MAT157: Analysis I — Tutorial 15

Topics: Integration techniques, Improper integrals.

Question 1. Compute each of the following integrals.

(a) $\displaystyle\int_2^{57} \frac{1}{x\log(x)^3}\,\mathrm{d}x$ (b) $\displaystyle\int \arctan(x)\,\mathrm{d}x$ (c) $\displaystyle\int_{-1}^{2} \frac{e^{1/x}}{x^2}\,\mathrm{d}x$

Question 2. Suppose $f, g : [a, \infty) \to \mathbb{R}$ are integrable and non-negative. Prove that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0 \quad \text{and} \quad \int_a^\infty g(x)\,\mathrm{d}x = 157 \quad \implies \quad \int_a^\infty f(x)\,\mathrm{d}x \quad \text{converges}$$

Question 3. In this question, we will prove an important lemma in Statistical Learning Theory, Stein's identity. Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded, differentiable function, $\mu \in \mathbb{R}$ and $\sigma > 0$.

(a) Compute $\displaystyle\int \frac{(x-\mu)}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}\mathrm{d}x$. *Hint:* Make a substitution.

(b) Using integration by parts, prove that

$$\int_{\mathbb{R}} g(x)\frac{(x-\mu)}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}\mathrm{d}x = \sigma^2 \int_{\mathbb{R}} \frac{g'(x)}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}\mathrm{d}x$$

Question 4. Let $f, g : [a, b] \to \mathbb{R}$ be such that f is continuous and g is C^1 and increasing. Show that there is a point $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,\mathrm{d}x = g(a)\int_a^c f(x)\,\mathrm{d}x + g(b)\int_c^b f(x)\,\mathrm{d}x.$$

Bonus Problem. If $f : [a, b] \to \mathbb{R}$ is integrable and strictly positive everywhere, is it true that $\displaystyle\int_a^b f > 0$?