MAT157: Analysis I — Tutorial 20

Topics: Sequences of functions and uniform convergence.

Question 1. Explain briefly why uniform convergence implies pointwise convergence. Conversely, for each $n \in \mathbb{N}$, consider $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n$, and $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$$

Prove that $(f_n) \to f$ pointwise, and that the convergence in *not* uniform.

Question 2. Given a sequence of functions (f_n) on $I \subseteq \mathbb{R}$, we will say that (f_n) converges to f *supremely* on I if the sequence (b_n) defined by

$$b_n = \sup_{x \in I} |f_n(x) - f(x)|$$

converges to 0. Prove that $(f_n) \to f$ uniformly on I if and only if $(f_n) \to f$ supremely on I.

Question 3. Provide an example showing that integrability is *not* preserved by pointwise convergence. *Hint:* Construct a sequence of functions on $[0,1]$ each with finitely many discontinuities that converges to the characteristic function of $\mathbb{Q} \cap [0,1]$.

Question 4. Let (f_n) be a sequence of C^1 functions on $[a,b]$, and $f, g : [a,b] \to \mathbb{R}$. Prove if $(f_n) \to f$ pointwise and $(f_n') \to g$ uniformly, then f is C^1 on $[a,b]$ and $f' = g$. *Note:* On the assignment, you're proving a stronger version of this result, where we are not assuming f is C^1, only that f is differentiable.

Bonus Problem. Recall we say that a function $f : S \to \mathbb{R}$ is *Lipshcitz* on S if there is $M > 0$ for which $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in S$

(a) Prove that Lipschitz functions are uniformly continuous, and provide an example showing that the converse is not true.

(b) Prove that uniform convergence preserves uniform continuity.

(c) More generally, Suppose (f_n) is a sequence of Lipschitz functions on $[a,b]$, each with factor $M_n > 0$, and $(f_n) \to f$ uniformly. If $\sup_n M_n$ exists, prove that f is Lipschitz. (Note that this is still true if we only assume pointwise convergence)

(d) Can the same be said if we no longer assume $\sup_n M_n$ exists? Prove the result or provide a counterexample. *Note:* This is tricky ☺