MAT157: Analysis I — Tutorial 21

Topics: Series of functions and uniform convergence.

Question 1. Let (f_n) be a sequence of functions on I for which $\sum_n f_n(x)$ is uniformly Cauchy on I. Prove that

$$\lim_{n\to\infty} \sup_{x\in I} |f_n(x)| = 0$$

Hint: Use question 2 from last week and mimic the proof of the test for divergence done in lecture.

Question 2. Here we will study the relationship between series convergence and integration.

(a) Let (f_n) be a sequence of integrable functions on $[a,b]$, for which $\sum_{n=1}^{\infty} f_n(x) \to f(x)$ uniformly on $[a,b]$. Prove that f is integrable and

$$\int_a^b f(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,\mathrm{d}x$$

(b) Show that $\displaystyle\int_0^{\pi} \left(\sum_{n=1}^{\infty} \frac{n\sin(nx)}{e^n} \right) \mathrm{d}x = \frac{2e}{e^2-1}$.

Question 3. Here we will formulate a relationship between series convergence and differentiation.

(a) Let (f_n) be a sequence of differentiable functions on $[a,b]$. Using the result of Assignment 16 Question 2, formulate conditions on (f_n) to guarantee that we can write

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=1}^{\infty} f_n(x) \right) = \sum_{n=1}^{\infty} f_n'(x)$$

(b) Find an expression for $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=1}^{\infty} \frac{1}{n^2} \arctan\left(\frac{x}{n}\right) \right)$ on $[1,10]$.