MAT157: Analysis I — Tutorial 13

Topics: More Integration and Measure.

Question 1. Define the set $E = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$. Prove that $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & x \in E \\ 0 & \text{otherwise} \end{cases}$$

is integrable and compute $\displaystyle\int_0^1 f$. *Hint:* Where is f discontinuous?

Question 2. Let $E \subseteq \mathbb{R}$ be a measurable set, and for $c \in \mathbb{R}$, define the translated set

$$c + E = \{c + e : e \in E\}$$

Prove that $c + E$ is measurable and that $\mu(c + E) = \mu(E)$.

Question 3. Let $h : [a,b] \to \mathbb{R}$ be a bounded function. For an interval $I \subseteq [a,b]$, we define the *wiggle of h on I* as

$$\omega(h, I) = \sup_{x,y \in I} |h(x) - h(y)|$$

(a) For any interval $I \subseteq [a,b]$, prove that the wiggle of h coincides with the wobble of h defined in Assignment 9.

(b) Let $I \subseteq [a,b]$. Prove that there is $M > 0$ for which $|h(x)^2 - h(y)^2| \leq M|h(x) - h(y)|$ for all $x, y \in I$.

(c) Conclude that if $h : [a,b] \to \mathbb{R}$ is integrable, then so is h^2.

(d) Prove that the product of integrable functions $f, g : [a,b] \to \mathbb{R}$ is integrable. *Hint:* Expand $(f+g)^2$.

Bonus Problem. Given a set $E \subseteq \mathbb{R}$, we say that E is *Gordon measurable* if there is an interval $[a,b]$ containing E for which χ_E is integrable on $[a,b]$, and the Gordon measure of E is given by

$$\sigma(E) := \int_a^b \chi_E$$

(a) Prove that the Gordon measure is independent of the interval containing E.

(b) Prove that a set $E \subseteq \mathbb{R}$ is Gordon measurable if and only if Jordan measurable, and $\sigma(E) = \mu(E)$.