}{x_0 y_0}\right) = \frac{1}{2y_0} \geq \frac{1}{2} = \varepsilon_0$$

This emphasizes the importance that the interval be closed. Conversely, we show why the interval needs to be bounded

Example 2. The function $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = x^2$ is continuous and not uniformly continuous.

Proof. Let $c \in [0, \infty)$, once again it follows from the limit laws that f is continuous at c at hence continuous on $[0, \infty)$, we show f is not uniformly continuous. Pick $\varepsilon_0 = 1$ and let $\delta > 0$ be given. Set $x_0 = \frac{1}{\delta}$ and $y_0 = x_0 + \frac{\delta}{2}$. Then $|x_0 - y_0| = \frac{\delta}{2} < \delta$, and

$$|f(x_0) - f(y_0)| = x_0\delta + \frac{\delta^2}{4} > x_0\delta = 1 = \varepsilon_0$$

Thus, f is not uniformly continuous.

Moving on to the second of the hard theorems, which is likely the most intuitive, and is the easiest to prove of the three.

Theorem 2. If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a)f(b) < 0$, there is $c \in (a, b)$ such that $f(c) = 0$.

Proof. The condition $f(a)f(b) < 0$ says that one of the endpoints is positive and the other negative, let's assume $f(a) < 0 < f(b)$ (otherwise apply the following to $g := -f$). Define the set

$$U = \{c \in [a, b] : f(x) < 0 \quad \forall x \in [a, c]\}$$

As $f(a) < 0$ by assumption and $[a, a] = \{a\}$, it follows that $a \in U$, moreover $U \subseteq [a, b]$ by construction, and so $\alpha := \sup(U)$ exists by completeness. We show $\alpha \in (a, b)$ such that $f(\alpha) = 0$. Suppose $f(\alpha) > 0$, the left continuity of f at α (we can't guarantee two-sided continuity at α since

$\alpha = b$ is still technically possible), there is $\delta_1 > 0$ such that $f(x) > 0$ for all $x \in (\alpha - \delta_1, \alpha]$. By the criterion for suprema, there is $c_1 \in U$ such that $\alpha - \delta_1 < c_1 \leq \alpha$, and as $c_1 \in U, f(x) < 0$ for all $x \in [a, c_1]$, and hence $f(c_1) < 0$. But $c_1 \in (\alpha - \delta_1, \alpha]$, and so $f(c_1) > 0$, a contradiction. Thus $f(\alpha) \leq 0$ and hence $\alpha < b$. Suppose now that $f(\alpha) < 0$, by the continuity of f at $\alpha \in (a, b)$, there is $\delta_2 > 0$ such that $f(x) < 0$ for all $x \in (\alpha - \delta_2, \alpha + \delta_2)$, and hence for all $x \in \left[\alpha - \frac{\delta_2}{2}, \alpha + \frac{\delta_2}{2}\right]$. Again by there criterion for suprema, there is $c_2 \in U$ such that $\alpha - \delta_2 < c_2 \leq \alpha$. As $c_2 \in U, f(x) < 0$ for $x \in [a, c_2]$ and hence $f(x) < 0$ for $x \in \left[a, \alpha + \frac{\delta_2}{2}\right]$. Thus $f(x) < 0$ for all $x \in \left[a, \alpha + \frac{\delta_2}{2}\right]$, and hence $a + \frac{\delta_2}{2} \in U$, contradicting $\alpha = \sup(U)$. Thus $\alpha \in (a, b)$ and $f(\alpha) = 0$.

Corollary 3. If $f : [a, b] \to \mathbb{R}$ is continuous, and d is any value between $f(a)$ and $f(b)$, there is $c \in [a, b]$ such that $f(c) = d$.

Proof. Fix some d between $f(a)$ and $f(b)$. If $d = f(a)$ or $d = f(b)$, we're done, so suppose d lies strictly between them. Define $g := f - d$, so that g is continuous on $[a, b]$ and $g(a)g(b) < 0$. By Theorem 2, there is $c \in (a, b)$ such that $g(c) = 0$, namely $f(c) = d$.

