MAT157: Analysis I — Tutorial 16

Topics: Improper integrals, Sequences

Question 1. In this question, we will prove that $\int_2^\infty \frac{\sin(x)}{\log(x)}\,\mathrm{d}x$ converges conditionally.

(a) Prove that $\int_2^\infty \frac{\sin(x)}{\log(x)}\,\mathrm{d}x$ converges. *Hint:* Start with integration by parts, and prove the remaining integral is absolutely convergent by comparison.

(b) Assuming that $\int_2^\infty \frac{|\sin(x)|}{x}\,\mathrm{d}x$ diverges, which you will prove in your next assignment, argue that $\int_2^\infty \frac{|\sin(x)|}{\log(x)}\,\mathrm{d}x$ diverges.

Question 2. Suppose $S \subseteq \mathbb{R}$ is a nonempty set. We say that (a_n) is a *sequence in* S if $a_n \in S$ for all $n \in \mathbb{N}$. Let S be a finite, nonempty set, and (a_n) a sequence in S.

(a) If $(a_n) \to a$, prove that $a \in S$.

(b) Prove that (a_n) is convergent if and only if it is *eventually constant*, namely there is $a \in \mathbb{R}$ and $N \in \mathbb{N}$ for which $a_n = a$ for all $n \geq N$. (Past a certain point, all of the terms are the same)

Question 3. Suppose (a_n) is a sequence, let $(\overline{a}_n)$ denote the sequence of averages of (a_n), namely

$$\overline{a}_n = \frac{1}{n}\sum_{i=1}^{n} a_i = \frac{a_1 + \cdots + a_n}{n}$$

Prove that $(\overline{a}_n) \to a$.

Bonus Problem. Consider the sequence $(a_n) = \left(\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\right)$. Determine if a_n converges, and if so, what its limit is. *Hint:* Monotone convergence.