MAT157: Analysis I — Tutorial 3

Topics: Suprema/Infima, Density, Dedekind cuts.

Question 1. Find the suprema/infima of the following sets, if they exist.

(a) $[0, 1)$.

(b) $(0, 1]$.

(c) $\left\{\frac{1}{2n+1} : n \in \mathbb{Z}\right\}$.

(d) $\left\{(-1)^n - \frac{1}{n} : n \in \mathbb{N}\right\}$.

(e) $\emptyset$.

(f) $\mathbb{Q} \cap [0, 1)$.

Question 2. Let $A \subseteq \mathbb{R}$ be nonempty and bounded below. Prove that $\inf(A)$ exists, and that $\inf(A) = -\sup(-A)$ where $-A = \{-a : a \in A\}$. *Note:* This shows that we do not need to take an analogous completeness axiom for infima.

Question 3. Let $A \subseteq \mathbb{R}$ be nonempty, bounded below, and let m be a lower bound for A. Prove that $m = \inf(A)$ if and only if for every $\varepsilon > 0$, there is $a \in A$ such that $a < m + \varepsilon$.

Question 4. Let $A \subseteq \mathbb{R}$ be nonempty, bounded above, and let $c \in \mathbb{R}$. We define $c+A = \{c+a : a \in A\}$. Show that $\sup(c + A) = c + \sup(A)$.

Question 5. In this question, we prove that the irrationals $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

(a) Let $q \in \mathbb{Q} \setminus \{0\}$. Show that $q\sqrt{2} \notin \mathbb{Q}$.

(b) Let (a, b) be an open interval. Argue that $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ contains a non-zero rational number.

(c) Use part (a) to conclude that $(a, b) \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$.

Question 6. Here's another way to prove that $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

(a) Show that $|(a, b)| = |(0, 1)|$ for any open interval (a, b).

(b) Show that any open interval (a, b) must contain an irrational by contradiction.