Moving on to the last and final hard theorem, recall that a function $f : D \to \mathbb{R}$ is bounded on D if there is $M > 0$ for which $|f(x)| \leq M$ for all $x \in D$.

Theorem 3. If $f : [a, b] \to \mathbb{R}$ is continuous then f is bounded.

Proof. We will take a similar approach in the proof, namely define the set

$$U = \{c \in [a, b] : \exists M > 0 \text{ such that } |f(x)| \leq M \ \forall x \in [a, c]\}$$

namely U is the collection of points c for which f is bounded on $[a, c]$. We show $U = [a, b]$. Note that $[a, a] = \{a\}$, and so f is bounded trivially on $[a, a]$, namely $a \in U$, moreover $U \subseteq [a, b]$ by construction, and so $\alpha := \sup(U)$ exists by completeness. Note that $\alpha \leq b$ by construction, suppose $\alpha < b$ so that $\alpha \in (a, b)$. By the continuity of f at α, there is $\delta_1 > 0$ and $M_1 > 0$ such that $|f(x)| \leq M_1$ for all $x \in (\alpha - \delta_1, \alpha + \delta_1)$, and hence for all $x \in \left[\alpha - \frac{\delta_1}{2}, \alpha + \frac{\delta_1}{2}\right]$. By the criterion for suprema, there is $c_1 \in U$ such that $\alpha - \delta_1 < c_1 \leq \alpha$, and so there is $M_2 > 0$ such that $|f(x)| \leq M_2$ for all $x \in [a, c_1]$, and hence for all $x \in \left[a, \alpha - \frac{\delta_1}{2}\right]$. Thus, $|f(x)| \leq \max\{M_1, M_2\}$ for all $x \in \left[a, \alpha + \frac{\delta_1}{2}\right]$, namely $\alpha + \frac{\delta_1}{2} \in U$, contradicting $\alpha = \sup(U)$, and so $\alpha = b$. All that remains to show is that $b \in U$. By the left continuity at b, there is $\delta > 0$ and $M > 0$ such that $|f(x)| \leq M$ for all $x \in (b - \delta, b]$ and hence for all $x \in \left[b - \frac{\delta}{2}, b\right]$. As $b = \alpha$, there is $c \in U$ such that $c \in (b - \delta, b]$, and so there is $N > 0$ such that $|f(x)| \leq N$ for all $x \in [a, c]$, and hence for all $x \in \left[a, b - \frac{\delta}{2}\right]$. Thus, $|f(x)| \leq \max\{M, N\}$ for all $x \in [a, b]$, and so f is bounded.

Note that this theorem tells us that if $f : [a, b] \to \mathbb{R}$ is continuous, then we can meaningfully write

$$\sup_{x \in [a,b]} f(x) = \sup\{f(x) : x \in [a, b]\} \quad \text{and} \quad \inf_{x \in [a,b]} f(x) = \inf\{f(x) : x \in [a, b]\}$$

A consequence of this result, which will be very useful, is that the supremum and infimum are *witnessed*, namely there are $x_m, x_M \in [a, b]$ such that $f(x_m) = \sup(f)$ and $f(x_m) = \inf(f)$.

Corollary 4. If $f : [a, b] \to \mathbb{R}$ is continuous, then there are points $x_m, x_M \in [a, b]$ such that $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in [a, b]$.

Proof. By Theorem 3, $M = \sup\{f(x) : x \in [a,b]\}$ and $m = \inf\{f(x) : x \in [a,b]\}$ exist as f is bounded. Suppose $M \neq f(x)$ for all $x \in [a,b]$ so that $g(x) := \frac{1}{M-f(x)}$ is well defined and hence continuous on $[a,b]$. Let $N > 0$, by the criterion for supremum, there is $x_0 \in [a,b]$ such that

$$M - \frac{1}{N} < f(x_0) \leq M \quad \iff \quad 0 \leq N < \frac{1}{M - f(x_0)} = g(x_0)$$

Namely g is unbounded, contradicting Theorem 3. Similarly, suppose $m \neq f(x)$ for all $x \in [a,b]$ so that $h(x) := \frac{1}{f(x)-m}$ is well defined and hence continuous on $[a,b]$. Let $N > 0$, by the criterion for infimum, there is $x_1 \in [a,b]$ such that

$$m \leq f(x_1) < m + \frac{1}{N} \quad \iff \quad 0 \leq N < \frac{1}{f(x_1) - m} = h(x_1)$$

Namely h is unbounded, contradicting Theorem 3. Thus, there is $x_m, x_M \in [a,b]$ such that $m = f(x_m)$ and $M = f(x_M)$.

