MAT157: Analysis I — Tutorial 6

Topics: Continuity.

Question 1. Define the function $f : \mathbb{R} \to \mathbb{R}$ as follows

$$f(x) = \begin{cases} \frac{3x}{7-2x} & x < 2 \\ k & x = 2 \\ \frac{x^2}{4} + 1 & x > 2 \end{cases}$$

Determine the value of k that makes f continuous at $x = 2$ and prove your claim (prove that the left and right hand limits of f at 2 coincide with your choice of k)

Question 2. Suppose $g : \mathbb{R} \to \mathbb{R}$ is continuous at 0 with $g(0) = 0$. If $f : \mathbb{R} \to \mathbb{R}$ is a function for which there is $r > 0$ such that $|f(x)| \leq \sqrt{|g(x)|}$ for all $x \in (-r, r)$, prove that f is continuous at 0.

Question 3. Suppose $f, g, h : \mathbb{R} \to \mathbb{R}$ are functions.

(a) If h is continuous at $a \in \mathbb{R}$ such that $h(a) \neq 0$, prove that there is $\delta > 0$ such that $h(x) \neq 0$ for all $x \in (a - \delta, a + \delta)$. *Hint:* Assume without loss of generality that $h(a) > 0$.

(b) Suppose f and g are continuous, and $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Is it true that $f(x) = g(x)$ for all $x \in \mathbb{R}$? *Hint:* Define $h(x) = f(x) - g(x)$ and think about part (a).

Bonus Problem. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function such that $I = \{f(x) : x \in \mathbb{R}\}$ is dense. Prove that f is everywhere continuous.