MAT157: Analysis I — Tutorial 17

Topics: Sequences, Properties of sequences

Question 1. Let (a_n) be a sequence, and denote by $(a_{2n}), (a_{2n-1})$ the sequences of even and odd indexed terms, respectively. Namely $(a_{2n}) = (a_2, a_4, a_6, \dots)$ and $(a_{2n-1}) = (a_1, a_3, a_5, \dots)$.

(a) Prove that $(a_n) \to a$ if and only if $(a_{2n}) \to a$ and $(a_{2n-1}) \to a$.

(b) Suppose $(a_{2n}) \to a, (a_{2n-1}) \to b$, and $m = \frac{a+b}{2}$. Prove that (b_n), defined by $b_n = |a_n - m|$, converges and find its limit.

Question 2. Let $\{I_n\}_{n\in\mathbb{N}}$ be a collection of closed intervals $I_n = [a_n, b_n]$, for which $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$. If $(b_n - a_n) \to 0$. Prove that there is $p \in \mathbb{R}$ for which

$$\bigcap_{n\in\mathbb{N}} I_n = \{p\}$$

Hint: By Assignment 2 question 2, you know that the intersection is non-empty, so it suffices to prove that it contains only one element.

Question 3. Suppose $x_1, \dots, x_p \in \mathbb{R}$, and for $n \in \mathbb{N}$, define $a_n = \sqrt[n]{|x_1|^n + \cdots + |x_p|^n}$. Prove that $(a_n) \to \max\{|x_1|, \dots, |x_p|\}$. *Hint:* Write $\max\{|x_1|, \dots, |x_p|\} = |x_j|$ for some index j, and use the squeeze theorem.

Question 4. Let S be a non-empty, bounded set, so that $\sup(S), \inf(S)$ exist. Prove that there are sequences (a_n) and (b_n) in S for which $(a_n) \to \sup(S)$ and $(b_n) \to \inf(S)$.

Bonus Problem

To start things off, we will introduce a "new" definition, that of a *Cauchy sequence.*

Definition 1. A sequence (a_n) is *Cauchy* if for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ for which $|a_n - a_m| < \varepsilon$ for all $n, m \geq N$.

Take a moment to parse the definition and understand what its saying, as well as what, if any, differences there are between it and the definition of convergence introduced in lecture.

Question 1. Describe what a Cauchy sequence is, in your own words. Give an example of a Cauchy sequence.

Question 2. (a) Suppose (a_n) is a Cauchy sequence. Prove that (a_n) is bounded.

(b) Suppose (b_n) is a sequence and $(b_n) \to b$. Prove that (b_n) is Cauchy.

The next result is a rather famous result in Analysis, known as the *Bolzano-Weierstrass Theorem.*

Theorem 2. If (a_n) is a bounded sequence, then there is $a \in \mathbb{R}$ and a subsequence (a_{n_k}) for which $(a_{n_k}) \to a$.

Proof. The proof is slightly technical, but I will scaffold it as follows:

(a) Set $M = \sup\{|a_n| : n \in \mathbb{N}\}$ which exists by assumption. Consider $[-M, 0]$ and $[0, M]$. Argue that at least one of these intervals must contain infinitely many elements of (a_n), and call it I_1. Similarly, bisect I_1, and argue that at least one of the two halves must contain infinitely many elements of (a_n), call that half I_2.

(b) Recursively, assuming we have I_n, we can bisect I_n, and denote I_{n+1} as the half of I_n containing infinitely many elements of (a_n) (being sure to repeat the argument as to why such a half exists).

(c) Argue that (I_n) is a nested collection of closed intervals, and determine $\ell(I_n)$, the length of I_n, concluding that Proposition 2 can be applied. Set $\{p\} = \bigcap_{n \in \mathbb{N}} I_n$.

(d) Construct a subsequence (a_{n_k}) of (a_n) (For each k, you'd want to choose some $a_{n_k} \in I_k$) such that $(a_{n_k}) \to p$ and conclude the result. (Remember that the subindices must be strictly increasing!)

$\square$

We're now ready to prove the main result. The statement is an if and only if, and one of the directions you proved in Question 2, namely that convergent sequences are Cauchy. We will focus on proving the converse using the results we have thus far.

Theorem 3. A sequence (a_n) is convergent if and only if it is Cauchy.

Proof. $(\Rightarrow)$ Done in Question 2.

$(\Leftarrow)$ By Question 2, (a_n) is bounded, and hence by Theorem 3, there is $a \in \mathbb{R}$ and a subsequence such that $(a_{n_k}) \to a$. Prove that $(a_n) \to a$. $\square$