It's worth noting that like Theorem 1, dropping the conditions of closed and bounded on the domain will result in continuous functions that are not bounded. The same functions from Example 1 and Example 2 will be continuous and not bounded.

Topological Analogues

Throughout, (X, d_X) and (Y, d_Y) denote metric spaces, and we write $f : (X, d_X) \to (Y, d_Y)$ as $f : X \to Y$ for simplicity, with the understanding that d_X and d_Y are the metrics for X and Y, respectively. To start off, we recall the definition of continuity for functions on metric spaces.

Definition 5. A function $f : X \to Y$ is continuous if for all $a \in X$ and for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $x \in X, d_X(x,a) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon$.

It's worth noting that we can rewrite the implication as $\forall x \in X, x \in B_\delta(a) \Rightarrow f(x) \in B_\varepsilon(f(a))$, or even as $B_\delta(a) \subseteq f^{-1}(B_\varepsilon(f(a)))$. With this, we can introduce an equivalent notion of continuity.

Theorem 4. A function $f : X \to Y$ is continuous if and only if for any open set $U \subseteq Y, f^{-1}(U) \subseteq X$ is open.

Proof. $(\Rightarrow)$ Suppose f is continuous, fix an open set $U \subseteq Y$ and let $a \in f^{-1}(U)$. As $f(a) \in U$, there is $r > 0$ such that $B_r(f(a)) \subseteq U$, and as f is continuous at a, there is $\delta > 0$ such that for all $x \in X, x \in B_\delta(a) \Rightarrow f(x) \in B_r(f(a)) \subseteq U$. Thus, if $x \in B_\delta(a) \Rightarrow f(x) \in U$, and so $B_\delta(a) \subseteq f^{-1}(U)$, namely $f^{-1}(U)$ is open.

$(\Leftarrow)$ Conversely, suppose the pre-image of open sets is open, fix $a \in X$ and let $\varepsilon > 0$ be given. As $B_\varepsilon(f(a)) \subseteq Y$ is open, $f^{-1}(B_\varepsilon(f(a))) \subseteq X$ is open by assumption, and so as $a \in f^{-1}(B_\varepsilon(f(a)))$, there is $\delta > 0$ such that $B_\delta(a) \subseteq f^{-1}(B_\varepsilon(f(a)))$, namely if $x \in B_\delta(a)$, then $f(x) \in B_\varepsilon(f(a))$. Thus, f is continuous at a, and hence f is continuous.

Now that we have a nicer notion of continuity, we can state and prove the (metric space) topological analogues of the three hard theorems. Before doing so, we introduce an analogous notion of uniform continuity, as well as compactness and connectedness.

Definition 6. A function $f : X \to Y$ is uniformly continuous if for all $\varepsilon > 0$, there is $\delta > 0$ such that for all $x, y \in X, d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

Definition 7. As set $K \subseteq X$ is compact if for any collection of open sets $\{C_i : i \in I\}$ such that $K \subseteq \cup_I C_i$ (such a collection is called an open cover of K), there is finitely many such that $K \subseteq C_{i_1} \cup \cdots \cup C_{i_n}$ (such a collection is called a finite subcover). In other works, K is compact if every open cover admits a finite subcover.

Definition 8. A set $S \subseteq X$ is disconnected if there are disjoint, relatively open (i.e open in the subspace topology of S) sets $S_1, S_2 \subseteq S$ such that $S_1 \cup S_2 = S$. Such a pair (S_1, S_2) is called a disconnection of S. A set is connected if it is not disconnected, namely if it does not admit a disconnection.

Theorem 5. If $K \subseteq X$ is compact and $f : K \to Y$ is continuous, then f is uniformly continuous.

