MAT157: Analysis I — Tutorial 11

Topics: Mean value theorem and friends

Question 1. Suppose $f : (0, 3) \to \mathbb{R}$ is differentiable such that $f(1) = -1$ and $f'(x) > 1$ for all $x \in (0, 3)$. Prove that f has a root.

Question 2. Suppose $f : [a, c] \to \mathbb{R}$ is continuous, and for $b \in (a, c)$, f is differentiable on $(a, b) \cup (b, c)$ and not differentiable at b. If $f'(x) > 0$ for all $x \in (a, b) \cup (b, c)$, prove that f is increasing on (a, c).

Question 3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $[a, b]$ and $f'(a)f'(b) < 0$. Prove that there is $c \in (a, b)$ such that $f'(c) = 0$. *Note:* f may not be C^1, so IVT will not work in this case. *Hint:* Assume without loss of generality $f'(a) < 0 < f'(b)$. Can either of a and b be points where f attains a maximum or minimum on $[a, b]$?

Question 4. Suppose $a \in \mathbb{R}$ and $f : [a, \infty) \to \mathbb{R}$ is continuous, and differentiable on (a, ∞). If

$$\lim_{x \to \infty} f(x) = f(a)$$

Prove that there is $c > a$ such that $f'(c) = 0$.

Bonus Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function for which

$$\lim_{x \to \infty} f(x) = 157 \quad \text{and} \quad \lim_{x \to \infty} f'(x) = L$$

Determine the value of L and prove your result.

If you want a challenging *computational* question (for some reason), try the following :

Bonus Problem. Suppose $f : (0, \infty) \to \mathbb{R}$ satisfies $f'(x) = \frac{1}{x}$ for all $x > 0$, and $f(1) = 0$. Compute

$$\lim_{x \to 0} \left(\frac{1}{f(x + \sqrt{x^2 + 1})} - \frac{1}{f(x + 1)} \right)$$