MAT157: Analysis I — Tutorial 14

Topics: FTC, Logs and Exponentials.

Question 1. Let $f(x) = \int_1^{5x^3} \sin(t^3)\,\mathrm{d}t$. Compute $\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{f(x^2)}^{157} \frac{1}{1+t^7}\,\mathrm{d}t\right]$. *Hint:* Write this as a composition of functions and use the chain rule.

Question 2. Consider the logarithm function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \log(x) = \int_1^x \frac{1}{t}\,\mathrm{d}t$.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $a \in \mathbb{R}$ with $f(a) > 0$. Find an expression for the derivative of $\log(f(x))$ at $x = a$.

(b) Determine all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$(f(x))^2 - 157 = \int_0^x ((f(t))^2 + (f'(t))^2)\,\mathrm{d}t$$

for all $x \in \mathbb{R}$. *Hint:* Differentiate both sides.

Question 3. Consider the exponential function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$.

(a) For a fixed $n \in \mathbb{N}$, determine the n^{th} order Taylor polynomial $p_{n,0}(x)$ for f centered at 0.

(b) Determine an expression for $\int_0^x p_{n,0}(t)\,\mathrm{d}t$ for $n \in \mathbb{N}$ and $x > 0$.

(c) Let $r_{n,0}$ denote the remainder of the approximation from part (a), namely $r_{n,0}(x) = f(x) - p_{n,0}(x)$. Prove that $r_{1,0}(x) > 0$ for $x > 0$.

(d) Use parts (b) and (c) to conclude that $r_{n,0}(x) > 0$ for all $x > 0$ and $n \in \mathbb{N}$. *Hint:* Induction.

Note: The statement in (d) result holds for all $x \in \mathbb{R}$, try to adapt the results for $x < 0$.

Bonus Problem. Suppose $f : [a, b] \to [c, d]$ is a positive valued function for which $\log(f)$ is integrable on $[\log(c), \log(d)]$. Is it true that f is integrable on $[a, b]$? Prove the result or provide a counterexample.