MAT157: Analysis I — Tutorial 8

Topics: Value theorems, Derivatives.

Question 1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, and there is $\alpha > 0$ such that $f(x) = f(x + \alpha)$ for all $x \in \mathbb{R}$. Prove that there is $c \in \left[0, \frac{\alpha}{2}\right]$ such that $f(c) = f\left(c + \frac{\alpha}{2}\right)$.

Question 2. In this question, we will extend the intermediate value theorem.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and define

$$a = \lim_{x \to \infty} f(x), \quad b = \lim_{x \to -\infty} f(x).$$

If $ab < 0$, prove that there is $c \in \mathbb{R}$ such that $f(c) = 0$. *Hint:* if $a > 0$, show that $f(x_0) > 0$ for a sufficiently large x_0.

Question 3. Find all continuous functions $f : [0, 1] \to \mathbb{Q}$ such that $f(0) = 157$.

Question 4. Suppose U is an open neighbourhood of c, and $f : U \to \mathbb{R}$ is a function.

(a) If f is differentiable at c, prove that $f'(c) = \lim_{h \to 0} \dfrac{f(c) - f(c - h)}{h}$.

(b) If $\lim_{h \to 0} \dfrac{f(c) - f(c - h)}{h}$ exists, must f be differentiable at c?

Bonus Problem. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and injective. Must f be monotone?