MAT157: Analysis I — Tutorial 10

Topics: Chain rule, Inverse function theorem.

Question 1. Compute the derivatives of each of the following functions using the chain rule.

(a) $f(x) = \left(1 + x^2 + \sqrt{5x+1} + \frac{1}{\sqrt[3]{2x}}\right)^{10}$

(b) $g(x) = \sin\left(x + \frac{1}{x}\sin\left(\frac{1}{x}\sin\left(\frac{1}{x}\right)\right)\right)$

(c) Find $p'(0)$ where $p(x) = \underbrace{(s \circ \cdots \circ s)}_{57 \text{ times}}(x)$, $s(0) = 0$ and $s'(0) = 2$.

Question 2. Suppose $U, V \subseteq \mathbb{R}$ are open intervals and $f : U \to V$ is a C^1 bijection for which $f'(x) \neq 0$ for all $x \in U$. Prove that f^{-1} is C^1 on V.

Question 3. In this question we will derive a formula for the derivative of arctan. Define the function $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ by $f(x) = \tan(x)$ so that f is a bijection with inverse $f^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ given by $f^{-1}(x) = \arctan(x)$. Show that f^{-1} is differentiable on $\mathbb{R}$ and find a formula for its derivative.

Question 4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is invertible. Under what conditions on f is f^{-1} twice differentiable on $\mathbb{R}$? What is $(f^{-1})''(x)$?

Bonus Problem. On term test 2, many claimed that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $p \in \mathbb{R}$ with $f'(p) > 0$, then f is increasing in a neighbourhood of p. We will show by example that this is unfortunately not the case. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that f is everywhere differentiable and that $f'(0) > 0$.

(b) Show that if a function $g : (a, b) \to \mathbb{R}$ is differentiable and increasing, then $g'(c) \geq 0$ for all $c \in (a, b)$

(c) Use part (b) to show that f is not increasing in any open neighbourhood of 0, namely, f is not increasing on $(-\varepsilon, \varepsilon)$ for any $\varepsilon > 0$.