Proof. Let $\varepsilon > 0$ be given, by the continuity of f, for each $a \in K$ there is $\delta_a > 0$ such that for all $x \in K, d_X(x, a) < \delta_a \Rightarrow d_Y(f(x), f(a)) < \varepsilon$. Define $\rho_a = \frac{\delta_a}{2}, C_a = B_{\rho_a}(a)$ and consider $\mathcal{C} = \{C_a : a \in K\}$. $\mathcal{C}$ covers K by construction, and so as K is compact, $\mathcal{C}$ admits a finite subcover $\mathcal{C}' = \{C_{a_1}, \ldots C_{a_n}\}$. Define $\delta = \min\{\rho_{a_1}, \ldots, \rho_{a_n}\}$, fix $x, y \in K$ and suppose $d_X(x, y) < \delta$. As $\mathcal{C}'$ covers K, there are indices i and j such that $x \in C_{a_i}$ and $y \in C_{a_j}$, namely $d_X(x, a_i) < \rho_{a_i}$ and $d_X(y, a_j) < \rho_{a_j}$. By the triangle inequality, $d_X(x, a_j) \leq d_X(x, y) + d_X(y, a_j) < \delta + \rho_{a_j} \leq 2\rho_{a_j} = \delta_{a_j}$. Thus, by the continuity of f at a_j, it follows by the triangle inequality that

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f(a_j)) + d_Y(f(y), f(a_j)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, f is uniformly continuous.

Theorem 6. If $S \subseteq X$ is connected and $f : X \to Y$ is continuous, then $f(S)$ is connected.

Proof. Suppose towards a contradiction $f(S)$ admits a disconnection (T_1, T_2). As each T_i is relatively open in $f(S)$, there is $U_i \subseteq Y$ for which $T_i = U_i \cap f(S)$. As f is continuous and each $U_i \subseteq Y$ is open, $f^{-1}(U_i)$ is open by Theorem 4. Consider $S_i = f^{-1}(U_i) \cap S$. Each S_i is relatively open in S by construction, we show (S_1, S_2) forms a disconnection of S. Suppose $x \in S_1 \cap S_2 = f^{-1}(U_1) \cap f^{-1}(U_2) \cap S$, namely $x \in f^{-1}(U_1 \cap U_2)$, and so $f(x) \in U_1 \cap U_2 \cap f(S) = T_1 \cap T_2 = \emptyset$, and so $S_1 \cap S_2 = \emptyset$. Lastly, $S_1 \cup S_2 \subseteq S$ is immediate, suppose $x \in S$, then $f(x) \in f(S) = T_1 \cup T_2$, so without loss of generality $f(x) \in T_1 = U_1 \cap f(S)$, and it follows that $f(x) \in U_1$ and hence $x \in f^{-1}(U_1) \cap S = S_1$. Thus (S_1, S_2) forms a disconnection of S, contradicting the connectedness of S. Thus, $f(S)$ is connected.

Theorem 7. If $K \subseteq X$ is compact and $f : X \to Y$ is continuous, then $f(K)$ is compact.

Proof. Fix an open cover $\mathcal{C} = \{U_i : i \in I\}$ of $f(K)$. As f is continuous and each $U_i \subseteq Y$ is open, $f^{-1}(U_i)$ is open by Theorem 4. Consider $\mathcal{P} = \{f^{-1}(U_i) : i \in I\}$, let $x \in K$, then $f(x) \in f(K)$ and hence there is $i \in I$ for which $f(x) \in U_i$, namely $x \in f^{-1}(U_i)$. Thus, $\mathcal{P}$ covers K, and as K is compact, $\mathcal{P}$ admits a finite subcover $\mathcal{P}' = \{f^{-1}(U_{i_1}), \ldots, f^{-1}(U_{i_n})\}$. We show that $\mathcal{C}' = \{U_{i_1}, \ldots, U_{i_n}\}$ is a finite subcover for $f(K)$. Indeed, fix $y \in f(K)$ and write $y = f(x)$ for $x \in K$. As $x \in K$, there is a subindex k such that $x \in f^{-1}(U_{i_k})$, and hence $y \in f(f^{-1}(U_{i_k})) \subseteq U_{i_k}$. Thus $\mathcal{C}'$ covers $f(K)$, and hence $f(K)$ is compact.