MAT157: Analysis I — Tutorial 1

Topics: Logic, Quantifiers, Sets and Functions.

Logic and Quantifiers

Question 1. Express each of the following English statements in logical symbols.

1. For every real number, there exists a real number such that their product is 1.
2. There is a real number that is smaller than the square of any natural number.
3. If a and b are real numbers, and $a-b$ is negative, then a must be greater than b.
4. There is an integer that is its own square, and is neither one nor zero.
5. The product of two integers is zero if and only if at least one of them is zero.

Question 2. Write, using logical symbols, the negation of each of the following statements without using $\neg$.

1. There is a largest natural number.
2. $(\exists M > 0)(\forall x \in [-1, \infty))[\sqrt{x+1} > M]$
3. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})[m < 2^n < m+1]$
4. $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[|x-2| < \delta \Rightarrow |x^2-4| < \varepsilon]$
5. For any three real numbers x, y and z, if the distance from x to y and the distance from y to z are both less than 5, then the distance from x to z is less than 5.

Question 3. Determine whether each of the following statements are true or false. If you believe a statement is true, provide a proof, and if false, prove the negatation is true. If the statement is written in English, express it in logical symbols as well.

1. $(\exists M > 0)(\forall x \in (0, \infty))[\sqrt{x} + 2 \geq M]$
2. There is a largest natural number.
3. $(\exists x \in (0,1))(\forall y \in (0,1))[y \leq x]$
4. If x is a solution to $x^3 + 5x = 40$, then x must be less than 3.
5. $(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})[a \leq b \iff (\forall \varepsilon > 0)[a < b + \varepsilon]]$

Sets and Functions

Question 4. Fix a universal set U, and let $A, B, C \subseteq U$ be sets. Prove each of the following statements.

1. $(A^c)^c = A$.
2. $A \subseteq B \iff B^c \subseteq A^c$.
3. $A \subseteq B \iff A \cup B = B$.
4. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
5. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Question 5. Let $f : B \to C$ and $g : A \to B$ be functions.

1. Use the contrapositive to establish an equivalent definition of injectivity.
2. If f and g are surjective, prove that $f \circ g$ is surjective.
3. If $f \circ g$ is surjective, prove that f must be surjective.

Bonus Problem. Once again, let $f : B \to C$ and $g : A \to B$ be functions. In lecture, you showed that if $f \circ g$ is injective, then g must be injective. Show by example that f need not be injective. More precisely, construct sets A, B, C and functions $f : B \to C, g : A \to B$ for which $f \circ g$ is injective and f is *not* injective.