MAT157: Analysis I — Tutorial 7

Topics: Uniform Continuity.

Question 1.

- Briefly describe the differences between continuity and uniform continuity.
- Show that if $f : U \to \mathbb{R}$ is uniformly continuous, then f is continuous.
- Find a function $f : U \to \mathbb{R}$ that is continuous but not uniformly continuous.

Question 2. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Question 3. In lecture, you proved that continuous functions on closed and bounded intervals are uniformly continuous. We will examine why the conditions on the domain are necessary. Consider $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$, and $g : [0, \infty) \to \mathbb{R}$ defined by $g(x) = x^2$.

(a) Prove that f and g are continuous. *Hint:* Limit laws.

(b) Prove that f is *not* uniformly continuous.

(c) Prove that g is *not* uniformly continuous.

(d) **Bonus:** Given $\alpha, \beta > 0$, define $f_\alpha : (0, \alpha) \to \mathbb{R}$, $f_\beta : [\beta, \infty) \to \mathbb{R}$ by $f_\alpha(x), f_\beta(x) = \frac{1}{x}$. Prove that f_α is *not* uniformly continuous for any $\alpha > 0$, and f_β is uniformly continuous for any $\beta > 0$.

Question 4. Prove that $f(x) = \sqrt{x^2 + 1}$ is uniformly continuous on $(0, 1)$. *Hint:* Make use of the theorem mentioned in Question 3.

Bonus Problem. Recall the following result from lecture: "If $f : [a, b] \to \mathbb{R}$ is continuous, then f is uniformly continuous." To prove this, for a given $\varepsilon > 0$ we defined the set

$$C_\varepsilon = \{c \in [a, b] : \exists \delta > 0, \forall x, y \in [a, c]; |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon\}$$

and you proved that $b = \sup C_\varepsilon$. We will complete the proof of the main result by showing that $b \in C_\varepsilon$, namely f is uniformly continuous on $[a, b]$.

(a) Argue that there is $\delta_1 > 0$ such that $|f(x) - f(b)| < \frac{\varepsilon}{2}$ for all $x \in (b - \delta_1, b]$.

(b) Argue that there is $c_0 \in C_\varepsilon$ such that $b - \frac{\delta_1}{2} < c \leq b$, and conclude that there is $\delta_2 > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in [a, c_0]$ such that $|x - y| < \delta_2$.

(c) Define $\delta = \min\left\{\frac{\delta_1}{2}, \delta_2\right\}$, and fix $x, y \in [a, b]$ such that $|x - y| < \delta$. If $x, y \in \left[a, b - \frac{\delta_1}{2}\right]$, argue that $|f(x) - f(y)| < \varepsilon$.

(d) Assume, without loss of generality, that $x \in \left[b - \frac{\delta_1}{2}, b\right]$. Argue that $|y - b| < \delta_1$ and conclude that $|f(x) - f(y)| < \varepsilon$.