MAT157: Analysis I — Tutorial 5

Topics: One sided limits, Limits at infinity and Infinite limits.

Question 1. Let $a \in \mathbb{R}$ and $f : U \to \mathbb{R}$ a function defined on an open interval containing a, except possibly at a. Write out the definitions of

$$\lim_{x \to a^+} f(x) = L \quad \text{and} \quad \lim_{x \to a^-} f(x) = L$$

and prove that $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$.

Question 2. Using the appropriate definition, prove each of the following.

(a) $\lim_{x \to 4^+} x\sqrt{x-4} = 0$.

(b) $\lim_{x \to \infty} \left(\frac{6x-2}{3x+7} \right) = 2$.

(c) $\lim_{x \to 1^-} \frac{x}{x^2-1} = -\infty$.

Question 3. Let $g : (0,1) \to \mathbb{R}$ be a bounded function such that $\lim_{x \to 0^+} g(x) = 0$. Define

$$f : (0,1) \to \mathbb{R}, \qquad f(x) = \begin{cases} g(x) & x \neq \frac{1}{n} \\ n^2 & x = \frac{1}{n}. \end{cases}$$

(a) Prove that $\lim_{x \to 0^+} f(x) \neq \infty$ and $\lim_{x \to 0^+} f(x) \neq 0$.

(b) **Bonus:** Prove that $\lim_{x \to 0^+} f(x)$ does not exist.

Question 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) Prove that $\lim_{x \to \infty} f(x) = L$ if and only if $\lim_{x \to 0^+} f\left(\frac{1}{x}\right) = L$.

(b) If $\lim_{x \to \infty} f(x) = L$, is it true that $\lim_{x \to 0} f\left(\frac{1}{x}\right) = L$? Provide a proof or a counter-example.