MAT157: Analysis I — Tutorial 9

Topics: Derivatives, Differentiation rules

Question 1. Let $\chi_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ be the characteristic function of $\mathbb{Q}$, and define $f(x) = x\chi_{\mathbb{Q}}(x)$ and $g(x) = x^2\chi_{\mathbb{Q}}(x)$.

(a) Prove that f is not differentiable at 0.

(b) Prove that g is differentiable at 0. Is g differentiable at any other points? *Hint:* Is g continuous at any other points?

(c) Let $\alpha > 0$, and define $f_\alpha(x) = x^\alpha \chi_{\mathbb{Q}}(x)$. For which values of $\alpha > 0$ is f_α differentiable at 0?

Question 2. Find the derivatives of each of the following functions

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + x^{157} + 3$.

(b) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \dfrac{x^{157} + 1}{x^2 + 2}$.

(c) $h : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $h(x) = \dfrac{1 + x^2 \cos(x)}{x + \sin(x)}$.

Question 3. Consider a new definition of differentiability. We say that a function $f : U \to \mathbb{R}$ is *super-differentiable* at $c \in U$ if there is a function $\phi : U \to \mathbb{R}$ such that ϕ is continuous at c and $f(x) = f(c) + (x - c)\phi(x)$ for all $x \in U$, in which case the super derivative of f at c is $f^\circ(c) = \phi(c)$. Is there a relationship between regular differentiability and super differentiability?

Bonus Problem. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable on $\mathbb{R} \setminus \mathbb{Z}$. *Hint:* Think about $|x|$ and start by drawing a picture of what such a function would look like.