MAT157: Analysis I — Tutorial 18

Topics: Series and convergence tests.

Notational Convention: In the spirit of laziness, we will write $\sum_n a_n$ in place of $\sum_{n=1}^{\infty} a_n$, and explicitly include the indices when relevant.

Question 1. Suppose (a_n) is a sequence for which $\sum_n |a_n|$ converges. Prove that $\sum_n \frac{\sqrt{|a_n|}}{n}$ converges.

Question 2. Let a_n be a sequence of non-negative real numbers, and set $b_n = \frac{a_n}{1+a_n}$. Prove that $\sum_n a_n$ converges if and only if $\sum_n b_n$ converges. *Hint:* prove that if $(b_n) \to 0$, then $(a_n) \to 0$.

Question 3. Let (a_n) be a strictly positive sequence for which $\sum_n a_n$ diverges, and let $S_n = \sum_{k=1}^{n} a_k$ denote the partial sums of $\sum_n a_n$. Prove that $\sum_n b_n$ converges, where $b_n = \frac{a_n}{S_n^2}$.

Question 4. Suppose (a_n) and (b_n) are positive sequences such that $\left(\frac{a_n}{b_n}\right) \to 0$.

(a) Prove that if $\sum_n b_n$ converges, then $\sum_n a_n$ converges.

(b) Formulate and prove an analogous statement in the case that $\left(\frac{a_n}{b_n}\right) \to \infty$.

Bonus Problem. Compute
$$\sum_{k=0}^{\infty} \left[\frac{\displaystyle\int_2^4 \frac{\sqrt{\log(9-x)}}{\sqrt{\log(9-x)} + \sqrt{\log(3+x)}}\,dx}{\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n(n+1)} \int_{2n}^{4n+2} \frac{\sqrt{\log(9n+3-x)}}{\sqrt{\log(9n+3-x)} + \sqrt{\log(3n+1+x)}}\,dx} \right]^{k+1}$$