Question 1. Define the functions $f, g : [0, 2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x \neq 1 \\ 157 & x = 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 157 & x \notin \mathbb{Q} \end{cases}$$

Prove that f is integrable with $\int_0^2 f = 0$. Prove that g is not integrable.

Question 2. For a bounded function $f : [a, b] \to \mathbb{R}$, we define the *upper and lower integrals of f*, respectively, by

$$U(f) = \inf_P U(f, P) \quad \text{and} \quad L(f) = \sup_P L(f, P)$$

If $f : [a, b] \to \mathbb{R}$ is a bounded function for which $U(f) = L(f)$, prove that f is integrable.

Question 3. If $f : [a, b] \to \mathbb{R}$ is monotone, prove that f is integrable.

Bonus Problem. Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions such that f is integrable on $[t, b]$ for all $t \in (a, b)$, and g is integrable on $[a, s]$ for all $s \in (a, b)$ Prove that f and g are integrable, and that

$$\int_a^b f = \lim_{t \to a^+} \int_t^b f \quad \text{and} \quad \int_a^b g = \lim_{s \to b^-} \int_a^s f$$